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Algebraic stability of zigzag persistence modules

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The stability theorem for persistent homology is a central result in topological data analysis. While the original formulation of the result concerns the persistence barcodes of $\mathbb{R}$–valued functions, the result was later cast in a more general algebraic form, in the language of *persistence modules* and *interleavings*. We establish an analogue of this algebraic stability theorem for zigzag persistence modules. To do so, we functorially extend each zigzag persistence module to a two-dimensional persistence module, and establish an algebraic stability theorem for these extensions. One part of our argument yields a stability result for free two-dimensional persistence modules. As an application of our main theorem, we strengthen a result of Bauer et al on the stability of the persistent homology of Reeb graphs. Our main result also yields an alternative proof of the stability theorem for level set persistent homology of Carlsson et al.

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1 Introduction

1.1 Background

Persistence modules Let **Vec** denote the category of vector spaces over some fixed field k, and let **vec** denote the subcategory of finite-dimensional vector spaces. We define a *persistence module* to be a functor $M\colon \mathbb{P} \to \mathbf{Vec}$ for $\mathbb{P}$ a poset. We will often refer to such M as a $\mathbb{P}$*–indexed module*. If M takes values in **vec**, we say M is *pointwise finite-dimensional (pfd)*. The $\mathbb{P}$–indexed persistence modules form a category $\mathbf{Vec}^{\mathbb{P}}$ whose morphisms are the natural transformations.

Persistence modules are the basic algebraic objects of study in the theory of persistent homology. The theory begins with the study of 1*-D persistence modules*, ie functors $\mathbb{R} \to \mathbf{Vec}$ or $\mathbb{Z} \to \mathbf{Vec}$, where $\mathbb{R}$ and $\mathbb{Z}$ are taken to have the usual total orders. The structure theorem for 1-D persistence modules given by Webb [44] and Crawley-Boevey [25] tells us that the isomorphism type of a pfd 1-D persistence module M is completely described by a collection $\mathcal{B}(M)$ of intervals in $\mathbb{R}$, called the *barcode of* M. $\mathcal{B}(M)$ specifies the decomposition of M into indecomposable summands.

Persistent homology In topological data analysis, one often studies a data set by associating to the data a persistence module. To do so, we first associate to our data a *filtration*, ie a functor $\mathcal{F}\colon \mathbb{R} \to \mathbf{Top}$ such that the map $\mathcal{F}_a \to \mathcal{F}_b$ is an inclusion whenever $a \leq b$. For example, if our data is an $\mathbb{R}$–valued function $\gamma\colon T \to \mathbb{R}$ for T a topological space, we may take $\mathcal{F}$ to be the *sublevel set filtration* $\mathcal{S}^{\uparrow}(\gamma)$, defined by

$$\mathcal{S}^{\uparrow}(\gamma)_a = \{y \in T \mid \gamma(y) \leq a\}, \quad a \in \mathbb{R}.$$

Since $\mathcal{S}^{\uparrow}(\gamma)_a \subset \mathcal{S}^{\uparrow}(\gamma)_b$ whenever $a \leq b$, this indeed gives a filtration. If our data set is instead a point cloud, we often consider a *Vietoris–Rips* or *Čech* filtration; see eg the survey article of Carlsson [10] for details.

Letting $H_i\colon \mathbf{Top} \to \mathbf{Vec}$ denote the i^{th} singular homology functor with coefficients in k, we obtain a (typically pfd) persistence module $H_i\mathcal{F}$ for any $i \geq 0$. The barcodes $\mathcal{B}(H_i\mathcal{F})$ serve as concise descriptors of the coarse-scale, global, nonlinear geometric structure of the data set. These descriptors have been applied to many problems in science and engineering, for example to natural scene statistics (in the work of Carlsson, Ishkhanov, de Silva and Zomorodian [11]), evolutionary biology (Chan, Carlsson and Rabadan [16]), periodicity detection in gene expression data (Perea, Deckard, Haase and Harer [39]), sensor networks (de Silva and Ghrist [41]) and clustering (Chazal, de Silva and Oudot [21]).

Stability The *stability theorem* for persistent homology guarantees that in several settings, the barcode descriptors of data are stable with respect to perturbations of the data. The original formulation of the stability theorem, given by Cohen-Steiner, Edelsbrunner and Harer [23], concerns the persistent homology of $\mathbb{R}$–valued functions, and is formulated with respect to a standard metric d_b on barcodes called the *bottleneck distance*, which we define in Section 2.3. In the generality provided by Chazal, Cohen-Steiner, Glisse, Guibas and Oudot [17], the result is as follows:

Theorem 1.1 (stability of persistent homology for functions [23; 17]) *For T a topological space, $i \geq 0$ and functions $\gamma, \kappa \colon T \to \mathbb{R}$ such that $H_i\mathcal{S}^{\uparrow}(\gamma)$ and $H_i\mathcal{S}^{\uparrow}(\kappa)$ are pfd, we have*

$$d_b\big(\mathcal{B}(H_i\mathcal{S}^{\uparrow}(\gamma)), \mathcal{B}(H_i\mathcal{S}^{\uparrow}(\kappa))\big) \leq d_\infty(\gamma, \kappa),$$

where $d_\infty(\gamma,\kappa) = \sup_{x\in T} |\gamma(x) - \kappa(x)|$.

As a corollary of Theorem 1.1, Chazal et al obtain a stability theorem for persistent homology of Rips and Čech filtrations on finite metric spaces; see Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [18] and Chazal, de Silva and Oudot [21].

Algebraic stability A purely algebraic formulation of the stability theorem was introduced by Chazal et al [17], generalizing the stability results for $\mathbb{R}$–valued functions and point cloud data. This *algebraic stability theorem* asserts that an *ϵ–interleaving* (a pair of "approximately inverse" morphisms) between pfd 1-D persistence modules M and N induces an *ϵ–matching* (approximate isomorphism) between the barcodes $\mathcal{B}(M)$ and $\mathcal{B}(N)$. It was observed by Lesnick [33] that the converse of this result also holds: given an ϵ–matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$, we can easily construct an ϵ–interleaving between M and N. The algebraic stability theorem and its converse are together known as the *isometry theorem*; see Theorem 2.11 for the precise statement.

A slightly weaker formulation of the isometry theorem establishes a relationship between the *interleaving distance* (a pseudometric on persistence modules) and the bottleneck distance. It says that the interleaving distance between M and N is equal to the bottleneck distance between $\mathcal{B}(M)$ and $\mathcal{B}(N)$.

The algebraic stability theorem is perhaps the central theorem in the theory of persistent homology; it provides the core mathematical justification for the use of persistent homology in the study of noisy data. The theorem is used, in one form or another, in nearly all available results on the approximation, inference and estimation of persistent homology.

Induced matching theorem Bauer and Lesnick [3] showed that the algebraic stability theorem, ostensibly a result about pairs of morphisms of persistence modules, is in fact an immediate corollary of a general result about single morphisms of persistence modules. This result, called the *induced matching theorem*, concerns a simple, explicit map χ sending each morphism $f\colon M \to N$ of pfd 1-D persistence modules to a matching $\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$. The theorem tells us that the quality of this matching is tightly controlled by the lengths of the longest intervals in $\mathcal{B}(\ker f)$ and $\mathcal{B}(\operatorname{coker} f)$; see Theorem 2.12.

Zigzag modules For posets $\mathbb{A}$ and $\mathbb{B}$, the product poset $\mathbb{A}\times\mathbb{B}$ is defined by taking $(a,b)\le(a',b')$ if and only $a\le a'$ and $b\le b'$. Let $\mathbb{A}^{\mathrm{op}}$ denote the opposite poset of $\mathbb{A}$.

Zigzag modules are natural generalizations of $\mathbb{Z}$–indexed modules which have received much attention from the topological data analysis community; see in particular the foundational work of Carlsson and de Silva [12], Carlsson, de Silva and Morozov [13] and Bendich, Edelsbrunner, Morozov and Patel [5]. Zigzag modules are functors $\mathbb{ZZ}\to\mathbf{Vec}$, where $\mathbb{ZZ}$ is the subposet of $\mathbb{Z}^{\mathrm{op}}\times\mathbb{Z}$ given by

$$\mathbb{ZZ} := \big\{(i,j) \mid i\in\mathbb{Z},\ j\in\{i,i-1\}\big\}.$$

A structure theorem for pfd zigzag modules gives us a definition of barcode for these modules closely analogous to the one for 1-D persistence modules. In the special case where all but a finite number of the vector spaces are trivial, this is a classical result due to Gabriel [31]. A proof of the general structure theorem appears in a recent paper of Botnan [8].

$\mathbb{U}$–indexed modules Let $\mathbb{U}$ denote the subposet of $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$ consisting of objects (a,b) with $a\le b$. $\mathbb{U}$–indexed modules arise naturally as refinements of the sublevel set persistent homology modules introduced above: Given a function $\gamma\colon T\to\mathbb{R}$ with T a topological space, we obtain a functor $\mathcal{S}(\gamma)\colon\mathbb{U}\to\mathbf{Top}$, the *interlevel set filtration of* γ, by taking $\mathcal{S}(\gamma)_{(a,b)}=\gamma^{-1}([a,b])$, with $\mathcal{S}(\gamma)_{(a,b)}\to\mathcal{S}(\gamma)_{(c,d)}$ the inclusion map whenever $c\le a\le b\le d$. For $i\ge 0$, $H_i\mathcal{S}(\gamma)$ is clearly a $\mathbb{U}$–indexed module. It can be shown that if γ is continuous or bounded below, then $H_i\mathcal{S}(\gamma)$ determines $H_i\mathcal{S}^{\uparrow}(\gamma)$.

We will be especially interested in the case of functions γ of *Morse type*. These are certain generalizations of Morse functions for which each $H_i\mathcal{S}(\gamma)$ is completely determined by its restriction to a discrete subposet of $\mathbb{U}$; see Section 4.2 for the definition.

$\mathbb{U}$–indexed modules also arise naturally in a different (but related) way: in Section 4.1, we use Kan extensions to define a fully faithful functor $E\colon \mathbf{Vec}^{\mathbb{ZZ}} \to \mathbf{Vec}^{\mathbb{U}}$. This functor appears implicitly in recent work on interlevel set persistent homology [13; 5].

Block-decomposable modules In general, the algebraic structure of a $\mathbb{U}$–indexed module can be very complicated. As a result, there is no nice definition of a barcode available for such a module in general; see the work of Carlsson and Zomorodian [15] and also Lesnick and Wright [34, Section 1.4]. However, if M is a $\mathbb{U}$–indexed module such that either

(1) $M \cong H_i(\mathcal{S}(\gamma))$ for $\gamma\colon T \to \mathbb{R}$ of Morse type, or

(2) $M \cong E(V)$ for V a pfd zigzag module,

then M decomposes into especially simple indecomposable summands, which we call *block modules*; see Sections 3 and 4. We call any $\mathbb{U}$–indexed module that decomposes into block modules *block-decomposable*.

We may define the barcode $\mathcal{B}(M)$ of a block-decomposable module M in much the same way that we do for 1-D and zigzag modules. The barcode of a block-decomposable module is a collection of simple convex regions in $\mathbb{R}^2$ called *blocks*; see Section 3 for the definition and an illustration.

Level set barcodes The intersection of any block with the diagonal $y = x$ is either empty or an interval. Thus, for M block-decomposable, intersecting each block in $\mathcal{B}(M)$ with the line $y = x$ and identifying this line with $\mathbb{R}$, we obtain a collection $\operatorname{diag}\mathcal{B}(M)$ of intervals in $\mathbb{R}$. For $\gamma\colon T \to \mathbb{R}$ of Morse type, we call

$$\mathcal{L}_i(\gamma) := \operatorname{diag}\mathcal{B}(H_i\mathcal{S}(\gamma))$$

the i^{th} level set barcode of γ. Level set barcodes were introduced in [13]. $\mathcal{L}_i(\gamma)$ tracks how homological features are born and die as one sweeps across the level sets of γ.

Theorem 1.2 (stability of level set barcodes [13]) *For T a topological space, maps $\gamma, \kappa\colon T \to \mathbb{R}$ of Morse type and $i \geq 0$,*

$$d_b(\mathcal{L}_i(\gamma), \mathcal{L}_i(\kappa)) \leq d_\infty(\gamma, \kappa).$$

1.2 Our results: algebraic stability for zigzag and block-decomposable modules

The ϵ–interleavings and the interleaving distance d_I are readily defined on $\mathbb{U}$–indexed persistence modules. Furthermore, we will see in Section 2.3 that we can define

ϵ–matchings and a bottleneck distance d_b for the barcodes of block-decomposable modules in much the same way we do for 1-D persistence modules. Given this, it is natural to wonder whether an algebraic stability result holds for block-decomposable modules. Our Proposition 2.13 and Theorem 3.3 give the following such result:

Theorem

(i) *If there exists an ϵ–interleaving between pfd block-decomposable modules M and N, then there exists a $\frac{5}{2}\epsilon$–matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$.*

(ii) *Conversely, if there exists an ϵ–matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$, then there exists an ϵ–interleaving between M and N.*

In particular,

$$d_I(M,N) \leq d_b(\mathcal{B}(M),\mathcal{B}(N)) \leq \tfrac{5}{2} d_I(M,N).$$

The proof of (ii) is trivial. We refer to (i) as the *block stability theorem*. The block stability theorem was conjectured (independently) by Ulrich Bauer and Dmitriy Morozov, who were motivated by an application to the stability of Reeb graphs described below. Discussions with Bauer and Morozov inspired this work.

We show in Section 4.1 that by way of the functor $E\colon \mathbf{Vec}^{\mathbb{ZZ}} \to \mathbf{Vec}^{\mathbb{U}}$, our forward and converse algebraic stability results for block-decomposable modules specialize to corresponding algebraic stability results for zigzag modules. Prior to our work, the problem of establishing an algebraic stability theorem for zigzag modules was well known among researchers working on the theoretical foundations of topological data analysis, and was mentioned in the literature in several places; see Lesnick [33], Oudot [37], Oudot and Sheehy [38] and the mention of the more general problem of "hard stability" in the work of Bubenik, de Silva and Scott [9].

We obtain the block stability theorem as a corollary of induced matching results for block-decomposable modules analogous to those known to hold in 1-D. As part of the proof, we establish an induced matching theorem for free 2-D persistence modules; this yields an isometry theorem for such modules as a corollary.

The block stability theorem yields an alternative proof of Theorem 1.2, the stability result for level set persistent homology. In contrast to the earlier proof, our proof does not require us to consider extended persistence or relative homology.

Algebraic stability of constructible sheaves over $\mathbb{R}$ Interleavings and barcodes can be defined for pfd (co)sheaves of vector spaces over $\mathbb{R}$ that are constructible with respect to a locally finite partition of $\mathbb{R}$, much as we define them for block-decomposable persistence modules; see the thesis of Curry [26] and the subsequent related work by Curry and Patel [27]. As a corollary, the block stability theorem yields a similar algebraic stability theorem for such (co)sheaves. However, we will not explicitly consider (co)sheaves in this paper.

1.3 Stability of the persistent homology of Reeb graphs

We briefly describe the application of the block stability theorem to Reeb graphs; details are given in Section 4.3.

We define a Reeb graph to be a continuous function $\gamma\colon G \to \mathbb{R}$ of Morse type, where G is a topological graph and the level sets of γ are discrete. A well-known construction associates a Reeb graph, $\mathrm{Reeb}(\kappa)$, to an $\mathbb{R}$–valued function κ of Morse type. These invariants of $\mathbb{R}$–valued functions are readily computed and easy to visualize. As such, they are popular objects of study in computational geometry and topology, and have found many applications in data visualization and exploratory data analysis. In particular, the topological data analysis tool Mapper, introduced by Carlsson et al and commercialized by Ayasdi, constructs certain discrete approximations to Reeb graphs from point cloud data; see Singh, Mémoli and Carlsson [43].

If we want to study the stability of Reeb graphs and Mapper in the presence of noise, we need a good metric on Reeb graphs. In the last few years, several works have introduced such metrics and have studied their stability properties; see Bauer, Ge and Wang [2], Bauer, Munch and Wang [4], Di Fabio and Landi [28] and de Silva, Munch and Patel [42]. In particular, the last work presents an appealing definition of the interleaving distance d_I on Reeb graphs.

Bauer, Ge and Wang [2] observe that the 0^{th} level set barcode $\mathcal{L}_0(\gamma)$ of a Reeb graph γ encodes all nontrivial persistent homology information in the Reeb graph. A basic question about d_I, then, is whether Reeb graphs which are close with respect to d_I have close 0^{th} level set barcodes. Building on a result of [2], Bauer, Munch and Wang [4] recently provided an affirmative answer to this question. A simple formulation of their result says that for Reeb graphs γ and κ,

$$d_b(\mathcal{L}_0(\gamma), \mathcal{L}_0(\kappa)) \leq 9 d_I(\gamma, \kappa).$$

A somewhat stronger formulation of the result can be given using the language of extended persistence; see [4].

As an easy corollary of the block stability theorem, our Theorem 4.13 gives an improvement of the result of [4]:

$$d_b(\mathcal{L}_0(\gamma), \mathcal{L}_0(\kappa)) \leq 5 d_I(\gamma, \kappa). \tag{1}$$

1.4 Bjerkevik's related work

The version of the block stability theorem we establish here is not tight. To prove the result, we show that it suffices to establish the result for each of four subtypes of block-decomposable modules. Our algebraic stability results for three of the four subtypes are tight, but our result for the remaining subtype, denoted by type $\boldsymbol{o}$, turns out to be weaker than the optimal one by factor of $\frac{5}{2}$.

Following the release of the first version of this paper, Håvard Bakke Bjerkevik [6] has obtained a tight algebraic stability result for modules of type $\boldsymbol{o}$, via an elegant new argument. Together with our arguments in Section 7, this gives a tight form of the block stability theorem. As a corollary, our stability results for zigzag modules strengthen correspondingly to an isometry theorem for zigzag modules, and the constant in our stability result for the level set persistent homology of Reeb graphs is lowered from 5 to 2, which is tight. (On the other hand, the problem of giving a tight *single-morphism* algebraic stability result remains open; see Section 9.) Notably, the approach of [6] also adapts readily to give algebraic stability results for some other types of modules to which our approach does not readily extend, such as *rectangle-decomposable* persistence modules; see Section 9.

The main advantage of the approach to block stability taken in our paper, relative to that of [6], is that by extending the induced matching approach to algebraic stability, our approach provides explicit matchings of barcodes. In 1-D, the induced matching approach is very intuitive, and it is natural to study how the simple, explicit constructions of that approach extend to block-decomposable modules; our work makes clear both what can be done in this direction and where one encounters difficulties. We imagine that there could be a way to strengthen our arguments to recover the optimal constants for the block stability theorem obtained in [6], via explicit matchings. However, this would require further technical advances; see the end of Section 9.

Outline

Section 2 of this paper reviews algebraic aspects of persistent homology, introducing generalized definitions of barcodes and the bottleneck distance along the way. In

Section 3, we introduce block-decomposable modules and their barcodes, and state the block stability theorem. Section 4 presents our applications of the block stability theorem, including our treatment of algebraic stability for zigzag modules.

Sections 5, 6 and 7 are devoted to the proof of the block stability theorem. Section 5 introduces a way of decomposing a monomorphism of 2-D persistence modules. Using this decomposition, Section 6 proves the induced matching theorem for free 2-D persistence modules, as well as a similar induced matching result of a more technical nature for a class of 2-D persistence modules we call R_ϵ–free. Section 7 applies the results of Section 6 to prove the block stability theorem.

Section 8 gives an easy extension of the block stability theorem to a slightly more general class of modules, and speculates on an application of this to the stability of level set persistence for non-Morse-type functions. We conclude in Section 9 with a brief exploration of the problem of further generalizing the results of this paper.

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2 Preliminaries

For $\mathbb{P}$ a poset and $\mathcal{C}$ an arbitrary category, $M\colon \mathbb{P} \to \mathcal{C}$ a functor and $a, b \in \mathbb{P}$, let $M_a = M(a)$ and let $\varphi_M(a,b)\colon M_a \to M_b$ denote the morphism $M(a \le b)$.

2.1 Barcodes of interval-decomposable persistence modules

An *interval* of $\mathbb{P}$ is a subset $\mathcal{J} \subset \mathbb{P}$ such that:

(1) $\mathcal{J}$ is nonempty.

(2) If $a, c \in \mathcal{J}$ and $a \leq b \leq c$, then $b \in \mathcal{J}$.

(3) **Connectivity** For any $a, c \in \mathcal{J}$, there is a sequence $a = b_0, b_1, \dots, b_l = c$ of elements of $\mathcal{J}$ with b_i and b_{i+1} comparable for $0 \leq i \leq l-1$.

We refer to a multiset of intervals in $\mathbb{P}$ as a *barcode* (*over* $\mathbb{P}$).

Definition 2.1 For $\mathcal{J}$ an interval in $\mathbb{P}$, the interval module $I^{\mathcal{J}}$ is the $\mathbb{P}$–indexed module such that

$$I_a^{\mathcal{J}} = \begin{cases} k & \text{if } a \in \mathcal{J}, \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi_{I^{\mathcal{J}}}(a,b) = \begin{cases} \mathrm{id}_k & \text{if } a \leq b \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We say a persistence module M is *decomposable* if it can be written as $M \cong V \oplus W$ for nontrivial persistence modules V and W; otherwise, we say that M is *indecomposable*.

Proposition 2.2 *$I^{\mathcal{J}}$ is indecomposable.*

Proof For M a persistence module, let $\mathrm{End}(M)$ denote the k–vector space of endomorphisms of M. An endomorphism of $I^{\mathcal{J}}$ acts locally by multiplication, so it follows by commutativity and connectivity that $\mathrm{End}(I^{\mathcal{J}}) \cong k$. Assume that $I^{\mathcal{J}} \cong M \oplus N$ for persistence modules M and N. Then $\mathrm{End}(M) \oplus \mathrm{End}(N)$ is a subspace of $\mathrm{End}(M \oplus N) \cong \mathrm{End}(I^{\mathcal{J}}) \cong k$. The only subspaces of k are 0 and k, so either $\mathrm{End}(M) = 0$ or $\mathrm{End}(N) = 0$, implying that either M or N is trivial. □

A $\mathbb{P}$–indexed module M is *interval-decomposable* if there exists a (possibly infinite) multiset $\mathcal{B}(M)$ of intervals in $\mathbb{P}$ such that

$$M \cong \bigoplus_{\mathcal{J} \in \mathcal{B}(M)} I^{\mathcal{J}}.$$

Since the endomorphism rings of interval persistence modules are local (in fact, isomorphic to k), it follows from the Azumaya–Krull–Remak–Schmidt theorem [1] that the multiset $\mathcal{B}(M)$ is uniquely defined. We call $\mathcal{B}(M)$ the *barcode* of M.

Theorem 2.3 (structure of 1-D and zigzag persistence modules [8; 25]) *Suppose M is a pfd $\mathbb{P}$–indexed module for $\mathbb{P} \in \{\mathbb{R}, \mathbb{Z}, \mathbb{ZZ}\}$. Then M is interval-decomposable.*

Remark 2.4 For $\mathbb{ZZ}$–indexed modules, this structure theorem has typically appeared in the TDA literature under an additional finiteness assumption — see [12], for example. However, a proof of the general result as stated above can be found in [8].

2.2 Multidimensional persistence modules and interleavings

Multidimensional persistence modules For $n \geq 1$, let $\mathbb{R}^n$ denote the poset obtained by taking the product of $\mathbb{R}$ with itself n times. $\mathbb{R}^n$–indexed modules are known in the TDA literature as *n–dimensional persistence modules*. They arise naturally in the study of data with noise or nonuniformities in density; see eg [15; 19; 34].

Remark 2.5 The analogue of Theorem 2.3 does not hold for $\mathbb{P} = \mathbb{R}^n$ when $n \geq 2$. Indeed, it is a basic lesson from the representation theory of quivers that an arbitrary $\mathbb{P}$–indexed module M is interval-decomposable only for very special choices of $\mathbb{P}$.

Interleavings of $\mathbb{R}^n$–indexed functors For $\mathcal{C}$ an arbitrary category and $u \in \mathbb{R}^n$, define the *u–shift functor* $(-)(u)\colon \mathcal{C}^{\mathbb{R}^n} \to \mathcal{C}^{\mathbb{R}^n}$ on objects by $M(u)_a = M_{u+a}$, together with the obvious internal morphisms, and on morphisms $f\colon M \to N$ by $f(u)_a = f(u+a)\colon M(u)_a \to N(u)_a$. For $u \in [0,\infty)^n$, let $\varphi_M^u\colon M \to M(u)$ be the morphism whose restriction to each M_a is the linear map $\varphi_M(a, a+u)$. For $\epsilon \in [0,\infty)$ we will abuse notation slightly by letting $(-)(\epsilon)$ denote the $\epsilon(1,\dots,1)$–shift functor and letting φ_M^ϵ denote $\varphi_M^{\epsilon(1,\dots,1)}$.

Definition 2.6 Given $\epsilon \in [0,\infty)$, an ϵ–interleaving between $M, N\colon \mathbb{R}^n \to \mathcal{C}$ is a pair of morphisms $f\colon M \to N(\epsilon)$ and $g\colon N \to M(\epsilon)$ such that

$$g(\epsilon) \circ f = \varphi_M^{2\epsilon}, \quad f(\epsilon) \circ g = \varphi_N^{2\epsilon}.$$

We call f and g *ϵ–interleaving morphisms*. If there exists an ϵ–interleaving between M and N, we say M and N are ϵ–interleaved. The *interleaving distance*

$$d_I\colon \mathrm{Ob}(\mathcal{C}^{\mathbb{R}^n}) \times \mathrm{Ob}(\mathcal{C}^{\mathbb{R}^n}) \to [0,\infty]$$

is given by

$$d_I(M,N) = \inf\{\epsilon \geq 0 \mid M \text{ and } N \text{ are } \epsilon\text{–interleaved}\}.$$

The distance d_I is an *extended pseudometric*; that is, d_I is symmetric, d_I satisfies the triangle inequality and $d_I(M,M) = 0$ for all $\mathbb{R}^n$–indexed modules M.

