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Scl in free products

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We study stable commutator length (scl) in free products via surface maps into a wedge of spaces. We prove that scl is piecewise rational linear if it vanishes on each factor of the free product, generalizing a theorem of Danny Calegari. We further prove that the property of isometric embedding with respect to scl is preserved under taking free products. The method of proof gives a way to compute scl in free products which lets us generalize and derive in a new way several well-known formulas. Finally we show independently and in a new approach that scl in free products of cyclic groups behaves in a piecewise quasirational way when the word is fixed but the orders of factors vary, previously proved by Timothy Susse, settling a conjecture of Alden Walker.

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1 Introduction

Let G be a group and g be an element of the commutator subgroup $[G,G]$. The *commutator length* of g, denoted by $\mathrm{cl}(g)$, is the minimal number n such that $g = [a_1,b_1][a_2,b_2]\cdots[a_n,b_n]$ for some $a_i, b_i \in G$, and the *stable commutator length* of g, denoted by $\mathrm{scl}(g)$, is the limit $\lim_{n\to\infty} \mathrm{cl}(g^n)/n$, which always exists by subadditivity.

It is obvious from the definition that scl has the following basic properties:

(1) **Monotone** For any homomorphism $\phi\colon G \to H$ and $g \in [G,G]$, we have $\mathrm{scl}_G(g) \geq \mathrm{scl}_H(\phi(g))$.

(2) **Characteristic** For any $\phi \in \mathrm{Aut}(G)$ and $g \in [G,G]$, $\mathrm{scl}(g) = \mathrm{scl}(\phi(g))$.

It follows that the *spectrum*, the set of values that scl_G takes, is a group invariant. However, scl is notoriously difficult to compute unless it is known to vanish. Thus many interesting questions about the spectrum are extremely hard to answer.

1.1 Main results

Gromov [13] asked whether the spectrum is rational (or perhaps algebraic) when G is finitely presented. A counterexample was found by Zhuang [16]. On the other hand,

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Calegari [5] showed that scl is rational and can be computed efficiently in a free group by interpreting and studying scl in terms of surface maps. He later showed in [6] that a modification of the geometric argument proves rationality of scl in free products of abelian groups. We generalize this latter result, substantially weakening the assumption that the factors are abelian.

Theorem A (rationality) *For $G = *_\lambda G_\lambda$ with $\mathrm{scl}_{G_\lambda} \equiv 0$ for each λ, scl is piecewise rational linear in G.*

This holds, for example, when all G_λ are amenable. See Remark 4.10 for a list of groups having vanishing scl.

A homomorphism $\phi\colon G \to H$ for which $\mathrm{scl}_H(\phi(c)) = \mathrm{scl}_G(c)$ for all chains c (see Section 2) is said to be *isometric* for scl. Injections admitting a retraction are isometric. It is shown by Calegari and Walker [7] that random homomorphisms between free groups are isometric for scl. In this paper, we show that isometric embeddings (meaning injective and isometric) are preserved under taking free products:

Theorem B (isometric embedding) *If $f_\lambda\colon H_\lambda \to G_\lambda$ is a family of isometric embeddings, then so is the induced map $f\colon *_\lambda H_\lambda \to *_\lambda G_\lambda$.*

A spin-off of the techniques used in the proof is a new method to compute scl; we give examples in Section 5.

In particular, these techniques give new insights for scl in families. It was proved by Calegari and Walker [9] that for free products of free abelian groups, certain families of words $w(n)$ (called surgery families) are eventually quasirational in n. A similar question was studied by Walker. For any fixed rational chain c in F_n and any $\boldsymbol{o} = (o_1, o_2, \dots, o_n)$ with $o_i \geq 2$, let $c_{\boldsymbol{o}}$ be the image of w under the natural homomorphism $\phi\colon F_n \to *_i\, \mathbb{Z}/o_i\mathbb{Z}$. How does $\mathrm{scl}(c_{\boldsymbol{o}})$ vary as a function of $\boldsymbol{o}$?

It was observed experimentally by Walker [15] that $\mathrm{scl}(c_{\boldsymbol{o}})$ exhibits interesting periodic behavior, and he conjectured that the result is piecewise quasilinear in $1/o_i$ (see Conjecture 6.1). In Section 6 we give a counterexample, but prove a weaker version: $\mathrm{scl}(c_{\boldsymbol{o}})$ is piecewise quasirational in $\boldsymbol{o}$ (see Theorem 6.4). It was pointed out by Timothy Susse that he had proved this weaker version earlier in [14, Corollary 4.14] using a different approach.

It is worth mentioning that the method in this paper can be used to generalize and give a new approach to the spectral gap theorem by Duncan and Howie [12], which will be discussed in another paper [10].

Contents of the paper

We first give basic definitions in Section 2. Then in Section 3 we introduce a way, following [6], to use a finite-dimensional polyhedral cone to encode surface maps into a wedge of spaces with given boundary information. The encoding loses information, so in Section 4 we study a nonlinear optimization problem on the fibers. This reduces the computation of scl to a lattice point problem, which we solve, deducing Theorems A and B. When scl vanishes in each factor, the nonlinearity comes from *disk vectors*, which become complicated compared to the abelian case discussed in [6]. In Section 5, we apply our method to give generalizations and new proofs of old results, where we also prove a formula conjectured by Alden Walker [15]. Finally in Section 6 we give a counterexample to Walker's conjecture and prove a weaker version.

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2 Background

In this section we give the definitions and basic facts about scl that we will use. All of these can be found in [4].

Definition 2.1 Let S be a compact surface. Define

$$\chi^-(S)=\sum_i \min(0,\chi(S_i)),$$

where the S_i are the components of S and χ is the Euler characteristic. Equivalently, $\chi^-(S)$ is the Euler characteristic of S after removing disk and sphere components.

Definition 2.2 Let $g_j\in G$ for $1\le j\le k$ be elements that sum to 0 in $H_1(G;\mathbb{R})$. Let K be a $K(G,1)$. For all j, let $\gamma_j\colon S^1\to K$ be a loop representing the conjugacy class of g_j and $L = \bigsqcup_j S^1$. A compact oriented surface S together with a map $f\colon S \to K$ is called *admissible* of degree $n(S) \geq 1$ if the following diagram commutes:

$$\begin{tikzcd} \partial S \arrow[r, "i"] \arrow[d, "\partial f"'] & S \arrow[d, "f"] \\ L \arrow[r, "\bigsqcup \gamma_j"] & K \end{tikzcd}$$

where i is the inclusion map and $\partial f_*[\partial S] = n(S)[L]$.

Define

$$\mathrm{scl}(g_1 + g_2 + \cdots + g_k) = \inf_S \frac{-\chi^-(S)}{2n(S)}$$

over all admissible surfaces.

If $k = 1$, the geometric definition agrees with the algebraic one [4, Proposition 2.10]. We (informally) say a surface map is *efficient* if $-\chi^-(S)/2n(S)$ is close to $\mathrm{scl}\bigl(\sum g_i\bigr)$.

Remark 2.3 A priori the degrees on different components of ∂S could have opposite signs. Such an admissible surface can be replaced by another one that is at least as efficient as S, by taking suitable finite covers and gluing components with opposite orientations together. Thus one may restrict attention to *monotone* admissible surfaces, ie those where ∂f is orientation-preserving on each component [4, Proposition 2.13].

Recall the complex of real group chains $(C_*(G;\mathbb{R}), \partial)$ whose homology is $H_*(G;\mathbb{R})$, the real group homology of G. In the sequel, we write $B_1(G)$ for $B_1(G;\mathbb{R})$, the 1–boundaries. Scl is defined on integral 1–boundaries, and has a unique continuous linear extension to a pseudonorm on $B_1(G)$, which vanishes on

$$H(G) := \mathrm{span}_{\mathbb{R}}\langle ng - g^n, g - hgh^{-1}\rangle \leq B_1(G),$$

so scl descends to a pseudonorm on the quotient. See [4] for details.

Definition 2.4 Let $B_1^H(G) = B_1(G)/H(G)$. We say scl is *piecewise rational linear* if it is piecewise rational linear on every finite-dimensional rational subspace of $B_1^H(G)$. We say a group homomorphism $f\colon G_1 \to G_2$ is an *isometric embedding* if f is injective and the induced map $f\colon B_1^H(G_1) \to B_1^H(G_2)$ preserves scl, ie $\mathrm{scl}_{G_1}(c) = \mathrm{scl}_{G_2}(f(c))$ for all $c \in B_1^H(G_1)$.

The simplest isometric embeddings come from retracts.

Proposition 2.5 *If $i\colon H \to G$ and $r\colon G \to H$ are group homomorphisms such that $r \circ i = \mathrm{id}_H$, then i is an isometric embedding.*

This follows immediately from monotonicity of scl.

Remark 2.6 In particular, the calculation of scl in a free product of infinitely many groups reduces to computations in the free product of finitely many groups.

3 Encoding surface maps as vectors

In this section, we introduce the method from [6] to encode admissible surface maps into a wedge of spaces as vectors in a finite-dimensional rational polyhedron.

In the sequel, fix $G = A * B$ to be a free product of two groups A and B. Since every finite-dimensional rational subspace of $B_1^H(G)$ is a rational subspace of $\langle Z\rangle \cap B_1^H(G)$ for some finite subset Z of nontrivial conjugacy classes in G, we fix such a Z and study the restriction of scl to $\langle Z\rangle \cap B_1^H(G)$. We assume that there are no torsion elements in Z since $ng = g^n = 1$ in $B_1^H(G)$ if g is of order n.

Let K_A and K_B be a $K(A,1)$ and $K(B,1)$, respectively, and then $K = K_A \vee K_B$ is a $K(G,1)$ with wedge point $*$. By choosing appropriate loops to represent elements of Z, we get an oriented closed 1–manifold L (one component for each element of Z) together with a map $\Gamma\colon L \to K$ such that, for each component L_i,

(1) either $\Gamma(L_i)$ is disjoint from $*$ and thus contained entirely in K_A or K_B (referred to as *self-loops*);

(2) or $\Gamma^{-1}(*) \cap L_i$ cuts L_i into finitely many intervals, each mapped alternately to a based loop in one of K_A or K_B.

Therefore, $L \backslash \Gamma^{-1}(*)$ has finitely many components, each taken to a loop contained in one of K_A and K_B (see Figure 1). Let $T(A)$ and $T(B)$ be the set of components taken to K_A and K_B, respectively.

Now, for any surface $f\colon S \to K$ without sphere component and admissible for an integral class in $\langle Z\rangle \cap B_1^H(G)$, we may assume up to a homotopy that $\partial f\colon \partial S \to L$ is a (possibly disconnected) covering map, and assume f is transverse to $*$, ie $F := f^{-1}(*)$ is a finite disjoint union of embedded loops and proper arcs. We may also assume (by Remark 2.3) $\partial f\colon \partial S \to L$ is orientation-preserving on each component.

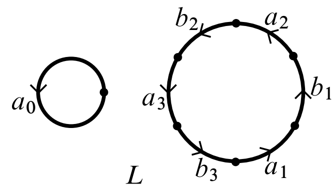

Figure 1: The 1–manifold L when $Z=\{a_0, a_1b_1a_2b_2a_3b_3\}$; the component on the left is a self-loop.

We can eliminate loops in F by repeating the following procedure: eliminate all null-homotopic loops in F by homotopy (innermost first), and then compress an essential loop in F. This procedure does not increase $-\chi^-(S)$, does not create sphere components and must terminate after finite repetitions since the number of loops in F decreases. Each proper arc in F is essential in S since ∂f is a covering. So, from now on, we assume that F consists of (essential) proper arcs.

Let S_A and S_B be $f^{-1}(K_A)$ and $f^{-1}(K_B)$, respectively; we focus on S_A in the rest of this section.

Now S_A is a surface that

1. possibly has corners,
2. has no sphere component, and
3. each component of ∂S_A either covers a self-loop mapped to K_A or can be decomposed into arcs alternating between those mapped to $*$ (components of F) and those mapped to elements in $T(A)$ (see Figure 2).

