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Knot Floer homology and Khovanov–Rozansky homology for singular links

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The (untwisted) oriented cube of resolutions for knot Floer homology assigns a complex $C_F(S)$ to a singular resolution S of a knot K. Manolescu conjectured that when S is in braid position, the homology $H_*(C_F(S))$ is isomorphic to the HOMFLY-PT homology of S. Together with a naturality condition on the induced edge maps, this conjecture would prove the existence of a spectral sequence from HOMFLY-PT homology to knot Floer homology. Using a basepoint filtration on $C_F(S)$, a recursion formula for HOMFLY-PT homology and additional $\mathfrak{sl}_n$–like differentials on $C_F(S)$, we prove Manolescu's conjecture. The naturality condition remains open.

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1 Introduction

The last few decades have seen tremendous growth within the field of knot theory. Many new knot invariants have been constructed, including the categorifications of several classical knot polynomials. These categorifications typically take the form of a multigraded homology theory, whose graded Euler characteristic returns the polynomial in question.

Some of the most notable categorifications include HOMFLY-PT homology, $\mathfrak{sl}_n$ homology and knot Floer homology, whose graded Euler characteristics return the HOMFLY-PT polynomial, the $\mathfrak{sl}_n$ polynomial and the Alexander polynomial, respectively. The HOMFLY-PT polynomial of a link L is a two variable polynomial $P_H(a,q)(L)$, and it is determined by the skein relation

$$aP_H(a,q)(L_+) - a^{-1}P_H(a,q)(L_-) = (q-q^{-1})P_H(a,q)(L_0),$$

where L_+, L_- and L_0 are identical except at one crossing, where L_+ has a positive crossing, L_- has a negative crossing and L_0 has the oriented smoothing (see Freyd, Yetter, Hoste, Lickorish, Millett and Ocneanu [4] and Przytycki and Traczyk [16]). Together with the normalization $P_H(a,q)(\text{unknot}) = 1$, this relation uniquely determines the HOMFLY-PT polynomial.

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We can obtain a single-variable polynomial invariant $P_n(q)(L)$ by setting $a = q^n$ in the HOMFLY-PT polynomial. For $n \geq 1$, $P_n(q)(L)$ gives the $\mathfrak{sl}_n$ polynomial, and setting $n = 0$ gives the Alexander polynomial. The most popular of the $\mathfrak{sl}_n$ polynomials is the Jones polynomial, which is obtained by setting $n = 2$, and $\mathfrak{sl}_2$ homology is isomorphic to Khovanov's original categorification of the Jones polynomial, known as Khovanov homology [8]. An explicit description of this isomorphism is described by Hughes [6].

The fact that the $\mathfrak{sl}_n$ and Alexander polynomials are specializations of the HOMFLY-PT polynomial led to the following conjecture of Dunfield, Gukov and Rasmussen:

Conjecture 1.1 [3] *For all $n \geq 1$, there is a spectral sequence from HOMFLY-PT homology to $\mathfrak{sl}_n$ homology, and there is a spectral sequence from HOMFLY-PT homology to knot Floer homology.*

Rasmussen was able to use similarities in the constructions of HOMFLY-PT homology and $\mathfrak{sl}_n$ homology to prove the first part of this conjecture. In particular, he showed [18] that there are a family of spectral sequences $E_k(n)$ for $n \geq 1$ such that the E_2 page is HOMFLY-PT homology and the E_∞ page is $\mathfrak{sl}_n$ homology.

Unfortunately, due to the fundamental differences between Khovanov–Rozansky homology and knot Floer homology, the second part of Conjecture 1.1 has remained unsolved for the last decade. This conjectured spectral sequence from HOMFLY-PT homology to knot Floer homology will be the focus of this paper.

It turns out that all three of these homology theories can be constructed via an oriented cube of resolutions. Let $C_H(D, d_0^H + d_1^H)$ denote the cube of resolutions for HOMFLY-PT homology, where d_i^H denotes the component of the differential which increases the cube grading by i. (For both HOMFLY-PT and $\mathfrak{sl}_n$ homology, the higher face maps d_i with $i \geq 2$ are zero.) The HOMFLY-PT homology $H_H(K)$ is defined to be

$$H_H(K) = H_*\big(H_*(C_H(D), d_0^H), (d_1^H)^*\big),$$

where D is a braid diagram of the knot K and $(d_1^H)^*$ is the induced map on homology. HOMFLY-PT homology is a knot invariant, which means that the homology does not depend on the choice of braid diagram for K. Note that this homology can be equivalently viewed as the E_2 page of the spectral sequence induced by the cube filtration on the HOMFLY-PT complex.

The $\mathfrak{sl}_n$ homology $H_n(K)$ is defined in the same way. If $C_n(D, d_0^{(n)} + d_1^{(n)})$ denotes the cube of resolutions for $\mathfrak{sl}_n$ homology, then $H_n(K)$ is given by

$$H_n(K) = H_*\big(H_*(C_n(D), d_0^{(n)}), (d_1^{(n)})^*\big),$$

where, again, D is a diagram for K (not necessarily a braid), and the homology is independent of the choice of D.

Knot Floer homology is a completely different story. There is not a standard way to define the complex $CFK^-(K)$, as it depends on a choice of Heegaard diagram for K, and there are many different ways to make this choice for a knot K. However, the chain homotopy type of $CFK^-(K)$ does not depend on the choice of diagram.

Using a particular choice of Heegaard diagram together with a bit of algebra, Oszváth and Szabó developed an oriented cube of resolutions for knot Floer homology with twisted coefficients [15]. This construction was modified by Manolescu [12] to give an untwisted cube of resolutions for knot Floer homology, which is chain homotopy equivalent to $CFK^-(K)$. We will denote this complex by $(C_F(D), d_0^F + d_1^F + \cdots + d_k^F)$. Unlike the HOMFLY-PT and $\mathfrak{sl}_n$ homology, the knot Floer homology $HFK^-(K)$ is the *total* homology of this complex:

$$HFK^-(K) \cong H_*(C_F(D), d_0^F + d_1^F + \cdots + d_k^F).$$

For all of these complexes, each vertex in the cube of resolutions can be viewed as a complex corresponding to the complete resolution S at that vertex. We will denote these complexes by $C_H(S)$, $C_n(S)$ and $C_F(S)$, with the corresponding homologies given by $H_H(S)$, $H_n(S)$ and $H_F(S)$.

Manolescu made the following conjecture:

Conjecture 1.2 [12] *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

An immediate consequence of this conjecture is an isomorphism

$$H_*(C_H(D), d_0^H) \cong H_*(C_F(D), d_0^F),$$

which maps each vertex in the HOMFLY-PT cube of resolutions to the same vertex in the knot Floer cube of resolutions. As discussed in [12], one would obtain a spectral sequence from HOMFLY-PT homology to knot Floer homology if the induced edge maps agree. In other words, if f is the isomorphism between them, then the existence of such a spectral sequence would follow from the square below being commutative:

$$\begin{array}{ccc} H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \\ \downarrow (d_1^H)^* & & \downarrow (d_1^F)^* \\ H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \end{array}$$

This idea can also be explained in terms of the spectral sequences induced by the cube filtrations on $C_H(D)$ and $C_F(D)$. Letting $E_k^H(D)$ and $E_k^F(D)$ denote the two spectral sequences, we see that the HOMFLY-PT homology is given by $E_2^H(D)$ and the knot Floer homology is given by $E_\infty^F(D)$. Manolescu's conjecture would imply an isomorphism $E_1^H(D) \cong E_1^F(D)$, and the induced edge maps commuting with this isomorphism would imply that $E_2^H(D) \cong E_2^F(D)$. This would give a spectral sequence whose E_2 page is isomorphic to HOMFLY-PT homology and whose E_∞ page is $HFK^-(K)$.

Manolescu showed that for a connected singular braid S, both $H_H(S)$ and $H_F(S)$ have a purely algebraic formulation in terms of Tor groups. Letting R denote the polynomial ring $\mathbb{Q}[U_1, \dots, U_k]$, where k is the number of edges in the singular braid S, he showed that there are ideals L, Q and N in R such that

$$H_H(S) \cong \mathrm{Tor}_R(R/L, R/Q) \quad \text{and} \quad H_F(S) \cong \mathrm{Tor}_R(R/L, R/N).$$

These Tor groups can naturally be viewed as bigraded vector spaces, where the dimension in each bigrading is finite. Thus, for S connected, Conjecture 1.2 is equivalent to an isomorphism of bigraded vector spaces

$$\mathrm{Tor}_R(R/L, R/Q) \cong \mathrm{Tor}_R(R/L, R/N).$$

Unfortunately, these Tor groups turned out to be difficult to compare due to the nonlocal nature of the ideal N.

In this paper, we will prove Conjecture 1.2 using a very different approach. First, we define an additional family of differentials on $C_F(S)$ for all $n \geq 1$. We will denote this complex by $C_{F(n)}$.

Theorem 1.3 *For all $n \geq 1$, there is an isomorphism $H_*(C_{F(n)}(S)) \cong H_{n+1}(S)$.*

The differential on $C_{F(n)}$ can be filtered by the Alexander grading, and when we only consider those differentials which preserve the Alexander grading, we get back the complex $C_F(S)$. Thus, using the Alexander filtration on $C_{F(n)}$, we get the following:

Corollary 1.4 *For all $n \geq 2$, there is a spectral sequence which starts at $H_F(S)$ and converges to $H_n(S)$.*

For all $n \geq 1$, there is also a known spectral sequence which starts at $H_H(S)$ and converges to $H_n(S)$; see Rasmussen [18]. Thus, $H_H(S)$ and $H_F(S)$ are both "limits" of $\mathfrak{sl}_n$ homology (in a suitable sense).

We are able to use these additional differentials together with a basepoint filtration to prove Conjecture 1.2.

Theorem 1.5 *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

Corollary 1.6 *Let D be a braid diagram and $E_2^F(D)$ the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. Then the graded Euler characteristic of $E_2^F(D)$ is the* HOMFLY-PT *polynomial.*

This corollary provides evidence for the conjecture that $E_2^F(D)$ is in fact isomorphic to HOMFLY-PT homology.

Remark 1.7 The spectral sequences from $H_F(S)$ to $H_n(S)$ may seem strange in the context of the conjectured spectral sequence from Khovanov homology to knot Floer homology, since, assuming this conjecture is true, we would have rank inequalities among the reduced homologies

$$\mathrm{rk}(\overline{H}_H(K)) \geq \mathrm{rk}(\overline{H}_n(K)) \geq \mathrm{rk}(\overline{H}_{n-1}(K)) \geq \cdots \geq \mathrm{rk}(\overline{H}_2(K)) \geq \mathrm{rk}(\widehat{HFK}(K))$$

for any knot K. The intuition behind this is that knot Floer homology has large homology at each vertex in the cube of resolutions, but unlike HOMFLY-PT and $\mathfrak{sl}_n$ homology, it has higher face maps which allow the total homology of a knot to still be smaller.

Outline of the paper

In Section 2 we give background on HOMFLY-PT and $\mathfrak{sl}_n$ homology for singular braids. In Section 3 we describe a recursion formula for $\mathfrak{sl}_n$ homology known as the composition product — see Wagner [21] — and prove a generalization to HOMFLY-PT homology. In Section 4 we describe the knot Floer homology of singular braids and define a filtration on the associated complex. The main theorems are proved in this section using the filtration together with the composition product formulas from Section 3.

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2 The Khovanov–Rozansky homology of singular links

2.1 Singular resolutions and the ground ring

A complete resolution S of a knot K in braid position can be viewed as an oriented planar graph with the following properties:

(1) All vertices are either 2–valent or 4–valent.

(2) The number of incoming edges is equal to the number of outgoing edges at each vertex.

(3) If Z is an oriented cycle in S, then the unique disc $D \subset \mathbb{R}^2$ with boundary Z intersects the center of the braid.

Let $e_1, \dots, e_k$ denote the edges of S. To each edge e_i, we assign an indeterminate U_i. All three homology theories will be defined over the ground ring $R = \mathbb{Q}[U_1, \dots, U_k]$.