Interleavings and ϵ–trivial (co)kernels For $u \in [0,\infty)^n$, we say an n-D persistence module M is *u–trivial* if $\varphi_M^u = 0$. For $\epsilon \in [0,\infty)$, we say M is ϵ–trivial if M is $(\epsilon,\epsilon,\dots,\epsilon)$–trivial. Note that M is 2ϵ–trivial if and only if M is ϵ–interleaved with 0.

Remark 2.7 It is an easy exercise to show that if $f\colon M \to N(\epsilon)$ is an ϵ–interleaving morphism, then $\ker f$ and $\operatorname{coker} f$ are each 2ϵ–trivial. For $n=1$, the converse is also true. For $n>1$, only a weaker converse holds: if $f\colon M \to N(\epsilon)$ has 2ϵ–trivial (co)kernel, then f is a 2ϵ–interleaving morphism, but it may not be the case that M and N are ϵ'–interleaved for any $\epsilon' < 2\epsilon$; see [3] for details.

Duals of persistence modules Dualizing each vector space and each linear map in a $\mathbb{P}$–indexed module M yields a $\mathbb{P}^{\mathrm{op}}$–indexed module M^*. As in the case of finite-dimensional vector spaces, when M is pfd, M^{**} is canonically isomorphic to M. Moreover, given a map $f\colon M \to N$ of $\mathbb{P}$–indexed modules, we have a dual map $f^*\colon N^* \to M^*$. This gives a functor

$$(-)^*\colon \mathbf{Vec}^{\mathbb{P}} \to \mathbf{Vec}^{\mathbb{P}^{\mathrm{op}}}.$$

We omit the proof of the following:

Proposition 2.8

(i) *If $f\colon M \to N$ is a morphism of $\mathbb{R}^n$–indexed modules with ϵ–trivial kernel, then f^* has ϵ–trivial cokernel.*

(ii) *Dually, if f has ϵ–trivial cokernel, then f^* has ϵ–trivial kernel.*

2.3 The isometry theorem

Matchings A *matching* σ between multisets S and T (written as $\sigma\colon S \nrightarrow T$) is a bijection $\sigma\colon S \supseteq S' \to T' \subset T$. Formally, we regard σ as a relation $\sigma \subset S \times T$ where $(s,t) \in \sigma$ if and only if $s \in S'$ and $\sigma(s) = t$. We call S' and T' the *coimage* and *image* of σ, respectively, and denote them by $\operatorname{coim}\sigma$ and $\operatorname{im}\sigma$. If $w \in \operatorname{coim}\sigma \cup \operatorname{im}\sigma$, we say that σ *matches* w. We say that σ is *bijective* if $S' = S$ and $T' = T$.

For two matchings $\sigma\colon S \nrightarrow R$ and $\tau\colon R \nrightarrow T$ we define the *composite matching* $\tau\circ\sigma\colon S \nrightarrow T$ by taking $(s,t) \in \tau\circ\sigma$ if and only if $(r,t) \in \tau$ and $(s,r) \in \sigma$ for some $r \in R$.

Generalized ϵ–matchings and bottleneck distance We now introduce a generalization of the bottleneck distance to barcodes over $\mathbb{R}^n$.

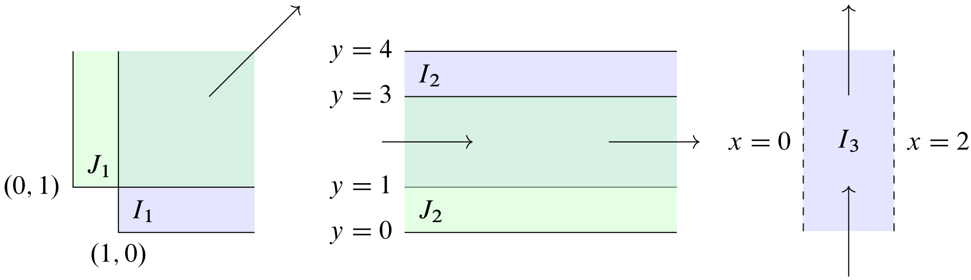


Figure 1: The 1–matching σ of Example 2.9. Left: σ matches the quadrants I_1 and J_1. Center: σ matches the horizontal strips I_2 and J_2. Right: σ does not match the vertical strip I_3.

We say intervals $\mathcal{J},\mathcal{K}\subset\mathbb{R}^n$ are ϵ–interleaved if $I^{\mathcal{J}}$ and $I^{\mathcal{K}}$ are ϵ–interleaved. Similarly, we say $\mathcal{J}$ is ϵ–trivial if $I^{\mathcal{J}}$ is ϵ–trivial, ie if for each $a\in\mathcal{J}$, $a+\epsilon(1,\dots,1)\notin\mathcal{J}$. For $\mathcal{C}$ a barcode over $\mathbb{R}^n$ and $\epsilon\geq 0$, define $\mathcal{C}_\epsilon\subset\mathcal{C}$ to be the multiset of intervals in $\mathcal{C}$ that are not ϵ–trivial.

Define an *ϵ–matching* between barcodes $\mathcal{C}$ and $\mathcal{D}$ to be a matching $\sigma\colon\mathcal{C}\nrightarrow\mathcal{D}$ satisfying the following properties:

(1) $\mathcal{C}_{2\epsilon}\subset\operatorname{coim}\sigma$ and $\mathcal{D}_{2\epsilon}\subset\operatorname{im}\sigma$.

(2) If $\sigma(\mathcal{J})=\mathcal{K}$, then $\mathcal{J}$ and $\mathcal{K}$ are ϵ–interleaved.

For barcodes $\mathcal{C}$ and $\mathcal{D}$, we define the bottleneck distance d_b by

$$d_b(\mathcal{C},\mathcal{D})=\inf\{\epsilon\in[0,\infty)\mid\text{there exists an }\epsilon\text{–matching between }\mathcal{C}\text{ and }\mathcal{D}\}.$$

It is not hard to check that d_b is an extended pseudometric. In particular, it satisfies the triangle inequality.

Example 2.9 Let $\mathcal{C}=\{I_1,I_2,I_3\}$ and $\mathcal{D}=\{J_1,J_2\}$, where

$$\begin{aligned} I_1&=\{a\in\mathbb{R}^2\mid a\geq(1,0)\}, & J_1&=\{a\in\mathbb{R}^2\mid a\geq(0,1)\},\\ I_2&=\{(a_1,a_2)\in\mathbb{R}^2\mid 1\leq a_2\leq 4\}, & J_2&=\{(a_1,a_2)\in\mathbb{R}^2\mid 0\leq a_2\leq 3\},\\ I_3&=\{(a_1,a_2)\in\mathbb{R}^2\mid 0<a_1<2\}. \end{aligned}$$

Observe that I_i and J_i are 1–interleaved for $i\in\{1,2\}$ and that I_3 is 2–trivial. Thus, the matching $\sigma\colon\mathcal{C}\nrightarrow\mathcal{D}$ defined by $\sigma=\{(I_1,J_1),(I_2,J_2)\}$ is a 1–matching. See Figure 1 for an illustration.

ϵ–Matchings of barcodes over $\mathbb{R}$ For $\mathcal{J} \subset \mathbb{R}$ an interval and $\epsilon \geq 0$, let the interval $\text{thk}^\epsilon(\mathcal{J})$ be given by

$$\text{thk}^\epsilon(\mathcal{J}) = \{a \in \mathbb{R} \mid |a-b| \leq \epsilon \text{ for some } b \in \mathcal{J}\}.$$

It is easy to check that intervals $\mathcal{J}, \mathcal{K} \subset \mathbb{R}$ are ϵ–interleaved if and only if either $\mathcal{J} \subset \text{thk}^\epsilon(\mathcal{K})$ and $\mathcal{K} \subset \text{thk}^\epsilon(\mathcal{J})$, or $\mathcal{J}$ and $\mathcal{K}$ are both 2ϵ–trivial. Moreover, $\mathcal{J}$ is 2ϵ–trivial if and only if $\mathcal{J}$ is strictly contained in the interval $[a, a+2\epsilon]$ for some $a \in \mathbb{R}$. This gives us a concrete description of ϵ–matchings of barcodes over $\mathbb{R}$.

Remark 2.10 In the 1-D setting, our definition of ϵ–matching is slightly different from the one given in [3], because it allows us to match 2ϵ–trivial intervals that are far away from each other. However, this difference turns out to be of no importance; in particular, it is easy to see that the two definitions of ϵ–matching yield equivalent definitions of d_b.

The isometry theorem In its strong formulation for pfd persistence modules [3], the isometry theorem says the following:

Theorem 2.11 (isometry [17; 33; 20; 3]) *Pfd $\mathbb{R}$–indexed persistence modules M and N are ϵ–interleaved if and only if there exists an ϵ-matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$. In particular,*

$$d_I(M,N) = d_b(\mathcal{B}(M), \mathcal{B}(N)).$$

See also [20] or [3] for a version of the isometry theorem which applies to a more general class of 1-D persistence modules called q–tame.

The induced matching theorem As noted in the introduction, the induced matching theorem [3] concerns a simple map χ sending each morphism $f\colon M \to N$ of pfd $\mathbb{R}$–indexed modules to a matching $\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$. We will not need the full strength of the induced matching theorem, and so to minimize the amount of notation we introduce, we present a slightly weaker version of the result.

For $a \in \mathbb{R}$, let $\langle a, a\rangle = [a, a]$. For $a < b \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\langle a, b\rangle \subset \mathbb{R}$ denote an interval in $\mathbb{R}$ with left endpoint a and right endpoint b. Thus,

- $\langle a, \infty\rangle$ denotes either of the intervals (a, ∞), $[a, \infty)$;
- $\langle -\infty, a\rangle$ denotes either of the intervals $(-\infty, a)$; $(-\infty, a]$,
- $\langle a, b\rangle$ denotes one of the intervals (a, b), $[a, b]$, $[a, b)$, $(a, b]$.

Theorem 2.12 (induced matchings) *Let $f\colon M \to N$ be a morphism of pfd $\mathbb{R}$–indexed modules and assume that $\chi(f)\langle a,b\rangle = \langle a',b'\rangle$. Then:*

(i) $a' \leq a \leq b' \leq b$.

(ii) *If f has ϵ–trivial kernel, then $\chi(f)$ matches each interval in $\mathcal{B}(M)_\epsilon$ and $|b-b'| \leq \epsilon$.*

(iii) *Dually, if f has δ–trivial cokernel, then $\chi(f)$ matches each interval in $\mathcal{B}(N)_\delta$ and $|a-a'| \leq \delta$.*

Converse algebraic stability One direction of Theorem 2.11 generalizes immediately to interval-decomposable $\mathbb{R}^n$–indexed modules; given the way we have defined ϵ–matchings, the proof is essentially trivial.

Proposition 2.13 (converse algebraic stability) *Given interval-decomposable $\mathbb{R}^n$–indexed modules M and N, if there exists an ϵ–matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$, then M and N are ϵ–interleaved. In particular,*

$$d_I(M,N) \leq d_b(\mathcal{B}(M),\mathcal{B}(N)).$$

2.4 $\mathbb{U}$–indexed modules as 2-D persistence modules

Recalling the definition of $\mathbb{U}$ from Section 1, we define a functor

$$\mathrm{emb}\colon \mathbf{Vec}^{\mathbb{U}} \to \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}},$$

given on objects M by taking $\mathrm{emb}(M)$ to be trivial outside of $\mathbb{U}$; explicitly, we define $\mathrm{emb}(M)$ by

$$\mathrm{emb}(M)_{(a,b)} = \begin{cases} M_{(a,b)} & \text{if } a \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{\mathrm{emb}(M)}((a,b),(c,d)) = \begin{cases} \varphi_M((a,b),(c,d)) & \text{if } c \leq a \leq b \leq d, \\ 0 & \text{otherwise,} \end{cases}$$

with the action of emb on morphisms defined in the obvious way. Clearly, emb is fully faithful, so by way of this functor, we may regard $\mathbf{Vec}^{\mathbb{U}}$ as full subcategory of $\mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}}$.

Remark 2.14 ($\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed and $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed modules) The isomorphism $\mathbb{R} \to \mathbb{R}^{\mathrm{op}}$ sending each $a \in \mathbb{R}$ to $-a$ induces an isomorphism $\mathbb{R}^{\mathrm{op}} \times \mathbb{R} \to \mathbb{R}^2$. This in turn induces an isomorphism $\mathbf{Vec}^{\mathbb{R}^2} \to \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}}$. By way of these isomorphisms, all the definitions introduced in Section 2.2 in the $\mathbb{R}^n$–indexed case, eg of ϵ–interleavings and ϵ–matchings, carry over to the $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed setting. Similarly, they carry over to the $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed setting.

2.5 Kan extensions

In several places in this paper, we introduce functors $\mathbf{Vec}^{\mathbb{A}} \to \mathbf{Vec}^{\mathbb{B}}$ for distinct posets $\mathbb{A}$ and $\mathbb{B}$, as we have in Section 2.4 above. For this, it will be convenient to adopt the language of Kan extensions. We now briefly review Kan extensions in the specific setting of interest to us, giving concrete formulae in terms of limits and colimits. See [35] for the standard, fully general definition of Kan extensions.

Given a functor of posets $F\colon \mathbb{A} \to \mathbb{B}$ and $b \in \mathbb{B}$, let

$$\mathbb{A}[F \leq b] := \{a \in \mathbb{A} \mid F(a) \leq b\}.$$

Define $\mathbb{A}[F \geq b] \subset \mathbb{A}$ analogously.

Given a persistence module $M\colon \mathbb{A} \to \mathbf{Vec}$, one defines a persistence module

$$\mathrm{Lan}_F(M)\colon \mathbb{B} \to \mathbf{Vec},$$

called the *left Kan extension of M along F*, by taking

$$\mathrm{Lan}_F(M)(b) = \varinjlim M|_{\mathbb{A}[F \leq b]},$$

with the internal maps $\mathrm{Lan}_F(b) \to \mathrm{Lan}_F(b')$ given by universality of colimits for all $b \leq b'$. Given $M, N\colon \mathbb{A} \to \mathbf{Vec}$ and a natural transformation $f\colon M \to N$, universality of colimits also yields an induced morphism $\mathrm{Lan}_F(f)\colon \mathrm{Lan}_F(M) \to \mathrm{Lan}_F(N)$. We thus obtain a functor $\mathrm{Lan}_F(-)\colon \mathbf{Vec}^{\mathbb{A}} \to \mathbf{Vec}^{\mathbb{B}}$.

Example 2.15 For emb the functor defined in Section 2.4, letting $\mathrm{e}\colon \mathbb{U} \hookrightarrow \mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ denote the inclusion, we have $\mathrm{emb} = \mathrm{Lan}_{\mathrm{e}}(-)$.

Dually, one also defines a persistence module $\mathrm{Ran}_F(M)\colon \mathbb{A} \to \mathbf{Vec}$, the *right Kan extension of M along F*, by taking

$$\mathrm{Ran}_F(M)(b) = \varprojlim M|_{\mathbb{A}[F \geq b]},$$

with the internal maps given by universality of limits. As with left Kan extensions, this definition is functorial, so we obtain a functor $\mathrm{Ran}_F(-)\colon \mathbf{Vec}^{\mathbb{A}} \to \mathbf{Vec}^{\mathbb{B}}$.

Proposition 2.16

(i) $\mathrm{Lan}_F(-)$ *preserves direct sums, ie for any indexing set* $\mathcal{A}$ *and persistence modules* $\{M_i\colon \mathbb{A} \to \mathbf{Vec}\}_{i \in \mathcal{A}}$*, we have*

$$\mathrm{Lan}_F\Big(\bigoplus_i M_i\Big) \cong \bigoplus_i \mathrm{Lan}_F(M_i).$$

(ii) *Dually,* $\mathrm{Ran}_F(-)$ *preserves direct products, ie for any persistence modules* $\{M_i\colon \mathbb{A} \to \mathbf{Vec}\}_{i \in \mathcal{A}}$*, we have*

$$\mathrm{Ran}_F\Big(\prod_i M_i\Big) \cong \prod_i \mathrm{Ran}_F(M_i).$$

Proof This follows directly from standard category theory results: $\mathrm{Lan}_F(-)$ is left adjoint to the restriction $\mathbf{Vec}^{\mathbb{B}} \to \mathbf{Vec}^{\mathbb{A}}$ along F; see for example [40, (1.1)]. Since $\mathrm{Lan}_F(-)$ is a left adjoint, it preserves coproducts [35, Theorem V.5.1]. This establishes (i) and the dual argument establishes (ii). □

Remark 2.17 Given an indexing set $\mathcal{A}$ and persistence modules

$$\{M_i\colon \mathbb{A} \to \mathbf{vec}\}_{i \in \mathcal{A}},$$

if $\bigoplus_i M_i$ is pfd, then

$$\bigoplus_i M_i = \prod_i M_i.$$

It follows that if, in Proposition 2.16(ii), both $\bigoplus_i M_i$ and $\bigoplus_i \mathrm{Ran}_F(M_i)$ are pfd, then

$$\mathrm{Ran}_F\Big(\bigoplus_i M_i\Big) \cong \bigoplus_i \mathrm{Ran}_F(M_i).$$

3 The block stability theorem

In general, a $\mathbb{U}$–indexed module does not decompose into a direct sum of interval modules. However, as noted in the introduction, we shall restrict our attention to $\mathbb{U}$–indexed modules called *block-decomposables* which admit a particularly simple decomposition.

Blocks For any interval $\mathcal{J}$ in $\mathbb{R}$, we define an interval $\mathcal{J}_{\mathrm{BL}}$ in $\mathbb{U}$ as follows:

$$(a,b)_{\mathrm{BL}} := \{(x,y) \in \mathbb{U} \mid a < x,\ y < b\} \quad \text{for } a < b \in \mathbb{R} \cup \{-\infty, \infty\},$$
$$[a,b)_{\mathrm{BL}} := \{(x,y) \in \mathbb{U} \mid a \le y < b\} \quad \text{for } a < b \in \mathbb{R} \cup \{\infty\},$$

$$(a,b]_{\mathrm{BL}} := \{(x,y) \in \mathbb{U} \mid a < x \leq b\} \qquad \text{for } a < b \in \mathbb{R} \cup \{-\infty\},$$
$$[a,b]_{\mathrm{BL}} := \{(x,y) \in \mathbb{U} \mid x \leq b,\ y \geq a\} \quad \text{for } a \leq b \in \mathbb{R}.$$

In addition, for $a < b \in \mathbb{R}$, we define an interval

$$[b,a]_{\mathrm{BL}} := \{(x,y) \in \mathbb{U} \mid x \leq a < b \leq y\}.$$

We call an interval in $\mathbb{U}$ having one of the five forms above a *block*, and we let BL denote the set of all blocks. Each of the five types of blocks is depicted in Figure 2.

For $a,b \in \mathbb{R} \cup \{\pm\infty\}$, let $\langle a,b\rangle_{\mathrm{BL}}$ denote a block of the form $(a,b)_{\mathrm{BL}}$, $[a,b)_{\mathrm{BL}}$, $(a,b]_{\mathrm{BL}}$ or $[a,b]_{\mathrm{BL}}$. For example, for $a \in \mathbb{R}$, $\langle a,\infty\rangle_{\mathrm{BL}}$ is either $[a,\infty)_{\mathrm{BL}}$ or $(a,\infty)_{\mathrm{BL}}$, and $\langle b,a\rangle_{\mathrm{BL}} = [b,a]_{\mathrm{BL}}$ for $a < b \in \mathbb{R}$,.

Block barcodes We call a multiset of blocks a *block barcode*. Note that in view of Remark 2.14, ϵ–matchings and the bottleneck distance d_b between block barcodes are well defined.

Partitions of block barcodes It will be convenient to partition BL into four subsets, as follows:

$$\mathrm{BL}^{\boldsymbol{o}} := \{(a,b)_{\mathrm{BL}} \mid a < b \in \mathbb{R}\},$$
$$\mathrm{BL}^{\boldsymbol{co}} := \{[a,b)_{\mathrm{BL}} \mid a < b \in \mathbb{R}\} \cup \{(-\infty,b)_{\mathrm{BL}} \mid b \in \mathbb{R}\},$$
$$\mathrm{BL}^{\boldsymbol{oc}} := \{(a,b]_{\mathrm{BL}} \mid a < b \in \mathbb{R}\} \cup \{(a,\infty)_{\mathrm{BL}} \mid a \in \mathbb{R}\},$$
$$\mathrm{BL}^{\boldsymbol{c}} := \{[a,b]_{\mathrm{BL}} \mid a,b \in \mathbb{R}\} \cup \{[a,\infty)_{\mathrm{BL}} \mid a \in \mathbb{R}\} \cup \{(-\infty,b]_{\mathrm{BL}} \mid a \in \mathbb{R}\} \cup \{(-\infty,\infty)_{\mathrm{BL}}\}.$$

If $\star \in \{\boldsymbol{o}, \boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}\}$ and $\mathcal{J} \in \mathrm{BL}^\star$, we say $\mathcal{J}$ is *is of type* $\star$. For example, $[0,1]_{\mathrm{BL}}$ and $(-\infty,\infty)_{\mathrm{BL}}$ are both of type $\boldsymbol{c}$.

For $\mathcal{B}$ a block barcode and $\star \in \{\boldsymbol{o}, \boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}\}$, we let $\mathcal{B}^\star$ denote the multisubset of blocks in $\mathcal{B}$ of type $\mathrm{BL}^\star$.

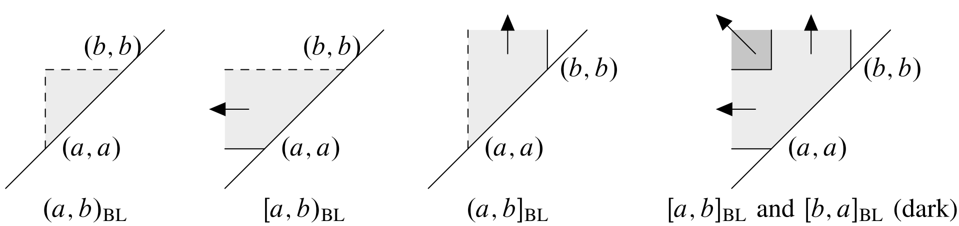


Figure 2: The five different types of blocks

ϵ–Matchings of block barcodes The following result, whose straightforward proof we omit, yields a concrete description of an ϵ–matching of block barcodes:

Lemma 3.1

(i) *$\langle a,b\rangle_{\mathrm{BL}}$ is 2ϵ–trivial if and only if one of the following is true:*
- *$\langle a,b\rangle_{\mathrm{BL}}$ is of type **co** or **oc**, and $b-a\le 2\epsilon$.*
- *$\langle a,b\rangle_{\mathrm{BL}}$ is of type **o** and $b-a\le 4\epsilon$.*

(ii) *Blocks $\langle a,b\rangle_{\mathrm{BL}}$ and $\langle a',b'\rangle_{\mathrm{BL}}$ are ϵ–interleaved if and only if either $\langle a,b\rangle_{\mathrm{BL}}$ and $\langle a',b'\rangle_{\mathrm{BL}}$ are of the same type and*

$$|a-a'|\le\epsilon,\quad |b-b'|\le\epsilon,$$

or both $\langle a,b\rangle_{\mathrm{BL}}$ and $\langle a',b'\rangle_{\mathrm{BL}}$ are 2ϵ–trivial.

Diagonals of block barcodes Let $D\colon \mathbb{R}\to\mathbb{R}^2$ denote the diagonal map $D(t)=(t,t)$ and, for any block $\mathcal{J}\subset\mathbb{U}$, let $\operatorname{diag}\mathcal{J}=D(\mathbb{R})\cap\mathcal{J}$. Note that for any interval $\mathcal{I}\subset\mathbb{R}$,

$$\operatorname{diag}\mathcal{I}_{\mathrm{BL}}=D(\mathcal{I}).$$

In this sense, $\mathcal{I}_{\mathrm{BL}}$ is labeled by its intersection with the diagonal.

For $\mathcal{B}$ a block barcode, we define $\operatorname{diag}\mathcal{B}$, the diagonal of $\mathcal{B}$, to be the barcode over $\mathbb{R}$ given by

$$\operatorname{diag}\mathcal{B}=\{\operatorname{diag}\mathcal{J}\mid \mathcal{J}\in\mathcal{B},\ \operatorname{diag}\mathcal{J}\ne\varnothing\}.$$

Proposition 3.2 *For block barcodes $\mathcal{B}$ and $\mathbb{C}$:*

(i) *An ϵ–matching $\sigma\colon\mathcal{B}\nrightarrow\mathbb{C}$ induces a 2ϵ–matching $\operatorname{diag}\sigma\colon\operatorname{diag}\mathcal{B}\nrightarrow\operatorname{diag}\mathbb{C}$. In particular,*

$$d_b(\operatorname{diag}\mathcal{B},\operatorname{diag}\mathbb{C})\le 2d_b(\mathcal{B},\mathbb{C}).$$

(ii) *If, additionally, σ matches each interval $(a,b)_{\mathrm{BL}}$ in $\mathcal{B}^{\boldsymbol{o}}_\epsilon\cup\mathbb{C}^{\boldsymbol{o}}_\epsilon$ to an interval $(a',b')_{\mathrm{BL}}$ with*

$$|a-a'|\le\epsilon\quad\text{and}\quad|b-b'|\le\epsilon,$$

then $\operatorname{diag}\sigma$ is an ϵ–matching.

Proof This is immediate from Lemma 3.1 and the definition of an ϵ–matching. □

Block-decomposable modules It follows from Proposition 2.2 that for any block $\mathcal{J}$, the $\mathbb{U}$–indexed interval module $I^{\mathcal{J}}$ is indecomposable; we call $I^{\mathcal{J}}$ a *block module*. We say a $\mathbb{U}$–indexed module is *block-decomposable* if it decomposes into a direct sum of block modules. We say an $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed module M is block-decomposable if $M = \mathrm{emb}(N)$ for N block-decomposable.

