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The eta-inverted sphere over the rationals

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We calculate the motivic stable homotopy groups of the two-complete sphere spectrum after inverting multiplication by the Hopf map η over fields of cohomological dimension at most 2 with characteristic different from 2 (this includes the p–adic fields $\mathbb{Q}_p$ and the finite fields $\mathbb{F}_q$ of odd characteristic) and the field of rational numbers; the ring structure is also determined.

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1 Introduction

Guillou and Isaksen [7] laid the foundation for calculating $\pi_{**}(\mathbb{1}_2^\wedge)[\eta^{-1}]$, the motivic stable homotopy groups of the two-complete sphere spectrum after inverting multiplication by η, over the complex numbers using the h_1–inverted motivic Adams spectral sequence. They conjectured a pattern of differentials in the h_1–inverted motivic Adams spectral sequence and identified the E_∞ page of the spectral sequence assuming the conjecture. Shortly after Guillou and Isaksen's paper appeared, Andrews and Miller [2] proved Guillou and Isaksen's conjecture. All together, these results show $\pi_{**}(\mathbb{1}_2^\wedge)[\eta^{-1}] \cong \mathbb{F}_2[\eta^{\pm 1}, \mu, \varepsilon]/(\varepsilon^2)$ [7, Conjecture 1.3]. Guillou and Isaksen [8] then analyzed the h_1–inverted motivic Adams spectral sequence over the real numbers and gave a complete calculation of the ring $\pi_{**}(\mathbb{1}_2^\wedge)[\eta^{-1}]$ over the base field $\mathbb{R}$.

The subject of this paper is the calculation of $\pi_{**}(\mathbb{1}_2^\wedge)[\eta^{-1}]$ over the field of rational numbers $\mathbb{Q}$ and fields F with $\mathrm{cd}_2(F) \leq 2$ and characteristic different from 2, such as the p–adic fields $\mathbb{Q}_p$ and finite fields $\mathbb{F}_q$ of odd characteristic. We write $\pi_{s,w}(\mathbb{1}_2^\wedge)(F)$ for the stable homotopy group $\mathcal{SH}(F)(\Sigma^{s,w}\mathbb{1}, \mathbb{1}_2^\wedge)$ and frequently abbreviate this to $\pi_{s,w}(\mathbb{1}_2^\wedge)$ if the base field F is clear from context.

We write $\mathfrak{M}(F)$ for the motivic Adams spectral sequence at the prime 2 over the field F at the motivic sphere spectrum. This spectral sequence has E_2 page given by $\mathfrak{M}(F)_2^{f,s,w} = \mathrm{Ext}_{\mathcal{A}_{**}(F)}^{f,s+f,w}(H_{**}(F), H_{**}(F))$ and conditionally converges:

$$\mathfrak{M}(F)^{f,s,w} \implies \pi_{s,w}(\mathbb{1}_H^\wedge)(F),$$

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where $\mathbb{1}^\wedge_H$ is the H–nilpotent completion of the sphere spectrum defined by Bousfield [4, Section 5]. Hu, Kriz and Ormsby [9, Theorem 1] proved that $\mathbb{1}^\wedge_H$ is weakly equivalent to $\mathbb{1}^\wedge_2$ over a field F of characteristic 0 with $\mathrm{cd}_2\big(F\big[\sqrt{-1}\big]\big) < \infty$. Wilson and Østvær [20, Proposition 5.10] note that the same argument works over fields over positive characteristic under the assumption that $\mathrm{cd}_2\big(F\big[\sqrt{-1}\big]\big) < \infty$.

Given the conditionally convergent spectral sequence $\mathfrak{M}(F) \Rightarrow \pi_{**}(\mathbb{1}^\wedge_2)(F)$ and the fact that $\eta \in \pi_{1,1}(\mathbb{1}^\wedge_2)(F)$ is detected by $h_1 \in \mathfrak{M}(F)^{1,1,1}$, one can try to calculate $\pi_{**}(\mathbb{1}^\wedge_2)(F)[\eta^{-1}]$ using the h_1–inverted spectral sequence, defined as the following colimit of spectral sequences:

$$\mathfrak{M}(F)[h_1^{-1}] = \mathrm{colim}(\mathfrak{M}(F) \xrightarrow{h_1\cdot} \mathfrak{M}(F) \xrightarrow{h_1\cdot} \mathfrak{M}(F) \cdots).$$

It is not obvious that $\mathfrak{M}(F)[h_1^{-1}]$ converges to $\pi_{**}(\mathbb{1}^\wedge_2)(F)[\eta^{-1}]$. Guillou and Isaksen show that it does converge for the complex numbers $\mathbb{C}$ in [7, Section 6] and the real numbers $\mathbb{R}$ in [8, Section 5]. We address convergence for more general fields in Section 2.

The Milnor–Witt t–stem of $\mathbb{1}^\wedge_2$ over F is the group $\widehat{\Pi}_t(F) = \bigoplus_{k\in\mathbb{Z}} \pi_{k+t,k}(\mathbb{1}^\wedge_2)(F)$. Note that $\widehat{\Pi}_0(F)$ is a ring and $\widehat{\Pi}_t(F)$ is a $\widehat{\Pi}_0(F)$–module. Our main results will be stated in terms of Milnor–Witt stems and the Witt group of quadratic forms $W(F)$. In many cases, the two-complete η–inverted Milnor–Witt 0–stem can be can be described in terms of the Witt ring of quadratic forms of the field.

Proposition 1 *If F is a field for which the Witt group of quadratic forms $W(F)$ is finitely generated or $W(F)$ has bounded 2–torsion exponent (if $F = \mathbb{Q}$, for example), then there is an isomorphism $\widehat{\Pi}_0(F)[\eta^{-1}] \cong W(F)^\wedge_2[\eta^{\pm 1}]$.*

Proof Morel [14] has shown there is an isomorphism $\pi_{n,n}(\mathbb{1}) \cong W(F)$ for $n \geq 1$. For $n \geq 1$ the homotopy group $\pi_{n,n}(\mathbb{1}^\wedge_2)$ fits into the exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/2^\infty, W(F)) \to \pi_{n,n}(\mathbb{1}^\wedge_2) \to \mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_{n-1,n}(\mathbb{1})) \to 0$$

by [9, Equation (2)]. But as $\pi_{n-1,n}\mathbb{1} = 0$ by Morel's connectivity theorem [14, Theorem 1.18], there is an isomorphism $\mathrm{Ext}(\mathbb{Z}/2^\infty, W(F)) \to \pi_{n,n}(\mathbb{1}^\wedge_2)$.

If $W(F)$ is finitely generated, there is an isomorphism $\mathrm{Ext}(\mathbb{Z}/2^\infty, W(F)) \cong W(F)^\wedge_2$ by a result of Bousfield and Kan [5, Chapter VI, Section 2.1], hence $\pi_{n,n}(\mathbb{1}^\wedge_2) \cong W(F)^\wedge_2$. If $W(F)$ has bounded 2–torsion exponent, then ${}_{2^n}W(F) = {}_{2^m}W(F)$ for all n and m sufficiently large. The Mittag-Leffler condition is satisfied for the tower $\{{}_{2^n}W(F)\}$, hence $\varprojlim^1 {}_{2^n}W(F) \cong \varprojlim^1 \operatorname{Hom}(\mathbb{Z}/2^n, W(F))$ vanishes. By the short exact sequence of Weibel [19, Application 3.5.10],

$$0 \to \varprojlim^1 \operatorname{Hom}(\mathbb{Z}/2^n, W(F)) \to \operatorname{Ext}(\mathbb{Z}/2^\infty, W(F)) \to W(F)^\wedge_2 \to 0,$$

there is an isomorphism $\operatorname{Ext}(\mathbb{Z}/2^\infty, W(F)) \cong W(F)^\wedge_2$, and so $\pi_{n,n}(\mathbb{1}^\wedge_2) \cong W(F)^\wedge_2$.

Finally, there is an isomorphism $\widehat{\Pi}_0(F)[\eta^{-1}] \cong W(F)^\wedge_2[\eta^{\pm 1}]$ since for any class $\alpha \in \widehat{\Pi}_0(F)$ and n sufficiently large, the class $\eta^n\alpha$ is an element of $\pi_{k,k}\mathbb{1} \cong W(F)$ with $k \geq 1$. □

For finite fields $\mathbb{F}_q$, the Milnor–Witt 0–stem is now determined by the calculation of the Witt group of finite fields, a standard reference being Scharlau [17, Chapter 2, Theorem 3.3]:

$$\widehat{\Pi}_0(\mathbb{F}_q)[\eta^{-1}] \cong W(\mathbb{F}_q)^\wedge_2[\eta^{\pm 1}] = \begin{cases} \mathbb{Z}/2[\eta^{\pm 1}, u]/u^2 & \text{if } q \equiv 1 \bmod 4, \\ \mathbb{Z}/4[\eta^{\pm 1}] & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

We find in Theorem 9 that for a field F with $\operatorname{cd}_2(F) \leq 2$ and characteristic different from 2, the two-complete η–inverted Milnor–Witt stems take the following form:

$$\widehat{\Pi}_t(F)[\eta^{-1}] \cong \begin{cases} W(F)^\wedge_2[\eta^{\pm 1}] & \text{if } t \geq 0 \text{ and either } t \equiv 3 \bmod 4 \text{ or } t \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a complete calculation of $\widehat{\Pi}_*(\mathbb{F}_q)[\eta^{-1}]$ for the finite fields $\mathbb{F}_q$ of odd characteristic.

Theorem 19 calculates the ring $\widehat{\Pi}_*(\mathbb{Q})[\eta^{-1}]$. In particular, the Milnor–Witt stems are

$$\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}] \cong \begin{cases} W(\mathbb{Q})^\wedge_2[\eta^{\pm 1}] & \text{if } t = 0, \\ W(\mathbb{Q})^\wedge_2[\eta^{\pm 1}]/2^{n+1} & \text{if } t \geq 0,\ t \equiv 3 \bmod 4,\ n = \nu_2(t+1), \\ M & \text{if } t \equiv 0 \bmod 4,\ t \geq 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $\nu_2(t+1)$ is the 2–adic valuation of $t+1$ and M is the submodule of $W(\mathbb{Q})^\wedge_2[\eta^{\pm 1}]$ defined in Definition 18.

The method of proof for the calculations over $\mathbb{Q}$ of Theorem 19 follows the strategy employed by Ormsby and Østvær [15] to calculate the homotopy groups of $BP\langle n\rangle$ over $\mathbb{Q}$. First, for each completion $\mathbb{Q}_\nu$ of $\mathbb{Q}$ one uses the ρ–Bockstein spectral sequence to calculate $\operatorname{Ext}(\mathbb{Q}_\nu)[h_1^{-1}]$ and then the motivic Adams spectral sequence to calculate $\pi_{**}(\mathbb{1}^\wedge_2)(\mathbb{Q}_\nu)[\eta^{-1}]$. We next follow the motivic Hasse principle to identify the differentials in the ρ–Bockstein spectral sequence and the motivic Adams spectral sequence over $\mathbb{Q}$ by comparing these spectral sequences with the associated spectral sequences over the completions.

