Algebraic & Geometric Topology 18 (2018) 1799–1822

Comparing 4–manifolds in the pants complex via trisections

GABRIEL ISLAMBOULI

Given two smooth, oriented, closed 4–manifolds, M_1 and M_2, we construct two invariants, $D^P(M_1, M_2)$ and $D(M_1, M_2)$, coming from distances in the pants complex and the dual curve complex, respectively. To do this, we adapt work of Johnson on Heegaard splittings of 3–manifolds to the trisections of 4–manifolds introduced by Gay and Kirby. Our main results are that the invariants are independent of the choices made throughout the process, as well as interpretations of "nearby" manifolds. This naturally leads to various graphs of 4–manifolds coming from unbalanced trisections, and we briefly explore their properties.

57M15, 57M99

1 Introduction

In [6], Johnson uses two closely related simplicial complexes associated to surfaces in order to define invariants of 3–manifolds. In particular, using Heegaard splittings, Johnson [6] defines distances between two 3–manifolds in the pants complex and the dual curve complex, which are independent of the particular Heegaard splittings chosen. An interesting interpretation of the distance between M_1 and M_2 in the dual curve complex is that it is equal to the minimum number of components of a link $L \subset M_1$ such that Dehn surgery along L produces M_2.

Through the recent work of Gay and Kirby [2], trisections of 4–manifolds have arisen as an analogue to Heegaard splittings. A (g,k)–trisection of a 4–manifold X is a decomposition into three pieces, $X = X_1 \cup X_2 \cup X_3$, where each X_i is diffeomorphic to $\natural^k S^1 \times D^3$, and is equipped with a genus-g Heegaard splitting of $\#^k\, S^1 \times S^2$ on its boundary. The intersection of the three pieces, $X_1 \cap X_2 \cap X_3$, is a surface of genus g, called the *trisection surface*. One of the nice features of a trisection is that it encodes all of the data of a smooth 4–manifold as curves on the trisection surface. We are therefore able to access the numerous complexes associated to surfaces to address questions about 4–manifolds.

Published: 9 April 2018 DOI: 10.2140/agt.2018.18.1799

We seek to adapt the work of [6] to 4–manifolds through trisections. One of the key observations which allows us to do this is that, if we have two 4–manifolds equipped with (g,k)–trisections for the same g and k, we may cut out a chosen X_i from each of them, and glue them together in a way which respects the Heegaard splitting on the boundary of X_i. This gives a way to view all relevant curves on a single surface, and hence compare them in the chosen complex. This allows us to define two nontrivial distances between trisections: $D(T_1,T_2)$ and $D^P(T_1,T_2)$.

If T is a genus-h trisection and $g = h + 3n$ for $n \in \mathbb{N}$, there is a natural way to construct a genus-g trisection of the same 4–manifold, which we call a stabilization of T and denote by T^g. The main theorem of the paper is the following:

Theorem 2.5 *Let M_1 and M_2 have trisections T_1 and T_2, respectively. Then the limit $\lim_{g\to\infty} D(T_1^g,T_2^g)$ is well defined and depends only on the underlying manifolds, M_1 and M_2.*

We also show the analogous result in the pants complex, and this allows us to define two natural number–valued invariants of two 4–manifolds, $D(M_1,M_2)$ and $D^P(M_1,M_2)$. Sections 3 and 4 are dedicated to exploring properties of these invariants. In Section 3, we find various upper and lower bounds. For example, if $\sigma(M)$ denotes the signature of M, we obtain the following inequality:

Proposition 3.3 $$D(M_1,M_2) \geq \tfrac{1}{2}|\sigma(M_1)-\sigma(M_2)|.$$

Section 4 consists of interpretations of nearby manifolds in terms of Kirby calculus. We first show that manifolds which are close in the pants complex have very similar Kirby diagrams. More precisely, we show the following:

Theorem 4.3 *If $D^P(M_1,M_2) = 1$, then M_1 and M_2 have Kirby diagrams which are identical, except for the framing on some 2–handle.*

We also show that manifolds with similar Kirby diagrams are close in the pants complex, which is encompassed in the following theorem:

Theorem 4.5 *Let M_1 and M_2 be nondiffeomorphic 4–manifolds with the same Euler characteristic which have Kirby diagrams K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2–handle, where the framing differs by 1, then $D^P(M_1,M_2) = 1$.*

Our line of inquiry in constructing these invariants leads naturally to the construction of various graphs of 4–manifolds coming from subgraphs of the pants complex and the dual curve complex. Section 5 is dedicated to carefully defining these graphs and obtaining some connectivity results.

Acknowledgements

The author would like to thank David Gay and Alex Zupan for the helpful conversation which made the author aware of [6], and the referee for many suggestions which improved the exposition of the article. The author would particularly like to thank his advisor, Slava Krushkal, for his many helpful comments and patience throughout the preparation of this paper. This research was supported in part by NSF grant DMS-1612159.

1.1 Simplicial complexes associated to surfaces

The most commonly used complex associated to a surface is the curve complex. It has proven to be a useful tool in investigating the structure of the mapping class group of an orientable surface. We recall the definition here.

Definition 1.1 Given a closed, orientable surface, Σ, of genus $g \geq 2$, the *curve complex* of Σ, denoted by $C(\Sigma)$, is a simplicial complex built out of simple closed curves on Σ. Each isotopy class of essential simple closed curves corresponds to a vertex. A collection of n vertices spans an $(n-1)$–simplex if the corresponding curves can be isotoped to be pairwise disjoint.

In his seminal work in [5], Hempel used the curve complex to give an invariant of Heegaard splittings generalizing the notions of reducibility, weak reducibility, and the disjoint curve property. While Hempel's distance is an indispensable tool for investigating the structure of Heegaard splittings of a 3–manifold, it is unlikely to be useful for constructing invariants of manifolds. This is due to the fact that the invariant completely collapses when a Heegaard splitting is stabilized. Our setup for trisections will have similar problems, so we consider the dual of the curve complex.

Definition 1.2 Given a closed, orientable surface of genus $g \geq 2$, the *dual curve complex* of Σ, denoted by $C^*(\Sigma)$, is the simplicial complex whose vertices correspond to maximal-dimensional simplices of $C(\Sigma)$. Two vertices in $C^*(\Sigma)$ have an edge between them if the corresponding maximal-dimensional simplices of $C(\Sigma)$ share a codimension-1 face.

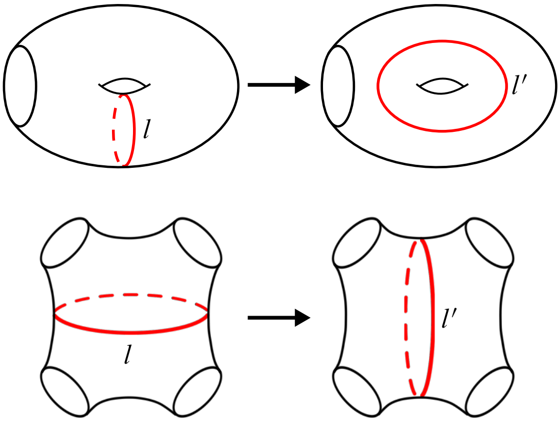

Figure 1: Above: An S–move in the pants complex. Below: An A–move in the pants complex.

For a closed, orientable surface of genus $g \geq 2$, maximal-dimensional simplices in $C(\Sigma)$ are of dimension $3g-4$ and correspond to a set of $3g-3$ simple closed curves, whose union separates the surface into pairs of pants. An edge in the dual curve complex therefore corresponds to starting with one pants decomposition of a surface and replacing one curve in order to obtain another pants decomposition of the surface. If, instead of allowing arbitrary curve replacements, we insist that curves are replaced in the simplest way possible, we obtain the pants complex.

Definition 1.3 Given a surface Σ, the *pants complex* of Σ, denoted by $P(\Sigma)$, is the simplicial complex whose vertices correspond to isotopy classes of pants decompositions of Σ. Two vertices v and v' in $P(\Sigma)$ are connected by an edge if the corresponding pants decompositions only differ in one curve, and the two different curves intersect minimally. That is, if $l \in v$ and $l' \in v'$ with $l \neq l'$, then either l and l' lie on a punctured torus with $|l \cap l'| = 1$ or l and l' lie on a four-punctured sphere with $|l \cap l'| = 2$. In the case that l and l' lie on a punctured torus, we say that v and v' are related by an *S–move*. If l and l' lie on a four punctured sphere we say that v and v' are related by an *A–move*. See Figure 1 for an illustration of these moves.

There is a natural map from the pants complex into the dual curve complex which is bijective on vertices and injective on edges. In [4], Hatcher and Thurston prove that the pants complex is connected, so the aforementioned map shows that the dual curve complex is connected. We therefore get naturally defined metrics on the 1–skeleton of these complexes.