With these definitions, we may work interchangeably with block-decomposable modules over $\mathbb{U}$ and their embeddings under emb. We will work primarily in the $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed setting. In particular, we will understand an ϵ–interleaving between $\mathbb{U}$–indexed modules M and N to be an ϵ–interleaving between $\mathrm{emb}(M)$ and $\mathrm{emb}(N)$; see Remark 2.14. Explicitly then, an ϵ–interleaving between M and N consists of two collections of linear maps

$$\{f_{a,b}\colon M_{a,b} \to N_{a-\epsilon,b+\epsilon}\}_{a\leq b\in\mathbb{R}} \quad \text{and} \quad \{g_{a,b}\colon N_{a,b} \to M_{a-\epsilon,b+\epsilon}\}_{a\leq b\in\mathbb{R}}$$

satisfying the obvious commutativity conditions with each other and with the internal maps of M and N.

Block stability We now state the main result of this paper, which establishes a relationship between the interleaving distance and bottleneck distance on block-decomposable modules:

Theorem 3.3 (block stability theorem) *Let M and N be ϵ–interleaved pfd block-decomposable modules. Then there exists a matching $\chi\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$ that matches each block in*

$$\mathcal{B}(M)^{\boldsymbol{c}} \cup \mathcal{B}(M)^{\boldsymbol{o}}_{5\epsilon} \cup \mathcal{B}(M)^{\boldsymbol{co}}_{2\epsilon} \cup \mathcal{B}(M)^{\boldsymbol{oc}}_{2\epsilon} \quad \text{and} \quad \mathcal{B}(N)^{\boldsymbol{c}} \cup \mathcal{B}(N)^{\boldsymbol{o}}_{5\epsilon} \cup \mathcal{B}(N)^{\boldsymbol{co}}_{2\epsilon} \cup \mathcal{B}(N)^{\boldsymbol{oc}}_{2\epsilon},$$

such that if $\chi(\mathcal{I}) = \mathcal{J}$, then $\mathcal{I}$ and $\mathcal{J}$ are ϵ–interleaved and of the same type. In particular, χ is a $\frac{5}{2}\epsilon$–matching.

We give the proof of Theorem 3.3 in Sections 5, 6 and 7.

4 Applications of the block stability theorem

Before turning to the proof of the block stability theorem, we consider three applications. First, we explain how the block stability theorem induces an algebraic stability theorem for zigzag modules. Next, we show how the stability result for level set zigzag persistence of [13] follows from the block stability theorem. Last, we explain the application to the stability of Reeb graphs.

4.1 Algebraic stability of zigzag persistence modules

In this section, we define the fully faithful functor E sending each zigzag module to a block-decomposable module, first mentioned in Section 1. We use E to define interleaving and bottleneck distances for zigzag modules and their barcodes. With these definitions, the block stability theorem and its converse extend trivially to zigzag modules.

Our functor E is closely analogous to the functor sending a cellular cosheaf over $\mathbb{R}$ to a constructible cosheaf over $\mathbb{R}$; see [26].

Block extensions of zigzags Let $\iota\colon \mathbb{ZZ} \hookrightarrow \mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ denote the inclusion, and let

$$(-)|_{\mathbb{U}}\colon \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}} \to \mathbf{Vec}^{\mathbb{U}}$$

denote the restriction. We define the *block extension functor* $E\colon \mathbf{Vec}^{\mathbb{ZZ}} \to \mathbf{Vec}^{\mathbb{U}}$ by

$$E := (-)|_{\mathbb{U}} \circ \mathrm{Lan}_\iota(-).$$

Figure 3 illustrates the action of E on objects.

Intervals in the zigzag category We partition the intervals of $\mathbb{ZZ}$ into four types; letting $<$ denote the partial order on $\mathbb{Z}^2$ (not on $\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}$), these are given as follows:

$$\begin{aligned}
(b,d)_{\mathbb{ZZ}} &:= \{(i,j) \in \mathbb{ZZ} \mid (b,b) < (i,j) < (d,d)\} && \text{for } b < d \in \mathbb{Z} \cup \{-\infty, \infty\},\\
[b,d)_{\mathbb{ZZ}} &:= \{(i,j) \in \mathbb{ZZ} \mid (b,b) \leq (i,j) < (d,d)\} && \text{for } b < d \in \mathbb{Z} \cup \{\infty\},\\
(b,d]_{\mathbb{ZZ}} &:= \{(i,j) \in \mathbb{ZZ} \mid (b,b) < (i,j) \leq (d,d)\} && \text{for } b < d \in \mathbb{Z} \cup \{-\infty\},\\
[b,d]_{\mathbb{ZZ}} &:= \{(i,j) \in \mathbb{ZZ} \mid (b,b) \leq (i,j) \leq (d,d)\} && \text{for } b \leq d \in \mathbb{Z}.
\end{aligned}$$

We shall let $\langle b,d\rangle_{\mathbb{ZZ}}$ denote any of the intervals above.

Properties of the block extension functor The following lemma is illustrated by Figure 4. The proof is left to the reader.

Lemma 4.1 *The block extension functor sends interval modules to block interval modules. Specifically, for any zigzag interval $\langle b,d\rangle_{\mathbb{ZZ}}$,*

$$E(I^{\langle b,d\rangle_{\mathbb{ZZ}}}) \cong I^{\langle b,d\rangle_{\mathrm{BL}}}.$$

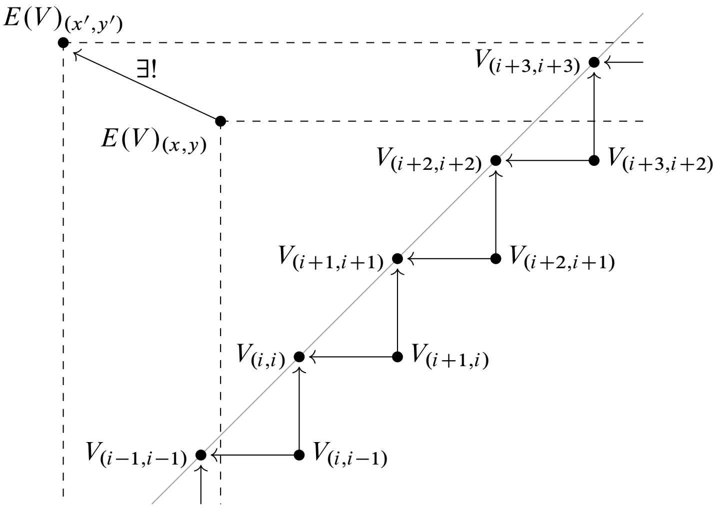


Figure 3: The vector space $E(V)_{(x,y)}$ is the colimit of the restriction of V to indices contained in the box with upper-left corner (x, y).

Proposition 4.2 *For any pfd zigzag module V, $E(V)$ is block-decomposable, and we have a bijective matching $\mathcal{B}(V) \leftrightarrow \mathcal{B}(E(V))$ which matches each zigzag interval $\langle b, d\rangle_{\mathbb{ZZ}}$ to the block interval $\langle b, d\rangle_{\mathrm{BL}}$.*

Proof By Proposition 2.16, $\mathrm{Lan}_\iota(-)$ preserves direct sums. Clearly $(-)|_{\mathbb{U}}$ preserves direct sums as well, so $E = (-)|_{\mathbb{U}} \circ \mathrm{Lan}_\iota(-)$ also preserves direct sums. The result now follows from Theorem 2.3 and Lemma 4.1. □

The following result, not used elsewhere in the paper, describes an additional sense in which E preserves the structure of $\mathbf{Vec}^{\mathbb{ZZ}}$:

Proposition 4.3 *$E\colon \mathbf{Vec}^{\mathbb{ZZ}} \to \mathbf{Vec}^{\mathbb{U}}$ is fully faithful.*

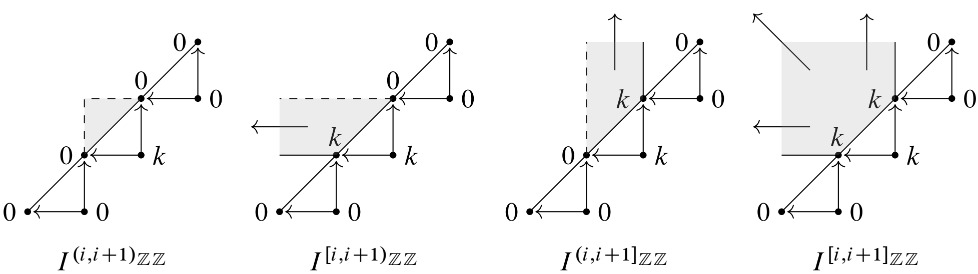


Figure 4: Extension to block interval modules of the four different types of zigzag interval modules. Compare with Figure 2.

Proof $\mathrm{Lan}_\iota(-)$ is left adjoint to the restriction functor $(-)|_\iota\colon \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}} \to \mathbf{Vec}^{\mathbb{ZZ}}$ [40, (1.1)]. The reader may easily verify that $(-)|_\iota \circ \mathrm{Lan}_\iota(-) \cong \mathrm{id}_{\mathbf{vec}^{\mathbb{ZZ}}}$; this also follows from [35, Corollary X.3.3]. Hence, $\mathrm{Lan}_\iota(-)$ is fully faithful [32]. It is easy to check that this property is preserved by postcomposition with $(-)|_{\mathbb{U}}$. □

Algebraic stability of zigzag modules

Definition 4.4 We define the interleaving and bottleneck distances on pfd zigzag persistence modules and their barcodes by

$$d_I(V,W) := d_I(E(V),E(W)), \quad d_b^{\mathbb{ZZ}}(\mathcal{B}(V),\mathcal{B}(W)) := d_b\big(\mathcal{B}(E(V)),\mathcal{B}(E(W))\big).$$

Given these definitions, we get forward and converse algebraic stability results for zigzags immediately from Theorem 3.3 and Proposition 2.13.

Remark 4.5 The interleaving distance on zigzag modules defined in this section is in fact an extension of the usual interleaving distance on $\mathbb{Z}$–indexed modules: We have an obvious fully faithful functor $D\colon \mathbf{Vec}^{\mathbb{Z}} \to \mathbf{Vec}^{\mathbb{ZZ}}$ which sends a $\mathbb{Z}$–indexed module to a zigzag module by taking all leftwards arrows to be isomorphisms; that is, for V a zigzag module, we take

$$D(V)_{(i,i)} = D(V)_{(i+1,i)} = V_i, \qquad \begin{aligned}\varphi_{D(V)}((i+1,i),(i,i)) &= \mathrm{id}_{V_i},\\ \varphi_{D(V)}((i,i-1),(i,i)) &= \varphi_V(i-1,i).\end{aligned}$$

The ordinary interleaving distance can be defined on $\mathbb{Z}$–indexed modules just as for $\mathbb{R}$–indexed modules, and it can be checked that D preserves interleaving distances.

4.2 Stability of (inter)level set persistence

We next explain how the stability of level set and interlevel set zigzag persistence, as established in [13; 5], follows from the block stability theorem. To begin, we introduce the necessary definitions, following [13].

Interlevel set persistent homology For T a topological space, we say a continuous function $\gamma\colon T \to \mathbb{R}$ is of *Morse type* if:

(1) There exists a strictly increasing function $\mathcal{G}\colon \mathbb{ZZ} \to \mathbb{R}$ such that

$$\lim_{z\to\pm\infty} \mathcal{G}_z = \pm\infty$$

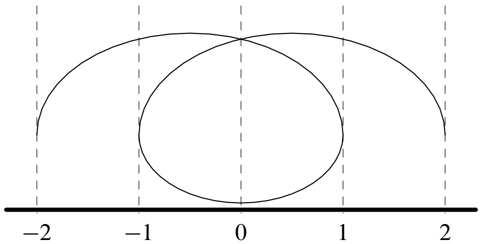


Figure 5: The immersed curve T of Examples 4.6 and 4.10

and such that for each open interval $I = (\mathcal{G}_z, \mathcal{G}_{z+1})$, $\gamma^{-1}(I)$ is homeomorphic to a product $I \times Y$ with γ the projection onto I. Note that Y may be different for different choices of I.

(2) Each homeomorphism $h\colon I \times Y \to \gamma^{-1}(I)$ extends to a continuous function

$$\bar{h}\colon \bar{I} \times Y \to \gamma^{-1}(\bar{I}) = \mathcal{S}(\gamma)_{\bar{I}},$$

where $\bar{I}$ denotes the closure of I.

(3) $\dim H_i(\gamma^{-1}(t)) < \infty$ for all $t \in \mathbb{R}$ and $i \geq 0$.

Example 4.6 Let T be the immersed curve in $\mathbb{R}^2$ depicted in Figure 5 and let $\gamma\colon T \to \mathbb{R}$ denote the projection onto the x–axis. Then γ is of Morse type; we may take the function $\mathcal{G}\colon \mathbb{Z} \to \mathbb{R}$ to be the usual inclusion.

Structure of interlevel set persistent homology Recall the definition of the interlevel set filtration $\mathcal{S}(\gamma)$ from Section 1.

Theorem 4.7 [13; 5] *For $\gamma\colon T \to \mathbb{R}$ of Morse type and $i \geq 0$:*

(i) *$H_i\mathcal{S}(\gamma)$ is block-decomposable, so $\mathcal{B}_i(\gamma) := \mathcal{B}(H_i\mathcal{S}(\gamma))$ is well defined.*

(ii) *There is a one-to-one correspondence between blocks $[b,a]_{\mathrm{BL}} \in \mathcal{B}_{i+1}(\gamma)$ with $a < b$ and blocks $(a,b)_{\mathrm{BL}} \in \mathcal{B}_i(\gamma)$.*

Theorem 4.7(i) is proven by appealing to the structure theorem for zigzag persistence modules and exploiting the connection between block-decomposable and zigzag persistence modules. Theorem 4.7(ii) is an application of the Mayer–Vietoris theorem.

Remarks 4.8

(1) In fact, Theorem 4.7 is proven in [13; 5] under an additional finiteness assumption. In view of the structure theorem for modules over infinite zigzags given in [8], the finiteness assumption is not necessary.

(2) Theorem 4.7 admits an extension to a *relative interlevel set persistence*; see [13; 5]. We will consider only the absolute version of the theorem here.

(3) $B_i(\gamma)$ can be computed in practice by doing an extended persistence or zigzag persistent homology computation, and appealing to the formulae in [13].

Level set barcodes Recall from Section 1 that the barcode $\mathcal{L}_i(\gamma) := \operatorname{diag} \mathcal{B}_i(\gamma)$ is called the i^{th} *level set (zigzag) barcode of* γ.

Remark 4.9 In view of Theorem 4.7 (ii), the block barcodes $\{\mathcal{B}_i(\gamma)\}_{i\geq 0}$ and the level set barcodes $\{\mathcal{L}_i(\gamma)\}_{i\geq 0}$ determine each other, so there is no loss in passing from interlevel set (block) barcodes to level set barcodes, as long as we consider homology in all degrees.

Example 4.10 It can be shown that for $\gamma\colon T \to \mathbb{R}$ as in Example 4.6,

$$\mathcal{B}_0(\gamma) = \{[-2,2]_{\text{BL}}, (-1,1)_{\text{BL}}, [-1,0)_{\text{BL}}, (0,1]_{\text{BL}}\}.$$

Thus, the 0^{th} level set barcode of γ is

$$\mathcal{L}_0(\gamma) = \{[-2,2], (-1,1), [-1,0), (0,1]\}.$$

Stability of level set persistence The stability theorem for level set persistence first appeared in [13]. The original proof is an application of the stability of extended persistence [24], and hence can be seen as an application of algebraic stability for 1-D persistence modules. We now give a different proof, based on the block stability theorem, which avoids consideration of extended persistence and relative homology.

Theorem 4.11 (stability of (inter)level set persistence) *Let $\gamma, \kappa\colon T \to \mathbb{R}$ be of Morse type and let $\epsilon = d_\infty(\gamma,\kappa)$. Then, for all $i \geq 0$,*

$$d_b(\mathcal{B}_i(\gamma), \mathcal{B}_i(\kappa)) \leq \epsilon,$$
$$d_b(\mathcal{L}_i(\gamma), \mathcal{L}_i(\kappa)) \leq \epsilon.$$

Proof For all $x \leq y$, we have inclusions

$$\mathcal{S}(\gamma)_{(x,y)} \subset \mathcal{S}(\kappa)_{(x-\epsilon,y+\epsilon)} \subset \mathcal{S}(\gamma)_{(x-2\epsilon,y+2\epsilon)},$$
$$\mathcal{S}(\kappa)_{(x,y)} \subset \mathcal{S}(\gamma)_{(x-\epsilon,y+\epsilon)} \subset \mathcal{S}(\kappa)_{(x-2\epsilon,y+2\epsilon)},$$

By the functoriality of H_i, these induce an ϵ–interleaving between $H_i\mathcal{S}(\gamma)$ and $H_i\mathcal{S}(\kappa)$. Applying Theorem 3.3, we obtain ϵ–matchings between $\mathcal{B}_i(\gamma)^\star$ and $\mathcal{B}_i(\kappa)^\star$

for $\star \in \{\boldsymbol{c}, \boldsymbol{oc}, \boldsymbol{co}\}$ and a $\frac{5}{2}\epsilon$–matching between $\mathcal{B}_i(\gamma)^{\boldsymbol{o}}$ and $\mathcal{B}_i(\kappa)^{\boldsymbol{o}}$. To establish the theorem, we will in fact need an ϵ–matching between $\mathcal{B}_i(\gamma)^{\boldsymbol{o}}$ and $\mathcal{B}_i(\kappa)^{\boldsymbol{o}}$ which matches each interval in $(a,b)_{\mathrm{BL}} \in \mathcal{B}_i(\gamma)^{\boldsymbol{o}}_\epsilon \cup \mathcal{B}_i(\kappa)^{\boldsymbol{o}}_\epsilon$ to an interval $(a',b')_{\mathrm{BL}}$ with

$$|a-a'| \le \epsilon \quad \text{and} \quad |b-b'| \le \epsilon,$$

as in the statement of Proposition 3.2(ii). We obtain this as follows: Let

$$\chi\colon \mathcal{B}_{i+1}(\gamma)^{\boldsymbol{c}} \to \mathcal{B}_{i+1}(\kappa)^{\boldsymbol{c}}$$

denote the ϵ–matching provided by Theorem 3.3 and note that χ is bijective. Then Theorem 4.7(ii) gives us injections

$$i_1\colon \mathcal{B}_i(\gamma)^{\boldsymbol{o}} \hookrightarrow \mathcal{B}_{i+1}(\gamma)^{\boldsymbol{c}} \quad \text{and} \quad i_2\colon \mathcal{B}_i(\kappa)^{\boldsymbol{o}} \hookrightarrow \mathcal{B}_{i+1}(\kappa)^{\boldsymbol{c}}.$$

By composition, we get a matching

$$i_2^{-1} \circ \chi \circ i_1\colon \mathcal{B}_i(\gamma)^{\boldsymbol{o}} \nrightarrow \mathcal{B}_i(\kappa)^{\boldsymbol{o}},$$

where i_2^{-1} denotes the reverse of the matching i_2.

The composition $\chi \circ i_1$ matches each block $(a,b)_{\mathrm{BL}} \in \mathcal{B}_i(\gamma)^{\boldsymbol{o}}_\epsilon$ to a block

$$\chi[b,a]_{\mathrm{BL}} = [b',a']_{\mathrm{BL}} \in \mathcal{B}_{i+1}(\kappa)^{\boldsymbol{c}}$$

with

$$|a-a'| \le \epsilon \quad \text{and} \quad |b-b'| \le \epsilon.$$

Since $b-a > 2\epsilon$, we have in particular that

$$a' \le a+\epsilon < b-\epsilon \le b'.$$

Thus, $[b',a']_{\mathrm{BL}} \in \operatorname{im} i_2$, and

$$i_2^{-1} \circ \chi \circ i_1(a,b)_{\mathrm{BL}} = (a',b')_{\mathrm{BL}} \in \mathcal{B}_i(\kappa)^{\boldsymbol{o}}.$$

This shows that

$$\mathcal{B}_i(\gamma)^{\boldsymbol{o}}_\epsilon \in \operatorname{coim} i_2^{-1} \circ \chi \circ i_1.$$

Applying the same argument in the opposite direction, we obtain that

$$\mathcal{B}_i(\kappa)^{\boldsymbol{o}}_\epsilon \in \operatorname{im} i_2^{-1} \circ \chi \circ i_1$$

and that $i_2^{-1} \circ \chi \circ i_1$ is an ϵ–matching as in the statement of Proposition 3.2(ii). Applying Proposition 3.2(ii), the result now follows. □

In Section 8, we discuss the stability problem for interlevel set and level set persistent homology in the case that our functions are not of Morse type.

4.3 Interleaving stability of Reeb graphs

This section applies the block stability theorem to strengthen the result of [4] on the interleaving stability of Reeb graphs. To begin, we review Reeb graphs and their interleavings. Our discussion loosely follows [42], which gives an in-depth treatment of the categorical interpretation of Reeb graphs; see that paper for more details.

Reeb graphs Recall from Section 1.3 that we define a Reeb graph to be a continuous function $\gamma\colon G \to \mathbb{R}$ of Morse type, where G is a topological graph and the level sets of γ are discrete.

We associate a Reeb graph, $\mathrm{Reeb}(\kappa)$, to any function $\kappa\colon T \to \mathbb{R}$ of Morse type, in the following way: Define an equivalence relation on T by taking $x \sim y$ if and only if x and y lie in the same connected component of $\kappa^{-1}(s)$ for some $s \in \mathbb{R}$, and let $T/{\sim}$ denote the resulting quotient space. Then κ descends to a continuous function

$$\mathrm{Reeb}(\kappa)\colon T/{\sim} \to \mathbb{R}.$$

It is easy to check that $\mathrm{Reeb}(\kappa)$ is indeed a Reeb graph as defined above.

Interleavings of Reeb graphs In essentially the same way that we defined the functor $\mathrm{emb}\colon \mathbf{Vec}^{\mathbb{U}} \to \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}}$ in Section 2.4, we can define a functor

$$\mathrm{emb}\colon \mathbf{Set}^{\mathbb{U}} \to \mathbf{Set}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}}.$$

Namely, for $M\colon \mathbb{U} \to \mathbf{Set}$, we take $\mathrm{emb}(M)|_{\mathbb{U}} = M$, and we take $\mathrm{emb}(M)_{(a,b)} = \varnothing$ whenever $b < a$. We define an ϵ–interleaving of Reeb graphs γ and κ to be an ϵ–interleaving between $\mathrm{emb} \circ \pi_0 \circ \mathcal{S}(\gamma)$ and $\mathrm{emb} \circ \pi_0 \circ \mathcal{S}(\kappa)$, where $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ denotes the path-components functor.

Remark 4.12 The definition of interleaving of Reeb graphs introduced in [42] is slightly different from ours, in that the definition of [42] is given in terms of the inverse images under γ of bounded open intervals, rather than bounded closed intervals. It is easy to see, however, that the interleaving distances associated with the two definitions are equal.

Interlevel persistence of Reeb graphs As noted in Section 1.3, the following stability result for the persistent homology of Reeb graphs strengthens the result of Bauer, Munch and Wang [4].

Theorem 4.13 *For ϵ–interleaved Reeb graphs γ and κ of Morse type,*

$$d_b(\mathcal{L}_0(\gamma), \mathcal{L}_0(\kappa)) \leq 5\, d_I(\gamma, \kappa).$$

Proof Note that we have isomorphisms

$$H_0 \circ \mathrm{emb} \circ \pi_0 \circ S(\gamma) \cong \mathrm{emb} \circ H_0 \circ \pi_0 \circ S(\gamma) \cong \mathrm{emb} \circ H_0 \circ S(\gamma),$$

and similarly for κ. Thus, by functoriality of H_0, an ϵ–interleaving between γ and κ induces an ϵ–interleaving between $H_0\mathcal{S}(\gamma)$ and $H_0\mathcal{S}(\kappa)$. Applying Theorem 3.3 and Proposition 3.2 to this interleaving gives the desired result. □

5 Decomposition of monomorphisms with small cokernel

We now begin developing the technical machinery needed to prove the block stability theorem and our induced matching theorem for free 2-D persistence modules.

A morphism $f\colon M \to N$ of persistence modules is a *monomorphism* if each map of vector spaces $f_a\colon M_a \to N_a$ is an injection and an *epimorphism* if each f_a is a surjection. This section concerns the decomposition of a monomorphism of 2-D persistence modules with ϵ–trivial cokernel into a pair of simpler monomorphisms whose cokernels are each short-lived in one of the two coordinate directions.

To give the reader a sense of the role that these decompositions play in our arguments, let us recall that in the induced matching approach to proving algebraic stability in the 1-D case, one associates a matching $\chi(f)\colon \mathcal{B}(M) \to \mathcal{B}(N)$ to a morphism $f\colon M \to N$ of pfd 1-D persistence modules. To do so, one considers the epi-mono decomposition of f

$$M \twoheadrightarrow \mathrm{im}\, f \hookrightarrow N; \tag{2}$$

$\chi(f)$ is defined as the composition of canonical matchings

$$\mathcal{B}(M) \nrightarrow \mathcal{B}(\mathrm{im}\, f) \nrightarrow \mathcal{B}(N).$$

In the present paper, we use the decompositions introduced in this section in an analogous way, to define matchings between the barcodes of free or block-decomposable modules.

5.1 Definition and first properties of our decomposition

Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ denote the standard basis vectors in $\mathbb{R}^2$.

