Algebraic & Geometric Topology 18 (2018) 2471–2495

On the virtually cyclic dimension of mapping class groups of punctured spheres

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We calculate the virtually cyclic dimension of the mapping class group of a sphere with at most six punctures. As an immediate consequence, we obtain the virtually cyclic dimension of the mapping class group of the twice-holed torus and of the closed genus-two surface.

For spheres with an arbitrary number of punctures, we give a new upper bound for the virtually cyclic dimension of their mapping class group, improving the recent bound of Degrijse and Petrosyan (2015).

20F65, 55R35; 20F36

1 Introduction

Given a discrete group G, a family $\mathcal{F}$ of subgroups of G is a set of subgroups of G which is closed under conjugation and taking subgroups. Of particular interest here are the families $\mathcal{FIN}_G$ and $\mathcal{VC}_G$, which consist, respectively, of finite and virtually cyclic subgroups of G.

A model for the *classifying space* $E_{\mathcal{F}}G$ *of the family* $\mathcal{F}$ is a G–CW–complex X such that the fixed-point set of $H \in \mathcal{F}$ is contractible, and is empty whenever $H \notin \mathcal{F}$. Using standard terminology, we denote $E_{\mathcal{FIN}}G$ by $\underline{E}G$, and $E_{\mathcal{VC}}G$ by $\underline{\underline{E}}G$. The study of models for these families finds a large part of its motivation in the Baum–Connes and Farrell–Jones conjectures, respectively.

Although a model for the space $E_{\mathcal{F}}G$ always exists, it need not be finite-dimensional. The smallest possible dimension of a model of $E_{\mathcal{F}}G$ is called the *geometric dimension of* G *for the family* $\mathcal{F}$, and is usually denoted by $\mathrm{gd}_{\mathcal{F}} G$. Again using standard terminology, we will write $\underline{\mathrm{gd}}\, G = \mathrm{gd}_{\mathcal{FIN}}\, G$ and $\underline{\underline{\mathrm{gd}}}\, G = \mathrm{gd}_{\mathcal{VC}}\, G$, and refer to them as

Published: 1 May 2018 DOI: 10.2140/agt.2018.18.2471

the *proper geometric dimension* and the *virtually cyclic dimension* of G, respectively. For many families of groups these two numbers are related by the inequality

$$\underline{\underline{\mathrm{gd}}}\, G \leq \underline{\mathrm{gd}}\, G + 1, \tag{1}$$

although it is known not to be true in general; see Example 6.5 of Degrijse and Petrosyan [5]. Classes of groups for which it does hold include CAT(0) groups (see Lück [17]), hyperbolic groups (see Juan-Pineda and Leary [12]), standard braid groups (see Flores and González-Meneses [8]), and groups satisfying a certain property (Max) (see Lück and Weiermann [18, Theorem 5.8]), which roughly states that every infinite virtually cyclic subgroup is contained in a unique maximal such subgroup (see Section 5).

In this note we investigate the relation between $\underline{\underline{\mathrm{gd}}}\, G$ and $\underline{\mathrm{gd}}\, G$ for the mapping class group $\mathrm{Mod}(S)$ of a connected, orientable surface S, mainly in the case when S has genus zero. We stress that mapping class groups do not fall in any of the categories above; however, they contain finite-index subgroups with property (Max) (see Juan-Pineda and Trujillo-Negrete [13, Proposition 5.1]); compare with Lemma 5.5 below. For these subgroups the inequality (1) holds, although this does not say anything about whether this is the case for the whole group.

We will denote by $S^n_{g,b}$ the connected orientable surface of genus g, with b boundary components and n punctures. If $b = 0$, we will omit b from the notation.

At this point, we remark that the proper geometric dimension of $\mathrm{Mod}(S^n_g)$ is known (see Aramayona and Martínez-Pérez [1]) to coincide with its virtual cohomological dimension, which in turn was computed by Harer [10] and is an explicit linear function of g and n (in the particular case when $g = 0$, it is equal to $n - 3$). Our main result is as follows:

Theorem 1.1 *Let $n \in \{5, 6\}$. Then $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^n_0) = \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_0) + 1 = n - 2$.*

We remark that $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^n_0)$ is zero for $n \leq 3$, and the group $\mathrm{Mod}(S^4_0)$ is virtually free, so $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^4_0) = 2$; this follows from Juan-Pineda and Leary [12]. As an immediate corollary of Theorem 1.1, we will obtain:

Corollary 1.2 *If $S \in \{S^2_1, S^0_2\}$, then $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S) = \underline{\mathrm{gd}}\, \mathrm{Mod}(S) + 1$.*

The explicit values are $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^2_1) = 3$ and $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^0_2) = 4$. Since $\mathrm{Mod}(S^0_1) \simeq \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic, $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^0_1) = 2$; see [12].

As a further corollary of Theorem 1.1 we calculate the exact value of the virtually cyclic dimension of the spherical braid group B_n on n strands for $n \in \{5, 6\}$. Indeed, using the classical fact that B_n is a finite extension of $\mathrm{Mod}(S_0^n)$, we will obtain:

Corollary 1.3 *If $n \in \{5, 6\}$, then $\underline{\underline{\mathrm{gd}}}(B_n) = \underline{\mathrm{gd}}(B_n) + 1 = n - 2$.*

This latter result should be compared with a recent theorem of Flores and González-Meneses [8], which proves the analogous statement for braid groups of the disk, with an arbitrary number of strands.

In order to prove Theorem 1.1, we will use a result of Lück and Weiermann [18], stated as Theorem 3.1 below, which relates the virtually cyclic dimension of a group G to the proper dimension of certain subgroups associated to infinite-order elements of G. We then use the Nielsen–Thurston classification of mapping classes and a case-by-case analysis to bound the dimension of such subgroups.

Remark 1.4 It is unlikely that our arguments could be generalized to arbitrary surfaces; see Remark 4.2 below for more details. In spite of this, the interested reader can check that an immediate adaptation of the proof of Theorem 1.1 for $n = 6$ gives a direct proof of Corollary 1.2, as well as of the analogous statement for $\mathrm{Mod}(S_1^3)$. In particular, we obtain that inequality (1) is in fact an equality for all surfaces S_g^n for which $3g - 3 + n \leq 3$.

For a general number of punctures, a recent result of Degrijse and Petrosyan [6] gives a bound for $\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_g^n)$ which is linear in g and n; see Theorem 5.1 below. In the particular case when $g = 0$, this bound takes the form

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_0^n) \leq 3n - 8 = 3 \cdot \underline{\mathrm{gd}}\,\mathrm{Mod}(S_0^n) + 1. \tag{2}$$

Using the aforementioned result of Lück and Weiermann [18] with a theorem of Cameron, Solomon and Turull [4], we will prove the following slightly improved bound:

Theorem 1.5 *Suppose $n \geq 4$. Let b_n be the number of ones in the binary expression of n. Then*

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_0^n) \leq n - 4 + \left[\frac{3n-1}{2}\right] - b_n,$$

where $[\,\cdot\,]$ denotes integer part.

Remark 1.6 We have that $3n - 8 \geq n - 4 + \left[\frac{3n-1}{2}\right] - b_n$ for all $n \geq 4$, and that the inequality is strict for $n \geq 5$.

Pure mapping class groups of spheres As we will observe in Lemma 5.5 pure mapping class groups of spheres have property (Max), and hence inequality (1) is satisfied. Moreover, we will remark in Proposition 5.4 that in this case we get an equality, in fact.

Surfaces with boundary It follows from the definition that $\mathrm{Mod}(S)$ is torsion-free whenever S has boundary. Combining this with a number of results by various authors, quickly yields that (1) holds for surfaces with boundary. The argument is essentially contained in the paper by Flores and González-Meneses [8]; we offer a short account in the appendix.

Acknowledgements Aramayona was partially supported by grants RYC-2013-13008 and MTM2015-67781. Juan-Pineda and Trujillo-Negrete were partially supported by CONCAYT FORDECYT 265667. We would like to thank Yago Antolín, John Guaschi, Conchita Martínez and Luis Paris for conversations. We are also grateful to the referee for comments and suggestions.