Definition 1.4 Let v_1 and v_2 be two vertices in $C^*(\Sigma)$. The *dual distance*, $D(v_1, v_2)$, is the length of the minimal path between v_1 and v_2 in the dual curve complex. Similarly, if v_1 and v_2 are two vertices in $P(\Sigma)$, the *pants distance* $D^P(v_1, v_2)$ is the length of the minimal path between v_1 and v_2 in the pants complex.

Since the pants complex appears as a subcomplex of the dual curve complex, we get the inequality $D^P(v_1, v_2) \geq D(v_1, v_2)$. This inequality should be kept in mind when bounds are discussed later in the paper.

1.2 Trisections

To fix notation, we briefly summarize the relevant notions in the theory of trisections of 4–manifolds. For a more detailed account, the reader is referred to [2].

Definition 1.5 A (g,k)*–trisection* of a smooth, closed 4–manifold M is a decomposition $M = X_1 \cup X_2 \cup X_3$ such that:

- $X_i \cong \natural^k S^1 \times D^3$.
- $X_i \cap X_j = H_{ij}$ is a genus-g handlebody for $i \neq j$.
- $\partial X_i = H_{ij} \cup H_{ik}$ is a genus-g Heegaard splitting for $\partial X_i \cong \#^k\, S^1 \times S^2$.
- $X_1 \cap X_2 \cap X_3$ is a closed, orientable, genus-g surface.

In [2], Gay and Kirby show that every smooth, closed, 4–manifold admits a trisection. At times, it will be useful to relax the condition that all of the X_i are diffeomorphic to the same 4–dimensional handlebody. In particular, we allow $X_i \cong \natural^{k_i} S^1 \times D^3$ where, for $i \neq j$, it is possible that $k_i \neq k_j$. In this case, we insist that $\partial X_i = H_{ij} \cup H_{ik}$ is a genus-g Heegaard splitting for $\partial X_i \cong \#^{k_i}\, S^1 \times S^2$. We will call this more general setup an *unbalanced* $(g; k_1, k_2, k_3)$*–trisection*. Note that S^4 has unbalanced trisections with parameters $(1; 1, 0, 0)$, $(1; 0, 1, 0)$ and $(1; 0, 0, 1)$. These can be used to balance trisections by taking connected sums (see Definition 3.8 of [10] for more details on this construction). Unless otherwise noted, all trisections will be assumed to be balanced.

The union $H_{12} \cup H_{23} \cup H_{31}$ is called the *spine* of the trisection. Note that if we thicken the spine of the trisection by taking the product of the surface with D^2 and the product of handlebodies with D^1, then we are left with a 4–manifold with three boundary components each diffeomorphic to $\#^k\, S^1 \times S^2$. Recovering the original 4–manifold amounts to gluing back in three copies of $\natural^k S^1 \times D^3$. By [8], this can only be done in one way. In other words, the spine uniquely determines the trisection. The spine, in turn, is uniquely determined by three cut systems for the handlebodies which pairwise form Heegaard diagrams for $\#^k S^1 \times S^2$. Thus, a trisected 4–manifold is completely determined by these cut systems drawn on the trisection surface Σ. We refer to the trisection surface, together with the three cut systems, as a *trisection diagram*.

If T_1 is a (g_1, k_1)–trisection and T_2 is a (g_2, k_2)–trisection, we may form their connected sum $T_1 \# T_2$, which inherits the structure of a (g_1+g_2, k_1+k_2)–trisection. On the level of diagrams, this amounts to taking the connected sum of trisection diagrams for T_1 and T_2. S^4 has a $(3,1)$–trisection, so, if T is a (g,k)–trisection for M^4, we may form a $(g+3, k+1)$–trisection for M by taking a connected sum with the aforementioned trisection for S^4. The resulting $(g+3, k+1)$–trisection is called a *stabilization* of T. We may also take the connected sum of T with one of the unbalanced genus-1 trisections of S^4. The resulting (possibly unbalanced) trisection is called an *i–stabilization* of T, where $i \in \{1,2,3\}$ indicates that we are summing with the unbalanced genus-1 trisection of S^4 where $k_i \neq 0$. Let T be a genus-h trisection and let $g = h + 3n$ for some $n \in \mathbb{N}$. We denote by T^g the $(h+3n, k+n)$–trisection obtained by stabilizing T n times to a genus-g trisection. If T has spine $H_{12} \cup H_{23} \cup H_{31}$ and trisection surface Σ, we will denote the spine and trisection surface of T^g by $H_{12}^g \cup H_{23}^g \cup H_{31}^g$ and Σ^g. The following theorem will be essential for extending invariants of trisections to invariants for 4–manifolds. It can be seen as the analogue to the Reidemeister–Singer theorem [11; 12] for trisections.

Theorem 1.6 [2, Theorem 11] *If T_1 and T_2 are trisections of the same manifold X, then there exists a natural number n such that T_1^n and T_2^n are isotopic as trisections. That is, if $T_1^n = X_1 \cup X_2 \cup X_3$ and $T_2^n = Y_1 \cup Y_2 \cup Y_3$, then there exists a self-diffeomorphism f of X isotopic to the identity such that $f(X_i) = Y_i$.*

2 Distances of trisections

Let (T_1, Σ_1) and (T_2, Σ_2) be two (g,k)–trisections with corresponding spines $H_{\alpha_1} \cup H_{\beta_1} \cup H_{\gamma_1}$ and $H_{\alpha_2} \cup H_{\beta_2} \cup H_{\gamma_2}$. Both $H_{\alpha_1} \cup H_{\beta_1}$ and $H_{\alpha_2} \cup H_{\beta_2}$ are genus-g Heegaard splittings for $\#^k S^1 \times S^2$. Waldhausen's theorem [13] therefore asserts that both of these are in fact the unique genus-g splitting of $\#^k S^1 \times S^2$. Therefore, there exists a map $\phi\colon H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ such that $\phi(H_{\alpha_1}) = H_{\alpha_2}$ and $\phi(H_{\beta_1}) = H_{\beta_2}$. Such a map induces isometries on the various complexes associated to Σ_1 and Σ_2, and we will denote the induced isometry by $\widehat{\phi}$.

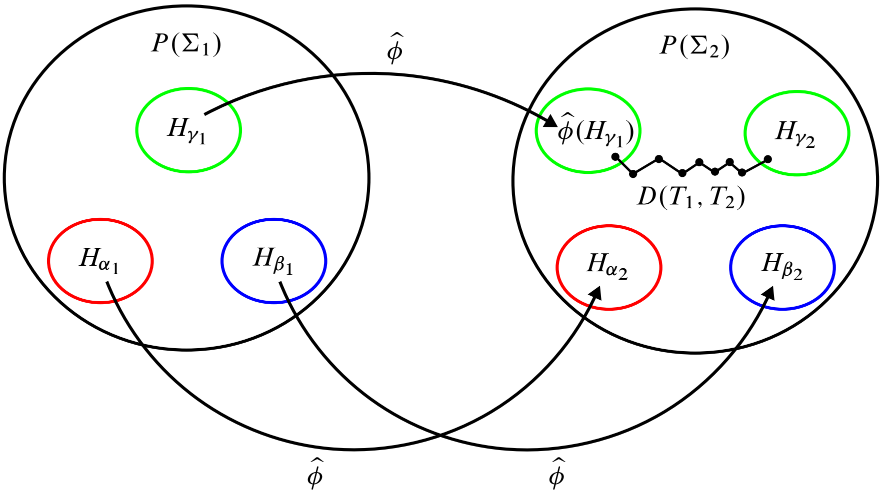

Figure 2: The distance between two trisections is the minimum distance between the sets $\widehat{\phi}(H_{\gamma_1})$ and H_{γ_2}

If we fix an identification of both $H_{\alpha_1} \cup H_{\beta_1}$ and $H_{\alpha_2} \cup H_{\beta_2}$ with $\#^k S^1 \times S^2$, all such maps (up to isotopy with $\phi_t(\Sigma_1) = \Sigma_2$) can be identified with the mapping class group of the Heegaard splitting, which we denote by $\mathrm{Mod}(\#^k S^1 \times S^2, \Sigma_g)$. The mapping class groups of Heegaard splittings have been studied extensively and can be quite complicated. For example, when $g > k$, the group $\mathrm{Mod}(\#^k S^1 \times S^2, \Sigma_g)$ will always have pseudo-Anosov elements [7].

We say a vertex $v \in C^*(\Sigma)$ (or $P(\Sigma)$) *defines a handlebody*, H, if all of the curves in the pants decomposition associated to v bound disks in H. Equivalently, attaching 3–dimensional 2–handles to Σ along the curves of v and filling in the resulting 2–sphere boundary components with 3–balls produces H. We are now ready to define the main objects of study.