The result of Ormsby, Röndigs and Østvær [16, Proof of Theorem 1.5] shows that the vanishing $\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}] = 0$ when $t \equiv 1, 2 \bmod 4$ occurs systematically for all formally real fields F with $\mathrm{cd}_2(F[i]) < \infty$. Their calculation is used in this paper to show that $\widehat{\Pi}_1(\mathbb{Q})[\eta^{-1}]$ vanishes, as it is unclear whether or not the motivic Adams spectral sequence over $\mathbb{Q}$ converges strongly in Milnor–Witt stem 1.

Ananyevskiy, Levine and Panin [1] investigate the η–inverted sphere spectrum $\mathbb{1}[\eta^{-1}]$ over fields F of characteristic different from 2. They find that the stable homotopy sheaf $\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}\mathbb{1}[\eta^{-1}]$ is isomorphic to the sheaf $\underline{W}[\eta^{\pm 1}]$, where $\underline{W}$ is the Nisnevich sheaf associated to the presheaf of Witt groups (the Witt group $W(X)$ of an algebraic variety X is defined by Knebusch [12, Chapter I, Section 5]). The consequence of this for calculating stable homotopy groups is that

$$\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(\mathbb{1}[\eta^{-1}])(F) \cong W(F)[\eta^{\pm 1}]$$

for all fields F of characteristic different from 2. In addition to this absolute statement about the η–inverted Milnor–Witt 0–stem, they identify the rationalization of $\mathbb{1}[\eta^{-1}]$ with an object in the heart of the homotopy t–structure on $\mathcal{SH}(F)$ [1, Theorem 3.4] and find that the sheaf $\pi_{s,w}(\mathbb{1}[\eta^{-1}]_{\mathbb{Q}})$ takes the following form:

$$\pi_{s,w}(\mathbb{1}[\eta^{-1}]_{\mathbb{Q}}) = \begin{cases} \underline{W}_{\mathbb{Q}} & \text{if } s = w, \\ 0 & \text{otherwise.} \end{cases}$$

The calculations in this paper are about the η–inverted 2–complete sphere spectrum $\mathbb{1}^{\wedge}_2[\eta^{-1}]$ in contrast to Ananyevskiy, Levine and Panin's results about $\mathbb{1}[\eta^{-1}]$ and $\mathbb{1}[\eta^{-1}]_{\mathbb{Q}}$.

We will follow the grading conventions for $\mathrm{Ext}(F) = \mathrm{Ext}_{\mathcal{A}_{**}(F)}(H_{**}(F), H_{**}(F))$ employed by Guillou and Isaksen [7, Section 2.1]. In particular, for a class $x \in \mathrm{Ext}(F)$ in Adams filtration f, stem s, and weight w, the Milnor–Witt stem of x is $t = s - w$ and the Chow weight of x is $c = s + f - 2w$. We will write degrees as $\deg(x) = (f, t, c)$ unless otherwise specified. We will frequently use the isomorphism $\mathrm{Ext}(\mathbb{C})[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4, v_2, v_3, \dots]$ with $\deg(v_1^4) = (4, 4, 4)$ and $\deg(v_n) = (1, 2^n - 1, 1)$ established in [7, Theorem 3.4], and we adopt the convention of writing P for v_1^4.

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2 Convergence of the h_1–inverted motivic Adams spectral sequence

We refer the reader to Boardman's notes on spectral sequences [3] for the terminology concerning their convergence. The notion of a map of filtered groups compatible with a map of spectral sequences is defined by Weibel in [19, page 126].

Consider a collection of cohomologically graded spectral sequences $({}^iE_r, {}^id_r)$ for $i \in \mathbb{N}$, where ${}^iE_r^s = 0$ for all $s < 0$ and all r, and each spectral sequence ${}^iE$ converges strongly to an abelian group ${}^iG$ filtered by ${}^iF_s$. Here r indicates the page of the spectral sequence and s indicates the internal degree. We will omit these subscripts and superscripts where it is convenient. These assumptions on our spectral sequences precisely mean the following conditions hold:

(1) The filtration is exhaustive with ${}^iF_0 = {}^iG$:

$$ {}^iG = {}^iF_0 \supseteq {}^iF_1 \supseteq {}^iF_2 \supseteq \cdots . $$

(2) For all $s \in \mathbb{Z}$ there are isomorphisms ${}^iE^s_\infty \cong {}^iF_s / {}^iF_{s+1}$.

(3) The filtration is Hausdorff: $\bigcap_j {}^iF_j = 0$.

(4) The filtration is complete: $\varprojlim_j {}^iG/{}^iF_j \cong {}^iG$.

We assume the spectral sequences are $\mathbb{Z}^k$–graded in addition to the internal grading s. In practice, k is either 1 or 2.

Let ${}^if\colon {}^iE \to {}^{i+1}E$ be a directed system of maps of spectral sequences compatible with ${}^ig\colon {}^iG \to {}^{i+1}G$ of degree $a \in \mathbb{Z}^k$. The colimit of the directed system ${}^if\colon {}^iE \to {}^{i+1}E$ is again a spectral sequence $\widetilde{E} = \operatorname{colim} {}^iE$ that converges weakly to $\widetilde{G} = \operatorname{colim} {}^iG$ filtered by $\widetilde{F}_j = \operatorname{colim} {}^iF_j$. Under what circumstances can we guarantee $\widetilde{E}$ converges strongly to $\widetilde{G}$?

The filtration $\widetilde{F}_j$ of $\widetilde{G}$ may fail to be Hausdorff or complete. To see how the Hausdorff condition may fail, consider ${}^iE = \mathbb{Z}$, ${}^iG = \mathbb{Z}$, and ${}^iF_j = 0$ for $j \geq i$ and ${}^iF_j = {}^iG$ for $j < i$. If we define maps ${}^if = 0$ and ${}^ig(1) = 1$, then $\widetilde{E} = 0$, $\widetilde{G} = \mathbb{Z}$, and $\widetilde{F}_j = \mathbb{Z}$ for all $j \geq 0$. The issue is that the class $1 \in {}^iG$ is in filtration i, which shows $1 \in \widetilde{G}$ is in filtration i for all natural numbers i.

Completeness can fail if each ${}^iG$ has a finite filtration but the colimit $\widetilde{G}$ has an infinite filtration. Consider ${}^iG = \mathbb{Z}$, ${}^iF_j = 2^j\mathbb{Z}$ for $j \leq i$ and ${}^iF_j = 0$ for $j > i$ with the map ${}^ig(1) = 1$. Then $\widetilde{G} = \mathbb{Z}$, yet $\widetilde{E}^s \cong \widetilde{F}_s/\widetilde{F}_{s+1} \cong \mathbb{Z}/2$ for all $s \geq 0$, and $\varprojlim_s \widetilde{G}/\widetilde{F}_s$ is isomorphic to the 2–adic integers $\mathbb{Z}_2$.

Definition 2 Consider a directed system of $\mathbb{Z}^k$–graded spectral sequences ${}^if\colon {}^iE \to {}^{i+1}E$ of degree a, that is, for all $i \in \mathbb{N}$ and $x \in {}^iE$ the degree of ${}^if(x)$ is $a + \deg(x)$, and ${}^if$ does not change the internal degree s. The directed system ${}^if$ has a horizontal vanishing line of height N in the direction a if for any degree b there exists $K \in \mathbb{N}$ for which the groups ${}^iE^{s,b+ia}$ vanish for all $i > K$ and $s > N$.

The term horizontal vanishing line comes from the special case where for all i we have ${}^iE = E$ and $E = \bigoplus_{s,p} E^{s,p}$ is a $\mathbb{Z}$–graded spectral sequence in p with internal degree s. If one makes a chart for ${}^iE$ where the vertical axis is the internal degree s and the horizontal axis is p, a horizontal vanishing line of height N in the direction 1 says that $E^{s,p}$ vanishes when (s,p) is above the horizontal line $s = N$ and p is sufficiently large.

Proposition 3 *Suppose ${}^if\colon {}^iE \to {}^{i+1}E$ is a directed system of $\mathbb{Z}^k$–graded spectral sequences of degree a with a horizontal vanishing line of height N in the direction a. The colimit spectral sequence $\widetilde{E}$ then converges strongly to $\widetilde{G}$ with respect to the filtration $\widetilde{F}$.*

Proof We first show the filtration $\widetilde{F}$ of $\widetilde{G}$ is complete. For $b \in \mathbb{Z}^k$, the degree-b component of $\widetilde{G}$ is

$$\widetilde{G}^b = \operatorname{colim}({}^0G^b \to {}^1G^{b+a} \to \cdots \to {}^iG^{b+ia} \to \cdots).$$

The assumption that there is a horizontal vanishing line of height N in the direction a implies for all $i > K$ the filtration of ${}^iG^{b+ia}$ is finite. This is because the filtration of ${}^iG$ is Hausdorff and ${}^iE^{s,b+ia}$ vanishes for $s > N$, so ${}^iF_j$ is trivial for $i > K$

and $j > N$. Since finite limits and directed colimits commute, it follows that

$$\begin{aligned}\varprojlim_j (\widetilde{G}^b/\widetilde{F}_j^b) &\cong \varprojlim_j (\operatorname*{colim}_{i>K} {}^iG/{}^iF_j)\\ &\cong \operatorname*{colim}_{i>K}\bigl(\varprojlim_j {}^iG/{}^iF_j\bigr)\\ &\cong \operatorname*{colim}_{i>K} {}^iG^{b+ia}\\ &\cong \widetilde{G}^b.\end{aligned}$$

We now show the filtration $\widetilde{F}_j$ of $\widetilde{G}$ is Hausdorff. Let $x \in \widetilde{G}^b$ be a nonzero element. Lemma 4 shows there is some ${}^ix \in {}^iG^{b+ia}$ which maps to $x \in \widetilde{G}^b$ for which ${}^ix$ is detected by ${}^iy \in {}^iE_r^{s,b+ia}$ and ${}^{i+k+1}y$ is nonzero for all $k \in \mathbb{N}$. Since ${}^if$ is compatible with ${}^ig$, it follows that ${}^{i+k+1}y$ detects ${}^{i+k+1}x = {}^{i+k}g \circ \cdots \circ {}^ig({}^ix)$ for all $k \in \mathbb{N}$. Furthermore, ${}^{i+k}x \in {}^{i+k}G$ is nonzero for all $k \in \mathbb{N}$, and so ${}^{i+k}y$ survives to ${}^{i+k}E_\infty^{s,b+(i+k)a}$. Our assumption that the spectral sequences ${}^iE$ converge strongly to ${}^iG$ means that

$${}^{i+k}E_\infty^{s,b+(i+k)a} \cong {}^{i+k}F_s/{}^{i+k}F_{s+1}.$$

Hence every class ${}^{i+k}x$ is in filtration s but not $s+1$, so that $x \in \widetilde{F}_s$ but $x \notin \widetilde{F}_{s+1}$. It now follows that the filtration $\widetilde{F}$ of $\widetilde{G}$ is Hausdorff. □

Lemma 4 *Under the conditions of Proposition 3, consider a nonzero element $x \in \widetilde{G}^b$. There exists some ${}^ix \in {}^iG^{b+ia}$ that maps to $x \in \widetilde{G}^b$ for which ${}^ix$ is detected by ${}^iy \in {}^iE_r^{s,b+ia}$ and*

$${}^{i+k+1}y = {}^{i+k}f \circ \cdots \circ {}^if({}^iy) \in {}^{i+k+1}E_r^{s,b+(i+k+1)a}$$

is nonzero for all $k \in \mathbb{N}$.