2 Mapping class groups and braid groups

In this section we give some preliminaries on mapping class groups and their relation with braid groups. We refer the reader to [7] for a thorough discussion on these and related topics.

2.1 Mapping class groups

Let S be a (possibly disconnected) orientable surface with empty boundary and whose every connected component has negative Euler characteristic, so that S supports a complete hyperbolic metric of finite area. Sometimes it will be convenient to regard (some of) the punctures of S as marked points, and we will switch between the two points of view without further mention. As mentioned above, we will write S_g^n to denote the connected surface of genus g with n marked points.

The *mapping class group* $\mathrm{Mod}(S)$ is the group of isotopy classes of self-homeomorphisms of S; elements of $\mathrm{Mod}(S)$ are called *mapping classes*. The *pure mapping class group* $\mathrm{PMod}(S)$ is the subgroup of $\mathrm{Mod}(S)$ whose elements send every marked point to itself; observe that $\mathrm{PMod}(S)$ has finite index in $\mathrm{Mod}(S)$.

Since we will deal mainly with surfaces of genus zero, from now on we will restrict our attention to the case of $S = S_0^n$, with $n \geq 3$.

2.1.1 Curves and multicurves By a *curve* on S_0^n we mean the (free) isotopy class of a simple closed curve that does not bound a disk with at most one marked point. A *multicurve* is then a set of curves that *pairwise disjoint*, ie they may be realized in a disjoint manner on S_0^n. An easy counting argument shows that a maximal multicurve on S_0^n has $n-3$ elements.

2.1.2 Nielsen–Thurston classification We say that $f \in \mathrm{Mod}(S_0^n)$ is *reducible* if there exists a multicurve $\sigma \subset S_0^n$ such that $f(\sigma) = \sigma$; otherwise, we say that f is *irreducible*. A notable example of a reducible element is the *Dehn twist* T_α about the curve α; see [7] for definitions and properties of Dehn twists. Finally, we note that finite-order elements of $\mathrm{Mod}(S_0^n)$ may be reducible or irreducible.

The celebrated *Nielsen–Thurston classification* of mapping classes asserts that an irreducible element of infinite order has a representative which is a *pseudo-Anosov* homeomorphism; see [7, Chapter 5] for details. For this reason, irreducible mapping classes of infinite order are normally referred to as *pseudo-Anosov* mapping classes.

2.1.3 Canonical reduction system Note that, in general, a reducible mapping class may fix more than one multicurve. For this reason, we define the *canonical reduction system* of a mapping class as the intersection of all the maximal (with respect to inclusion) multicurves that it fixes. For instance, the canonical reduction system of the Dehn twist T_α is equal to α.

2.1.4 The cutting homomorphism Let σ be a multicurve on S_0^n, and consider $(\mathrm{Mod}(S_0^n))_\sigma = \{g \in \mathrm{Mod}(S_0^n) \mid g(\sigma) = \sigma\}$. Denote by $S_0^n - \sigma$ the (disconnected) surface which results from removing from S_0^n a closed regular neighborhood of each element of σ. Write $S_0^n - \sigma = Y_1 \sqcup \cdots \sqcup Y_k$, observing that each Y_j is a sphere with marked points. There is an obvious surjective homomorphism

$$(\mathrm{Mod}(S_0^n))_\sigma \to \mathrm{Mod}\Big(\bigsqcup_i Y_i, \sigma\Big),$$

called the *cutting homomorphism* associated to σ. Here, $\mathrm{Mod}\big(\bigsqcup_i Y_i, \sigma\big)$ denotes the subgroup of $\mathrm{Mod}\big(\bigsqcup_i Y_i\big)$ whose elements preserve the set of punctures of $\bigsqcup_i Y_i$ that correspond to elements σ. The cutting homomorphism fits in a short exact sequence

$$1 \to T_\sigma \to (\mathrm{Mod}(S_0^n))_\sigma \to \mathrm{Mod}\Big(\bigsqcup_i Y_i, \sigma\Big) \to 1, \tag{3}$$

where T_σ is the free abelian group generated by the Dehn twists along the elements of σ.

Armed with these definitions, we can give a *canonical form* for elements of $\mathrm{PMod}(S_0^n)$. More concretely, let $f \in \mathrm{PMod}(S_0^n)$, and write σ for its canonical reduction system, so that $f \in (\mathrm{Mod}(S_0^n))_\sigma$. Again, let $S_0^n - \sigma = Y_1 \sqcup \cdots \sqcup Y_k$. Since f is pure, it follows that $f(\alpha) = \alpha$ for every $\alpha \in \sigma$; also, $f(Y_i) = Y_i$ for every i. From this discussion, and using the Nielsen–Thurston classification, we have deduced:

Lemma 2.1 *With the notation above, the image of* $f \in \mathrm{PMod}(S_0^n)$ *under the cutting homomorphism* (3) *belongs to* $\mathrm{PMod}(Y_1) \times \cdots \times \mathrm{PMod}(Y_k)$. *Moreover, the projection of this image onto each factor is either the identity or pseudo-Anosov.*

2.1.5 Normalizers We will use the following well-known result about normalizers of pseudo-Anosov elements; see [20]:

Lemma 2.2 *Let* $f \in \mathrm{Mod}(S_0^n)$ *be a pseudo-Anosov. Then its normalizer* $N_{\mathrm{Mod}(S_0^n)}(f)$ *is virtually cyclic.*

It is also possible to describe the normalizer of a *multitwist*, which is defined as the product of Dehn twists along a set of pairwise disjoint curves. Indeed, observe that, for any $f \in \mathrm{Mod}(S_0^n)$, we have $f T_\sigma f^{-1} = T_{f(\sigma)}$. In particular, we obtain:

Lemma 2.3 *For any multicurve* σ, $N_{\mathrm{Mod}(S_0^n)}(T_\sigma) = \mathrm{Mod}(S_0^n)_\sigma$.

2.2 Braid groups

Given $n \geq 0$, we denote by F_n the *configuration space* of n distinct points on a sphere. Note that the symmetric group Σ_n acts on F_n by permutation the coordinates; the quotient space $J_n = F_n/\Sigma_n$ may then be regarded as the configuration space of n *unordered* points on the sphere. Birman [2, Proposition 1.1] proved that the natural projection $F_n \to J_n$ is a regular $(n!)$–fold covering map.

We define the *n–strand spherical braid group* as $B_n = \pi_1(J_n)$, and its *pure* subgroup as $P_n = \pi_1(F_n) < \pi_1(J_n)$.

As mentioned in the introduction, braid groups are strongly related to mapping class groups of spheres. More concretely, for $n \geq 3$ there is a short exact sequence (see, for instance, [7, Section 9.4.2])

$$1 \to \mathbb{Z}_2 \to B_n \to \mathrm{Mod}(S_0^n) \to 1, \tag{4}$$

where $\mathbb{Z}_2$ is generated by the full twist braid, Δ_n, of B_n and it generates the center of B_n. In turn, for pure braid groups we have

$$1 \to \mathbb{Z}_2 \to P_n \to \mathrm{PMod}(S_0^n) \to 1. \tag{5}$$

3 General results on geometric dimension

In this section we introduce the main ingredient in our proofs, namely the result of Lück and Weiermann [18] stated as Theorem 3.1 below.