Definition 2.1 Let M_1 and M_2 be two 4–manifolds equipped with (g,k)–trisections T_1 and T_2. The *dual distance* between T_1 and T_2, $D(T_1, T_2)$, is

$$\min\{D(\widehat{\phi}(v_1), v_2) \mid v_1 \text{ defines } H_{\gamma_1} \text{ and } v_2 \text{ defines } H_{\gamma_2}\}.$$

Similarly, the *pants distance*, $D^P(T_1, T_2)$, is

$$\min\{D^P(\widehat{\phi}(v_1), v_2) \mid v_1 \text{ defines } H_{\gamma_1} \text{ and } v_2 \text{ defines } H_{\gamma_2}\}.$$

Here, the minimum is taken over all orientation-preserving maps $\phi\colon H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ such that $\phi(H_{\alpha_1}) = H_{\alpha_2}$ and $\phi(H_{\beta_1}) = H_{\beta_2}$ as well as all vertices defining

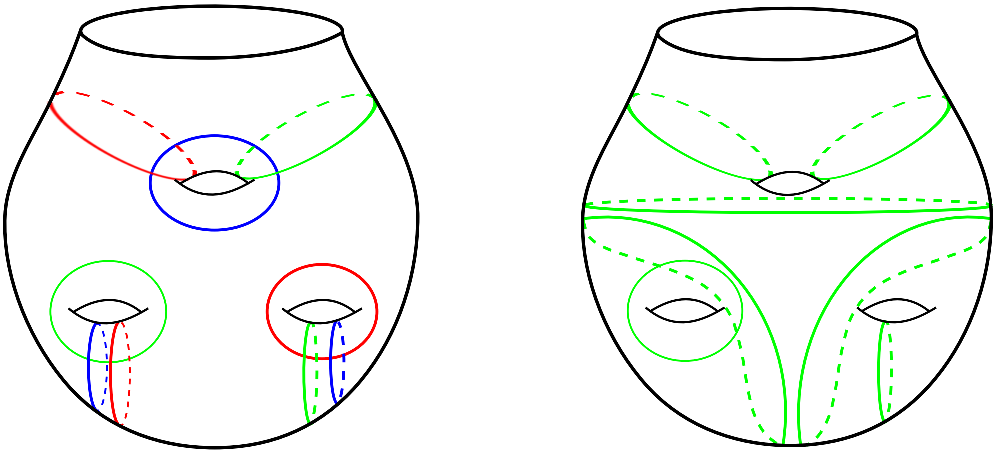

Figure 3: Left: Stabilizing a trisection amounts to gluing this diagram onto a punctured trisection diagram. Right: A pants decomposition for one handlebody of the stabilizing surface

the handlebodies in the respective complexes. See Figure 2 for an illustration of the definition. Since these distances are natural number–valued, they give well-defined invariants of two (g,k)–trisections. Furthermore, if either distance is 0, then the distance-minimizing map extends to a homeomorphism of spines, which means that T_1 and T_2 are in fact diffeomorphic trisections. Since there are many manifolds admitting a (g,k)–trisection for a given g and k with $g \neq k$ (see for example [1; 9]), this distance is nontrivial.

We now seek to extend these distances of particular trisections to well-defined distances of 4–manifolds. To do this, we need to understand how the distance behaves under stabilization. If (T,Σ) is a trisection, we may stabilize T by puncturing Σ in a disk and gluing on the stabilizing surface shown in Figure 3. From this point of view, it is clear that we should begin by understanding paths in the complexes associated to $\Sigma \backslash D^2$. The following key lemma treating this case is contained in Lemma 15 of [6].

Lemma 2.2 *Let* $v_1, v_2, \dots, v_n$ *be a minimal path in* $C^*(\Sigma)$ *(resp.* $P(\Sigma)$*) between two handlebodies* H_1 *and* H_2*. Then there exists a disk* $D \subset \Sigma$ *and a path* $v_1', v_2', \dots, v_m'$ *in* $C^*(\Sigma \backslash D)$ *(resp.* $P(\Sigma \backslash D)$*) with* $m \leq 2n$ *such that after capping off* $\Sigma \backslash D$ *with a disk, every loop in* v_1' *bounds a disk in* H_1 *and every loop in* v_m' *bounds a disk in* H_2*. Moreover, if there is some loop which is never moved in the path from* v_1 *to* v_n*, then there exists a disk* D *and a path* $v_1', v_2', \dots, v_m'$ *satisfying the previous conclusions with* $m = n$*.*

In what follows, we will adapt the work of [6] to prove that the invariants of trisections behave well under stabilization. In order to aid exposition, we will only explicitly treat the case of the distance in the dual curve complex. It should be clear after the fact that by using the part of Lemma 2.2 pertaining to the pants complex, all of the arguments go through virtually unchanged.

Lemma 2.3 *Let* (T_1, Σ_1) *and* (T_2, Σ_2) *be* (g,k)*–trisections, and let* T_1^h *and* T_2^h *be their genus-*h *stabilizations. Then* $D(T_1^h, T_2^h) \leq 2D(T_1, T_2)$.

Proof If $D(T_1, T_2) = n$ then there is a map $\phi\colon H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ and a path $v_1, v_2, \dots, v_n$ in $C^*(\Sigma_2)$ such that v_1 defines $\phi(H_{\gamma_1})$ and v_n defines H_{γ_2}. Let $l_i^1, \dots, l_i^m$ be the loops corresponding to the pants decomposition given by v_i. Let D be a disk in the annulus formed by two parallel copies of l_1^1. Consider the pants decomposition for $\Sigma_2 \backslash D$ consisting of $l_1^1, \dots, l_1^m, l_1^{m+1}$, where l_1^{m+1} is the parallel copy of l_1^1 on Σ_2 lying on the other side of D on $\Sigma_2 \backslash D$. Let v_1' be the corresponding vertex of $C^*(\Sigma_2 \backslash D)$.

By Lemma 2.2, there is a path from v_1' to a vertex w such that if $l \in w$, then after capping off $\Sigma_2 \backslash D$ with a disk, l is isotopic to some loop in v_n. Moreover, this path is of length at most $2n$.

We first treat the case of a single stabilization. Consider the stabilization of Σ_1 produced by cutting out the disk $\phi^{-1}(D)$ gluing on a stabilizing surface to the resulting boundary component. Then $H_{\gamma_1}^{g+3}$ has a pants decomposition given by the curves in $\phi^{-1}(v_1')$ along with $\phi^{-1}(\partial D)$ and the pants decomposition for the stabilizing surface shown in Figure 3. By gluing on a stabilizing surface to ∂D, we can extend ϕ to a map $\phi^{g+3}\colon \Sigma_1^{g+3} \to \Sigma_2^{g+3}$ such that $\phi^{g+3}(w) \cup \partial D \cup \phi^{g+3}$(the curves shown in Figure 3) is a pants decomposition for $H_{\gamma 2}^{g+3}$. Since the path in $C^*(\Sigma_1 \backslash D)$ from v_1' to w takes place away from the stabilizing surface, it corresponds to a path in $C^*(\Sigma_2^{g+3})$ such that $D(T_1^{g+3}, T_2^{g+3}) \leq 2D(T_1, T_2)$. To achieve the more general result, simply connect-sum multiple stabilizing surfaces first before connect-summing with the given trisection surfaces. □

Lemma 2.4 *For sufficiently large* g, $D(T_1^h, T_2^h) \leq D(T_1^g, T_2^g)$ *when* $h \geq g$.