Proof There is some ${}^jx \in {}^jG$ which maps to $x \in \widetilde{G}$. The classes

$${}^{j+k}x = {}^{j+k-1}g \circ \cdots \circ {}^jg({}^jx)$$

are nonzero for all $k \geq 1$ and are therefore detected by some class ${}^{j+k}y \in {}^{j+k}E^{s_k,*}$. The vanishing line implies $s_k \leq N$ for k sufficiently large, and the compatibility of the maps ${}^jf$ with ${}^jg$ implies s_k is a nondecreasing function of k. Hence s_k is eventually constant, say for all $k+j \geq i$. Then ${}^ix$ has the desired property. □

These results can be applied to inverting multiplication by h_1 in the motivic Adams spectral sequence at the prime 2 after reindexing the filtration. For a field F, write ${}^iE^f$

for $\mathfrak{M}(F)^{f+i}$. Here f is the internal degree of the spectral sequence ("f" for "filtration"). With this convention, the maps $h_1\cdot\colon {}^iE \to {}^{i+1}E$ form a directed system of spectral sequences which is compatible with the maps $\eta\colon {}^iG \to {}^{i+1}G$, where ${}^if_j = F_{j+i}(\pi_{**}(\mathbb{1}_2^\wedge))$. The degree of multiplication by η is $(s,w) = (1,1)$, where s is the stem and w the weight. A horizontal vanishing line of height N in the direction $(1,1)$ is equivalent to the following condition: for any (s,w) there exists k such that for all $i > k$ and $f > N$ the group $\mathfrak{M}(F)^{f+i,s+i,w+i}$ vanishes. In the usual manner of drawing a chart for $\mathfrak{M}(F)$, such as those made by Isaksen [10], the horizontal vanishing line for the system ${}^iE^f$ is transformed into a line of slope 1.

Such vanishing conditions occur over $\mathbb{R}$ in positive Milnor–Witt stems as proved by Guillou and Isaksen [8, Lemma 5.1]. Over $\mathbb{R}$ it suffices to take $N = 1$, but one must take larger values for other fields. For fields of cohomological dimension at most 2 and number fields, take $N = 3$ for the positive Milnor–Witt stems.

Corollary 5 *The h_1–inverted motivic Adams spectral sequence over fields of cohomological dimension at most 2 and the field of rational numbers $\mathbb{Q}$ converges strongly to $\pi_{s,w}(\mathbb{1}_2^\wedge)[\eta^{-1}]$ when $s - w > 1$.*

Proof Consider the directed system ${}^iE^f = \mathfrak{M}(F)^{f+i}$ with maps $h_1\cdot\colon {}^iE \to {}^{i+1}E$ as described above. With $N = 3$, the vanishing conditions required for Proposition 3 are satisfied for fields F of cohomological dimension at most 2. The ρ–Bockstein spectral sequence for such a field has E_1 page $H_{**}(F) \otimes_{\mathbb{F}_2[\tau]} \mathrm{Ext}(\mathbb{C})$ and converges off to $\mathrm{Ext}(F)$. The E_1 page of the ρ–Bockstein spectral sequence has the claimed vanishing line in positive Milnor–Witt stem; hence $\mathrm{Ext}(F)$ does too.

An argument similar to the one given by Guillou and Isaksen in [8, Lemma 5.1] establishes a vanishing line over $\mathbb{Q}$ in positive Milnor–Witt stems with $N = 3$. Their choice of A works just as well over $\mathbb{Q}$ (A corresponds to k when $s = 0$ in the notation above) because the ρ–inverted Hopf algebroid $(H_{**}(\mathbb{Q})[\rho^{-1}], \mathcal{A}_{**}(\mathbb{Q})[\rho^{-1}])$ is isomorphic to the ρ–inverted Hopf algebroid over $\mathbb{R}$. Their argument with the ρ–Bockstein spectral sequence must only be modified to account for y being of the form $y = \alpha\tilde{y}$ with $\tilde{y} \in \mathrm{Ext}(\mathbb{C})$, and $\alpha \in H_{i,i}(\mathbb{Q})$ with $i \le 2$ and α not divisible by ρ.

The motivic Adams spectral sequence for fields F with $\mathrm{cd}_2(F) \le 2$ and $\mathbb{Q}$ converges conditionally to $\pi_{**}(\mathbb{1}_2^\wedge)$ by [9, Theorem 1] of Hu, Kriz and Ormsby. The vanishing line described above ensures that it also converges strongly in Milnor–Witt stem at least 2, as in such degrees $d_r = 0$ for r sufficiently large. Hence we get the convergence result of the h_1–inverted Adams spectral sequences. □

3 Fields of cohomological dimension at most 2

Let F be a field of 2–cohomological dimension at most 2. The mod 2 Milnor K–theory of such a field satisfies $k_n^M(F) = 0$ for $n \geq 3$. We first calculate $\mathrm{Ext}(F)[h_1^{-1}]$ using the ρ–Bockstein spectral sequence and then observe that the structure of $\mathfrak{M}(F)_2 \cong \mathrm{Ext}(F)[h_1^{-1}]$ forces the h_1–inverted motivic Adams spectral sequence to collapse at the E_2 page. See Figures 1 and 2 for a depiction of the ρ–Bockstein spectral sequence E_1 and E_∞ pages up to Milnor–Witt stem 24.

Proposition 6 *For F a field with $\mathrm{cd}_2(F) \leq 2$, the E_2 page of $\mathfrak{M}(F)[h_1^{-1}]$ is*

$$\mathrm{Ext}(F)[h_1^{-1}] \cong k_*^M(F) \otimes \mathrm{Ext}(\mathbb{C})[h_1^{-1}].$$

Proof If -1 is a square in F, it follows that $\mathrm{Ext}(F) \cong H_{**}(F) \otimes_{\mathbb{F}_2[\tau]} \mathrm{Ext}(\mathbb{C})$ by an argument similar to [20, Proposition 7.1]. The class τ is killed after inverting h_1, so the result follows in this case.

If -1 is not a square in F, use the ρ–Bockstein spectral sequence. The E_1 page of the h_1–inverted ρ–Bockstein spectral sequence is

$$E_1^{\epsilon,*,*,*} \cong \rho^\epsilon k_*^M(F)/\rho^{\epsilon+1} k_*^M(F) \otimes \mathrm{Ext}(\mathbb{C})[h_1^{-1}],$$

and the d_r differential has degree $(r, 1, -1, 0)$ with respect to the grading (ϵ, f, t, c).

The differentials d_r with $r \geq 1$ vanish on the generators $P = v_1^4$ and v_n for $n \geq 2$ of $\mathrm{Ext}(\mathbb{C})[h_1^{-1}]$ for degree reasons. Any nonzero class $x \in \mathrm{Ext}(\mathbb{C})[h_1^{-1}]$ has $t + c \equiv 0 \bmod 4$, but the degree of $d_r(x)$ satisfies $t + c \equiv 3 \bmod 4$. If F has cohomological dimension at most 2, then any nonzero class in the h_1–inverted ρ–Bockstein spectral sequence satisfies $t + c \not\equiv 3 \bmod 4$. Hence the h_1–inverted ρ–Bockstein spectral sequence collapses. There is no possibility for hidden extensions, so the proposition follows. □

Proposition 7 *If $\overline{F}$ is an algebraically closed field of characteristic different from 2, the η–inverted motivic homotopy groups of spheres over $\overline{F}$ are given by*

$$\pi_{**}(\mathbb{1}_2^\wedge)(\overline{F})[\eta^{-1}] \cong \pi_{**}(\mathbb{1}_2^\wedge)(\mathbb{C})[\eta^{-1}] \cong \mathbb{F}_2[\eta^{\pm 1}, \mu, \varepsilon]/(\varepsilon^2),$$

where $\mu \in \pi_{9,5}(\mathbb{1}_2^\wedge)$ is the unique homotopy class detected by Ph_1 and $\varepsilon \in \pi_{8,5}(\mathbb{1}_2^\wedge)$ is the unique homotopy class detected by c_0.

