

# Macfarlane hyperbolic 3–manifolds

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We identify and study a class of hyperbolic 3–manifolds (which we call Macfarlane manifolds) whose quaternion algebras admit a geometric interpretation analogous to Hamilton’s classical model for Euclidean rotations. We characterize these manifolds arithmetically, and show that infinitely many commensurability classes of them arise in diverse topological and arithmetic settings. We then use this perspective to introduce a new method for computing their Dirichlet domains. We give similar results for a class of hyperbolic surfaces and explore their occurrence as subsurfaces of Macfarlane manifolds.

[11R52](#), [57M27](#), [57M99](#)

## 1 Introduction

Quaternion algebras over complex number fields arise as arithmetic invariants of complete orientable finite-volume hyperbolic 3–manifolds. Quaternion algebras over totally real number fields are similarly associated to immersed totally geodesic hyperbolic subsurfaces of these manifolds. The arithmetic properties of the quaternion algebras can be analyzed to yield geometric and topological information about the manifolds and their commensurability classes. For background information on this topic, see Maclachlan and Reid [16], Neumann and Reid [20] and Takeuchi [28].

In this paper we introduce an alternative geometric interpretation of these algebras, recalling that they are a generalization of the classical quaternions  $\mathbb{H}$  of Hamilton. The author [22] elaborated on a classical idea of Macfarlane [15] to show how an involution on the complex quaternion algebra can be used to realize the action of  $\text{Isom}^+(\mathfrak{H}^3)$  multiplicatively, similarly to the classical use of the standard involution on  $\mathbb{H}$  to realize the action of  $\text{Isom}^+(S^2)$ . Here we generalize this to a class of quaternion algebras over complex number fields and characterize them by an arithmetic condition. We define Macfarlane manifolds as those having these algebras as their invariants.

We establish the existence of arithmetic and nonarithmetic Macfarlane manifolds and, in each of these classes, infinitely many noncommensurable compact and noncompact

examples. We then develop a new algorithm for computing Dirichlet domains of Macfarlane manifolds and their immersed totally geodesic hyperbolic subsurfaces, using the duality between points and isometries that comes from the quaternionic structure.

## Main results

Let  $X$  be a complete orientable finite-volume hyperbolic 3-manifold. Let  $K$  and  $\mathcal{B}$  be the trace field and quaternion algebra of  $X$ , respectively. We say that  $X$  is *Macfarlane* if  $K$  is an imaginary quadratic extension of a (not necessarily totally) real field  $F$ , and the nontrivial element of the Galois group  $\mathfrak{G}(K : F)$  preserves the ramification set of  $\mathcal{B}$  and acts on it with no fixed points. We then show (in [Theorem 3.4](#) and [Corollary 4.2](#)) that this is equivalent to the existence of an involution  $\dagger$  on  $\mathcal{B}$  which, with the quaternion norm, naturally gives rise to a 3-dimensional hyperboloid  $\mathcal{I}_\Gamma \subset \mathcal{B}$  over  $F$ . Moreover,  $\dagger$  is unique and the action of  $\pi_1(X)$  by orientation-preserving isometries of  $\mathfrak{H}^3$  can be written quaternionically as

$$\pi_1(X) \curvearrowright \mathcal{I}_\Gamma, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

By comparison, via Hamilton's classical result one can use the standard involution  $*$  on  $\mathbb{H}$  to realize  $\text{Isom}^+(S^2)$  quaternionically [\[22\]](#) as

$$\text{PH}^1 \curvearrowright \mathbb{H}_0^1, \quad (\gamma, p) \mapsto \gamma p \gamma^*.$$

In [Section 4](#), we describe an adaptation of the main result for hyperbolic surfaces and show how, in certain instances, an immersion of a surface in a 3-manifold is sufficient for the 3-manifold to be Macfarlane. We then study other topological and arithmetic conditions under which Macfarlane manifolds arise, culminating in the following theorem.

- Theorem 1.1** (1) *Every arithmetic, noncompact manifold  $X$  is finitely covered by a Macfarlane manifold, where the index of the cover is at most  $|H(X, \mathbb{Z}/2)|$ .*
- (2) *Every arithmetic, compact manifold containing an immersed, closed totally geodesic subsurface is finitely covered by a Macfarlane manifold, where the index of the cover is at most  $|H(X, \mathbb{Z}/2)|$ .*
- (3) *Among the nonarithmetic manifolds, there exist infinitely many noncommensurable Macfarlane manifolds in each category of noncompact and compact.*

In [Section 5](#), we use the quaternion model to improve on existing algorithms for computing Dirichlet domains and illustrate this with some basic examples.

## 2 Preliminaries

See Bonahon [5] and Ratcliffe [23] as general references for preliminary information on hyperbolic geometry and its relevance to Kleinian and Fuchsian groups. See Quinn [22] for preliminary information on algebras with involution, the standard Macfarlane space and some additional historical context. See Voight [30] for a comprehensive treatment of quaternion algebras.

### 2.1 Quaternion algebras

Let  $K$  be a field with  $\text{char}(K) \neq 2$ . In this article  $K$  will usually be one of  $\mathbb{R}, \mathbb{C}$ , a  $p$ -adic field, or a number field. Throughout this article, a number field will always mean some  $K \subset \mathbb{C}$  with  $[K : \mathbb{Q}] < \infty$  and with a fixed embedding into  $\mathbb{C}$ , ie a *concrete number field* (in the sense of Neumann [19]).

**Definition 2.1** Let  $a, b \in K^\times$ , which we call the *structure constants* of the algebra. The *quaternion algebra*  $\left(\frac{a,b}{K}\right)$  is the associative  $K$ -algebra (with unity)  $K \oplus Ki \oplus Kj \oplus Kij$ , with multiplication rules  $i^2 = a$ ,  $j^2 = b$  and  $ij = -ji$ .

- (1) The *quaternion conjugate* of  $q$  is  $q^* := w - xi - yj - zk$ .
- (2) The (*reduced*) *norm* of  $q$  is  $n(q) := qq^* = w^2 - ax^2 - by^2 + abz^2$ .
- (3) The (*reduced*) *trace* of  $q$  is  $\text{tr}(q) := q + q^* = 2w$ .
- (4)  $q$  is a *pure quaternion* when  $\text{tr}(q) = 0$ .

We indicate conditions on the trace via subscript; for instance, given a subset  $E \subset \left(\frac{a,b}{K}\right)$ , we write  $E_0 = \{q \in E \mid \text{tr}(q) = 0\}$  and  $E_+ = \{q \in E \mid \text{tr}(q) > 0\}$ . We indicate conditions on the norm via superscript, for instance  $E^1 = \{q \in E \mid n(q) = 1\}$ .

**Proposition 2.2** [30, Section 3.3] *If  $K \subset \mathbb{C}$ , then there exists a faithful matrix representation of  $\left(\frac{a,b}{K}\right)$  into  $M_2(\mathbb{C})$ . Moreover, under any such representation,  $n$  and  $\text{tr}$  correspond to the matrix determinant and trace, respectively.*

We will be interested in the  $K$ -algebra isomorphism class of  $\left(\frac{a,b}{K}\right)$  (preserving the embedding of  $K$ ), which is not uniquely determined by the structure constants  $a$  and  $b$ .

**Theorem 2.3** [17, Section 2] (1) *Either  $\left(\frac{a,b}{K}\right) \cong M_2(K)$ , or  $\left(\frac{a,b}{K}\right)$  is a division algebra.*

- (2)  $\left(\frac{a,b}{K}\right) \cong M_2(K)$  if and only if there exist  $x, y \in K$  such that  $ax^2 + by^2 = 1$ .
- (3) For all  $x, y \in K^\times$ , we have  $\left(\frac{a,b}{K}\right) \cong \left(\frac{b,a}{K}\right) \cong \left(\frac{ax^2, by^2}{K}\right)$ .

It follows that for any  $K$ , there is the quaternion  $K$ -algebra  $(\frac{1,1}{K}) \cong M_2(K)$ . Moreover, if  $(\frac{a,b}{K})$  is not a division algebra, then its isomorphism class is unique and can be represented by  $(\frac{1,1}{K})$ . Thus we now focus on quaternion division algebras.

**Example 2.4** See [17, Sections 2.6 and 2.7] for proofs of (3) and (4) below.

- (1) Over  $\mathbb{R}$ , the only quaternion division algebra up to isomorphism is  $\mathbb{H} := (\frac{-1,-1}{\mathbb{R}})$ , Hamilton’s quaternions.
- (2) There are no quaternion division algebras over  $\mathbb{C}$ .
- (3) Over each  $p$ -adic field, there is a unique quaternion division algebra up to isomorphism.
- (4) Over each (concrete) number field, there are infinitely many nonisomorphic quaternion division algebras.

This raises the question of how to tell, when  $K$  is a number field, whether or not two quaternion  $K$ -algebras are isomorphic. This can be done by investigating the local algebras with respect to the places of  $K$ , in the following sense.

Let  $K$  be a (concrete) number field and let  $\mathcal{B} = (\frac{a,b}{K})$ . For a place  $v$  of  $K$ , let  $K_v$  be the completion of  $K$  with respect to  $v$ . To each  $v$ , we associate an embedding  $\sigma: K \hookrightarrow K_v$  as described in the following paragraph, and then define the localization of  $\mathcal{B}$  with respect to  $v$  as  $\mathcal{B}_v := \mathcal{B} \otimes_{\sigma} K_v$ .

If  $v$  is infinite (ie Archimedean), then it corresponds (up to complex conjugation) to an embedding of  $K$  into  $\mathbb{C}$  fixing  $\mathbb{Q}$ , under which the completion of the image of  $K$  is either  $\mathbb{R}$  or  $\mathbb{C}$ , and we define  $\sigma$  as the corresponding embedding. So if  $\sigma(K) \subset \mathbb{R}$  then  $\mathcal{B}_v = (\frac{\sigma(a),\sigma(b)}{\mathbb{R}})$ , which is isomorphic to  $\mathbb{H}$  or  $M_2(\mathbb{R})$ . If  $\sigma(K) \not\subset \mathbb{R}$  then  $\mathcal{B}_v = (\frac{\sigma(a),\sigma(b)}{\mathbb{C}})$ , which is always isomorphic to  $M_2(\mathbb{C})$ . If  $v$  is finite (ie non-Archimedean), then it corresponds to a prime ideal  $\mathfrak{P} \triangleleft \mathbb{Z}_K$ , where  $\mathbb{Z}_K$  is the ring of integers of  $K$ . In this case we define  $\sigma$  as the identity embedding into the corresponding  $p$ -adic field  $K_{\mathfrak{P}}$ ; thus  $\mathcal{B}_v := (\frac{a,b}{K_{\mathfrak{P}}})$ .