Proof By the previous lemma, $D(T_1^g, T_2^g) \leq 2D(T_1, T_2)$. Choose g so that $3g-3 > 2D(T_1, T_2)$. Since a pants decomposition of Σ_2^g consists of $3g-3$ loops, it follows that some loop is never moved in the path on Σ^g. In this case, we conclude by Lemma 2.2

that paths on Σ^g from $\widehat{\phi}^g(H^g_{\gamma_1})$ to $H^g_{\gamma_2}$ lift to paths of the same length on Σ^h from $\widehat{\phi}^h(H^h_{\gamma_1})$ to $H^h_{\gamma_2}$. □

Theorem 2.5 *Let M_1 and M_2 have trisections T_1 and T_2, respectively. Then the limit $\lim_{g\to\infty} D(T_1^g, T_2^g)$ is well defined and depends only on the underlying manifolds, M_1 and M_2.*

Proof Since the sequence $D(T_1^g, T_2^g)$ is natural number–valued and nonincreasing for sufficiently large g, it converges. Furthermore, by Theorem 1.6, any two trisections of the same manifold have a common stabilization fixing the labels of the handlebodies. Therefore, if T_1 and T_3 are distinct trisections of M_1, and T_2 and T_4 are distinct trisections of M_2, then there exists an h such that T_1^h is isotopic to T_3^h and T_2^h is isotopic to T_4^h. Then, for $g > h$ we have that $D(T_1^g, T_2^g) = D(T_3^g, T_4^g)$, so that $\lim_{g\to\infty} D(T_1^g, T_2^g) = \lim_{g\to\infty} D(T_3^g, T_4^g)$. □

Remark 2.6 The reader may be concerned that the definitions of D and D^P seem to distinguish between which third of the trisection is labeled X_1, whereas we have seemingly defined an invariant of a 4–manifold which is not sensitive to this information. However, Theorem 2.5 does actually encompass this case. Suppose M_1 has two trisections of the form $T_1 = (X_1, X_2, X_3)$ and $T_2 = (Y_1, Y_2, Y_3)$ such that $Y_i = X_{i-1}$ with indices taken mod 3. In [2], it is shown that any two trisections of the same manifold have a common stabilization fixing the labels of the sectors. We therefore have a map $f\colon M^4 \to M^4$ isotopic to the identity such that $f(Y_i^h) = X_i^h$.

We are now justified in making the following definitions:

Definition 2.7 Let M_1 and M_2 be two 4–manifolds which have (g,k)–trisections for the same g and k. The *dual distance* between M_1 and M_2 is $\lim_{g\to\infty} D(T_1^g, T_2^g)$, where T_1 is any trisection of M_1 and T_2 is any trisection of M_2 with the same parameters as T_1. Similarly, the *pants distance* between two 4–manifolds is given by $\lim_{g\to\infty} D^P(T_1^g, T_2^g)$.

Remark 2.8 In the 3–manifold case, any two pants decompositions will determine a 3–manifold, so that minimal paths between two 3–manifolds pass through intermediary 3–manifolds. This nice property simplifies many of the arguments in [6]. In our setup, we can not guarantee that H_{α_2}, H_{β_2} and the handlebody determined by an intermediary vertex in a minimal path between $\widehat{\phi}(H_{\gamma_1})$ and H_{γ_2} will still pairwise form Heegaard splittings for $\#^k S^1 \times S^2$. Therefore, these three handlebodies may not form the spine of a trisection, and there may be no way to uniquely obtain a closed 4–manifold from this information. This leads to two natural questions:

(1) Can we pass between trisections through paths whose intermediary vertices form trisections?

(2) Is there any significance to the three handlebodies which occur in paths between two trisections?

It is clear from the definitions that trisections T_1 and T_2 can only be compared when $(g_1, k_1) = (g_2, k_2)$. However, it is not immediately obvious when two manifolds can be compared. A necessary and sufficient condition for comparing M_1 and M_2 is that both manifolds have a (g,k)–trisection for some g and k. If a 4–manifold has a (g,k)–trisection, the Euler characteristic is given by $\chi(M) = 2 + g - 3k$, so it is necessary that $\chi(M_1) = \chi(M_2)$. The following straightforward lemma shows that this is also a sufficient condition.

Lemma 2.9 *$D(M_1, M_2)$ and $D^P(M_1, M_2)$ are well defined whenever $\chi(M_1) = \chi(M_2)$.*

Proof Let M_1 have a (g_1, k_1)–trisection T_1 and let M_2 have a (g_2, k_2)–trisection T_2. Now, $2 + g_1 - 3k_1 = 2 + g_2 - 3k_2$ since $\chi(M_1) = \chi(M_2)$. Without loss of generality, assume $k_1 > k_2$. Then, by stabilizing T_2 $k_1 - k_2$ times we get a new trisection T_2' of M_2, with $k_2' = k_1$ and $g_2' = g_2 + 3(k_1 - k_2) = g_2 + (g_1 - g_2) = g_1$, hence these two trisections can be compared. □

Remark 2.10 If we don't insist on the trisections being balanced, we may compare any two 4–manifolds, regardless of their Euler characteristics. To do this, let M_1 and M_2 be 4–manifolds with corresponding trisections, T_1 and T_2. We may perform 1–stabilizations until they have the same k_1. Next, perform 2–stabilizations until both trisections have the same genus. We now have two trisections, $T_1' = X_1 \cup X_2 \cup X_3$ and $T_2' = Y_1 \cup Y_2 \cup Y_3$, such that both ∂X_1 and ∂Y_1 are diffeomorphic to $\#^{k_1} S^1 \times S^2$. Moreover, both ∂X_1 and ∂Y_1 have the structure of a genus-g Heegaard splitting, so, as before, there are diffeomorphisms respecting the structure of the Heegaard splittings. This allows us to carry through with the definition of the distance between trisections virtually unchanged.

It is a quick corollary of Theorem 1.6 that unbalanced trisections of the same 4–manifold with the same parameters (ie (g, k_1, k_2, k_3)) become isotopic after some number of balanced stabilizations. This allows us to carry through with Theorem 2.5 and define a distance between any two manifolds which stabilizes as the initial trisections are stabilized. A slight caveat is that this will, in general, depend on the values of the k_i. Nevertheless, we can still get a well-defined distance, depending only on the underlying manifolds, by further minimizing over all choices of k_i.

It is natural to ask whether these distances induce a metric on the set of 4–manifolds with the same Euler characteristic. It follows quickly from the definition that these distances are 0 if and only if the manifolds are diffeomorphic. These distances are also symmetric, since if ϕ is the distance-minimizing map for $D(M_1, M_2)$ (resp. $D^P(M_1, M_2)$), then ϕ^{-1} minimizes the distance $D(M_2, M_1)$ (resp. $D^P(M_2, M_1)$). The triangle inequality, however, is likely false. This is due to the fact that we are minimizing over handlebody sets of infinite diameter in the complexes. Given three handlebodies H_1, H_2 and H_3, the representative of H_2 closest to H_1 may be far away from the representative of H_2 closest to H_3.

3 Some bounds

Lemma 3.1 *If $D(M_1, M_2) = n$ and $D(M_3, M_4) = m$, then $D(M_1 \# M_3, M_2 \# M_4) \leq n + m$.*

Proof Stabilize trisections of M_1 and M_2 to genus-g trisections, (T_1^g, Σ_1^g) and (T_2^g, Σ_2^g), with $3g - 3 > n$. Also, stabilize trisections of M_3 and M_4 to genus-h trisections, (T_3^h, Σ_3^h) and (T_2^h, Σ_2^h), with $3h - 3 > m$. Since a pants decomposition for Σ_2^g has $3g - 3$ loops, it follows that some loop in the path from $\widehat{\phi}^g(H_{\gamma_1}^g)$ to $H_{\gamma_2}^g$ is never moved, where $\widehat{\phi}$ is the distance-minimizing map. Let $v_1', v_2', \dots, v_n'$ be the path in the dual curve complex (or the pants complex) guaranteed by Lemma 2.2 on $C^*(\Sigma_2^g \backslash D)$, and let $w_1', w_2', \dots, w_n'$ be the path guaranteed by the same lemma on $C^*(\Sigma_4^h \backslash D')$.

Form the connect sum $\Sigma_2^g \# \Sigma_4^h$ along the disks D and D'. Let $v_i' \cup w_j'$ be the pants decomposition of $\Sigma_2^g \# \Sigma_4^h$ consisting of the pants decomposition for Σ_2^g induced by v_i', the pants decomposition for Σ_4^h induced by w_j', along with the additional curve $\partial D = \partial D'$. Then the path $v_1' \cup w_1', v_2' \cup w_1', \dots, v_n' \cup w_1', v_n' \cup w_2', \dots, v_n' \cup w_m'$ is a path of length $n + m$ from $M_1 \# M_3$ to $M_2 \# M_4$. □

Corollary 3.2 *For any N with $\chi(N) = 2$, $D(M_1 \# N, M_2) \leq D(M_1, M_2) + D(N, S^4)$.*

The question of whether $D(M_1 \# M_3, M_2 \# M_4) = n + m$ is quite easily shown to be false. For example, if M_1 and M_2 are homeomorphic — but not diffeomorphic — 4–manifolds that become diffeomorphic after a single connected sum with $S^2 \times S^2$, then $D(M_1, M_2) \neq 0$ whereas $D(M_1 \# (S^2 \times S^2), M_2 \# (S^2 \times S^2)) = 0$.