For $f\colon M \to N$ a morphism of $\mathbb{R}^2$–indexed modules and $\epsilon \in [0,\infty]$, we define a factorization

(3) $$\operatorname{im} f \xhookrightarrow{f_1^\epsilon} L^\epsilon(f) \xhookrightarrow{f_2^\epsilon} N$$

of the inclusion $\operatorname{im} f \hookrightarrow N$ as follows: For $\epsilon \in [0,\infty)$ and $a \in \mathbb{R}^n$, let

$$L^\epsilon(f)_a := \{n \in N_a \mid \varphi_N(a, a+\epsilon\boldsymbol{e}_1)(n) \in \operatorname{im} f\}.$$

Noting that $L^\epsilon(f) \subset L^{\epsilon'}(f)$ whenever $\epsilon < \epsilon' < \infty$, we let

$$L^\infty(f) := \bigcup_{\epsilon \geq 0} L^\epsilon(f) = \{n \in N_a \mid \varphi_N(a, a+\epsilon\boldsymbol{e}_1)(n) \in \operatorname{im} f \text{ for some } \epsilon \in [0,\infty)\}.$$

We call the modules $L^\epsilon(f)$ *interpolants*. The following lemma is immediate:

Lemma 5.1 *For all* $\epsilon \in [0,\infty)$*:*

(i) f_1^ϵ *has* $\epsilon\boldsymbol{e}_1$*–trivial cokernel.*

(ii) *If* f *has* ϵ*–trivial cokernel, then* f_2^ϵ *has* $\epsilon\boldsymbol{e}_2$*–trivial cokernel.*

Remark 5.2 The factorizations (3) dualize in the expected way, yielding factorizations of the epimorphism $M \twoheadrightarrow \operatorname{im} f$ associated to f and a dual version of Lemma 5.1. However, in this paper, rather than work explicitly with such decompositions of epimorphisms, we will simply dualize the epimorphisms we encounter to obtain monomorphisms, and work with the decompositions (3).

Interpolants between free and R_ϵ–free modules The remainder of this section is devoted to the proof of two results describing the structure of the interpolants $L^\epsilon(f)$ in special cases. The first of these, Proposition 5.19, tells us that when f is a monomorphism of pfd free $\mathbb{R}^2$–indexed modules, then $L^\infty(f)$ is also free. This result is a main step in our proof of the induced matching theorem for free modules (Theorem 6.7). The second result, Proposition 5.22, is a more technical variant of Proposition 5.19 concerning monomorphisms of R_ϵ*–free* modules. An R_ϵ–free module is one obtained from a pfd free $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed module by setting to 0 all vector spaces below the diagonal line $y = x + 2\epsilon$; see Definition 5.20. Proposition 5.22 plays a role in part of our proof of the block stability theorem analogous that of Proposition 5.19 in the proof of Theorem 6.7.

Our strategy for proving Propositions 5.19 and 5.22 centers around the computation of *(multigraded) Betti numbers*, standard invariants of $\mathbb{Z}^2$–indexed modules in commutative algebra. The starting point for our approach is the simple observation that the first Betti number of a finitely generated $\mathbb{Z}^2$–indexed module M is 0 if and only if M is free.

Because we work with $\mathbb{R}^2$–indexed modules and do not assume our modules to be finitely generated, our arguments in this section are necessarily somewhat technical. The reader may find it helpful to consider how these arguments simplify in the finitely generated, $\mathbb{Z}^2$–indexed setting.

5.2 Free 2-D persistence modules and Betti numbers

To prepare for the main results of this section, we review some standard definitions and facts about 2-D persistence modules. Though we restrict attention to the 2-D setting, everything we say here in Section 5.2 extends immediately to n-D persistence modules.

Free modules For $a \in \mathbb{R}^2$, define the interval

$$a^{\llcorner} := \{b \in \mathbb{R}^2 \mid a \le b\}.$$

We say an $\mathbb{R}^2$–indexed module F is *free* if there is a multiset $\xi(F)$ in $\mathbb{R}^2$ such that

$$F \cong \bigoplus_{a \in \xi(F)} I^{a^{\llcorner}}.$$

Note that since the barcode $\mathcal{B}(F)$ is uniquely defined, the multiset $\xi(F)$ is unique.

We say an $\mathbb{R}^2$–indexed module is *finitely generated* if it is isomorphic to a quotient of a free $\mathbb{R}^2$–indexed module F with $B(F)$ finite.

Free $\mathbb{Z}^2$–indexed modules are defined in the analogous way; for F a free $\mathbb{Z}^2$–indexed module, the invariant $\xi(F)$ is defined as a multiset in $\mathbb{Z}^2$.

Remark 5.3 Later we shall consider free $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed and $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed modules. These are the interval-indecomposable modules with barcodes consisting, respectively, of intervals of the form

$$(a_1,a_2)^{\lrcorner} := \{(b_1,b_2) \in \mathbb{R}^2 \mid a_1 \ge b_1 \text{ and } a_2 \le b_2\},$$
$$(a_1,a_2)^{\ulcorner} := \{(b_1,b_2) \in \mathbb{R}^2 \mid a_1 \le b_1 \text{ and } a_2 \ge b_2\}.$$

A *basis* for a free $\mathbb{R}^2$–indexed module F is a set $\mathcal{W} \subset \bigcup_{a \in \mathbb{R}^2} F_a$ such that any element $m \in F_d$ can be uniquely expressed as a finite sum

$$m = c_1 \varphi_F(d_1, d)(w_1) + \cdots + c_l \varphi_F(d_l, d)(w_l) \tag{4}$$

for $w_i \in \mathcal{W} \cap F_{d_i}$ and scalars $c_i \in k$. For $w \in \mathcal{W} \cap F_a$, we write $\deg(w) = a$. Clearly, a basis exists for any free $\mathbb{R}^n$–indexed module.

We leave the proof of the following as an easy exercise.

Lemma 5.4 *If $f\colon M \to N$ is a monomorphism of free $\mathbb{R}^2$–indexed persistence modules with ϵ–trivial cokernel, then $L^\infty(f) = L^\epsilon(f)$.*

Lemma 5.5 *If $f\colon M \to N$ is a morphism of finitely generated free $\mathbb{R}^2$–indexed modules, then $L^\infty(f) = L^\epsilon(f)$ for some finite ϵ.*

Proof Since N is finitely generated, there exists $b \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, $N_{(a,t)} = 0$ whenever $a \le b$. In addition, since M is finitely generated and N is free, there exists $c > b$ such that $\varphi^{\operatorname{im} f}((d,t),(d',t))$ is an isomorphism for any $t \in \mathbb{R}$ whenever $c \le d \le d'$. It is easy to check that $L^\infty(f) = L^{c-b}(f)$. □

Example 5.6 We give an example of a monomorphism $f\colon M \hookrightarrow N$ of free modules with 2–trivial cokernel and its interpolant $L^\infty(f)$. The example is illustrated in Figure 6. Let

$$M = I^{(3,1)^\llcorner} \oplus I^{(2,2)^\llcorner} \quad \text{and} \quad N = I^{(2,0)^\llcorner} \oplus I^{(0,1)^\llcorner}.$$

For $a \le b \in \mathbb{R}^2$, let $j_a^b\colon I^{b^\llcorner} \hookrightarrow I^{a^\llcorner}$ denote the inclusion and let $f\colon M \hookrightarrow N$ be the monomorphism given in matrix form by

$$\begin{pmatrix} j_{(2,0)}^{(3,1)} & j_{(2,0)}^{(2,2)} \\ j_{(0,1)}^{(3,1)} & 0 \end{pmatrix}.$$

It's easy to see that $L^\infty(f) \cong I^{(2,1)^\llcorner} \oplus I^{(0,2)^\llcorner}$, so $L^\infty(f)$ is indeed free, as guaranteed by Proposition 5.19 below. Note that, as guaranteed by Lemma 5.4, $L^\infty(f) = L^2(f)$. Note also that lower edges of the intervals in $\mathcal{B}(M)$ and $\mathcal{B}(L^\infty(f))$ lie on the same horizontal lines, while the left edges the intervals in $\mathcal{B}(L^\infty(f))$ and $\mathcal{B}(N)$ lie on the same vertical lines. As shown in Section 6, this is true whenever f is a monomorphism of free $\mathbb{R}^2$–indexed modules with ϵ–trivial cokernel for some finite ϵ.

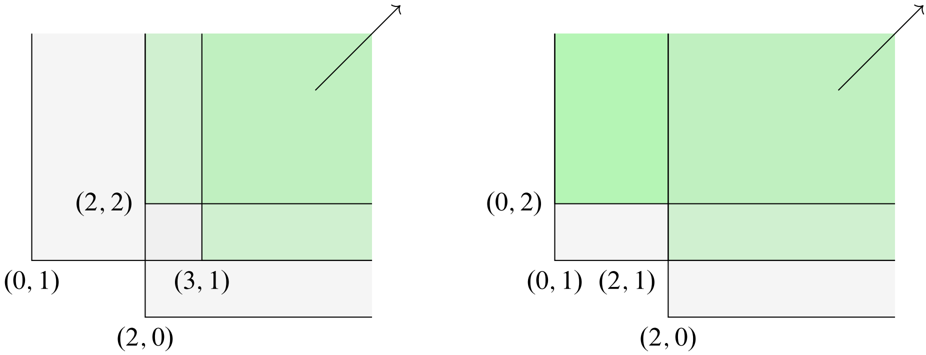


Figure 6: The barcodes of the free modules M, N and $L^\infty(f)$ of Example 5.6. Left: the modules M (green) and N (gray outside of the support of M). Right: the modules $L^\infty(f)$ (green) and N (gray outside of the support of $L^\infty(f)$).

Bigraded modules We define a *bigraded* module to be a $k[x_1,x_2]$–module M equipped with a direct sum decomposition as a k–vector space $M \cong \bigoplus_{a\in\mathbb{Z}^2} M_a$ such that the action of $k[x_1,x_2]$ on M satisfies $x_i(M_a) \subset M_{a+\boldsymbol{e}_i}$ for all $a \in \mathbb{Z}^2$ and $i \in \{1,2\}$. The bigraded modules form a category, where the morphisms $f\colon M \to N$ are module homomorphisms such that $f(M_a) \subset N_a$ for all $a \in \mathbb{Z}^2$. There is an obvious isomorphism between $\mathbf{Vec}^{\mathbb{Z}^2}$ and the category of bigraded modules. Thus, we may regard $\mathbb{Z}^2$–indexed modules as modules, in the usual sense.

Minimal resolutions We next give a brief introduction to minimal free resolutions of finitely generated $\mathbb{Z}^2$–indexed persistence modules. For more details, consult [29; 36].

A *free resolution* of a $\mathbb{Z}^2$–indexed module M is an exact sequence

$$\boldsymbol{F} = \cdots \xrightarrow{d_3} F^2 \xrightarrow{d_2} F^1 \xrightarrow{d_1} F^0$$

of free $\mathbb{Z}^2$–indexed modules with $M \cong \operatorname{coker} d_1$. We say $\boldsymbol{F}$ is *minimal* if $\operatorname{im} d_i \subset IF_{i-1}$ for every i, where $I = \langle x_1, x_2\rangle$ is the maximal graded ideal of $k[x_1,x_2]$.

Theorem 5.7 [29, Theorems 19.4 and 20.2] *For any finitely generated $\mathbb{Z}^2$–indexed module M:*

(i) *There exists a minimal free resolution $\boldsymbol{F}$ of M with each F^i finitely generated.*

(ii) *If $\boldsymbol{F}$ and $\boldsymbol{G}$ are minimal free resolutions of M, then there is an isomorphism $\boldsymbol{F} \to \boldsymbol{G}$ inducing the identity map on M.*

For the remainder of Section 5.2, let M be a finitely generated $\mathbb{Z}^2$–indexed module.

Betti numbers For $i \geq 0$ and $a \in \mathbb{Z}^2$, we define a nonnegative integer $\xi_i(M)_a$, the i^{th} *Betti number of M at degree a*, by choosing a minimal free resolution $\boldsymbol{F}$ for M and letting $\xi_i(M)_a$ be the number of copies of a in $\xi(F^i)$. It follows from Theorem 5.7(ii) that this definition of $\xi_i(M)_a$ is independent of the choice of $\boldsymbol{F}$, and is thus well formed.

Observe that $\xi_1(M)_a = 0$ for all $a \in \mathbb{Z}^2$ if and only if M is free.

A Koszul homology formula For $z \in \mathbb{Z}^2$, we define the $\mathbb{Z}^2$–indexed module $M(z)$ to be the shift of M by z, exactly as we did for $\mathbb{R}^2$–indexed modules in Section 2.2. For any $a = (a_1, a_2) \in \mathbb{N}^2$, we have a short chain complex

$$M(-a_1\boldsymbol{e}_1 - a_2\boldsymbol{e}_2) \xrightarrow{\kappa^a} M(-a_1\boldsymbol{e}_1) \oplus M(-a_2\boldsymbol{e}_2) \xrightarrow{\gamma^a} M, \tag{5}$$

where

$$\kappa^a|_{M(-a_1\boldsymbol{e}_1-a_2\boldsymbol{e}_2)}(m) = (-x_2^{a_2}m, x_1^{a_1}m), \quad \gamma^a|_{M(-a_i\boldsymbol{e}_i)}(q) = x_i^{a_i}q.$$

We will sometimes write κ^a and γ^a as κ^a_M and γ^a_M, respectively. In addition, we abbreviate $\kappa^{(1,1)}$ and $\gamma^{(1,1)}$ by κ and γ.

The following commutative algebra result tells us that the first Betti number can be computed locally in terms of γ and κ:

Theorem 5.8 [30, Proposition 2.7] *For any $z \in \mathbb{Z}^2$,*

$$\xi_1(M)_z = \dim \ker \gamma_z / \operatorname{im} \kappa_z.$$

Eisenbud [30] establishes Theorem 5.8 in the slightly different setting of $\mathbb{Z}$–graded $k[x_1, x_2]$–modules, ie where $k[x_1, x_2]$ is given the standard grading

$$\deg(x_1^{r_1} x_2^{r_2}) = r_1 + r_2.$$

However, the proof in our case is essentially the same.

Remark 5.9 One can extend the short chain complex (5) to a chain complex whose i^{th} homology gives the i^{th} Betti number of M for all $i \geq 0$. Namely,

$$\xi_i(M)_a = \dim H_i(M \otimes K_\bullet)_a,$$

where $K_\bullet$, the *Koszul complex*, is a minimal free resolution of k as a $k[x_1, x_2]$–module. For more on this see [30].

We conclude this subsection with a technical result which will be useful to us later, leaving the easy proof to the reader:

Lemma 5.10 *If M is free, then for any $a \in \mathbb{N}^2$,*

$$\ker \gamma_M^a = \operatorname{im} \kappa_M^a.$$

5.3 Continuous extensions of discrete persistence modules

We wish to use Theorem 5.8 to study the first Betti numbers of the interpolants $L^\epsilon(f)$ in the decomposition (3). However, Theorem 5.8 applies to finitely generated $\mathbb{Z}^2$–indexed modules, whereas the modules $L^\epsilon(f)$ are $\mathbb{R}^2$–indexed and, in the settings of interest to us, need not be finitely generated. To bridge the gap between the $\mathbb{Z}^2$– and $\mathbb{R}^2$–indexed settings, we use left Kan extensions.

Grid functions We define an *(injective) 2-D grid* to be a function $\mathcal{G}\colon \mathbb{Z}^2 \to \mathbb{R}^2$ given by

$$\mathcal{G}(z_1, z_2) = (\mathcal{G}_1(z_1), \mathcal{G}_2(z_2))$$

for strictly increasing functions $\mathcal{G}_i\colon \mathbb{Z} \to \mathbb{R}$ with $\lim_{i\to-\infty} = -\infty$ and $\lim_{i\to\infty} = \infty$. Define $\operatorname{fl}_\mathcal{G}\colon \mathbb{R}^2 \to \operatorname{im}(\mathcal{G})$ by

$$\operatorname{fl}_\mathcal{G}(t) = \max\{s \in \operatorname{im}(\mathcal{G}) \mid s \leq t\}.$$

Continuous extensions For $\mathcal{G}$ a 2-D grid, we let $E_\mathcal{G}$ denote the functor

$$\operatorname{Lan}_\mathcal{G}(-)\colon \mathbf{Vec}^{\mathbb{Z}^2} \to \mathbf{Vec}^{\mathbb{R}^2};$$

equivalently, but more concretely, we may specify $E_\mathcal{G}$ as follows:

(1) For M a $\mathbb{Z}^2$–indexed persistence module and $a, b \in \mathbb{R}^2$,

$$E_\mathcal{G}(M)_a = M_y, \quad \varphi_{E_\mathcal{G}(M)}(a,b) = \varphi_M(y,z),$$

where $y, z \in \mathbb{Z}^2$ are given by $\mathcal{G}(y) = \operatorname{fl}_\mathcal{G}(a)$ and $\mathcal{G}(z) = \operatorname{fl}_\mathcal{G}(b)$.

(2) The action of $E_\mathcal{G}$ on morphisms is the obvious one.

Let

$$(-)|_\mathcal{G}\colon \mathbf{Vec}^{\mathbb{R}^2} \to \mathbf{Vec}^{\mathbb{Z}^2}$$

denote the restriction along $\mathcal{G}$.

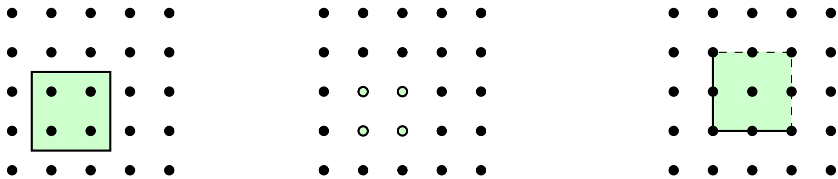


Figure 7: The restriction and continuous extension of Example 5.11

Example 5.11 Let $\mathcal{G}(z) = z$ for all $z \in \mathbb{Z}^2$ and define rectangles J and K by

$$J = \{(a,b) \mid 0.5 \le a \le 2.5,\ 0.5 \le b \le 2.5\}, \quad K = \{(a,b) \mid 1 \le a < 3,\ 1 \le b < 3\}.$$

Then $E_{\mathcal{G}}(\mathcal{I}^J|_{\mathcal{G}}) = \mathcal{I}^K$; see Figure 7.

Interpolants of a morphism between free modules as continuous extensions

Lemma 5.12 *If F is a free $\mathbb{R}^2$–indexed module and $\mathcal{G}\colon \mathbb{Z}^2 \to \mathbb{R}^2$ is a 2-D grid such that $d \in \operatorname{im}\mathcal{G}$ whenever $d \in \xi(F)$, then $\varphi_F(\mathrm{fl}_{\mathcal{G}}(a), a)$ is an isomorphism for all $a \in \mathbb{R}^2$.*

Proof Let $b = \mathrm{fl}_{\mathcal{G}}(a)$. The map $\varphi_F(b,a)$ is an injection since F is free, so it suffices to show that $\varphi_F(b,a)$ is a surjection. Assume that $n \in F_a$ and $n \notin \operatorname{im}\varphi_F(b,a)$. Then there must exist $d \in \xi(F)$ such that $d \le a$ and $d_l > b_l$ for at least one $l \in \{1,2\}$. Assuming $l = 1$, then the point

$$b' = (d_1, b_2)$$

is in $\operatorname{im}\mathcal{G}$ and $b < b' \le a$, contradicting the maximality of b, and similarly if $l = 2$. □

Proposition 5.13 *For $f\colon M \to N$ a morphism of finitely generated free $\mathbb{R}^2$–indexed modules and $\epsilon \in [0,\infty)$, let*

$$W_1 := \{a_1 \mid a \in \xi(M) \cup \xi(N)\} \cup \{a_1 - \epsilon \mid a \in \xi(M)\},$$
$$W_2 := \{a_2 \mid a \in \xi(M) \cup \xi(N)\}.$$

If $\mathcal{G}\colon \mathbb{Z}^2 \to \mathbb{R}^2$ is a 2-D grid whose image contains $W_1 \times W_2$, then

$$L^\epsilon(f) \cong E_{\mathcal{G}}(L^\epsilon(f)|_{\mathcal{G}}).$$

Proof It suffices to show that for all $a \in \mathbb{R}^2$,

$$\varphi_{L^\epsilon(f)}(\mathrm{fl}_{\mathcal{G}}(a), a)\colon L^\epsilon(f)_{\mathrm{fl}_{\mathcal{G}}(a)} \to L^\epsilon(f)_a$$

is an isomorphism.

Let $b = \mathrm{fl}_{\mathcal{G}}(a)$. By Lemma 5.12, $\varphi_N(b,a)$ is an isomorphism. Moreover, an argument similar to the proof of Lemma 5.12 shows that $\varphi_M(b+\epsilon \boldsymbol{e}_1, a+\epsilon \boldsymbol{e}_1)$ is an isomorphism: Let $m \in M_{a+\epsilon \boldsymbol{e}_1}$ and assume that $m \notin \operatorname{im}\varphi_M(b+\epsilon \boldsymbol{e}_1, a+\epsilon \boldsymbol{e}_1)$. Then, as above, there must exist $d \in \xi(M)$ such that $d \leq a+\epsilon \boldsymbol{e}_1$ and $d_l > (b+\epsilon \boldsymbol{e}_1)_l$ for at least one $l \in \{1,2\}$, contradicting the maximality of b.

Since $L^\epsilon(f)$ is a submodule of N, the map $\varphi_{L^\epsilon(f)}(b,a)$ is injective. Let $n \in L^\epsilon(f)_a$ with $\varphi_N(a, a+\epsilon \boldsymbol{e}_1)(n) = f(m)$. Since $\varphi_N(b,a)$ and $\varphi_M(b+\epsilon \boldsymbol{e}_1, a+\epsilon \boldsymbol{e}_1)$ are isomorphisms, there exist $n' \in N_b$ and $m' \in M_{b+\epsilon \boldsymbol{e}_1}$ with $n = \varphi_N(b,a)(n')$ and $m = \varphi_M(b+\epsilon \boldsymbol{e}_1, a+\epsilon \boldsymbol{e}_1)(m')$. The commutativity of f and injectivity of $\varphi_N(b+\epsilon \boldsymbol{e}_1, a+\epsilon \boldsymbol{e}_1)$ imply that $\varphi_N(b, b+\epsilon \boldsymbol{e}_1)(n') = f(m')$, and thus $n' \in L^\epsilon(f)_b$. This shows that $\varphi_{L^\epsilon(f)}(b,a)$ is surjective, and hence an isomorphism. □

5.4 Trivial first Betti numbers and freeness of interpolants

Lemma 5.14 *For $f\colon M \to N$ and $\mathcal{G}$ as in Proposition 5.13:*

(i) *$L^\epsilon(f)|_{\mathcal{G}}$ is finitely generated.*

(ii) *$\xi_1(L^\epsilon(f)|_{\mathcal{G}})_z = 0$ whenever $\mathcal{G}(z) \leq a - \epsilon \boldsymbol{e}_1$ for some $a \in \mathbb{R}^2$ with f_a an injection.*

Proof (i) This holds because $L^\epsilon(f)|_{\mathcal{G}}$ is a submodule of the finitely generated persistence module $N|_{\mathcal{G}}$; the standard result that a submodule of a finitely generated module over a Noetherian ring is itself finitely generated [29] also holds in the bigraded case.

To prove (ii), let us simplify notation by writing

$$\mathcal{L} = L^\epsilon(f)|_{\mathcal{G}}, \quad \mathcal{N} = N|_{\mathcal{G}}, \quad \mathcal{M} = M|_{\mathcal{G}}, \quad \mathfrak{f} = f|_{\mathcal{G}}.$$

Assume without loss of generality that $z = 0$. We will prove that $\xi_1(\mathcal{L})_0 = 0$ by showing that the quotient $\ker\gamma_{\mathcal{L}}/\operatorname{im}\kappa_{\mathcal{L}}$ of Theorem 5.8 vanishes at 0.

For $y \in \mathbb{Z}^2$, let y^+ denote the maximum element of $\mathbb{Z}^2$ with $\mathcal{G}(y^+) \leq \mathcal{G}(y) + \epsilon \boldsymbol{e}_1$. Note that by Lemma 5.12, for $y \in \mathbb{Z}^2$ and $v \in \mathcal{N}_y$, we have $v \in \mathcal{L}_y$ if and only if $\varphi_{\mathcal{N}}(y, y^+)(v) \in \operatorname{im} \mathfrak{f}_{y^+}$.

Note that, in view of the way we define grid functions, the y–coordinates of 0^+ and $(-\boldsymbol{e}_1)^+$ are equal, as are the y–coordinates of $(-\boldsymbol{e}_2)^+$ and $(-\boldsymbol{e}_1 - \boldsymbol{e}_2)^+$. Symmetrically, the x–coordinates of 0^+ and $(-\boldsymbol{e}_2)^+$ are equal, as are the x–coordinates of $(-\boldsymbol{e}_1)^+$ and $(-\boldsymbol{e}_1 - \boldsymbol{e}_2)^+$.

Let $b = (0_1^+ - (-\boldsymbol{e}_1)_1^+, 0_2^+ - (-\boldsymbol{e}_2)_2^+)$, and let

$$\gamma_{\mathcal{N}}^+ = (\gamma_{\mathcal{N}}^b)_{0^+}\colon \bigoplus_{j\in\{1,2\}} \mathcal{N}_{(-\boldsymbol{e}_j)^+} \to \mathcal{N}_{0^+},$$
$$\kappa_{\mathcal{N}}^+ = (\kappa_{\mathcal{N}}^b)_{0^+}\colon \mathcal{N}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+} \to \bigoplus_{j\in\{1,2\}} \mathcal{N}_{(-\boldsymbol{e}_j)^+}.$$

Define $\gamma_{\mathcal{M}}^+$ and $\kappa_{\mathcal{M}}^+$ analogously.