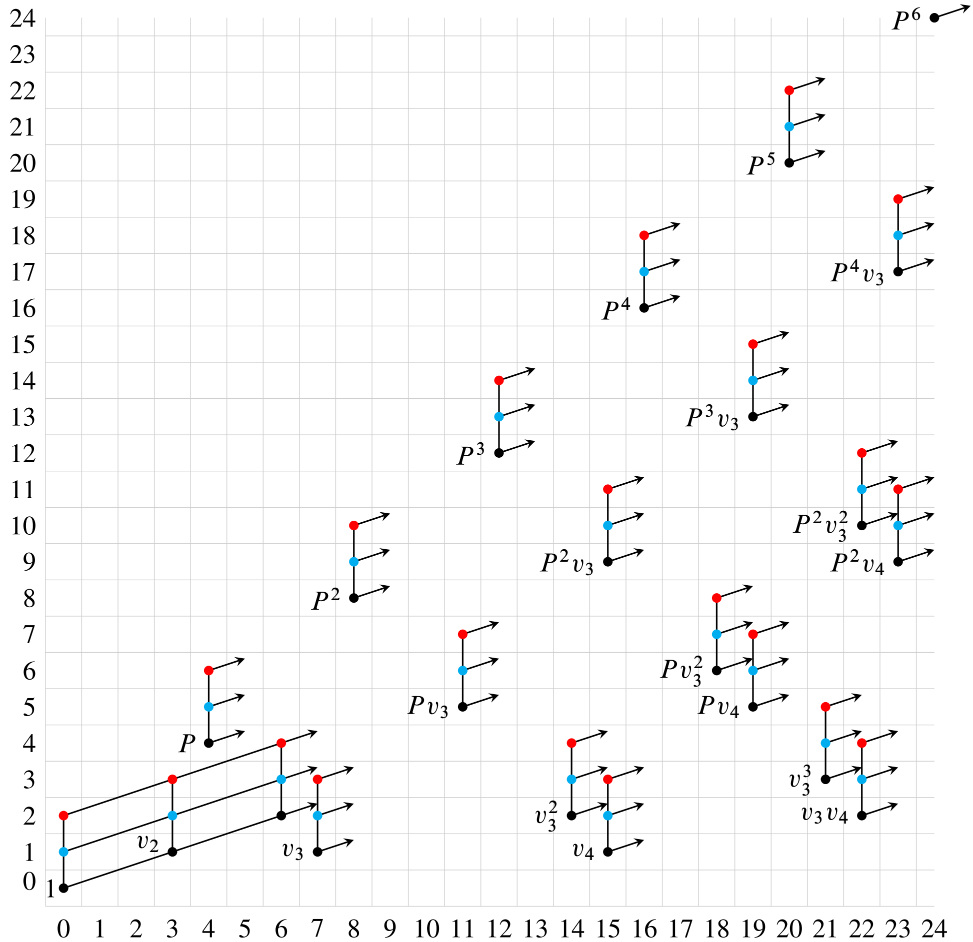

Figure 1: The E_1 page of the h_1–inverted ρ–Bockstein spectral sequence over a field F with $\mathrm{cd}_2(F)=2$ up to Milnor–Witt stem 24. Solid vertical lines indicate possible ρ–multiplications that depend on the field. Black dots represent the group $\mathbb{Z}/2[h_1^{\pm 1}]$, blue dots represent $k_1^M(F)[h_1^{\pm 1}]$, and red dots represent $k_2^M(F)[h_1^{\pm 1}]$. Solid lines of slope $\frac{1}{3}$ indicate multiplication by v_2 and arrows in this direction represent a tower of nonzero v_2 multiples. The horizontal axis t is the Milnor–Witt stem and the vertical axis c is the Chow weight, while the Adams filtration is suppressed.

Proof If $\bar{F}$ has characteristic zero, there is an isomorphism $\mathfrak{M}(\bar{F}) \cong \mathfrak{M}(\mathbb{C})$ by the proof of [20, Lemma 6.4]. If $\bar{F}$ has positive characteristic, the change of characteristic argument [20, Corollary 6.1] comparing $\mathfrak{M}(\bar{F})$ to $\mathfrak{M}(\mathbb{C})$ via the motivic Adams spectral sequence over the ring of Witt vectors of $\bar{F}$ shows there is an isomorphism of

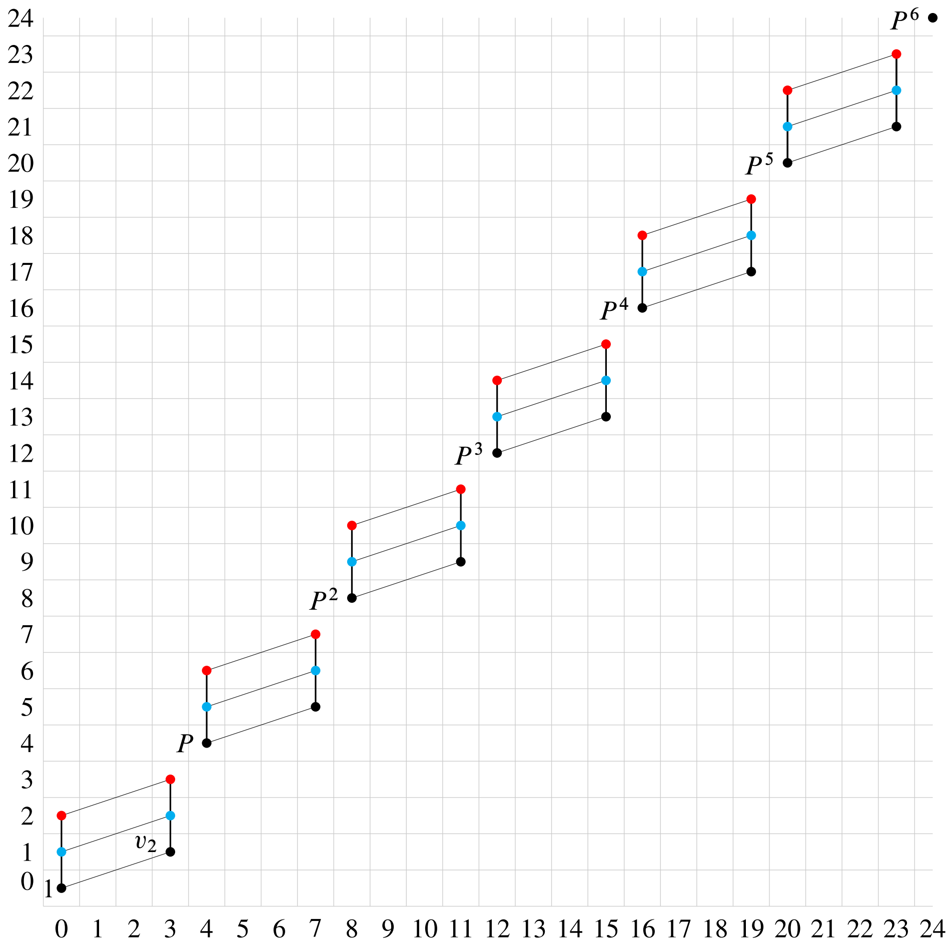

Figure 2: The E_∞ page of the h_1–inverted motivic Adams spectral sequence over a field F with $\mathrm{cd}_2(F) \le 2$ up to Milnor–Witt stem 24. The notational conventions of Figure 1 apply here.

spectral sequences $\mathfrak{M}(\bar{F}) \cong \mathfrak{M}(\mathbb{C})$. This isomorphism propagates to an isomorphism after inverting multiplication by h_1. The now resolved conjecture of Guillou and Isaksen in [7, Conjecture 1.3] gives the explicit description. □

Proposition 8 *The d_2 differentials for the h_1–inverted motivic Adams spectral sequence for a field F with characteristic different from 2 and $\mathrm{cd}_2(F) \le 2$ follow from $d_2(v_n) = h_1^2 v_{n-1}^2$ for $n \ge 3$ and $d_2(x) = 0$ for $x \in k_*^M(F)$ by using the Leibniz rule. Furthermore, $\mathfrak{M}(F)[h_1^{-1}]$ collapses at the E_3 page.*

Proof The inclusion of F into its algebraic closure $\bar{F}$ induces a map of spectral sequences

$$\Phi\colon \mathfrak{M}(F)[h_1^{-1}] \to \mathfrak{M}(\bar{F})[h_1^{-1}] \cong \mathfrak{M}(\mathbb{C})[h_1^{-1}].$$

Andrews and Miller [2, Theorem 9.15] have proved that in $\mathfrak{M}(\mathbb{C})[h_1^{-1}]$ there are differentials $d_2(v_n) = h_1^2 v_{n-1}^2$ for all $n \geq 3$. It follows that in $\mathfrak{M}(F)[h_1^{-1}]$ we must have $d_2(v_n) = h_1^2 v_{n-1}^2$ up to some element in the kernel of the comparison map Φ. A class $x \in \ker(\Phi)$ satisfies $t + c \equiv 1 \bmod 4$ or $t + c \equiv 2 \bmod 4$, whereas $d_2(v_n)$ satisfies $t + c \equiv 0 \bmod 4$. Hence $d_2(v_n) = h_1^2 v_{n-1}^2$ is true on the nose. That the spectral sequence collapses at the E_3 page follows by degree reasons. □

Theorem 9 *For a field F with $\mathrm{cd}_2(F) \leq 2$ and characteristic different from 2, the two-complete η–inverted Milnor–Witt stems of F are*

$$\widehat{\Pi}_t(F)[\eta^{-1}] \cong \begin{cases} W(F)^\wedge_2[\eta^{\pm 1}] & \text{for } t \geq 0 \text{ and either } t \equiv 3 \bmod 4 \text{ or } t \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

$\widehat{\Pi}_*(F)[\eta^{-1}]$ is the polynomial ring over $W(F)^\wedge_2[\eta^{\pm 1}]$ on two classes $\{v_2\}$ and $\{P\}$ in Milnor–Witt stems 3 and 4 respectively, subject to the relation $\{v_2\}^2 = 0$.

Proof $\widehat{\Pi}_0(F)[\eta^{-1}]$ is shown to be $W(F)^\wedge_2[\eta^{\pm 1}]$ in Proposition 1. The remaining stems and ring structure follow from the calculation of the h_1–inverted motivic Adams spectral sequence over F whose differentials are determined in Proposition 8. □

We now identify some classes in $\pi_{**}(\mathbb{1}^\wedge_2)(\mathbb{F}_q)$ for finite fields $\mathbb{F}_q$ using the analysis of the motivic Adams spectral sequence by Wilson and Østvær [20]. Over a finite field $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$, define $\varepsilon \in \pi_{8,5}(\mathbb{1}^\wedge_2) \cong (\mathbb{Z}/2)^4$ to be a class detected by c_0. The class ε is uniquely determined modulo $u\eta\varepsilon$. Write μ for a class in $\pi_{9,5}(\mathbb{1}^\wedge_2) \cong (\mathbb{Z}/2)^4$ detected by Ph_1. The class μ is uniquely determined modulo $u\eta\mu$.

Over a finite field $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, there is an isomorphism $\pi_{8,5}(\mathbb{1}^\wedge_2) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$. Recall there is a Hopf map $\sigma \in \pi_{7,4}(\mathbb{1}^\wedge_2)$ defined by Dugger and Isaksen in [6]. The class $\eta\sigma$ generates an order-four cyclic subgroup of $\pi_{8,5}(\mathbb{1}^\wedge_2)$; define $\varepsilon \in \pi_{8,5}(\mathbb{1}^\wedge_2)$ by the property that ε generates $\pi_{8,5}(\mathbb{1}^\wedge_2)/(\eta\sigma)$. The class ε is detected by c_0 and well defined up to an odd multiple. Further, there is an isomorphism $\pi_{9,5}(\mathbb{1}^\wedge_2) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2$. Define μ to be a class of order four that is detected by Ph_1; the class μ is uniquely defined up to an odd multiple.