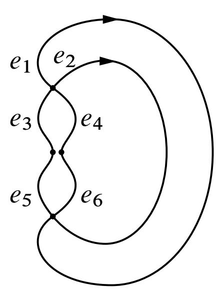

Figure 1: An example of a singular braid diagram

2.2 HOMFLY-PT homology and $\mathfrak{sl}_n$ homology

This section will give a brief description of the HOMFLY-PT and $\mathfrak{sl}_n$ homologies as defined in [9; 10]. We will use the grading conventions from [18], though we will leave out the overall grading shifts coming from the braid number. The reader can refer to these resources for further background. The HOMFLY-PT and $\mathfrak{sl}_n$ complexes have the same generators, with the $\mathfrak{sl}_n$ complex having strictly more differentials than the HOMFLY-PT complex (see Remark 2.1). For this reason, we will start by defining the HOMFLY-PT complex, then we will describe the additional differentials to make the $\mathfrak{sl}_n$ complex.

Remark 2.1 In all of our homology theories, the differential is a map $d\colon C \to C$ with $d^2 = 0$, where C is free R–module. However, we will also commonly refer to a particular component of d as a "differential" as well. For example, if the coefficient of y in the basis expansion of $d(x)$ is $a \in R$, then we will say that there is a differential from x to y with coefficient a.

The HOMFLY-PT complex for links comes equipped with a triple-grading, and the $\mathfrak{sl}_n$ complex with a bigrading. One of the gradings in both theories, however, comes from the height in the cube of resolutions, so it will be fixed for a single resolution. The HOMFLY-PT complex will therefore come with a bigrading, and the $\mathfrak{sl}_n$ complex with a single grading. For the HOMFLY-PT complex, these gradings are called the quantum grading, denoted gr_q, and the horizontal grading, denoted gr_h. Multiplication by the U_i increases the quantum grading by 2 and preserves the horizontal grading.

Let $V_2(S)$ denote the 2–valent vertices in S and $V_4(S)$ the 4–valent vertices of S. For vertices v in $V_2(S)$, there is a unique outgoing edge e_i and a unique incoming edge e_j. Define $L(v)$ to be the linear term $U_i - U_j$. Similarly, for vertices v in $V_4(S)$ there are two outgoing edges e_i and e_j and two incoming edges e_k and e_l. We define $L(v)$ to be the linear term $U_i + U_j - U_k - U_l$ and $Q(v)$ to be the quadratic term $U_iU_j - U_kU_l$.

The HOMFLY-PT complex is a tensor product of complexes $C_H(v)$ for each vertex v. For v in $V_2(S)$, $C_H(v)$ is defined as

$$R\{0,-2\} \xrightarrow{L(v)} R\{0,0\},$$

where $R\{i,j\}$ refers to the free R–module of rank 1 shifted by i in gr_q and by j in gr_h, so that the generator lies in bigrading $\{i,j\}$. For v in $V_4(S)$, $C_H(v)$ is defined as in Figure 2.

$$\begin{CD} R\{1,-4\} @>{L(v)}>> R\{1,-2\} \\ @V{Q(v)}VV @VV{Q(v)}V \\ R\{-1,-2\} @>{-L(v)}>> R\{-1,0\} \end{CD}$$

Figure 2: The HOMFLY-PT complex at a 4–valent vertex v

Note that the differential is homogeneous of degree $\{2,2\}$. The HOMFLY-PT complex for the singular diagram S is given by

$$C_H(S) = \bigotimes_{v \in S} C_H(v),$$

where the tensor product is taken over R and the HOMFLY-PT homology $H_H(S)$ is the homology of $C_H(S)$.

We will now define the additional differentials which give $\mathfrak{sl}_n$ homology. For a vertex v in S with outgoing edges E_{out} and incoming edges E_{in}, let the potential w_n be given by

$$w_n(v) = \sum_{e_i \in E_{\text{out}}} U_i^{n+1} - \sum_{e_j \in E_{\text{in}}} U_j^{n+1}.$$

For v in $V_2(S)$, let $u_1(v)$ be the unique element in R such that $u_1(v)L(v) = w_n(v)$. For v in $V_4(S)$, we can choose $u_1(v)$ and $u_2(v)$ such that

$$u_1(v)L(v) + u_2(v)Q(v) = w_n(v).$$

Unlike the 2–valent case, the choice is not unique, but the reader can refer to [9, page 5] for the precise choice. (It is not relevant for our discussion.)

For each vertex v, we will add new differentials to $C_H(v)$ to make a new complex $C_n(V)$. For v in $V_2(S)$, $C_n(v)$ is given by

$$R\{0,-2\} \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\{0,0\}$$

and for v in $V_4(S)$, $C_n(v)$ is given by Figure 3.

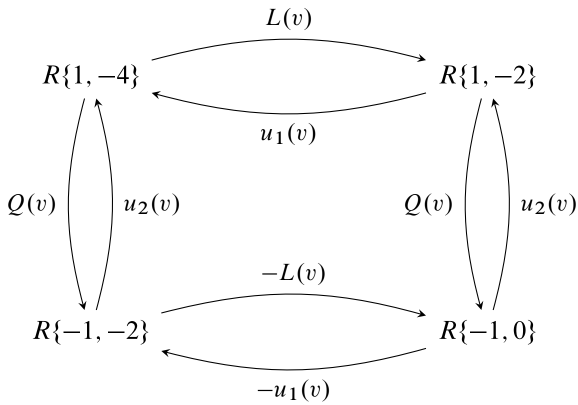

Figure 3: The $\mathfrak{sl}_n$ complex at a 4–valent vertex v

Observe that for both types of vertices, the differential on $C_n(v)$ satisfies $d^2 = w_n(v)I$. Such a complex is called a matrix factorization with potential w_n. Since d^2 is nonzero, its homology is not well defined. However, we are interested in the tensor product of $C_n(v)$ over all vertices v in S. Define the $\mathfrak{sl}_n$ complex $C_n(S)$ by

$$C_n(S) = \bigotimes_{v \in S} C_n(v),$$

where again the tensor product is taken over R.

As mentioned above, the HOMFLY-PT differentials are homogeneous of degree $\{2, 2\}$. These differentials are denoted by d_+. The new differentials, those with coefficients $u_1(v)$ and $u_2(v)$, are homogeneous of degree $\{2n, -2\}$. These are denoted by d_-. The total differential $d_{\mathrm{tot}} = d_+ + d_-$ is not homogeneous in this bigrading. However, if we look at the grading $\mathrm{gr}_n = \mathrm{gr}_q + \frac{1}{2}(n-1)\,\mathrm{gr}_h$, then d_{tot} is homogeneous of degree $n+1$.

Additionally, $d_{\mathrm{tot}}^2 = 0$. This can be seen from the fact that the potential is additive under tensor product, so $d_{\mathrm{tot}}^2 = \sum_{v \in S} w_n(v)$. The sum must be zero because each edge e_i is an outgoing edge for one vertex, which will contribute U_i^{n+1}, and an incoming edge for another vertex, which will contribute $-U_i^{n+1}$.

This shows that $C_n(S)$ is a well-defined chain complex which is homogeneous with respect to the grading gr_n. We define the $\mathfrak{sl}_n$ homology $H_n(S)$ to be the homology of this complex.

Remark 2.2 The definitions given here correspond to the unreduced theories in [18] as opposed to the middle or reduced homologies. To translate between our definition

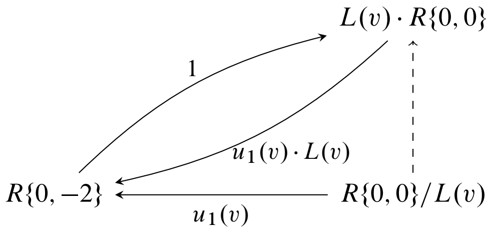

Figure 4

and Rasmussen's, it suffices to show they are the same for connected diagrams S, as in both theories disjoint union corresponds to tensor product over $\mathbb{Q}$.

Let v_0 be a bivalent vertex in S (insert one if necessary). The linear relations coming from all of the vertices besides v form a regular sequence in R — this is shown in Rasmussen's proof of Lemma 3.11 in [18]. We can write our complex $C_n(S)$ as

$$C_n(S) = C_n^Q(S) \otimes_R \big(R\{0,-2\} \underset{u_1(v_0)}{\overset{L(v_0)}{\rightleftarrows}} R\{0,0\}\big) \otimes_R C_n^L(S - v_0),$$

where $C_n^Q(S)$ is the Koszul complex on the terms $(Q(v), u_2(v))$ for all 4–valent vertices v, and $C_n^L(S - v_0)$ is the Koszul complex on the terms $(L(v), u_1(v))$ for all vertices except v_0 (see [18, Definition 3.7] for the definition of a Koszul complex in the context of $\mathbb{Z}$–graded matrix factorizations). Each Koszul factor in $C_n^L(S - v_0)$ is given by

$$R\{0,-2\} \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\{0,0\}$$

for some vertex v. This complex can be rewritten as in Figure 4, where the dashed line refers to the fact that multiplication by $L(v)$ in $R\{0,0\}/L(v)$ maps into $L(v)\cdot R\{0,0\}$ in the obvious way (ie it is recording the R–module structure, not the differential).

After tensoring this matrix factorization with the remaining terms to obtain $C_n(S)$, the map by 1 in this diagram gives a contractible subcomplex, so our complex $C_n(S)$ is chain homotopy equivalent to the complex in which the linear Koszul factor is replaced with the quotient complex $R\{0,0\}/L(v)$. Since all of the elements $L(v)$ in $C_n^L(S - v_0)$ form a regular sequence, we can do this recursively until $C_n^L(S - v_0)$ has been replaced with the complex $R/L = R\{0,0\}/\{L(v) = 0 \text{ for } v \neq v_0\}$. (The fact that the elements form a regular sequence guarantees that the maps labeled by 1 in the diagram are always isomorphisms rather than just surjections, making the subcomplex contractible at each stage.)

Since $L(v_0)$ is the negative of the sum of the $L(v)$ for $v \neq v_0$, $L(v_0)$ is in the ideal L as well. Thus, we have shown that

$$C_n(S) \cong C_n^Q(S) \otimes_R (R\{0,-2\} \underset{u_1(v_0)}{\overset{0}{\rightleftarrows}} R\{0,0\}) \otimes_R R/L.$$

This corresponds to the definition of the unreduced theory in [18, Definition 2.14]. The middle theory can be obtained by dropping the Koszul complex $R\{0,-2\} \underset{u_1(v_0)}{\overset{0}{\rightleftarrows}} R\{0,0\}$.

Remark 2.3 This argument that Koszul complexes on regular sequences can be simplified to quotients R/I where I is the ideal generated by the elements of the regular sequence will be repeated throughout the paper. When we are using this argument, we will simply say that we are canceling the corresponding differentials. The above discussion describes why no new differentials arise as a result of this cancelation.

2.3 Rasmussen's spectral sequences

Rasmussen [18] showed that there is a family of spectral sequences $E_k(n)$ which start at HOMFLY-PT homology and converge to $\mathfrak{sl}_n$ homology. The existence of these spectral sequences is somewhat difficult to prove for the case of knots and links, but the argument is much simpler for fully singular diagrams.

With respect to the horizontal grading, d_+ has grading 2 and d_- has grading -2. If we take homology with respect to d_+, all induced differentials decrease the horizontal grading. Thus, there is a well-defined spectral sequence $(E_k(n), d_k(n))$, where d_k is defined to be the part of d_{tot} which increases the horizontal grading by $2-4k$.

The E_1 page of this spectral sequence is $H_*(C_n(S), d_+)$, which is exactly the definition of HOMFLY-PT homology. Since $C_n(S)$ is bounded in horizontal grading, the E_∞ page is the homology with respect to d_{tot}, or $\mathfrak{sl}_n$ homology. It turns out (Corollary 2.7) that this spectral sequence collapses at the E_2 page, given by

$$H_*(H_*(C_n(S), d_+), d_-^*).$$

We will denote this page by $H^\pm(C_n(S))$. Note that $H^\pm(C_n(S))$ is bigraded, as both d_+ and d_- are homogeneous. In order to see why this spectral sequence collapses, we must first introduce the rotation number.

Definition 2.4 Given a (possibly singular) diagram D, let D' denote the diagram obtained by replacing each crossing or singularization with the oriented smoothing. The resulting diagram is a collection of oriented circles. These circles are called the *Seifert circles* of D.

Definition 2.5 The *rotation number* of a (possibly singular) diagram D is the sum of the signs of the Seifert circles, with a circle contributing a $+1$ if it is oriented counterclockwise and a -1 if it is oriented clockwise.