Let $\mathcal{S}_A$ be the set of surface maps to K_A satisfying the conditions above.

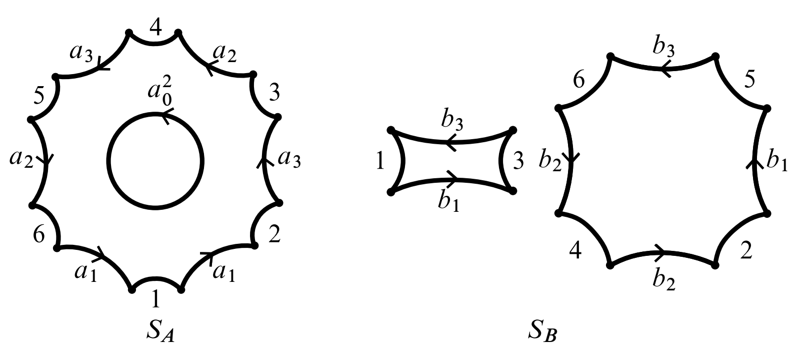

Figure 2: An example of S_A and S_B; components of F are labeled by numbers and arcs with the same label are identified after gluing.

Note that the corners of S_A are exactly $F \cap \partial S$, so the orbifold Euler characteristic of S_A is

$$\chi_o(S_A) := \chi(S_A) - \tfrac{1}{4}\,\#(\text{corners}) = \chi(S_A) - \tfrac{1}{2}\,\#(\text{components of } F).$$

Since S can be obtained by gluing S_A and S_B along F, we have

$$\chi(S) = \chi_o(S_A) + \chi_o(S_B).$$

Also note that each component of F with orientation induced from S_A goes from one element of $T(A)$ to another, and thus can be encoded as an ordered pair of these two elements of $T(A)$. Although elements of $T(A)$ corresponding to self-loops do not appear in this way, it is convenient to encode a component of ∂S_A that covers a self-loop τ with degree n as $n(\tau,\tau)$. Thus we define

$$T_2(A) = \{(\tau,\tau') \in T(A)^2 \mid \tau = \tau' \text{ if one of them corresponds to a self-loop}\}.$$

Let $C_1(A)$ and $C_2(A)$ be the $\mathbb{R}$–vector spaces with bases $T(A)$ and $T_2(A)$, respectively. Then we can encode the surface S_A as a vector $v(S_A)$ in $C_2(A)$ as follows: each component of F is encoded as an element of $T_2(A)$ described as above, each component of ∂S_A that covers some self-loop τ with degree n is encoded as $n(\tau,\tau)$, and $v(S_A)$ is defined to be the sum of these vectors in $C_2(A)$.

Obviously, $v(S_A)$ is a nonnegative integer vector in $C_2(A)$, and it satisfies two more linear constraints. Define a (rational) linear map $\partial\colon C_2(A) \to C_1(A)$ by $\partial(\tau,\tau') = \tau - \tau'$. Then $\partial \circ v(S_A) = 0$ since every boundary component of S_A closes up. Define $h\colon C_2(A) \to H_1(A) \otimes \mathbb{R}$ by $h(\tau,\tau') = \frac{1}{2}(\tau + \tau')$, where $H_1(A)$ is the abelianization of A. Then $h \circ v(S_A)$ is just the image of $[\partial S_A]$ in $H_1(A;\mathbb{R})$, which is 0 since it bounds S_A.

Definition 3.1 Let V_A be the convex rational polyhedral cone of nonnegative vectors $v \in C_2(A)$ satisfying $\partial(v) = 0$ and $h(v) = 0$.

The discussion above shows that $v(S_A)$ is an integer vector in V_A for any $S_A \in \mathcal{S}_A$. Conversely, for any integer vector $v \in V_A$, since $\partial(v) = 0$, the sum $\sum \frac{1}{2}(\tau+\tau')$ actually defines an integral homology class in $H_1(A;\mathbb{Z})$, whose image under $H_1(A;\mathbb{Z}) \to H_1(A;\mathbb{Z}) \otimes \mathbb{R} \cong H_1(A;\mathbb{R})$ is $h(v) = 0$. Hence there is a positive integer n such that the integral homology class given by nv is trivial and thus bounds some (actually many) surface(s) in $\mathcal{S}_A$. The same thing holds for rational vectors in V_A. We summarize this as a lemma for later use.

Lemma 3.2 (Calegari [6]) *The vector $v(S_A)$ is integral in V_A. Conversely, for any rational vector $v \in V_A$, there is an integer $n \geq 1$ such that $nv = v(S_A)$ for some $S_A \in \mathcal{S}_A$.*

Such an encoding reduces the huge space of admissible surfaces to a finite-dimensional space. However, this reduction comes at a cost. There are many different surfaces S_A encoded as the same $v(S_A)$. Thus we are led to the following optimization problem: given a rational vector v in V_A, what is the infimum of $-\chi_o(S_A)/n(S_A)$ over all surfaces S_A with $v(S_A) = n(S_A)v$ for some n? We address this in the next section.

4 Nonlinear optimization

Now we study the optimization problem discussed above. We follow [6], except that there are significant new issues because the factors are nonabelian. The key observation is Lemma 4.7.

Definition 4.1 For any rational vector $v \in V_A$, define

$$\chi_{o,A}(v) = \sup\left\{ \frac{\chi_o(S_A)}{n} \;\middle|\; v(S_A) = nv \text{ for some } n \in \mathbb{N},\ S_A \in \mathcal{S}_A \right\}.$$

As we saw in Section 3, $\chi_o(S_A) = \chi(S_A) - \frac{1}{2}\,\#(\text{components of } F)$. The number of components of F is a linear function $|v|$ in $v(S_A)$ defined as follows: on the basis, $|(\tau, \tau')|$ is 1 if $\tau \neq \tau'$, and is 0 if otherwise; then extend by linearity. Notice that $|v|$ is just the L^1 norm if there is no self-loop.

Therefore,

$$\chi_{o,A}(v) = -\tfrac{1}{2}|v| + \sup\left\{ \frac{\chi(S_A)}{n} \;\middle|\; v(S_A) = nv \text{ for some } n \in \mathbb{N} \right\}. \tag{4-1}$$

Note that the second term is quite similar to the definition of $-2\,\mathrm{scl}$, but S_A could have disk components and it could be admissible for different chains in $B_1^H(A)$. We first deal with disk components.

4.1 Disk vectors

Definition 4.2 We call $v \in V_A$ a *disk vector* if v encodes some disk. Denote the set of disk vectors by $\mathcal{D}_A$. For $v \in V_A$, we say $v = v' + \sum t_i d_i$ is an *admissible expression* if $v' \in V_A$, $t_i \geq 0$ and $d_i \in \mathcal{D}_A$. Define

$$\kappa_A(v) = \sup\left\{ \sum t_i \;\middle|\; v = v' + \sum t_i d_i \text{ is an admissible expression} \right\}.$$

Roughly speaking, $\kappa_A(v)$ is the maximal "number" of disk vectors that can be subtracted from v. In the case where scl vanishes on $B_1^H(A)$, we have the following lemma:

Lemma 4.3 *If* scl *vanishes on* $B_1^H(A)$*, then* $\chi_{o,A}(v) = -\frac{1}{2}|v| + \kappa_A(v)$ *for any rational vector* $v \in V_A$*.*

Proof This is equivalent to showing that

$$\sup\left\{\frac{\chi(S_A)}{n} \;\middle|\; v(S_A) = nv \text{ for some } n \in \mathbb{N}\right\} = \kappa_A(v)$$

by (4-1). Suppose $v(S_A) = nv$ for some $n \in \mathbb{N}$, and let $D_1, \dots, D_k$ be the disk components of S_A and $S_A = S'_A \sqcup \left(\bigsqcup D_i\right)$. Then $\chi(S_A) = \chi^-(S_A) + k \le k$ and $v = v(S'_A)/n + \sum v(D_i)/n$ is an admissible expression. Then $\kappa_A(v) \ge k/n$ and thus $\chi(S_A)/n \le \kappa_A(v)$. This proves the "$\le$" direction.

Conversely, for any given $\epsilon > 0$, there is an admissible expression $v = v' + \sum t_i d_i$, where $\left|\kappa_A(v) - \sum t_i\right| < \epsilon$. We may assume that each t_i is rational, and then v' is also rational since v is. Hence there is an integer $n \ge 1$ such that each nt_i is an integer and $nv' = v(S'_A)$ for some S'_A by Lemma 3.2. Now $\partial S'_A$ defines a chain c in $B_1(A)$ where scl vanishes. Thus we can find some S''_A such that $\partial S''_A = Nc$, $v(S''_A) = Nv(S'_A) = Nnv'$ and $|-\chi^-(S''_A)/N| < \epsilon$. Also find disks D_i such that $v(D_i) = d_i$, and take Nnt_i copies of D_i for each i. Finally take S_A to be the disjoint union of all these disks and S''_A. Then $v(S_A) = Nnv' + Nn\sum t_i d_i = Nnv$ and $\chi(S_A)/Nn = \chi(S''_A)/Nn + \sum t_i \ge \chi^-(S''_A)/Nn + (\kappa_A(v) - \epsilon) \ge -\epsilon/n + \kappa_A(v) - \epsilon$. Since ϵ is arbitrary, this proves the other direction. $\square$

This motivates the study of $\kappa_A(v)$ since $|v|$ is already linear on V_A.

Recall the following standard notions that we will use. In a vector space X, the convex hull of a subset E, denoted by $\mathrm{conv}(E)$, is the smallest convex set containing E. The Minkowski sum of two subsets E and F is the set

$$E + F := \{x + y \mid x \in E \text{ and } y \in F\}.$$

A function f defined on a convex subset $E \subset X$ is concave if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for any $\lambda \in [0,1]$ and any $x, y \in E$. Finally, a function f defined on a cone $C \subset X$ centered at the origin is homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda \ge 0$ and $x \in C$.

The following lemma is the same as [6, Lemma 3.10]. The proof is standard, so we omit it.

Lemma 4.4 *The function κ_A is a nonnegative concave homogeneous function on V_A. The subset of V_A on which $\kappa_A = 1$ is the boundary of* $\mathrm{conv}(\mathcal{D}_A) + V_A$ *in* V_A.

4.2 Key observation

Now we come to the key observation that makes it possible to generalize the result in [6] to our rationality theorem.

In [6], essentially using that A is free abelian, $\mathcal{D}_A$ is determined explicitly as integer points lying in some open faces of V_A. Then $\mathrm{conv}(\mathcal{D}_A) + V_A$, which is a subset of V_A, is shown to be a finite-sided rational convex polyhedron using such an explicit description of $\mathcal{D}_A$, which also produces an effective algorithm to compute scl in that case. However, the set $\mathcal{D}_A$ of disk vectors could be very complicated and hard to determine explicitly in general.

The following example illustrates how complicated $\mathcal{D}_A$ could be, even when A is the simplest nonabelian group. The study of this example was initiated in an unpublished note by Timothy Susse, who did computer experiments and gave conjectural pictures of the result.

Example 4.5 Let $A = \mathcal{H}_3(\mathbb{Z}) = \langle x, y, z \mid z = [x, y],\ [x, z] = [y, z] = 1\rangle$ be the 3–dimensional Heisenberg group, which is 2–step nilpotent, and $[A, A] = \langle z\rangle$. Suppose $T(A) = \{a, b, c\}$ for some $a, b, c \in A\backslash\{\mathrm{id}\}$ such that $abc = z^m$ for some $m \in \mathbb{Z}$, which occurs if we consider $g = a\alpha b\beta c\gamma \in [G, G]$ with $G = A * B$ and α, β and γ in some group B.