In addition, let

$$\varphi_\bullet = \bigoplus_{j\in\{1,2\}} \varphi_{\mathcal{N}}(-\boldsymbol{e}_j, (-\boldsymbol{e}_j)^+)\colon \bigoplus_{j\in\{1,2\}} \mathcal{N}_{-\boldsymbol{e}_j} \to \bigoplus_{j\in\{1,2\}} \mathcal{N}_{(-\boldsymbol{e}_j)^+},$$
$$\varphi_{\bullet\bullet} = \varphi_{\mathcal{N}}(-\boldsymbol{e}_1-\boldsymbol{e}_2, (-\boldsymbol{e}_1-\boldsymbol{e}_2)^+)\colon \mathcal{N}_{-\boldsymbol{e}_1-\boldsymbol{e}_2} \to \mathcal{N}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+},$$
$$\mathfrak{f}_\bullet = \bigoplus_{j\in\{1,2\}} \mathfrak{f}_{(-\boldsymbol{e}_j)^+}\colon \bigoplus_{j\in\{1,2\}} \mathcal{M}_{(-\boldsymbol{e}_j)^+} \to \bigoplus_{j\in\{1,2\}} \mathcal{N}_{(-\boldsymbol{e}_j)^+},$$
$$\mathfrak{f}_{\bullet\bullet} = \mathfrak{f}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+}\colon \mathcal{M}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+} \to \mathcal{N}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+}.$$

Consider the commutative diagram of vector spaces

$$\begin{array}{ccccc}
\mathcal{L}_{-\boldsymbol{e}_1-\boldsymbol{e}_2} & \xrightarrow{(\kappa_{\mathcal{L}})_0} & \bigoplus_{j\in\{1,2\}} \mathcal{L}_{-\boldsymbol{e}_j} & \xrightarrow{(\gamma_{\mathcal{L}})_0} & \mathcal{L}_0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{N}_{-\boldsymbol{e}_1-\boldsymbol{e}_2} & \xrightarrow{(\kappa_{\mathcal{N}})_0} & \bigoplus_{j\in\{1,2\}} \mathcal{N}_{-\boldsymbol{e}_j} & \xrightarrow{(\gamma_{\mathcal{N}})_0} & \mathcal{N}_0 \\
\big\downarrow{\scriptstyle \varphi_{\bullet\bullet}} & & \big\downarrow{\scriptstyle \varphi_\bullet} & & \big\downarrow{\scriptstyle \varphi_{\mathcal{N}}(0,0^+)} \\
\mathcal{N}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+} & \xrightarrow{\kappa_{\mathcal{N}}^+} & \bigoplus_{j\in\{1,2\}} \mathcal{N}_{(-\boldsymbol{e}_j)^+} & \xrightarrow{\gamma_{\mathcal{N}}^+} & \mathcal{N}_{0^+} \\
\big\uparrow{\scriptstyle \mathfrak{f}_{\bullet\bullet}} & & \big\uparrow{\scriptstyle \mathfrak{f}_\bullet} & & \big\uparrow{\scriptstyle \mathfrak{f}_{0^+}} \\
\mathcal{M}_{(-\boldsymbol{e}_1-\boldsymbol{e}_2)^+} & \xrightarrow{\kappa_{\mathcal{M}}^+} & \bigoplus_{j\in\{1,2\}} \mathcal{M}_{(-\boldsymbol{e}_j)^+} & \xrightarrow{\gamma_{\mathcal{M}}^+} & \mathcal{M}_{0^+}
\end{array}$$

We have $\mathcal{G}(0^+) \le \mathcal{G}(0) + \epsilon \boldsymbol{e}_1 \le a$, where the second inequality holds by assumption, so since f_a is an injection and M is free, $\mathfrak{f}_{0^+}$ is an injection as well.

Let $l \in \ker(\gamma_{\mathcal{L}})_0$ and observe that $l \in \ker \gamma_{\mathcal{N}}$ by commutativity of the top-right square. Thus, since the second row of the diagram is exact by Theorem 5.8, there exists

$$l' \in \mathcal{N}_{-\boldsymbol{e}_1-\boldsymbol{e}_2}$$

such that $\kappa_{\mathcal{N}}(l') = l$. To establish the result, it suffices to show that

$$l' \in \mathcal{L}_{-\boldsymbol{e}_1 - \boldsymbol{e}_2}$$

or, equivalently, that $\varphi_{\bullet\bullet}(l') \in \operatorname{im} \mathfrak{f}_{\bullet\bullet}$.

There exists $m' \in \bigoplus_{j\in\{1,2\}} \mathcal{M}_{(-\boldsymbol{e}_j)+}$ such that $\mathfrak{f}_{\bullet}(m') = \varphi_{\bullet}(l)$. By the injectivity of $\mathfrak{f}_{0+}$ and the commutativity of the middle-right and bottom-right squares in the diagram above, $m' \in \ker \gamma^{+}_{\mathcal{M}}$. It follows from Lemma 5.10 that the bottom row of the diagram is exact, so there exists m such that $\kappa^{+}_{\mathcal{M}}(m) = m'$. Moreover, commutativity of the bottom-left square yields

$$\kappa^{+}_{\mathcal{N}} \circ \mathfrak{f}_{\bullet\bullet}(m) = \mathfrak{f}_{\bullet} \circ \kappa^{+}_{\mathcal{M}}(m) = \varphi_{\bullet}(l).$$

On the other hand, from the definition of l', we have that

$$\varphi_{\bullet}(l) = \varphi_{\bullet} \circ \kappa_{\mathcal{N}}(l') = \kappa^{+}_{\mathcal{N}} \circ \varphi_{\bullet\bullet}(l').$$

The injectivity of $\kappa^{+}_{\mathcal{N}}$ implies that $\mathfrak{f}_{\bullet\bullet}(m) = \varphi_{\bullet\bullet}(l')$, and (ii) follows. □

For M a $\mathbb{Z}^2$–indexed or $\mathbb{R}^2$–indexed persistence module, we define a *presentation* of M to be a morphism $\Phi\colon F^1 \to F^0$ of free persistence modules with $M \cong \operatorname{coker} \Phi$. When M is $\mathbb{Z}^2$–indexed, we'll say Φ is *minimal* if $\operatorname{im} \Phi \subset IF_0$.

From Lemma 5.14, we obtain the following:

Lemma 5.15

(i) *For $f\colon M \to N$ a morphism of finitely generated free $\mathbb{R}^2$–indexed modules and $\epsilon \in [0,\infty)$, there exists a presentation $\Phi\colon F^1 \to F^0$ of $L^{\epsilon}(f)$ with F^0 and F^1 finitely generated such that $F^1_v \cong 0$ whenever $v \le a - \epsilon \boldsymbol{e}_1$ for some $a \in \mathbb{R}^2$ with f_a an injection.*

(ii) *In particular, if f is a monomorphism, then $L^{\epsilon}(f)$ is free.*

Proof For $\mathcal{G}$ a 2-D grid as above, Lemma 5.14(i) tells us that $L^{\epsilon}(f)|_{\mathcal{G}}$ is finitely generated. Thus, by Theorem 5.7(i) there exists a minimal presentation

$$\Phi'\colon G^1 \to G^0$$

for $L^{\epsilon}(f)|_{\mathcal{G}}$. The functor $E_{\mathcal{G}}$ is easily seen to be exact, so by Proposition 5.13, applying this functor to Φ' yields a presentation

$$E_{\mathcal{G}}(\Phi')\colon E(G^1) \to E(G^0)$$

for $L^\epsilon(f)$. We take $\Phi = E_{\mathcal{G}}(\Phi')$ and $F^i = E_{\mathcal{G}}(G^i)$ for $i = 0, 1$. Since G_0 and G_1 are finitely generated, the same is true for F_0 and F_1.

If f_a is an injection, then in view of Lemma 5.14(ii), $G^1_z \cong 0$ for all $z \in \mathbb{Z}^2$ with $\mathcal{G}(z) \le a - \epsilon \boldsymbol{e}_1$. If $v \le a - \epsilon \boldsymbol{e}_1$, then clearly $\mathrm{fl}_{\mathcal{G}}(v) \le a - \epsilon \boldsymbol{e}_1$, and we thus have

$$F^1_v \cong F^1_{\mathrm{fl}_{\mathcal{G}}(v)} \cong G^1_{\mathcal{G}^{-1}(\mathrm{fl}_{\mathcal{G}}(v))} \cong 0.$$

This gives (i). (ii) follows immediately from (i). □

Persistence modules free below a For $a \in \mathbb{R}^2$, let $\mathbb{R}^2_{\le a}$ denote the subposet of $\mathbb{R}^2$ with objects $\{v \in \mathbb{R}^2 \mid v \le a\}$. We say that an $\mathbb{R}^2$–indexed module M is *free below a* if there exists a free $\mathbb{R}^2$–indexed module F such that the restrictions of M and F to $\mathbb{R}^2_{\le a}$ are isomorphic.

Let M^a denote the $\mathbb{R}^2$–indexed module for which $M^a_v = M_{\min(a,v)}$, where

$$\min(a, v) = (\min(a_1, v_1), \min(a_2, v_2)),$$

with the internal morphisms in M^a induced by those of M. A morphism $f\colon M \to N$ induces a morphism $f^a\colon M^a \to N^a$ in an obvious way.

We omit the following lemma's easy proof:

Lemma 5.16 *If M is free below a, then M^a is free.*

Lemma 5.17 *For f a morphism of finitely generated free $\mathbb{R}^2$–indexed modules, $\epsilon \in [0, \infty)$, and $a \in \mathbb{R}^2$ with f_a an injection, we have that $L^\epsilon(f)^{a-\epsilon \boldsymbol{e}_1}$ is free.*

Proof For $\Phi\colon F^1 \to F^0$ a presentation for $L^\epsilon(f)$ as in Lemma 5.15(i), the restrictions of $L^\epsilon(f)$ and F^0 to $\mathbb{R}^2_{\le a-\epsilon \boldsymbol{e}_1}$ are isomorphic. Thus, $L^\epsilon(f)$ is free below $a - \epsilon \boldsymbol{e}_1$. The result now follows from Lemma 5.16. □

Example 5.18 We give an example showing that Lemma 5.17 is sharp, in the sense that under the hypotheses of the lemma, $L^\epsilon(f)^b$ is not necessarily free for $b > a - \epsilon \boldsymbol{e}_1$. Let $M = I^{(2,0)^\llcorner} \oplus I^{(0,2)^\llcorner}$, $N = I^{(0,0)^\llcorner}$ and let $f\colon M \to N$ be any morphism whose restriction to both the first and second summands of M is a monomorphism. Then $L^1(f) = I^{(1,0)^\llcorner \cup (0,2)^\llcorner} \subset N$. Note that $f_{(2,2)}$ is not an injection, but f_c is an injection for any $c < (2, 2)$. We have that $L^1(f)^{(1,2)}$ is not free, but $L^1(f)^c$ is free for any $c < (1, 2)$.

Here is the first main result of this section; see Example 5.6 and Figure 6 for an illustration.

Proposition 5.19 *If $f\colon M\to N$ is a monomorphism of pfd free $\mathbb{R}^2$–indexed modules, then $L^\infty(f)$ is free.*

Proof For $j\in\{0,1,2,\dots\}$, let $a_j=(j,j)$. Note that $f^{a_j}\colon M^{a_j}\to N^{a_j}$ is a monomorphism of finitely generated free persistence modules. By Lemma 5.5, $L^\infty(f^{a_j})=L^\epsilon(f^{a_j})$ for some finite ϵ. Lemmas 5.15(ii) and 5.16 then imply that $L^\infty(f^{a_j})^{a_j}=L_i^\infty(f)^{a_j}$ is free.

Letting $L^j:=L^\infty(f)^{a_j}$, note that there is a canonical monomorphism $L^j\hookrightarrow L^{j+1}$, so we may identify L^j with a submodule of L^{j+1}, and $\varinjlim L^j\cong L^\infty(f)$. We inductively define a basis $\mathcal{W}_j$ for each L^j such that $\mathcal{W}_j\subset\mathcal{W}_{j+1}$: Take $\mathcal{W}_0$ to be any basis for L^0. Now assume that we have defined $\mathcal{W}_j$. If $\mathcal{W}'$ is any basis for L^{j+1} then

$$\mathcal{W}''=\{w'\in\mathcal{W}'\mid \deg(w')\le a_j\}$$

is a basis for L^j. Hence, $\mathcal{W}_{j+1}=\mathcal{W}_j\cup(\mathcal{W}'-\mathcal{W}'')$ is a basis for L^{j+1} with $\mathcal{W}_j\subset\mathcal{W}_{j+1}$. Clearly,

$$\mathcal{W}_0\cup(\mathcal{W}_1-\mathcal{W}_0)\cup(\mathcal{W}_2-\mathcal{W}_1)\cup\cdots$$

is a basis for $\varinjlim L_j$, so $\varinjlim L_j$ is free. □

5.5 Interpolants of R_ϵ–free modules

For $\epsilon\ge 0$, define an endofunctor R_ϵ on $\mathbf{Vec}^{\mathbb{R}\times\mathbb{R}^{\mathrm{op}}}$ by

$$R_\epsilon(M)_{(s,t)}=\begin{cases}M_{(s,t)} & \text{for all } t-s>2\epsilon,\\ 0 & \text{otherwise,}\end{cases}$$

with the internal maps $\varphi_{R_\epsilon(M)}(-,-)$ and the action of R_ϵ on morphisms defined in the obvious way. Note that we have a canonical epimorphism $M\twoheadrightarrow R_\epsilon(M)$.

Definition 5.20 We say that an $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed module M is *R_ϵ–free* if $M\cong R_\epsilon(F_M)$ for F_M a pfd free $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed module.

Observe that an R_ϵ–free module M is interval-decomposable, with

$$\mathcal{B}(M)=\{(a,b)^{\triangledown}_\epsilon\mid(a,b)^{\ulcorner}\in\mathcal{B}(F_M),\ (a,b)^{\triangledown}_\epsilon\ne\varnothing\},$$

where

$$(a,b)^{\triangledown}_\epsilon=\{(s,t)\in(a,b)^{\ulcorner}\mid t-s>2\epsilon\};$$

see Figure 8.

We omit the easy proof of the following:

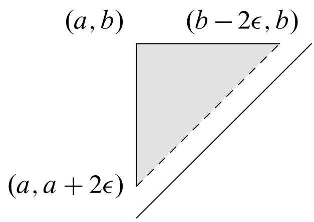


Figure 8: An interval $(a,b)_\epsilon^{\triangledown}$

Lemma 5.21 *M is R_ϵ–free if and only if $R_\epsilon(M)\cong M$ and there exists a set*

$$\mathcal{W}\subset \bigcup_{t-s>2\epsilon} M_{(s,t)}$$

such that for any $(s,t)\in\mathbb{R}^2$ with $t-s>2\epsilon$ and $m\in M_{(s,t)}$, m can be uniquely expressed as a linear combination of elements of $\mathcal{W}$, as in (4).

In analogy with the free case, we call the set $\mathcal{W}$ above an R_ϵ*–basis*. Finally, we come to the second main result of this section:

Proposition 5.22 *Let $f\colon M\to N$ be a monomorphism of R_ϵ–free persistence modules. Then $R_{3\epsilon/2}(L^\epsilon(f))$ is $R_{3\epsilon/2}$–free.*

Proof Let $\alpha_M\colon M\to R_\epsilon(F_M)$ and $\alpha_N\colon N\to R_\epsilon(F_N)$ be isomorphisms. The map $\alpha_N\circ f\circ\alpha_M^{-1}\colon R_\epsilon(F_M)\to R_\epsilon(F_N)$ lifts to a map $\tilde f\colon F_M\to F_N$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \xrightarrow[\cong]{\alpha_M} & R_\epsilon(F_M) & \twoheadleftarrow & F_M \\ \downarrow f & & \downarrow R_\epsilon(\tilde f) & & \downarrow \tilde f \\ N & \xrightarrow[\cong]{\alpha_N} & R_\epsilon(F_N) & \twoheadleftarrow & F_N \end{array}$$

Observe that

$$R_{3\epsilon/2}(L^\epsilon(f))\cong R_{3\epsilon/2}(L^\epsilon(R_\epsilon(\tilde f)))=R_{3\epsilon/2}(L^\epsilon(\tilde f)),$$

where the isomorphism on the left follows from commutativity of the left square in the diagram. Hence, it suffices to show that $R_{3\epsilon/2}(L^\epsilon(\tilde f))$ is $R_{3\epsilon/2}$–free. Our argument is similar to the proof of Proposition 5.19.

Let $a^j=(j,-j)$ for $j\in\{0,1,2,\dots\}$. We first show that $R_{3\epsilon/2}(L^\epsilon(\tilde f^{a^j}))$ is $R_{3\epsilon/2}$–free. Note that since F_M and F_N are pfd, $\tilde f^{a^j}\colon F_M^{a^j}\to F_N^{a^j}$ is a morphism of finitely generated free persistence modules. By commutativity of the above diagram, $R_\epsilon(\tilde f)$ is a monomorphism, ie $\tilde f_{(s,t)}$ is an injection for $t-s>2\epsilon$. Further, $\tilde f^{a^j}_{(s,t)}$ is also

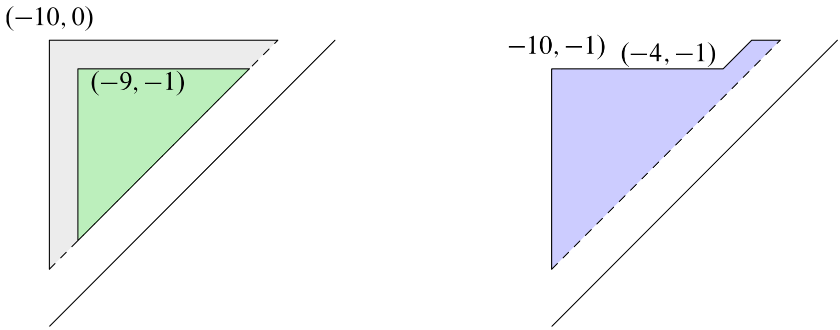


Figure 9: Illustration of Example 5.23 in the case $\epsilon = 1$. Left: the support of the R_ϵ–free persistence modules M (green) and N (gray). Right: the support of $L^\epsilon(f)$.

an injection for $t - s > 2\epsilon$. To see this, note that there exist $u \leq s$ and $v \geq t$ with $\widetilde{f}^{a_j}_{(s,t)} = \widetilde{f}_{(u,v)}$. We have $v - u \geq t - s > 2\epsilon$, so $\widetilde{f}^{a_j}_{(s,t)} = \widetilde{f}_{(u,v)}$ is injective.

By Lemma 5.15 then, there exists a presentation $\Phi\colon F^1 \to F^0$ for $L^\epsilon(\widetilde{f}^{a_j})$ with F^0 finitely generated such that $F^1_{(s,t)} = 0$ whenever $t - s > 3\epsilon$. Thus, $R_{3\epsilon/2}(L^\epsilon(\widetilde{f}^{a_j}))$ is $R_{3\epsilon/2}$–free, as claimed.

For any $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$–indexed module Q such that $R_{3\epsilon/2}(Q)$ is $R_{3\epsilon/2}$–free, $R_{3\epsilon/2}(Q^a)$ is also $R_{3\epsilon/2}$–free for all $a \in \mathbb{R}^2$: If $R_{3\epsilon/2}(Q) \cong R_{3\epsilon/2}(F)$ for F pfd and free, then

$$R_{3\epsilon/2}(Q^a) \cong R_{3\epsilon/2}(F^a).$$

Thus, since $R_{3\epsilon/2}(L^\epsilon(\widetilde{f}^{a_j}))$ is $R_{3\epsilon/2}$–free, $R_{3\epsilon/2}(L^\epsilon(\widetilde{f}^{a_j})^{a_j-\epsilon e_1})$ is $R_{3\epsilon/2}$–free as well. Moreover,

$$L^j := R_{3\epsilon/2}(L^\epsilon(\widetilde{f})^{a_j-\epsilon e_1}) = R_{3\epsilon/2}(L^\epsilon(\widetilde{f}^{a_j})^{a_j-\epsilon e_1}),$$

so L^j is also $R_{3\epsilon/2}$–free.

Note that we have a canonical monomorphism $L^j \hookrightarrow L^{j+1}$ and that $\varinjlim L^j \cong R_{3\epsilon/2}(L^\epsilon(\widetilde{f}))$. By choosing an $R_{3\epsilon/2}$–basis for each L^j, we may inductively construct an $R_{3\epsilon/2}$–basis for $R_{3\epsilon/2}(L^\epsilon(\widetilde{f}))$ precisely as in the proof of Proposition 5.19. By Lemma 5.21 then, $R_{3\epsilon/2}(L^\epsilon(\widetilde{f}))$ is $R_{3\epsilon/2}$–free. □

Example 5.23 The previous lemma is tight: Let $M = I^{(-9\epsilon,-\epsilon)^{\triangledown}_{\epsilon}}$, $N = I^{(-10\epsilon,0)^{\triangledown}_{\epsilon}}$, and let $f\colon M \to N$ be any nonzero morphism. Then f is a monomorphism of R_ϵ–free persistence modules, but the persistence module $R_{3\epsilon/2-\delta}(L^\epsilon(f))$ is not $R_{3\epsilon/2-\delta}$–free for any $\delta > 0$; see Figure 9.

6 Induced matching theorem for free multidimensional persistence modules

Let $f\colon M \to N$ be a morphism of $\mathbb{R}^2$–indexed modules. For $i \in \{1,2\}$, let

$$o(i) := \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \end{cases}$$

and for $a \in \mathbb{R}$, define the line

$$\mathbb{T}^i_a := \{t\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)} \mid t \in \mathbb{R}\}.$$

In Section 6.1, we associate to each such line a morphism $\bar{f}$ of 1-D persistence modules derived from f. When M and N are free, intervals in the barcodes of the domain and codomain of $\bar{f}$ correspond, respectively, to intervals in $\mathcal{B}(M)$ and $\mathcal{B}(N)$ with an edge lying on $\mathbb{T}^i_a$. We prove that when f is a monomorphism with ϵ–trivial cokernel, then so is $\bar{f}$.

In Section 6.2, we use the morphisms $\bar{f}$, together with the decomposition (3) and the 1-D induced matchings of [3], to define the matching

$$\chi(f)\colon \mathcal{B}(M) \to \mathcal{B}(N)$$

induced by a monomorphism $f\colon M \to N$ of pfd free modules. We use this matching to formulate our induced matching theorem for free modules.

In Section 6.3, we establish a similar induced matching result for monomorphisms of R_ϵ–free modules.

6.1 Induced morphisms of 1-D persistence modules

For $c = (c_1, c_2) \in \mathbb{R}^2$, $i \in \{1,2\}$ and $a \in \mathbb{R}$, we write $c < \mathbb{T}^i_a$ if $c_{o(i)} < a$. For M an $\mathbb{R}^2$–indexed module, define the submodule $M'' \subset M$ by

$$M''_b = \{m \in M_b \mid m \in \operatorname{im} \varphi_M(c,b) \text{ for some } c < \mathbb{T}^i_a\}, \tag{6}$$

and let $M' := M/M''$. Note that if M is free, then M' and M'' are both free, and

$$\mathcal{B}(M') = \{c^{\llcorner} \in \mathcal{B}(M) \mid c \not< \mathbb{T}^i_a\},$$
$$\mathcal{B}(M'') = \{c^{\llcorner} \in \mathcal{B}(M) \mid c < \mathbb{T}^i_a\}.$$

Given a morphism $f\colon M \to N$, we have that $f(M'') \subset N''$, so f induces a morphism $f'\colon M' \to N'$. Restricting f' to the line $\mathbb{T}^i_a$, we obtain a morphism of 1-D persistence modules

$$\bar{f} := f'|_{\mathbb{T}^i_a} \colon M'|_{\mathbb{T}^i_a} \to N'|_{\mathbb{T}^i_a}. \tag{7}$$

For the next two lemmas, f' and $\bar{f}$ are understood to be defined with respect to a fixed choice of $a \in \mathbb{R}$.

Lemma 6.1 *If f has $\epsilon \boldsymbol{e}_i$–trivial cokernel then $\bar{f}$ has ϵ–trivial cokernel.*

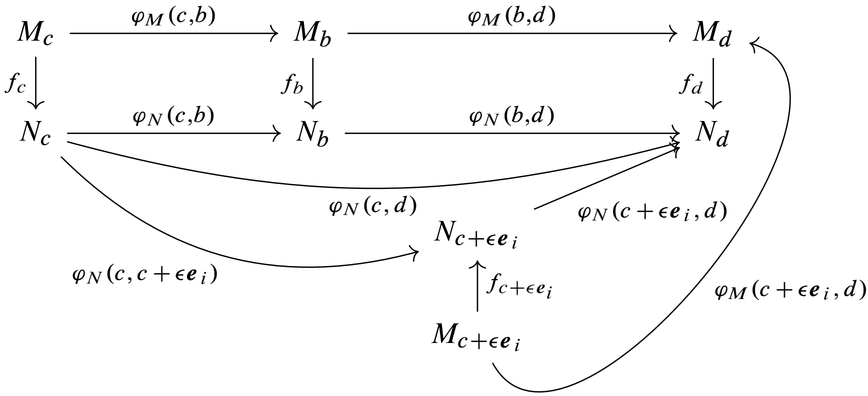
Proof We show that f' has $\epsilon \boldsymbol{e}_i$–trivial cokernel; the result then follows by restricting the indexing poset to $\mathbb{T}^i_a$. For any $b \in \mathbb{R}^2$ and $n \in N_b$, let $[n] \in N'_b$ denote the corresponding coset. Suppose that $m \in M_{b+\boldsymbol{e}_i}$ satisfies $f(m) = \varphi_N(b, b+\epsilon\boldsymbol{e}_i)(n)$. Then

$$f'[m] = [f(m)] = [\varphi_N(b, b+\epsilon\boldsymbol{e}_i)(n)] = \varphi_{N'}(b, b+\epsilon\boldsymbol{e}_i)[n]. \qquad \square$$

Lemma 6.2 *Assume that M is free below $t\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)}$* (see the end of Section 5) *and that f has $\epsilon\boldsymbol{e}_i$–trivial cokernel. If $f_{t\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)}}$ is injective, then $\bar{f}_{t-\epsilon}$ is injective. In particular, if f is a monomorphism of free modules, then $\bar{f}$ is a monomorphism.*

Proof Let $b = (t-\epsilon)\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)}$. Then $\bar{f}_{t-\epsilon} = f'_b$, so we need to show that f'_b is injective, ie that for any $m \in M_b$ with $f_b(m) \in N''_b$, we have $m \in M''_b$.