The fact that all higher differentials are trivial follows from the following lemma:

Lemma 2.6 [18] *The homology $H^{\pm}(C_n(S))$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$, where $r(S)$ is the rotation number of S. Since S is a singular braid oriented clockwise, $r(S)$ is the negative of the number of strands in S.*

Since none of the higher differentials preserve the horizontal grading, they must all be trivial, causing the spectral sequence to collapse.

Corollary 2.7 *Viewing $H^{\pm}(C_n(S))$ as singly graded with grading gr_n, there is a graded isomorphism $H^{\pm}(C_n(S)) \cong H_n(S)$.*

Remark 2.8 The reader familiar with [18] may note that our homology lies in $\mathrm{gr}_h = 2r(S)$, while Rasmussen's lies in $\mathrm{gr}_h = 1 + r(S)$. This difference comes from the fact that our homology is unreduced, which decreases the grading by 1, and because we are leaving out the overall grading shift of $-r(S)$ in [18, Definition 2.14].

3 A recursion formula for the Khovanov–Rozansky homology of singular links

3.1 The composition product

3.1.1 Jaeger's formula The first composition product formula was defined by Jaeger in [7]. In order to discuss the composition product, we must first define labelings of a diagram. Let K be a knot with corresponding diagram D. Viewing D as an oriented 4–valent graph, we say that a subset S of the edges of D is a homological cycle if at each vertex in D the number of incoming edges in S is equal to the number of outgoing edges in S. A *labeling* f of the diagram D is a function from the set of edges in D to the set $\{1, 2\}$ such that $f^{-1}(1)$ is a homological cycle. (Note that $f^{-1}(1)$ is a homological cycle if and only if $f^{-1}(2)$ is a homological cycle.)

Figure 5: Nonadmissible labelings

We will place a restriction on which homological cycles are allowed. A homological cycle is said to make a turn at a crossing c if it has one incoming edge at c and one outgoing edge at c and those edges are not diagonal from one another. Let $T(f)$ denote the number of turns of the labeling f. A labeling f is *admissible* if $f^{-1}(1)$ doesn't make any left turns at positive crossings or right turns at negative crossings.

Since the homological cycle $f^{-1}(1)$ uniquely determines the labeling f, we will say that a homological cycle Z is admissible if the unique labeling f with $f^{-1}(1)=Z$ is admissible. The two cycles $f^{-1}(1)$ and $f^{-1}(2)$ can both be viewed as diagrams of links if we retain the crossing information whenever one of them contains all four edges at a crossing, and forget it otherwise. We will refer to these diagrams as $D_{f,1}$ and $D_{f,2}$, respectively. Note that $r(D_{f,1})+r(D_{f,2})=r(D)$.

With the HOMFLY-PT polynomial P_H as defined in the introduction, define

$$P'_H(a,q,D)=\left(\frac{a-a^{-1}}{q-q^{-1}}\right)(a^{w(D)})P_H(a,q,D).$$

Note that P'_H is invariant under Reidemeister II and III moves, but performing a Reidemeister I move changes the writhe, so one picks up a factor of a or a^{-1} depending on the sign of the crossing. With this normalization, $P'_H(\mathcal{O})=(a-a^{-1})/(q-q^{-1})$, and $P'_H(\varnothing)=1$, where $\varnothing$ denotes the empty diagram and $\mathcal{O}$ denotes the crossingless diagram for the unknot. Jaeger's composition product can be stated as

$$\sum_{f \text{ admissible}} (q-q^{-1})^{T(f)} a_1^{r(D_{f,2})} a_2^{-r(D_{f,1})} P'_H(a_1,q,D_{f,1}) P'_H(a_2,q,D_{f,2}) = P'_H(a_1a_2,q,D). \tag{1}$$

The proof of this formula is combinatorial in nature — one can show that it behaves properly under Reidemeister moves and that it satisfies the necessary skein relation via local computations. In fact, Jaeger [7] showed that this formula is invariant under all Reidemeister moves, so the formula holds for arbitrary diagrams D instead of just braid diagrams.

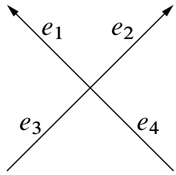

Figure 6: A labeled 4–valent vertex

3.1.2 The composition product for singular graphs

Defining

$$P'_n(q,D) = P'_H(q^n,q,D),$$

the composition product formula can be specialized to the $\mathfrak{sl}_n$ polynomials to give

$$\sum_{f \text{ admissible}} (q-q^{-1})^{T(f)} q^{mr(D_{f,1})-nr(D_{f,2})} P'_n(q,D_{f,1}) P'_m(q,D_{f,2}) = P'_{m+n}(q,D). \tag{2}$$

This formula was extended by Wagner to singular braids in the following way. If S is a singular braid, we can define labelings of S in the same way as labelings for knots. We will drop the admissibility condition at 4–valent vertices since they no longer correspond to positive or negative crossings. Note that as with the nonsingular diagrams, we have $r(S_{f,1})+r(S_{f,2})=r(S)$. Given a labeling f of S, let $T_1(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_1 and e_3 in Figure 6. Similarly, let $T_2(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_2 and e_4. With this terminology, the composition product for singular braids can be stated as

$$P'_{m+n}(S) = \sum_{f \in L(S)} q^{\sigma_{m,n}(f)} P'_n(S_{f,1}) P'_m(S_{f,2}),$$

where $P'_k(S)$ is the unreduced $\mathfrak{sl}_k$ polynomial of S,

$$\sigma_{m,n}(f) = T_2(S_{f,1}) - T_1(S_{f,1}) + mr(S_{f,1}) - nr(S_{f,2})$$

and $r(S_{f,i})$ is the negative of the number of strands in the singular braid $S_{f,i}$.

Remark 3.1 The $\mathfrak{sl}_k$ polynomials for singular braids are defined by the skein relations

$$P_k(D_x) = qP_k(D_0) - q^k P_k(D_+) = q^{-1}P_k(D_0) - q^{-k}P_k(D_-)$$

and

$$P'_k(D_x) = qP'_k(D_0) - P'_k(D_+) = q^{-1}P'_k(D_0) - P'_k(D_-),$$

where D_+, D_-, D_x and D_0 are diagrams which are the same away from a crossing, but locally D_+ has a positive crossing, D_- has a negative crossing, D_x has the singularization and D_0 has the oriented smoothing. For a fully singular graph S, the writhe is zero, so $P'_k(S) = (q^k - q^{-k})/(q - q^{-1})P_k(S)$. Thus, $P'_k(S)$ is categorified by the unreduced $\mathfrak{sl}_k$ homology of S, and $P_k(S)$ is categorified by the reduced $\mathfrak{sl}_k$ homology of S.

3.2 The categorification for $\mathfrak{sl}_n$ homology

Wagner further showed that the composition product formula is true on the level of categorifications:

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{\sigma_{m,n}(f)\} \quad \text{(with polynomial grading)},$$

where the $\mathfrak{sl}_n$ homology groups are singly graded, using the polynomial grading gr_n. In order for us to generalize this theorem, it will be useful to have a bigraded version of it using our $(\mathrm{gr}_q, \mathrm{gr}_h)$ gradings. Since the bigraded $H_n(S)$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$, and $\mathrm{gr}_n = \mathrm{gr}_q + \frac{1}{2}(n-1)\,\mathrm{gr}_h$, the q–grading is uniquely determined by gr_n:

$$\mathrm{gr}_n = \mathrm{gr}_q + (n-1)r(S).$$

Thus, if we view each $H_n(S)$ as singly graded, where the grading is gr_q instead, the formula becomes

$$\begin{aligned} &H_{m+n}(S)\{(n+m-1)r(S)\} \\ &\qquad = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{\sigma_{m,n}(f) + (n-1)r(S_{f,1}) + (m-1)r(S_{f,2})\}. \end{aligned}$$

This can be simplified to

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2nr(S_{f,2})\}.$$

Finally, to make this bigraded, we need to add in the horizontal grading. Since $H_{m+n}(S)$ lies in horizontal grading $2r(S)$, $H_n(S_{f,1})$ lies in horizontal grading $2r(S_{f,1})$, and $H_m(S_{f,2})$ lies in horizontal grading $2r(S_{f,2})$, we see that the tensor product

$$H_n(S_{f,1}) \otimes H_m(S_{f,2})$$

always lies in horizontal grading $2r(S_{f,1}) + 2r(S_{f,2})$. Since $r(S_{f,1}) + r(S_{f,2}) = r(S)$, it follows that the tensor product lies in grading $2r(S)$ — the same grading as $H_{m+n}(S)$.

Thus, we can add the horizontal grading to the composition product formula, with no grading shift required for each labeling f. The bigraded formula is then

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2nr(S_{f,2}), 0\},$$

where the bigrading is given by $(\mathrm{gr}_q, \mathrm{gr}_h)$.

3.3 A categorification for HOMFLY-PT homology

In relating these formulas to knot Floer homology, we will be most interested in the case when $n = 1$. In this case, the previous formula becomes

$$(3) \quad H_{m+1}(S) = \bigoplus_{f \in L(S)} H_1(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}.$$

In this section we will prove a generalization of this formula to HOMFLY-PT homology. Letting $H_H(S)$ denote the HOMFLY-PT homology with the standard bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$, define $H_H(S)\langle k\rangle$ to be HOMFLY-PT homology with a new grading $(\mathrm{gr}_q + k\,\mathrm{gr}_h, \mathrm{gr}_h)$.

Theorem 3.2 *There is an isomorphism of bigraded vector spaces*

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\} \cong H_H(S)\langle 1\rangle.$$

The proof of this theorem will rely heavily the fact that for all $n \geq 1$, there is a differential $d_-(n)$ on $H_H(S)$ which is homogeneous with bigrading $\{2n, -2\}$, and

$$H_*(H_H(S), d_-(n)) \cong H_n(S),$$

where we are viewing $H_n(S)$ as a bigraded vector space. (This is the homology $H_n^{\pm}(S)$ from Section 2.3.) Recall that as a bigraded vector space, $H_n(S)$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$.

The theorem will be proved in two parts. First we will show that any bigraded vector space with certain algebraic properties must be isomorphic to $H_H(S)\langle k\rangle$, and then we will show that our construction satisfies those properties for $k = 1$.

Lemma 3.3 *Let $H(S)$ denote a bigraded vector space over $\mathbb{Q}$ with the following properties:*

(1) *$H(S)$ is bounded above and below in horizontal grading.*

(2) *$H(S)$ is bounded below in q–grading.*

(3) *$H(S)$ is finite-dimensional in each bigrading.*

(4) *For each $n \geq n_0$, there is a differential d_n on $H(S)$ which is homogeneous with bigrading $\{2n, -2\}$ such that $H_*(H(S), d_n) \cong H_n(S)$, where the isomorphism is as bigraded vector spaces.*

Then $H(S) \cong H_H(S)$.

Note that HOMFLY-PT homology itself satisfies these conditions, so they are not vacuous.

Proof Let $H^{i,j}$ denote the homology in bigrading $\{i, j\}$, and similarly for $H_H^{i,j}(S)$. The lemma states that for any integers i and j, $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$.

Suppose for some i and j, the dimensions do not agree. Since $H(S)$ is bounded in horizontal grading, we can take choose i_0 and j_0 so that j_0 is minimized subject to the constraint that $\dim(H^{i_0,j_0}(S)) \neq \dim(H_H^{i_0,j_0}(S))$. Note that the choice for i_0 may not be unique.

Since both $H(S)$ and $H_H(S)$ are bounded below in q–grading, there exists a constant $a(S)$ such that for all $i \leq a(S)$, $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S)) = 0$.

Choose $n > \max(i_0 - a, n_0)$. We will now use the fact that $H_*(H(S), d_n) \cong H_n(S)$ as bigraded vector spaces. We can rewrite $\mathfrak{sl}_n$ homology as the homology of $H_H(S)$ with respect to $d_-(n)$:

$$H_*(H(S), d_n) \cong H_*(H_H(S), d_-(n)).$$

The differentials on both of these complexes have bigrading $\{2n, -2\}$. We can put an equivalence relation on $\mathbb{Z}^2$, where $(i, j) \sim (i', j')$ if $(i - i', j - j') = k(2n, -2)$, and both complexes must split according to this equivalence relation. In other words, for a fixed equivalence class A, the sum

$$\bigoplus_{(i,j)\in A} H^{i,j}(S)$$

gives a subcomplex of $(H(S), d_n)$, and similarly for $(H_H(S), d_-(n))$.