3.1 The main tool

Let G be a group, and $\mathcal{C}_G^\infty$ the family of infinite, virtually cyclic subgroups of G. After [17; 18], we define an equivalence relation $\sim$ on $\mathcal{C}_G^\infty$ by

$$C \sim D \iff |C \cap D| = \infty. \tag{6}$$

Let $[\mathcal{C}_G^\infty]$ denote the set of equivalence classes and by $[C]$ the equivalence class of $C \in \mathcal{C}_G^\infty$. The normalizer of $[C]$ is defined as

$$N_G[C] := \{g \in G : |gCg^{-1} \cap C| = \infty\}; \tag{7}$$

in other words, it is the commensurator of C in G. We define the following family of subgroups of $N_G[C]$:

$$\mathcal{G}_G[C] = \{H \in \mathcal{VC}_{N_G[C]} : |H : H \cap C| < \infty\} \cup \mathcal{FIN}_{N_G[C]}. \tag{8}$$

After all these definitions, we are ready to give Lück and Weiermann's bound from [18]:

Theorem 3.1 [18, Theorem 2.3] *Let $\mathcal{C}_G^\infty$ and $\sim$ be as above. Let $\mathfrak{I}$ be a complete system of representatives $[H]$ of the G–orbits in $[\mathcal{C}_G^\infty]$ under the G–action coming from conjugation. Suppose there exists $d \in \mathbb{N}$ such that*

(1) $\underline{\mathrm{gd}}\, G \leq d$,

(2) $\underline{\mathrm{gd}}\, N_G[H] \leq d - 1$, *and*

(3) $\mathrm{gd}_{\mathcal{G}[H]}\, N_G[H] \leq d$

for each $[H] \in \mathfrak{I}$. Then $\underline{\underline{\mathrm{gd}}}\, G \leq d$.

We stress that, in [18], Lück and Weiermann construct an explicit model for the classifying space of G with respect to the family of virtually cyclic subgroups, although this construction will not be needed here.

Under certain circumstances, Theorem 3.1 becomes a lot easier to work with, as we now explain. First, we need the following definition:

Definition 3.2 (property (C)) A group G has *property* (C) if, whenever $f, g \in G$ are elements of infinite order with $gf^mg^{-1} = f^k$, we have that $|m| = |k|$.

We remark that if G has property (C), then [17, Lemma 4.2] yields that for any $C \in \mathcal{C}_G^\infty$,

$$N_G(C) \subseteq N_G(2!C) \subseteq N_G(3!C) \subseteq \cdots,$$

where $k!C = \{h^{k!} \mid h \in C\}$ and $N_G[C] = \bigcup_{k \geq 1} N_G(k!C)$.

We also need:

Definition 3.3 (uniqueness of roots) A group G has the property of *uniqueness of roots* if $f, g \in G$ are such that $f^n = g^n$ for some n, then $f = g$.

Armed with these definition, we give the following easy consequence of Theorem 3.1:

Proposition 3.4 *Suppose G satisfies property* (C) *and has a finite-index normal subgroup H with the property of uniqueness of roots. If for any $C \in \mathcal{C}_H^\infty$ we have*

(i) $\underline{\mathrm{gd}}\, G \leq d$,

(ii) $\underline{\mathrm{gd}}\, N_G(C) \leq d - 1$,

(iii) $\underline{\mathrm{gd}}\, W_G(C) \leq d$,

where $W_G(C) = N_G(C)/C$, then $\underline{\underline{\mathrm{gd}}}\, G \leq d$.

Proof We will use Theorem 3.1. Since H is a normal subgroup of finite index with the property of uniqueness of roots, we have $N_G(D) = N_G(tD)$ for any $D \in \mathcal{C}_H^\infty$ and any $t \in \mathbb{Z} \setminus \{0\}$.

Let $C \in \mathcal{C}_G^\infty$. Combining this with [17, Lemma 4.2], we have that $N_G[C] = N_G(k!C)$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $k!C \in \mathcal{C}_H^\infty$. Thus we may assume that $C \in \mathcal{C}_H^\infty$ and $N_G[C] = N_G(C)$. Further, a model for $\underline{E} W_G(C)$ is a model for $E_{\mathcal{G}_G} N_G[C]$ with the action given from the projection $p\colon N_G(C) \to W_G(C)$. Applying Theorem 3.1, we conclude the proof. □

We finish this subsection with the following definition from [18], which will be used later:

Definition 3.5 (property (Max)) We say that a group G satisfies (Max) if every subgroup $H \in \mathcal{VC}_G \setminus \mathcal{FIN}_G$ is contained in a unique $H_{\max} \in \mathcal{VC}_G \setminus \mathcal{FIN}_G$ which is maximal in $\mathcal{VC}_G \setminus \mathcal{FIN}_G$.

Remark 3.6 Let G and H be as in Proposition 3.4. If, in addition, H is a torsion-free group that satisfies property (Max), then for $D \in \mathcal{C}_H^\infty$, we have $N_G(D) = N_G(D_{\max})$; this follows from the property of uniqueness of roots and because H is a normal subgroup of finite index in G. In that case, in Proposition 3.4 we may assume that $C \in \mathcal{C}_H^\infty$ is maximal in C_H.

3.2 On proper geometric dimension

In light of Proposition 3.4, in order to estimate the virtually cyclic dimension, one needs to be able to estimate proper geometric dimension. With this motivation, we now present a number of known results about proper geometric dimension.

First, an immediate consequence of the definition of proper geometric dimension is that, for any two groups G_1 and G_2, one has

$$\underline{\mathrm{gd}}(G_1 \times G_2) \leq \underline{\mathrm{gd}}\, G_1 + \underline{\mathrm{gd}}\, G_2. \tag{9}$$

Another observation is that if H is a subgroup of a group G, then

$$\underline{\mathrm{gd}}\, H \leq \underline{\mathrm{gd}}\, G. \tag{10}$$

Next, a result of Karrass, Pietrowski and Solitar [14] implies that the Bass–Serre tree of a virtually free group G is a model for $\underline{E}G$. In other words, we have:

Lemma 3.7 *Let G be a virtually free group. Then $\underline{\mathrm{gd}}\, G \leq 1$, with equality if and only if G is infinite.*

The next theorem, due to Lück [15], gives a relation between the geometric dimension of a group and that of finite-index subgroups:

Theorem 3.8 [15, Theorem 2.4] *If $H \subseteq G$ is a subgroup of finite index n, then $\underline{\mathrm{gd}}\, G \leq \underline{\mathrm{gd}}\, H \cdot n$.*

We will also need to be able to bound the proper geometric dimension of certain extensions of groups. In this direction, we will use the next result, which is a consequence of [16, Theorem 5.16]:

Theorem 3.9 *Let $1 \to H \to G \to K \to 1$ be an exact sequence of groups. Suppose that H has the property that for any group $\widetilde{H}$ which contains H as subgroup of finite index, $\underline{\mathrm{gd}}\, \widetilde{H} \leq n$. If $\underline{\mathrm{gd}}\, K \leq k$, then $\underline{\mathrm{gd}}\, G \leq n + k$.*

Finally, we will make use the following well-known result [9, Proposition 2.6] in order to prove Corollaries 1.2 and 1.3:

Lemma 3.10 *Suppose* $\underline{\underline{\mathrm{gd}}}\, G \geq 3$. *Let* $1 \to F \to G \to H \to 1$ *be a short exact sequence of groups, where* F *is finite. Then* $\underline{\underline{\mathrm{gd}}}\, G = \underline{\underline{\mathrm{gd}}}\, H$.

4 Proof of Theorem 1.1

In this section we prove Theorem 1.1, using Proposition 3.4 for $\mathrm{Mod}(S_0^n)$ with $n \leq 6$. In order to do so, we first remark that the second and third authors showed that $\mathrm{Mod}(S_g^n)$ has property (C) [13], and that [3, Theorem 6.1] implies that $\mathrm{PMod}(S_0^n)$ has unique roots; we recall that $\mathrm{PMod}(S_0^n)$ has finite index in $\mathrm{Mod}(S_0^n)$. Next, we will use the following special case of the main result of [1], combined with Harer's calculation [10] of the virtual cohomological dimension of the mapping class group:

Theorem 4.1 *For every* n, $\underline{\mathrm{gd}}\, \mathrm{Mod}(S_0^n) = n - 3$.

In light of this result, inequality (10) implies that

$$\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^n)}(f) \leq n - 3,$$

for any $f \in \mathrm{Mod}(S_0^n)$. We will show in Lemma 5.5 that $\mathrm{PMod}(S_0^n)$ has property (Max). Therefore, by Remark 3.6, the proof of Theorem 1.1 boils down to proving that

$$\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^n)}(f) \leq n - 2$$

for every infinite-order element $f \in \mathrm{PMod}(S_0^n)$ such that $\langle f \rangle$ is maximal. We will do so using a case-by-case analysis depending on the Nielsen–Thurston type of such a mapping class. We have separated the proof in the cases $n = 5$ and $n = 6$, since the combinatorial possibilities are different in these two cases.