Corollary 10 *For a finite field $\mathbb{F}_q$ with q odd, the η–inverted Milnor–Witt stems are as follows:*

$$\widehat{\Pi}_n(\mathbb{F}_q)[\eta^{-1}] \cong \begin{cases} W(\mathbb{F}_q)^{\wedge}_2[\eta^{\pm 1}] & \text{if } n \geq 0 \text{ and either } n \equiv 3 \bmod 4 \text{ or } n \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

The classes μ and ε generate $\widehat{\Pi}_*(\mathbb{F}_q)[\eta^{-1}]$ as an algebra over $\widehat{\Pi}_0(\mathbb{F}_q)[\eta^{-1}]$, subject to the relation $\varepsilon^2 = 0$. When $q \equiv 3 \bmod 4$ this shows $\widehat{\Pi}_*(\mathbb{F}_q) \cong \mathbb{Z}/4[\eta^{\pm 1}, \mu, \epsilon]/\epsilon^2$, and for $q \equiv 1 \bmod 4$ we have $\widehat{\Pi}_*(\mathbb{F}_q) \cong \mathbb{Z}/2[\eta^{\pm 1}, u, \mu, \epsilon]/(u^2, \epsilon^2)$.

Proof The mod 2 Milnor K–theory of a finite field with odd characteristic is given by $k^M_*(\mathbb{F}_q) = \mathbb{F}_2[u]/u^2$, where u is the class of a nonsquare element of $\mathbb{F}_q^\times$. If $q \equiv 3 \bmod 4$ then $u = \rho = [-1]$. As $h_1\rho$ in $\mathfrak{M}(\mathbb{F}_q)[h_1^{-1}]$ detects multiplication by 2 in $\pi_{**}(\mathbb{1}^{\wedge}_2)$, we arrive at the claimed group structure. The product structure is clear given the products in the h_1–inverted motivic Adams spectral sequence. □

Corollary 11 *The η–inverted Milnor–Witt stems for a p–adic field $\mathbb{Q}_p$ are as follows:*

$$\widehat{\Pi}_0(\mathbb{Q}_p)[\eta^{-1}] \cong W(\mathbb{Q}_p)^{\wedge}_2[\eta^{\pm 1}] \cong \begin{cases} \mathbb{Z}/2[\eta^{\pm 1}, u, \pi]/(u^2, \pi^2) & \text{if } p \equiv 1 \bmod 4, \\ (\mathbb{Z}/4 \oplus \mathbb{Z}/4)[\eta^{\pm 1}] & \text{if } p \equiv 3 \bmod 4, \\ (\mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2)[\eta^{\pm 1}] & \text{if } p = 2, \end{cases}$$

$$\widehat{\Pi}_n(\mathbb{Q}_p)[\eta^{-1}] \cong \begin{cases} W(\mathbb{Q}_p)^{\wedge}_2[\eta^{\pm 1}] & \text{if } n \geq 0 \text{ and either } n \equiv 3 \bmod 4 \text{ or } n \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof The mod 2 Milnor K–theory of the p–adic fields can be calculated from the result of Milnor [13, Lemma 4.6] in addition to the description of the Witt ring for p–adic fields which is discussed by Serre in [18]. Explicitly, the mod 2 Milnor K–theory of a p–adic field is

$$k^M_*(\mathbb{Q}_p) = \begin{cases} \mathbb{Z}/2[\pi, u]/(\pi^2, u^2) & \text{if } p \equiv 1 \bmod 4, \\ \mathbb{Z}/2[\pi, \rho]/(\rho^2, \rho\pi + \pi^2) & \text{if } p \equiv 3 \bmod 4, \\ \mathbb{Z}/2[\pi, \rho, u]/(\rho^3, u^2, \pi^2, \rho u, \rho\pi, \rho^2 + u\pi) & \text{if } p = 2, \end{cases}$$

where $\pi = [p]$, $\rho = [-1]$, u is the class of a lift of a nonsquare in $\mathbb{F}_p^\times$ when $p \equiv 1 \bmod 4$, and $u = [5]$ when $p = 2$. □

4 The field of rational numbers

We approach the calculation of $\widehat{\Pi}_*(\mathbb{Q})[\eta^{-1}]$ with the strategy suggested by the motivic Hasse principle, following the method of Ormsby and Østvær in [15]. That is, we

product	conditions
$[\ell][2] = 0$	$\ell = 2$ or $\ell = -1$
$[-1][\ell] = a_\ell$	$\ell = -1$ or ℓ prime and $\ell \equiv 3 \mod 4$
$[-1][\ell] = 0$	ℓ prime and $\ell \equiv 1 \mod 4$
$[\ell][q] = (q \mid \ell)a_\ell + (\ell \mid q)a_q$	ℓ and q odd primes
$[2][q] = \left(\frac{1}{8}(q^2-1) \mod 2\right)a_q$	q odd prime

Table 1: Products in $k_*^M(\mathbb{Q})$

analyze the h_1–inverted motivic Adams spectral sequence for $\mathbb{Q}$ using our knowledge of the h_1–inverted motivic Adams spectral sequence over the completions of $\mathbb{Q}$.

We fix our notation for $k_*^M(\mathbb{Q})$. The mod 2 Milnor K–theory of $\mathbb{Q}$ is generated by the classes $[-1]$ and $[p]$ for p a prime. Milnor shows in [13, Lemma A.1] that there is a short exact sequence

$$0 \to k_2^M(\mathbb{Q}) \to \bigoplus k_2^M(\mathbb{Q}_\nu) \to \mathbb{Z}/2 \to 0,$$

where the summation is over all completions $\mathbb{Q}_\nu$ of $\mathbb{Q}$. For every completion $\mathbb{Q}_\nu$ of $\mathbb{Q}$ there is an isomorphism $k_2^M(\mathbb{Q}_\nu) \cong \mathbb{Z}/2$; write e_ν for the image of 1 under the canonical map $k_2^M(\mathbb{Q}_\nu) \to \bigoplus k_2^M(\mathbb{Q}_\nu)$. For ℓ an odd prime or -1, write a_ℓ for the class in $k_2^M(\mathbb{Q})$ that maps to $e_\ell + e_2$ in $\bigoplus k_2^M(\mathbb{Q}_\nu)$. For $n \geq 3$ the class ρ^n generates $k_n^M(\mathbb{Q})$. The product structure in $k_*^M(\mathbb{Q})$ can be deduced from the products given in Table 1; we write $(q \mid \ell)$ for the Legendre symbol that takes values in the additive group $\mathbb{Z}/2$.

Lemma 12 *The E_1 page of the h_1–inverted ρ–Bockstein spectral sequence over $\mathbb{Q}$ is the $\mathbb{Z}/2$–algebra*

$$\mathfrak{B}(\mathbb{Q})[h_1^{-1}] \cong \bigoplus_{n\in\mathbb{N}} \rho^n k_*^M(\mathbb{Q})/\rho^{n+1} \otimes_{\mathbb{F}_2} \mathrm{Ext}(\mathbb{C})[h_1^{-1}].$$

The class ρ^n is in filtration $\epsilon = n$ for all $n \in \mathbb{N}$, for $\ell \equiv 3 \mod 4$ a prime a_ℓ is in filtration 1, for $\ell \equiv 1 \mod 4$ a prime a_p is in filtration 0, and $[p]$ for p a prime is in filtration 0. The r^{th} differential d_r for the ρ–Bockstein spectral sequence has degree $(\epsilon, f, t, c) = (r, 1, -1, 0)$. See Figure 3 for a chart of the E_1 page up to Milnor–Witt stem 15.

Proof The ρ–Bockstein spectral sequence arises from filtering the cobar complex $\mathcal{C}^*(\mathbb{Q})$ by powers of ρ. The s^{th} term of the cobar complex is

$$\mathcal{C}^s(\mathbb{Q}) = H_{**}(\mathbb{Q}) \otimes \mathcal{A}_{**}(\mathbb{Q})^{\otimes s},$$

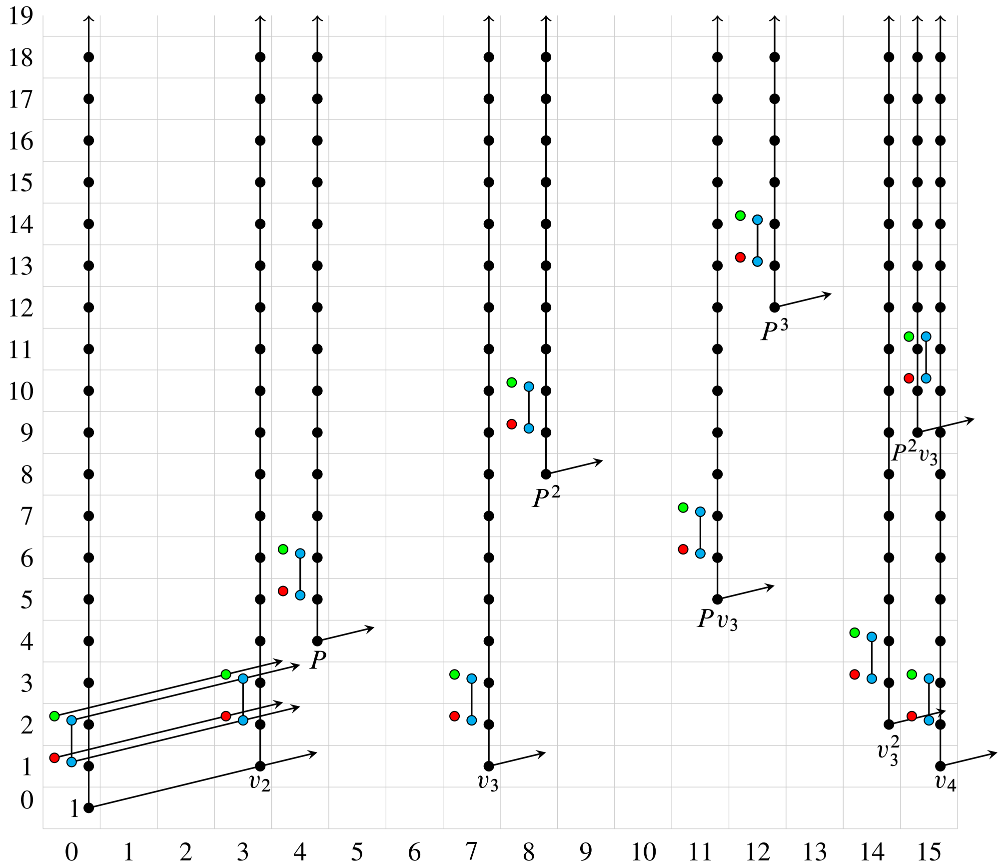

Figure 3: The E_1 page of the h_1–inverted ρ–Bockstein spectral sequence over $\mathbb{Q}$ up to Milnor–Witt stem 15. Black dots represent the group $\mathbb{Z}/2[h_1^{\pm 1}]$, blue dots represent $\bigoplus_{p\equiv 3\,(4)}\mathbb{Z}/2[h_1^{\pm 1}]$, red dots represent $\bigoplus_{p\equiv 1\,(4),p=2}\mathbb{Z}/2[h_1^{\pm 1}]$, and green dots represent $\bigoplus_{p\equiv 1\,(4)}\mathbb{Z}/2[h_1^{\pm 1}]$. Solid vertical lines indicate multiplication by ρ, and a vertical arrow means that the tower of ρ–multiplications continues indefinitely. Every dot supports an infinite tower of v_2–multiples, however we only indicate this with lines and arrows of slope $\frac{1}{3}$ on the classes of $\mathrm{Ext}(\mathbb{C})[h_1^{-1}]$ and $k_*^M(\mathbb{Q})$. The horizontal axis t is the Milnor–Witt stem, and the vertical axis c is the Chow weight, while the Adams filtration is suppressed.