Let us look at a 2–dimensional subcone of V_A spanned by $P = (a, b) + (b, c) + (c, a)$ and $N = (a, c) + (b, a) + (c, b)$, and find all $(u, v) \in \mathbb{Z}^2_+$ such that $uP + vN$ is a disk vector. For fixed (u, v), the vector $uP + vN$ is a disk vector if and only if there is some cyclic word w in a, b and c such that

(1) w represents id in A;

(2) w contains u copies of each of ab, bc, ca and v copies of ac, cb and ba as subwords.

Notice that since $abc = z^m$, the element a commutes with bc and cb, and similarly for b and c; we also have $[a, b] = [b, c] = [c, a] = z^n$ for some n. Any cyclic word with equal number (say, k) of a, b, c in it can be written uniquely as $(abc)^k[a, b]^r$ for some $r \in \mathbb{Z}$ by moving letters around. One can prove by induction (see the appendix) that for fixed (u, v), and the set of cyclic words satisfying the restriction (2) above, the

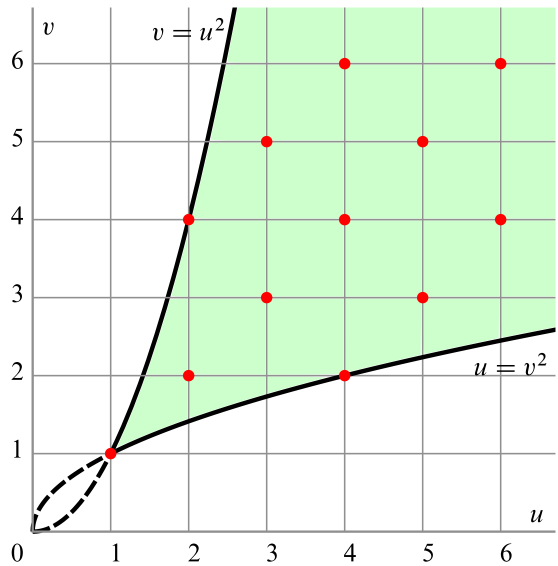

Figure 3: Disk vectors in a subcone of V_A with $A = \mathcal{H}_3(\mathbb{Z})$

set of r that can appear is

$$S_{u,v} = \begin{cases} \left[-\frac{1}{2}v(v+1),\, \frac{1}{2}v(v-3)\right] \cap \mathbb{Z} & \text{if } u = v, \\ \left[-\frac{1}{2}v(v+1),\, \frac{1}{2}v(v-1)\right] \cap \mathbb{Z} & \text{if } u > v, \\ \left[-\frac{1}{2}u(u+1) - v,\, \frac{1}{2}u(u-1) - v\right] \cap \mathbb{Z} & \text{if } u < v. \end{cases} \tag{4-2}$$

Therefore, $uP + vN$ is a disk vector if and only if $(abc)^{u+v}[a,b]^r = \mathrm{id}$ for some $r \in S_{u,v}$, or equivalently $m(u+v) + nr = 0$ has a solution for $r \in S_{u,v}$. For example, if $\frac{m}{n} = \frac{1}{2}$, the set of (u,v) for which $uP + vN$ is a disk vector is

$$\{(u,v) \in \mathbb{Z}_+^2 \mid 1 \le v \le u \le v^2 \text{ or } u < v \le u^2, \text{ and } u \equiv v \bmod 2\},$$

which is the set of integer points in the shaded region in Figure 3, bounded by two parabolas, such that the two coordinates have the same parity.

Nevertheless, Corollary 4.9, a consequence of our key lemma, Lemma 4.7, shows that $\mathrm{conv}(\mathcal{D}_A) + V_A$ is always a *finite-sided* rational convex cone no matter how complicated $\mathcal{D}_A$ is. The key reason is that an integer point in a rational cone cannot be too close to a given face unless it lies on that face.

We first recall some standard definitions.

Definition 4.6 A convex *polyhedral cone* in $\mathbb{R}^n$ is a set $C = \{x \mid f_i(x) \ge 0 \text{ for all } i \in I\}$ where each $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is a linear map and I is *finite*. In addition, C is *rational* if the f_i can be chosen to be rational. We say C is *simplicial* if the f_i can be chosen to be linearly independent in $(\mathbb{R}^n)^*$, or, equivalently, C is the convex cone spanned by some linearly independent vectors.

Similarly, a convex *polyhedron* in $\mathbb{R}^n$ is a set $P = \{x \mid f_i(x) \geq \alpha_i \text{ for all } i \in I\}$ where each $f_i \colon \mathbb{R}^n \to \mathbb{R}$ is a linear map, each α_i is real and I is *finite*. In addition, P is *rational* if the f_i and α_i can be chosen to be rational.

It follows that if C is simplicial and rational, then C is the convex cone spanned by some linearly independent rational vectors. Here is the key observation.

Lemma 4.7 *Let C be a rational polyhedral cone in $\mathbb{R}^n$ and let D be a subset of $C \cap \left(\frac{1}{L} \cdot \mathbb{Z}\right)^n$ for some $L \in \mathbb{Z}_+$. Then there is a finite subset D' of D such that $D + C = D' + C$.*

Proof The claim is trivially true if D is empty. From now on we assume D to be nonempty. We first reduce the problem to the case where C is simplicial. Decompose $C = \bigcup C_i$ as the union of finitely many simplicial rational cones C_i for $i = 1, \dots, k$. Suppose the claim is true for simplicial rational cones. Letting $D_i = D \cap C_i$ and applying the claim to each pair (C_i, D_i), we get some finite sets D_i' such that $D_i' + C_i = D_i + C_i$. Now let $D' = \bigcup D_i'$. It suffices to show that $D + C \subset D' + C$. Actually, for each point $d + c \in D + C$ where $d \in D$ and $c \in C$, we have $d \in D_i$ for some i since $D = \bigcup D_i$. Now $D_i \subset D_i + C_i = D_i' + C_i$, so there exists $d' \in D_i' \subset D'$ and $c' \in C_i \subset C$ such that $d = d' + c'$, and thus $d + c = d' + (c' + c)$ lies in $D' + C$.

Therefore, we only need to show the claim for any simplicial rational cone C, which can be further reduced as follows to the case where C is the first orthant of $\mathbb{R}^n$. Let c_i for $i = 1, \dots, k$ be the linearly independent rational vectors that span C. Extend this to a rational basis of $\mathbb{R}^n$ and take a linear transformation f of $\mathbb{R}^n$ by sending c_i to e_i, where $\{e_i\}_{i=1}^n$ is the standard basis. In terms of matrices (with respect to the basis e_i), f is an $n \times n$ matrix with rational entries. Let N be the lcm of the denominators of the entries. Then the image of $\left(\frac{1}{L} \cdot \mathbb{Z}\right)^n$ under f lies in $\left(\frac{1}{LN} \cdot \mathbb{Z}\right)^n$, so $f(D)$ is a subset of $\left(\frac{1}{LN} \cdot \mathbb{Z}\right)^n$.

Thus we only need to show the claim for

$$C = \{x = (x_1, \dots, x_n) \mid x_i \geq 0 \text{ for } i = 1, \dots, k,\ x_i = 0 \text{ for } i > k\}.$$

Up to applying the map $v \mapsto Lv$ (our statement is irrelevant to the scale), we assume without loss of generality that $L = 1$ in the sequel, ie D lies in the integer lattice. Now we may ignore e_i for $i > k$. Thus we assume without loss of generality that C is

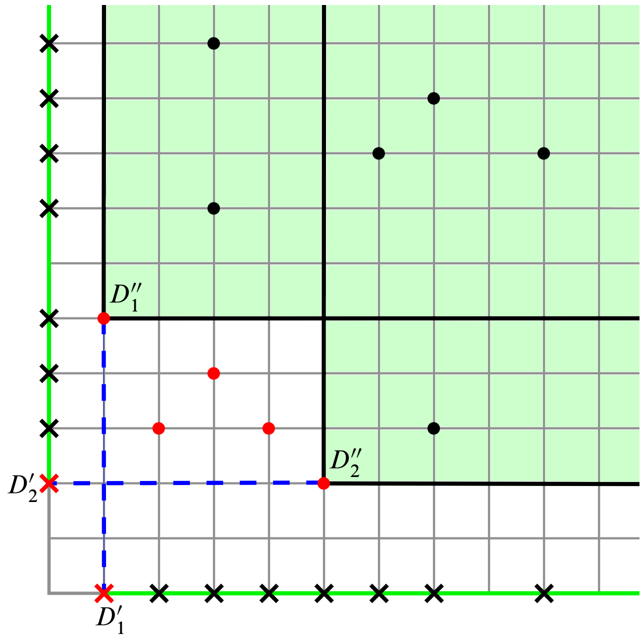

Figure 4: An illustration of our induction argument in the case $n = 2$ and $C = \mathbb{R}^2_{\geq 0}$. The dots are points in D and the crosses on the axes are D_1 and D_2; the red crosses are D'_1 and D'_2, whose lifts are D''_1 and D''_2. The shaded region $D'' + C$ contains the majority (black dots) of D, so we can take D' to be the red dots.

the first orthant of $\mathbb{R}^n$ and proceed by induction on the dimension n (note that n is actually the dimension of C, not of the underlying space we started with). See Figure 4 for an illustration of our induction in a special case.

The base case $n = 1$ is obvious. For the inductive step, fix any $i \in \{1, \dots, n\}$, and let $F_i = \{x \in C \mid x_i = 0\}$ be the i^{th} face of C and p_i be the projection from C to F_i. Let $D_i = p_i(D)$, which lies in the integer lattice and F_i. Thus, by the induction hypothesis (applied to (F_i, D_i)), there is a finite set $D'_i \subset D_i$ such that $D'_i + F_i = D_i + F_i$. For each $x' \in D'_i$, choose some $x'' \in D$ such that $p_i(x'') = x'$. Hence there is a finite set $D''_i \subset D$ which projects to D'_i under p_i. Note the following simple but crucial fact: for any $x, y \in C$, if $y_i \geq x_i$, then y lies in $x + C$ if and only if $p_i(y) \in p_i(x) + F_i$. Thus, if we take $M_i = \max\{x_i \mid x \in D''_i\}$, then for any point y with $y_i \geq M_i$, we have $y \in D''_i + C$ if and only if $p_i(y) \in D'_i + F_i = D_i + F_i$. Therefore, if $y_i \geq M_i$ and $y \in D$, then $p_i(y) \in D_i \subset D_i + F_i$, which means $y \in D''_i + C$. In other words, if $y \in D \backslash (D''_i + C)$, then $0 \leq y_i < M_i$.

Now let i range from 1 to n and take $D'' = \bigcup D_i''$. By what we showed above, if $y \in D \backslash (D''+C) = D \backslash \big[\bigcup (D_i''+C)\big]$, then $0 \le y_i < M_i$ for each i. Hence, $D \backslash (D''+C)$ is a bounded subset of $\mathbb{Z}^n$ and therefore finite. Take $D' = D'' \cup [D \backslash (D''+C)]$, and the claim follows. □

Corollary 4.8 *Let C and D be as in Lemma 4.7. Then* $\mathrm{conv}(C+D) = \mathrm{conv}(D)+C$ *is a (closed) rational polyhedron.*

Proof We have $\mathrm{conv}(C+D) = \mathrm{conv}(D)+C$ since the Minkowski sum commutes with taking convex hull and C is convex. The same consideration and Lemma 4.7 implies $\mathrm{conv}(C+D) = \mathrm{conv}(C+D') = \mathrm{conv}(D')+C$. Now $\mathrm{conv}(D')+C$ is a rational polyhedron since it is the Minkowski sum of two rational polyhedra (see for example the proof of [1, Theorem 3.5]). □

Applying this to $C = V_A$ and $D = \mathcal{D}_A$, we get the following corollary:

Corollary 4.9 *For any $G = A * B$ and Z as in the setup in Section 3, the set* $\mathrm{conv}(\mathcal{D}_A)+V_A$ *is a (closed) rational polyhedron. Thus κ_A is the minimum of finitely many rational linear functions. If* scl *vanishes on $B_1^H(A)$, then $\chi_{o,A}$* (originally defined on rational vectors in V_A) *has a (unique) continuous extension $\kappa_A - \frac{1}{2}|\cdot|$, which is the minimum of finitely many rational linear functions.*

Proof The first assertion follows immediately from Corollary 4.8. Let $\{f_i \mid i \in I\}$ be a finite subset of rational linear functions defining the rational polyhedron $\mathrm{conv}(\mathcal{D}_A)+V_A$ such that each $\{f_i = 1\}$ contains some top-dimensional face of the boundary of $\mathrm{conv}(\mathcal{D}_A)+V_A$ in V_A. Combining with Lemma 4.4, we have $\kappa_A(x) = \min_i\{f_i(x)\}$. Combining with Lemma 4.3, we get the last assertion. □

4.3 Rationality theorem

Corollary 4.9 generalizes [6, Lemma 3.12] by weakening the assumption "(free) abelian" to "scl vanishes". Now, following the argument in [6], we get our first main result:

Theorem A (rationality) *If G_λ ($\lambda \in \Lambda$) is a family of groups where* scl *vanishes on each G_λ, then* scl *is piecewise rational linear on the free product $G = *_\lambda G_\lambda$.*

Given Corollary 4.9, the proof is the same as that in [6]. We include it for completeness.