Remark 4.2 As hinted in Remark 1.4, our methods are unlikely to carry over to spheres with an arbitrary number of punctures. The main reason is that the image of the cutting homomorphism (3) is in general a complicated group, namely the semidirect product of a symmetric group and the direct product of mapping class groups of the corresponding subsurfaces. In particular, it is not clear that one can effectively control the value of $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^n)}(f)$ for infinite-order elements $f \in \mathrm{PMod}(S_0^n)$ such that $\langle f \rangle$ is maximal. However, in the cases $n \in \{5, 6\}$ this image is easy to describe, and the value of the geometric dimension is amenable to our computations.

On the other hand, we stress that essentially the same analysis as in the case $n = 6$ will give a direct proof of Corollary 1.2, as well as the analogous statement for S_1^3.

4.1 The case of the five-punctured sphere

As indicated above, we need to prove

$$\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) \leq 3$$

for every infinite-order element $f \in \mathrm{PMod}(S_0^5)$ such that $\langle f \rangle$ is maximal. There are two cases to consider:

Case 1 (f is pseudo-Anosov) Here, Lemma 2.2 implies that $N_{\mathrm{Mod}(S_0^5)}(f)$ is virtually cyclic, in which case $\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^5)}(f) = 1$ and $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) = 0$.

Case 2 (f is reducible) Let σ be its canonical reduction system. We distinguish the following further cases, depending on whether σ has one or two elements.

Subcase 2(a) (σ has exactly one element) Write $\sigma = \{\alpha\}$, observing that $S_0^5 \setminus \alpha = S_0^3 \sqcup S_0^4$. Let ρ be the cutting homomorphism (3) associated to σ. Suppose first that $\rho(f)$ is trivial, so that $f \in \langle T_\alpha \rangle$. In fact, since $\langle f \rangle$ is assumed to be maximal, we obtain that $f = T_\alpha$. By Lemma 2.3, $N_{\mathrm{Mod}(S_{0,5})}(T_\alpha) = \mathrm{Mod}(S_{0,5})_\alpha$, and thus we have

$$1 \to \langle T_\alpha \rangle \to N_{\mathrm{Mod}(S_0^5)}(f) \to \mathrm{Mod}(S_0^3, q_1) \times \mathrm{Mod}(S_0^4, q_2) \to 1, \tag{11}$$

where the punctures q_1 and q_2 are those that appear when the surface is cut along α (see Figure 1). Therefore,

$$W_{\mathrm{Mod}(S_0^5)}(f) \simeq \mathrm{Mod}(S_0^3, q_1) \times \mathrm{Mod}(S_0^4, q_2). \tag{12}$$

Since $\mathrm{Mod}(S_0^3)$ is finite and $\mathrm{Mod}(S_0^4)$ is virtually free, the combination of Lemma 3.7 with equations (9) and (10) implies that $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) = 1$, as desired.

Suppose now that $\rho(f)$ is not trivial, so that the restriction of f to $\mathrm{Mod}(S_0^4, q_2)$ (using the notation above) is a pseudo-Anosov, which we denote by f_2. In this case, we have

$$1 \to \langle T_\alpha \rangle \to N_{\mathrm{Mod}(S_0^5)}(f) \to \mathrm{Mod}(S_0^3, q_1) \times N_{\mathrm{Mod}(S_0^4, q_2)}(f_2) \to 1. \tag{13}$$

By Lemma 2.2, $N_{\mathrm{Mod}(S_0^4, q_2)}(f_2)$ is virtually cyclic, hence taking quotients we obtain

$$1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^5)}(f) \to F \to 1, \tag{14}$$

where F is a finite group. In other words, $W_{\mathrm{Mod}(S_0^5)}(f)$ is virtually cyclic, and thus $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) = 1$ by Lemma 3.7.

Subcase 2(b) (σ has two elements) Write $\sigma = \{\alpha, \beta\}$ and $S \setminus \sigma = Y_1 \sqcup Y_2 \sqcup Y_3$. Note that Y_j is homeomorphic to S_0^3 for $j = 1, 2, 3$.

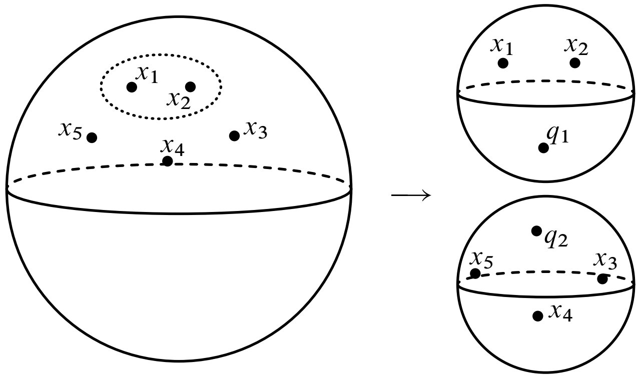

Figure 1: Cutting along a curve

Since f is pure, it follows that $f \in \langle T_\alpha, T_\beta \rangle$. Therefore, the normalizer of f in $\mathrm{Mod}(S_0^5)$ coincides with $\mathrm{Mod}(S_0^5)_\sigma$, by Lemma 2.3. The cutting homomorphism (3) reads

$$1 \to \langle T_\alpha, T_\beta \rangle \to \mathrm{Mod}(S_0^5)_\sigma \to \mathrm{Mod}(Y_1 \sqcup Y_2 \sqcup Y_3, \sigma) \to 1. \tag{15}$$

Since $\mathrm{Mod}(Y_1 \sqcup Y_2 \sqcup Y_3)$ is a finite group, we obtain

$$1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^5)}(f) \to F',$$

where F' is a finite group. Therefore, $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) = 1$, again by Lemma 3.7. This finishes the proof of Theorem 1.1 in the case $n = 5$.

4.2 The case of the six-punctured sphere

We now prove Theorem 1.1 in the case $n = 6$. Again, it suffices to prove that

$$\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^5)}(f) \leq 4$$

for every $f \in \mathrm{PMod}(S_0^6)$ of infinite order such that $\langle f \rangle$ is maximal. Let f be such an element. As in the case $n = 5$, if f is pseudo-Anosov, then $\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^6)}(f) = 1$ and $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) = 0$. Therefore, from now on we assume that f is reducible. Write σ for the canonical reduction system of f, noting that $1 \leq |\sigma| \leq 3$.

Case 1 $(|\sigma| = 1)$ We write $\sigma = \{\alpha_1\}$, and distinguish the following subcases:

Subcase 1(i) (α_1 bounds a disk with exactly two punctures) In this case, $S_0^6 \setminus \alpha_1 = S_0^3 \sqcup S_0^5$, and the cutting homomorphism (3) associated to σ reads

$$1 \to \langle T_{\alpha_1} \rangle \to \mathrm{Mod}(S_0^6)_\sigma \to \mathrm{Mod}(S_0^3 \sqcup S_0^5, \sigma) \to 1. \tag{16}$$

Since the two components of $S_0^6 \setminus \alpha_1$ are not homeomorphic (or by Lemma 2.1) we deduce that $\rho_\sigma(\mathrm{Mod}(S_0^6)_\sigma) = \mathrm{Mod}(S_0^3, q_1) \times \mathrm{Mod}(S_0^5, q_2)$, where q_1 and q_2 are the punctures that appear when cutting S_0^6 along α_1. Restricting this sequence to $N_{\mathrm{Mod}(S_0^6)}(f)$ and observing that $\mathrm{Mod}(S_0^3, q_1) \cong \mathbb{Z}_2$, we obtain

$$1 \to \langle T_{\alpha_1} \rangle \to N_{\mathrm{Mod}(S_0^6)}(f) \to \mathbb{Z}_2 \times \mathrm{Mod}(S_0^5, q_2) \to 1. \tag{17}$$

Suppose first that f has no pseudo-Anosov components; in other words, the projection of f under the cutting homomorphism is trivial. In this case, f is central in $N_{\mathrm{Mod}(S_0^6)}(f)$, and from (17) we obtain

$$W_{\mathrm{Mod}(S_0^6)}(f) \cong \mathbb{Z}_2 \times \mathrm{Mod}(S_0^5, q_2).$$

Note that any model for $\underline{E}G(\mathrm{Mod}(S_0^5))$ is also a model for $\underline{E}G(\mathbb{Z}_2 \times \mathrm{Mod}(S_0^5))$ also; thus $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \le \underline{\mathrm{gd}}\, \mathrm{Mod}(S_0^5) \le 2$.