Since $f_b(m) \in N''_b$, we have $f_b(m) = \varphi_N(c, b)(n)$ for some $c \in \mathbb{R}^2$ with $c_i \le (t-\epsilon)$, $c_{o(i)} < a$ and $n \in N_c$. Let $d := t\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)}$. Since f has $\epsilon\boldsymbol{e}_i$–trivial cokernel, there exists $m' \in M''_{c+\epsilon\boldsymbol{e}_i}$ such that $f_{c+\epsilon\boldsymbol{e}_i}(m') = \varphi_N(c, c+\epsilon\boldsymbol{e}_i)(n)$. This, together with the commutative diagram



yields the chain of equalities

$$\begin{aligned} f_d \circ \varphi_M(b,d)(m) &= \varphi_N(b,d) \circ f_b(m) \\ &= \varphi_N(b,d) \circ \varphi_N(c,b)(n) \\ &= \varphi_N(c,d)(n) \\ &= \varphi_N(c+\epsilon \boldsymbol{e}_i, d) \circ \varphi_N(c, c+\epsilon \boldsymbol{e}_i)(n) \\ &= \varphi_N(c+\epsilon \boldsymbol{e}_i, d) \circ f_{c+\epsilon \boldsymbol{e}_i}(m') \\ &= f_d \circ \varphi_M(c+\epsilon \boldsymbol{e}_i, d)(m'). \end{aligned}$$

The injectivity of f_d implies

$$\varphi_M(b,d)(m) = \varphi_M(c+\epsilon \boldsymbol{e}_i, d)(m').$$

Since M is free below d, it follows that $m \in \operatorname{im} \varphi_M(e,b)$, where

$$e = \min(c_i + \epsilon, (t-\epsilon))\boldsymbol{e}_i + c_{o(i)}\boldsymbol{e}_{o(i)}.$$

Since $e < \mathbb{T}_a^i$, we thus have $m \in M_b''$, as desired. □

The next example shows that the previous proposition is tight.

Example 6.3 Let $M = I^{(0,1)^{\llcorner}} \oplus I^{(1,0)^{\llcorner}}$, $N = I^{(0,0)^{\llcorner}}$, and $f\colon M \to N$ be any morphism that is injective on each of the summands of M. Then f has $\boldsymbol{e}_1$–trivial cokernel and f is injective on $(t,1)$ for all $t<1$. It is easy to see that $M'|_{\mathbb{T}_1^1} \cong I^{[0,\infty)}$ and $N'|_{\mathbb{T}_1^1} = 0$. Hence $\bar{f}_s\colon M'|_{\mathbb{T}_1^1} \to N'|_{\mathbb{T}_1^1}$ is not injective for any $s \in [0,\infty)$.

6.2 Induced matchings of free 2-D persistence modules

For M a pfd $\mathbb{R}^2$–indexed module and $a \in \mathbb{R}$, let $\mathcal{B}(M;i,a) := \mathcal{B}(M'|_{\mathbb{T}_a^i})$. For $f\colon M \to N$ a morphism of pfd $\mathbb{R}^2$–indexed modules, let

$$\chi(f;i,a) := \chi(\bar{f})\colon \mathcal{B}(M;i,a) \nrightarrow \mathcal{B}(N;i,a),$$

where $\chi(\bar{f})$ is the matching induced by $\bar{f}\colon M'|_{\mathbb{T}_a^i} \to N'|_{\mathbb{T}_a^i}$; see Section 2.3. The matchings $\chi(f;i,a)$ assemble into a matching

$$\bigsqcup_{a\in\mathbb{R}} \chi(f;i,a)\colon \bigsqcup_{a\in\mathbb{R}} \mathcal{B}(M;i,a) \nrightarrow \bigsqcup_{a\in\mathbb{R}} \mathcal{B}(N;i,a). \tag{8}$$

Definition 6.4 (direction-i induced matchings) Assume that M and N are free. We then have a bijection $\bigsqcup_{a\in\mathbb{R}} \mathcal{B}(M;i,a) \to \mathcal{B}(M)$ matching $[t,\infty) \in \mathcal{B}(M;i,a)$

to $(t\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)})^{\llcorner} \in \mathcal{B}(M)$, and similarly for N. By way of these bijections, the matching (8) induces a matching

$$\chi(f;i)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N).$$

We call this *the direction-i matching induced by f*.

Definition 6.5 (induced matchings for monomorphisms of free $\mathbb{R}^2$–indexed modules) Now assume that $f\colon M \hookrightarrow N$ is a monomorphism of pfd free $\mathbb{R}^2$–indexed persistence modules. We decompose f as in Section 5.1:

$$M \cong \operatorname{im} f \xhookrightarrow{f_1} L^\infty(f) \xhookrightarrow{f_2} N. \tag{9}$$

We define $\chi(f)$, the matching induced by f, as

$$\chi(f) := \chi(f_2;2) \circ \chi(f_1;1)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$$

Example 6.6 Let $f\colon M \to N$ be as in Example 5.6. We have

$$\chi(f_1;1)(3,1)^{\llcorner} = (2,1)^{\llcorner}, \quad \chi(f_1;1)(2,2)^{\llcorner} = (0,2)^{\llcorner},$$
$$\chi(f_2;2)(2,1)^{\llcorner} = (2,0)^{\llcorner}, \quad \chi(f_2;2)(0,2)^{\llcorner} = (0,1)^{\llcorner}.$$

Therefore,

$$\chi(f)(3,1)^{\llcorner} = (2,0)^{\llcorner}, \quad \chi(f)(2,2)^{\llcorner} = (0,1)^{\llcorner}.$$

Theorem 6.7 (induced matchings of free modules) *Let $f\colon M \to N$ be a monomorphism of pfd free $\mathbb{R}^2$–indexed modules and assume that $\chi(f)(b^{\llcorner}) = b'^{\llcorner}$. Then:*

(i) *$\chi(f)$ matches each interval in $\mathcal{B}(M)$ and $b' \leq b$.*

(ii) *If f has ϵ–trivial cokernel for some $\epsilon \in [0,\infty)$, then $\chi(f)$ also matches each interval in $\mathcal{B}(N)$, and $\|b-b'\|_\infty \leq \epsilon$.*

Proof To streamline our exposition, we give the proof of both (i) and (ii) under the assumption that f has ϵ–trivial cokernel; our argument adapts immediately to give a proof of (i) in the case that f does not have ϵ–trivial cokernel for any finite ϵ.

$L^\infty(f)$ is free by Proposition 5.19. By Lemma 5.4, $L^\infty(f) = L^\epsilon(f)$, so by Lemma 5.1, for $i \in \{1,2\}$ the inclusion f_i has $\epsilon\boldsymbol{e}_i$–trivial cokernel. For convenience, we introduce the notation

$$L_0 := \operatorname{im} f, \quad L_1 := L^\infty(f) \quad \text{and} \quad L_2 := N.$$

For $a \in \mathbb{R}$, Lemmas 6.1 and 6.2 imply that

$$\bar{f} = f_i'|_{\mathbb{T}_a^i} \colon L_{i-1}'|_{\mathbb{T}_a^i} \to L_i'|_{\mathbb{T}_a^i}$$

is a monomorphism with ϵ–trivial cokernel. From Theorem 2.12 it follows that

$$\chi(\bar{f}) \colon \mathcal{B}(L_{i-1}; i, a) \nrightarrow \mathcal{B}(L_i; i, a)$$

is a bijective matching such that $\chi(\bar{f})[b,\infty) = [b',\infty)$, where $b - \epsilon \leq b' \leq b$. Thus, the direction-$i$ matching

$$\chi(f_i; i) \colon \mathcal{B}(L_{i-1}) \nrightarrow \mathcal{B}(L_i)$$

is bijective and matches $(b\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)})^{\llcorner}$ to $(b'\boldsymbol{e}_i + a\boldsymbol{e}_{o(i)})^{\llcorner}$.

Hence, $\chi(f) \colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$ is a bijective matching with the desired properties. □

We omit the easy proof of the following:

Proposition 6.8 *For free $\mathbb{R}^2$–indexed modules M and N, a matching $\sigma \colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$ is an ϵ–matching if and only if it is bijective and for all $b^{\llcorner} \in \mathcal{B}(M)$, $\sigma(b^{\llcorner}) = b'^{\llcorner}$ with $\|b - b'\|_\infty \leq \epsilon$.*

Define a bijection $r_\epsilon \colon \mathcal{B}(N(\epsilon)) \to \mathcal{B}(N)$ by $r_\epsilon(b^{\llcorner}) = (b + (\epsilon,\epsilon))^{\llcorner}$.

Corollary 6.9 (isometry theorem for free $\mathbb{R}^2$–indexed modules) *Pfd free $\mathbb{R}^2$–indexed modules M and N are ϵ–interleaved if and only if there exists an ϵ–matching between $\mathcal{B}(M)$ and $\mathcal{B}(N)$.*

Proof An ϵ–interleaving morphism $f \colon M \to N(\epsilon)$ is a monomorphism with 2ϵ–trivial cokernel. By Theorem 6.7(ii), $\chi(f) \colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N(\epsilon))$ is a bijective matching such that $\chi(f)(b^{\llcorner}) = b'^{\llcorner}$, where $b - (2\epsilon, 2\epsilon) \leq b' \leq b$. The composition

$$r_\epsilon \circ \chi(f) \colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$$

is a bijective matching such that for all $b^{\llcorner} \in \mathcal{B}(M)$, $\sigma(b^{\llcorner}) = b'^{\llcorner}$ with $\|b - b'\|_\infty < \epsilon$. Thus, by Proposition 6.8, $r_\epsilon \circ \chi(f)$ is an ϵ–matching.

The converse is a special case of Proposition 2.13. □

The difficulty of defining induced matchings for free $\mathbb{R}^3$–indexed modules An algebraic stability theorem for free $\mathbb{R}^n$–indexed modules for any n is established in [6], generalizing our Corollary 6.9. We imagine that Theorem 6.7 can be correspondingly generalized to an induced matching theorem for free $\mathbb{R}^n$–indexed modules for any n. However, the construction of induced matchings given here does not generalize directly to $n \geq 3$. To explain, the decomposition (9) does generalize to a decomposition

$$M \xhookrightarrow{f_1} L_1^\infty(f) \xhookrightarrow{f_2} \cdots \xhookrightarrow{f_{n-1}} L_{n-1}^\infty(f) \xhookrightarrow{f_n} N$$

of a monomorphism $f\colon M \hookrightarrow N$ of free $\mathbb{R}^n$–indexed modules, where for $a \in \mathbb{R}^n$ and $\boldsymbol{e}_{[i]} := \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_i$,

$$L_i^\infty(f)_a := \{n \in N_a \mid \varphi_N(a, a + \epsilon \boldsymbol{e}_{[i]})(n) \in \operatorname{im} f \text{ for some } \epsilon \in [0, \infty)\},$$

and each f_i is the inclusion. As in the 2-D case, if f has ϵ–trivial cokernel, then each f_i has $\epsilon \boldsymbol{e}_i$–trivial cokernel. However, the next example shows that in contrast to the $n = 2$ case, $L_i^\infty(f)$ needn't be free for $n \geq 3$.

Example 6.10 Take N to be the free $\mathbb{R}^3$–indexed module with generators a, b, and c at respective grades $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $M \subset N$ be the free submodule generated by

$$\{a - c \in N_{(1,0,1)},\ a - b \in N_{(1,1,0)},\ a \in N_{(1,1,1)}\},$$

where by slight abuse of notation, we use the same label for a generator and its image under an internal map in N. Let $f\colon M \hookrightarrow N$ be the inclusion. Then

$$\{a - c \in N_{(1,0,1)},\ a - b \in N_{(1,1,0)},\ b \in N_{(0,1,1)},\ c \in N_{(0,1,1)}\}$$

is a minimal set of generators for $L_1^\infty(f)$; clearly, $L_1^\infty(f)$ is not free.

When each $L_i^\infty(f)$ is free, the construction of this section does extend to give an induced matching $\chi(f)\colon \mathcal{B}(M) \to \mathcal{B}(N)$ with the desired properties. However, when one or more of the $L_i^\infty(f)$ is not free, the construction breaks down. Thus, a new idea is needed to extend our definition of induced matchings to free $\mathbb{R}^n$–indexed modules for $n \geq 3$.

6.3 Matchings induced by monomorphisms of R_ϵ–free modules

Suppose $f\colon M \to N$ is a morphism of R_ϵ–free $\mathbb{R} \times \mathbb{R}^{\mathrm{op}}$–indexed modules. Then, for $i \in \{1, 2\}$, we can define the direction-i matching

$$\chi(f; i)\colon \mathcal{B}(M) \to \mathcal{B}(N)$$

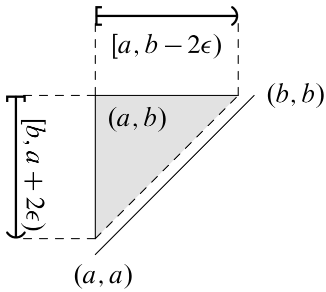


Figure 10: An illustration of the matchings (10) for a single choice of interval in $\mathcal{B}(M)$

in essentially the same way we did for free modules in Definition 6.4. To see this, note that as illustrated in Figure 10, for an R_ϵ–free module M, we may define bijective matchings

$$\bigsqcup_{b\in\mathbb{R}} \mathcal{B}(M;1,b) \nrightarrow \mathcal{B}(M), \quad \bigsqcup_{a\in\mathbb{R}} \mathcal{B}(M;2,a) \nrightarrow \mathcal{B}(M) \tag{10}$$

by matching both $[a, b-2\epsilon) \in \mathcal{B}(M;1,b)$ and $[b, a+2\epsilon) \in \mathcal{B}(M;2,a)$ to $(a,b)^{\triangledown}_{\epsilon}$. The construction of Definition 6.4 now carries over.

Now let $f\colon M \to N$ be a monomorphism of R_ϵ–free modules with ϵ–trivial cokernel. Consider the decomposition of f given by (3):

$$M \cong \operatorname{im} f \xhookrightarrow{f_1} L \xhookrightarrow{f_2} N,$$

where $L = L^\epsilon(f)$.

For the remainder of this section, we write the functor $R_{3\epsilon/2}$ simply as R. Note that $RL := R(L)$ is $R_{3\epsilon/2}$–free by Proposition 5.22. Hence, we have the following sequence of $R_{3\epsilon/2}$–free modules

$$RM \xrightarrow[\cong]{Rf} R(\operatorname{im} f) \xhookrightarrow{Rf_1} RL \xhookrightarrow{Rf_2} RN,$$

where Rf_1 and Rf_2 have $\epsilon \boldsymbol{e}_1$ and $\epsilon(-\boldsymbol{e}_2)$–trivial cokernel, respectively. For simplicity, we let $g := Rf_1 \circ Rf$ and $h := Rf_2$.

We define $\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$, the *matching induced by* f, as the composite matching

$$\mathcal{B}(M) \nrightarrow \mathcal{B}(RM) \xrightarrow{\chi(g;1)} \mathcal{B}(RL) \xrightarrow{\chi(h;2)} \mathcal{B}(RN) \nrightarrow \mathcal{B}(N), \tag{11}$$

where the matchings $\mathcal{B}(M) \nrightarrow \mathcal{B}(RM)$ and $\mathcal{B}(RN) \nrightarrow \mathcal{B}(N)$ are the obvious ones, ie the ones matching each interval $(a,b)^{\triangledown}_{3\epsilon/2}$ to $(a,b)^{\triangledown}_{\epsilon}$.

Proposition 6.11 *The composite matching*

$$\sigma := \chi(h;2) \circ \chi(g;1) \colon \mathcal{B}(RM) \nrightarrow \mathcal{B}(RN)$$

satisfies:

(1) $\sigma((a,b)^{\triangledown}_{3\epsilon/2}) = (a',b')^{\triangledown}_{3\epsilon/2}$, *where* $a' \le a \le a' + \epsilon$ *and* $b' - \epsilon \le b \le b'$.

(2) $\mathcal{B}(RM)_{\epsilon/2} \subset \operatorname{coim} \sigma$ *and* $\mathcal{B}(RN)_{\epsilon} \subset \operatorname{im} \sigma$.

Proof First, note that for any $R_{3\epsilon/2}$–free module Q and $b \in \mathbb{R}$, each interval in $\mathcal{B}(Q;1,b)$ is of the form $[a, b-3\epsilon)$.

Let $\overline{g}$ be the morphism of 1-D persistence modules associated to g for the point $b\boldsymbol{e}_2 \in \mathbb{L}_1$. Then $\overline{g}$ has ϵ–trivial cokernel by Lemma 6.1. Further, $\overline{g}_t$ is an injection for all $t < b - 4\epsilon$ by Lemma 6.2, so in particular $\overline{g}$ has ϵ–trivial kernel. By Theorem 2.12, then, the matching

$$\chi(\overline{g}) \colon \mathcal{B}(RM;1,b) \nrightarrow \mathcal{B}(RL;1,b)$$

satisfies:

(1) $\{[a, b-3\epsilon) \in \mathcal{B}(RM;1,b) \mid a < b - 4\epsilon\} \subset \operatorname{coim} \chi(\overline{g})$.

(2) $\{[a, b-3\epsilon) \in \mathcal{B}(RL;1,b) \mid a < b - 4\epsilon\} \subset \operatorname{im} \chi(\overline{g})$.

(3) $\chi(\overline{g})[a_1, b-3\epsilon) = [a_2, b-3\epsilon)$ where $a_2 \le a_1 \le a_2 + \epsilon$.

For Q any $R_{3\epsilon/2}$–free module and $(a,b)^{\triangledown}_{3\epsilon/2} \in \mathcal{B}(Q)$, $a < b - 4\epsilon$ if and only if $(a,b)^{\triangledown}_{3\epsilon/2} \in \mathcal{B}(Q)_{\epsilon/2}$. Thus, the direction-1 matching $\chi(g;1)$ satisfies:

(1) $\chi(g;1)((a_1,b)^{\triangledown}_{3\epsilon/2}) = (a_2,b)^{\triangledown}_{3\epsilon/2}$, where $a_2 \le a_1 \le a_2 + \epsilon$.

(2) $\mathcal{B}(RM)_{\epsilon/2} \subset \operatorname{coim} \chi(g;1)$ and $\mathcal{B}(RL)_{\epsilon/2} \subset \operatorname{im} \chi(g;1)$.

By the symmetric argument, the direction-2 matching $\chi(h;2)$ satisfies:

(1) $\chi(h;2)((a,b_1)^{\triangledown}_{3\epsilon/2}) = (a,b_2)^{\triangledown}_{3\epsilon/2}$, where $b_2 - \epsilon \le b_1 \le b_2$.

(2) $\mathcal{B}(RL)_{\epsilon/2} \subset \operatorname{coim} \chi(h;2)$ and $\mathcal{B}(RN)_{\epsilon/2} \subset \operatorname{im} \chi(h;2)$.

It follows that $\sigma((a,b)^{\triangledown}_{3\epsilon/2}) = (a',b')^{\triangledown}_{3\epsilon/2}$, where $a' \le a \le a' + \epsilon$ and $b' - \epsilon \le b \le b'$, as desired. Moreover,

$$\mathcal{B}(RN)_{\epsilon} \subset \chi(h;2)(\mathcal{B}(RL)_{\epsilon/2}) \subset \chi(h;2)(\operatorname{im} \chi(g;1) \cap \operatorname{coim} \chi(h;2)) = \operatorname{im} \sigma,$$

and

$$\chi(g;1)(\mathcal{B}(RM)_{\epsilon/2}) \subset \mathcal{B}(RL)_{\epsilon/2} \subset \operatorname{coim} \chi(h;2).$$

The latter shows that $\mathcal{B}(RM)_{\epsilon/2} \subset \operatorname{coim} \sigma$. □

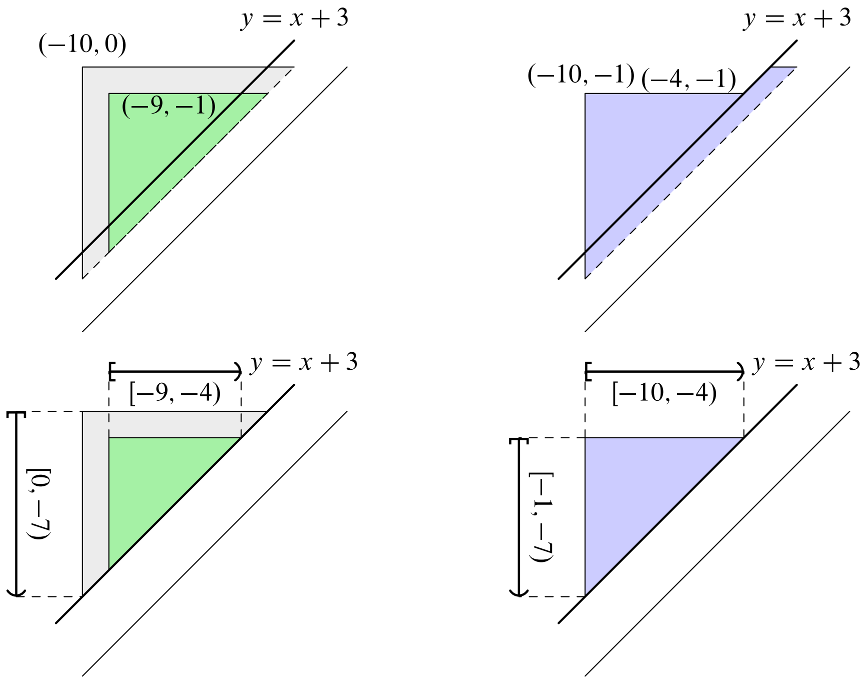


Figure 11: The intervals arising in the construction of the induced matching $\chi(f)$ of Example 6.13. Top-left: supports of the R_1–free modules M (green) and N (gray). Top-right: support of $L := L^1(f)$. Bottom-left: supports of the $R_{3/2}$–free modules RM (green) and RN (gray); also shown are $\mathcal{B}(RM; 1, -1) = \{[-9, -4)\}$ and $\mathcal{B}(RN; 2, -10) = \{[0, -7)\}$. Bottom-right: support of the $R_{3/2}$–free module RL; also shown are $\mathcal{B}(RL; 1, -1) = \{[-10, -4)\}$ and $\mathcal{B}(RL; 2, -10) = \{[-1, -7)\}$.

Corollary 6.12 (induced matchings of R_ϵ–free modules) *For $f\colon M \to N$ a monomorphism of R_ϵ–free modules with ϵ–trivial cokernel, the induced matching*

$$\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$$

satisfies:

(1) $\chi((a,b)^{\triangledown}_{\epsilon}) = (a',b')^{\triangledown}_{\epsilon}$, *where* $a' \le a \le a' + \epsilon$ *and* $b' - \epsilon \le b \le b'$.

(2) $\mathcal{B}(M)_\epsilon \subset \operatorname{coim}\chi$ *and* $\mathcal{B}(N)_{\frac{3}{2}\epsilon} \subset \operatorname{im}\chi$.

Example 6.13 We consider the induced matching $\chi(f)$ defined above in the case where f is the monomorphism of R_ϵ–free modules of Example 5.23 and $\epsilon = 1$. Note that f has 1–trivial cokernel. The intervals involved in the construction of the matching are illustrated in Figure 11. We see that in this example, the sequence of matchings (11) defining $\chi(f)$ is of the form

$$\{(-9,-1)^{\triangledown}_{1}\} \nrightarrow \{(-9,-1)^{\triangledown}_{3/2}\} \xrightarrow{\chi(g;1)} \{(-10,-1)^{\triangledown}_{3/2}\} \xrightarrow{\chi(h;2)} \{(-10,0)^{\triangledown}_{3/2}\} \nrightarrow \{(-10,0)^{\triangledown}_{1}\}.$$

Thus, each barcode in the sequence consists of a single interval.

We observe that in fact, each matching in this sequence is a bijection, so the same is true of composition $\chi(f)$. This is clearly true for the first and last matchings in the sequence. As illustrated in Figure 11,

$$\bigsqcup_{b\in\mathbb{R}} \mathcal{B}(RM;1,b) = \mathcal{B}(RM;1,-1) = \{[-9,-4)\},$$
$$\bigsqcup_{b\in\mathbb{R}} \mathcal{B}(RL;1,b) = \mathcal{B}(RL;1,-1) = \{[-10,-4)\},$$
$$\bigsqcup_{a\in\mathbb{R}} \mathcal{B}(RL;2,a) = \mathcal{B}(RL;2,-10) = \{[-1,-7)\},$$
$$\bigsqcup_{a\in\mathbb{R}} \mathcal{B}(RN;2,a) = \mathcal{B}(RN;2,-10) = \{[0,-7)\}.$$

Thus, we have

$$\chi(g;1,-1)([-9,-4)) = [-10,-4) \quad \text{and} \quad \chi(h;2,-10)([-1,-7)) = [0,-7),$$

so under the bijections of (10) we get the bijective matchings

$$\chi(g;1)((-9,-1)^{\triangledown}_{3/2}) = (-10,-1)^{\triangledown}_{3/2} \quad \text{and} \quad \chi(h;2)((-10,-1)^{\triangledown}_{3/2}) = (-10,0)^{\triangledown}_{3/2}.$$

7 Proof of the block stability theorem

In this section, we complete the proof of our main stability result for block-decomposable modules. Throughout, we regard block-decomposable modules as $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed modules.

7.1 Decomposition of interleavings

Definition 7.1 For a block-decomposable module M, we choose summands

$$M^{\boldsymbol{o}} \cong \bigoplus_{(a,b)_{\mathrm{BL}}\in\mathcal{B}(M)^{\boldsymbol{o}}} I^{(a,b)_{\mathrm{BL}}}, \qquad M^{\boldsymbol{co}} \cong \bigoplus_{\langle a,b\rangle_{\mathrm{BL}}\in\mathcal{B}(M)^{\boldsymbol{co}}} I^{\langle a,b\rangle_{\mathrm{BL}}},$$
$$M^{\boldsymbol{oc}} \cong \bigoplus_{\langle a,b\rangle_{\mathrm{BL}}\in\mathcal{B}(M)^{\boldsymbol{oc}}} I^{\langle a,b\rangle_{\mathrm{BL}}}, \qquad M^{\boldsymbol{c}} \cong \bigoplus_{\langle a,b\rangle_{\mathrm{BL}}\in\mathcal{B}(M)^{\boldsymbol{c}}} I^{\langle a,b\rangle_{\mathrm{BL}}}$$

such that $M = M^{\boldsymbol{o}} \oplus M^{\boldsymbol{co}} \oplus M^{\boldsymbol{oc}} \oplus M^{\boldsymbol{c}}$. For $\star \in \{\boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}, \boldsymbol{o}\}$, we say M *is of type* $\star$ if M is pfd and $M = M^{\star}$.