Proof We first focus on the case of $G = A * B$.

As in Section 3, fix a finite subset Z of nontrivial conjugacy classes in G and define the 1–manifold L. Also let $T_2(A)$, V_A and $\chi_{o,A}$ be as above, and similarly define these for B. Let $Y \subset V_A \times V_B$ be the set of pairs (v_A, v_B) that can be "glued up": for any $(\tau, \tau') \in T_2(A)$ with τ not a self-loop (then neither is τ'), there is a unique $(\sigma, \sigma') \in T_2(B)$ such that σ' is the oriented arc in the 1–manifold L following τ, and τ' follows σ; we require the (τ, τ')–coordinate of v_A to equal the (σ, σ')–coordinate of v_B for any such (τ, τ') and (σ, σ'). Then Y is still a rational cone. Define $\chi_o(v_A, v_B) = \chi_{o,A}(v_A) + \chi_{o,B}(v_B)$ for any $(v_A, v_B) \in Y$. Then, by Corollary 4.9, χ_o is the minimum of finitely many rational linear functions. Finally define $d\colon Y \to H_1(L)$ to be the unique rational linear map such that $d(y) = \partial f_*(\partial S)$ in $H_1(L)$ whenever $y = (v(S_A), v(S_B)) \in Y$ for some surface $S = S_A + S_B$.

Now, for any $l \in H_1(L)$ that corresponds to a chain $z \in \langle Z \rangle \cap B_1^H(G)$, let $Y_l = d^{-1}(l) \subset Y$. Then we have

$$\operatorname{scl}(z) = -\max_{y \in Y_l} \tfrac{1}{2} \chi_o(y). \tag{4-3}$$

Notice that Y_l is a finite-sided convex polyhedron since Y is, and it is rational if l is. Now $\chi_o(y) = \min f_i(y)$ for some finite collection of rational linear functions f_i. Then maximizing $\chi_o(y)$ for $y \in Y_l$ can be solved by introducing a slack variable z and maximizing z subject to the rational linear constraints $z \le f_i(y)$ and $y \in Y_l$. This is a linear programming problem in (y, z). Then it follows that scl is piecewise rational linear on $\langle Z \rangle \cap B_1^H(G)$. Since Z is arbitrary, the conclusion follows.

For the general case, since every rational subspace only involves finitely many factors, it suffices to show that the conclusion holds when Λ is finite according to Remark 2.6. Now if Λ is finite, we can build $K(G,1)$ by gluing up the $K(G_\lambda, 1)$ so that no three factors are attached at the same point. This guarantees that the transversality argument we used to cut the surfaces still applies and that the surfaces S_{G_λ} are glued up in a simple way. Then define $T_2(G_\lambda)$, V_{G_λ} and χ_{o,G_λ} as before. Similarly define Y by writing down the suitable gluing condition. Then the same argument above shows that scl is piecewise rational linear. □

Remark 4.10 Many groups have vanishing scl. There are three main sources:

(1) Small groups such as amenable groups, which include finite groups and solvable groups.

(2) Irreducible lattices of higher rank Lie groups (see [4, Theorem 5.26] for a precise statement).

(3) Some transformation groups such as $\mathrm{Homeo}^+(S^1)$ [4, Theorem 2.43], subgroups of $\mathrm{PL}^+(I)$ [3, Theorem A], $\mathrm{Homeo}_c(\mathbb{R}^n)$ and Thompson–Stein groups $T_{p,q}$ with $\gcd(p-1,q-1)=1$ (see [16, Lemma 3.6] or [4, Lemma 5.15]).

Remark 4.11 The proof actually gives a method to determine scl in free products when scl vanishes on each factor. It produces an algorithm as long as one can determine the vertices of the convex cone $\mathrm{conv}(V_{A_\lambda}+\mathcal{D}_{A_\lambda})$, which seems hard in general since it requires some knowledge of $\mathcal{D}_{A_\lambda}$. The method, however, is still helpful to study scl in families.

Remark 4.12 When considering Y_l, it is redundant in the following sense to impose $h=0$ in the definition of V_A (see Definition 3.1). If we define V'_A to be nonnegative vectors $v\in C_2(A)$ satisfying $\partial(v)=0$, then V_A is the subpolyhedral cone of V'_A on which $h=0$. We can similarly define Y and the linear map d using V'_A instead of V_A, and denote them by Y' and d'. It turns out that if $l\in H_1(L)$ corresponds to a homologically trivial chain in $B_1(G)$, then $d'^{-1}(l)\subset Y'$ coincides with Y_l.

Corollary 4.13 *Let $f_\lambda\colon A_\lambda\to B_\lambda$ be a family of injective group homomorphisms. If* scl *vanishes on each A_λ and B_λ, then the induced map $f\colon *_\lambda A_\lambda\to *_\lambda B_\lambda$ is an isometric embedding with respect to* scl. *More precisely, for any $c\in B_1^H(*_\lambda A_\lambda)$, we have* $\mathrm{scl}(c)=\mathrm{scl}(f(c))$.

Proof Again this reduces to the case of finitely many factors. Now run the process above on both sides using V'_{A_λ} and V'_{B_λ} as in Remark 4.12 instead of V_{A_λ} and V_{B_λ}. Then f_λ induces a bijection between $T(A_\lambda)$ and $T(B_\lambda)$ and similarly between $T_2(A_\lambda)$ and $T_2(B_\lambda)$. These give rise to an isomorphism $(f_\lambda)_*$ between $C_2(A_\lambda)$ and $C_2(B_\lambda)$ that takes V'_{A_λ} to V'_{B_λ} isomorphically. The map $(f_\lambda)_*$ may not restrict to an isomorphism between V_{A_λ} and V_{B_λ} when the induced map of f_λ on group homology is not injective. Injectivity of f_λ ensures that $(f_\lambda)_*$ restricts to a bijection between $\mathcal{D}_{A_\lambda}$ and $\mathcal{D}_{B_\lambda}$, and thus κ_{A_λ} is the pullback of κ_{B_λ} by $(f_\lambda)_*$. Then the computation of scl on two sides are results of the same linear programming problem, so f is isometric for scl. □

The condition that scl vanishes on each A_λ and B_λ ensures that each f_λ is isometric. Thus it is natural to ask whether it is enough to get the conclusion only assuming each f_λ to be an isometric embedding. Our second main result confirms this. To prove it, we will reveal how scl in free products is determined when scl does not necessarily vanish on each factor.

4.4 Scl in general free products

Now we return to the general case. Similar to the special case discussed above, we need an analog of Lemma 4.3 to reveal the structure of $\chi_{o,A}(v)$. Unlike the case where scl vanishes on factors, the second term on the right-hand side of (4-1) cannot be computed via κ_A as in (4-1) any more. With notations as before and $\mathcal{D}_A$ defined as in Definition 4.2, we make the following definition:

Definition 4.14 For any rational $v \in V_A$, define

$$\operatorname{pscl}_A(v) := \inf\left\{ \frac{-\chi^-(S_A)}{2n} \;\middle|\; v(S_A) = nv \text{ for some } n \in \mathbb{N} \right\}.$$

Let $\operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A) := \left\{\sum t_i d_i \mid t_i \in \mathbb{Q},\ t_i \ge 0,\ d_i \in \mathcal{D}_A\right\}$. Equivalently, $\operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A)$ is the set of rational points in $\operatorname{cone}(\mathcal{D}_A)$. We use the convention that $\operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A) = \{0\}$ if $\mathcal{D}_A = \varnothing$. For any $x \in \operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A)$, define

$$\eta_A(x) := \sup\left\{ \sum t_i \;\middle|\; x = \sum t_i d_i \text{ with } t_i \in \mathbb{Q},\ t_i \ge 0,\ d_i \in \mathcal{D}_A \right\}.$$

Lemma 4.15 *For any rational $v \in V_A$, we have*

$$\begin{aligned}\chi_{o,A}(v) &= -\tfrac{1}{2}|v| + \sup\{-2\operatorname{pscl}_A(v-d) + \eta_A(d) \mid d \in \operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A),\ v-d \in V_A\}\\ &\le -\tfrac{1}{2}|v| + \kappa_A(v).\end{aligned}$$

Proof We first prove the equality in a similar way as we did for Lemma 4.3. Let

$$\begin{aligned}L &= \sup\left\{ \frac{\chi(S_A)}{n} \;\middle|\; v(S_A) = nv \text{ for some } n \in \mathbb{N} \right\},\\ R &= \sup\{-2\operatorname{pscl}_A(v-d) + \eta_A(d) \mid d \in \operatorname{cone}_{\mathbb{Q}}(\mathcal{D}_A), v-d \in V_A\}.\end{aligned}$$

By (4-1), we just need to show $L = R$.

On the one hand, if $v(S_A) = nv$, let $D_1, \dots, D_k$ be the disk components of S_A and $S_A = S'_A \sqcup \left(\bigsqcup D_i\right)$. Then

$$\chi(S_A) = \chi^-(S'_A) + k \le -2n\operatorname{pscl}_A(v-d) + k \le -2n\operatorname{pscl}_A(v-d) + n\eta_A(d),$$

where $d = \sum v(D_i)/n$. This proves $L \le R$.

On the other hand, for any given $\epsilon > 0$, there exists $d \in \mathrm{cone}_{\mathbb{Q}}(\mathcal{D}_A)$ such that $v' = v - d \in V_A$ and $-2\,\mathrm{pscl}_A(v') + \eta_A(d) > R - \epsilon$. Then we can write $d = \sum t_i d_i$ with t_i nonnegative rational and $d_i \in \mathcal{D}_A$ such that $\sum t_i > \eta_A(d) - \epsilon$. There also exist an integer $n \geq 1$ and a surface S'_A such that $v(S'_A) = nv'$ and $\frac{1}{2n}\chi^-(S'_A) > -\mathrm{pscl}_A(v') - \epsilon$. Up to replacing S'_A by a bunch of copies of itself, we may assume that each nt_i is an integer. Now take disks D_i such that $v(D_i) = d_i$. Take nt_i copies of D_i for each i and let S_A be the disjoint union of these disks together with S'_A. Then $v(S_A) = nv' + n\sum t_i d_i = nv$ and

$$\frac{\chi(S_A)}{n} = \frac{\chi(S'_A)}{n} + \sum t_i \geq \frac{\chi^-(S'_A)}{n} + \sum t_i > -2\,\mathrm{pscl}_A(v') + \eta_A(d) - 3\epsilon > R - 4\epsilon.$$

This shows $L \geq R$ and finishes the proof of the equality part.