Thus, we may assume that the restriction of f to the S_0^5–component of $S_0^6 \setminus \alpha_1$ is pseudo-Anosov. In this case, Lemma 2.2 and (16) yield

$$1 \to \mathbb{Z} \to N_{\mathrm{Mod}(S_0^6)}(f) \to \mathbb{Z}_2 \times V \to 1, \tag{18}$$

where $V \subseteq N_{\mathrm{Mod}(S_0^5)}(f_2)$ is infinite and virtually cyclic. Thus, taking quotients,

$$1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \to \mathbb{Z}_2 \times F \to 1, \tag{19}$$

and hence $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \le 1$, as desired.

Subcase 1(ii) (each component of $S_0^6 \setminus \alpha_1$ contains three punctures) In this case, $S_0^6 \setminus \alpha_1 = S_0^4 \sqcup S_0^4$, and thus

$$\mathrm{Mod}(S_0^4 \sqcup S_0^4) \overset{\psi}{\simeq} (\mathrm{Mod}(S_0^4) \times \mathrm{Mod}(S_0^4)) \rtimes \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ is generated by a mapping class that interchanges the two components of $S_0^6 \setminus \alpha_1$. Furthermore, the image of $\mathrm{Mod}(S_0^6)_\sigma$ under the cutting homomorphism (3) is equal to

$$\rho_\sigma(\mathrm{Mod}(S_0^6)_\sigma) \overset{\psi\rho_\sigma}{\simeq} (\mathrm{Mod}(S_0^4, q_1) \times \mathrm{Mod}(S_0^4, q_2)) \rtimes \mathbb{Z}_2,$$

where q_1 and q_2 are again the new punctures of $S_0^6 \setminus \alpha_1$. Let $\mathrm{Mod}(S_0^6)_\sigma^* \subseteq \mathrm{Mod}(S_0^6)_\sigma$ be the subgroup whose elements do not permute the components of $S_0^6 \setminus \alpha_1$, and let

$\rho_\sigma^* := \rho_\sigma|_{\mathrm{Mod}(S_0^6)_\sigma^*}$. We have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \langle T_{\alpha_1}\rangle & \longrightarrow & \mathrm{Mod}(S_0^6)_\sigma^* & \xrightarrow{\psi\rho_\sigma^*} & \mathrm{Mod}(S_0^4,q_1)\times\mathrm{Mod}(S_0^4,q_2) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \text{inclusion}} & & \Big\downarrow & & \\
1 & \longrightarrow & \langle T_{\alpha_1}\rangle & \longrightarrow & \mathrm{Mod}(S_0^6)_\sigma & \xrightarrow{\psi\rho_\sigma} & (\mathrm{Mod}(S_0^4,q_1)\times\mathrm{Mod}(S_0^4,q_2))\rtimes\mathbb{Z}_2 & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbb{Z}_2 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array} \tag{20}$$

With the above diagram in mind, we distinguish the following two cases, depending on the image of f under the cutting homomorphism associated to σ:

(a) Suppose first that f has no pseudo-Anosov components. Then $N_{\mathrm{Mod}(S_0^6)}(f) = \mathrm{Mod}(S_0^6)_\sigma$. Since $\mathrm{Mod}(S_0^4)$ is virtually free, (9) and Lemma 3.7 imply that

$$\underline{\mathrm{gd}}(\mathrm{Mod}(S_0^4,q_1)\times\mathrm{Mod}(S_0^4,q_2)) \le 2,$$

which in turn yields

$$\underline{\mathrm{gd}}\big((\mathrm{Mod}(S_0^4,q_1)\times\mathrm{Mod}(S_0^4,q_2))\rtimes\mathbb{Z}_2\big) \le 4,$$

by Theorem 3.8. Finally, using Theorem 3.9 and Lemma 3.7, we obtain

$$\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^6)}(f) \le 5 \quad \text{and} \quad \underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \le 4,$$

as desired.

(b) Now suppose that f has at least one pseudo-Anosov component; equivalently, assume that $\psi\rho_\sigma(f) = (f_1, f_2, \mathrm{Id}_{\mathbb{Z}_2})$ is not trivial. Again, there are two cases to consider.

Suppose first that there is (g_1, g_2, γ) in the image $\psi\rho_\sigma(N_{\mathrm{Mod}(S_0^6)}(f))$ with $\gamma \neq \mathrm{Id}_{\mathbb{Z}_2}$. In particular, f_1 is conjugate to $f_2^{\pm 1}$, and hence both f_1 and f_2 are pseudo-Anosov. Let

$$N_{\mathrm{Mod}(S_0^6)}(f)^* = N_{\mathrm{Mod}(S_0^6)}(f) \cap \mathrm{Mod}(S_0^6)_\sigma^*.$$

By restricting the diagram (20) we have

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \langle T_{\alpha_1} \rangle & \longrightarrow & N_{\mathrm{Mod}(S_0^6)}(f)^* & \xrightarrow{\psi\rho_\sigma^*} & V_1 \times V_2 & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \text{inclusion}} & & \Big\downarrow & & \\
1 & \longrightarrow & \langle T_{\alpha_1} \rangle & \longrightarrow & N_{\mathrm{Mod}(S_0^6)}(f) & \xrightarrow{\psi\rho_\sigma} & (V_1 \times V_2) \rtimes \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbb{Z}_2 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}
\tag{21}
$$

where $V_1 \times V_2 \subseteq N_{\mathrm{Mod}(S_0^4, q_1)}(f_1) \times N_{\mathrm{Mod}(S_0^4, q_2)}(f_2)$, which is a product of virtually cyclic subgroups by Lemma 2.2. Taking quotients in (21) we obtain

$$1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \to V_3 \to 1, \tag{22}$$

where V_3 is a virtually cyclic subgroup. In particular, this implies that

$$\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2, \tag{23}$$

using Theorem 3.9. This finishes the proof of the case under consideration.

Next, suppose that $\psi\rho_\sigma(g) = (g_1, g_2, \mathrm{Id}_{\mathbb{Z}_2})$ for any element $g \in N_{\mathrm{Mod}(S_0^6)}(f)$, and thus $N_{\mathrm{Mod}(S_0^6)}(f) = N_{\mathrm{Mod}(S_0^6)}(f)^*$. Hence $N_{\mathrm{Mod}(S_0^6)}(f)$ and $W_{\mathrm{Mod}(S_0^6)}(f)$ fit into the short exact sequences

$$
\begin{gathered}
1 \to \mathbb{Z} \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\psi\rho_\sigma} V_1 \times \mathrm{Mod}(S_0^4, q_2) \to 1, \\
1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \to F \times \mathrm{Mod}(S_0^4, q_2) \to 1
\end{gathered}
\tag{24}
$$

in the case when f_1 is pseudo-Anosov and f_2 is the identity, or into

$$
\begin{gathered}
1 \to \mathbb{Z} \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\psi\rho_\sigma} V_1 \times V_2 \to 1, \\
1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \to V_3 \to 1
\end{gathered}
\tag{25}
$$

when both f_1 and f_2 are pseudo-Anosov; here, V_1, V_2 and V_3 are virtually cyclic subgroups and F is finite. Proceeding as above, in both cases (24) and (25) we conclude that $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2$. This finishes the discussion of Case 1.

Case 2 ($|\sigma| = 2$) Write $\sigma = \{\alpha_1, \alpha_2\}$. Again, there are some cases to consider, depending on the topological type of α_1 and α_2.