where the tensor products are taken over $H_{**}(\mathbb{Q})$, taking care to use the left and right actions of $H_{**}(\mathbb{Q})$ on $\mathcal{A}_{**}(\mathbb{Q})$ arising from the left and right units η_L and η_R. Any class $a[x_1|\cdots|x_s]$ can be reduced to a sum of monomials $b[y_1|\cdots|y_s]$, where each y_i is a monomial in $\mathbb{Z}/2[\tau_0,\tau_1,\dots,\xi_1,\xi_2,\dots]$. The class τ is killed after inverting h_1,

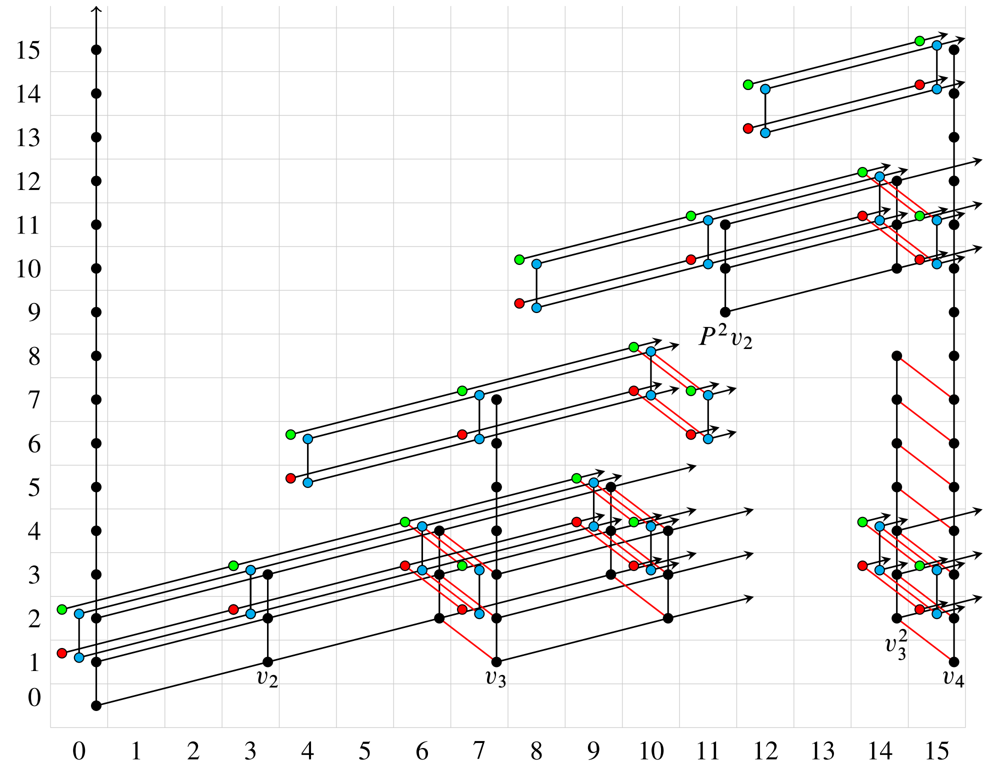

Figure 4: The E_∞ page of $\mathfrak{B}(\mathbb{Q})[h_1^{-1}]$, which is also $\operatorname{Ext}(\mathbb{Q})[h_1^{-1}]$, up to Milnor–Witt stem 15. Black dots represent the group $\mathbb{Z}/2[h_1^{\pm 1}]$, blue dots represent $\bigoplus_{p\equiv 3\,(4)}\mathbb{Z}/2[h_1^{\pm 1}]$, red dots represent $\bigoplus_{p\equiv 1\,(4),p=2}\mathbb{Z}/2[h_1^{\pm 1}]$, and green dots represent $\bigoplus_{p\equiv 1\,(4)}\mathbb{Z}/2[h_1^{\pm 1}]$. Solid vertical lines indicate multiplication by ρ, and a vertical arrow means that the tower of ρ–multiplications continues indefinitely. Multiplication by v_2 is indicated by lines of slope $\frac{1}{3}$, and an arrow of slope $\frac{1}{3}$ indicates that the class supports a tower of v_2–multiplications. The d_2 differentials of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ are indicated with red lines of slope -1.

hence every element of $\mathcal{C}^s(\mathbb{Q})$ is a sum of monomials $b[y_1|\cdots|y_s]$, where each y_i is a monomial in $\mathbb{Z}/2[\tau_0,\tau_1,\dots,\xi_1,\xi_2,\dots]$ and $b\in k_*^M(\mathbb{Q})$. The filtration of the cobar complex now is determined by the filtration of $k_*^M(\mathbb{Q})$ by powers of ρ. □

Proposition 13 *The differentials for the h_1–inverted ρ–Bockstein spectral sequence over $\mathbb{Q}$ are determined by $d_{2^n-1}(v_1^{2^n}) = h_1^{2^n}\rho^{2^n-1}v_n$ for $n\geq 2$ and $d_r(v_n)=0$ for $r\geq 1$ and $n\geq 2$.* (See Figure 4 for a chart of the E_∞ page up to Milnor–Witt stem 15).

class	(f,t,c)	ρ–torsion	conditions
$h_1^{\pm 1}$	$(\pm 1,0,0)$	∞	
ρ	$(0,0,1)$	∞	
$[2]+\rho$	$(0,0,1)$	1	
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	1	ℓ prime, $\ell \equiv 1\,(4)$, $k \geq 0$
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	2	ℓ prime, $p \equiv 3\,(4)$, $k \geq 0$
$[2]P^k$	$(0,0,1)+k(4,4,4)$	1	$k \geq 1$
$P^{2k}v_2$	$(1,3,1)+k(8,8,8)$	3	$k \geq 0$
$P^{4k}v_3$	$(1,7,1)+k(16,16,16)$	7	$k \geq 0$
$P^{8k}v_4$	$(1,15,1)+k(32,32,32)$	15	$k \geq 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: Generators of $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$

Proof The injection $k_*^M(\mathbb{Q}) \to \prod_\nu k_*^M(\mathbb{Q}_\nu)$ extends to an injection of h_1–inverted ρ–Bockstein spectral sequences at the E_1 page:

$$\mathfrak{B}(\mathbb{Q})[h_1^{-1}] \to \prod_\nu \mathfrak{B}(\mathbb{Q}_\nu)[h_1^{-1}].$$

The differentials in the h_1–inverted ρ–Bockstein spectral sequence over $\mathbb{Q}_p$ vanish for all primes p. Only the differentials in $\mathfrak{B}(\mathbb{R})[h_1^{-1}]$ contribute to the differentials over $\mathbb{Q}$, and these were identified by Guillou and Isaksen in [8, Lemma 3.1]. □

Proposition 14 *The h_1–inverted ρ–Bockstein spectral sequence for $\mathbb{Q}$ converges strongly to* $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$, *and there are no hidden extensions.*

Proof The h_1–inverted ρ–Bockstein spectral sequence is isomorphic to the ρ–Bockstein spectral sequence obtained by filtering the $[\xi_1]$–inverted cobar complex $\mathcal{C}^*(\mathbb{Q})$, hence it converges strongly to $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$. Guillou and Isaksen have shown that there are no hidden extensions in the h_1–inverted ρ–Bockstein spectral sequence over $\mathbb{R}$ [8, Proposition 4.9] and there are no hidden extensions in the h_1–inverted ρ–Bockstein spectral sequence over the other completions of $\mathbb{Q}$ by Proposition 6. We therefore conclude there are no hidden extensions since the Hasse map embeds $\mathfrak{B}(\mathbb{Q})[h_1^{-1}]_\infty \Rightarrow \mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ into $\prod_\nu \mathfrak{B}(\mathbb{Q}_\nu)[h_1^{-1}]_\infty \Rightarrow \prod_\nu \mathrm{Ext}(\mathbb{Q}_\nu)[h_1^{-1}]$. □

Corollary 15 $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ *is generated by the classes in Table 2. The relations among these generators over $k_*^M(\mathbb{Q})$ include: $[\ell]P^k \cdot [q]P^j = [\ell]\cdot[q]P^{k+j}$ for ℓ, q primes and $k, j \geq 0$; $[\ell]P^{2^{n-1}k} \cdot v_n = [\ell]\cdot P^{2^{n-1}k}v_n$ for ℓ a prime, $n \geq 2$, and $k \geq 0$; the*

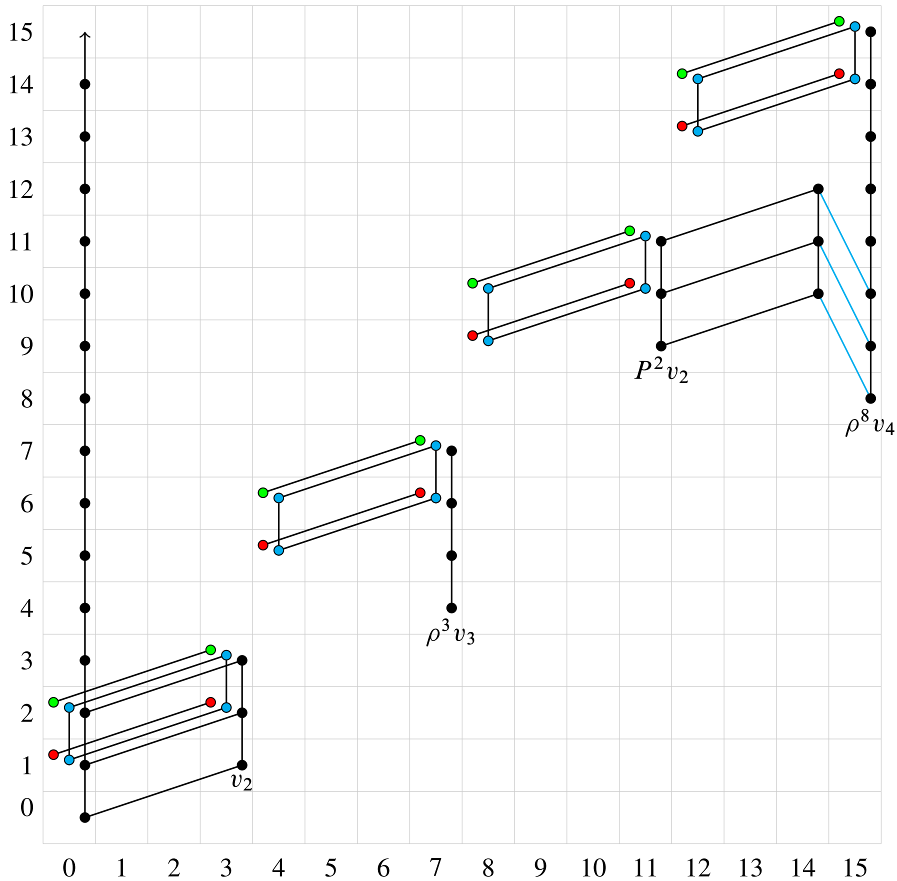

Figure 5: The E_3 page of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ up to Milnor–Witt stem 15. The d_3 differentials are indicated with blue lines of slope -2. The notational conventions of Figure 4 apply here.

vanishing of the product of three or more generators of the form $[\ell]P^k$; and the relations which set the ρ–torsion of the generators.