To show the inequality, suppose $d \in \mathrm{cone}_{\mathbb{Q}}(\mathcal{D}_A)$ and $v - d \in V_A$. For any $\epsilon > 0$, we can write $d = \sum t_i d_i$ with $\eta_A(d) < \sum t_i + \epsilon$. Thus the admissible expression $v = (v - d) + \sum t_i d_i$ shows

$$\kappa_A(v) \geq \sum t_i > \eta_A(d) - \epsilon \geq -2\,\mathrm{pscl}_A(v - d) + \eta_A(d) - \epsilon.$$

Since ϵ is arbitrary, the desired inequality holds. □

Now we can describe how scl is determined in general free products. Let $G = *_i G_i$ be the free product of finitely many groups. Consider a finite set Z of conjugacy classes (without torsion elements) in G and build the 1–manifold L as before. Define V_{G_i}, $\mathcal{D}_{G_i}$, η_{G_i} and pscl_{G_i} as above. Then define Y, as in the proof of Theorem A, to be the rational polyhedron in $\prod V_{G_i}$ consisting of tuples of vectors from V_{G_i} that can be "glued up", and define χ_o (only for rational points in Y) to be the sum of χ_{o,G_i} evaluated on the i^{th} coordinate. Now, for any rational $l \in H_1(L)$ that corresponds to a rational chain $z \in \langle Z \rangle \cap B_1^H(G)$, let $Y_l = d^{-1}(l) \subset Y$.

Lemma 4.16 *With notations as above,*

$$\mathrm{scl}(z) = \inf_{\text{rational } y \in Y_l} -\tfrac{1}{2}\chi_o(y).$$

The proof follows exactly as before. By ignoring the contribution from scl in factor groups, we get the following estimate:

Corollary 4.17 *With notations as above,*

$$2 \cdot \mathrm{scl}(z) \geq \inf_{y = (v_{G_i}) \in Y_l} \sum_i \left[\tfrac{1}{2}|v_{G_i}| - \kappa_{G_i}(v_{G_i})\right],$$

and equality holds if $\mathrm{scl}_{G_i} \equiv 0$ *for all* i.

Proof The inequality follows from Lemma 4.16 and the inequality in Lemma 4.15 together with the fact that κ_{G_i} has a continuous extension to irrational points (Corollary 4.9). Using Lemma 4.3 instead of Lemma 4.15, we get the equality part. $\square$

Using Lemma 4.16, we can generalize Corollary 4.13 to our second main result.

Theorem B (isometric embedding) *If $f_\lambda\colon H_\lambda \to G_\lambda$ is a family of isometric embeddings with respect to* scl, *then the induced map $f\colon *H_\lambda \to *G_\lambda$ is also an isometric embedding.*

Proof The proof is almost the same as that of Corollary 4.13. First reduce to finite free products, and then apply Lemma 4.16 on both sides accordingly with V'_{H_i} and V'_{G_i} instead as in Remark 4.12 to avoid assuming that f_i induces an injective map on homology. As in the proof of Corollary 4.13, each f_i induces an isomorphism of V'_{H_i} and V'_{G_i} that takes $\mathcal{D}_{H_i}$ bijectively to $\mathcal{D}_{G_i}$ by injectivity of f_i, and thus η_{H_i} is the pullback of η_{G_i}. Since f_i preserves scl, we see pscl_{H_i} is the pullback of pscl_{G_i}. It follows that χ_{o,H_i} is the pullback of χ_{o,G_i} and thus the computations of scl (for rational chains) on both sides are obtained by solving the same optimization problem. Hence, f preserves scl. $\square$

Remark 4.18 Alternatively, one can prove this theorem in a more direct way. Here is an outline:

Reduce to finite free products and then to the case $A * B \to A' * B'$ by induction. It suffices to show $\mathrm{scl}(c) \le \mathrm{scl}'(c')$ for integral homologically trivial chains c where $c' = f(c)$, $\mathrm{scl} = \mathrm{scl}_{A*B}$ and $\mathrm{scl}' = \mathrm{scl}_{A'*B'}$. Take any surface S' mapped into $A' * B'$ that approximates $\mathrm{scl}'(c')$ well, and decompose it as in Section 3 into pieces $S_{A'}$ and $S_{B'}$. Disk components of $S_{A'}$ "factor through" A by injectivity of $A \to A'$. The boundary of the union of the other components defines a chain on A' that can be pulled back to a homologically trivial (for the same reason explained in Remark 4.12) chain on A with identical scl by assumption. Then find a surface that approximates the scl of this chain well, take a finite cover if necessary and take the union with multiple copies of the disk components from S'_A, and we obtain a surface S_A with $-\chi_o(S_A) \le m(-\chi_o(S'_A) + \epsilon)$ and such that $v(S_A)$ corresponds to $mv(S'_A)$, using the notation in Section 3. Do the same thing for B. Then glue up S_A and S_B (after taking suitable finite covers) to get a surface S mapped to $A * B$ that winds around c and is almost as efficient as S'. Thus, $\mathrm{scl}(c) \le \mathrm{scl}'(c')$.

5 Applications: generalizations and new proofs of old results

In this section, we apply the isometric embedding theorem and the computational methods we have developed to get generalizations and new proofs of old results.

We start with a simple corollary of Theorem B.

Corollary 5.1 *Let $g_\lambda \in G_\lambda$, $G = *_\lambda G_\lambda$ and $f_\lambda\colon \langle g_\lambda\rangle \to G_\lambda$ be the inclusion. Then the induced map $f\colon *_\lambda \langle g_\lambda\rangle \to G$ is an isometric embedding. In particular, if $g_\lambda \in G_\lambda$ has order k_λ, then the spectrum of scl_G contains the spectrum of scl on $*_\lambda(\mathbb{Z}/k_\lambda\mathbb{Z})$. Here $k_\lambda \geq 2$ could be ∞, in which case $\mathbb{Z}/k_\lambda\mathbb{Z}$ would be $\mathbb{Z}$; and the "spectrum" could refer to the values scl takes on B_1^H or the commutator subgroups.*

Proof Simply note that f_λ is an isometric embedding even if $\mathrm{scl}_{G_\lambda}(g_\lambda) > 0$, because the definition of an isometric embedding is that f_i induces an isometric map $B_1^H(\langle g_\lambda\rangle) \to B_1^H(G_\lambda)$, and $B_1^H(\langle g_\lambda\rangle) = B_1^H(\mathbb{Z}/k_\lambda\mathbb{Z}) = 0$ since $\mathbb{Z}/k_\lambda\mathbb{Z}$ is abelian. □

Remark 5.2 Scl in free products of cyclic groups has been studied in [5; 6; 7]. The program `scallop` [8] can compute scl on specific chains.

This allows us to generalize results about the scl spectrum in free groups (or free products of cyclic groups) to general free products. Here is an example.

Corollary 5.3 *Let $G = *_{\lambda\in\Lambda} G_\lambda$ with $|\Lambda| \geq 2$ and suppose at least two of the G_λ contain elements of infinite order. Then the image of $[G,G]$ under scl contains elements congruent to every element of $\mathbb{Q}$ mod $\mathbb{Z}$. Moreover, it contains a well-ordered sequence of values with ordinal type ω^ω.*

Proof This follows from [7, Corollary 3.19], which states that the conclusion is true for nonabelian free groups, and our Corollary 5.1. □

Now we give three examples to illustrate how Lemma 4.16 works. We first deduce the following product formula, which was originally stated not quite correctly in [2] (but the proof is still valid for elements of infinite order) and later corrected and proved in [4] for the general case.

Proposition 5.4 (product formula) *Let $G = A * B$, and let $a \in A$ and $b \in B$ be nontrivial elements. Suppose a and b are of order n_a and n_b* (which could be ∞, in which case $1/\infty = 0$ by convention). *Then*

$$\mathrm{scl}_G(ab) = \mathrm{scl}_A(a) + \mathrm{scl}_B(b) + \frac{1}{2}\Big(1 - \frac{1}{n_a} - \frac{1}{n_b}\Big).$$

Proof If the image of a in $H_1(A;\mathbb{R})$ is not zero, then both sides are ∞ by convention, and similarly for b. Now we assume this is not the case. Using the notations in previous sections, let $Z = \{ab\}$; then L is just an oriented circle, $V_A = \{t(a,a) \mid t \geq 0\}$ and $\mathrm{pscl}_A(t(a,a)) = \mathrm{scl}_A(a)$ by homogeneity of scl. If n_a is finite, then $\mathrm{scl}_A(a) = 0$, $\mathcal{D}_A = \{kn_a(a,a) \mid k \in \mathbb{Z}_+\}$ and thus $\eta_A(t(a,a)) = t/n_a$, $\chi_{o,A}(t(a,a)) = -\frac{1}{2}t + t/n_a$ by Lemma 4.15. If $n_a = \infty$, then $\mathcal{D}_A = \varnothing$ and $\mathrm{cone}_{\mathbb{Q}}(\mathcal{D}_A) = \{0\}$, and thus $\chi_{o,A}(t(a,a)) = -\frac{1}{2}t - 2t\,\mathrm{scl}_A(a)$ by Lemma 4.15. Then $\chi_{o,A}(t(a,a)) = -\frac{1}{2}t - 2t\,\mathrm{scl}_A(a) + t/n_a$ is valid in both cases. Similarly we get $\chi_{o,B}$. Now the "glue-up" condition on $V_A \times V_B$ simply requires $s = t$ for $(t(a,a), s(b,b))$, so $Y = \{(t(a,a), t(b,b)) \mid t \geq 0\}$. Then the fundamental class $l \in H_1(L)$ corresponds to the chain ab. Thus, Y_l is a singleton $\{((a,a),(b,b))\}$ and $\chi_o\big(((a,a),(b,b))\big) = -1 - 2\,\mathrm{scl}_A(a) - 2\,\mathrm{scl}_B(b) + 1/n_a + 1/n_b$. Therefore, by Lemma 4.16, we get

$$\mathrm{scl}_G(ab) = -\frac{\chi_o\big(((a,a),(b,b))\big)}{2} = \mathrm{scl}_A(a) + \mathrm{scl}_B(b) + \frac{1}{2}\Big(1 - \frac{1}{n_a} - \frac{1}{n_b}\Big). \qquad \square$$

The following self-product formula is an analog of the product formula. When $B = \mathbb{Z}$ and t is the generator, it is proved in [4].

Proposition 5.5 (generalized self-product formula) *Let A and B be groups and $x, y \in A$ and $t \in B$ be elements of infinite order. Then*

$$\mathrm{scl}_{A*B}(xtyt^{-1}) = \mathrm{scl}_A(x+y) + \tfrac{1}{2}.$$

Proof Again both sides are ∞ if $x + y$ is not 0 in $H_1(A)$. Thus we assume this is not the case. The result easily follows from the original self-product formula and Theorem B by considering id: $A \to A$ and the inclusion $i\colon \langle t\rangle \to B$. But we prove it using the computational tool above, which gives a new proof.

We apply Lemma 4.16 to calculate the left-hand side. Using notations as before, if $Z = \{xtyt^{-1}\}$, then L is an oriented circle, and $C_2(B)$ consists of vectors of the form $v_b = b_{11}(t,t) + b_{12}(t,t^{-1}) + b_{21}(t^{-1},t) + b_{22}(t^{-1},t^{-1})$, where we encode the coefficients into a 2×2 matrix $b = (b_{ij})$. Then, by the definition of V'_B (Remark 4.12), $\partial(v_b) = 0$

requires the sum of entries in the i^{th} row to equal that of those in the i^{th} column for all i, which is $b_{12}=b_{21}$ in this case. Similarly, $V'_A=\{u_a \mid a_{12}=a_{21},\ a_{ij}\geq 0\}$, where $u_a=a_{11}(x,x)+a_{12}(x,y)+a_{21}(y,x)+a_{22}(y,y)$. If $(u_a,v_b)\in Y'$, the glue-up condition requires $a_{11}=b_{12}$, $a_{12}=b_{11}$, $a_{21}=b_{22}$ and $a_{22}=b_{21}$. In other words, b is the matrix we get by interchanging the columns of a. Finally $(u_a,v_b)\in Y_l$ requires in addition that each row of b (and a) sums up to 1. Together with $\partial(v_b)=0$, this implies that $b_{12}=b_{21}=1-b_{11}=1-b_{22}$.