Subcase 2(i) (α_i bounds a disc with exactly two punctures for $i = 1, 2$) Observe that $S_0^6 \setminus (\alpha_1 \cup \alpha_2) = S_0^4 \sqcup S_0^3 \sqcup S_0^3$. The cutting homomorphism (3) yields the exact sequence

$$1 \to \langle T_{\alpha_1}, T_{\alpha_2} \rangle \to \mathrm{Mod}(S_0^6)_\sigma \to \mathrm{Mod}(S_0^4 \sqcup S_0^3 \sqcup S_0^3, \sigma) \to 1, \tag{26}$$

noting that $\langle T_{\alpha_1}, T_{\alpha_2} \rangle \simeq \mathbb{Z}^2$. Observe that

$$\begin{aligned} \mathrm{Mod}(S_0^4 \sqcup S_0^3 \sqcup S_0^3, \sigma) &\overset{\psi}{\simeq} \mathrm{Mod}(S_0^4, q_1, q_2) \times (\mathrm{Mod}(S_0^3, q_3) \times \mathrm{Mod}(S_0^3, q_4)) \rtimes \mathbb{Z}_2 \\ &\overset{\phi}{\simeq} \mathrm{Mod}(S_0^4, q_1, q_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2. \end{aligned} \tag{27}$$

Again, there are different cases depending on the image of f under the cutting homomorphism. In this direction, suppose first that $f = T_{\alpha_1}^{k_1} T_{\alpha_2}^{k_2}$ with $\gcd(k_1, k_2) = 1$. In this case $N_{\mathrm{Mod}(S_0^6)}(f) = \mathrm{Mod}(S_0^6)_\sigma$, and we have

$$1 \to \mathbb{Z}^2 \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\phi\psi\rho_\sigma} \mathrm{Mod}(S_0^4, q_1, q_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \to 1, \tag{28}$$

$$1 \to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \to \mathrm{Mod}(S_0^4, q_1, q_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \to 1. \tag{29}$$

From the exact sequences (28) and (29), plus (9) and Theorem 3.9, we conclude that $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2$, as desired.

Suppose now that $\psi\rho_\sigma(f) = (f_1, \mathrm{Id}) \in \mathrm{Mod}(S_0^4, q_1, q_2) \times F$, with f_1 is pseudo-Anosov. Then

$$1 \to \mathbb{Z}^2 \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\phi\psi\rho_\sigma} V \times F \to 1,$$

$$1 \to \mathbb{Z}^2 \to W_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\psi\rho_\sigma} F \times F' \to 1,$$

where F and F' are finite groups and V is virtually cyclic. By Theorem 3.9 we conclude

$$\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^6)}(f) \leq 3 \quad \text{and} \quad \underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^6)}(f) \leq 2,$$

as desired.

Subcase 2(ii) (α_1 bounds a disc with exactly two punctures and α_2 bounds a disc with three punctures) In this case, the cutting homomorphism (3) again gives

$$1 \to \langle T_{\alpha_1}, T_{\alpha_2} \rangle \to \mathrm{Mod}(S_0^6)_\sigma \to \mathrm{Mod}(S_0^4 \sqcup S_0^3 \sqcup S_0^3, \sigma) \to 1.$$

However, in this case we have

$$\begin{aligned} \psi\rho_\sigma(\mathrm{Mod}(S_0^6)_\sigma) &= \mathrm{Mod}(S_0^3, q_1) \times \mathrm{Mod}(S_0^3, q_2, q_3) \times \mathrm{Mod}(S_0^4, q_4) \\ &\overset{\nu}{\simeq} \mathbb{Z}_2 \times \mathrm{Mod}(S_0^4, q_4). \end{aligned}$$

Again, we distinguish two cases depending on the image of f under the cutting homomorphism. First, assume that $f = T_{\alpha_1}^{k_1} T_{\alpha_2}^{k_2}$ with $\gcd(k_1, k_2) = 1$. Then $N_{\mathrm{Mod}(S_0^6)}(f) = \mathrm{Mod}(S_0^6)_\sigma$, and therefore we have the sequences

$$1 \to \mathbb{Z}^2 \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\nu\rho_\sigma} \mathbb{Z}_2 \times \mathrm{Mod}(S_0^4, q_4) \to 1,$$
$$1 \to\to \mathbb{Z} \to W_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\nu\rho_\sigma} \mathbb{Z}_2 \times \mathrm{Mod}(S_0^4, q_4) \to 1.$$

From these sequences, we conclude that

$$\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^6)}(f) \leq 3 \quad \text{and} \quad \underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2,$$

as desired.

Suppose now that $\nu\rho_\sigma f = (\mathrm{Id}_{\mathbb{Z}_2}, f_1)$, where f_1 is pseudo-Anosov. From Lemma 2.2 we have the sequences

$$1 \to \mathbb{Z}^2 \to N_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\nu\rho_\sigma} V \to 1,$$
$$1 \to \mathbb{Z}^2 \to W_{\mathrm{Mod}(S_0^6)}(f) \xrightarrow{\nu\rho_\sigma} F \to 1,$$

where V' is a virtually cyclic subgroup and F is finite. Therefore $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2$ again. This finishes the discussion of Case 2.

Case 3 $(|\sigma| = 3)$ Write $\sigma = \{\alpha_1, \alpha_2, \alpha_3\}$, observing that $S_0^6 \setminus (\alpha_1 \cup \alpha_2 \cup \alpha_3)$ is the disjoint union of four copies of S_0^3. Thus the cutting homomorphism (3) gives

$$1 \to \langle T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3} \rangle \to \mathrm{Mod}(S_0^6)_\sigma \xrightarrow{\rho_\sigma} \mathrm{Mod}(S_0^3 \sqcup S_0^3 \sqcup S_0^3 \sqcup S_0^3, \sigma) \to 1,$$

observing that $\langle T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3} \rangle \simeq \mathbb{Z}^3$. Note that $\mathrm{Mod}(S_0^3 \sqcup S_0^3 \sqcup S_0^3 \sqcup S_0^3)$ is a finite subgroup and that f is in the kernel of ρ_σ; moreover, $f = T_{\alpha_1}^{k_1} T_{\alpha_2}^{k_2} T_{\alpha}^{k_3}$ with $\gcd(k_1, k_2, k_3) = 1$. Therefore we have the sequences

$$1 \to \mathbb{Z}^3 \to N_{\mathrm{Mod}(S_0^6)}(f) \to F \to 1,$$
$$1 \to \mathbb{Z}^2 \to W_{\mathrm{Mod}(S_0^6)}(f) \to F \to 1.$$

In particular, $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^6)}(f) \leq 2$, as desired. This finishes the discussion of Case 3, and also the proof of Theorem 1.1. □

4.3 Proof of Corollaries 1.2 and 1.3

We now explain how to prove Corollaries 1.2 and 1.3. First, the latter follows immediately from the combination of Theorem 1.1, (4) and Lemma 3.10.

Now, Corollary 1.2 follows along the same lines, recalling that there are short exact sequences

$$1 \to \mathbb{Z}_2 \to \mathrm{Mod}(S_1^2) \to \mathrm{Mod}(S_0^5) \to 1,$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Mod}(S_2^0) \to \mathrm{Mod}(S_0^6) \to 1;$$

in both cases, the $\mathbb{Z}_2$ is generated by a *hyperelliptic involution*; see [7].

5 A general bound

In this section we prove Theorem 1.5. Before doing so, we remark that Degrijse and Petrosyan [6] have recently given the following bound for the virtually cyclic dimension of $\mathrm{Mod}(S_g^n)$:

Theorem 5.1 [6] *Let* $g, n \geq 0$ *with* $3g-3+n \geq 1$. *Then*

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_g^n) \leq 9g+3n-8.$$

The above result is stated in [6] for closed surfaces only; however the argument remains valid in full generality. For completeness we include a sketch here, which uses known facts about the geometry of the Weil–Petersson metric on Teichmüller space. We refer the reader to [21] for a thorough discussion on these and many other topics.