We next seek to prove a lower bound on the distance between two manifolds based on the difference of their signatures. To this end, we briefly discuss how this information can be recovered from a trisection. Given a genus-g surface Σ, choose a symplectic basis for $H_1(\Sigma_g, \mathbb{R})$. That is, a basis $\{a_1, b_1, \dots, a_g, b_g\}$ such that for all i and j, $|a_i \cap a_j| = |b_i \cap b_j| = 0$ and $|a_i \cap b_j| = \delta_{ij}$. Let ω be the associated symplectic form on $\mathbb{R}^{2g}$.

Given a trisection with spine $H_1 \cup H_2 \cup H_3$, we get three Lagrangian subspaces of $H_1(\Sigma_g, \mathbb{R})$, given by $L_i = \ker\bigl(i_*\colon H_1(\Sigma_g, \mathbb{R}) \to H_1(H_i, \mathbb{R})\bigr)$. We may define a symmetric bilinear form, q, on $L_1 \oplus L_2 \oplus L_3$ by $q((x_1, x_2, x_3), (y_1, y_2, y_3)) = \omega(x_1, y_2) + \omega(y_1, x_2) + \omega(x_2, y_3) + \omega(y_2, y_3) + \omega(x_3, y_1) + \omega(y_3, x_1) + \omega(x_3, y_1)$. In [2], it is observed that the signature of the matrix associated to this bilinear form is the signature of the original 4–manifold. While intermediary vertices in minimal paths between handlebodies will not always define trisections, the signature of this matrix will always be well defined. As a result, we obtain the following proposition:

Proposition 3.3 $D(M_1, M_2) \geq \frac{1}{2}|\sigma(M_1) - \sigma(M_2)|$.

Proof Suppose $D(M_1, M_2) = n$ with a path $v_1, \dots, v_n$ such that v_1 defines $\phi(H_{\gamma_1})$ and v_n defines H_{γ_2}. At each vertex, we have a triple of handlebodies determined by H_{α_2}, H_{β_2} and the handlebody determined by v_i. These in turn determine three Lagrangian subspaces of $H_1(\Sigma_2, \mathbb{R})$, L_1, L_2 and L_{v_i}. Going from v_i to v_{i+1} involves changing a single curve in the pants decomposition so that L_{v_i} and $L_{v_{i+1}}$ have bases in $H_1(\Sigma_2, \mathbb{R})$ which are the same except for possibly one vector.

Let M_i be the matrix corresponding to the symmetric bilinear form q_i on $L_1 \oplus L_2 \oplus L_{v_i}$. Then M_i has real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{3g}$. Let $a_1, \dots, a_g$ be a basis for L_{v_i}. In going from L_{v_i} to $L_{v_{i+1}}$, it is possible that none of the basis vectors are changed, in which case the signature of the matrix is obviously unchanged. It is also possible that one vector, say a_j, is changed. Let M_i' be the matrix obtained by deleting the row and column corresponding to a_j, and let $\lambda_1' \leq \lambda_2' \leq \cdots \leq \lambda_{3g-1}'$ be its eigenvalues. By the Cauchy interlacing theorem, $\lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \lambda_2' \leq \cdots \leq \lambda_{3g-1}' \leq \lambda_{3g}$, so that $|\sigma(M_i) - \sigma(M_i')| \leq 1$. Similarly, we may obtain M_i' by deleting a row and column of M_{i+1}, so that $|\sigma(M_{i+1}) - \sigma(M_i')| \leq 1$. Therefore, $|\sigma(M_{i+1}) - \sigma(M_i)| \leq 2$. The result immediately follows. $\square$

Comparing the standard $(g,0)$–trisections of $\#^g\, \mathbb{C}P^2$ and $\#^g\, \overline{\mathbb{C}P^2}$ one can see that this bound is sharp. Moreover, we may conclude that

$$\lim_{g\to\infty} D\big(\#^g\, \mathbb{C}P^2, \#^g\, \overline{\mathbb{C}P^2}\big) = \infty.$$

4 Nearby manifolds

We next seek to build some intuition as to what it means when 4–manifolds are close to each other with respect to our distances. We first consider an illustrative example. The top of Figure 4 shows trisection diagrams T_1 for $S^2 \times S^2$ and T_2 for $S^2 \mathbin{\tilde{\times}} S^2$, where the α and β handlebodies are identical. Below that are pants decompositions for the γ handlebodies, which are identical except for in one curve. The curves which are different intersect exactly once, showing that $D^P(T_1, T_2) = 1$. Moreover, we have two curves in the pants decomposition which never move, so, by Lemma 2.2, the path lifts to new paths of distance 1 on all stabilizations. Since these manifolds are nondiffeomorphic, we may therefore conclude that $D^P(S^2 \times S^2, S^2 \mathbin{\tilde{\times}} S^2) = 1$.

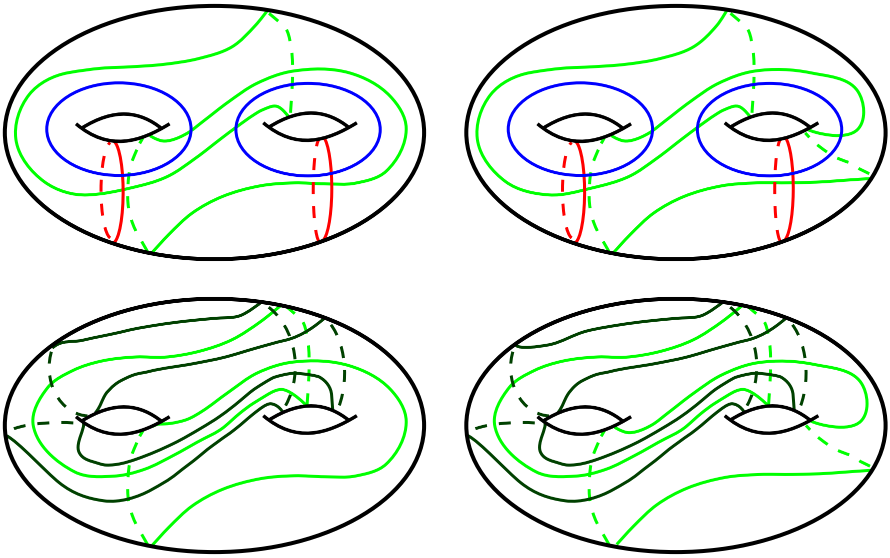

Figure 4: $S^2 \times S^2$ and $S^2 \mathbin{\tilde{\times}} S^2$ are distance 1 in the pants complex.

In [2], it is shown how to obtain a Kirby diagram from a trisection diagram, and these particular trisection diagrams give rise to Kirby diagrams which are identical except for in the framing of a 2–handle. We seek to show that this is in fact the case in general. That is, if $D^P(M_1, M_2) = 1$ then M_1 and M_2 have Kirby diagrams which are identical except for in the framing of some 2–handle. To do this, we first consider what it means for two handlebodies to be distance 1 apart in the pants complex.

Lemma 4.1 *Let H_1 and H_2 be two genus-g handlebodies with boundary Σ. If $D^P(H_1, H_2) = 1$ then $H_1 \cup_\Sigma H_2 \cong \#^{g-1} S^1 \times S^2$.*

Proof Let $v_1, v_2 \in P(\Sigma)$ define H_1 and H_2, respectively, with $D^P(v_1, v_2) = 1$. The pants decompositions corresponding to these vertices are exactly the same except for some loops $l_1 \in v_1$ and $l_2 \in v_2$. Moreover, since A–moves do not change the handlebody and $H_1 \neq H_2$, we know that l_1 and l_2 lie in a punctured torus with $|l_1 \cap l_2| = 1$. We may therefore build a Heegaard diagram for $H_1 \cup_\Sigma H_2$ consisting of $g-1$ identical loops in both v_1 and v_2, along with l_1 and l_2. It is easy to see that this is a once-stabilized splitting for $\#^{g-1} S^1 \times S^2$. □

Genus-g Heegaard splittings of $\#^{g-1} S^1 \times S^2$ are in some sense the second-most simple Heegaard splittings in a given genus after $\#^g S^1 \times S^2$. Genus-g trisections where two of the handlebodies form $\#^g S^1 \times S^2$ are easily shown to be diffeomorphic to $\#^g S^1 \times S^3$. Given these facts, it would be reasonable to assume that genus-g trisections where two of the handlebodies form $\#^{g-1} S^1 \times S^2$ are also relatively simple. The following theorem of [10] pertaining to unbalanced trisections makes this precise.