In summary, $Y_l=\{(u_{M(\alpha)},v_{M(1-\alpha)}) \mid \alpha\in[0,1]\}$, where

$$M(x)=\begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}.$$

Now $\mathrm{pscl}_B(v_{M(1-\alpha)})=0$ since all t and t^{-1} will cancel. Since t has infinite order, $\mathcal{D}_B+V'_B=\{(t,t^{-1})+(t^{-1},t)\}+V'_B$, which implies $\eta_B\bigl(\beta[(t,t^{-1})+(t^{-1},t)]\bigr)=\beta$ and, further, $\eta_B(v_{M(1-\alpha)})=\alpha$. Thus, $\chi_{o,B}(v_{M(1-\alpha)})=-1+\alpha$ by Lemma 4.15. For $\chi_{o,A}$, it is more straightforward to use (4-1). Thus we have

$$\chi_o(u_{M(\alpha)},v_{M(1-\alpha)})=-2+\alpha+\sup\left\{\frac{\chi(S_A)}{n} \;\middle|\; v(S_A)=n\cdot u_{M(\alpha)}\right\}.$$

Therefore, by Lemma 4.16, we only need to show

$$1+2\,\mathrm{scl}_A(x+y)=\inf_{\alpha\in[0,1]\cap\mathbb{Q}}\left\{2-\alpha+\inf\left\{-\frac{\chi(S_A)}{n} \;\middle|\; v(S_A)=n\cdot u_{M(\alpha)}\right\}\right\}.$$

Letting $\alpha=1$, we have $u_{M(1)}=(x,x)+(y,y)$ and thus

$$\inf\left\{-\frac{\chi(S_A)}{n} \;\middle|\; v(S_A)=n\cdot u_{M(1)}\right\}=2\,\mathrm{scl}_A(x+y)$$

since S_A has no disk components because x and y have infinite order. This gives the "$\geq$" direction.

Conversely, we just need to show that $2\,\mathrm{scl}_A(x+y)\leq 1-\alpha-\chi(S_A)/n$ always holds. In fact, since $v(S_A)=nu_{M(\alpha)}=n(1-\alpha)[(x,y)+(y,x)]+n\alpha[(x,x)+(y,y)]$, there are $2n(1-\alpha)$ edges on the boundary of S_A, mapped to the wedge point $*$, that sit in-between an x and a y. Half of these edges are from x to y (referred to as an (x,y)–edge) and the other half are from y to x (referred to as a (y,x)–edge). Whenever we have an (x,y)–edge and a (y,x)–edge that lie on the same boundary component, we glue a rectangle to the surface with one edge glued to the (x,y)–edge and its opposite edge glued to the (y,x)–edge, and let f map the rectangle to the wedge point. Such a surgery increases $-\chi$ by 1. Repeating the process we get a new surface S'_A such that

$-\chi(S'_A) = -\chi(S_A) + n(1-\alpha)$ and each boundary component either winds around x several times or around y. This implies that S'_A has no disk components since x and y have infinite order and $\partial S'_A$ winds around each of x and y n times in total. Thus, $1-\alpha-\chi(S_A)/n = -\chi^-(S'_A)/n \geq \mathrm{scl}_A(x+y)$. This completes the proof. □

Finally we prove the following formula, which was conjectured for free products of cyclic groups and proved for $G = \mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$ by Walker [15]. It was pointed out by Susse that in the case of free products of cyclic groups, this is equivalent to [14, Proposition 4.1], which he proved by considering certain amalgams of abelian groups.

Proposition 5.6 *Let $G = A * B$, $a \in A\backslash\{\mathrm{id}\}$ and $b \in B\backslash\{\mathrm{id}\}$. Then*

$$\mathrm{scl}_G([a,b]) = \frac{1}{2} - \frac{1}{k},$$

where $2 \leq k \leq +\infty$ is the minimum of the orders of a and b.

Proof By Theorem B, we may assume $A = \langle a \rangle$ and $B = \langle b \rangle$. Let k_a and k_b be the orders of a and b, respectively.

Similar to the proof of Proposition 5.5, we have

$$Y_l = \{(u_{M(\alpha)}, v_{M(1-\alpha)}) \mid \alpha \in [0,1]\},$$

where

$$M(x) = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}$$

and we get

$$\kappa_A(u_{M(\alpha)}) = 1 - \alpha + \frac{2\alpha}{k_a}, \quad \kappa_B(v_{M(1-\alpha)}) = \alpha + \frac{2(1-\alpha)}{k_b}.$$

Therefore,

$$-\chi_o(u_{M(\alpha)}, v_{M(1-\alpha)}) = 2 - 1 - \frac{2\alpha}{k_a} - \frac{2(1-\alpha)}{k_b} = 1 - \frac{2\alpha}{k_a} - \frac{2(1-\alpha)}{k_b},$$

which has maximum $1 - 2/k$ for $\alpha \in [0,1]$, and thus

$$\mathrm{scl}_G([a,b]) = \frac{1}{2} - \frac{1}{k}.$$ □

6 Walker's conjecture

Fix a rational chain c in F_n. For any $\boldsymbol{o}=(o_1,o_2,\dots,o_n)$ with $o_i\geq 2$; let $c_{\boldsymbol{o}}$ be the image of c under the natural homomorphism $\phi\colon F_n\to *_i\,\mathbb{Z}/o_i\mathbb{Z}$. It is natural to ask: how does $\mathrm{scl}(c_{\boldsymbol{o}})$ depend on $\boldsymbol{o}$?

Based on computer experiments, Walker conjectured in [15] the following formulas:

$$
\begin{aligned}
&c=aba^{-2}b^{-2}+ab, && \mathrm{scl}(c_{\boldsymbol{o}})=\frac{2}{3}-\frac{\{\frac{2}{3},\frac{1}{2}\}}{\min(o_1,o_2)} && \text{if } \min(o_1,o_2)\geq 2,\\
&c=aba^{-3}b^{-3}, && \mathrm{scl}(c_{\boldsymbol{o}})=\frac{3}{4}-\frac{1}{o_1}-\frac{1}{o_2} && \text{if } \min(o_1,o_2)\geq 7,\\
&c=a^2ba^{-1}b^{-1}a^{-2}bab^{-1}, && \mathrm{scl}(c_{\boldsymbol{o}})=\frac{1}{2}-\frac{\{2,1\}}{o_1} && \text{if } \min(o_1,o_2)\geq 3,\\
&c=aba^2b^2a^3b^3a^{-5}b^{-5}, && \mathrm{scl}(c_{\boldsymbol{o}})=1-\frac{1}{2o_1}-\frac{1}{2o_2} && \text{if } \min(o_1,o_2)\geq 6,
\end{aligned}
$$

where o_1 and o_2 are the orders of a and b, respectively, and brackets indicate that the coefficients depend on congruence classes: for example, $\{2,1\}/o_1$ means $2/o_1$ if $o_1\equiv 0 \bmod 2$ and $1/o_1$ if $o_1\equiv 1 \bmod 2$.

Motivated by this, Walker proposed the following conjecture:

Conjecture 6.1 (Walker [15]) For any fixed chain c in F_n, $\mathrm{scl}(c_{\boldsymbol{o}})$ is piecewise quasilinear in $1/o_i$, ie there are some $p\in\mathbb{Z}_+$, and a finite partition of $\mathbb{Z}^n_{\geq 2}$, such that on each piece, fixing any congruence class of each o_i mod p, $\mathrm{scl}(c_{\boldsymbol{o}})$ is linear in $1/o_i$.

Computer experiments suggest that this conjecture is false.

Example 6.2 Conjecturally, for $n=2$ and $c=aba^{-2}b^{-2}a^2b^2a^{-1}b^{-1}$ with $\frac{1}{2}o_2>o_1>10$,

$$
\mathrm{scl}(c_{\boldsymbol{o}})=\begin{cases}
1-3(o_1-1)/(o_1(o_1+1)) & \text{if } o_1\equiv 1,3,5 \bmod 6,\\
1-3/o_1 & \text{if } o_1\equiv 0 \bmod 6,\\
1-15/(5o_1+8) & \text{if } o_1\equiv 2 \bmod 6,\\
1-3/(o_1+2) & \text{if } o_1\equiv 4 \bmod 6.
\end{cases}
$$

This is verified by the computer program `scallop` [8] for $o_2=100$ and $10<o_1<50$. We see from this example that the denominator could be a higher-degree polynomial in o_1, and even when it is linear in o_1, it could be inhomogeneous.

To seriously disprove the conjecture, it suffices to verify a special case of the formula above:

Proposition 6.3 *For $n=2$ and $c=aba^{-2}b^{-2}a^2b^2a^{-1}b^{-1}$, there exist constants $r,s\geq 1$ such that when $o_1=6K+3$, $o_2=6L+3$ with $L\geq rK$ and $K\geq s$, we have*

$$\operatorname{scl}(c_{\boldsymbol{o}})=1-\frac{3(o_1-1)}{o_1(o_1+1)}.$$

We prove the "$\leq$" direction and give an outline of the proof for the other direction.

Proof Follow the notations in Section 4 and apply our method to $G=A*B$ with $A=\mathbb{Z}/o_1\mathbb{Z}$ and $B=\mathbb{Z}/o_2\mathbb{Z}$. Then $T(A)=\{a,a^{-2},a^2,a^{-1}\}$. Let

$$\begin{aligned}
v_A=\;&\frac{1}{3K+2}[(a,a^{-1})+(a^{-1},a)]+\frac{1}{3K+2}[(a^2,a^{-2})+(a^{-2},a^2)]\\
&+\frac{K}{(2K+1)(3K+2)}[(6K+3)(a,a)]+\frac{1}{(6K+3)(3K+2)}[(6K+3)(a^{-1},a^{-1})]\\
&+\frac{1}{3K+2}[(a,a^2)+3K(a^2,a^2)+(a^2,a)]\\
&+\frac{1}{3K(3K+2)}[(a^{-1},a^{-2})+3K(a^{-2},a^{-2})+(a^{-2},a^{-1})]\\
&+\frac{9K^2-1}{K(3K+2)(6K+3)}[(2K+1)(a^{-2},a^{-1})+(2K+1)(a^{-2},a^{-1})],
\end{aligned}$$

where each bracket is a disk vector. In particular, we know $v_A\in V_A$ and

$$\begin{aligned}
\kappa_A(v_A)&\geq\frac{3}{3K+2}+\frac{K}{(2K+1)(3K+2)}+\frac{1}{(6K+3)(3K+2)}\\
&\qquad+\frac{1}{3K(3K+2)}+\frac{9K^2-1}{K(3K+2)(6K+3)}\\
&=\frac{30K+12}{(3K+2)(6K+3)}.
\end{aligned}$$

Similarly, let

$$\begin{aligned}
v_B=\;&\frac{3K}{3K+2}[(b,b^{-1})+(b^{-1},b)]+\frac{3K}{3K+2}[(b^2,b^{-2})+(b^{-2},b^2)]\\
&+\frac{1}{3K+2}[(b^2,b^{-1})+(b^{-1},b^{-1})+(b^{-1},b^2)]\\
&+\frac{1}{3K+2}[(b,b^{-2})+(b^{-2},b^{-2})+(b^{-2},b)+(b,b^2)+(b^2,b)]\in V_B,
\end{aligned}$$

where each bracket is a disk vector and

$$\kappa_B(v_B) \geq 2 - \frac{2}{3K+2}.$$

One can check that v_A and v_B satisfy the gluing condition and $(v_A, v_B) \in Y_l$, where l is the fundamental class of the loop representing the chain c. Therefore, by Corollary 4.17 (the equality part), for any $K, L \geq 0$, we have

$$\mathrm{scl}(c_{\boldsymbol{o}}) \leq \frac{1}{2}[2-\kappa_A(v_A)] + \frac{1}{2}[2-\kappa_B(v_B)] \leq 1 - \frac{18K+6}{(6K+3)(6K+4)} = 1 - \frac{3(o_1-1)}{o_1(o_1+1)}.$$