- Definition 2.5**
- (1)  $\mathcal{B}$  is *ramified* if it is a division algebra, and is *split* if  $\mathcal{B} \cong M_2(K)$ .
  - (2)  $\mathcal{B}$  is *ramified at  $v$*  (respectively, *split at  $v$* ) if  $\mathcal{B}_v$  is *ramified* (respectively, *split*).
  - (3)  $\text{Ram}(\mathcal{B})$  is the set of real embeddings and prime ideals that correspond to the places where  $\mathcal{B}$  is ramified.

The set  $\text{Ram}(\mathcal{B})$  provides the desired classification of isomorphism classes of quaternion algebras over number fields as follows.

- Theorem 2.6** [30, Section 14.6] (1)  $\mathcal{B}$  is split if and only if  $\text{Ram}(\mathcal{B}) = \emptyset$ .  
 (2)  $\text{Ram}(\mathcal{B})$  uniquely determines the isomorphism class of  $\mathcal{B}$ .  
 (3)  $\text{Ram}(\mathcal{B})$  is a finite set of even cardinality, and every such set of places of  $K$  occurs as  $\text{Ram}(\mathcal{B})$  for some  $\mathcal{B}$ .

**2.1.1 Involutions on quaternion algebras** We will have need to think about involutions on  $\mathcal{B}$  besides quaternion conjugation, so we include some basic notions of more general such involutions.

**Definition 2.7** An *involution* on  $\mathcal{B}$  is a map  $\star: \mathcal{B} \rightarrow \mathcal{B}$ ,  $x \mapsto x^\star$ , such that for all  $x, y \in \mathcal{B}$ ,

- (1)  $(x + y)^\star = x^\star + y^\star$ ,
- (2)  $(xy)^\star = y^\star x^\star$ , and
- (3)  $(x^\star)^\star = x$ .

**Definition 2.8** For a subset  $E \subset \mathcal{B}$ , the *set of symmetric elements of  $\star$  in  $E$*  is  $\text{Sym}(E, \star) := \{x \in E \mid x^\star = x\}$ .

- Definition 2.9** (1) An involution  $\star$  is *of the first kind* if  $K = \text{Sym}(K, \star)$ .  
 (2) An involution  $\star$  is *standard* if it is of the first kind and  $xx^\star \in K$  for all  $x \in A$ .  
 (3) An involution  $\star$  is *of the second kind* if  $K \neq \text{Sym}(K, \star)$ .

**Proposition 2.10** [14, Section I.2] Let  $\star$  be an involution on  $\mathcal{B}$ .

- (1)  $1^\star = 1$ .
- (2) If  $\star$  is of the first kind, then it is  $K$ -linear.
- (3) If  $\star$  is of the second kind, then  $[K : \text{Sym}(K, \star)] = 2$ .

## 2.2 The arithmetic of hyperbolic 3-manifolds

Let  $X$  be a complete orientable finite-volume hyperbolic 3-manifold, and from here on this is what we will mean by *manifold*. Then  $\pi_1(X) \cong \Gamma < \text{PSL}_2(\mathbb{C})$  for some discrete group  $\Gamma$  (ie  $\Gamma$  is a Kleinian group). Let  $\hat{\Gamma} := \{\pm\gamma \mid \{\pm\gamma\} \in \Gamma\} < \text{SL}_2(\mathbb{C})$ .

**Definition 2.11** (1) The *trace field* of  $\Gamma$  is  $K\Gamma := \mathbb{Q}(\{\text{tr}(\gamma) \mid \gamma \in \hat{\Gamma}\})$ .

(2) The *quaternion algebra* of  $\Gamma$  is  $B\Gamma := \{\sum_{\ell=1}^n t_\ell \gamma_\ell \mid t_\ell \in K\Gamma, \gamma_\ell \in \hat{\Gamma}, n \in \mathbb{N}\}$ .

**Remark 2.12** In the literature these are usually denoted by  $k_0\Gamma$  and  $A_0\Gamma$ , but we write them differently to avoid confusion with the notation for pure quaternions.

Then  $K\Gamma$  is a concrete number field, which, recall, means it is a fixed subfield of  $\mathbb{C}$ , and  $B\Gamma$  is a quaternion algebra over  $K\Gamma$  [16]. By Mostow–Prasad rigidity, these are manifold invariants in the sense that if  $\Gamma$  and  $\Gamma'$  are two discrete faithful representations of  $\pi_1(X)$ , then  $K\Gamma = K\Gamma'$  and  $B\Gamma \cong B\Gamma'$  via a  $K\Gamma$ -algebra isomorphism (though the converse does not hold). So we may also refer to them as the *trace field* and *quaternion algebra* of  $X$  up to homeomorphism.

**Definition 2.13** Let  $\Gamma^{(2)} := \langle \gamma^2 \mid \gamma \in \Gamma \rangle$ .

(1) The *invariant trace field* of  $\Gamma$  is  $k\Gamma := K\Gamma^{(2)}$ .

(2) The *invariant quaternion algebra* of  $\Gamma$  is  $A\Gamma := B\Gamma^{(2)}$ .

These likewise are invariants of  $X$ , but have the stronger property of being commensurability invariants. That is, if  $\Gamma$  is commensurable up to conjugation to some Kleinian group  $\Gamma'$ , then  $k\Gamma = k\Gamma'$  and  $A\Gamma \cong A\Gamma'$  (though the converse does not hold) [20].

We call  $X$  *arithmetic* if  $\Gamma$  is an arithmetic group in the sense of [6], but this admits the following alternative characterization, which is more useful for our purposes.

**Definition 2.14** [16] (1)  $\Gamma$  (or  $X$ ) is *derived from a quaternion algebra* if there exists a quaternion algebra  $\mathcal{B}$  over a field  $K$  with exactly one complex place  $\sigma$ , such that  $\mathcal{B}$  is ramified at every real place of  $K$ , and there exists an order  $\mathcal{O} \subset \mathcal{B}$  such that  $\Gamma$  is isomorphic to a finite-index subgroup of  $\text{PO}^1 := \mathcal{O}^1 / \pm 1$ .

(2)  $\Gamma$  (or  $X$ ) is *arithmetic* if it is commensurable up to conjugation to one that is derived from a quaternion algebra.

If  $\Gamma$  is derived from a quaternion algebra  $\mathcal{B}$  over a field  $K$ , then  $K = K\Gamma = k\Gamma$  and  $B\Gamma \cong B\Gamma \cong A\Gamma$ . If  $\Gamma$  is arithmetic, then  $\Gamma^{(2)}$  is derived from a quaternion algebra. In general,  $\Gamma^{(2)}$  is a finite-index subgroup of  $\Gamma$  [20].

While  $k\Gamma$  and  $A\Gamma$  are generally more suitable to the application of arithmetic, we will work instead with  $B\Gamma$  so that we may take advantage of the natural embedding

$\widehat{\Gamma} \leftrightarrow B\Gamma$ . (To simplify notation, and where it will not cause confusion, we will often refer to an element  $\{\pm\gamma\} \in \widehat{\Gamma}$  by a representative  $\gamma \in \Gamma$ .) Often,  $A\Gamma$  and  $B\Gamma$  coincide (though not always [24]).

**Proposition 2.15** (1) *We have  $k\Gamma = K\Gamma$  if and only if  $A\Gamma \cong B\Gamma$ .*

(2) *If  $\mathcal{H}^3/\Gamma$  is a knot or link complement, then  $A\Gamma \cong B\Gamma$ .*

**Proof** We prove (1); see [17, Section 4.2] for (2). The reverse implication is immediate. For the forward implication, note that  $A\Gamma \subset B\Gamma$  and both are 4-dimensional vector spaces, so if they are over the same field, then they must be the same. □

We now collect some important properties of these invariants.

**Theorem 2.16** [17, Section 8.2] (1) *If  $X$  is noncompact, then  $B\Gamma \cong (\frac{1,1}{k\Gamma})$  and  $A\Gamma \cong (\frac{1,1}{k\Gamma})$ .*

(2) *If  $X$  is noncompact and arithmetic, then there exists  $d \in \mathbb{N}$  such that  $k\Gamma = \mathbb{Q}(\sqrt{-d})$ .*

(3) *If  $X$  is compact and arithmetic, then  $A\Gamma$  is a division algebra.*

### 2.3 Dirichlet domains

We conclude our preliminary discussion, following [5, Section 7.4], by introducing the type of fundamental domain we will be interested in for our application in Section 5. Let  $\mathcal{X}$  be a model for hyperbolic 3-space upon which  $\Gamma$  acts faithfully.

**Definition 2.17** The *Dirichlet domain* for the action of  $\Gamma$  on  $\mathcal{X}$  centered at  $c$  is

$$\mathcal{D}_\Gamma(c) := \{p \in \mathcal{X} \mid d(c, p) \leq d(c, \gamma(p)) \text{ for all } \gamma \in \Gamma \setminus \text{Stab}_\Gamma(c)\}.$$

As long as  $\text{Stab}_\Gamma(c) = \{1\}$ , the Dirichlet domain  $\mathcal{D}_\Gamma(c)$  is a fundamental domain for the action of  $\Gamma$  on  $\mathcal{X}$ , and since for us  $\Gamma$  is torsion-free, this the case for all  $c$ . There is a more explicit characterization of  $\mathcal{D}_\Gamma(c)$ , for which the following notation is useful.

**Definition 2.18** For each  $\gamma \in \Gamma$ , let

- (1)  $g(\gamma)$  be the geodesic segment from  $c$  to  $\gamma(c)$ ,
- (2)  $\tilde{s}(\gamma)$  be the complete geodesic hyperplane perpendicularly bisecting  $g(\gamma)$ , and
- (3)  $E(\gamma) \subset \mathcal{X}$  be the half-space  $\{p \in \mathcal{X} \mid d(c, p) \leq d(c, \gamma(p))\}$  (so  $\tilde{s}(\gamma) = \partial E(\gamma)$ ).

Since  $\Gamma$  is geometrically finite, there is some finite minimal set  $S \subset \Gamma$  such that

$$\mathcal{D}_\Gamma(c) = \bigcap_{\gamma \in S} E(\gamma).$$

We say that  $\gamma$  contributes a side to  $\mathcal{D}_\Gamma(c)$  if  $\gamma \in S$  and for each of these, let  $s(\gamma) := \tilde{s}(\gamma) \cap \partial \mathcal{D}_\Gamma(c)$ , which we call the side contributed by  $\gamma$ . The idea is to understand  $X$  by studying  $\mathcal{D}_\Gamma(c)$  equipped with side-pairing maps on its boundary. These side-pairings are given by applying  $\gamma^{-1}$  to the side contributed by  $\gamma$ , for each  $\gamma \in S$ .

### 3 Macfarlane quaternion algebras and $\text{Isom}^+(\mathfrak{H}^3)$

Our goal in this section is to show that the arithmetic definition of Macfarlane manifolds admits the geometric interpretation of containing a hyperboloid model tailor-made for the action of  $\Gamma$ , made precise in Theorem 3.4. Let  $K$  denote a fixed subfield of  $\mathbb{C}$ , and let  $\mathcal{B}$  denote a quaternion algebra over  $K$ .