Theorem 4.2 [10, Theorem 1.2] *Suppose that M admits a $(g; g-1, k_2, k_3)$–trisection T, and let $k' = \max\{k_2, k_3\}$. Then M is diffeomorphic to either $\#^{k'} S^1 \times S^3$ or to the connect sum of $\#^{k'} S^1 \times S^3$ with one of either $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$, and T is a connect sum of genus-1 trisections.*

Theorem 4.3 *If $D^P(M_1, M_2) = 1$, then M_1 and M_2 have Kirby diagrams which are identical, except for the framing on some 2–handle.*

Proof Suppose the distance has stabilized in T_1 for M_1 and T_2 for M_2, where T_1 and T_2 are (g,k)–trisections. We will construct two new manifolds from these trisections following the schematic found in Figure 5. Consider the manifold obtained by removing X_1 from T_1 and X_1' from T_2, and gluing the resulting two manifolds

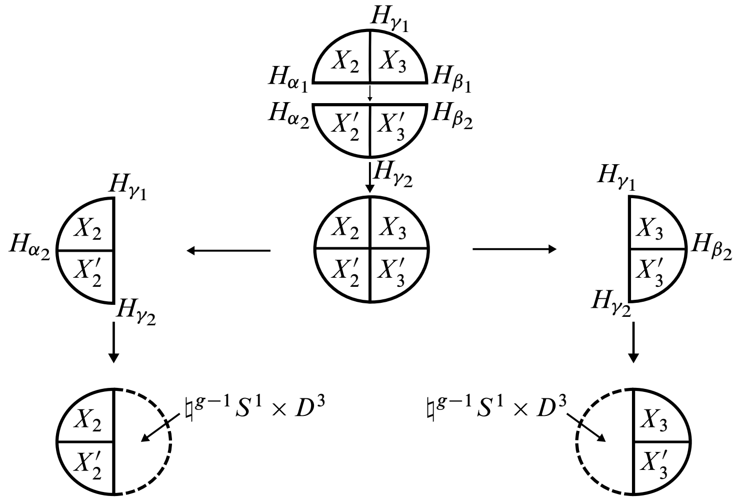

Figure 5: A schematic of the construction in Theorem 4.3

by the distance-minimizing map. Since $D^P(M_1, M_2) = 1$, Lemma 4.1 implies that the 3–manifold $H_{\gamma_1} \cup H_{\gamma_2}$ is diffeomorphic to $\#^{g-1} S^1 \times S^2$. We may cut the resulting 4–manifold along $H_{\gamma_1} \cup H_{\gamma_2}$ to obtain two 4–manifolds each with boundary $\#^{g-1} S^1 \times S^2$. We may fill in each of the resulting manifolds with boundary with $\natural^{g-1} S^1 \times D^3$ in order to obtain two trisected, closed 4–manifolds.

We will focus on the manifold with trisection spine $H_\alpha \cup H_{\gamma_1} \cup H_{\gamma_2}$. This closed 4–manifold inherits the structure of a $(g; g-1, g-k, g-k)$–trisection. By Theorem 4.2, this trisection is a connect sum of genus-1 trisections. In particular, there are curves $l_1, \dots, l_g$, all bounding disks in H_α, H_{γ_1} and H_{γ_2}, which cut Σ into g once-punctured tori and a sphere with g holes. We may also ensure that each of these tori contain one α, γ_1 and γ_2 curve, so that in all but one particular torus, the γ_1 and γ_2 curves are identical.

Let α^0, γ_1^0 and γ_2^0 be the three curves on the same punctured torus with $\gamma_1^0 \neq \gamma_2^0$. By virtue of the classification of genus-1 trisections, as well as the fact that $\gamma_1^0 \neq \gamma_2^0$, these three curves either form a diagram for $\mathbb{C}P^2$ or for S^4. However, if both T_1 and T_2 are balanced trisections, the three curves must form $\mathbb{C}P^2$, for, otherwise, $H_\alpha \cup H_\beta$ is $\#^k S^1 \times S^2$ whereas $H_\alpha \cup H_{\gamma'}$ is $\#^{k\pm 1} S^1 \times S^2$. After a diffeomorphism, α^0, γ_1^0 and γ_2^0 form the trisection diagram for $\mathbb{C}P^2$ shown in Figure 6. From here, it can be seen that γ_1^0 and γ_2^0 are related by a Dehn twist about a curve bounding a disk in H_α, so that after pushing them into H_α they become isotopic.

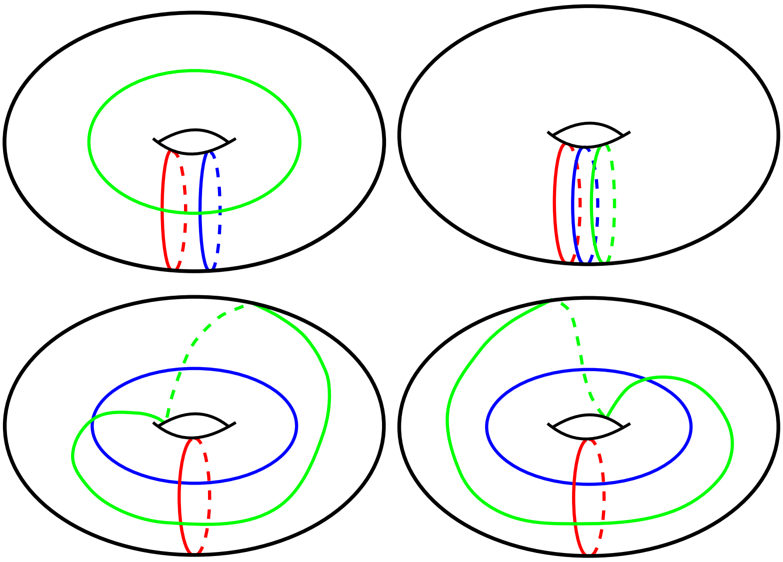

Figure 6: The genus-1 trisections S^4 (top-left), $S^1 \times S^3$ (top-right), $\mathbb{C}P^2$ (bottom-left) and $\overline{\mathbb{C}P^2}$ (bottom-right)

We may now take a diffeomorphism of the surface and perform handle slides of the α and β curves so that the α and β curves form the standard Heegaard diagram for $\#^k S^1 \times S^2$. By pushing the $g-k$ γ_1 and γ_2 curves dual to α curves into H_α, and giving them the surface framing, we obtain framed links L_1 and L_2 in $H_\alpha \cup H_\beta$. On page 3104 of [2], it is observed that the L_i are the framed attaching links for the 2–handles in a handle decomposition for M_i. Note that $g-k-1$ of these curves are identical, and the final curves have been shown to be isotopic in H_α, which completes the argument. □

Remark 4.4 The construction of $D^P(T_1, T_2)$ can be generalized to encompass unbalanced trisections where one of the k_i agrees on each trisection. We may then mimic the proof of Theorem 4.3 to study adjacent manifolds represented by unbalanced trisections. The proof goes through unchanged except that we must also consider the possibility that α^0, γ_1^0 and γ_2^0 form the unbalanced trisection diagram for S^4 shown in Figure 6. In this case, the γ curve parallel to the α curve does not play a role in the induced Kirby diagram whereas the γ curve dual to the α curve manifests itself as a 2–handle. We may therefore conclude that distance 1 in this more general case corresponds to either changing a handle framing by 1 (the balanced case) or adding or removing a 2–handle.

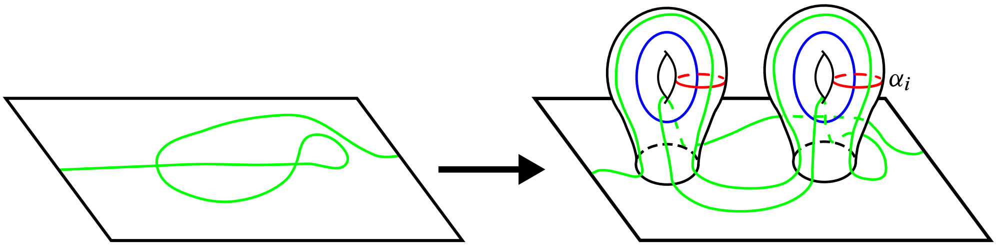

Figure 7: Resolving a Reidemeister 2 move of the attaching link on the Heegaard surface

We now seek to prove a partial converse to Theorem 4.3. We begin by understanding how to obtain a trisection diagram from a Kirby diagram. To do this, we follow the proof for the existence of trisections given in [2], while taking a little extra care on the particular diagram constructed. Take a Kirby diagram for M with k_1 1–handles and k_2 2–handles. The 0– and 1–handles form $\natural^{k_1} S^1 \times D^3$ and have boundary $\#^{k_1} S^1 \times S^2$. We may take a genus-k_1 Heegaard splitting for this boundary and draw k_1 parallel α and β curves on the surface which bound disks in both handlebodies. Now the framed attaching link for the 2–handles projects onto the Heegaard surface, with perhaps a few crossings. Do Reidemeister 1 moves on the link on the surface to make the surface framing match the handle framing, and do a Reidemeister 2 move on each component to make sure it has at least one self-crossing. Stabilize the Heegaard surface at each of the crossings to resolve them, resolving the self-crossings as in Figures 7 and 8. By construction, for each component of the link L_i, we may choose a dual α curve, α_i,