Proof The generators can be determined by comparing $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ to $\mathrm{Ext}(\mathbb{R})[h_1^{-1}]$, and the latter was determined by Guillou and Isaksen in [8, Theorem 4.10]. The relations stated are present in the ρ–Bockstein spectral sequence and persist to $\mathrm{Ext}(\mathbb{Q})$. □

The differentials in $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$, the h_1–inverted Adams spectral sequence over $\mathbb{Q}$, are determined by the differentials obtained from the comparison to $\mathbb{Q}_p$ and $\mathbb{R}$. See Figures 4 and 5 for a depiction of the E_2 and E_3 pages up to Milnor–Witt stem 15.

class	(f,t,c)	ρ–torsion	conditions
$h_1^{\pm 1}$	$(\pm 1,0,0)$	∞	
ρ	$(0,0,1)$	∞	
$[2]+\rho$	$(0,0,1)$	1	
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	1	ℓ prime, $\ell\equiv 1\,(4),\ k\geq 0$
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	2	ℓ prime, $\ell\equiv 3\,(4),\ k\geq 0$
$[2]P^k$	$(0,0,1)+k(4,4,4)$	1	$k\geq 1$
$P^{2k}v_2$	$(1,3,1)+k(8,8,8)$	3	$k\geq 0$
$\rho^3P^{4k}v_3$	$(1,7,4)+k(16,16,16)$	4	$k\geq 0$
$\rho^7P^{8k}v_4$	$(1,15,8)+k(32,32,32)$	8	$k\geq 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P^{4(2j+1)}v_3^2$	$(2,14,2)+(2j+1)(4,4,4)$	7	$j\geq 0$
$P^{8(2j+1)}v_4^2$	$(2,30,2)+(2j+1)(4,4,4)$	15	$j\geq 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: Generators of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]_3$

Proposition 16 *The d_2 differential in $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ is determined by the Leibniz rule from the equations $d_2(P^{2^{n-1}k}v_n)=P^{2^{n-1}k}v_{n-1}^2$ for $k\geq 0$ and $n\geq 3$ and the vanishing of d_2 on the remaining generators. For $r\geq 3$, the differential d_r in $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ is determined by the Leibniz rule from the equations*

$$d_r(\rho^{2^n-2^{n-r+2}-r+2}P^{2^{n-1}k}v_n)=P^{2^{n-1}k+2^{n-2}-2^{n-r}}v_{n-r+1}^2$$

for $n\geq r+1$ and the vanishing of d_r on the remaining generators.

Proof $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ injects into the product $\prod_\nu \mathrm{Ext}(\mathbb{Q}_\nu)[h_1^{-1}]$ under the base-change maps obtained from $\mathbb{Q}\to\mathbb{Q}_\nu$. The map is seen to be injective by the explicit calculation of $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ given in Corollary 15, $\mathrm{Ext}(\mathbb{Q}_p)[h_1^{-1}]$ in Proposition 6, and $\mathrm{Ext}(\mathbb{R})[h_1^{-1}]$ in [8, Theorem 4.10]. The differentials $d_2(v_n)=v_{n-1}^2$ for $n\geq 3$ over $\mathbb{Q}_p$ imply that the class $d_2(P^{2^{n-1}k}v_n)$ must map to $d_2(P^{2^{n-1}k}v_n)=P^{2^{n-1}k}v_{n-1}^2$ in $\mathrm{Ext}(\mathbb{Q}_p)[h_1^{-1}]$. Comparison to $\mathbb{R}$ also shows that the differential $d_2(P^{2^{n-1}k}v_n)$ maps to $P^{2^{n-1}k}v_{n-1}^2$ in $\mathrm{Ext}(\mathbb{R})[h_1^{-1}]$ for $n\geq 3$, as determined by Guillou and Isaksen [8, Lemma 5.2]. The differential d_2 over $\mathbb{Q}$ vanishes on the classes $[\ell]P^k$ by comparison to $\mathbb{Q}_p$ for all p. Finally, d_2 vanishes on all elements of $k_*^M(\mathbb{Q})$ for degree reasons. This accounts for all of irreducible classes of $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$; the generators for $\mathfrak{M}(\mathbb{Q})_3[h_1^{-1}]$ are given in Table 3. Note that the classes $P^{2(2j+1)}v_2^2$ also survive but decompose as the product $P^{2(2j+1)}v_2\cdot v_2$.

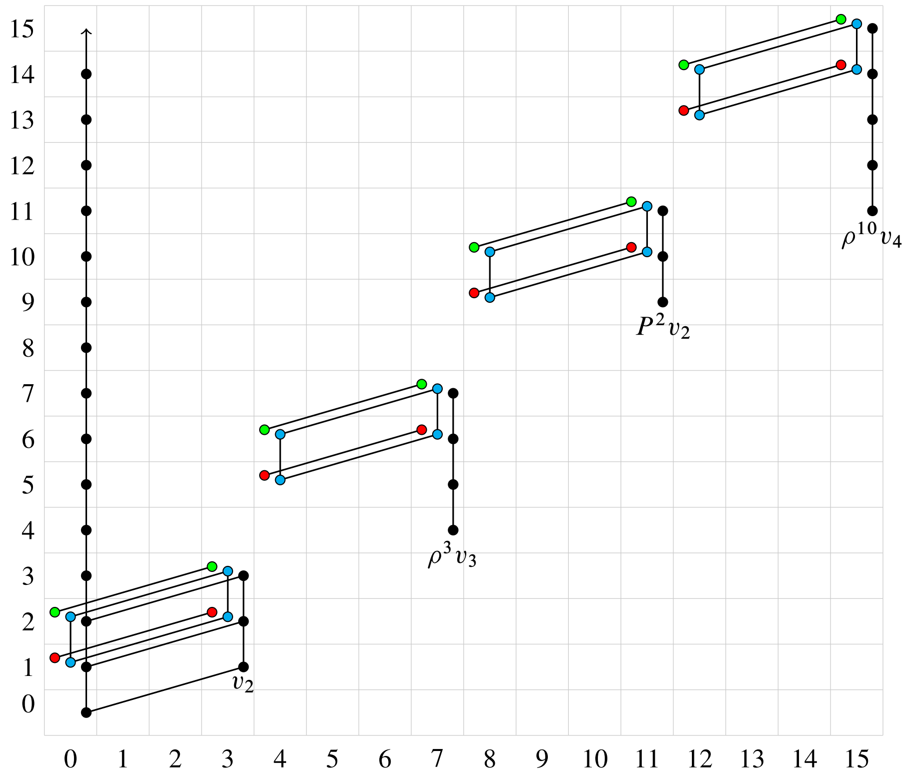

Figure 6: The E_∞ page of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ up to Milnor–Witt stem 15. The notational conventions of Figure 4 apply here.

The Hasse map

$$\mathfrak{M}(\mathbb{Q})_3[h_1^{-1}] \to \prod_{v} \mathfrak{M}(\mathbb{Q}_v)_3[h_1^{-1}]$$

is still injective. Over the p–adic fields, all further differentials vanish, and over $\mathbb{R}$ the differentials are determined by Guillou and Isaksen [8, Lemma 5.8]; these comparisons determine the remaining differentials. □

Proposition 17 *The E_∞ page of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ is generated over $k_*^M(\mathbb{Q})$ by the classes $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$ for $k \geq 0$ and $n \geq 2$. Such a class has degree*

$$(2^{n+1}k+1,\ 2^{n+1}k+2^n-1,\ 2^{n+1}k+2^n-n-1),$$

and its ρ–torsion is $n+1$. (See Table 4 for some low-degree generators and Figure 6 for a chart of the E_∞ page up to Milnor–Witt stem 15).

class	(f,t,c)	ρ–torsion	conditions
$h_1^{\pm 1}$	$(\pm 1, 0, 0)$	∞	
ρ	$(0,0,1)$	∞	
$[2]+\rho$	$(0,0,1)$	1	
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	1	ℓ prime, $\ell \equiv 1\,(4)$, $k \geq 0$
$[\ell]P^k$	$(0,0,1)+k(4,4,4)$	2	ℓ prime, $\ell \equiv 3\,(4)$, $k \geq 0$
$[2]P^k$	$(0,0,1)+k(4,4,4)$	1	$k \geq 1$
$P^{2k}v_2$	$(1,3,1)+k(8,8,8)$	3	$k \geq 0$
$\rho^3 P^{4k} v_3$	$(1,7,4)+k(16,16,16)$	4	$k \geq 0$
$\rho^{10} P^{8k} v_4$	$(1,15,11)+k(32,32,32)$	5	$k \geq 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: Generators of $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]_\infty$

Proof This is a consequence of the differential analysis of Proposition 16 and the result of Guillou and Isaksen [8, Proposition 5.9]. □

Definition 18 Let M be the submodule of $W(\mathbb{Q})^\wedge_2[\eta^{\pm 1}]$ generated by the rank-one forms $\ell \cdot X^2$ for ℓ a prime. As an abelian group, M is isomorphic to

$$\mathbb{Z}/2 \oplus \bigoplus_{p \equiv 3\,(4)} \mathbb{Z}/4 \oplus \bigoplus_{p\equiv 1\,(4)} (\mathbb{Z}/2)^2[\eta^{\pm 1}].$$

Following the notational convention of Guillou and Isaksen [8, Section 7], write $P^{2^{n-1}k}\lambda_n$ for a class in $\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}]$ detected by $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$ in $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$, where $n \geq 2$, $k \geq 0$ and $t = 2^{n+1}k + 2^n - 1$. Also, we abuse notation and write $[\ell]P^k$ for a class in $\widehat{\Pi}_{4k}(\mathbb{Q})[\eta^{-1}]$ detected by the class of the same name in $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$.