**Definition 3.1** We call  $\mathcal{B}$  Macfarlane if

- (1) there exist  $F \subset \mathbb{R}$  and  $d \in F^+$  such that  $K = F(\sqrt{-d})$ , and
- (2) the nontrivial element  $\sigma$  of  $\mathfrak{G}(K : F)$  preserves  $\text{Ram}(\mathcal{B})$  and for all  $v \in \text{Ram}(\mathcal{B})$ ,  $\sigma(v) \neq v$ .

The manifold  $X$  with Kleinian group  $\Gamma$  is Macfarlane if  $B\Gamma$  is Macfarlane.

**Remark 3.2** Since  $F$  is a fixed subfield of  $\mathbb{R}$ , the set  $F^+ := \{f \in F \mid f > 0\}$  is well defined. In the case where  $F$  is a (recall, concrete) number field, we are only concerned with the positivity of its elements under the identity embedding into  $\mathbb{R}$ ; ie we form  $F^+$  the same way regardless of whether or not  $F$  is totally real.

**Example 3.3** (1)  $(\frac{1,1}{\mathbb{C}})$  is Macfarlane because  $\mathbb{C} = \mathbb{R}(\sqrt{-1})$  and  $\text{Ram}(\frac{1,1}{\mathbb{C}}) = \emptyset$ .

(2) The figure-8 knot complement and its quaternion algebra  $(\frac{1,1}{\mathbb{Q}(\sqrt{-3})})$  are Macfarlane.

(3) The quaternion algebra  $\mathcal{B}$  over  $\mathbb{Q}(\sqrt{-5})$  that has ramification set  $\text{Ram}(\mathcal{B}) = \{(3, 1 + \sqrt{-5}), (3, 1 - \sqrt{-5})\}$  is Macfarlane because  $\sigma$  permutes the ramified places.

(4) The quaternion algebra

$$\mathcal{B} = \left( \frac{\sqrt{2}, \sqrt{2}}{\mathbb{Q}\sqrt{1-\sqrt{2}}} \right)$$



is Macfarlane. To see this, take  $F = \mathbb{Q}(\sqrt{2})$  and  $d = \sqrt{2} - 1$ , and notice that  $\text{Ram}(\mathcal{B})$  consists of the pair of conjugate real embeddings that take  $\sqrt{1 - \sqrt{2}}$  to  $\pm\sqrt{1 + \sqrt{2}}$ .

We now state our main result.

**Theorem 3.4**  $\mathcal{B}$  is Macfarlane if and only if it admits an involution  $\dagger$  such that  $\text{Sym}(\mathcal{B}, \dagger)$  (which we denote by  $\mathcal{M}$ ), equipped with the restriction of the quaternion norm, is a quadratic space of signature  $(1, 3)$  over  $\text{Sym}(K, \dagger)$ .

Moreover,  $\dagger$  is unique, and letting  $\mathcal{M}_+^1 = \{p \in \mathcal{M} \mid \text{tr}(p) > 0, n(p) = 1\}$ , a faithful action of  $\text{PB}^1$  upon  $\mathfrak{H}^3$  by orientation-preserving isometries is defined by the group action

$$\mu_{\mathcal{B}}: \text{PB}^1 \curvearrowright \mathcal{M}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

**Remark 3.5** The isomorphism class of  $\mathcal{B}$ , as a quaternion algebra over the fixed subfield  $K \subset \mathbb{C}$ , does not include nonidentity embeddings  $K \hookrightarrow \mathbb{C}$ . Thus the signature of  $\text{Sym}(\mathcal{B}, \dagger)$  over  $\text{Sym}(K, \dagger)$  is well defined as long as  $\text{Sym}(K, \dagger)$  is real.

**Definition 3.6** The space  $\mathcal{M}$  as defined in Theorem 3.4 is called a *Macfarlane space*.

### Proof of Theorem 3.4

First we show that the existence of an involution as in the theorem is equivalent to a condition on the field and structure constants (as in Definition 2.1) of the algebra, up to isomorphism. Recall from Theorem 2.3 that even over a fixed field  $F(\sqrt{-d})$ , the structure constants  $a$  and  $b$  are not unique up to  $F(\sqrt{-d})$ -algebra isomorphism, nor are their signs. The idea is to show that there is a representative of this isomorphism class which normalizes the Macfarlane space to a convenient form.

**Lemma 3.7**  $\mathcal{B}$  admits an involution with the properties described in Theorem 3.4 if and only if  $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$  for some  $F \subset \mathbb{R}$  and  $a, b, d \in F^+$ .

The reverse direction of this, in the case where  $\mathcal{B} = \left(\frac{a,b}{F(\sqrt{-d})}\right)$ , is Theorem 7.2 of [22]. This generalizes to  $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$  because an isomorphism between quaternion algebras is also a quadratic space isometry with respect to the quaternion norms [30, Section 5.2], thus it transfers the multiplicative structure, the involution and the Macfarlane space.

So it suffices to prove the forward direction, and we do this via a series of claims. Let  $\mathcal{B}$  be a quaternion algebra over a field  $K$  and suppose  $\mathcal{B}$  admits an involution  $\dagger$  with the properties described in [Theorem 3.4](#).

**Claim 3.8**  $K$  is of the form  $F(\sqrt{-d})$ , where  $F = \text{Sym}(K, \dagger) \subset \mathbb{R}$  and  $d \in F^+$ , and  $\dagger|_K$  acts as complex conjugation.

**Proof** If  $K$  were real, then  $n$  would be a quadratic form of signature  $(2, 2)$ , making it impossible for  $\mathcal{B}$  to contain a subspace of signature  $(1, 3)$ , thus  $K \not\subseteq \mathbb{R}$ . On the other hand, for a space to have nontrivial signature over  $\text{Sym}(K, \dagger)$ , we must have  $\text{Sym}(K, \dagger) \subset \mathbb{R}$ . This means  $\dagger$  is an involution of the second kind, which implies  $[K : \text{Sym}(K, \dagger)] = 2$  by [Proposition 2.10](#), as desired.

We now show that  $\dagger|_K$  acts as complex conjugation. Since  $-d \in F = \text{Sym}(K, \dagger)$ ,

$$(\sqrt{-d}^\dagger)^2 = (\sqrt{-d}^2)^\dagger = (-d)^\dagger = -d,$$

thus  $\sqrt{-d}^\dagger = \pm\sqrt{-d}$ . Since  $\sqrt{-d} \notin \text{Sym}(K, \dagger)$ , this leaves  $\sqrt{-d}^\dagger = -\sqrt{-d}$ .  $\square$

Write  $\mathcal{M} = \text{Sym}(\mathcal{B}, \dagger)$ . We are going to use the signature of  $n|_{\mathcal{M}}$  to prove that  $\mathcal{B}$  has real structure parameters up to isomorphism, but a priori we do not know what  $\mathcal{M}$  is. So we will first need to establish that  $\mathcal{M}$  includes enough linearly independent elements of  $\mathcal{B}$ , in the following sense.

**Claim 3.9**  $\text{Span}_K(\mathcal{M}) = \mathcal{B}$ .

**Proof** We know that  $F \subset \mathcal{M}$  and  $\text{Span}_K(F) = K$ , so it suffices to prove that  $\text{Span}_K(\mathcal{M}_0) = \mathcal{B}_0$ .

Let  $E = \{s_1, s_2, s_3\}$  be a basis for  $\mathcal{M}_0$  over  $F$  and assume by way of contradiction that  $E$  is not linearly independent over  $K$ . Then there exists  $k_\ell \in K$  such that  $\sum_{\ell=1}^3 k_\ell s_\ell = 0$ . Since  $K = F(\sqrt{-d})$ , we have that each  $k_\ell = f_{\ell,1} + f_{\ell,2}\sqrt{-d}$  for some  $f_{\ell,1}, f_{\ell,2} \in F$ . Substituting these into  $\sum_{\ell=1}^3 k_\ell s_\ell = 0$  and rearranging terms, we get

$$f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = -\sqrt{-d}(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3).$$

But  $f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3$  and  $f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3$  both lie in  $\mathcal{M}$ , so are fixed by  $\dagger$ ; meanwhile by the previous claim,  $\sqrt{-d}^\dagger = -\sqrt{-d}$ . So applying  $\dagger$  to both sides of the equation gives

$$f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = \sqrt{-d}(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3).$$

Adding the last two displayed equations then gives that  $f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3 = 0$ . Since  $f_{1,2}, f_{2,2}, f_{3,2} \in F$ , this contradicts that  $E$  is a basis for  $\mathcal{M}_0$  over  $F$ .

We conclude that  $E$  is linearly independent over  $K$ , giving

$$\dim_K(\text{Span}_K(E)) = \dim_K(\text{Span}_K(\mathcal{M}_0)) = 3,$$

which forces  $\text{Span}_K(\mathcal{M}_0) = \mathcal{B}_0$ . □

**Claim 3.10**  $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$  for some  $a, b \in F^+$ .

**Proof** The norm  $n|_{\mathcal{M}}$  is a real-valued quadratic form of signature  $(1, 3)$ , so there exists an orthogonal basis  $D$  for  $\mathcal{M}$  such that the Gram matrix for  $n|_{\mathcal{M}}$  with respect to  $D$  is a diagonal matrix  $G_{n|_{\mathcal{M}}}^D$ , with diagonal of the form  $(f_1, -f_2, -f_3, -f_4)$  for some  $f_\ell \in F^+$ . Since  $\text{Span}_K(\mathcal{M}) = \mathcal{B}$ , this same  $D$  is also an orthogonal basis for  $\mathcal{B}$  over  $K$ .

Let  $C$  be the standard basis  $\{1, i, j, ij\}$  for  $\mathcal{B}$ . Then  $C$  is another orthogonal basis for  $\mathcal{B}$  over  $K$  and, in particular, the Gram matrix  $G_n^C$  for  $n$  with respect to  $C$  is the diagonal matrix with diagonal  $(1, -a, -b, ab)$ .

Now while  $G_{n|_{\mathcal{M}}}^D$  and  $G_n^C$  are not congruent over  $F$ , they are congruent over  $K$  because  $D$  and  $C$  are both bases for  $\mathcal{B}$ , ie there exists  $\delta \in \text{GL}_4(K)$  such that

$$\delta G_n^D \delta^T = G_n^C.$$

But since  $G_n^D$  and  $G_n^C$  are diagonal and nonzero on their diagonals,  $\delta$  must also be diagonal and nonzero on its diagonal, ie there exists  $x_\ell \in K^\times$  such that  $\delta$  is the diagonal matrix with diagonal  $(x_1, x_2, x_3, x_4)$ . Plugging in to the last displayed equation and solving for the  $f_\ell$  gives

$$f_1 = \frac{1}{x_1^2}, \quad f_2 = \frac{a}{x_2^2}, \quad f_3 = \frac{b}{x_3^2}, \quad f_4 = \frac{-ab}{x_4^2}.$$

Now let  $\mathcal{B}' = \left(\frac{f_2, f_3}{F(\sqrt{-d})}\right)$  and recall that  $f_2, f_3 \in F^+$ . Then  $\mathcal{B}$  has the desired form because, by [Theorem 2.3](#),

$$\mathcal{B}' \cong \left(\frac{f_2 x_2^2, f_3 x_3^2}{F(\sqrt{-d})}\right) = \mathcal{B}. \quad \square$$

This completes the proof of [Lemma 3.7](#).