Consider the summand corresponding to the equivalence class of (i_0, j_0). For the complex $(H(S), d_n)$, this summand looks like

$$\cdots \xrightarrow{d_n} H^{i_0-2n, j_0+2}(S) \xrightarrow{d_n} H^{i_0,j_0}(S) \xrightarrow{d_n} H^{i_0+2n, j_0-2}(S) \xrightarrow{d_n} \cdots$$

and, for $(H_H(S), d_-(n))$,

$$\cdots \xrightarrow{d_-(n)} H_H^{i_0-2n,j_0+2}(S) \xrightarrow{d_-(n)} H_H^{i_0,j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n,j_0-2}(S) \xrightarrow{d_-(n)} \cdots .$$

Now, since the complex is bounded below in q–grading and we've chosen n sufficiently large, all of the chain groups before the (i_0, j_0) summand are trivial, so the complexes become

$$\cdots 0 \xrightarrow{d_n} 0 \xrightarrow{d_n} H^{i_0,j_0}(S) \xrightarrow{d_n} H^{i_0+2n,j_0-2}(S) \xrightarrow{d_n} \cdots$$

and

$$\cdots 0 \xrightarrow{d_-(n)} 0 \xrightarrow{d_-(n)} H_H^{i_0,j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n,j_0-2}(S) \xrightarrow{d_-(n)} \cdots .$$

Since both complexes are bounded in horizontal grading, the two chain complexes are both finitely generated. They therefore have a well-defined Euler characteristic, and since the homologies are isomorphic, the Euler characteristics must be the same. Since the alternating sum of the dimension of homology is the same as the alternating sum of the dimension of the chain groups themselves, we have

$$\sum_{k=0}^{N} (-1)^k \dim(H^{i_0+2nk,j_0-2k}(S)) = \sum_{k=0}^{N} (-1)^k \dim(H_H^{i_0+2nk,j_0-2k}(S)).$$

Furthermore, we know that there is an equality $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$ for $j < j_0$. Thus, the two subcomplexes above have the same dimension in all of the bigradings except bigrading (i_0, j_0). But this means that for $k \geq 1$, the terms in the two sums are equal, which forces the $k = 0$ terms to be equal. This contradicts our assumption that $\dim(H^{i_0,j_0}(S)) \neq \dim(H_H^{i_0,j_0}(S))$, proving the isomorphism. □

Corollary 3.4 *Let $H(S)$ be a bigraded vector space over $\mathbb{Q}$ that satisfies the conditions in Lemma 3.3 with one difference — instead of d_n having bigrading $\{2n, -2\}$, it has bigrading $\{2n-2k, -2\}$. Then $H(S) \cong H_H(S)\langle k\rangle$.*

Proof After the change in grading $(\mathrm{gr}_q, \mathrm{gr}_h) \mapsto (\mathrm{gr}_q + k\,\mathrm{gr}_h, \mathrm{gr}_h)$, the conditions of Lemma 3.3 are satisfied, so this homology is isomorphic to $H_H(S)$. Shifting back to the original gradings proves the corollary. □

Proof of Theorem 3.2 To prove Theorem 3.2, we need to show that our homology

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}$$

satisfies the conditions of this corollary for $k = 1$. It satisfies Lemma 3.3(1)–(3) since each of the summands do, so we just need to define the d_n differentials.

We will define d_n for $n \geq 2$ as follows. It will preserve the direct sum decomposition, and it will act on each $H_1(S_{f,1}) \otimes H_H(S_{f,2})$ summand by $1 \otimes d_-(n-1)$, where $d_-(n-1)$ is the standard $\mathfrak{sl}_{n-1}$ differential on the HOMFLY-PT homology of $S_{f,2}$.

Since $H_*(H_H(S_{f,2}), d_-(n-1)) \cong H_{n-1}(S_{f,2})$, the homology with respect to d_n is

$$\bigoplus_f H_1(S_{f,1}) \otimes H_{n-1}(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}.$$

From Section 3.2, we know that this sum is isomorphic as a bigraded vector space to $H_n(S)$. The differential d_n has bigrading $\{2n-2, -2\}$, so applying Corollary 3.4, this proves Theorem 3.2. □

4 The knot Floer complex at a vertex in the cube of resolutions

4.1 Definition of the complex

We will assume that the reader is familiar with Heegaard diagrams and knot Floer homology. For background on the subject, refer to [13; 14; 17]. The oriented cube of resolutions for *HFK* was originally defined with twisted coefficients by Ozsváth and Szabó, and they noted some similarities between their complex and HOMFLY-PT homology [15]. The complex was further studied by Gilmore, who reframed the relationships in terms of framed trivalent graphs [5].

However, we will be dealing with the untwisted version defined by Manolescu [12]. This is in some ways the most natural version, as it doesn't involve twisted coefficients, and the total homology of the complex is the usual knot Floer homology. The oriented cube of resolutions uses the Heegaard diagram $\mathcal{H}(S)$ shown in Figure 7.

Note that in [12], there is a marked bivalent vertex on the leftmost strand in the braid at which an α and a β circle are removed. Our picture is slightly different — instead of having a marked edge, we place an additional X and O outside of our braid; we will denote them by X_{new} and O_{new}. Since discs are not allowed to pass through X_{new}, this can be viewed as puncturing the sphere, making our diagram a truly planar diagram. We will also set U_{new} equal to zero to avoid increasing the ground ring.

The knot Floer complex corresponding to this Heegaard diagram is denoted by $CFK^-(S)$. There are several versions of knot Floer homology, and the minus sign

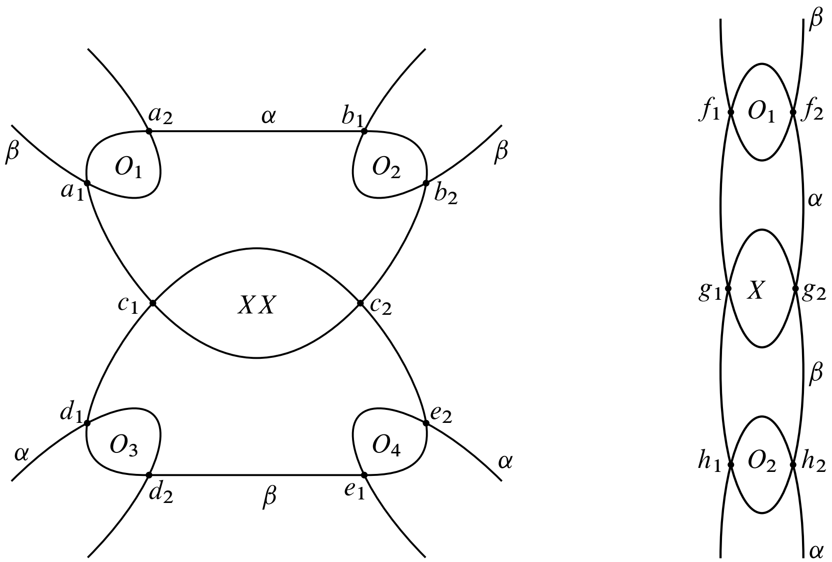

Figure 7: The local Heegaard diagram for a singular link at a 4–valent vertex (left) and a 2–valent vertex (right)

refers to the fact that none of the U_i in the ground ring will be set to 0 (except for U_{new}). For consistency with the HOMFLY-PT and $\mathfrak{sl}_n$ definitions, we will be working with $\mathbb{Q}$ coefficients.

We can relate the complex $CFK^-(S)$ to the complex from Manolescu's Heegaard diagram as follows. Place a bivalent vertex on the leftmost strand of the braid, and let α_1 and β_1 be the corresponding α and β circles. We can handleslide α_1 over all the other α circles and β_1 over all of the other β circles, so that we are left with Manolescu's diagram together with the diagram shown in Figure 8. The complex for this diagram is homotopy equivalent to the tensor product of Manolescu's complex with $H_*(S^1)$, so its homology has twice the rank of Manolescu's. (This can also be seen by Manolescu's formula for disjoint union of singular braids [12, page 198].)

Remark 4.1 Strictly speaking, this complex also depends on some auxiliary information, including a choice of a path of almost complex structures and orientations on the moduli spaces of the holomorphic disks, called a *system of orientations*. There are many choices for orienting these moduli spaces, but it was shown by Alishahi and Eftekhary that there is always a system of orientations such that Maslov index 2 α–degenerations come with positive sign, and the β–degenerations come with negative sign (see [1, Section 5.1]). Note that CFK^- is a special case of their sutured HF^-

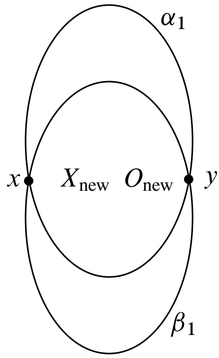

Figure 8: The extra unknot outside the braid diagram

construction. Sarkar [20] showed that any two systems of orientations satisfying this property give chain homotopy equivalent complexes. We will leave out any discussion about complex structure, since none of the discs counted in our computations depend on the complex structure.

The complex ascribed to a vertex in the cube of resolutions, which we will denote by $C_F(S)$, is the tensor product of $CFK^-(S)$ with a certain Koszul complex. Using the terminology from the previous section, we can define $C_F(S)$ as

$$C_F(S) = CFK^-(S) \otimes \bigotimes_{v \in V_4(S)} (R \xrightarrow{L(v)} R).$$

We denote the Koszul complex by $K(S)$. The bigrading on $C_F(S)$ will be described in Section 4.2.3.

4.1.1 The generators of $CFK^-(S)$ In order to understand the homology of $C_F(S)$, we are going to need some tools for understanding $CFK^-(S)$. Let $E(S)$ denote the set of edges of S, and let x be a generator of the complex $CFK^-(S)$ (ie an n–tuple of intersection points of the α and β curves). We ascribe a subset Z of E to the generator x as follows.

Each O_i in the Heegaard diagram is contained in a unique minimal bigon. The boundary of this bigon contains two intersection points — if either of these intersection points are in the n–tuple x, then e_i is in Z. For example, in Figure 7, left, there are five types of generators: (a,d), (a,e), (b,d), (b,e) and (c). The underlying sets of edges of these generators are e_1e_3, e_1e_4, e_2e_3, e_2e_4 and $\varnothing$, respectively.

As observed in [15], Z must satisfy two conditions. First, for any vertex v in S, the number of incoming edges in Z must equal the number of outgoing edges in Z, and second, Z cannot contain all four edges at any 4–valent vertex in S. In other words, Z must be a disjoint union of oriented circles contained in S. We call such a set of edges a *multicycle*. Note that multicycles differ from the homological cycles in Section 3.1 in that multicycles cannot contain all four edges at a vertex.

Let $CFK^-(Z)$ denote the R–module spanned by generators x where the multicycle underlying x is Z, and let $C_F(Z)$ be the tensor product of $CFK^-(Z)$ with the Kozsul complex:

$$C_F(Z) = CFK^-(Z) \otimes \bigotimes_{v \in V_4(S)} (R \xrightarrow{L(v)} R).$$

4.2 The filtered complex and the spectral sequence from HOMFLY-PT homology to *HFK*

4.2.1 A filtration on $CFK^-(S)$ It turns out that there is a filtration on $CFK^-(S)$ that divides generators according to their underlying cycles. In other words, if there is a filtration-preserving differential from x to y, then x and y have the same underlying cycle.

In [15, Section 3], Ozsváth and Szabó choose a distinguished generator x_0 of $CFK^-(S)$ corresponding to the empty cycle. Using Figure 9, this generator is defined to contain the intersection point c_1 at each 4–valent vertex and the intersection point g_1 at each bivalent vertex.

The filtration is induced by placing additional basepoints p_i in our Heegaard diagram, as shown in Figure 9. The markings p_i are in canonical bijection with regions in $\mathbb{R}^2 - S$. Let N denote the number of such regions.

Definition 4.2 The *basepoint filtration* is a map

$$\mathcal{F}\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}^N.$$

The i^{th} component of $\mathcal{F}(x)$ is defined to be the multiplicity of ϕ at p_i, where ϕ is a homotopy class in $\pi_2(x, x_0)$ with multiplicity 0 at X_{new}. The map $\mathcal{F}$ extends to a (multi)filtration on $CFK^-(S)$ with multiplication by U_j preserving the filtration level.