For $f\colon M \to N$ a morphism of block-decomposable modules and $\star, \dagger \in \{\boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}, \boldsymbol{o}\}$, let $f^{\star,\dagger}\colon M^{\star} \to N^{\dagger}$ denote the morphism obtained by precomposing f with the inclusion $M^{\star} \hookrightarrow M$ and postcomposing with the projection $N \twoheadrightarrow N^{\dagger}$.

Lemma 7.2 *For block-decomposable modules M and N,* $\mathrm{Hom}(M^{\star}, N^{\dagger}) = 0$ *whenever*

$$(\star, \dagger) \in \{(\boldsymbol{o}, \boldsymbol{co}), (\boldsymbol{o}, \boldsymbol{oc}), (\boldsymbol{o}, \boldsymbol{c}), (\boldsymbol{co}, \boldsymbol{oc}), (\boldsymbol{co}, \boldsymbol{c}), (\boldsymbol{oc}, \boldsymbol{co}), (\boldsymbol{oc}, \boldsymbol{c})\}.$$

Proof We show that $\mathrm{Hom}(M^{\boldsymbol{o}}, N^{\boldsymbol{co}}) = 0$. Similar arguments apply to the remaining cases. It suffices to consider the case that M and N are indecomposables. Assume to the contrary that we have $f \in \mathrm{Hom}(I^{(a,b)_{\mathrm{BL}}}, I^{[c,d)_{\mathrm{BL}}})$ with $f_{(x,y)} \neq 0$. Then

$$a < x \leq y < b, \quad y < d,$$

and by choosing $x' < a$ we obtain the commutative diagram

$$\begin{array}{ccc} k = I^{(a,b)_{\mathrm{BL}}}_{(x,y)} & \longrightarrow & I^{(a,b)_{\mathrm{BL}}}_{(x',y)} = 0 \\ {\scriptstyle f_{(x,y)}}\big\downarrow & & \big\downarrow{\scriptstyle f_{(x',y)}} \\ k = I^{[c,d)_{\mathrm{BL}}}_{(x,y)} & \xrightarrow{\mathrm{id}} & I^{[c,d)_{\mathrm{BL}}}_{(x',y)} = k \end{array}$$

contradicting that $f_{(x,y)} \neq 0$. This shows that $\mathrm{Hom}(I^{(a,b)_{\mathrm{BL}}}, I^{[c,d)_{\mathrm{BL}}}) = 0$. The same argument shows that $\mathrm{Hom}(I^{(a,b)_{\mathrm{BL}}}, I^{(-\infty,d)_{\mathrm{BL}}}) = 0$. □

Proposition 7.3 *If $f\colon M \to N(\epsilon)$ is an ϵ–interleaving morphism, then so is $f^{\star,\star}$ for any $\star \in \{\boldsymbol{o}, \boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}\}$. In particular, $f^{\star,\star}$ has 2ϵ–trivial kernel and cokernel.*

Proof Let $g\colon N \to M(\epsilon)$ be such that f and g form an ϵ–interleaving. By decomposing M and N as in Definition 7.1 and applying Lemma 7.2, we can express f in matrix form as

$$f = \left[\begin{array}{cccc|c} M^{\boldsymbol{o}} & M^{\boldsymbol{co}} & M^{\boldsymbol{oc}} & M^{\boldsymbol{c}} & \\ \hline f^{\boldsymbol{o},\boldsymbol{o}} & f^{\boldsymbol{co},\boldsymbol{o}} & f^{\boldsymbol{oc},\boldsymbol{o}} & f^{\boldsymbol{c},\boldsymbol{o}} & N^{\boldsymbol{o}}(\epsilon) \\ 0 & f^{\boldsymbol{co},\boldsymbol{co}} & 0 & f^{\boldsymbol{c},\boldsymbol{co}} & N^{\boldsymbol{co}}(\epsilon) \\ 0 & 0 & f^{\boldsymbol{oc},\boldsymbol{oc}} & f^{\boldsymbol{c},\boldsymbol{oc}} & N^{\boldsymbol{oc}}(\epsilon) \\ 0 & 0 & 0 & f^{\boldsymbol{c},\boldsymbol{c}} & N^{\boldsymbol{c}}(\epsilon) \end{array}\right],$$

and similarly for $g(\epsilon)$. Since $g(\epsilon)\circ f=\varphi^{2\epsilon}_{M^o}\oplus\varphi^{2\epsilon}_{M^{co}}\oplus\varphi^{2\epsilon}_{M^{oc}}\oplus\varphi^{2\epsilon}_{M^c}$, we may write $g(\epsilon)\circ f$ in matrix form as

$$g(\epsilon)\circ f=\left[\begin{array}{cccc|c} M^o & M^{co} & M^{oc} & M^c & \\ \hline g^{o,o}(\epsilon)\circ f^{o,o} & 0 & 0 & 0 & M^o(2\epsilon)\\ 0 & g^{co,co}(\epsilon)\circ f^{co,co} & 0 & 0 & M^{co}(2\epsilon)\\ 0 & 0 & g^{oc,oc}(\epsilon)\circ f^{oc,oc} & 0 & M^{oc}(2\epsilon)\\ 0 & 0 & 0 & g^{c,c}(\epsilon)\circ f^{c,c} & M^c(2\epsilon)\end{array}\right],$$

and the following equality is immediate:

$$(g^{o,o}(\epsilon)\circ f^{o,o})\oplus(g^{co,co}(\epsilon)\circ f^{co,co})\oplus(g^{oc,oc}(\epsilon)\circ f^{oc,oc})\oplus(g^{c,c}(\epsilon)\circ f^{c,c})\\ =\varphi^{2\epsilon}_{M^o}\oplus\varphi^{2\epsilon}_{M^{co}}\oplus\varphi^{2\epsilon}_{M^{oc}}\oplus\varphi^{2\epsilon}_{M^c}.$$

The result follows by applying the symmetric argument to the composition $f(\epsilon)\circ g$. □

Thus, we can study algebraic stability for block-decomposables by considering an interleaving morphism on each of four subtypes individually.

Remark 7.4 In view of Proposition 7.3, one might wonder whether ϵ–triviality of the (co)kernel of a morphism $f\colon M\to N$ is inherited by $f^{\star,\star}$ for $\star\in\{o,co,oc,c\}$. In fact, the answer is no: it can be shown that if $f\colon M\to N$ has ϵ–trivial kernel and cokernel, then so have the three morphisms $f^{c,c}$, $f^{co,co}$ and $f^{oc,oc}$, and the morphism $f^{o,o}$ has ϵ–trivial kernel and 2ϵ–trivial cokernel. This result is tight, as demonstrated by the following example.

Example 7.5 Let $M=I^{(0,\epsilon]_{\mathrm{BL}}}\oplus I^{[3\epsilon,4\epsilon)_{\mathrm{BL}}}$ and $N=I^{(0,4\epsilon)_{\mathrm{BL}}}$. Let $f_1\colon I^{(0,\epsilon]_{\mathrm{BL}}}\to N$ and $f_2\colon I^{[3\epsilon,4\epsilon)_{\mathrm{BL}}}\to N$ be any two nonzero morphisms and define $f(m_1,m_2)=f_1(m_1)+f_2(m_2)$. Then f has ϵ–trivial kernel and cokernel, but the cokernel of $0=M^o\to N^o=N$ is 2ϵ–trivial and not δ–trivial for any $\delta<2\epsilon$.

7.2 An induced matching theorem

We establish the block stability theorem (Theorem 3.3) by separating the interleaving morphism f into its four components via Proposition 7.3, and studying each of them independently. In fact, Theorem 3.3 is an easy corollary of Proposition 7.3 and the following result:

Theorem 7.6 (induced matchings of block-decomposables) *For any fixed $\star \in \{\boldsymbol{c}, \boldsymbol{o}, \boldsymbol{co}, \boldsymbol{oc}\}$, let M and N be block-decomposable modules of type $\star$, and let $f\colon M \to N$ be a morphism with ϵ–trivial kernel and cokernel. Then we can define an explicit matching*

$$\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$$

such that for $\chi(f)\langle a,b\rangle_{\mathrm{BL}} = \langle a',b'\rangle_{\mathrm{BL}}$:

(i) *If $\star = \boldsymbol{co}$, then $\mathcal{B}(M)_\epsilon \subset \operatorname{coim}\chi(f)$, $\mathcal{B}(N)_\epsilon \subset \operatorname{im}\chi(f)$ and*

$$a-\epsilon \le a' \le a, \quad b-\epsilon \le b' \le b.$$

(ii) *If $\star = \boldsymbol{oc}$, then $\mathcal{B}(M)_\epsilon \subset \operatorname{coim}\chi(f)$, $\mathcal{B}(N)_\epsilon \subset \operatorname{im}\chi(f)$ and*

$$a \le a' \le a+\epsilon, \quad b \le b' \le b+\epsilon.$$

(iii) *If $\star = \boldsymbol{c}$, then $\mathcal{B}(M) = \operatorname{coim}\chi(f)$, $\mathcal{B}(N) = \operatorname{im}\chi(f)$ and*

$$a-\epsilon \le a' \le a, \quad b \le b' \le b+\epsilon.$$

(iv) *If $\star = \boldsymbol{o}$, then $\mathcal{B}(M)_{\frac{5}{2}\epsilon} \subset \operatorname{coim}\chi(f)$, $\mathcal{B}(N)_{2\epsilon} \subset \operatorname{im}\chi(f)$ and*

$$a \le a' \le a+\epsilon, \quad b-\epsilon \le b' \le b.$$

Proof of Theorem 3.3 from Theorem 7.6 For Q an $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed module, let $\bar{R}(Q)$ denote the $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed module given by

$$\bar{R}(Q)_{(s,t)} = \begin{cases} Q_{(s,t)} & \text{for all } t-s \ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

with the internal maps $\varphi_{\bar{R}(Q)}(-,-)$ inherited from Q. We have an obvious morphism $\pi_Q\colon Q \to \bar{R}(Q)$.

If M and N are block-decomposable modules, then $\bar{R}(N(\epsilon))$ is block-decomposable. If $f\colon M \to N(\epsilon)$ is an ϵ–interleaving morphism, then as mentioned in Remark 2.7, f has 2ϵ–trivial kernel and cokernel, and the same is true for $\pi_{N(\epsilon)}\circ f\colon M \to \bar{R}(N(\epsilon))$. By Proposition 7.3 then, for $\star \in \{\boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}, \boldsymbol{o}\}$, $f^{\star,\star}\colon M^\star \to \bar{R}(N(\epsilon))^\star$ has 2ϵ–trivial kernel and cokernel as well.

Let $r_\epsilon^\star\colon \mathcal{B}(\bar{R}(N(\epsilon)))^\star \nrightarrow \mathcal{B}(N)^\star$ be the matching given by

$$r_\epsilon^\star\langle b,d\rangle_{\mathrm{BL}} = \begin{cases} \langle a+\epsilon, b+\epsilon\rangle_{\mathrm{BL}} & \text{if } \star = \boldsymbol{co},\\ \langle a-\epsilon, b-\epsilon\rangle_{\mathrm{BL}} & \text{if } \star = \boldsymbol{oc},\\ \langle a+\epsilon, b-\epsilon\rangle_{\mathrm{BL}} & \text{if } \star = \boldsymbol{c},\\ \langle a-\epsilon, b+\epsilon\rangle_{\mathrm{BL}} & \text{if } \star = \boldsymbol{o}.\end{cases}$$

If $\star \in \{\boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}\}$, then $r_\epsilon^\star$ is bijective; in the case that $\star = \boldsymbol{o}$, $r_\epsilon^{\boldsymbol{o}}$ matches all blocks of $\mathcal{B}(\bar{R}(N(\epsilon)))^{\boldsymbol{o}}$ and all blocks $(a,b)_{\mathrm{BL}} \in \mathcal{B}(N)^{\boldsymbol{o}}$ with $b - a > 2\epsilon$.

Let $\chi(\pi_{N(\epsilon)} \circ f^\star)\colon \mathcal{B}(M)^\star \nrightarrow \mathcal{B}(\bar{R}(N(\epsilon)))^\star$ be the matching given by Theorem 7.6. We define the matching $\chi\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$ in the statement of Theorem 3.3 as the (disjoint) union of the four matchings

$$\left\{r_\epsilon^\star \circ \chi(\pi_{N(\epsilon)} \circ f^\star)\colon \mathcal{B}(M)^\star \nrightarrow \mathcal{B}(N)^\star \mid \star \in \{\boldsymbol{co}, \boldsymbol{oc}, \boldsymbol{c}, \boldsymbol{o}\}\right\}.$$

By Theorem 7.6, the definitions of the matchings $r_\epsilon^\star$ and Lemma 3.1(i), it follows that χ has the desired properties. □

The remainder of this section is devoted to the proof of Theorem 7.6. The cases $\star \in \{\boldsymbol{co}, \boldsymbol{oc}\}$ can be understood in terms of an equivalence with $\mathbb{R}$–indexed persistence, whereas our proofs for the cases $\star \in \{\boldsymbol{c}, \boldsymbol{o}\}$ build on our results for free and R_ϵ–free modules from Section 6.

7.3 Proof of Theorem 7.6(i)–(ii)

As the arguments for Theorem 7.6(i)–(ii) are essentially identical, we will only prove (i). We shall see that the result follows easily from Theorem 2.12.

Note that if M is of type $\boldsymbol{co}$, the shift map $\varphi_M((x,y),(x',y))$ is an isomorphism for all $x' \le x$. Hence, there is a functorial way to identify M with an $\mathbb{R}$–indexed module $M^{\mathbf{Ord}}$: Define

$$M_t^{\mathbf{Ord}} := M_{(t,t)}, \quad \varphi_{M^{\mathbf{Ord}}}(t,t') := \varphi_M((t',t'),(t,t'))^{-1} \circ \varphi_M((t,t),(t,t')),$$

and for $f\colon M \to N$ a morphism of modules of type $\boldsymbol{co}$, define $f^{\mathbf{Ord}}\colon M^{\mathbf{Ord}} \to N^{\mathbf{Ord}}$ by

$$f_t^{\mathbf{Ord}} := f_{(t,t)}.$$

Lemma 7.7 *Let M and N be of type $\boldsymbol{co}$, and let $f\colon M \to N$ have ϵ–trivial kernel and cokernel. Then $f^{\mathbf{Ord}}$ has ϵ–trivial kernel and cokernel.*

Proof Since f has ϵ–trivial kernel, $\varphi_{\ker f}((t,t),(t-\epsilon,t+\epsilon)) = 0$, so since

$$\varphi_M((t,t+\epsilon),(t-\epsilon,t+\epsilon))$$

is an isomorphism, we also have

$$\varphi_{\ker f}((t,t),(t,t+\epsilon)) = 0.$$

Similarly,

$$\varphi_{\operatorname{coker} f}((t,t),(t,t+\epsilon)) = 0.$$

Thus, the two commutative diagrams

$$\begin{array}{ccc} \ker f_t^{\mathbf{Ord}} & \xrightarrow{=} & \ker f_{(t,t)} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{0}} \\ & & \ker f_{(t,t+\epsilon)} \\ & & \Big\downarrow{\scriptstyle \cong} \\ \ker f_{t+\epsilon}^{\mathbf{Ord}} & \xleftarrow[=]{} & \ker f_{(t+\epsilon,t+\epsilon)} \end{array} \qquad \begin{array}{ccc} \operatorname{coker} f_t^{\mathbf{Ord}} & \xrightarrow{=} & \operatorname{coker} f_{(t,t)} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{0}} \\ & & \operatorname{coker} f_{(t,t+\epsilon)}. \\ & & \Big\downarrow{\scriptstyle \cong} \\ \operatorname{coker} f_{t+\epsilon}^{\mathbf{Ord}} & \xleftarrow[=]{} & \operatorname{coker} f_{(t+\epsilon,t+\epsilon)} \end{array}$$

complete the proof. □

Proof of Theorem 7.6(i) It is easy to see that for $\langle a,b\rangle_{\mathrm{BL}}$ a block of type ***co***, $(I^{\langle a,b\rangle})^{\mathbf{Ord}} = I^{\langle a,b\rangle}$, and, more generally, that for any module Q of type ***co***, $Q^{\mathbf{Ord}} \cong \bigoplus_{\langle a,b\rangle_{\mathrm{BL}}\in\mathcal{B}(Q)} I^{\langle a,b\rangle}$. We therefore have a bijection $\mathcal{B}(Q) \to \mathcal{B}(Q^{\mathbf{Ord}})$ which matches $\langle a,b\rangle_{\mathrm{BL}}$ to $\langle a,b\rangle$. For $f\colon M \to N$ a morphism of modules of type ***co*** with ϵ–trivial kernel and cokernel, the matching $\chi(f^{\mathbf{Ord}})\colon \mathcal{B}(M^{\mathbf{Ord}}) \nrightarrow \mathcal{B}(N^{\mathbf{Ord}})$ thus induces a matching $\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$. It follows from Lemma 7.7 and Theorem 2.12 that $\chi(f)$ has the desired properties. □

7.4 Proof of Theorem 7.6(iii)

Further decomposition of a module of type *c* To prove Theorem 7.6(iii), we shall separately match the four types of closed intervals $[a,b]_{\mathrm{BL}}$, $(-\infty,b]_{\mathrm{BL}}$, $[a,\infty)_{\mathrm{BL}}$ and $(-\infty,\infty)_{\mathrm{BL}}$. First, much as we decomposed a block-decomposable module into four summands in Definition 7.1, we choose a further decomposition of a module M of type ***c*** into four submodules

$$M = M^{()} \oplus M^{(]} \oplus M^{[)} \oplus M^{[]},$$

where

$$M^{()} \cong \bigoplus_{(-\infty,\infty)_{\mathrm{BL}}\in\mathcal{B}(M)} I^{(-\infty,\infty)_{\mathrm{BL}}}, \quad M^{(]} \cong \bigoplus_{(-\infty,b]_{\mathrm{BL}}\in\mathcal{B}(M)} I^{(-\infty,b]_{\mathrm{BL}}},$$
$$M^{[)} \cong \bigoplus_{[a,\infty)_{\mathrm{BL}}\in\mathcal{B}(M)} I^{[a,\infty)_{\mathrm{BL}}}, \quad M^{[]} \cong \bigoplus_{[a,b]_{\mathrm{BL}}\in\mathcal{B}(M)} I^{[a,b]_{\mathrm{BL}}}.$$

For N of type $\boldsymbol{c}$ and $\dagger \in \{(), (], [), []\}$, we let $f^\dagger\colon M^\dagger \to N^\dagger$ be the morphism obtained by the composition $M^\dagger \hookrightarrow M \xrightarrow{f} N \twoheadrightarrow N^\dagger$, where the first morphism is inclusion and the last is projection.

Proposition 7.8 *If $f\colon M \to N$ is a monomorphism with ϵ–trivial cokernel, then so is $f^\dagger\colon M^\dagger \to N^\dagger$ for $\dagger \in \{(), (], [), []\}$.*

Proof We shall prove the result for $\dagger = ()$. The proofs of the three remaining cases are similar. Using an argument similar to the proof of Lemma 7.2, it is easy to see that $\operatorname{Hom}(M^{()}, N^\dagger) = 0$ for $\dagger \in \{(], [), []\}$. Hence, $f^{()}$ is a monomorphism.

Since coker f is ϵ–trivial, for any $y - x \geq 2\epsilon$ and $n \in N^{()}_{(x,y)}$, there exists $m \in M_{(x,y)}$ with $f(m) = n$. Write

$$m = m^{()} + m^{(]} + m^{[)} + m^{[]}$$

for $m^\dagger \in M^\dagger$. We shall argue that $m^{(]} = m^{[)} = m^{[]} = 0$, so that $f^{()}(m^{()}) = n$. It follows that coker $f^{()}$ is ϵ–trivial.

To arrive at a contradiction, assume that $m^{[]} \neq 0$. By the structure of $M^{[]}_{(x,y)}$, we may choose sufficiently large $x' > y$ such that for $y' = x' + 2\epsilon$, we have

$$\varphi_M((x,y),(x,y'))(m^{[]}) \notin \operatorname{im} \varphi_M((x',y'),(x,y')). \tag{12}$$

Consider the unique element $n' \in N^{()}_{(x',y')}$ such that

$$\varphi_N((x',y'),(x,y'))(n') = \varphi_N((x,y),(x,y'))(n).$$

Since coker f is ϵ–trivial, $n' \in \operatorname{im} f$. That is, there exists $m' \in M_{(x',y')}$ such that $n' = f_{(x',y')}(m')$. Hence,

$$\begin{aligned} f \circ \varphi_M((x',y'),(x,y'))(m') &= \varphi_N((x',y'),(x,y'))(n') \\ &= \varphi_N((x,y),(x,y'))(n) \\ &= f \circ \varphi_M((x,y),(x,y'))(m). \end{aligned}$$

This, together with the injectivity of f, implies

$$\varphi_M((x,y),(x,y'))(m) = \varphi_M((x',y'),(x,y'))(m').$$

Letting $m'^{[]}$ denote the component of m' in $M^{[]}_{(x',y')}$, it follows that

$$\varphi_M((x,y),(x,y'))(m^{[]}) = \varphi_M((x',y'),(x,y'))(m'^{[]}),$$

contradicting that $\varphi_M((x,y),(x,y'))(m^{[]}) \notin \operatorname{im} \varphi_M((x',y'),(x,y'))$. Thus, $m^{[]} = 0$. Similarly, one can show that $m^{(]} = m^{[)} = 0$. □

The matching $\chi(f)$ If M and N are of type $\mathbf{c}$ and $f\colon M \to N$ has ϵ–trivial kernel and cokernel, then in fact f is a monomorphism. By Proposition 7.8 we may split f into four monomorphisms $f^\dagger\colon M^\dagger \to N^\dagger$ with ϵ–trivial cokernel. We take the matching $\chi(f)\colon \mathcal{B}(M) \nrightarrow \mathcal{B}(N)$ to be the disjoint union of four matchings

$$\{\chi(f^\dagger)\colon \mathcal{B}(M^\dagger) \nrightarrow \mathcal{B}(N^\dagger) \mid \dagger \in \{\,(),(],[),[]\,\}\}.$$

For $\dagger \in \{\,(),(],[)\,\}$ we define the matching $\chi(f^\dagger)$ as follows:

() A morphism between $M^{()}$ and $N^{()}$ is a monomorphism with ϵ–trivial cokernel if and only if it is an isomorphism. Thus, $\mathcal{B}(M^{()}) = \mathcal{B}(N^{()})$; we take $\chi(f^{()})$ to be the identity.

[) $\varphi_{M^{[)}}((x,y),(x',y))$ is an isomorphism for all $x' \le x$, and similarly for $N^{[)}$, so we may define the matching $\chi(f^{[)})$ in essentially the same way we defined the induced matching of Theorem 7.6(i). The same argument used to prove Theorem 7.6(i) shows that $\chi(f^{[)})$ is bijective, and that if $\chi(f^{[)})[a,\infty)_{\mathrm{BL}} = [a',\infty)_{\mathrm{BL}}$, then $a-\epsilon \le a' \le a$.

(] We define $\chi(f^{(]})$ in essentially the same way as for $\chi(f^{[)})$. $\chi(f^{(]})$ is bijective, and if $\chi(f^{[)})(\infty,b]_{\mathrm{BL}} = (\infty,b']_{\mathrm{BL}}$, then $b-\epsilon \le b' \le b$.

To finish the proof of Theorem 7.6(iii), it remains to define the matching $\chi(f^{[]})$ and verify that if $\chi(f^{[)})[a,b]_{\mathrm{BL}} = [a',b']_{\mathrm{BL}}$, then

$$a \le a' \le a+\epsilon \quad \text{and} \quad b-\epsilon \le b' \le b.$$

In what follows, we define $\chi(f^{[]})$ via the induced matching construction for free 2-D persistence modules of Section 6.2.

The matching $\chi(f^{[]})$ Letting $\mathrm{e}\colon \mathbb{U} \hookrightarrow \mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ denote the inclusion, we define an endofunctor $\overleftarrow{(-)}$ on $\mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}}$ by

$$\overleftarrow{(-)} := \mathrm{Ran}_{\mathrm{e}}(-) \circ (-)|_{\mathbb{U}}.$$

Thus, for $(s,t) \in \mathbb{R}^2$ and $\overleftarrow{(s,t)} \subset \mathbb{U}$ given by

$$\overleftarrow{(s,t)} := \{(x,y) \in \mathbb{U} \mid x \le s,\ y \ge t\},$$

we have $\overleftarrow{M}_{(s,t)} = \varprojlim M|_{\overleftarrow{(s,t)}}$ for any $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed module M.

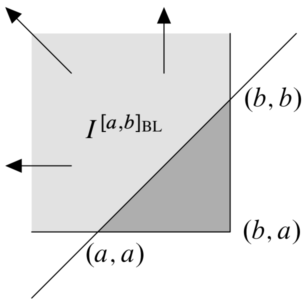


Figure 12: The image under $\overleftarrow{(-)}$ of the block module $I^{[a,b]_{\mathrm{BL}}}$ (in light gray) is a free module with a single generator at (b,a).

Properties of $\overleftarrow{(-)}$ on modules of type $c^{[\,]}$ The following lemma is illustrated by Figure 12; we omit the proof.

Lemma 7.9 *For any $a,b \in \mathbb{R}$, we have*

$$\overleftarrow{I^{[a,b]_{\mathrm{BL}}}} \cong I^{(b,a)^{\lrcorner}}.$$

We say a module M is *of type* $\boldsymbol{c}^{[\,]}$ if M is of type $\boldsymbol{c}$ and $M = M^{[\,]}$.