Now to complete the proof of [Theorem 3.4](#), we show that the condition on the isomorphism class of the symbol  $(\frac{a,b}{K})$  from [Lemma 3.7](#) is equivalent to the arithmetic characterization of Macfarlane quaternion algebras given by [Definition 3.1](#).

**Lemma 3.11** *Let  $\mathcal{B}$  be a quaternion algebra over  $K = F(\sqrt{-d})$ , where  $F \subset \mathbb{R}$  and  $d \in F^+$ . The nontrivial element of  $\mathfrak{G}(K : F)$  preserves  $\text{Ram}(\mathcal{B})$  with no fixed points if and only if there exist  $a, b \in F^+$  such that  $\mathcal{B} \cong (\frac{a,b}{K})$ .*

**Proof** With  $a, b, F$  and  $K$  as in the statement, we have  $(\frac{a,b}{K}) = (\frac{a,b}{F}) \otimes_F K$ . Also if  $a$  (or  $b$ ) is negative, then by [Theorem 2.3](#), we can replace it by  $-ad$  (or  $-bd$ ) without changing the isomorphism class. So it suffices to prove that the condition on  $\text{Ram}(\mathcal{B})$  is equivalent to the existence of a quaternion algebra  $\mathcal{A}$  over  $F$  such that  $\mathcal{B} \cong \mathcal{A} \otimes_F K$ .

If there is such an  $\mathcal{A}$ , then  $\text{Ram}(\mathcal{B})$  is the set of pairs of real (respectively, non-Archimedean) places  $v, v'$  of  $K$  associated with a real place in  $\text{Ram}(\mathcal{A})$  that splits into two real places of  $K$  (respectively, associated with a prime ideal of  $\mathbb{Z}_F$  that splits in  $\mathbb{Z}_K$ ). By [Theorem 2.3\(3\)](#), this sets up a bijection between quaternion  $K$ -algebras of the form  $\mathcal{A} \otimes_F K$  and sets of pairs of places arising as described. In this situation, the nontrivial element  $\sigma$  of  $\mathfrak{G}(K : F)$  will stabilize  $\text{Ram}(\mathcal{B})$ , and interchange  $v$  and  $v'$ . Conversely, if  $\sigma$  preserves and acts freely on  $\text{Ram}(\mathcal{B})$ , then there are no inert or ramified primes as these are fixed by  $\mathfrak{G}(K : F)$ , forcing the ramified places of  $K$  to be as described above. □

This completes the proof of [Theorem 3.4](#). The following consequence has computational advantages which will be exploited in [Section 5](#).

**Corollary 3.12** *If  $\mathcal{B}$  is Macfarlane, then there is an isomorphism  $\mathcal{B} \cong (\frac{a,b}{F(\sqrt{-d})})$ , where  $a, b, d \in F^+$ . In this case the Macfarlane space is*

$$\mathcal{M} = F \oplus Fi \oplus Fj \oplus \sqrt{-d}Fij,$$

and for  $q = w + xi + yj + zij \in \mathcal{B}$  with  $w, x, y, z \in F(\sqrt{-d})$ , the involution  $\dagger$  is given by

$$(3-1) \quad q^\dagger = \bar{w} + \bar{x}i + \bar{y}j - \bar{z}ij.$$

A final remark on [Theorem 3.4](#) is that even though we are using  $\mathcal{M}_+^1$  as a hyperboloid model for the group action, it is technically not a model for hyperbolic 3-space unless

$F = \mathbb{R}$ . In the cases of interest,  $F$  is a number field embedded in  $\mathbb{R}$ , and this gives us all we need to model a Macfarlane manifold. If a complete model for hyperbolic 3-space were desired, one is given by  $(\mathcal{M} \otimes_F \mathbb{R})_+^1$ , but this would lose the arithmetic structure that makes Macfarlane manifolds interesting.

## 4 Macfarlane manifolds

In this section we explore the various conditions under which Macfarlane manifolds arise. First we clarify how [Theorem 3.4](#) translates to the context of Kleinian groups. Then we look at an adaptation of our results to hyperbolic surfaces and see some settings in which hyperbolic subsurfaces imply that manifolds are Macfarlane. The remainder of the section culminates in a proof of [Theorem 1.1](#), about the diverse settings in which Macfarlane manifolds arise.

As in [Section 2.2](#), let  $X$  denote a complete orientable finite-volume hyperbolic 3-manifold with Kleinian group  $\Gamma \cong \pi_1(X)$ . Let  $K = K\Gamma$  and  $\mathcal{B} = B\Gamma$ .

**Definition 4.1** When  $X$  is Macfarlane and  $\mathcal{M} \subset \mathcal{B}$  is its Macfarlane space as in [Theorem 3.4](#), define  $\mathcal{I}_\Gamma := \mathcal{M}_+^1$  and call this a *quaternion hyperboloid model* for  $\Gamma$  (or  $X$ ).

It is immediate that  $\mathcal{I}_\Gamma$ , up to quadratic space isometry over  $K\Gamma$ , is a manifold invariant. By the definition of  $B\Gamma = \mathcal{B}$ , there is no confusion in speaking of  $\Gamma$  quaternionically, as lying in  $P\mathbb{B}^1$  rather than in  $\mathrm{PSL}_2(\mathbb{C})$ . In this way,  $\Gamma$  (up to choice of representatives in  $\hat{\Gamma}$ ) and  $\mathcal{I}_\Gamma$  are both subsets of  $\mathcal{B}$ , making sense of the following.

**Corollary 4.2** *If  $X$  is Macfarlane, then the action of  $\Gamma$  by orientation-preserving isometries of  $\mathfrak{H}^3$  is faithfully represented by*

$$\mu_\Gamma: \Gamma \curvearrowright \mathcal{I}_\Gamma, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

### 4.1 Hyperbolic surfaces and subsurfaces

The author showed in [\[22\]](#) how the representation of  $\mathrm{Isom}^+(\mathfrak{H}^3)$  in  $(\frac{1}{\mathbb{C}})$  restricts to a representation of  $\mathrm{Isom}^+(\mathfrak{H}^2)$  in  $(\frac{1}{\mathbb{R}})$ . Similarly, we seek analogs of [Theorem 3.4](#) and [Corollary 4.2](#) for hyperbolic surfaces. This is possible to some extent but we must proceed with care because of the following important differences between the 3-dimensional and 2-dimensional settings. For a complete orientable finite-volume hyperbolic surface  $S$ , the group  $\pi_1(S)$  admits discrete faithful representations into

$\mathrm{PSL}_2(\mathbb{R})$  but, in the absence of Mostow–Prasad rigidity, the trace of some fixed hyperbolic element of  $\pi_1(S)$  could take any value in  $\mathbb{R}^{>2}$  under these representations, so the trace field is no longer a manifold invariant. We resolve this by requiring  $S$  to have a fixed immersion into a hyperbolic 3–manifold  $X$  under which it is totally geodesic, which implies an injection  $\pi_1(S) \hookrightarrow \pi_1(X)$ . We can do this in such a way that, taking the discrete faithful representation  $\pi_1(X) \cong \Gamma < \mathrm{PSL}_2(\mathbb{C})$ , we represent  $\pi_1(S) \cong \Delta < \Gamma$  as a fixed group of matrices.

We then define the (invariant) trace field of  $\Delta$  and (invariant) quaternion algebra of  $\Delta$  on this fixed representation in the same way as in Definitions 2.11 and 2.13, and we denote them similarly by  $(k\Delta)$   $K\Delta$  and  $(A\Delta)$   $B\Delta$ , respectively. These now have properties similar to what we saw in the Kleinian setting:  $K\Delta$  and  $k\Delta$  are now fixed subfields of  $\mathbb{R}$ ,  $B\Delta$  and  $A\Delta$  are quaternion algebras over  $K\Delta$  and  $k\Delta$ , respectively [27], and  $A\Delta$  is a commensurability invariant [17, Sections 4.9 and 5.3.2].

The results from [22, Section 6] along with the proof of Lemma 3.7 then give the corollary below, after the following observations. The field  $K\Delta$  is real, and so now the involution  $\dagger$  is of the first kind. That is,  $\mathrm{Sym}(K\Delta, \dagger) = K\Delta$  and  $\mathrm{Sym}(B\Delta, \dagger)$  is comprised of  $K\Delta$  and the unique 2–dimensional negative-definite subspace with respect to the norm on  $B\Delta$ .

**Corollary 4.3** *If  $B\Delta \cong \left(\frac{a,b}{K\Delta}\right)$  for some  $a, b > 0$ , then it admits an involution  $\dagger$  such that  $\mathrm{Sym}(B\Delta, \dagger)$  (which we denote by  $\mathcal{L}$ ), equipped with the restriction of the quaternion norm, is a quadratic space of signature  $(1, 2)$  over  $K\Delta$ .*

Moreover,  $\dagger$  is unique and, letting  $\mathcal{L}_+^1 = \{p \in \mathcal{L} \mid \mathrm{tr}(p) > 0, n(p) = 1\}$ , a faithful action of  $\Delta$  upon  $\mathfrak{H}^2$  by orientation-preserving isometries is defined by the group action

$$\mu_\Delta: \Delta \curvearrowright \mathcal{L}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

**Definition 4.4** We call the space  $\mathcal{L} \subset B\Delta$  as above a *restricted Macfarlane space*, and we call the space  $\mathcal{I}_\Delta := \mathcal{L}_+^1$  a *quaternion hyperboloid model* for  $\Delta$ .

We next give a way of realizing Macfarlane 3–manifolds using totally geodesic sub-surfaces. There is a stronger version of this in the arithmetic setting, which will be done in the next subsection.