For the other direction, we only need to show that (v_A, v_B) constructed above achieves the maximum of the optimization problem

$$(P_0) \qquad \text{maximize } \kappa_A(u) + \kappa_B(w) \text{ subject to } (u, w) \in Y_l,$$

and that the estimates for $\kappa_A(v_A)$ and $\kappa_B(v_B)$ above are sharp. The key idea is to use duality of linear programming. Here is an outline:

(1) We linearize this optimization problem (P_0) in a way similar to [11]. On the "A" side, consider the directed graph (as in [6]) with vertex set $T(A)$ and directed edge set $T(A)^2$. Let SL_A be the set of directed simple (ie visiting each vertex at most once) loops. Each directed loop cyclically visiting vertices $a_1, \dots, a_n$ corresponds to a vector $\sum_{i=1}^n (a_i, a_{i+1})$ in V_A'. Then disk vectors can be written (not uniquely) as linear combinations of simple loops with nonnegative integral coefficients. One can enumerate disk vectors that are *extremal*, ie cannot be written as a convex combination of other disk vectors plus a nonnegative linear combination of simple loops. It turns out that there are finitely many (169) extremal disk vectors and each depends linearly on K, which is compatible with Lemma 6.7 below. Denote the set of extremal disk vectors by ED_A. Obtain SL_B and ED_B on the "B" side simply by substituting a and K by b and L, respectively since the two sides have the same structure. Then (P_0) can be linearized as:

$$(P) \qquad \text{maximize } f^T x \text{ subject to } Cx = b \text{ and } x \geq \mathbf{0} \text{ (entrywise)},$$

where $x = (x_i)$ and $f = (f_i)$ are indexed by $\mathrm{SL}_A \sqcup \mathrm{ED}_A \sqcup \mathrm{SL}_B \sqcup \mathrm{ED}_B$, $f_i = 1$ if $i \in \mathrm{ED}_A \sqcup \mathrm{ED}_B$ and $f_i = 0$ otherwise, and the constraint $Cx = b$ corresponds to gluing and normalization conditions.

(2) The way we decompose v_A and v_B into disk vectors gives rise to a feasible solution x_0 to (P). Our goal is to show that x_0 achieves the maximum. To accomplish

this, it suffices to find y_0 such that

$$C^T y_0 \geq f \text{ (entrywise) and } x_0^T C^T y_0 = x_0^T f.$$

This proves the maximality because

$$f^T x = x^T f \leq x^T C^T y_0 = b^T y_0 = x_0^T C^T y_0 = x_0^T f = f^T x_0.$$

One such y_0 (in an explicit formula involving K and L) can be guessed via results found by computers for small values of K and L (v_A and v_B are also found in this way). The constants r and s come into the statement because the author only checked $C^T y_0 \geq f$ when L/K and K are large enough.

We omit the details since it is tedious and takes too much space to enumerate the extremal disk vectors and check $C^T y_0 \geq f$. □

Nevertheless, a weaker version of Walker's conjecture is true:

Theorem 6.4 *For any fixed rational chain c in F_n, $\mathrm{scl}(c_{\boldsymbol{o}})$ is piecewise quasirational in $\boldsymbol{o}$, ie there are some $p \in \mathbb{Z}_+$ and a finite partition of $\mathbb{Z}^n_{\geq 2}$ such that on each piece, fixing any congruence class of each $o_i \bmod p$, $\mathrm{scl}(c_{\boldsymbol{o}})$ is in $\mathbb{Q}(\boldsymbol{o})$.*

Susse [14, Corollary 4.14] proved the same result by considering a fixed chain in a family of amalgamations of free abelian groups whose projection to the free product of cyclic groups preserves scl. Our proof is independent and new.

We focus on a single factor $A = \mathbb{Z}/k\mathbb{Z}$. Using notations as in Section 4, the key is to show that the vertices of $\mathrm{conv}(\mathcal{D}_A + V_A)$ behave nicely as k varies in congruence classes (see Lemma 6.7). Since $H_1(A;\mathbb{R})$=0, we have $h = 0$. Thus V_A consists of nonnegative vectors in $C_2(A) \cap \{\partial = 0\}$ and does not depend on k. However, $\mathcal{D}_A$ typically depends on k, and we denote it by D_k to emphasize the dependence.

We first describe D_k. For simplicity, we assume $n = 2$, and $c = a_1 b_1 \dots a_m b_m$ is a single word, but the proofs of the lemmas are the same for the general case. Consider the directed graph $X(A)$ with vertex set $T(A)$ and edge set $T_2(A)$. Then each $v \in V_A$ defines nonnegative weights on the directed edges, and its support, $\mathrm{supp}(v)$, is the subgraph of $X(A)$ consisting of edges with positive weights.

Let a and b be the generators of F_2 giving the free product structure. Then each a_i equals a^{t_i} for some $t_i \in \mathbb{Z} \setminus \{0\}$. Let $\widetilde{h}\colon C_2(A) \to \mathbb{R}$ be the linear map such that $\widetilde{h}(a_i, a_j) = \frac{1}{2}(t_i + t_j)$ for any $(a_i, a_j) \in T_2(A)$.

Then it is easy to see that D_k is the set of integer vectors v in V_A such that $\widetilde{h}(v) \in k\mathbb{Z}$ and $\mathrm{supp}(v)$ is connected and nonempty (see [6] for details).

We can decompose V_A into finitely many simplicial rational open cones, ie each is of the form

$$\left\{ \sum_{i=1}^{d} t_i v_i \,\middle|\, t_i > 0 \right\}$$

for some $d \geq 1$ and a set of linearly independent rational vectors v_i. Moreover, each simplicial rational cone can be decomposed into finitely many *unimodular* cones, ie where we can take the set of v_i to be unimodular, by Barvinok's theorem [1, Chapter 16]. So we first prove the following key lemma, leading to Lemma 6.7 and Theorem 6.4.

Lemma 6.5 *Let $V = \mathbb{R}^d_{>0}$ and let $f(x) = \sum a_i x_i$ with $a_i \in \mathbb{Q}$ be a rational linear function. Let $V_k = f^{-1}(k) \cap V$ and let E_k be the set of integer points in V_k. Then there are $M, p \in \mathbb{Z}_+$ such that:*

(1) *For each congruence class mod p, there are finitely many points $v_j(k) \in V$ that depend linearly on k such that $\mathrm{conv}(E_k + V_k) = \mathrm{conv}(\{v_j(k)\} + V_k)$ for any $k > M$ in this given congruence class.*

(2) *For each congruence class mod p, there is a finite set F_k of points depending linearly on k such that*

$$\mathrm{conv}\left(\bigcup_{t \in \mathbb{Z}_+} E_{tk} + V \right) = \mathrm{conv}(F_k + V)$$

for any $k > M$ in the given congruence class mod p. More precisely, we can take $F_k = \bigcup_{t=1}^{p} \{v_j(tk)\}$ for any $k > M$.

Lemma 6.5 is similar in spirit to the following special case of the main theorem of [9], which we will use in our proof.

Lemma 6.6 (Calegari and Walker [9]) *Let $\{\xi_i(k)\}$ be a finite set of points depending linearly on k; then there are $M, p \in \mathbb{Z}_+$ such that the (finitely many) vertices of the integer hull of $\mathrm{conv}(\xi_i(k))$ depend linearly on $k > M$ in each congruence class mod p.*

Proof of Lemma 6.5 We first prove (2) assuming (1). Notice that each $v_j(k)$ depends linearly on k and stays in V. Thus, if $k' > k > M$ and $k' \equiv k \mod p$, then $v_j(k') \in v_j(k) + V$. Also notice that any tk is congruent to some $t_0 k$ with $1 \leq t_0 \leq p$. Hence, the vertices of $\mathrm{conv}\big(\bigcup_{t\in\mathbb{Z}_+} E_{tk}+V\big)$ are contained in $F_k=\bigcup_{t=1}^{p}\{v_j(tk)\}$, and the assertion holds.

Now we prove (1). We may assume all the a_i are nonzero, otherwise we can do a dimension reduction. Let P and N be the set of indices such that a_i is positive or negative, respectively. If $P=\varnothing$, then $E_k=\varnothing$ and the problem is trivial, so we also assume $P\neq\varnothing$ in the sequel. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$. For any $i\in P$, let $\xi_i(k)=ke_i/a_i\in V_k$.

If $N=\varnothing$, ie all $a_i>0$, then V_k is the interior of the simplex with vertices $\{\xi_i(k)\}$ and its set of integer points E_k coincides with that of the polyhedron

$$\Delta_k:=\{(x_1,x_2,\dots,x_d)\mid x_i\geq 1\}\cap\mathrm{conv}\{\xi_i(k)\}.$$

When $k>\sum_i a_i$, Δ_k is the (compact) simplex with vertices

$$\left\{\sum_{j\neq i}e_j+\frac{k-\sum_{j\neq i}a_j}{a_i}e_i\right\}_{i=1}^{d}$$

depending linearly on k, so our assertion follows from Lemma 6.6.

Now also suppose $N\neq\varnothing$. Then $V_k=\mathrm{conv}\{\xi_i\mid i\in P\}+V_0$. We first deal with integer points in each ξ_i+V_0. Pick p such that $p/a_i\in\mathbb{Z}$. Then, for $k=tp+k_0$ with $0\leq k_0\leq p-1$ fixed, ξ_i+V_0 is $t(p/a_i)\cdot e_i+C$, where $C=(k_0/a_i)\cdot e_i+V_0$ is a translate of V_0 which does not depend on k. Therefore, in this congruence class, the integer hull of ξ_i+V_0 is just that of C translated by $t(p/a_i)e_i$, a vector depending linearly on k.

If $x\in V_k$ is not contained any ξ_i+V_0 (this does not happen for $|P|=1$, so we assume $|P|\geq 2$ below), then for each $i\in P$, we have $x_i\leq k/a_i$, and hence x lies in

$$C_k:=V_k\cap\left(\bigcap_{i\in P}\left\{x\;\middle|\;x_i\leq\frac{k}{a_i}\right\}\right).$$

The set of integer points in C_k coincides with that in

$$Q_k:=\left\{x\in V_k\;\middle|\;x_i\geq 1\text{ for all } i\text{ and } x_i\leq\frac{k}{a_i}\text{ for all } i\in P\right\}.$$

Q_k is compact since $1\leq x_i\leq k/a_i$ for any $i\in P$ and $x\in V_k$ implies

$$x_j\leq k\frac{|P|-1}{-a_j}\quad\text{for all } j\in N.$$

To see the vertices of Q_k, consider its decomposition into the level sets

$$Q_k^{(t)} := \Big\{ x \in Q_k \;\Big|\; \sum_{i\in P} a_i x_i = t \Big\}.$$

When $k(|P|-1) \geq \sum_{j\in N}(-a_j)$ and $k \geq \sum_{i=1}^d a_i$, the set $Q_k^{(t)}$ is nonempty if and only if $k - \sum_{j\in N} a_j \leq t \leq k|P|$. For such t, one can see that $Q_k^{(t)}$ is the product of

$$\Big\{ (x_i)_{i\in P} \;\Big|\; 1 \leq x_i \leq k/a_i \text{ for all } i \in P,\ \sum_{i\in P} a_i x_i = t \Big\}$$

(combinatorially a level set of a cube) and

$$\Big\{ (x_j)_{j\in N} \;\Big|\; x_j \geq 1 \text{ for all } j \in N,\ \sum_{j\in N} (-a_j x_j) = t - k \Big\}$$

(a simplex).