Lemma 4.3 *The filtration $\mathcal{F}$ is well defined, ie it does not depend on the choice of ϕ.*

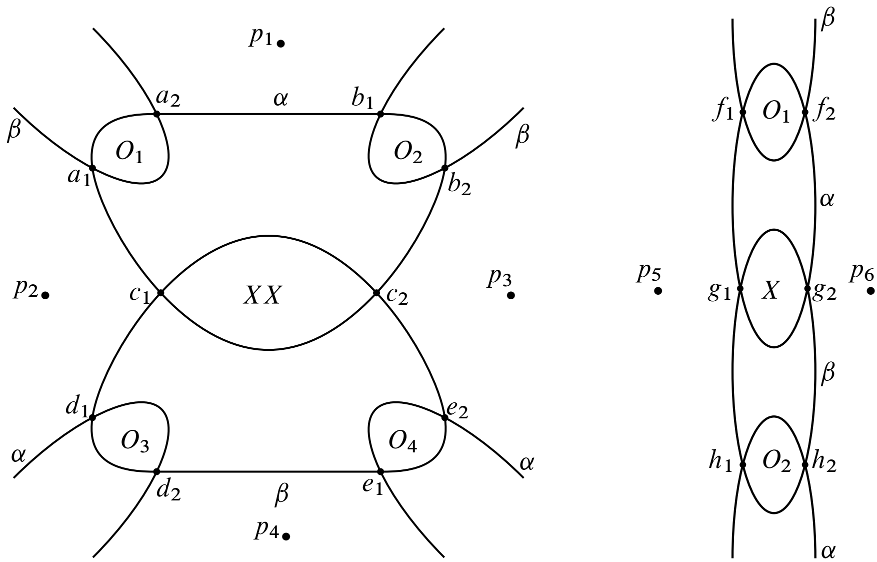

Figure 9: Local diagrams with additional markings

Proof Since any two such ϕ differ by a periodic domain, it is sufficient to show that any periodic domain with multiplicity 0 at X_{new} has multiplicity 0 at these markings. This follows from that fact that for any α or β circle, the markings and X_{new} lie on the same side. So, for any periodic domain, the multiplicity at any of these points is the same as that of X_{new}, which is required to be 0. □

Let $x(Z)$ denote the generator which, at each v not in Z, has the same intersection point as x_0 and for each e_i in Z has the leftmost generator on the bigon containing O_i.

Lemma 4.4 *If x is $CFK^-(S)$ with underlying cycle Z, then $\mathcal{F}(x) = \mathcal{F}(x(Z))$.*

Proof We will build a homotopy class $\phi \in \pi_2(x, x(Z))$ such that ϕ has multiplicity 0 at every p_i. Start with ϕ being constant on x. For each bivalent vertex v not in Z, if y has the left intersection point, we do nothing, and if x has the right intersection point, we add one of the two bigons which goes from the right intersection point to the left intersection point. In Figure 9, these are the bigons from g_2 to g_1 which pass through O_1 and O_2, respectively.

Similarly, for each 4–valent vertex v not in Z and each edge e in Z where x has a different intersection point from $x(Z)$, we can find a bigon from the x intersection point to the $x(Z)$ intersection point. Defining ϕ_1 to be the union of these bigons, we get a homotopy class from x to $x(Z)$ which does not pass through any of the basepoints. □

Define $F\colon \{\text{multicycles}\} \to \mathbb{Z}^N$ as follows. Let Z be a multicycle, and let C be a 2–chain in S^2 with boundary Z. If we require that C has multiplicity 0 on the outer region (the one corresponding to X_{new}), it is clear that this 2–chain is unique. Define the i^{th} component of $F(Z)$ to be the multiplicity of C at the region corresponding to p_i.

Lemma 4.5 *If x is a generator of $CFK^-(S)$ with underling cycle Z, then $\mathcal{F}(x) = F(Z)$.*

Proof We have already shown that $\mathcal{F}(x) = \mathcal{F}(x(Z))$, so it suffices to show that $\mathcal{F}(x(Z)) = F(Z)$.

Write Z in terms of its components $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_k$. For each i, Ozsváth and Szabó identify two Maslov index 1 homotopy classes $\phi_1^i(Z_i)$ and $\phi_2^i(Z_i)$ in $\pi_2(x(Z), x(Z - Z_i))$ (see [15, Section 3]). Let $\phi_1(Z)$ be the homotopy class in $\pi_2(x(Z), x_0)$ obtained by composing the $\phi_1^i(Z_i)$ homotopy classes. The multiplicities of $\phi_1(Z)$ at each basepoint are equal to the multiplicity of C in that region, so $\mathcal{F}(x(Z)) = F(Z)$. □

Remark 4.6 Although there were two choices of homotopy $\phi_1^i(Z_i)$ and $\phi_2^i(Z_i)$ for each component of the cycle Z_i, the two classes have the same multiplicity at all of the basepoints, so it doesn't matter which one we choose.

Corollary 4.7 *Two generators x and y of $CFK^-(Z)$ have the same filtration level if and only if they have the same underlying cycle.*

Proof If x and y have the same underlying cycle Z, then $\mathcal{F}(x) = F(Z) = \mathcal{F}(y)$. For the other direction, suppose $\mathcal{F}(x) = \mathcal{F}(y)$, and let Z_x and Z_y be their underlying cycles. Then $F(Z_x) = \mathcal{F}(x) = \mathcal{F}(y) = F(Z_y)$. But F is injective, so this implies $Z_x = Z_y$. □

We extend this filtration to $C_F(S)$ by placing the whole Koszul complex in filtration level $\{0, 0, \dots, 0\}$. The $\mathbb{Z}^N$–filtration can be turned into a $\mathbb{Z}$–filtration by summing over the components of $\mathcal{F}$. Let d_k denote the component of the differential on $C_F(S)$ which decreases the $\mathbb{Z}$–filtration by k.

Corollary 4.8 *The differential d_0 preserves $C_F(Z)$, ie it does not change the underlying cycle of a generator.*

This corollary tells us that the filtered complex $(C_F(S), d_0)$ splits as a direct sum

$$(C_F(S), d_0) = \bigoplus_Z (C_F(Z), d_0).$$

4.2.2 Homology of a cycle Before computing the homology $H(C_F(Z), d_0)$, we will need a definition. If S is a singular braid and Z is a multicycle in S, let $S - Z$ denote the diagram obtained by removing all edges in Z from S. Note that $S - Z$ is still a singular braid because Z is an oriented cycle in the graph.

Given a cycle Z, the complex $CFK^-(Z)$ is easy to compute. Each intersection point in the Heegaard diagram lies on a unique convex bigon (convex in the traditional planar geometry sense), and this bigon either contains an X, an XX or a O_i. There are canonical bijections between the O_i bigons and the edges e_i, between the X bigons and $V_2(S)$, and between the XX bigons and $V_4(S)$.

Given a generator x, let $W_2(x)$ denote the set of vertices at which x has an intersection point on one of the X bigons, and let $W_4(x)$ denote the set of vertices at which x has an intersection point on one of the XX bigons. $W_2(x)$ and $W_4(x)$ are uniquely determined by the underlying cycle Z of x. In particular, $W_2(x)$ and $W_4(x)$ are those vertices which are not endpoints of any edges in Z. We can therefore define $W_2(Z)$ and $W_4(Z)$ accordingly. Note that x is uniquely determined by a choice of one of the two corresponding intersection points at each edge e in $\mathbb{Z}$, each vertex v in $W_2(Z)$ and each vertex v in $W_4(Z)$.

The complex for a cycle Z can now be described as follows. Each edge e_i in Z corresponds to two intersection points, which are connected by a bigon containing O_i. These are the only filtered differentials involving these two intersection points, so $CFK^-(Z)$ is going to come with a tensor factor of the Koszul complex

$$\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R.$$

Remark 4.9 The fact that bigons are the only discs that contribute can be seen by a Maslov index argument — in fact, bigons are the only Maslov index one 2–chains in $\pi_2(x, y)$ for any two generators x and y in $CFK^-(Z)$. We show this in detail (and in greater generality) in Lemma 4.18.

Each vertex v in $W_2(Z)$ also corresponds to two intersection points. They are connected by two bigons, one which passes through O_i (where e_i is the outgoing edge from v)

and one which passes through O_j (where e_j is the incoming edge at v). These two bigons will give a coefficient of $\pm(U_i - U_j)$. Thus, we also get a tensor factor of the Koszul complex

$$\bigotimes_{v\in W_2(Z)} R \xrightarrow{L(v)} R.$$

Proving that the signs are correct requires slightly more advanced machinery, and will be discussed at the end of the section.

Finally, the vertices v in $W_4(Z)$ correspond to two intersection points, also connected by two bigons. One passes through O_i and O_j, where e_i and e_j are the outgoing edges of v, and the other passes through O_k and O_l, where e_k and e_l are the incoming edges at v. These two bigons will contribute a coefficient of $\pm(U_iU_j - U_kU_l)$, giving us the last Koszul complex

$$\bigotimes_{v\in W_4(Z)} R \xrightarrow{Q(v)} R.$$

These are all the generators and all the differentials, so the total complex is given by

$$CFK^-(Z) = \Big[\bigotimes_{e_i\in Z} R \xrightarrow{U_i} R\Big] \otimes \Big[\bigotimes_{v\in W_2(Z)} R \xrightarrow{L(v)} R\Big] \otimes \Big[\bigotimes_{v\in W_4(Z)} R \xrightarrow{Q(v)} R\Big] \tag{4}$$

and so the total complex for $C_F(Z)$ is given by

$$C_F(Z) = \Big[\bigotimes_{e_i\in Z} R \xrightarrow{U_i} R\Big] \otimes \Big[\bigotimes_{v\in W_2(Z)} R \xrightarrow{L(v)} R\Big] \otimes \Big[\bigotimes_{v\in W_4(Z)} R \xrightarrow{Q(v)} R\Big] \otimes \Big[\bigotimes_{v\in V_4(S)} R \xrightarrow{L(v)} R\Big].$$

Lemma 4.10 *The filtered homology $H_*(C_F(Z), d_0)$ is isomorphic to $H_H(S - Z)$.*

Proof The U_i in the first tensor product form a regular sequence in R, so we can cancel all of these differentials. The resulting complex is chain homotopy equivalent to the original complex over $\mathbb{Q}$. This has the effect of setting U_i equal to zero for all e_i in Z. Let R_Z be the quotient $R/\{U_i = 0 \text{ for } e_i \in Z\}$. Note that this is precisely the ground ring for the singular braid $S - Z$.

We are left with the complex

$$\Big[\bigotimes_{v\in W_2(Z)} R_Z \xrightarrow{L(v)} R_Z\Big] \otimes \Big[\bigotimes_{v\in W_4(Z)} R_Z \xrightarrow{Q(v)} R_Z\Big] \otimes \Big[\bigotimes_{v\in V_4(S)} R_Z \xrightarrow{L(v)} R_Z\Big].$$

$$\begin{array}{ccc} R_Z & \xrightarrow{\;L(v)\;} & R_Z \\ {\scriptstyle Q(v)}\big\downarrow & & \big\downarrow{\scriptstyle Q(v)} \\ R_Z & \xrightarrow{\;-L(v)\;} & R_Z \end{array}$$

Figure 10: The complex $C_F(Z)$ at a vertex $v \in V_4(S-Z)$

For each 4–valent vertex v in $S-Z$, we have tensor factors $R_Z \xrightarrow{L(v)} R_Z$ and $R_Z \xrightarrow{Q(v)} R_Z$, which together give a factor of the complex shown in Figure 10, which is precisely the HOMFLY-PT complex $C_H(v)$.

For 2–valent vertices v in $S-Z$, there are two possibilities to consider — v is 2–valent in S $(v \in W_2(Z))$, and v is 4–valent in S $(v \in V_4(S), v \notin W_4(Z))$. When v is 2–valent in S, we get the factor

$$R_Z \xrightarrow{L(v)} R_Z,$$

which is again the HOMFLY-PT complex $C_H(v)$ for $S-Z$. For v 4–valent in S, let e_i and e_j be the outgoing edges at v, and e_k and e_l the incoming edges at v. Since $S-Z$ is 2–valent at v, we know that Z must include one outgoing edge and one incoming edge. Without loss of generality, assume they are e_i and e_k. To avoid confusion, we will write out the terms of the linear elements, as $L(v)$ refers to $U_i+U_j-U_k-U_l$ in S, while $L(v)$ refers to U_j-U_l in $S-Z$.