Lemma 7.10 *For each module M of type $\boldsymbol{c}^{[\,]}$, $\overleftarrow{M}$ is pfd.*

Proof For $(s,t) \in \mathbb{R}^2$, let $v = (v_1, v_2)$, where $v_1 = \min(s,t)$ and $v_2 = \max(s,t)$. Note that for $a,b \in \mathbb{R}$, if $(s,t) \in (b,a)^{\lrcorner}$, then $v \in [a,b]_{\mathrm{BL}}$. In view of Lemma 7.9, then, it follows that

$$\dim\Bigg(\bigoplus_{\mathcal{J}\in\mathcal{B}(M)} \overleftarrow{I^{\mathcal{J}}}\Bigg)_{(s,t)} \le \dim\Bigg(\bigoplus_{\mathcal{J}\in\mathcal{B}(M)} I^{\mathcal{J}}\Bigg)_{v} = \dim M_v < \infty.$$

Thus, $\bigoplus_{\mathcal{J}\in\mathcal{B}(M)} \overleftarrow{I^{\mathcal{J}}}$ is pfd. Since M is also pfd, it follows from Remark 2.17 that

$$\overleftarrow{M} \cong \overleftarrow{\Bigg(\bigoplus_{\mathcal{J}\in\mathcal{B}(M)} I^{\mathcal{J}}\Bigg)} \cong \bigoplus_{\mathcal{J}\in\mathcal{B}(M)} \overleftarrow{I^{\mathcal{J}}}. \tag{13}$$

In particular, $\overleftarrow{M}$ is pfd. □

Proposition 7.11 *If M is of type $\boldsymbol{c}^{[\,]}$, then $\overleftarrow{M}$ is a free $\mathbb{R}^{\mathrm{op}} \times \mathbb{R}$–indexed module and*

$$\mathcal{B}(\overleftarrow{M}) = \{(b,a)^{\lrcorner} \mid [a,b]_{\mathrm{BL}} \in \mathcal{B}(M)\}.$$

Proof This follows immediately from Lemma 7.9 and (13). □

Proposition 7.12 *If M and N are of type $\mathbf{c}^{[]}$ and $g\colon M \to N$ is a monomorphism with ϵ–trivial cokernel, then $\overleftarrow{g}\colon \overleftarrow{M} \to \overleftarrow{N}$ is a monomorphism with ϵ–trivial cokernel.*

Proof We need to show that for each $(s,t) \in \mathbb{R}^2$, $\overleftarrow{g}_{(s,t)}$ is an injection, and

$$\operatorname{im} \varphi_{\overleftarrow{N}}((s,t),(s-\epsilon,t+\epsilon)) \subset \operatorname{im} \overleftarrow{g}_{(s-\epsilon,t+\epsilon)}.$$

First, assume that $s \le t$. The universality of limits yields canonical isomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
M_{(s-\epsilon,t+\epsilon)} & \multicolumn{5}{c}{\xrightarrow{\qquad\qquad\qquad\cong\qquad\qquad\qquad}} & \overleftarrow{M}_{(s-\epsilon,t+\epsilon)} \\
M_{(s-\epsilon,t+\epsilon)} & \longleftarrow & M_{(s,t)} & \xrightarrow{\cong} & \overleftarrow{M}_{(s,t)} & \longrightarrow & \overleftarrow{M}_{(s-\epsilon,t+\epsilon)} \\
\big\downarrow g_{(s-\epsilon,t+\epsilon)} & & \big\downarrow g_{(s,t)} & & \big\downarrow \overleftarrow{g}_{(s,t)} & & \big\downarrow \overleftarrow{g}_{(s-\epsilon,t+\epsilon)} \\
N_{(s-\epsilon,t+\epsilon)} & \longleftarrow & N_{(s,t)} & \xrightarrow{\cong} & \overleftarrow{N}_{(s,t)} & \longrightarrow & \overleftarrow{N}_{(s-\epsilon,t+\epsilon)} \\
N_{(s-\epsilon,t+\epsilon)} & \multicolumn{5}{c}{\xrightarrow[\qquad\qquad\qquad\cong\qquad\qquad\qquad]{}} & \overleftarrow{N}_{(s-\epsilon,t+\epsilon)}
\end{array}$$

It follows that $\overleftarrow{g}_{(s,t)}$ and $\varphi_{\overleftarrow{N}}((s,t),(s-\epsilon,t+\epsilon))$ have the required properties.

Next we consider the case $s > t$. If $\overleftarrow{g}_{(s,t)}(m) = 0$ then

$$\overleftarrow{g}_{(t,s)} \circ \varphi_{\overleftarrow{M}}((s,t),(t,s))(m) = 0$$

by commutativity. By the case $s \le t$ considered above and Proposition 7.11, the two morphisms in the latter composition are injective, so $m = 0$. Hence, $\overleftarrow{g}_{(s,t)}$ is injective.

Let $n \in \overleftarrow{N}_{(s,t)}$ and observe that there exist $m_1 \in \overleftarrow{M}_{(t-\epsilon,t+\epsilon)}$ and $m_2 \in \overleftarrow{M}_{(s-\epsilon,s+\epsilon)}$ such that

$$\begin{aligned}
\overleftarrow{g}(m_1) &= \big(\varphi_{\overleftarrow{N}}((t,t),(t-\epsilon,t+\epsilon)) \circ \varphi_{\overleftarrow{N}}((s,t),(t,t))\big)(n), \\
\overleftarrow{g}(m_2) &= \big(\varphi_{\overleftarrow{N}}((s,s),(s-\epsilon,s+\epsilon)) \circ \varphi_{\overleftarrow{N}}((s,t),(s,s))\big)(n).
\end{aligned}$$

This is true because $\overleftarrow{g}$ has ϵ–trivial cokernel when restricted to indices (s',t') for which $s' \le t'$.

As $\overleftarrow{M}$ is free and $\overleftarrow{g}_{(t-\epsilon,s+\epsilon)}$ is an injection, there exists an element $m \in \overleftarrow{M}_{(s-\epsilon,t+\epsilon)}$ such that

$$\begin{aligned}
\varphi_{\overleftarrow{M}}((s-\epsilon,t+\epsilon),(t-\epsilon,t+\epsilon))(m) &= m_1, \\
\varphi_{\overleftarrow{M}}((s-\epsilon,t+\epsilon),(s-\epsilon,s+\epsilon))(m) &= m_2.
\end{aligned}$$

Hence, $\overleftarrow{g}(m) = \varphi_{\overleftarrow{N}}((s,t),(s-\epsilon,t+\epsilon))(n)$ by commutativity and the injectivity of the internal maps in $\overleftarrow{N}$. □

Completion of the proof of Theorem 7.6(iii) Propositions 7.11 and 7.12 assure that

$$\overleftarrow{f}^{[]}\colon \overleftarrow{M}^{[]} \to \overleftarrow{N}^{[]}$$

is a monomorphism of free $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed modules with ϵ–trivial cokernel. By Theorem 6.7(ii),

$$\chi(\overleftarrow{f}^{[]})\colon \mathcal{B}(\overleftarrow{M}^{[]}) \nrightarrow \mathcal{B}(\overleftarrow{N}^{[]})$$

is a bijective matching such that if

$$\chi(\overleftarrow{f}^{[]})((a,b)^{\lrcorner}) = (a',b')^{\lrcorner},$$

then

$$a \le a' \le a+\epsilon \quad \text{and} \quad b-\epsilon \le b' \le b.$$

By Proposition 7.11, $\chi(\overleftarrow{f}^{[]})$ induces a bijective matching $\chi(f^{[]})\colon \mathcal{B}(M^{[]}) \nrightarrow \mathcal{B}(N^{[]})$ such that if

$$\chi(f^{[]})[a,b]_{\mathrm{BL}} = [a',b']_{\mathrm{BL}},$$

then

$$a \le a' \le a+\epsilon \quad \text{and} \quad b-\epsilon \le b' \le b. \qquad \square$$

7.5 Proof of Theorem 7.6(iv)

To prove Theorem 7.6(iv), we apply the induced matching theorem for R_ϵ–free modules in a way analogous to the way we applied the induced matching theorem for free 2-D persistence modules in the proof of Theorem 7.6(iii). First, we define a functor X_ϵ sending each module of type $\boldsymbol{o}$ to an R_ϵ–free module.

Definition of X_ϵ Let us extend $(\mathbb{R}^{\mathrm{op}}\times\mathbb{R})\times\{0,1\}$ to a poset $\mathbb{D}$ with the same underlying set by adding an arrow $(v,0)\to(w,1)$ if and only if $v<w$. For $i\in\{0,1\}$, let

$$\iota_i\colon \mathbb{R}^{\mathrm{op}}\times\mathbb{R} \hookrightarrow \mathbb{D}$$

denote the obvious map sending $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}$ to $\mathbb{R}\times\mathbb{R}^{\mathrm{op}}\times\{i\}$. We define

$$\overrightarrow{(-)} := (-)|_{\iota_1} \circ \mathrm{Lan}_{\iota_0}(-).$$

Thus, for $(s,t)\in\mathbb{R}^2$ and $\overrightarrow{(s,t)}\subset\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$ given by

$$\overrightarrow{(s,t)} := \{(x,y) \mid s<x \text{ and } y<t\},$$

we have $\overrightarrow{M}_{(s,t)} = \varinjlim M|_{\overrightarrow{(s,t)}}$ for any $\mathbb{R}^{\mathrm{op}}\times\mathbb{R}$–indexed persistence module M; the internal maps of $\overrightarrow{M}$ are given by the universality of colimits.

Define

$$X_\epsilon := R_\epsilon \circ (-)^* \circ \overrightarrow{(-)},$$

where

$$(-)^*\colon \mathbf{Vec}^{\mathbb{R}^{\mathrm{op}}\times\mathbb{R}} \to \mathbf{Vec}^{\mathbb{R}\times\mathbb{R}^{\mathrm{op}}}$$

denotes the dualization functor of Section 2.4.

Properties of X_ϵ on modules of type o

Lemma 7.13 *For M of type $\boldsymbol{o}$, $\delta > 0$ and $(s,t) \in \mathbb{R}^2$, there are a finite number of blocks $(a,b)_{\mathrm{BL}} \in \mathcal{B}(M)$ such that $a \le s$, $b \ge t$ and $b-a \ge \delta$.*

Proof Let $\#(s,t)$ denote the number of blocks $(a,b)_{\mathrm{BL}}$ with the specified properties. It is easy to check that since $M_{(s,s)}$ is finite-dimensional for all $s \in \mathbb{R}$, each $\#(s,s)$ is finite. If $s<t$, then

$$\#(s,t) \le \#\big(\tfrac{1}{2}(s+t), \tfrac{1}{2}(s+t)\big) < \infty.$$

If $s>t$, then, choosing a positive integer l such that $\min(s,t)+l\delta > \max(s,t)$, we have

$$\#(s,t) \le \sum_{i=0}^{l} \#(\min(s,t)+i\delta, \min(s,t)+i\delta) < \infty. \qquad \square$$

Proposition 7.14 *If M is of type $\boldsymbol{o}$ and $\delta>0$, then $X_\delta(M)$ is R_δ–free and*

$$\mathcal{B}(X_\delta(M)) = \{(a,b)^{\triangledown}_\delta \mid (a,b)_{\mathrm{BL}} \in \mathcal{B}(M)_\delta\}.$$

Proof As illustrated in Figure 13, $X_\delta(I^{(a,b)_{\mathrm{BL}}}) = I^{(a,b)^{\triangledown}_\delta}$ for all $a<b\in\mathbb{R}$. By Proposition 2.16(i), $\mathrm{Lan}_{\iota_0}(-)$ preserves direct sums. Clearly, $(-)|_{\iota_1}$, $(-)^*$ and R_δ also preserve direct sums, so the composition X_δ preserves direct sums as well. Hence,

$$X_\delta(M) \cong \bigoplus_{(a,b)_{\mathrm{BL}}\in\mathcal{B}(M)_\delta} X_\delta(I^{(a,b)_{\mathrm{BL}}}) = \bigoplus_{(a,b)_{\mathrm{BL}}\in\mathcal{B}(M)_\delta} I^{(a,b)^{\triangledown}_\delta}.$$

Thus, $\mathcal{B}(X_\delta(M))$ is as claimed.

To see that $X_\delta(M)$ is R_δ–free, let

$$F := \bigoplus_{(a,b)_{\mathrm{BL}}\in\mathcal{B}(M)_\delta} I^{(a,b)^{\ulcorner}}.$$

F is pfd by Lemma 7.13 , so since $X_\delta(M) \cong R_\delta(F)$, the result follows. $\square$

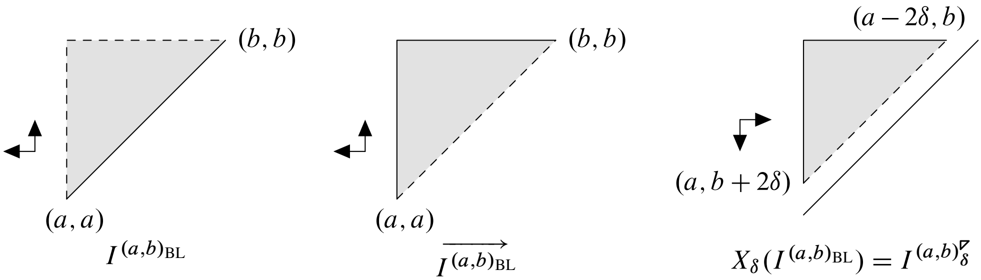


Figure 13: Applying $X_\delta(-)$ to $I^{(a,b)_{\mathrm{BL}}}$

Lemma 7.15 *Let M and N be of type $\boldsymbol{o}$, and let $f\colon M \to N$ be a morphism with ϵ–trivial cokernel. Then f is surjective at all indices (s,t) for which $t-s \geq 2\epsilon$.*

Proof Observe that $\varphi_N((s',t'),(s,t))$ is surjective whenever $t'-s' \geq 0$. If $n \in N_{(s,t)}$ is not in the image of f, then neither is any element in

$$\varphi_N\big(\big(\tfrac{1}{2}(s+t), \tfrac{1}{2}(s+t)\big),(s,t)\big)^{-1}(n) \neq \varnothing,$$

contradicting that f has ϵ–trivial cokernel. □

Proposition 7.16 *If M and N are of type $\boldsymbol{o}$ and $f\colon M \to N$ has ϵ–trivial kernel and cokernel, then $X_\epsilon(f)$ is a monomorphism with ϵ–trivial cokernel.*

Proof It follows from Lemma 7.13 that for any module Q of type $\boldsymbol{o}$ and $(s,t) \in \mathbb{R}^2$, there exists a $\eta > 0$ such that $\varphi_Q((s+\eta,t-\eta),(s+\eta',t-\eta'))$ is an isomorphism for all $0 < \eta' \leq \eta$. In particular, the natural map $Q_{(s+\eta,t-\eta)} \to \vec{Q}_{(s,t)}$ is an isomorphism. Applying this observation four times, we find that there exists $\eta > 0$ such that the leftmost and rightmost horizontal maps are isomorphisms in the following commutative diagram:

$$\begin{array}{ccccccc}
M_{(s-\epsilon+\eta,t-\epsilon-\eta)} & \xrightarrow{\cong} & \vec{M}_{(s-\epsilon,t+\epsilon)} & \longleftarrow & \vec{M}_{(s,t)} & \xleftarrow{\cong} & M_{(s+\eta,t-\eta)} \\
\downarrow{\scriptstyle f_{(s-\epsilon+\eta,t-\epsilon-\eta)}} & & \downarrow{\scriptstyle \vec{f}_{(s-\epsilon,t+\epsilon)}} & & \downarrow{\scriptstyle \vec{f}_{(s,t)}} & & \downarrow{\scriptstyle f_{(s+\eta,t-\eta)}} \\
N_{(s-\epsilon+\eta,t-\epsilon-\eta)} & \xrightarrow{\cong} & \vec{N}_{(s-\epsilon,t+\epsilon)} & \longleftarrow & \vec{N}_{(s,t)} & \xleftarrow{\cong} & N_{(s-\eta,t+\eta)}
\end{array}$$

This shows that $\vec{f}$ has ϵ–trivial kernel, and by Lemma 7.15, that $\vec{f}_{(s,t)}$ is surjective at all indices satisfying $t-s > 2\epsilon$. The result now follows from Proposition 2.8. □

Proof of Theorem 7.6(iv) Suppose M and N are of type $\boldsymbol{o}$ and $f\colon M \to N$ has ϵ–trivial kernel and cokernel. By Propositions 7.14 and 7.16,

$$X_\epsilon(f)\colon X_\epsilon(N) \to X_\epsilon(M)$$

is a monomorphism of R_ϵ–free persistence modules with ϵ–trivial cokernel. By Corollary 6.12 and Proposition 7.14, we obtain matchings

$$\mathcal{B}(M) \nrightarrow \mathcal{B}(X_\epsilon(M)) \nrightarrow \mathcal{B}(X_\epsilon(N)) \nrightarrow \mathcal{B}(N).$$

The composition of these is our desired matching. □

8 Stability of almost-block-decomposable modules

In this section, we present a simple extension of the block stability theorem to a slightly more general classes of modules, and discuss an application to the stability of (inter)level set persistent homology.

Recall our definition of a block from Section 3. We define an *almost-block* $\mathcal{J}$ to be an interval in $\mathbb{U}$ for which there exists a block $\mathcal{J}_{\mathrm{BL}}$ such that $d_I(I^{\mathcal{J}}, I^{\mathcal{J}_{\mathrm{BL}}}) = 0$. Some almost-blocks which are not blocks are shown in Figure 14. We say M is *almost-block-decomposable* if M is interval-decomposable, with each interval in $\mathcal{B}(M)$ an almost-block.

Corollary 8.1 (almost-block stability) *For pfd almost-block-decomposable modules M and N,*

$$d_I(M,N) \le d_b(\mathcal{B}(M),\mathcal{B}(N)) \le \tfrac{5}{2} d_I(M,N).$$

Sketch of proof For any $\delta > 0$, there exist pfd block-decomposable modules M' and N' with

$$d_I(M,M'),\ d_I(N,N'),\ d_b(M,M'),\ d_b(N,N') \le \delta.$$

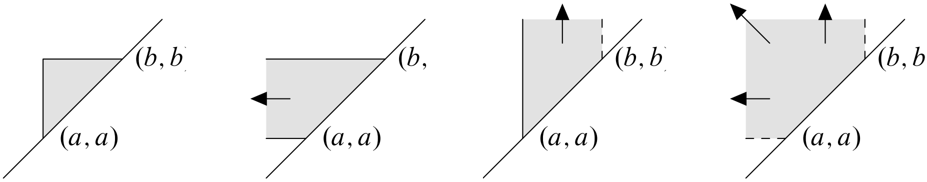


Figure 14: Four almost-blocks that are not blocks

Given this, the inequality $d_b(\mathcal{B}(M),\mathcal{B}(N)) \le \frac{5}{2} d_I(M,N)$ follows easily from Theorem 3.3 together with the triangle inequalities for d_I and d_b.

It follows from Proposition 2.13 that $d_I(M,N) \le d_b(\mathcal{B}(M),\mathcal{B}(N))$. □

Almost-block stability and interlevel set persistent homology Almost-block-decomposable persistence modules can arise as the interlevel set persistent homology of non-Morse functions, as the following example illustrates:

Example 8.2 The function $\gamma\colon (0,1) \to \mathbb{R}$ given by $\gamma(t) = t$ is not of Morse type. $H_0(\mathcal{S}(\gamma))$ is almost-block-decomposable but not block-decomposable; $\mathcal{B}(H_0(\mathcal{S}(\gamma)))$ consists of a single interval $\mathcal{J}$ with $d_b(\mathcal{J},[0,1]_{\mathrm{BL}}) = 0$.

In fact, we hypothesize that Theorem 4.7(i) generalizes as follows:

Conjecture 8.3 *For any topological space T and continuous function $\gamma\colon T \to \mathbb{R}$, if $H_i(\mathcal{S}(\gamma))$ is pfd then it is almost-block-decomposable.*

If Conjecture 8.3 is true, then the definition of level set barcodes of Section 4.2 extends to any $\mathbb{R}$–valued function with pfd interlevel set homology, and a stability result for the interlevel and level set barcodes of such functions follows immediately from Corollary 8.1.

Remark 8.4 In [14], Carlsson, de Silva, Kališnik and Morozov use the formalism of rectangle measures [20] to define level set barcodes of $\mathbb{R}$–valued functions in a general setting, and establish a stability result for these barcodes. Conjecture 8.3 is inspired by discussions with de Silva and Kališnik about that work.

Remark 8.5 Subsequent to the first iteration of this paper, Cochoy and Oudot [22] have established a structure theorem for a certain class of 2-D persistence modules which yields as corollaries two variants of Conjecture 8.3:

(i) Let $\mathcal{S}(\gamma)^\circ$ be the $\mathbb{U}$–indexed module given by $\mathcal{S}(\gamma)^\circ_{(a,b)} := \gamma^{-1}((a,b))$ if $a<b$, and $\mathcal{S}(\gamma)^\circ_{(a,b)} = 0$ otherwise. Then $H_i(\mathcal{S}(\gamma)^\circ)$ is almost-block-decomposable.

(ii) Let M be the $\mathbb{U}$–indexed module obtained from $H_i(\mathcal{S}(\gamma))$ by setting to 0 each vector space on the diagonal line $y=x$. Then M is almost-block-decomposable.

9 Discussion

Towards a general theory of algebraic stability In this paper, we have introduced an algebraic stability theorem for block-decomposable modules which, as an easy corollary, yields a stability result for zigzag modules. It is natural to ask whether our results generalize to an algebraic stability theorem for arbitrary interval-decomposable $\mathbb{R}^n$–indexed modules. In answer to this question, the following example shows that for interval-decomposable $\mathbb{R}^2$–indexed modules M and N, the ratio

$$\frac{d_b(\mathcal{B}(M), \mathcal{B}(N))}{d_I(M, N)}$$

can be arbitrarily large.

Example 9.1 For fixed $a \geq 0$, let $\mathcal{J}_1 \subset \mathbb{R}^2$ be the polygonal interval whose outer edge is specified by the sequence of vertices

$$(5, -a), \quad (9+a, -a), \quad (9+a, 4), \quad (6, 4), \quad (6, 6),$$
$$(4, 6), \quad (4, 9+a), \quad (-a, 9+a), \quad (-a, 5), \quad (5, 5).$$

Let $\mathcal{J}_2$ be the square interval with vertices $(6, 1-a)$, $(10+a, 1-a)$, $(10+a, 5)$, $(6, 5)$, and let $\mathcal{J}_3$ be the square with vertices $(1-a, 6)$, $(5, 6)$, $(5, 10+a)$, $(1-a, 10+a)$; see Figure 15. For $M = I^{\mathcal{J}_1}$ and $N = I^{\mathcal{J}_2} \oplus I^{J_3}$, we have

$$d_I(M, N) = 1, \quad d_b(\mathcal{B}(M), \mathcal{B}(M)) = 2 + \tfrac{1}{2}a.$$

Example 9.1 makes clear that to formulate a general algebraic stability result for interval-decomposable $\mathbb{R}^n$–indexed modules, we need either to constrain the shape of the intervals in our barcodes, or to work with a distance on barcodes other than the bottleneck distance.

Let us say an $\mathbb{R}^2$–indexed module M is *rectangle-decomposable* if M is interval-decomposable and $\mathcal{B}(M)$ is a collection of rectangles. A preliminary version of this paper [7] conjectured that the isometry theorem holds for interval-decomposable $\mathbb{R}^n$–indexed modules whose barcodes consist of convex intervals. However, Bjerkevik has subsequently given an example of rectangle-decomposable $\mathbb{R}^2$–indexed modules M and N with

$$d_b(\mathcal{B}(M), \mathcal{B}(N)) = 3 d_I(M, N),$$

disproving the conjecture [6]. We thus weaken the conjecture as follows:

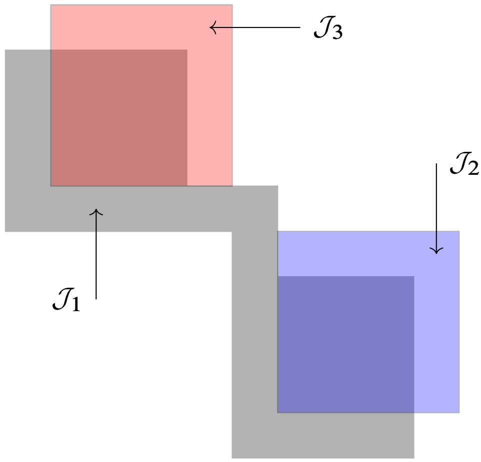


Figure 15: An illustration of Example 9.1 in the case $a = 0$

Conjecture 9.2 (generalized algebraic stability) *For each $n \in \{1, 2, \ldots\}$, there is a constant c_n such that for M and N interval-decomposable $\mathbb{R}^n$–indexed modules with each interval in $\mathcal{B}(M)$ and $\mathcal{B}(N)$ convex, we have*

$$d_b(\mathcal{B}(M), \mathcal{B}(N)) \leq c_n \, d_I(M, N).$$

Bjerkevik [6] provides positive answers to this conjecture in the case of free and rectangle-decomposable modules, using arguments similar to the one used there to strengthen the block stability theorem.

Single morphism algebraic stability We have proven the block stability theorem by way of an induced matching result for block-decomposable modules, Theorem 7.6. While Theorem 7.6(i)–(iii) are tight, Theorem 7.6(iv) (concerning modules of type $\boldsymbol{o}$) is not tight: A simple application of the tight form of the block stability theorem appearing in [6] gives that under the assumptions of Theorem 7.6(iv), there exists a 2–matching between the barcodes in question; this improves on the constant of $\frac{5}{2}$ appearing in Theorem 7.6(iv), albeit with matchings that are not explicitly given. On the other hand, for modules of type $\boldsymbol{o}$, the best lower bound we know for single morphism algebraic stability is $\frac{3}{2}$. The problem of establishing a tight single morphism algebraic stability result for block-decomposable modules thus remains open. The same problem is also of interest for more general interval-decomposable $\mathbb{R}^n$–indexed modules.

As with the proof of the induced matching theorem in 1-D given in [3], we have proven Theorem 7.6(iv) by factoring a morphism of block-decomposable persistence modules into morphisms with simpler structure, and then defining induced matchings for each of the factors. We wonder whether this strategy could be pushed further to yield stronger, more general single morphism stability results. The central difficulty is that the interpolating modules one obtains via our factorization are typically not interval-decomposable. In our study of block-decomposable modules, we have circumvented this issue by working with certain truncations of the interpolating modules which are interval-decomposable.

A potential alternative strategy would be to avoid truncation, and instead perturb our morphism $f\colon M \to N$ to obtain another morphism $f'\colon M \to N$ whose associated interpolants are interval-decomposable, while controlling the persistence of $\ker f'$ and $\operatorname{coker} f'$. It seems plausible that such an approach could yield stronger and more general results.

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