From this, we can see that the vertices of Q_k are of the form

$$\Big\{ x \;\Big|\; x_i = 1 \text{ or } \frac{k}{a_i} \text{ for all } i \in P \text{ and, for some } l \in N,\\ x_j = 1 \text{ for all } j \in N - \{l\} \text{ and } x_l = \frac{\sum_{i\neq l} a_i x_i - k}{-a_l} \geq 1 \Big\}$$

or

$$\Big\{ x \;\Big|\; x_j = 1 \text{ for all } j \in N \text{ and, for some } l \in P,\\ x_i = 1 \text{ or } \frac{k}{a_i} \text{ for all } i \in P - \{l\} \text{ and } x_l = \frac{k - \sum_{i\neq l} a_i x_i}{a_l} \in \Big[1, \frac{k}{a_l}\Big] \Big\},$$

each depending linearly on k, so Lemma 6.6 applies. Since V_k is the union of $\xi_i + V_0$ for $i \in P$ and C_k, and the integer hull of each part has vertices depending linearly on $k \gg 1$ in a congruence class, our assertion follows. □

Now we can prove the following result:

Lemma 6.7 *There are $M, p \in \mathbb{Z}_+$ such that for each congruence class mod p, there exist finitely many points $v_j \in V_A$, each depending* linearly *on k, such that* $\mathrm{conv}(D_k + V_A) = \mathrm{conv}(\{v_j\} + V_A)$ *for any $k > M$ in this given congruence class.*

Proof According to the discussion ahead of Lemma 6.5, we can express V_A as the union of top-dimensional faces (denote them by $V(i)$) of finitely many simplicial *unimodular* (Barvinok's theorem [1, Chapter 16]) rational cones, and the intersection of D_k with each $V(i)$ is either empty (when the support is disconnected) or exactly the integer points in $V(i)\cap\widetilde{h}^{-1}(k\mathbb{Z})$. Applying Lemma 6.5 to each $V(i)$ with $f=\widetilde{h}$ and $f=-\widetilde{h}$, respectively (together with the set $V(i)\cap\widetilde{h}^{-1}(0)$, which does not depend on k), we see that there are $M, p\in\mathbb{Z}_+$ such that for each congruence class mod p, there exist finitely many points $v_j(i)$ such that $\mathrm{conv}(D_k\cap V(i)+V(i))=\mathrm{conv}(\{v_j(i)\}+V(i))$ for any $k>M$ in this given congruence class. This completes the proof by taking the union, since there are only finitely many i's. □

Proof of Theorem 6.4 It follows from Lemma 6.7 that for $k\gg 1$ in a fixed congruence class mod p, κ_A is the minimum of finitely many linear functions each having coefficients in $\mathbb{Q}(k)$. Here A can be any factor group and k is the corresponding o_i. Therefore, if we fix the congruence classes of $o_i\gg 1$, combining with the proof of Theorem A, $\mathrm{scl}(c_{\boldsymbol{o}})$ is determined by minimizing, on a fixed compact convex set C, the maximum of finitely many linear functions f_j each having coefficients in $\mathbb{Q}(\boldsymbol{o})$. Thus we can find a finite polyhedral decomposition of C with vertices having $\mathbb{Q}(\boldsymbol{o})$–coordinates and $\max_j\{f_j\}$ linear on each piece. It follows that scl is the minimum of finitely many functions in $\mathbb{Q}(\boldsymbol{o})$, ie the values of $\max_j\{f_j\}$ on these finitely many vertices, and hence scl is piecewise $\mathbb{Q}(\boldsymbol{o})$. □

Appendix

Here we give a proof of (4-2). For convenience, we use $\#_s(w)$ to denote the number of subwords s inside w. Let $W_{u,v}$ be the set of cyclic words w in a, b and c such that w contains u copies of each of ab, bc and ca and v copies of ac, cb and ba as subwords.

For each $w\in W_{u,v}$, let $f(w)$ be the unique integer such that w can be written as $(abc)^k[a,b]^{f(w)}$ by moving letters around and using $[a,b]=[b,c]=[c,a]$. In Example 4.5, we defined $S_{u,v}$ to be the image of $W_{u,v}$ under f.

In order to prove the equation inductively, we first introduce a way to reduce the computation of $S_{u,v}$ to that of smaller indices.

For each $w\in W_{u,v}$, the letter a appears $u+v$ times in w. For convenience, we make the following definition:

Definition A.1 An a–connecting subword of w is the subword between two consecutive a's in w.

For example, if $abcba$ is a subword of w, then bcb is an a–connecting subword of w. We classify all a–connecting subwords and divide them into three categories:

(1) **Degree 1** $b(cb)^k c$ with $k \geq 0$.

(2) **Degree 0** $b(cb)^k$ or $c(bc)^k$ with $k \geq 0$.

(3) **Degree −1** $c(bc)^k b$ with $k \geq 0$.

Lemma A.2 *If there are two degree* 1 *a–connecting subwords in $w \in W_{u,v}$, then we can find $w_1, w_2 \in W_{u-1,v}$ such that*

$$f(w_1) \leq f(w) \leq f(w_2).$$

Proof Up to a cyclic permutation, $w = ab(cb)^k cRab(cb)^l cT$, where R and T are empty words or subwords starting with a and $k, l \geq 0$. Recall that abc is in the center and a commutes with bc, so we have $ab(cb)^l c = a^{-l}(abc)^{l+1}$ and

$$[ab(cb)^l c, R] = [a^{-l}, R] = [a, b]^{l(\#_c(R) - \#_b(R))}.$$

Thus,

$$\begin{aligned} w &= ab(cb)^k cab(cb)^l cRT \cdot [ab(cb)^l c, R]^{-1} \\ &= ab(cb)^k cab(cb)^l cRT \cdot [a, b]^{l(\#_b(R) - \#_c(R))}. \end{aligned}$$

Notice that $ab(cb)^k \underline{cab}(cb)^l cRT$ is still in $W_{u,v}$, and removing the underlined cab which is followed by c, we will get a word $w_1 = ab(cb)^{k+l} cRT \in W_{u-1,v}$ and $f(w_1) = f(w) + l(\#_b(R) - \#_c(R))$.

Similarly

$$w = Rab(cb)^k \underline{cab}(cb)^l cT \cdot [a, b]^{k(\#_c(R) - \#_b(R))},$$

so $w_2 = Rab(cb)^{k+l} cT \in W_{u-1,v}$ and $f(w_2) = f(w) - k(\#_b(R) - \#_c(R))$.

Hence, if $\#_b(R) - \#_c(R) \leq 0$, we are done; otherwise switch w_1 and w_2. □

Proof of (4-2) Notice the following symmetry: Reading a word $w \in W_{u,v}$ in reverse order gives a word $r(w) \in W_{v,u}$ and $f(r(w)) = -f(w) - (u + v)$. Thus,

$$S_{u,v} = -S_{v,u} - u - v.$$

According to Lemma A.2, if $w \in W_{u,v}$ has two degree 1 a–connecting subwords in w, then $f(w) \in S_{u-1,v}$ assuming that $S_{u-1,v}$ consists of integers in an interval. Similarly, by the symmetry above, if $w \in W_{u,v}$ has two degree -1 a–connecting subwords in w, then $f(w) \in S_{u,v-1}$ assuming that $S_{u,v-1}$ consists of integers in an interval.

First assume we have proved the equation for $u = v + 1$. We induct over $u - v$ to show that $S_{u,v} = S_{v+1,v}$ whenever $u \geq v + 1$. Suppose $w \in W_{u,v}$ with $u > v + 1$, notice that an a–connecting subword w_0 has degree d if and only if

$$[\#_{ab}(aw_0a) + \#_{ca}(aw_0a)] - [\#_{ac}(aw_0a) + \#_{ba}(aw_0a)] = 2d.$$

Also notice that if we sum the left-hand side of the equation above over all a–connecting subwords, we will get $2(u-v) \geq 4$. Hence, we conclude that there exist two degree 1 a–connecting subwords in w, and thus $f(w) \in S_{u-1,v}$ since $S_{u-1,v}$ consists of integers in an interval by the induction hypothesis. This shows that $S_{u,v} \subset S_{u-1,v}$, but the other inclusion is obvious: adding a copy of abc ahead of a letter a in $w \in S_{u-1,v}$ will result in a new word $w' \in S_{u,v}$ with $f(w') = f(w)$.

Therefore, using the symmetry, we only need to prove the equation for $S_{u,v}$ with $|u - v| \leq 1$, and we induct on $u + v$. The base cases are easy to check. We now show

$$S_{v+1,v} = \left[-\tfrac{1}{2}v(v+1), \tfrac{1}{2}v(v-1)\right] \cap \mathbb{Z}$$

assuming (4-2) holds for all $S_{u',v'}$ with $u' + v' < 2v + 1$ and $|u' - v'| \leq 1$.

Consider the family of words in $W_{v+1,v}$

$$w_k = a(bc)^{k+1}ac(bc)^{v-k}(ac)^{v-1}(ab)^v \quad \text{for } 0 \leq k \leq v.$$

A direct computation shows that $f(w_k) = \frac{1}{2}v(v-3) + k$. This, together with earlier arguments, shows that $S_{v,v} \cup \left[\frac{1}{2}v(v-3), \frac{1}{2}v(v-1)\right] \subset S_{v+1,v}$, and hence by the induction hypothesis, we have

$$\left[-\tfrac{1}{2}v(v+1), \tfrac{1}{2}v(v-1)\right] \cap \mathbb{Z} \subset S_{v+1,v}.$$

So we only need to show

$$\max(S_{v+1,v}) \leq \tfrac{1}{2}v(v-1) \text{ and } \min(S_{v+1,v}) \geq -\tfrac{1}{2}v(v+1).$$

Suppose $w \in W_{v+1,v}$ achieves $\max(S_{v+1,v}) \geq f(w_v) = \frac{1}{2}v(v-1)$; we see that:

(1) w does not contain subwords $abca$, $bcab$ or $cabc$, otherwise $f(w) \in S_{v,v}$, which has maximum $\frac{1}{2}v(v-3) < \frac{1}{2}v(v-1)$ by induction.

(2) w does not contain subwords $acba$, $bacb$ or $cbac$, otherwise $f(w) \in S_{v+1,v-1}$, and $S_{v+1,v-1} = S_{v,v-1}$ has maximum $\frac{1}{2}(v-1)(v-2) - v < \frac{1}{2}v(v-1)$ by induction.

(3) w does not contain the subword $abaca$, since it can be replaced by $acaba$ to get a new word $w' \in W_{v+1,v}$ with $f(w') > f(w)$.

(4) Only one a–connecting subword in w has degree 1 and the others have degree 0, otherwise there will be at least two degree 1 subwords (since the sum of degrees is 1), which implies (by Lemma A.2 and the induction hypothesis) $f(w) \le \max(S_{v,v})$, contradicting maximality.

Therefore, w must be of the form (up to replacing it by another that also achieves the max)

$$w = abc(bc)^k ac(bc)^{p_1} a \dots ac(bc)^{p_s} acabab(cb)^{q_1} ab(cb)^{q_t},$$

where $s, t \ge 0$, $k \ge 0$ and $p_i, q_j \ge 0$. Since $w \in W_{v+1,v}$, we see $s = t = v-1$ and $k + \sum p_i + \sum q_j = v$. A direct computation shows

$$w = (abc)(ab)^v (bc)^v (ab)^v [a,b]^e,$$

where $e = vk + \sum (v-i) p_i + \sum j q_j$. Maximizing $f(w)$ is the same as maximizing e, which requires $p_i = q_j = 0$ and $k = v$. Therefore, $w = w_v$, as we constructed, and $\max(S_{v+1,v}) \le f(w_v) = \frac{1}{2} v(v-1)$.

Similarly, we can show $\min(S_{v+1,v}) \ge -\frac{1}{2} v(v+1)$. Hence, (4-2) holds for $S_{v+1,v}$, and for $S_{v,v+1}$ by symmetry. The inductive step for $S_{v,v}$ is completely similar, so we omit it. This completes the proof. □

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