In $C_F(Z)$, we have the factor

$$R_Z \xrightarrow{U_i+U_j-U_k-U_l} R_Z.$$

In the HOMFLY-PT complex for $S-Z$, on the other hand, we have the factor

$$R_Z \xrightarrow{U_j-U_l} R_Z.$$

Fortunately, since e_i and e_k are in Z, U_i and U_k are zero in R_Z, so $U_i+U_j-U_k-U_l = U_j-U_l$, making the above complexes isomorphic.

Thus, after canceling the Koszul complex on the edges in Z, we get exactly the HOMFLY-PT complex for $S-Z$. It follows that $H_*(C_F(Z), d_0) \cong H_H(S-Z)$. □

Corollary 4.11 *The filtered homology decomposes as the direct sum*

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S-Z).$$

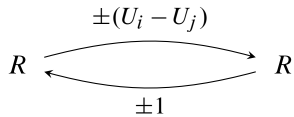

Figure 11: Local complex when allowing discs to pass through the X basepoint

Remark 4.12 (signs) Since we are discussing Koszul complexes, the $\pm$ in the terms $\pm(U_i - U_j)$ and $\pm(U_i U_j - U_k U_l)$ are not relevant. Some will have to come with positive signs and some with negative to make $d^2 = 0$, but where they are doesn't impact the chain homotopy type. What we need to show is that the two bigons in each case come with *different* signs.

Each two-valent vertex corresponds to a specific X marking in the diagram. This X lies within the same α circle as O_i and the same β circle as O_j. In CFK^-, we do not allow discs to pass through the X basepoints. However, if we do allow them to pass through only this X, we get a new complex. In this complex, d^2 is nonzero — instead, it is a multiple of the identity. This multiple is determined by the α and β degenerations, which will correspond to the α and β circles containing X. Since the α circle contains O_i, it gives a coefficient of U_i, and similarly, the β circle gives a coefficient of U_j. Since we chose a system of orientations such that the α and β degenerations come with opposite signs, this gives

$$d^2 = \pm(U_i - U_j)I.$$

Moreover, the additional differentials are also subject to the basepoint filtration, so we get

$$d_0^2 = \pm(U_i - U_j)I.$$

This X basepoint lies inside a minimal bigon, and this bigon now contributes to the differential with a coefficient of ± 1. The local contribution therefore must be the complex in Figure 11, so U_i and U_j must come with opposite sign.

The argument for the quadratic term is the same, only instead of allowing discs to pass through an X, we are allowing them to pass through an XX. The α degeneration is $U_i U_j$ and the β degeneration is $U_k U_l$, and they must come with opposite sign, so we get the complex in Figure 12, which proves that $U_i U_j$ and $U_k U_l$ come with opposite sign.

4.2.3 Gradings

The knot Floer complex comes equipped with two gradings: the Maslov grading M and the Alexander grading A. The differential decreases the Maslov

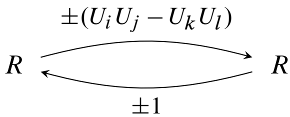

Figure 12: Local complex when allowing discs to pass through the XX basepoint

grading by 1 and preserves the Alexander grading. Multiplication by U_i decreases the Maslov grading by 2 and decreases the Alexander grading by 1.

Certain linear combinations of the Maslov and Alexander gradings return analogs of the quantum and horizontal gradings from the Khovanov–Rozansky complex. Let gr_q be given by $-2M + 2A$ and gr_h by $-2M + 4A$. Note that the knot Floer differential has bigrading $\{2, 2\}$ with respect to this differential and multiplication by U_i changes the bigrading by $\{2, 0\}$, the same as the Khovanov–Rozansky complex. Instead of the Maslov and Alexander gradings, we will henceforth use the quantum and horizontal gradings.

Before computing gradings, we need to introduce some terminology. For a multicycle Z, let $T_1(Z)$ denote the number of vertices $v \in V_4(S)$ at which Z contains the edges e_1 and e_3 in Figure 13. Similarly, let $D_1(Z)$ denote the number of vertices at which Z contains the edges e_1 and e_4, $D_2(Z)$ the number of vertices at which Z contains the edges e_2 and e_3, and $T_2(Z)$ the number of vertices at which Z contains the edges e_2 and e_4.

We will now compute the bigrading on the knot Floer complex to get a graded version of Corollary 4.11. Since until now we have only defined our complex up to an overall grading shift, the following definition pins down the absolute bigrading on $C_F(S)$. Recall that the subcomplex corresponding to the empty cycle $C_F(Z_\varnothing)$ is canonically isomorphic to the HOMFLY-PT complex $C_H(S)$, and with the new gradings $(\mathrm{gr}_q, \mathrm{gr}_h)$, they are isomorphic as bigraded complexes up to an overall grading shift.

Definition 4.13 We define the bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$ on $C_F(S)$ so that the subcomplex $C_F(Z_\varnothing)$ is isomorphic to $C_H(S)\{-2r(S), 0\}$ as bigraded chain complexes, where $Z_\varnothing$ is the empty cycle.

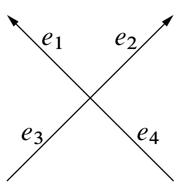

Figure 13: A labeled 4–valent vertex

Let Z denote a k–component multicycle in S. Viewing Z as a braid diagram for the k–component unlink, we can define $r(Z)$ to be the rotation number of Z, so $r(Z) = -k$. As before, let $x(Z)$ denote the generator corresponding to Z at the bottom of the Koszul complex (ie with the largest horizontal grading). Similarly, let x_0 denote the generator corresponding to the empty cycle with the largest horizontal grading. In the proof of Lemma 4.5 we utilize a set of k Maslov index one discs whose composition takes $x(Z)$ to x_0 defined by Ozsváth and Szabó [15]. This composition has coefficient which is a polynomial in R of degree $T_2(Z) + \frac{1}{2}(D_1(Z) + D_2(Z))$. Using the fact that differentials have bigrading $\{2, 2\}$ and that this is a composition of k differentials, we can see that $x(Z)$ and x_0 differ in grading by

$$\{2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}.$$

The bottom generator of the HOMFLY-PT complex has bigrading $\{-|V_4(S)|, 0\}$, so x_0 has bigrading $\{-2r(S) - |V_4(S)|, 0\}$. Thus, $x(Z)$ has bigrading

$$\begin{aligned}\{-2r(S) - |V_4(S)| + 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}\\ = \{-2r(S - Z) - |V_4(S)| + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}.\end{aligned}$$

The bottom generator of the HOMFLY-PT complex for $S - Z$ has bigrading equal to $\{-|V_4(S - Z)|, 0\}$, so we get

$$\begin{aligned}&H(C_F(Z), d_0) \cong \\ &H_H(S-Z)\{-2r(S-Z)+|V_4(S-Z)|-|V_4(S)|+2T_2(Z)+D_1(Z)+D_2(Z), 2r(Z)\}.\end{aligned}$$

The grading shift in this formula can be simplified somewhat. The quantity $|V_4(S-Z)| - |V_4(S)|$ is the negative of the number of 4–valent vertices in S at which Z contains two edges:

$$|V_4(S - Z)| - |V_4(S)| = -T_1(Z) - T_2(Z) - D_1(Z) - D_2(Z).$$

Thus, the formula becomes

$$H(C_F(Z), d_0) \cong H_H(S - Z)\{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

and we get a graded version of Corollary 4.11:

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S - Z)\{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}.$$

We will be able to connect this formula to the composition product with the following lemma:

Lemma 4.14 *Let f be a labeling of S. The $\mathfrak{sl}_1$ homology of $S_{f,1}$ is given by*

$$H_1(S_{f,1}) = \begin{cases} \mathbb{Q}\{0, 2r(S_{f,1})\} & \text{if } S_{f,1} \text{ is a multicycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Applying this lemma, the formula becomes

$$H_*(C_F(S), d_0) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}.$$

But by Theorem 3.2, this is isomorphic to $H_H(S)\langle 1\rangle$. Thus, we have proved the following theorem:

Theorem 4.15 *There is an isomorphism of bigraded groups*

$$H_*(C_F(S), d_0) \cong H_H(S)\langle 1\rangle.$$

Corollary 4.16 *There is a spectral sequence whose E_1 page is $H_H(S)\langle 1\rangle$ and which converges to $H_F(S)$.*

Proof This is just the spectral sequence induced by the basepoint filtration on $C_F(S)$. □

Manolescu's conjecture is thus equivalent to this spectral sequence collapsing at the E_1 page.

4.3 Additional differentials and the spectral sequences from *HFK* to $\mathfrak{sl}_n$

We are going to add differentials to the complex $C_F(S)$ so that the total homology is isomorphic to $H_{n+1}(S)$ for any $n \geq 1$. These new differentials do not preserve the Alexander grading, so using the Alexander grading as a filtration, this induces a spectral sequence from $H_F(S)$ to $H_{n+1}(S)$.

The complex $C_F(S)$ is constructed as a tensor product of complexes $CFK^-(S)$ and a Koszul complex $K(S)$ on linear elements:

$$C_F(S) = CFK^-(S) \otimes K(S).$$

The complex $CFK^-(S)$ does not count discs which pass through the X or XX markings. For the new differential, we are going to count these discs with certain polynomial coefficients.

Each X marking in the Heegaard diagram corresponds to a 2–valent vertex v in S. Whenever a holomorphic disc passes through this X with multiplicity k, it picks up a coefficient of $u_1(v)^k$. The only exception is the special marking X_{new}, at which we still require discs to have multiplicity 0. Similarly, each XX corresponds to a 4–valent vertex v in S. If a holomorphic disc passes through this XX with multiplicity k, it picks up a coefficient of $u_2(v)^k$. We will call this new complex $CFK_n^-(S)$.

Note that there is no guarantee that the differential on this complex squares to zero — in fact, it doesn't. To fix this, we will also modify the differential on the Koszul complex. Originally, it was given by

$$K(S) = \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R.$$

We are going to add in differentials to make it a matrix factorization:

$$K_n(S) = \bigotimes_{v \in V_4(S)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R.$$

The total complex $C_{F(n)}(S)$ is defined to be the tensor product of $CFK_n^-(S)$ and $K_n(S)$,

$$C_{F(n)}(S) = CFK_n^-(S) \otimes K_n(S).$$

Lemma 4.17 *The differential on $C_{F(n)}(S)$ satisfies $d^2 = 0$.*

Proof At each vertex v in S, we will show that d^2 has a contribution of $w_n(v)I$, with

$$w_n(v) = \sum_{e_i \in E_{\text{out}}} U_i^{n+1} - \sum_{e_j \in E_{\text{in}}} U_j^{n+1}.$$

The lemma will then follow from the fact that $\sum_{v \in S} w_n(v)I = 0$.

The quantity d^2 has two contributions, one from $CFK_n^-(S)$ and one from $K_n(S)$. The contribution from $K_n(S)$ can be computed directly to be

$$\sum_{v \in V_4(S)} L(v)u_1(v).$$

The contribution from $CFK_n^-(S)$ can be computed via the α and β degenerations. We orient the moduli spaces so that the α and β degenerations come with opposite signs, with the α degenerations being positive and the β degenerations negative (see Remark 4.1 for an explanation of the signs).