**Proposition 4.5** *If  $X$  contains an immersed closed totally geodesic surface, and its trace field is  $F(\sqrt{-d})$  for some  $F \subset \mathbb{R}$  (not necessarily totally real) and  $d \in F^+$  (under the fixed embedding), then  $X$  is Macfarlane.*

**Proof** Let  $S \subset X$  be a surface as in the hypothesis. Then  $\pi_1(S)$  has a Fuchsian representation  $\Delta < \text{PSL}_2(\mathbb{R})$  and  $\pi_1(X)$  has a Kleinian representation  $\Gamma < \text{PSL}_2(\mathbb{C})$  such that  $\Delta < \Gamma$ . Then  $K\Delta \subset K \cap \mathbb{R}$  is a totally real subfield. Therefore  $B\Delta \subset B\Gamma$  is a quaternion subalgebra over a subfield of  $F$ . Hence there exist  $a, b \in F$  so that  $B\Delta = \left(\frac{a,b}{K\Delta}\right)$ . Then  $B\Delta \otimes_{K\Delta} K\Gamma = \left(\frac{a,b}{F(\sqrt{-d})}\right) \subset B\Gamma$ , and since  $B\Gamma$  is a 4-dimensional vector space over the same field, we have  $B\Gamma = \left(\frac{a,b}{F(\sqrt{-d})}\right)$ . Thus  $X$  is Macfarlane by Lemma 3.11, where we recall that, as in the proof of that lemma, if  $a$  (or  $b$ ) is negative, we can replace it with  $-da$  (or  $-db$ ).  $\square$

**Remark 4.6** With an immersion as above, the action  $\mu_\Gamma$  as given in Corollary 4.2 restricts to the action  $\mu_\Delta$  as given in Corollary 4.3. An example using this will be shown in Section 5.3.

### 4.2 Arithmetic Macfarlane manifolds

In this subsection, we construct examples of arithmetic Macfarlane manifolds and prove parts (1) and (2) of Theorem 1.1.

Pending the following arguments, notice that if  $X$  is the initial arithmetic manifold, the bound on the index of its cover by a Macfarlane manifold is at most  $|H(X, \mathbb{Z}/2)|$  because this is the index of the group  $\Gamma^{(2)}$  in  $\Gamma$  and, as discussed in Section 2.2, the invariant quaternion algebra of  $\Gamma$  is the quaternion algebra of  $\Gamma^{(2)}$ . The rest of the argument is separated into the noncompact and compact cases.

**4.2.1 Noncompact arithmetic Macfarlane manifolds** Recall from Theorem 2.16 that a noncompact manifold has a split quaternion algebra, thus the structure parameters can always be taken as  $a = b = 1$ . So the manifold will be Macfarlane provided it satisfies our condition on the trace field. In the arithmetic case, this is easy to control.

**Lemma 4.7** *Every arithmetic noncompact manifold is finitely covered by a Macfarlane manifold.*

**Proof** Let  $\Gamma$  be arithmetic and noncocompact. Then  $\Gamma$  is commensurable to a Bianchi group  $\text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$  ( $d \in \mathbb{N}$  square-free), which has quaternion algebra  $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$  by Theorem 2.16; see also [17, Section 8.2]. This is Macfarlane by Lemma 3.7.  $\square$

If  $\Gamma$  additionally is derived from a quaternion algebra, then  $B\Gamma$  will also be  $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$  and thus Macfarlane itself. Otherwise we can only say that  $A\Gamma = \left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$ , which is

why above we work up to commensurability. It is very common for these groups to satisfy  $K\Gamma = k\Gamma$  (so that  $B\Gamma = A\Gamma$ , making them Macfarlane), but this is not always the case [24]. For example, the arithmetic manifold  $m009$  in the cusped census has invariant trace field  $\mathbb{Q}(\sqrt{-7})$  but its trace field is  $\mathbb{Q}(\sqrt{(5 - \sqrt{-7})/2})$ , which contains no degree 2 subfield with a real place (making it not Macfarlane).

**Example 4.8** Arithmetic link complements are Macfarlane by Proposition 2.15:

- (1) The figure-8 knot complement is Macfarlane with trace field  $\mathbb{Q}(\sqrt{-3})$ .
- (2) The Whitehead link is Macfarlane with trace field  $\mathbb{Q}(\sqrt{-1})$ .
- (3) The six-component chain link is Macfarlane with trace field  $\mathbb{Q}(\sqrt{-15})$ .

For more information on arithmetic link complements, see [12; 3; 11] and Section 9.2 of [17].

**4.2.2 Compact arithmetic Macfarlane manifolds** Quaternion algebras of compact manifolds can be split or can be among the infinitely many ramified possibilities over each possible trace field. If we look at groups derived from quaternion algebras, we can easily construct examples of compact arithmetic Macfarlane manifolds in the usual way that these groups are formed.

Start with a concrete number field  $K = F(\sqrt{-d})$  with  $F \subset \mathbb{R}$  and  $d \in F^+$ , where  $K$  has a unique complex place. Choose a pair  $a, b \in K \cap \mathbb{R}$  such that  $(\frac{a,b}{K})$  is ramified, take an order  $\mathcal{O} \subset (\frac{a,b}{K})$ , then let  $\Gamma = \text{PO}^1$  and  $X = \mathcal{I}^3 / \Gamma$ . Naturally, not every isomorphism class of quaternion algebras over  $K$  admits such a choice of  $a, b$  but there are infinitely many that do. For instance, for each split non-Archimedean place of  $F$  that splits into a pair  $v_1, v_2$  of places of  $K$  (of which there are infinitely many, by the Chebotarev density theorem), choose the quaternion algebra  $B$  over  $K$  with  $\text{Ram}(B) = \{v_1, v_2\}$ . Thus there are infinitely many noncommensurable compact arithmetic Macfarlane manifolds over each such  $K$ .

We now look at a geometric condition that gives compact arithmetic Macfarlane manifolds.

**Lemma 4.9** *If  $X$  is arithmetic, compact and contains an immersed closed totally geodesic surface, then  $X$  is finitely covered by a Macfarlane manifold.*

**Proof** Let  $S \subset X$  be a surface as in the hypothesis. Then  $\pi_1(S) \cong \Delta' < \Gamma$  for some Fuchsian group  $\Delta'$ . Then by [17, Theorem 9.5.2],  $\Delta' < \Delta$  for some arithmetic



Fuchsian group  $\Delta$  satisfying  $[k\Gamma : k\Delta] = 2$  and  $k\Delta = k\Gamma \cap \mathbb{R}$ . This implies that  $K\Gamma^{(2)} = K\Delta^{(2)}(\sqrt{-d})$  for some  $d \in K\Delta^{(2)}$ . Lastly, we know that  $\Gamma^{(2)}$  is a finite-index subgroup of  $\Gamma$ , ie  $X$  finitely covers  $\mathcal{H}^3 / \Gamma^{(2)}$ , which contains the immersed closed totally geodesic surface corresponding to  $\Delta^{(2)}$ , so apply [Proposition 4.5](#) to  $\Gamma^{(2)}$ .  $\square$

### 4.3 Nonarithmetic Macfarlane manifolds

We now complete the proof of [Theorem 1.1](#) by providing an abundance of commensurability classes of nonarithmetic examples.

**Lemma 4.10** *There are infinitely many commensurability classes of noncompact nonarithmetic Macfarlane manifolds.*

**Proof** Recall that the quaternion algebra of a noncompact manifold is necessarily split ([Theorem 2.16](#)), so it suffices to satisfy the condition on the trace field.

In [7], an infinite class of noncommensurable link complements are generated, all having invariant trace field  $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$ , by gluing along totally geodesic 4-punctured spheres. Since this field is not of the form  $\mathbb{Q}(\sqrt{-d})$  with  $d \in \mathbb{N}$  and these manifolds are noncompact, they are nonarithmetic. By [Proposition 2.15](#), since they are link complements, their trace fields are also  $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$ , which is of the desired form.  $\square$

To realize nonarithmetic compact Macfarlane manifolds, we start with an arithmetic compact Macfarlane manifold having immersed totally geodesic subsurfaces. Then we apply the technique of interbreeding introduced by Gromov and Piatetski-Shapiro [10]. This entails gluing together a pair of noncommensurable arithmetic manifolds along a pair of totally geodesic and isometric subsurfaces, resulting in a nonarithmetic manifold. We use a variation on this technique introduced by Agol [1] called inbreeding, whereby one instead glues together a pair of geodesic subsurfaces bounding noncommensurable submanifolds of the same arithmetic manifold.

**Lemma 4.11** *For every arithmetic Macfarlane manifold  $X$  containing an immersed closed totally geodesic surface, there exist infinitely many commensurability classes of nonarithmetic Macfarlane manifolds with the same quaternion algebra as  $X$ .*

**Proof** Let  $X$  be as in the statement and  $\Gamma \cong \pi_1(X)$  be a Kleinian group. Then  $X$  contains infinitely many commensurability classes of immersed closed totally geodesic arithmetic hyperbolic subsurfaces [17, Section 9.5]. Since Kleinian groups are

LERF (locally extended residually finite) [2], among these are infinitely many pairs of surfaces where each pair is isometric but noncommensurable [1, Section 4]. Let  $S_1$  and  $S_2$  be one of these pairs. Since these are arithmetic, they each correspond to a lattice in a quaternion subalgebra over a real subfield of  $F$ , and these lattices are noncommensurable because of the noncommensurability of  $S_1$  with  $S_2$ .

Deform the lattices so that  $S_1$  and  $S_2$  can be glued together via the identification map  $f: S_1 \rightarrow S_2$  as in [1] and form the amalgamated product  $X_{1,2} := X *_f X$ . Since we did this by a deformation and gluing of noncommensurable lattices,  $X_{1,2}$  will have a systole inducing a matrix of nonintegral trace in any discrete faithful Kleinian representation of  $\Gamma_{1,2} := \pi_1(X_{1,2})$ . Thus  $X_{1,2}$  is nonarithmetic. However the reflection involution through the identified subsurface in  $X_{1,2}$  lies in the commensurator of  $\Gamma$ . So we have  $K\Gamma = K\Gamma_{1,2}$  and  $B\Gamma \cong B\Gamma_{1,2}$  over  $K\Gamma$ .

Lastly, by [1, Section 4], there exists an infinite sequence of choices for pairs of surfaces  $S_\ell, S_m$  as above so that the injectivity radius of  $X_{\ell,m}$  gets arbitrarily small. But by Margulis’s lemma, there is a lower bound to the injectivity radius of any class of commensurable nonarithmetic manifolds. Thus in our sequence  $\{X_{\ell,m}\}$ , we enter a new commensurability class infinitely often as this radius approaches 0, giving infinitely many noncommensurable nonarithmetic manifolds all with quaternion algebra  $B\Gamma$  (up to  $K\Gamma$ -algebra isomorphism). □

## 5 Applications

In this section, we give an example of a computational advantage of the quaternion hyperboloid model. To encourage further exploration of this and other potential applications, we first provide a way of transferring data between the current approach and the more conventional upper half-space model. Then we introduce a tool that improves algorithms for finding Dirichlet domains of Macfarlane manifolds.