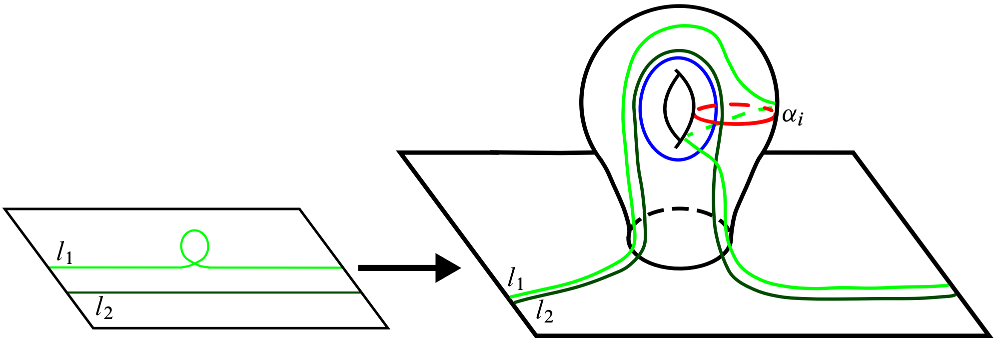

Figure 8: Resolving a Reidemeister 1 move to change the surface framing of l_1 by 1. Parallel curves such as l_2 can be sent over the stabilizing surface without twisting about α_i to preserve the surface framing.

that no other link component intersects. Then we may slide any other α curve α_j along L_i over α_i to eliminate any intersections between L_i and α_j.

Embedded on the Heegaard surface, we now see g α curves, g β curves and k_2 curves coming from the attaching link which are dual to k_2 α curves and disjoint from the rest of the α curves. We complete L to the set of γ curves by adding in $g-k_2$ curves parallel to each α curve which does not intersect any component of L. It is clear that the pairs of curves (α,β) and (α,γ) are Heegaard diagrams for the connect sum of some number of copies of $S^1\times S^2$. What is left to check is that the same holds for the pair (β,γ).

The γ curves define a handlebody, H_γ. Note that this handlebody is the result of pushing the γ curves dual to α curves into H_α and performing surface-framed Dehn surgery on them. But these dual curves come from the attaching link for the 2–handles of a closed 4–manifold. After attaching 2–handles along these curves pushed into H_α, H_α becomes H_γ, but H_β remains unchanged. Now H_γ and H_β form a Heegaard splitting for the boundary of the 3– and the 4–handles, so that the pair (β,γ) is indeed a Heegaard diagram for some number of copies of $S^1\times S^2$. We now have a possibly unbalanced trisection diagram for M, which we may balance by connect-summing with the genus-1 unbalanced trisection diagrams for S^4.

Theorem 4.5 *Let M_1 and M_2 be nondiffeomorphic 4–manifolds with the same Euler characteristic which have Kirby diagrams K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2–handle, where the framing differs by 1, then $D^P(M_1,M_2)=1$.*

Proof Let L_i be the framed attaching links for K_i and let l_i be the component of the L_i in which the framing differs. Without loss of generality, suppose that $|\mathrm{fr}(l_1)|>|\mathrm{fr}(l_2)|$ where $\mathrm{fr}(l_i)$ is the framing of l_i. Since K_1 and K_2 have the same 0– and 1–handles, we may project both attaching links onto the Heegaard surface for the boundary of the union of the 0– and 1–handles. Introduce self-intersections as previously described to obtain dual α curves, and to make the surface framing match the handle framing. This results in almost the same immersed curves on the Heegaard surface, except that l_1 has one more kink in it than l_2. Stabilize the Heegaard surface at all crossings, and, in the extra kink, send l_2 over the stabilizing genus without twisting, so as not to change the framing. See Figure 8 for an illustration of this process.

We must now choose dual α curves for each component of the link in order to eliminate intersections. Let α_i be the α curve in the stabilization where l_1 and l_2 differ. Choose

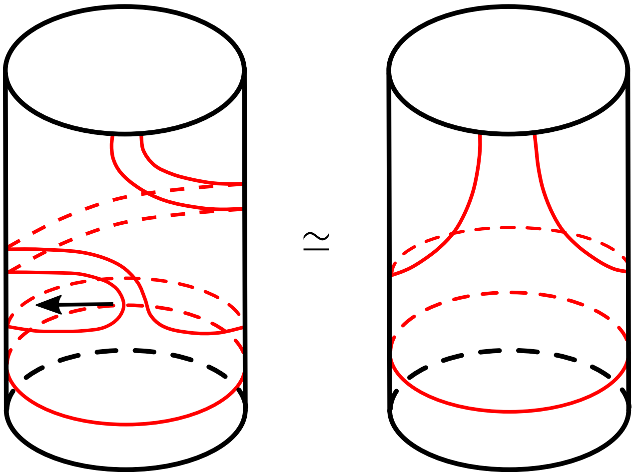

Figure 9: Dehn twisting the sliding arc about the target curve does not change the isotopy type of slid curve.

α_i to be the α curve dual to both l_1 and l_2 and choose arbitrary dual α curves for the rest of the components of the L_i. We now claim that eliminating the extra α intersections with L_i by sliding curves off over the dual α curves along arcs parallel to the link components results in identical α curves. Sliding any curve along a link component which is not l_1 or l_2 obviously results in the same curve, since we have constructed these curves to be identical. Moreover, sliding an α curve over α_i along l_1 is isotopic to sliding the α curve over α_i along l_2, as can be seen in Figure 9.

We now have possibly unbalanced trisection diagrams for M_1 and M_2 with identical α and β curves. We seek to show that the k_i for both of these manifolds are equal, so that we may connect sum with the same unbalanced trisections of S^4 in order to balance them. It is straightforward to show that a $(g;k_1,k_2,k_3)$–trisection has Euler characteristic $2+g-k_1-k_2-k_3$. First note that both of these trisections have the same genus. Furthermore, k_1 is the number of copies of $S^1\times S^2$ formed by the α and β curves, which is clearly the same for both trisections. In addition, k_3 comes from the α and γ curves, which we have constructed to be the same in both cases. Finally, the assumption that M_1 and M_2 have the same Euler characteristic ensures that these manifolds have equal k_3, so we may balance these trisections in an identical manner.

Finally, we complete both sets of γ curves to pants decompositions of the handlebodies to finish the argument. To this end, note that l_1 and l_2 intersect transversely in one point, so the boundary of a regular neighborhood of the curves bounds a disk in both handlebodies. This cuts off a punctured torus containing l_1 and l_2. Outside of this punctured torus, the γ handlebodies are identical and so we may complete them to an arbitrary pants decomposition. The resulting pants decompositions are easily seen to be 1 apart in the pants complex. □

We may also alter the framings of 2–handles by Dehn twisting about a chosen dual α curve which intersects no other component of the attaching link (recall that such curves may always be created by the introduction of self-crossings in the link component). The result of repeatedly Dehn twisting a link component about the given α curve may intersect our original link component many times, however both curves lie in the punctured torus filled by the α curve and the original link component. In addition, adding more Dehn twists to the sliding arc in Figure 9 does not change the isotopy type of the resulting curve, so that we may again eliminate intersections via isotopic handle slides of the α curves. These are all the essential ingredients to the following theorem, whose details we leave to the reader.

Theorem 4.6 *Let M_1 and M_2 be nondiffeomorphic 4–manifolds with the same Euler characteristic which have Kirby diagrams K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2–handle, then $D(M_1, M_2) = 1$.*

5 Complexes of trisections

We next seek to define a collection of graphs associated to trisections. Here, it is useful to consider the more general case of unbalanced trisections. Fix a surface, Σ, and two handlebodies, H_α and H_β, with boundary Σ, such that $H_\alpha \cup_\Sigma H_\beta \cong \#^{k_1} S^1 \times S^2$. We may identify the first two handlebodies in a $(g; k_1, k_2, k_3)$–trisection with $H_\alpha \cup_\Sigma H_\beta$. The third handlebody then gives rise to some handlebody subset of $P(\Sigma)$. We therefore have a subcomplex of the pants complex associated to any (possibly unbalanced) trisection with parameters $(g; k_1, -, -)$. This motivates the following definition:

Definition 5.1 Fix a genus-g surface Σ and two handlebodies H_α and H_β such that $H_\alpha \cup_\Sigma H_\beta \cong \#^{k_1} S^1 \times S^2$. Define the $(g; k_1, -, -)$ complex of trisections, $P(g, k_1)$, to be the full subgraph of the pants complex whose vertices are

$$\{\gamma \in P(\Sigma) \mid \gamma \text{ defines } H_\gamma,\ H_\alpha \cup_\Sigma H_\gamma \cong \#^{k_2} S^1 \times S^2 \text{ and } H_\beta \cup_\Sigma H_\gamma \cong \#^{k_3} S^1 \times S^2\}.$$

Definition 5.2 $\gamma \in P(g, k_1)$ is a *representative* for a trisection T if γ defines H_γ and $H_\alpha \cup H_\beta \cup H_\gamma$ is a spine for T. We say T_1 and T_2 are *adjacent* in $P(g, k_1)$ if they have representatives which are adjacent.