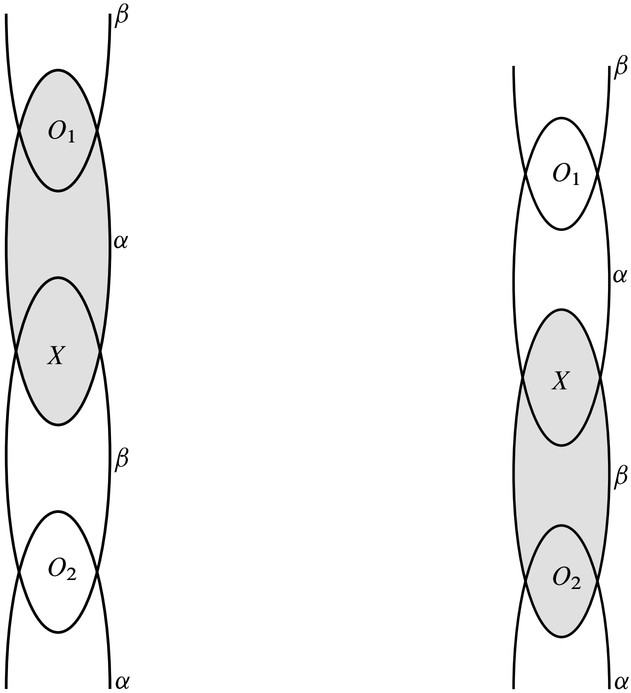

Figure 14: α–degenerations (left) and β–degenerations (right) at a bivalent vertex

At each 2–valent vertex v in S, we have one α circle and one β circle, shown in Figure 14 . The α circle contains U_i and X, and the X contributes coefficient $u_1(v)$, so the α degeneration contributes $U_i u_1(v)$. Similarly, the β circle contains U_j and X, so its contribution is $-U_j u_1(v)$. Thus, the net contribution at v is $(U_i - U_j)u_1(v)$. This can be simplified to $L(v)u_1(v) = w_n(v)$.

At each 4–valent vertex v in S, we also have one α circle and one β circle, shown in Figure 15. The α circle contains U_i, U_j and XX, and the XX contributes coefficient $u_2(v)$, so its contribution is $U_i U_j u_2(v)$. The β circle contains U_k, U_l and XX, so its contribution is $-U_k U_l u_2(v)$. Thus, the net contribution at v is $(U_i U_j - U_k U_l)u_2(v)$. This can be simplified to $Q(v)u_2(v)$.

Thus, counting the contribution from $K_n(S)$, we see that d^2 is given by

$$
\begin{aligned}
d^2 &= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} Q(v)u_2(v) + \sum_{v \in V_4(S)} L(v)u_1(v) \\
&= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} L(v)u_1(v) + Q(v)u_2(v)
\end{aligned}
$$

$$= \sum_{v\in V_2(S)} w_n(v) + \sum_{v\in V_4(S)} w_n(v)$$
$$= \sum_{v\in S} w_n(v) = 0.$$

This concludes the proof. □

We can extend the ($\mathbb{Z}$–valued) basepoint filtration from Section 4.2.1 to make $C_{F(n)}(S)$ a filtered complex — since we still require discs to have multiplicity 0 at X_{new}, the same argument works as in the proof of Lemma 4.3. As before, let d_i denote the differentials which decrease the filtration level by i. Since d_0 must preserve multicycles, the homology $H(C_{F(n)}(S), d_0)$ splits over the multicycles

$$H(C_{F(n)}(S), d_0) = \bigoplus_Z H(C_{F(n)}(Z), d_0).$$

We want to compute the complex $C_{F(n)}(Z)$. Recall that $C_F(Z)$ was computed to be

$$\Big[\bigotimes_{e_i\in Z} R\xrightarrow{U_i} R\Big]\otimes\Big[\bigotimes_{v\in W_2(Z)} R\xrightarrow{L(v)} R\Big]\otimes\Big[\bigotimes_{v\in W_4(Z)} R\xrightarrow{Q(v)} R\Big]\otimes\Big[\bigotimes_{v\in V_4(S)} R\xrightarrow{L(v)} R\Big].$$

We can therefore compute $C_{F(n)}(Z)$ by adding the new differentials to this complex. The only new discs in $CFK_n^-(Z)$ correspond to bigons containing X or XX basepoints (we will prove this in Lemma 4.18). For example, let e_i be an edge in Z, with x and y the two intersection points corresponding to e_i. When we weren't allowing discs to

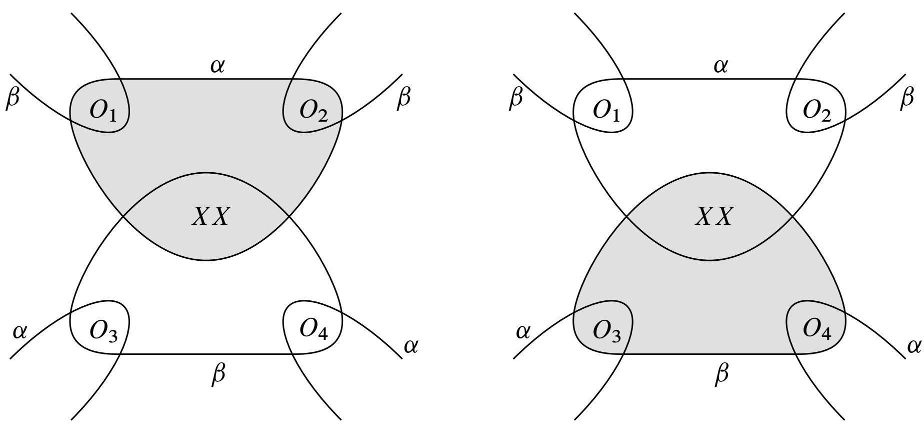

Figure 15: α–degenerations (left) and β–degenerations (right) at a four-valent vertex

Figure 16: Two new bigons from y to x

pass through X or XX, the only disc connecting x and y was the bigon containing U_i. This contributed the tensor factor of

$$R \xrightarrow{U_i} R.$$

However, when we allow discs to pass through X and XX, we get two new bigons which map from y to x, shown in Figure 16.

The type of contribution from these bigons depends on whether the endpoints of e_i are 2–valent or 4–valent. These two cases are shown in Figure 17. In either case, the contribution has a coefficient of degree n. We will denote the contribution from these new bigons at an edge e_i in Z by $p(e_i)$ (the precise polynomial will not be relevant

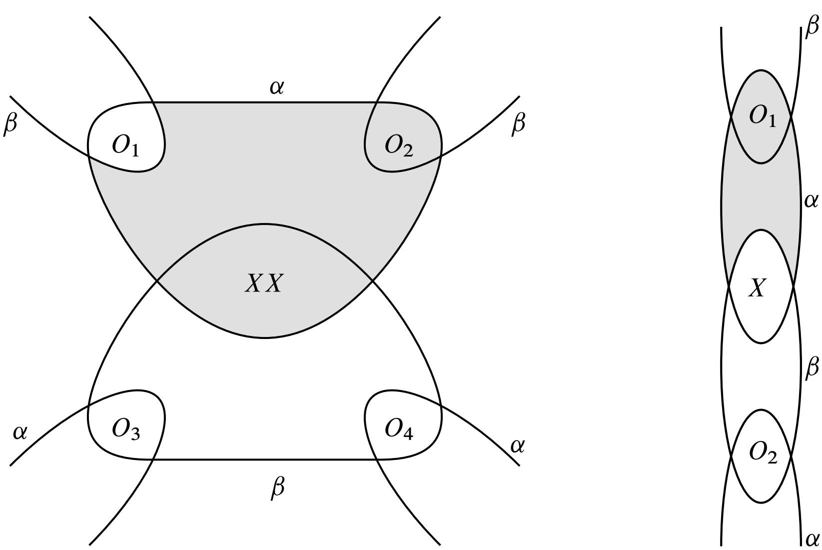

Figure 17: The two types of new bigons

for our computations). The tensor factor then becomes

$$R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R.$$

For a vertex v in $W_2(Z)$, there are two intersection points x and y corresponding to v. In $C_F(Z)$, they contributed a tensor factor of

$$R \xrightarrow{L(v)} R.$$

In $C_{F(n)}(Z)$, there is an extra differential corresponding to the bigon from y to x through X (See Figure 18). Since X carries a coefficient of $u_1(v)$, the factor becomes

$$R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R.$$

Similarly, for 4–valent vertices v in $W_4(Z)$, $C_F(Z)$ contains a tensor factor

$$R \xrightarrow{Q(v)} R.$$

In $C_{F(n)}(Z)$, there is an extra differential corresponding to the bigon through XX shown in Figure 18. The XX contributes a coefficient of $u_2(v)$, so the factor becomes

$$R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R.$$

Thus, only counting bigons, we get the following complex for $CFK_n^-(Z)$:

$$\left[\bigotimes_{e_i \in Z} R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R\right]. \tag{5}$$

We will now show that these are in fact the only differentials in the complex $CFK_n^-(Z)$.

Lemma 4.18 *All of the differentials in $CFK_n^-(Z)$ come from bigons in $\mathcal{H}(S)$.*

Proof Suppose x and y are two distinct generators of $CFK_n^-(Z)$ such that there is a Maslov index 1 homotopy class $\phi_{x,y} \in \pi_2(x, y)$ which does not pass through the basepoints p_i, and let $D(\phi_{x,y})$ denote the corresponding 2–chain in $\mathcal{H}(S)$. (See [19] for a discussion of the Maslov index.)

From (4), we see that when counting only bigons in the differential, $CFK^-(Z)$ can be written as a tensor product of Koszul complexes. Therefore, it is natural to view it as a hypercube in which the differentials from bigons correspond to oriented edges. The generators x and y are two of the vertices of this hypercube. The new bigons in $CFK_n^-(Z)$ also correspond to edges in the cube, but they have the opposite orientation

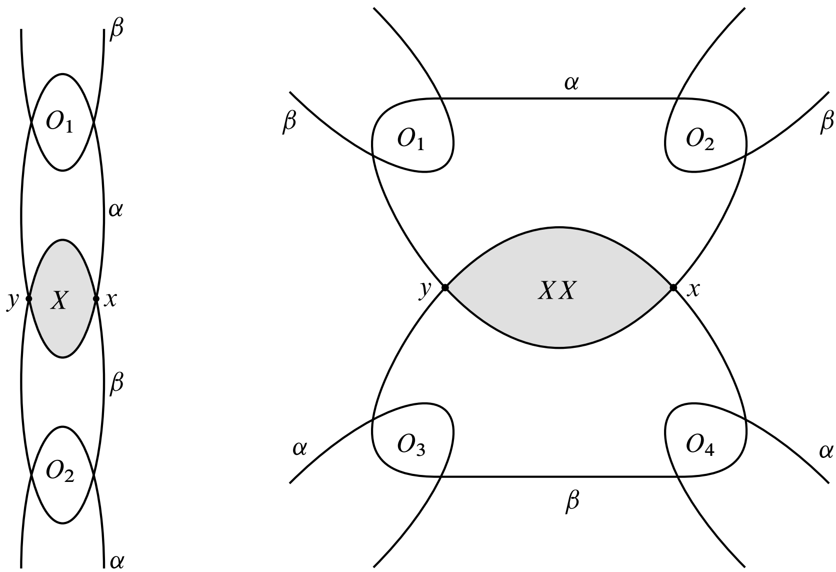

Figure 18: New differentials: the bigons passing through X (left) and XX (right)

compared with the edges coming from bigons in $CFK^-(Z)$. It is clear that this cube is fully connected in the sense that for any two vertices a and b, there is an oriented path from a to b.

Let γ be a path of minimal length from y to x — call this length l. Each edge in γ corresponds to a set of bigons (this set has either one element or two). Pick one bigon from each edge in γ, so that we have a sequence $B_1, \dots, B_l$. Since $D(\phi_{x,y})$ and $B_1, \dots, B_l$ all have Maslov index 1, the 2–chain $D(\phi_{x,y}) + B_1 + \cdots + B_l$ gives a Maslov index $l+1$ homotopy class in $\pi_2(x,x)$.

The only 2–chains which do not change the underlying generator are sums of periodic domains. The periodic domains with multiplicity 0 at the basepoints are the interiors of the α and β circles — let P_i denote these periodic domains. We can write

$$D(\phi_{x,y}) + B_1 + \cdots + B_l = \sum_{i=1}^{k} P_i.$$

We know that the sum on the right will have only positive coefficients because the total sum must have nonnegative multiplicity at every region in $\mathcal{H}(S)$, and each α or β circle contains a region which is not contained in any other α or β circles.

We see by looking at the Heegaard diagram that if two distinct B_i are contained in a single P_j, then the two B_i correspond to the same edge in the cube but with opposite orientations. Note that this is not true for arbitrary bigons in the $\mathcal{H}(S)$, but the B_i which can appear in the cube are restricted in that the two endpoints must correspond to intersection points of generators with underlying cycle Z.