Throughout this section, let  $X$  be Macfarlane. Choose  $\Gamma \cong \pi_1(X)$  so that  $\mathcal{B} := B\Gamma = \left(\frac{a,b}{F(\sqrt{-d})}\right)$  with  $F \subset \mathbb{R}$  and  $a, b, d \in F^+$  (guaranteed by Corollary 3.12), then identify  $\Gamma$  with its natural image in  $P\mathcal{B}^1$ . Let  $\dagger$  be the unique involution on  $\mathcal{B}$  so that  $\text{Sym}(\mathcal{B}, \dagger)$  is the Macfarlane space  $\mathcal{M}$ . Then

$$\mathcal{M} = F \oplus Fi \oplus Fj \oplus \sqrt{-d}Fij,$$

and  $\mathcal{I}_\Gamma = \mathcal{M}^1_+$  is a hyperboloid model for  $X$ . Recall the group action from Corollary 4.2:

$$\mu_\Gamma: \Gamma \times \mathcal{I}_\Gamma, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

### 5.1 Comparison with the upper half-space model

The following is an adaptation of [22, Theorem 5.2]; we omit the computational details.

The map

$$(5-1) \quad \rho_B: \mathcal{B} \hookrightarrow M_2(K(\sqrt{a}, \sqrt{b})),$$

$$w + xi + yj + zij \mapsto \begin{pmatrix} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{pmatrix},$$

is an injective  $F(\sqrt{-d})$ -algebra homomorphism, and let us also write  $\rho_B^{-1}$  to mean the inverse of the corestriction  $\rho|_{\rho(\mathcal{B})}: \mathcal{B} \xrightarrow{\cong} \rho_B(\mathcal{B}), q \mapsto \rho_B(q)$ . For future reference, this is

$$(5-2) \quad \rho_B^{-1}: \mathcal{B} \rightarrow M_2(K(\sqrt{a}, \sqrt{b})), \quad \begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \frac{v+s}{2} + \frac{v-s}{2\sqrt{a}}i + \frac{u+t}{2\sqrt{b}} + \frac{u-t}{2\sqrt{ab}}.$$

Write the upper half-space model as the subspace  $\mathcal{H}^3 = \mathbb{R} \oplus \mathbb{R}I \oplus \mathbb{R}^+J$  of Hamilton’s quaternions, where  $I^2 = J^2 = -1$  and  $IJ = -JI$ . Then

$$(5-3) \quad \iota_\Gamma: \mathcal{I}_\Gamma \rightarrow \mathcal{H}^3, \quad w + xi + yj + \sqrt{-d}zij \mapsto \frac{y\sqrt{b} + z\sqrt{abd}I + J}{w + x\sqrt{a}},$$

is an isometry such that the Möbius action  $\Gamma \times \mathcal{H}^3 \rightarrow \mathcal{H}^3$  equals  $\iota(\mu_\Gamma(\rho_A^{-1}(\cdot), \iota^{-1}(\cdot)))$ . That is, our quaternion representation  $\mu_\Gamma$  of the group action from Corollary 4.2 transfers to the usual Möbius action via  $\iota$  and  $\rho_B$ .

### 5.2 Quaternion Dirichlet domains

In this subsection we will see a shortcut to a method of computing Dirichlet domains, introduced by Page [21], for Macfarlane manifolds. We focus on Page’s algorithm because earlier ones have been specific to the noncompact arithmetic case [13; 4; 26; 25], or have required either arithmeticity [8; 29; 21] or compactness [9; 18], whereas, as we showed in Section 4, Macfarlane manifolds include examples from every combination of arithmetic and nonarithmetic with compact and noncompact.

While Page’s algorithm focuses on arithmetic Kleinian groups using their characterization within quaternion orders, it extends easily to arbitrary (complete orientable finite-covolume Kleinian) groups, as follows. Given a finite set of matrix generators for the group, Page [21, Section 2.4.1] indicates an efficient way of listing the elements of the group in order of increasing Frobenius norm. This has the effect of locating sides of a Dirichlet domain centered at  $(0, 0, 0)$  in the Poincaré ball model, in order of increasing distance from the center.

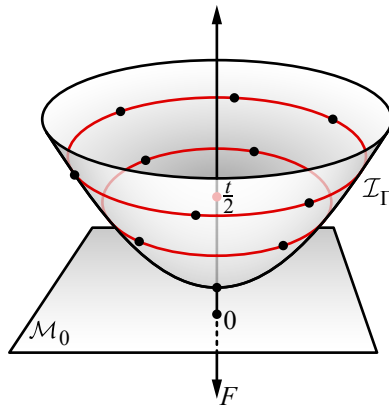


Figure 1: A 2–dimensional analog of grouping orbit points by trace

**5.2.1 Page’s algorithm translated to  $\mathcal{I}_\Gamma$**  Via the usual identification of the Poincaré ball model with  $\mathcal{H}^3$ , the point  $(0, 0, 0)$  maps to  $J \in \mathcal{H}^3$ , and  $\iota_\Gamma^{-1}$  from display (5-3) maps  $J$  to the point  $1 \in \mathcal{I}_\Gamma$ , which is the canonical choice for our center. So we will be computing the (canonical) Dirichlet domain  $\mathcal{D}_\Gamma(1) \subset \mathcal{I}_\Gamma$ .

An isometry  $\gamma \in \Gamma$  acts on  $1$  via  $\mu_\Gamma(\gamma, 1) = \gamma\gamma^\dagger$ , and we can interpret the Frobenius norm of  $\gamma$  as the square root of the trace of this image, as follows.

**Proposition 5.1** *If  $\gamma = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$ , then the trace of  $\rho_B^{-1}(\gamma)(1)$  is*

$$|r|^2 + |s|^2 + |u|^2 + |v|^2,$$

*the square of the Frobenius norm of  $\gamma$ .*

**Proof** With  $\gamma$  as in the statement,

$$\rho_B(\rho_B^{-1}(\gamma)(1)) = \gamma\gamma^\dagger = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} \bar{r} & \bar{u} \\ \bar{s} & \bar{v} \end{pmatrix} = \begin{pmatrix} |r|^2 + |s|^2 & r\bar{u} + s\bar{v} \\ u\bar{r} + v\bar{s} & |u|^2 + |v|^2 \end{pmatrix}.$$

Then  $\text{tr}(\rho_B^{-1}(\mu_\Gamma(\gamma, 1))) = \text{tr}(\gamma\gamma^\dagger) = |r|^2 + |s|^2 + |u|^2 + |v|^2$ . □

The set of points in  $\mathcal{I}_\Gamma$  of some fixed trace  $t$  forms an ellipsoid, in particular,

$$(5-4) \quad \left\{ \frac{1}{2}t + xi + yj + z\sqrt{-d}ij \mid x, y, z \in F, 4ax^2 + 4by^2 + 4abdz^2 = t^2 - 4 \right\}.$$

Since  $\Gamma$  is discrete, the set of points lying in each ellipsoid (which is compact) must be finite (Figure 1 illustrates a 2–dimensional version of this). Thus, in our setup, Page’s algorithm translates to searching for points in the orbit  $\text{Orb}_\Gamma(1) = \{\gamma\gamma^\dagger \mid \gamma \in \Gamma\}$  in



Let  $p = \frac{1}{2} \operatorname{tr}(q) + \lambda p_0 \in \tilde{g}$  in this notation; then since  $\gamma \in \mathcal{M} = \operatorname{Sym}(\mathcal{B}, \dagger)$ , we have

$$\mu_\Gamma(\gamma, q) = \gamma q \gamma^\dagger = \gamma q \gamma = \left(\frac{1}{2} \operatorname{tr}(\gamma) + \gamma_0\right) \left(\frac{1}{2} \operatorname{tr}(q) + \lambda \gamma_0\right) \left(\frac{1}{2} \operatorname{tr}(\gamma) + \gamma_0\right).$$

If we expand this, there will be scalars  $r, s, t, u \in F$  such that the expression has the form

$$\mu_\Gamma(\gamma, q) = r + s\gamma_0 + t\gamma_0^2 + u\gamma_0^3 = (r + t\gamma_0^2) + (s + u\gamma_0^2)\gamma_0.$$

But since  $\gamma_0^2 = \gamma_0(-\gamma_0^*) = -n(\gamma_0) \in F$ , this is indeed a point in  $\tilde{g}$  as characterized by (5-5).

Next we show that  $\gamma$  is the midpoint between 1 and  $\mu_\Gamma(\gamma, 1)$ . Let  $\delta$  be the hyperbolic translation along  $\tilde{g}$  such that  $\mu_\Gamma(\delta, 1) = \gamma$ . Since  $\gamma$  is a hyperbolic translation along the same geodesic, it commutes with  $\delta$ , giving  $\delta(\gamma) = \delta\gamma\delta^\dagger = \gamma\delta\delta^\dagger = \gamma\delta(1) = \gamma^2$ . Therefore  $d(1, \gamma)$  and  $d(\gamma, \gamma^2)$  both equal the translation length of  $\delta$ . This proves (2), and also shows that every purely hyperbolic translation along a geodesic through 1 occurs as a point on  $\mathcal{I}_\Gamma$ , completing the proof of (1).

For part (3), observe that if  $\gamma, \gamma^\dagger \in \Gamma$ , then  $\gamma(1) = \gamma\gamma^\dagger \in \Gamma$ . □

**Corollary 5.3** *If  $\gamma \in \Gamma \cap \mathcal{I}_\Gamma$ , then  $\tilde{s}(\gamma)$  perpendicularly bisects  $g(\gamma)$  at  $\gamma$ , so  $\gamma$  is the closest point to 1 on  $\tilde{s}(\gamma)$ .*

We can use this to shorten the computation of  $\mathcal{D}_\Gamma(1)$  by first approximating it by the bisectors of the elements of  $\Gamma \cap \mathcal{I}_\Gamma$ , and we are guaranteed that any region outside of those does not lie in  $\mathcal{D}_\Gamma(1)$ . As we do this, we can skip the computation of their Frobenius norm, and just look through the matrices in  $\Gamma \cap \mathcal{I}_\Gamma$  in order of increasing trace. We can also skip the usually longer computation of perpendicular bisectors for these points, instead simply taking the hyperplane whose closest point to 1 is  $\gamma \in \Gamma \cap \mathcal{I}_\Gamma$ . As we will see in the examples at the end of this section, this often gives a lot of information so that completing the Dirichlet domain computation is easy from there.

This method is especially effective in the event that  $\Gamma$  is closed under complex conjugation. However it is important to note that even in that case, an element  $\gamma$  one finds by searching through  $\Gamma \cap \mathcal{I}_\Gamma$  in order of increasing trace does not necessarily contribute a side to  $\mathcal{D}_\Gamma(1)$ , even if  $\tilde{s}(\gamma)$  (recalling the notation from Section 2.3) truncates the region computed so far. In particular, if  $\gamma \in \operatorname{Orb}_\Gamma(1)$ , then  $\gamma = \delta\delta^\dagger$  for some  $\delta \in \Gamma \setminus \mathcal{I}_\Gamma$ , in which case  $\tilde{s}(\delta)$  passes halfway between 1 and  $\gamma$ . (Notice that if  $\gamma \in \operatorname{Orb}_\Gamma(1)$  and  $\gamma = \delta\delta^\dagger$  for some  $\delta \in \Gamma \cap \mathcal{I}_\Gamma$ , then  $\gamma$  would not contribute a side since  $\tilde{s}(\gamma)$  would be excluded from the region by  $\tilde{s}(\delta)$ .) On the other hand, if

$\gamma \notin \text{Orb}_\Gamma(1)$  and does contribute a side to the region computed thus far, then it also contributes a side to  $\mathcal{D}_\Gamma(1)$  (at the very least, the point on that side which is closest to 1 will not be truncated by any other side).