Proof of Theorem 5.1 Denote by $T_{g,n}$ the Teichmüller space of S_g^n, which is homeomorphic to $\mathbb{R}^{6g+2n-6}$. Endow $T_{g,n}$ with its Weil–Petersson metric, on which $\mathrm{Mod}(S_g^n)$ acts by semisimple isometries. The metric completion $\overline{T}_{g,n}$ of $T_{g,n}$ is a complete separable CAT(0) space, and the action of $\mathrm{Mod}(S_g^n)$ on $T_{g,n}$ extends to a semisimple isometric action on $\overline{T}_{g,n}$. Moreover, the stabilizer of a point is a virtually abelian group of rank $\leq 3g+n-3$. At this point, Corollary 3(iii) of [6] implies that

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_g^n) = (6g+2n-6)+(3g+n-3)+1 = 9g+3n-8,$$

as desired □

We now proceed to prove Theorem 1.5. Again, the main tool will be Proposition 3.4, this time combined with a result of Martínez-Pérez [19]. Before stating the latter, we need the following definition. Let G be a group, and $F \in \mathcal{FIN}_G$ a finite subgroup. The *length* $l(F)$ of F is defined as the largest natural number k for which there is a chain $1 = F_0 < F_1 < \cdots < F_k = F$. The *length* of G is

$$l(G) = \sup\{l(F) \mid F \in \mathcal{FIN}_G\}.$$

Theorem 5.2 [19, Theorem 3.10, Lemma 3.9] *Suppose that* $3 \le \underline{\mathrm{gd}}\, G < \infty$. *If* $l(G)$ *is finite, then*

$$\underline{\mathrm{gd}}\, G \le \mathrm{vcd}\, G + l(G),$$

where $\mathrm{vcd}(\cdot)$ *denotes virtual cohomological dimension.*

We begin with the following lemma:

Lemma 5.3 *Suppose* $n \ge 4$. *Let* $f \in \mathrm{PMod}(S_0^n)$ *and suppose that* $\langle f \rangle$ *is maximal in* $\mathcal{C}^{\infty}_{\mathrm{PMod}(S_0^n)}$. *Then*

$$\underline{\mathrm{gd}}\, W_{\mathrm{PMod}(S_0^n)}(f) \le n-4.$$

Proof Suppose that f has canonical reduction system $\sigma = (\alpha_1, \dots, \alpha_k)$, the restriction $f|_{Y_j}$ is pseudo-Anosov for $j \in 1, \dots, r$, and $f|_{Y_i}$ is the identity for $i \in \{r+1, \dots, k+1\}$. Note that Y_j is a sphere with at least four punctures for $j \in \{1, \dots, r\}$. Thus, from the definition of the cutting homomorphism (3) and the comment after it, we have

$$1 \to \langle T_\sigma \rangle \to N_{\mathrm{PMod}(S_0^n)}(f) \xrightarrow{\rho_\sigma} U \times \prod_{i=r+1}^{k+1} \mathrm{PMod}(Y_i) \to 1, \tag{30}$$

where U is a finite-index subgroup of $\prod_{i=1}^{r} N_{\mathrm{PMod}(Y_i)}(f|_{Y_i}) \simeq \mathbb{Z}^r$. If $f \in \langle T_\sigma \rangle$, then $r = 0$ and

$$1 \to \langle T_\sigma \rangle / \langle f \rangle \to W_{\mathrm{PMod}(S_0^n)}(f) \xrightarrow{\widehat{\rho}_\sigma} \prod_{i=1}^{k+1} \mathrm{PMod}(Y_i) \to 1, \tag{31}$$

where $\langle T_\sigma \rangle / \langle f \rangle \simeq \mathbb{Z}^{k-1}$. By Theorem 3.9, (9) and [11, Corollary 10.5], we have

$$\begin{aligned}
\underline{\mathrm{gd}}\, W_{\mathrm{PMod}(S_0^n)}(f) &\le k-1 + \sum_{i=1}^{k+1} \underline{\mathrm{gd}}\, \mathrm{PMod}(Y_i) \\
&= k-1 + \sum_{i=1}^{k+1} \mathrm{vcd}\, \mathrm{PMod}(Y_i) \\
&= k-1+n-k-3 \\
&= n-4.
\end{aligned}$$

If, on the other hand, $f \notin \langle T_\sigma \rangle$, then $r \ge 1$. Let $\bar{f} = (f|_{Y_1}, \dots, f|_{Y_r})$; then

$$1 \to \langle T_\sigma \rangle \to W_{\mathrm{PMod}(S_0^n)}(f) \xrightarrow{\rho_\sigma} U/\langle \bar{f} \rangle \times \prod_{i=r+1}^{k+1} \mathrm{PMod}(Y_i) \to 1. \tag{32}$$

Note that $U/\langle \bar{f} \rangle$ is virtually $\mathbb{Z}^{r-1}$, and thus $\underline{\mathrm{gd}}(U/\langle \bar{f} \rangle) = r-1$. Again, applying Theorem 3.9, (9) and [11, Corollary 10.5] to (32), we have

$$\begin{aligned}\underline{\mathrm{gd}}\, W_{\mathrm{PMod}(S_0^n)}(f) &\le k+1+r-1+\sum_{i=r+1}^{k+1} \mathrm{vcd}\,\mathrm{PMod}(Y_i)\\ &\le k+r-1+\sum_{i=1}^{k+1} \mathrm{vcd}\,\mathrm{PMod}(Y_i) - r\\ &= n-4.\end{aligned}$$ □

We are finally in a position to prove Theorem 1.5:

Proof of Theorem 1.5 We will use Proposition 3.4. Let $\langle f \rangle$ maximal in $\mathcal{C}^{\infty}_{\mathrm{PMod}(S_0^n)}$, and note that $\underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S_0^n)}(f) \le \underline{\mathrm{gd}}\, \mathrm{Mod}(S_0^n) = n-3$, by [1]. We now give a bound for $\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^n)}(f)$. We have the exact sequence

$$1 \to N_{\mathrm{PMod}(S_0^n)}(f) \to N_{\mathrm{Mod}(S_0^n)}(f) \to A \to 1,$$

where $A \subseteq \Sigma_n$. Since $\langle f \rangle \subset \mathrm{PMod}(S_0^n)$ we have

$$1 \to W_{\mathrm{PMod}(S_0^n)}(f) \to W_{\mathrm{Mod}(S_0^n)}(f) \to A \to 1.$$

We remark that $W_{\mathrm{PMod}(S_0^n)}(f)$ is torsion-free since $\mathrm{PMod}(S_0^n)$ is, and $\langle f \rangle$ is maximal in $\mathcal{C}^{\infty}_{\mathrm{PMod}(S_0^n)}$. Then $l(W_{\mathrm{Mod}(S_0^n)}(f)) \le l(A) \le l(\Sigma_n)$. By a result of Cameron, Solomon and Turull [4, Theorem 1],

$$l(\Sigma_n) = \left[\frac{3n-1}{2}\right] - b_n.$$

At this point, Theorem 5.2 and Lemma 5.3 together imply

$$\begin{aligned}\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S_0^n)}(f) &\le \mathrm{vcd}\, W_{\mathrm{Mod}(S_0^n)}(f) + l(A)\\ &= \mathrm{vcd}\, W_{\mathrm{PMod}(S_0^n)}(f) + l(A)\\ &\le n-4+\left[\frac{3n-1}{2}\right] - b_n,\end{aligned}$$

where the equality holds since $\mathrm{PMod}(S_0^n)$ has finite index in $\mathrm{Mod}(S_0^n)$. □

5.1 Pure subgroups

As mentioned in the introduction, if one considers the pure mapping class group instead of the full mapping class group, then the situation is a lot easier. Indeed, after Harer's calculation of the virtual cohomological dimension of the (pure) mapping class group, we get:

Proposition 5.4 *Let* $n \geq 4$. *Then*

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{PMod}(S_0^n) = \underline{\mathrm{gd}}(\mathrm{PMod}(S_0^n)) + 1 = n - 2.$$

The main ingredient of the proof is property (Max).

Lemma 5.5 *Let* $n \geq 4$. *Then the pure mapping class group* $\mathrm{PMod}(S_0^n)$ *satisfies property* (Max).