Theorem 19 *The η–inverted Milnor–Witt 0–stem of $\mathbb{1}^\wedge_2$ over $\mathbb{Q}$ is*

$$\widehat{\Pi}_0(\mathbb{Q})[\eta^{-1}] \cong W(\mathbb{Q})^\wedge_2[\eta^{\pm 1}].$$

The t^{th} η–inverted Milnor–Witt stem of $\mathbb{1}^\wedge_2$ over $\mathbb{Q}$ is as follows:

$$\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}] \cong \begin{cases} \widehat{\Pi}_0(\mathbb{Q})[\eta^{-1}]/2^{n+1} & \text{if } t \geq 0,\ t \equiv 3 \bmod 4,\ n = \nu_2(t+1), \\ M & \text{if } t \equiv 0 \bmod 4,\ t \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\nu_2(x)$ is the 2–adic valuation of an integer x, and M is the $\widehat{\Pi}_0(\mathbb{Q})[\eta^{-1}]$–module of Definition 18.

The remaining product structure of $\widehat{\Pi}_*(\mathbb{Q})[\eta^{-1}]$ *is determined by the following relations: the product of any two generators with Milnor–Witt stem congruent to* $3 \bmod 4$ *is zero;* $[q]P^j \cdot [\ell]P^k = [q]\cdot[\ell]P^{j+k}$ *for all primes* ℓ *and* q *and* $k, j \geq 0$*;* $[q]\cdot[\ell]P^k = 0$ *if* q *is a prime or* -1 *and* $[q][\ell] = 0$ *in* $k_*^M(\mathbb{Q})$*.*

Proof The zero stem was calculated in Proposition 1 and [16, Proof of Theorem 1.5] shows the one stem vanishes. Proposition 17 identifies the structure of the E_∞ page of the h_1–inverted motivic Adams spectral sequence over $\mathbb{Q}$ and Corollary 5 shows that $\mathfrak{M}(\mathbb{Q})[h_1^{-1}]$ strongly converges to $\widehat{\Pi}_*(\mathbb{Q})[\eta^{-1}]$ in Milnor–Witt stem at least 2. The 2–extensions are resolved because ρh_1 detects multiplication by 2, from which the additive structure of the η–inverted stems follows.

The product structure in the E_∞ page of the h_1–inverted motivic Adams spectral sequence determines the $\widehat{\Pi}_0(\mathbb{Q})[\eta^{-1}]$–module structure of the stems $\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}]$ for $t \not\equiv 3 \bmod 4$ and $t \equiv 3 \bmod 8$. It only remains to identify the hidden product of $[\ell]$ for ℓ a prime with a class $P^{2^{n-1}k}\lambda_n$ of $\widehat{\Pi}_t(\mathbb{Q})[\eta^{-1}]$ for $u \geq 3$, $k \geq 0$; note that $n = \nu_2(t+1)$. Lemma 20 shows that the products $[\ell]\cdot P^{2^{n-1}k}\lambda_n$ and $a_\ell \cdot P^{2^{n-1}k}\lambda_n$ are always nonzero; hence the canonical map $\widehat{\Pi}_0(\mathbb{Q})[\eta^{-1}]/2^{n+1} \to \widehat{\Pi}_n(\mathbb{Q})[\eta^{-1}]$ is an isomorphism.

The product of any two generators with Milnor–Witt stem congruent to $3 \bmod 4$ is zero for degree reasons. The remaining products are detected in the motivic Adams spectral sequence. □

Lemma 20 *For* $n \geq 3$ *and* $k \geq 0$*, the products* $[\ell]\cdot P^{2^{n-1}k}\lambda_n$ *with* ℓ *a prime and* $a_\ell \cdot P^{2^{n-1}k}\lambda_n$ *with* ℓ *an odd prime in* $\widehat{\Pi}_*(\mathbb{Q})[\eta^{-1}]$ *are nonzero.*

Proof For $m \geq 0$ and ℓ a prime, the Massey product $\langle \rho P^{2m} v_2, \rho^2, [\ell]\rangle$ in $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ contains $[\ell]P^{2m+1}$ by Lemma 21 and this has no indeterminacy. The hypotheses of Moss's convergence theorem [11, Theorem 3.1.1] hold here;[1] hence $[\ell]P^{2m+1}$ detects a class of $\langle 2P^{2m}\lambda_2, 2^2, [\ell]\rangle$. The indeterminacy of this Toda bracket is $[\ell]\widehat{\Pi}_{8m+4}$, which is in higher filtration than $[\ell]P^{2m+1}$. We conclude $\langle 2P^{2m}\lambda_2, 2^2, [\ell]\rangle$ does not contain zero.

The Massey product $\langle v_2, \rho P^{2m} v_2, \rho^2\rangle$ can be shown to contain $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$ using the Adams differential

$$d_r(\rho^{2^n-n-4}P^{2^{n-1}k}v_n) = \rho P^{2m} v_2^2,$$

[1] Observe that for $r \geq 2$, $c > t$ and $t \equiv 3 \bmod 4$, the groups $E_2^{t,c}$ are trivial in the h_1–inverted motivic Adams spectral sequence over $\mathbb{Q}$.

where $n = \nu_2(m+1)+3$ and $r = n-1$. This Massey product has trivial indeterminacy. Moss's convergence theorem shows that $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$ detects a class of $\langle \lambda_2, 2P^{2m}\lambda_2, 2^2\rangle$;[2] hence $P^{2^{n-1}k}\lambda_n$ is in the Toda bracket $\langle \lambda_2, 2P^{2m}\lambda_2, 2^2\rangle$.

We now use the shuffle relation

$$\lambda_2\langle 2P^{2m}\lambda_2, 2^2, [\ell]\rangle = \langle \lambda_2, 2P^{2m}\lambda_2, 2^2\rangle[\ell].$$

Multiplication by λ_2 is an injection on the stems $\widehat{\Pi}_{4j}(\mathbb{Q})[\eta^{-1}] \to \widehat{\Pi}_{4j+3}(\mathbb{Q})[\eta^{-1}]$ by the product structure in the motivic Adams spectral sequence, hence the left-hand side of the shuffle relation does not contain zero. As $[\ell]\cdot P^{2^{n-1}k}\lambda_n$ is in the right-hand side of the shuffle relation, we conclude that $[\ell]\cdot P^{2^{n-1}k}\lambda_n$ is nonzero.

A similar argument using the shuffle relation

$$\lambda_2\langle 2P^{2k}v_2, 2^2, a_\ell\rangle = \langle \lambda_2, 2P^{2k}v_2, 2^2\rangle a_\ell$$

establishes the claim that $a_\ell \cdot P^{2^{u-1}k}$ is nonzero. □

Lemma 21 *Let $m \geq 0$ and ℓ a prime. The Massey product $\langle \rho P^{2m}v_2, \rho^2, [\ell]\rangle$ in* $\mathrm{Ext}(\mathbb{Q})[h_1^{-1}]$ *contains $[\ell]P^{2m+1}$ and has trivial indeterminacy.*

Proof The ρ–Bockstein spectral sequence differential $d_3(P^{2m+1}) = \rho^3 P^{2m}v_2$ shows that $\langle \rho P^{2m}v_2, \rho^2, [\ell]\rangle$ contains $[\ell]P^{2m+1}$; this Massey product has trivial indeterminacy. To verify the hypotheses of May's convergence theorem [11, Theorem 2.2.1], first note the degree of $\rho^2[\ell]$ is $\epsilon = 2$, $t = 0$, and $c = 1$. All ρ–Bockstein spectral sequence differentials vanish in this graded component. It remains to check d_R differentials on the graded piece with $\epsilon' \geq 3$, $t = 8m+3$, $c = 8m+4$ and $R > \epsilon'$ corresponding to $\rho^3 P^{2m}v_2$.

We now look for elements of the E_4 page of the ρ–Bockstein spectral sequence which land in degrees (t,c) for which $t+c \equiv 7 \bmod 8$. Given the description of

[2] The condition to check for $\rho P^{2m}v_2^2$ in Moss's theorem is that $d_{r'}\colon E_{r'}^{8m+7,c'} \to E_{r'}^{8m+6,c''}$ must be zero when $c' \leq 8m+1-\nu_2(m+1)$, $c'' \geq 8m+4$ and $r' = c''-c'$. Nonzero differentials are only possible in these Milnor–Witt stems on classes $\rho^? P^? v_n$; it follows that $\nu_2(m+1) = n-3$, so $r' \geq n$. But by Proposition 16, nonzero differentials on such classes occur only when $n \geq r'+1$. The condition to check for the element $\rho^3 P^{2m}v_2$ is that $d_{r'}\colon E_{r'}^{8m+4,c'} \to E_{r'}^{8m+3,c''}$ must be zero when $c' \leq 8m+2-\nu_2(m+1)$, $c'' \geq 8m+5$ and $r' = c''-c'$. The classes in Milnor–Witt stems 4 mod 8 are generated by the classes ρ, $h_1^{\pm 1}$ and $[\ell]P^k$. The Adams differentials vanish on these classes, so the hypotheses of Moss's theorem are true.

the generators of the E_4 page, it suffices to consider products of just v_n for $n \geq 3$, $[\ell]P$, v_2, and ρ. The sum $t+c$ mod 8 for each of these generators is 0, 1, 4, and 1, respectively. As $([\ell]P)^2 = 0$, $\rho^3 v_2 = 0$, and $\rho^2[\ell]P v_2 = 0$ in the E_4 page, the only nonzero product of these generators in degree (t,c) with $t + c \equiv 7$ mod 8 must be of the form $\rho^\epsilon v_3^{a_3} \cdots v_i^{a_i}$.

Suppose now that $\rho^\epsilon v_3^{a_3} \cdots v_i^{a_i}$ is in degree $t = 8m+3$, $c = 8m+4$ for some m. Let $A = \sum a_i$ and $j = \min\{x \mid a_x \neq 0\}$. Under these assumptions it follows that

$$8m+3 = \sum a_i(2^i - 1) \geq A(2^j - 1).$$

Hence $\epsilon \geq A(2^j - 2) + 1$. Note that A must be at least 2 in order for the Milnor–Witt stem t to be congruent to 3 modulo 8. It follows that if $R > \epsilon$, then $R > 2^j - 1$ and so the class $\rho^\epsilon v_3^{a_3} \cdots v_i^{a_i}$ is zero in the E_R page, by the relation $\rho^{2^j-1} v_j = 0$, which arises from a d_{2^j-1} differential. We conclude May's convergence theorem applies in this situation. It is straightforward to check that the indeterminacy is trivial. □

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