So invoking [Theorem 5.2](#) when  $\Gamma$  is closed under complex conjugation gives that each element of  $\Gamma \cap \mathcal{I}_\Gamma$  either contributes a side to  $\mathcal{D}_\Gamma(1)$  or lies on a side of  $\mathcal{D}_\Gamma(1)$  at the point where that side is closest to 1. If one cannot see which case a given  $\gamma$  falls into by some easier means, it is straightforward to check whether  $\gamma$  factors into  $\gamma = \delta\delta^\dagger$  as a word in the generators. This will be illustrated in the examples at the end of this section.

**5.2.3 Projective pure quaternions as slopes** We gain an additional tool by expanding on the idea from the proof of [Theorem 5.2](#) where we used pure quaternion parts to characterize geodesics through 1. For each point  $\mu_\Gamma(\gamma, 1) \in O_\Gamma$ , the geodesic ray that starts at 1 and passes through  $\mu_\Gamma(\gamma, 1)$  has a pure quaternion part which is a Euclidean ray (this can be seen in [Figure 2](#)). So we can identify these geodesic rays as follows.

**Definition 5.4** The *slope* of  $w + xi + yj + \sqrt{-d}zij \in O_\Gamma$  is  $[x, y, z] \in F^3/F^+$ .

As we search through  $\Gamma \cap \mathcal{I}_\Gamma$  (or through  $\text{Orb}_\Gamma(1)$ ) in order of trace, once we find an element having slope  $[x, y, z]$ , we know that any other element with that slope but higher trace cannot contribute a side to  $\mathcal{D}_\Gamma(1)$ . After all, for two bisectors of the same geodesic ray, one will be contained in the half-space of the other. So keeping track of these slopes shortens the algorithm further because every time the slope of some element in  $\Gamma \cap \mathcal{I}_\Gamma$  is the same as one already found, we know it will not contribute a side and can skip any more complicated means of determining this.

## 5.3 Examples

We conclude by clarifying these ideas with some simple worked examples. First we look at an implementation of the 2-dimensional analogy of these ideas, then at an application to noncompact arithmetic manifolds. Notably, the fundamental domains shown here were drawn by hand in a basic illustration program, as the computational method was simple enough that it did not require advanced software.

**5.3.1 A hyperbolic punctured torus** Let  $S$  be the hyperbolic punctured torus. Then  $\pi_1(S)$  can be represented by  $\Delta < \text{PSL}_2(\mathbb{Z})$ , where  $\Delta$  is the torsion-free subgroup of

the modular group. Then  $\Delta = \langle \gamma, \delta \rangle$ , where [5, page 116]

$$\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The quaternion algebra of  $\Delta$  is  $B\Delta = \left(\frac{1}{\mathbb{Q}}\right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ , where  $i^2 = j^2 = 1$  and  $ij = -ji$ , and let  $\mathcal{A} = B\Delta$ . Then  $\mathcal{A}$  contains the restricted Macfarlane space  $\mathcal{L} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j$ , and  $\mathcal{I}_\Delta := \mathcal{L}_+^1$  is a quaternion hyperboloid model for  $S$ .

Recall that  $\mathcal{A} \cong M_2(\mathbb{Q})$ . So the map (5-1) gives the  $\mathbb{Q}$ -algebra isomorphisms

$$\rho_{\mathcal{A}}: \mathcal{A} \rightarrow M_2(\mathbb{Q}), \quad w + xi + yj + zij \mapsto \begin{pmatrix} w-x & y-z \\ w+z & y+z \end{pmatrix},$$

$$\rho_{\mathcal{A}}^{-1}: M_2(\mathbb{Q}) \rightarrow \mathcal{A}, \quad \begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \frac{1}{2}(v+s) + \frac{1}{2}(v-s)i + \frac{1}{2}(u+t)j + \frac{1}{2}(u-t)ij,$$

the map (5-3) gives the isometry

$$\iota_\Delta: \mathcal{I}_\Delta \rightarrow \mathcal{H}^2, \quad w + xi + yj \mapsto \frac{y + J}{w + x},$$

and these transfer the Macfarlane model to the Möbius action of  $\Delta$  on  $\mathcal{H}^2$ . When the context is clear, we use  $\Delta = \langle \gamma, \delta \rangle$  to mean both the matrix group and the corresponding quaternion group under  $\rho_{\mathcal{A}}^{-1}$ , where

$$\gamma = \frac{3}{2} + \frac{1}{2}i + j \quad \text{and} \quad \delta = \frac{3}{2} + \frac{1}{2}i - j.$$

To implement the algorithm, we want to find the points in  $\text{Orb}_\Delta(1)$  in order of increasing trace. Since  $\Delta$  consists of all the nonelliptic elements of  $\text{PSL}_2(\mathbb{Z})$ , it is closed under transposition, therefore  $\text{Orb}_\Delta(1) \subset \Delta$ . Then for all  $t \in \mathbb{N}$ ,

$$\{q \in \text{Orb}_\Gamma(1) \mid \text{tr}(qq^\dagger) = t\} \subset \{q \in \Delta \cap \mathcal{I}_\Delta \mid \text{tr}(q) = t\}.$$

But using  $n(\Delta \cap \mathcal{I}_\Delta) = 1$ , and the fact that hyperbolic elements have traces in  $\mathbb{R} \setminus [-2, 2]$ , we can characterize these elements using a Diophantine equation where the only solutions for  $t$  lie in  $\mathbb{Z} \setminus \{0, \pm 1\}$ . In particular,

$$\{q \in \Delta \cap \mathcal{I}_\Delta \mid \text{tr}(q) = t\} = \left\{ \frac{1}{2}t + \frac{1}{2}(t - 2x)i + y \mid x^2 + y^2 = tx - 1, x, y \in \mathbb{Z} \right\}.$$

Once we know what this (finite) set is, we can find all matrices of trace  $t$  by checking whether each element can be written in the form  $q = ww^\dagger$  for some word  $w$  in the generators of  $\Delta$ . If it can, we get that  $\tilde{s}(q)$  (in the notation of Definition 2.18) passes halfway between 1 and  $q$ . If it cannot, we get that  $q^2 \in \text{Orb}_\Delta(1)$  and  $\tilde{s}(q^2)$  passes through  $q$  by Theorem 5.2.



**Table 1** lists some data from implementing this process, giving the points in  $\Delta \cap \mathcal{I}_\Delta$  up to trace 18. For each  $q \in \Delta \cap \mathcal{I}_\Delta$ , the direction of the corresponding geodesic ray is given by a normalized representative of the slope of  $q$  (in the sense of [Definition 5.4](#)). In the rightmost column, the corresponding points in  $\mathcal{H}^2$  under  $\iota_\Delta$  are given.

Notice that (as predicted by [Theorem 5.2](#)) if a point  $q$  in the chart does not lie in  $\text{Orb}_\Delta(1)$ , then later the point  $q^2$  does, and has the same slope. For example, the elements of  $\Delta \cap \mathcal{I}_\Delta$  at trace 3 lead to sides contributed by isometries as points at trace 6. The points in [Table 1](#) which lie in  $\text{Orb}_\Gamma(1)$  are in bold, and one can see that the sides they contributed had already been found in  $\Delta \cap \mathcal{I}_\Delta$  before they were reached in the orbit.

[Figure 3](#) shows in  $\mathcal{H}^2$  which sides are contributed at each trace (the table gives the trace of the isometry contributing the side, even though it was found earlier) until  $\mathcal{D}_\Delta(1)$  is complete, and illustrates how the induced side-pairings create a punctured torus.

**5.3.2 Noncompact arithmetic hyperbolic 3-manifolds** In [Section 4.2.1](#), we mentioned that a manifold in this class has a fundamental group which can be represented by a torsion-free finite-index subgroup of a Bianchi group  $\text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ ,  $d \in \mathbb{N}$  (square-free). Let  $\Gamma$  be such a group. Then  $B\Gamma = \left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$  is Macfarlane and we can find a Dirichlet domain for  $\Gamma$  by a similar method to that used in the previous subsection.

Since the only real traces occurring in  $\text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$  lie in  $\mathbb{Z}$ , the ellipse at trace  $t$  can only be nonempty when  $t \in \{2, 3, 4, \dots\}$ . Since  $\text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$  is closed under complex conjugation,  $\text{Orb}_\Gamma(1) \subset \text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ , so like in the previous example, the ellipse at trace  $t$  is a subset of  $\{\gamma \in \text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}) \mid \text{tr}(\gamma) = t\}$ . The points in  $\text{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$  of trace  $t$  correspond to solutions to the Diophantine equation arising from  $\det\left(\begin{smallmatrix} r & s \\ s & t-r \end{smallmatrix}\right) = 1$ , where  $r \in \mathbb{Z}$  and  $s \in \mathcal{O}_d$ . Once we find those, we can use the generators for  $\Gamma$  to determine which lie in the orbit and which are intersected by sides.

**Example 5.5** The fundamental group of the Whitehead link complement can be represented by the finite-index subgroup of  $\text{PSL}_2(\mathbb{Z}[\sqrt{-1}])$  generated by  $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ -1-\sqrt{-1} & 1 \end{pmatrix}$  [[17](#), page 62]. Suppressing some details, an implementation of the process described above yields the Dirichlet domain for this group illustrated in [Figure 4](#), where we view the faces from above in  $\mathcal{H}^3$  after applying  $\rho_{B\Gamma}$ , and indicate the traces of the isometries contributing each side.