Proof Let $C \subset D$ be an inclusion of infinite cyclic subgroups. Then the centralizers of C and D in $\mathrm{PMod}(S_0^n)$ are equal, since $\mathrm{PMod}(S_0^n)$ has unique roots. If C is generated by a pseudo-Anosov class, then its centralizer is cyclic, and in particular is the unique maximal cyclic subgroup that contains C. If C is generated by a reducible element f, by Lemma 2.1 and the case of pseudo-Anosov classes, we obtain a unique maximal cyclic subgroup that contains C. □

Proof of Proposition 5.4 As mentioned in the introduction, Lück and Weiermann [18, Theorem 5.8] proved that every group with property (Max) satisfies inequality (1). Now, a combination of Harer's calculation [10] of the virtual cohomological dimension of $\mathrm{Mod}(S_0^n)$ and [11, Corollary 10.5] yields $\underline{\mathrm{gd}}\,\mathrm{PMod}(S_0^n) = n - 3$. Therefore,

$$\underline{\underline{\mathrm{gd}}}\,\mathrm{PMod}(S_0^n) \leq n - 2.$$

Now, S_0^n contains $n-2$ disjoint essential curves $\alpha_1, \dots, \alpha_{n-3}$, and the subgroup $\langle T_{\alpha_1}, \dots, T_{\alpha_{n-3}} \rangle$ is isomorphic to $\mathbb{Z}^{n-3}$. By property (10), we conclude $\underline{\underline{\mathrm{gd}}}\,\mathrm{Mod}(S_0^n) \geq \underline{\underline{\mathrm{gd}}}\,\mathbb{Z}^{n-3} = n - 2$. □

Appendix: Surfaces with boundary

Finally, we explain how to establish inequality (1) in the case of mapping class groups of surfaces with nonempty boundary. As indicated in the introduction, the arguments appear in the paper of Flores and González-Meneses [8] in the case when the surface has genus zero. For completeness, we give a self-contained argument here.

For S a surface with nonempty boundary, its mapping class group $\mathrm{Mod}(S)$ is again defined as the group of isotopy classes of self-homeomorphisms of S, but this time the homeomorphisms and isotopies are required to fix each boundary component pointwise. As a byproduct of this definition, $\mathrm{Mod}(S)$ has no torsion.

The main ingredient will be the following result of Martínez-Pérez [19]; again, $\mathrm{vcd}(\cdot)$ denotes virtual cohomological dimension:

Theorem A.1 *Let G be a group such that any finite subgroup is nilpotent. Suppose $\mathrm{vcd}\, G < \infty$ and $\underline{\mathrm{gd}}\, G \geq 3$; then*

$$\underline{\mathrm{gd}}\, G \leq \max_{F \in \mathcal{FIN}_G} \{\mathrm{vcd}\, G + \mathrm{rk}(W_G F)\},$$

where $\mathrm{rk}(\cdot)$ denotes the biggest rank of a finite elementary abelian subgroup.

We denote by $\mathcal{S}^n_{g,b}$ the connected, orientable surface of genus g with n marked points and b boundary components. For $m \geq 3$, define the *congruence subgroup* $\mathrm{Mod}(S^n_{g,b})[m]$ as the finite-index subgroup of $\mathrm{Mod}(S^n_{g,b})$ consisting of those elements which act trivially on $H_1(S^n_{g,b}; \mathbb{Z}_m)$. It is known that $\mathrm{Mod}(S^n_{g,b})[m]$ has property (Max) [13, Proposition 5.11], and the property of uniqueness of roots [3]. We will make use of the following lemma:

Lemma A.2 *Let $b \geq 1$. If $g = 0$, suppose $b + n \geq 4$, and if $g \geq 1$, suppose $2g + b + n \geq 3$. Fix $m \geq 3$ and let $C \in \mathcal{C}^\infty_{\mathrm{Mod}(S^n_{g,b})[m]}$ maximal; then*

$$\underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S^n_{g,b})}(C) \leq \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b}) + 1. \tag{33}$$

Proof We will use Theorem A.1. First, the hypotheses imply that $\mathrm{Mod}(S^n_{g,b})$ contains $\mathbb{Z}^k$ with $k \geq 3$, and thus $\mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b}) + 1 \geq 3$. Also, observe that $\underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b}) = \mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b})$ since $\mathrm{Mod}(S^n_{g,b})$ has no torsion.

Let $C \in \mathcal{C}^\infty_{\mathrm{Mod}(S^n_{g,b})[m]}$ be maximal. Note that any finite subgroup of $W_{\mathrm{Mod}(S^n_{g,b})}(C)$ is of the form V/C where V is an infinite cyclic subgroup of $N_{\mathrm{Mod}(S^n_{g,b})}(C)$. Again, since $\mathrm{Mod}(S^n_{g,b})$ has no torsion, it follows that finite subgroups of $W_{\mathrm{Mod}(S^n_{g,b})}(C)$ are cyclic.

Write, for compactness, $Q = W_{\mathrm{Mod}(S^n_{g,b})}(C)$. Applying Theorem A.1 we have

$$\underline{\mathrm{gd}}\, Q \leq \max_{F \in \mathcal{FIN}_Q} \{\mathrm{vcd}\, Q + \mathrm{rk}(W_Q(F))\} = \mathrm{vcd}\, Q + 1. \tag{34}$$

We will give a bound for $\mathrm{vcd}\, Q$. Consider the short exact sequence

$$1 \to N_{\mathrm{Mod}(S^n_{g,b})[m]}(C) \to N_{\mathrm{Mod}(S^n_{g,b})}(C) \to K \to 1, \tag{35}$$

where K is a subgroup of the finite group $\mathrm{Aut}(H_1(S^n_{g,b},\mathbb{Z}_m))$. Passing to the quotient, we have

$$1 \to W_{\mathrm{Mod}(S^n_{g,b})[m]}(C) \to W_{\mathrm{Mod}(S^n_{g,b})}(C) \to K' \to 1, \tag{36}$$

where $K' \simeq K$. Now, C is maximal in $\mathcal{C}^{\infty}_{\mathrm{Mod}(S^n_{g,b})[m]}$, so $W_{\mathrm{Mod}(S^n_{g,b})[m]}(C)$ is torsion-free, and thus

$$\begin{aligned} \mathrm{vcd}\, W_{\mathrm{Mod}(S^n_{g,b})}(C) &= \mathrm{vcd}\, W_{\mathrm{Mod}(S^n_{g,b})[m]}(C) \\ &\le \underline{\mathrm{gd}}\, W_{\mathrm{Mod}(S^n_{g,b})[m]}(C) \\ &\le \underline{\mathrm{gd}}\, N_{\mathrm{Mod}(S^n_{g,b})[m]}(C) \\ &\le \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b}), \end{aligned} \tag{37}$$

where the equality holds since $\mathrm{Mod}(S^n_{g,b})[m]$ has finite index in $\mathrm{Mod}(S^n_{g,b})$, the second inequality is given in the proof of [18, Theorem 5.8], and the last inequality follows from subgroup inclusion. Thus the result follows. □

Finally, we have the desired bound for surfaces with boundary:

Proposition A.3 *Let $b \ge 1$. If $g = 0$, suppose $b+n \ge 4$, and if $g \ge 1$, suppose $2g+b+n \ge 3$. Then*

$$\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^n_{g,b}) \le \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b}) + 1,$$

and equality holds if $g \in \{0,1\}$.

Proof For the inequality $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^n_{g,b}) \le \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b}) + 1$, the proof is again an immediate consequence of Proposition 3.4, Remark 3.6 and Lemma A.2, with $d = \underline{\mathrm{gd}}\, \mathrm{Mod}(S^n_{g,b})$.

By [10, Theorem 4.1], $\mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b})$ and the maximal rank of an abelian subgroup of $\mathrm{Mod}(S^n_{g,b})$ are equal if and only if $g \in \{0,1\}$. Let Λ be that abelian subgroup of rank $\mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b})$; then $\underline{\underline{\mathrm{gd}}}\, \Lambda = \mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b}) + 1$, and therefore $\underline{\underline{\mathrm{gd}}}\, \mathrm{Mod}(S^n_{g,b}) \ge \mathrm{vcd}\, \mathrm{Mod}(S^n_{g,b}) + 1$. □

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Received: 22 August 2017 Revised: 4 January 2018