Note that a trisection has many representatives in $P(g, k_1)$. Not only are there infinitely many vertices in the pants complex defining the same handlebody, but multiple different handlebodies may represent the same trisection. For example, if $k_1 > 0$, there is some nonseparating curve which bounds disks in both H_α and H_β. A Dehn twist about this curve will usually change H_γ, but will give rise to a diffeomorphic trisection. More generally, we could take any element of the mapping class group $\mathrm{Mod}(H_\alpha \cup H_\beta, \Sigma)$ which does not extend across H_γ to produce similar results.

Lemma 5.3 *Let T be a stabilized trisection of M^4. Then there exists a trisection T' for $M \# \mathbb{C}P^2$ such that T and T' are adjacent in $P(g, k_1)$.*

The proof of the previous lemma is straightforward. We may in fact weaken the hypothesis that T is stabilized to the condition that T is 2– or 3–stabilized, but the lemma as stated will be sufficient for our needs. This lemma is useful to us because 4–manifolds can change drastically under connect sums with $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. The following corollary of Wall's theorem in [14] makes this precise.

Proposition 5.4 [3, Corollary 9.1.14] *Let M_1 and M_2 be simply connected 4–manifolds. Then there exist natural numbers l_1, l_2, m_1 and m_2 such that*

$$M_1 \# \#^{l_1} \mathbb{C}P^2 \# \#^{m_1} \overline{\mathbb{C}P^2} \quad \text{and} \quad M_2 \# \#^{l_2} \mathbb{C}P^2 \# \#^{m_2} \overline{\mathbb{C}P^2}$$

are diffeomorphic.

We are now well-equipped to prove the main proposition of this section.

Proposition 5.5 *Let M_1 and M_2 be simply connected, smooth, closed 4–manifolds. Then there exist $(g, k, -, -)$–trisections T_1 and T_2 for M_1 and M_2, respectively, such that T_1 and T_2 are in the same connected component of $P(g, k)$.*

Proof Take arbitrary trisections, T_1 of M_1 and T_2 of M_2. Now 1– and 2–stabilize them so that they have the same genus, g, and the same k_1. We will first calculate the number of stabilizations needed for the construction. Let l_1, l_2, m_1 and m_2 be as in Proposition 5.4. Let $a = \max\{l_1 + m_1, l_2 + m_2\}$. After 2–stabilizing T_1 and T_2

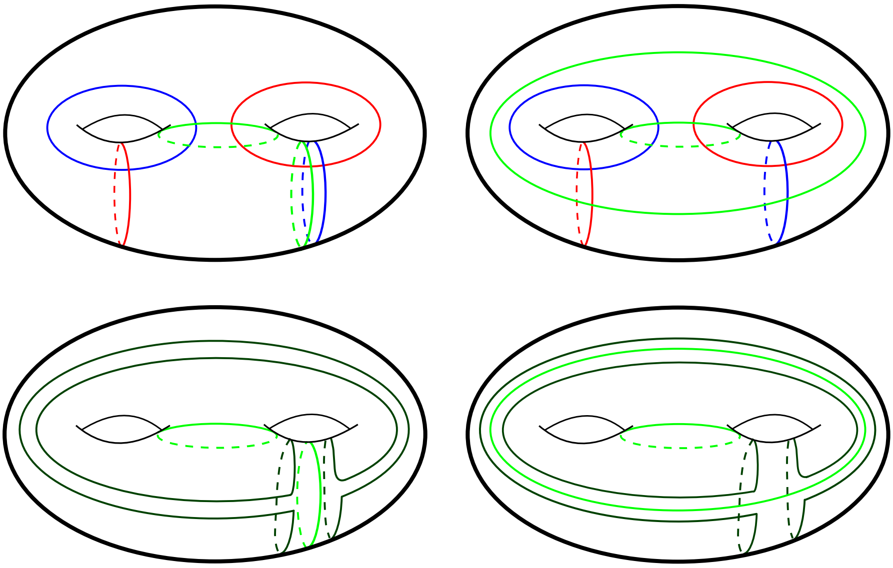

Figure 10: S^4 (left) is adjacent to $S^2 \times S^2$ (right) in $P(2,0)$.

a times, we may change each 2–stabilization into a summand of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ to obtain two (possibly different) trisections for the same 4–manifold. By Theorem 1.6, we may perform some number of balanced stabilizations on the resulting trisections until they are isotopic. Let b be the number of stabilizations needed to make the resulting trisections isotopic.

We claim that T_1^{g+a+3b} and T_2^{g+a+3b} can be connected in $P(g+a+3b, k_1+b)$. To see this, observe that, by Lemma 5.3, each 2–stabilization can be changed into an extra factor of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ adjacent to T_1^{g+a+3b} or T_2^{g+a+3b}. Changing each 2–stabilization in T_1^{g+a+3b} to the appropriate $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ summand successively leads to a path to a trisection of $M_1 \# \#^{l_1} \mathbb{C}P^2 \# \#^{m_1} \overline{\mathbb{C}P^2}$, which we know to be diffeomorphic to $M_2 \# \#^{l_2} \mathbb{C}P^2 \# \#^{m_2} \overline{\mathbb{C}P^2}$. Moreover, the constructed trisections have been stabilized enough to become isotopic. □

It is especially interesting to know which manifolds are adjacent to S^4, for if N is adjacent to S^4, then, for any M, we may stabilize a trisection to get an adjacent trisection for $M \# N$. It is straightforward to see that S^4 is adjacent to $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^1 \times S^3$. Furthermore, Figure 10 shows that S^4 is also adjacent to $S^2 \times S^2$. It is tempting to believe that this is a complete list of manifolds. In light of Remark 4.4, manifolds adjacent to S^4 correspond to starting with some (perhaps very complicated) Kirby diagram for S^4 and then changing the framing of some 2–handle, or adding/removing a 2–handle. We conclude with a question:

Question 5.6 Which 4–manifolds are adjacent to S^4?

References

[1] **R I Baykur**, **O Saeki**, *Simplifying indefinite fibrations on* 4*–manifolds*, preprint (2017) arXiv

[2] **D Gay**, **R Kirby**, *Trisecting* 4*–manifolds*, Geom. Topol. 20 (2016) 3097–3132 MR

[3] **R E Gompf**, **A I Stipsicz**, 4*–Manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc., Providence, RI (1999) MR

[4] **A Hatcher**, **W Thurston**, *A presentation for the mapping class group of a closed orientable surface*, Topology 19 (1980) 221–237 MR

[5] **J Hempel**, 3*–Manifolds as viewed from the curve complex*, Topology 40 (2001) 631–657 MR

[6] **J Johnson**, *Heegaard splittings and the pants complex*, Algebr. Geom. Topol. 6 (2006) 853–874 MR

[7] **J Johnson**, **H Rubinstein**, *Mapping class groups of Heegaard splittings*, J. Knot Theory Ramifications 22 (2013) art. id. 1350018 MR

[8] **F Laudenbach**, **V Poénaru**, *A note on* 4*–dimensional handlebodies*, Bull. Soc. Math. France 100 (1972) 337–344 MR

[9] **J Meier**, *Trisections and spun* 4*–manifolds*, preprint (2017) arXiv

[10] **J Meier**, **T Schirmer**, **A Zupan**, *Classification of trisections and the generalized property R conjecture*, Proc. Amer. Math. Soc. 144 (2016) 4983–4997 MR

[11] **K Reidemeister**, *Zur dreidimensionalen Topologie*, Abh. Math. Sem. Univ. Hamburg 9 (1933) 189–194 MR

[12] **J Singer**, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. 35 (1933) 88–111 MR

[13] **F Waldhausen**, *Heegaard-Zerlegungen der* 3*–Sphäre*, Topology 7 (1968) 195–203 MR

[14] **C T C Wall**, *On simply-connected* 4*–manifolds*, J. London Math. Soc. 39 (1964) 141–149 MR

Department of Mathematics, University of Virginia
Charlottesville, VA, United States

gfi8ps@virginia.edu

Received: 24 July 2017 Revised: 4 December 2017