trace	$q \in \mathcal{I}_\Delta \cap \Delta$	slope of $q$	$\rho_\Delta(q) \in \mathrm{PSL}_2(\mathbb{Z})$	$\iota_\Delta(q) \in \mathcal{H}^2$
2	1	–	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$J$
3	$\frac{3}{2} + \frac{1}{2}i \pm j$	$[1, \pm 2]$	$\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 2 \end{pmatrix}$	$\pm \frac{1}{2} + \frac{1}{2}J$
	$\frac{3}{2} - \frac{1}{2}i \pm j$	$[-1, \pm 2]$	$\begin{pmatrix} 2 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$	$\pm 1 + J$
6	$3 + 2i \pm 2j$	$[1, \pm 1]$	$\begin{pmatrix} 1 & \pm 2 \\ \pm 2 & 5 \end{pmatrix}$	$\pm \frac{2}{5} + \frac{1}{5}J$
	<b><math>3 - 2i \pm 2j</math></b>	$[-1, \pm 1]$	$\begin{pmatrix} 5 & \pm 2 \\ \pm 2 & 1 \end{pmatrix}$	<b><math>\pm 2 + J</math></b>
7	$\frac{7}{2} + \frac{3}{2}i \pm 3j$	$[1, \pm 2]$	$\begin{pmatrix} 2 & \pm 3 \\ \pm 3 & 5 \end{pmatrix}$	<b><math>\pm \frac{3}{5} + \frac{1}{5}J</math></b>
	$\frac{7}{2} - \frac{3}{2}i \pm 3j$	$[-1, \pm 2]$	$\begin{pmatrix} 5 & \pm 3 \\ \pm 3 & 2 \end{pmatrix}$	$\pm \frac{3}{2} + \frac{1}{2}J$
11	<b><math>\frac{11}{2} + \frac{9}{2}i \pm 3j</math></b>	$[3, \pm 2]$	$\begin{pmatrix} 1 & \pm 3 \\ \pm 3 & 10 \end{pmatrix}$	<b><math>\pm \frac{3}{10} + \frac{1}{10}J</math></b>
	$\frac{11}{2} - \frac{9}{2}i \pm 3j$	$[-3, \pm 2]$	$\begin{pmatrix} 10 & \pm 3 \\ \pm 3 & 1 \end{pmatrix}$	$\pm 3 + J$
15	$\frac{15}{2} + \frac{11}{2}i \pm 5j$	$[11, \pm 10]$	$\begin{pmatrix} 2 & \pm 5 \\ \pm 5 & 13 \end{pmatrix}$	$\pm \frac{5}{13} + \frac{1}{13}J$
	<b><math>\frac{15}{2} - \frac{11}{2}i \pm 5j</math></b>	$[-11, \pm 10]$	$\begin{pmatrix} 13 & \pm 5 \\ \pm 5 & 2 \end{pmatrix}$	<b><math>\pm \frac{5}{2} + \frac{1}{2}J</math></b>
	$\frac{15}{2} + \frac{5}{2}i \pm 7j$	$[5, \pm 14]$	$\begin{pmatrix} 5 & \pm 7 \\ \pm 7 & 10 \end{pmatrix}$	$\pm \frac{7}{10} + \frac{1}{10}J$
	$\frac{15}{2} - \frac{5}{2}i \pm 7j$	$[-5, \pm 14]$	$\begin{pmatrix} 10 & \pm 7 \\ \pm 7 & 5 \end{pmatrix}$	$\pm \frac{7}{5} + \frac{1}{5}J$
18	$9 + 8i \pm 4j$	$[2, \pm 1]$	$\begin{pmatrix} 1 & \pm 4 \\ \pm 4 & 17 \end{pmatrix}$	$\pm \frac{4}{17} + \frac{1}{17}J$
	$9 - 8i \pm 4j$	$[-2, \pm 1]$	$\begin{pmatrix} 17 & \pm 4 \\ \pm 4 & 1 \end{pmatrix}$	$\pm 4 + J$
	$9 + 4i \pm 8j$	$[1, \pm 2]$	$\begin{pmatrix} 5 & \pm 8 \\ \pm 8 & 13 \end{pmatrix}$	$\pm \frac{8}{13} + \frac{1}{13}J$
	$9 - 4i \pm 8j$	$[-1, \pm 2]$	$\begin{pmatrix} 13 & \pm 8 \\ \pm 8 & 5 \end{pmatrix}$	$\pm \frac{8}{5} + \frac{1}{5}J$

Table 1: A punctured torus group intersected with its own quaternion hyperboloid model. The points which lie in  $\mathrm{Orb}_\Gamma(1)$  are in bold.

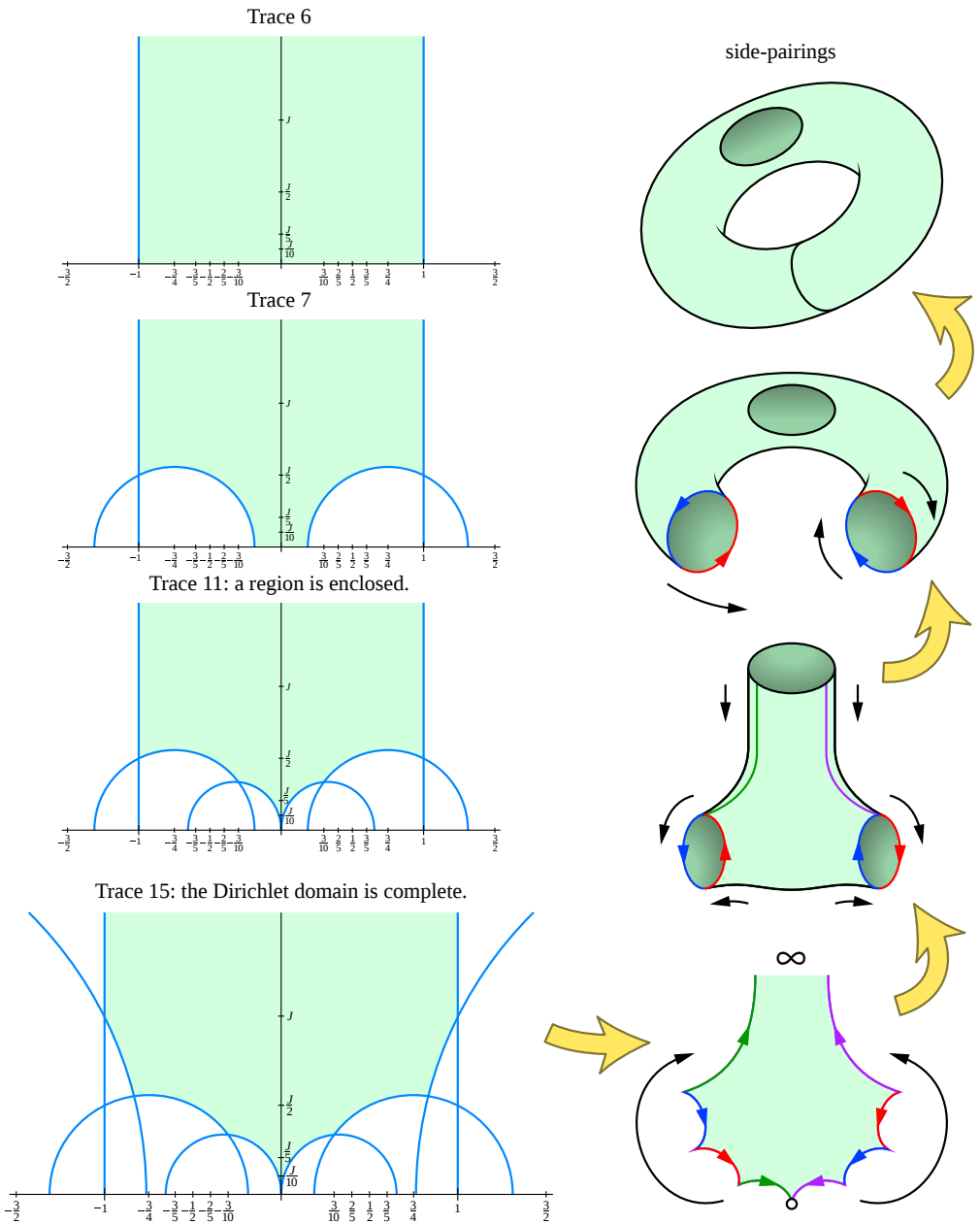
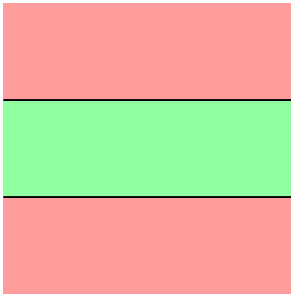
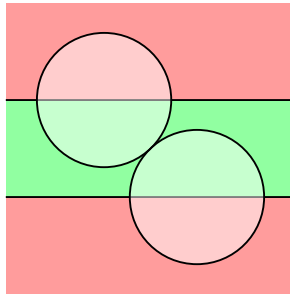


Figure 3: Dirichlet domain for a hyperbolic punctured torus

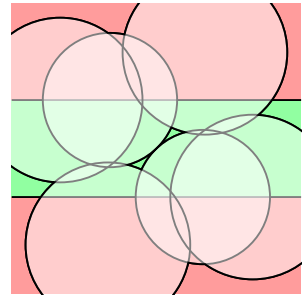
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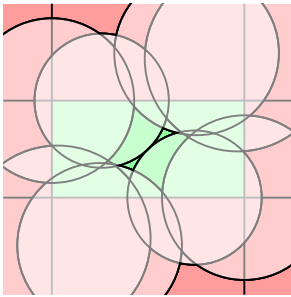
Trace 3: a pair of parallel half-planes.



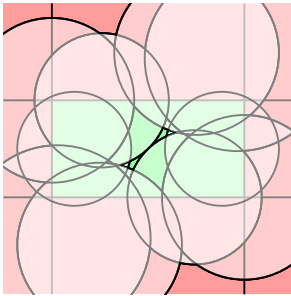
Trace 4: a pair of hemispheres tangent at zero.



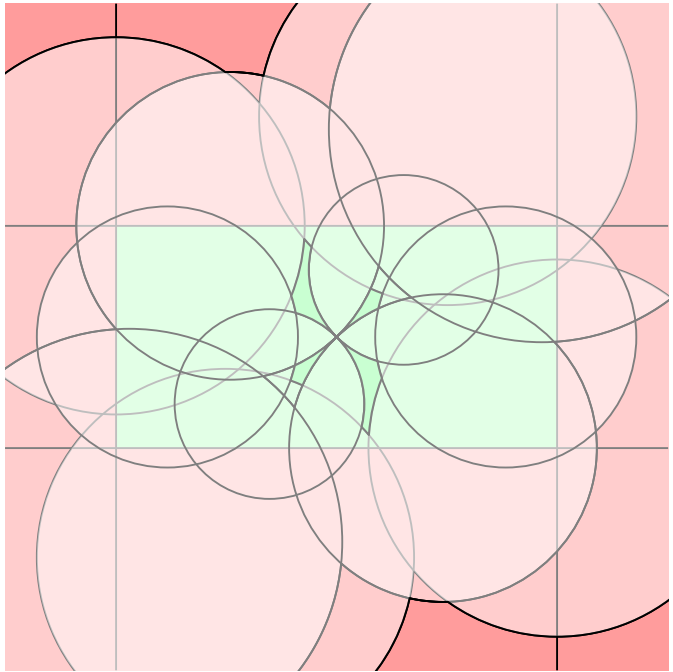
Trace 5: four overlapping hemispheres symmetric around the vertical axis.



Trace 6: two parallel half-planes completing a cusp at infinity and two larger hemispheres.



Trace 7: two smaller hemispheres centered on real axis further truncate the region.



The completed Dirichlet domain. At trace 10 a pair of hemispheres tangent at zero completed the cusp suggested at trace 4. Several other orbit points, starting at trace 6, are not included because they are too far away to contribute sides. After trace 10, this is true for all further orbit points.

Figure 4: Dirichlet domain for the Whitehead link complement

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