Since γ is a minimal-length path in the cube, γ does not traverse any edge in both directions. Thus, no two of the B_i are contained in a single P_j, so $k \geq l$. The Maslov index of the 2–chain on the left is $l+1$, while the Maslov index of the 2–chain on the right is $2k$, because each α– or β–degeneration has Maslov index 2. But the equality $l+1 = 2k$ with $k \geq l \geq 0$ forces $l = k = 1$. So we have

$$D(\phi_{x,y}) + B_1 = P_1,$$

where B_1 is a bigon and P_1 is the interior of an α or β circle. Then $D(\phi_{x,y}) = P_1 - B_1$. Since $D(\phi_{x,y})$ can have only nonnegative coefficients, $B_1 \subset P_1$. But removing a bigon from the interior of any α or β circle in $\mathcal{H}(S)$ results in another bigon, proving that $D(\phi_{x,y})$ is a bigon, as desired. □

It follows that $CFK_n^-(Z)$ is given by (5). Finally, the complex $K(S)$ gets changed to $K_n(S)$, so the whole complex for $C_{F(n)}(Z)$ can be written as

$$\Bigg[\bigotimes_{e_i \in Z} R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R\Bigg] \otimes \Bigg[\bigotimes_{v \in W_2(Z)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\Bigg] \otimes \Bigg[\bigotimes_{v \in W_4(Z)} R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R\Bigg] \otimes \Bigg[\bigotimes_{v \in V_4(S)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\Bigg].$$

Now that we have our complex computed, we want to compare its homology with $H_n(S-Z)$. We will denote the differentials which do not pass through any X or XX basepoints by d_{0+}, and the new differentials by d_{0-}. Observe that with respect to the bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$ introduced in Section 4.2.3, d_{0+} has bigrading $\{2,2\}$ and d_{0-} has bigrading $\{2n,-2\}$.

Lemma 4.19 *Up to an overall grading shift, $H_*\big(H_*(C_{F(n)}(Z), d_{0+}), d_{0-}^*\big)$ is isomorphic to $H_*\big(H_*(C_n(S-Z), d_+), d_-^*\big)$.*

Proof It follows from Lemma 4.10 that $H(C_{F(n)}(Z), d_{0+}) \cong H_H(S-Z)$. To complete the proof, we need to show that d_{0-}^* corresponds to the d_- differential under this isomorphism. For vertices which are 2–valent in both S and $S-Z$ (ie

$v \in W_2(Z)$), this is obvious. The same is true for vertices which are 4–valent in both S and $S-Z$ (ie $v \in W_4(Z)$).

The only identification which is nontrivial is that the d_{0-} differential corresponding to a vertex which is 4–valent in S but 2–valent in $S-Z$ is the same as the d_- differential on the 2–valent vertex in $S-Z$. Let e_i and e_j be the outgoing edges at v and e_k and e_l the incoming edges. The multicycle Z must contain one incoming and one outgoing edge — without loss of generality, assume Z contains e_i and e_k. The coefficient of the d_- differential is given by

$$\frac{U_j^{n+1} - U_l^{n+1}}{U_j - U_l},$$

while the coefficient of the d_{0-} differential is given by

$$\frac{U_i^{n+1} + U_j^{n+1} - U_k^{n+1} - U_l^{n+1} - Q(v)u_2(v)}{U_i + U_j - U_k - U_l}.$$

Recall that to achieve the isomorphism in Lemma 4.10, we first canceled the Koszul complex on the U_p for e_p in Z, as these elements formed a regular sequence. We therefore want to show that these two coefficients are equal in $R_Z = R/\{U_p = 0 \text{ for } e_p \in Z\}$. Substituting $U_i = U_k = 0$ into the above equation and noting that this causes $Q(v)$ to be zero, we get the desired equality. □

Define $H^{\pm}(C_{F(n)}(S)) = H_*\big(H_*(C_{F(n)}(S), d_{0+}), d_{0-}^*\big)$. Since both d_{0+} and d_{0-} are homogeneous with respect to the bigrading, this homology is bigraded as well. Applying the lemma and adding in the gradings from Section 4.2.3, we see that

$$H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_Z H^{\pm}(C_n(S-Z))\{-2r(S-Z)+T_2(Z)-T_1(Z), 2r(Z)\}. \tag{6}$$

Recall from Corollary 2.7 that $H^{\pm}(C_n(S-Z))$ lies in a single horizontal grading, namely $2r(S-Z)$. Adding in the shift, the homology corresponding to a multicycle Z must lie in horizontal grading $2r(S-Z) + 2r(Z) = 2r(S)$. But this does not depend on Z, so we have shown the following:

Lemma 4.20 *The homology $H^{\pm}(C_{F(n)}(S))$ lies in a single horizontal grading.*

The original differentials on $C_F(S)$ all have bigrading $\{2, 2\}$. The new differentials on $CFK_n^-(S)$ have bigrading $\{2 + 2k(n-1), 2 - 4k\}$, where k is the sum of the multiplicities of the holomorphic discs at all X and XX markings. The new differentials on $K_n(S)$ all have bigrading $\{2n, -2\}$. Thus, all differentials on $C_n(S)$ change the horizontal grading by $2 \pmod 4$. This implies that no induced differentials can have horizontal grading 0, which tells us that the remaining differentials on our complex are all trivial, giving us the following:

Lemma 4.21 *The total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to $H^{\pm}(C_{F(n)}(S))$.*

This isomorphism is singly graded with grading $\mathrm{gr}_n = \mathrm{gr}_q + \frac{1}{2}(n-1)\,\mathrm{gr}_h$, as the total differential on $C_{F(n)}(S)$ is homogeneous of degree $n+1$ with respect to this grading.

Going back to (6), we know that as bigraded vector spaces, we have the isomorphism

$$H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_Z H^{\pm}(C_n(S-Z))\{-2r(S-Z) + T_2(Z) - T_1(Z), 2r(Z)\}.$$

We can use Lemma 4.14 in the same way as in the previous section to convert this equation into a direct sum over labelings,

$$H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_n(S_{f,2})\{-2r(S_{f,2}) + T_2(S_{f,1}) - T_1(S_{f,1}), 0\}.$$

Applying the bigraded composition product formula (3), this gives an isomorphism of bigraded vector spaces

$$H^{\pm}(C_{F(n)}(S)) \cong H_{n+1}(S). \tag{7}$$

Since $H_*(C_{F(n)}(S), d) \cong H^{\pm}(C_{F(n)}(S))$ as graded vector spaces with grading gr_n, this gives an isomorphism

$$H_*(C_{F(n)}(S), d) \cong H_{n+1}(S),$$

where we are viewing $H_{n+1}(S)$ as singly graded, with grading gr_n. Since singly graded $\mathfrak{sl}_{n+1}$ homology is typically viewed with respect to the grading gr_{n+1}, this isn't quite what we want. Fortunately, since the homology is concentrated in horizontal grading $\mathrm{gr}_h = 2r(S)$, we see that $\mathrm{gr}_{n+1} = \mathrm{gr}_n + 2r(S)$.

Theorem 4.22 *The total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to the homology $H_{n+1}(S)\{2r(S)\}$, where $H_*(C_{F(n)}(S), d)$ has grading gr_n and $H_{n+1}(S)$ has grading gr_{n+1}.*

Remark 4.23 The grading shift by $2r(S)$ is only an artifact of passing from the grading gr_n to gr_{n+1}, and no grading shift is needed when comparing the bigraded complexes (see (7)).

This shift appears on the HOMFLY-PT side as well in the singly graded case. When looking at the spectral sequence from $H_H(S)\langle 1\rangle$ to $H_{n+1}(S)$ induced by d_n, the grading on $H_H(S)\langle 1\rangle$ is given by gr_n, while the grading on $H_{n+1}(S)$ is given by gr_{n+1}. Thus, with respect to these gradings, there is a spectral sequence from $H_H(S)\langle 1\rangle$ to $H_{n+1}(S)\{2r(S)\}$.

Corollary 4.24 *For all $n \geq 1$, there is a spectral sequence whose E_1 page is $H_F(S)$ which converges to $H_{n+1}(S)$.*

Proof All of the original differentials on $C_F(S)$ have Alexander grading 0. The new differentials on CFK_n^- have Alexander grading $k(-n-1)$, where k is the sum of the multiplicities of the disc at the X and XX basepoints, and the new differentials on the Koszul complex have Alexander grading $-n-1$. In particular, all of the new differentials strictly decrease the Alexander grading, so it induces a filtration with respect to which the filtered homology is $H_*(C_F(S))$. Thus, the corresponding spectral sequence has E_1 page $H_F(S)$, and converges to the total homology $H(C_{F(n)}(S)) \cong H_{n+1}(S)$. □

Remark 4.25 Using the tools from this section, we can give another proof of Theorem 4.15, which does not rely on the composition product formula for HOMFLY-PT homology. From (7), we have that

$$H_*(H_*(C_{F(n)}(S), d_{0+}), d_{0-}^*) \cong H_{n+1}(S).$$

Since $H_*(C_{F(n)}(S), d_{0+})$ is precisely $H_*(C_F(S), d_0)$, this shows that there a differential d_{n+1} on $H_*(C_F(S), d_0)$ satisfying the conditions of Corollary 3.4 for $k = 1$, where d_{n+1} is given by d_{0-}^*.

4.4 Proof of Manolescu's conjecture

At this point, we have three spectral sequences: one from HOMFLY-PT homology to $\mathfrak{sl}_n$ homology, one from HOMFLY-PT homology to knot Floer homology, and one from knot Floer homology to $\mathfrak{sl}_n$ homology. Diagrammatically, this looks like

$$\begin{array}{ccc} H_H(S)\langle 1\rangle & \longrightarrow & H_F(S) \\ & \searrow \quad \swarrow & \\ & H_n(S)\{2r(S)\} & \end{array}$$

where the arrows correspond to spectral sequences. Since both HOMFLY-PT homology and knot Floer homology have spectral sequences going to $\mathfrak{sl}_n$ homology for all $n \geq 2$, it is clear that they have a great deal in common. The conjecture of Manolescu is that they are in fact isomorphic. Since we have a spectral sequence from HOMFLY-PT homology to knot Floer homology, this is equivalent to the spectral sequence collapsing at the E_1 page. In this section, we will prove Manolescu's conjecture.

Theorem 4.26 *The* HOMFLY-PT *homology* $H_H(S)\langle 1\rangle$ *and knot Floer homology* $H_F(S)$ *are isomorphic as bigraded vector spaces.*

Proof We will start with the complex $C_{F(n)}$ from the previous section. There are two filtrations defined on this complex so far — the one induced by the Alexander grading, and the one induced by the basepoints. Consider the associated bigraded object from these two filtrations.

The differentials always change the Alexander grading by a multiple of $n+1$, so let d_{ij} denote those differentials which change the Alexander grading by $i(n+1)$ and change the basepoint grading by j.

Theorem 4.15 states that

$$H_*(C_{F(n)}(S), d_{00}) \cong H_H(S)\langle 1\rangle.$$

Since $H_F(S) \cong H_*(C_{F(n)}, d_{0*})$, our theorem is equivalent to d_{0k}^* being zero on $H_*(C_{F(n)}, d_{00})$. We will prove this by contradiction. In particular, suppose that some d_{0k}^* is nonzero, and let a denote the smallest such k.

Because a is minimal, it is clear that d_{0a}^* and d_{10}^* anticommute, as $d_{0a}^* \circ d_{10}^*$ and $d_{10}^* \circ d_{0a}^*$ are the only components of d^2 which change the basepoint filtration by a and the Alexander filtration by $n+1$. These are the differentials that we will be interested in, so we will rename them. We will write d_F instead of d_{0a}^* and d_n instead of d_{10}^*, since d_{10}^* depends on n. The differentials d_F and d_n both act on $H_*(C_{F(n)}, d_{00})$, so using the above isomorphism we will view them as acting on $H_H(S)\langle 1\rangle$.

We know from (7) that there is an isomorphism of bigraded vector spaces

$$H_*(H_H(S)\langle 1\rangle, d_n) \cong H_{n+1}(S).$$

To summarize our setup, we have a family of differentials d_n on $H_H(S)\langle 1\rangle$, each having bigrading $(2n, -2)$, such that the homology with respect to each is H_{n+1}, and there is a differential d_F on $H_H(S)\langle 1\rangle$ with bigrading $(2, 2)$ which is nontrivial and anticommutes with each d_n.

We know that the smallest horizontal grading in which $H_H(S)\langle 1\rangle$ is nontrivial is $2r(S)$, and that the homology $H_*(H_H(S)\langle 1\rangle, d_n)$ lies only in this horizontal grading. Let $h_{\min}$ be the minimal horizontal grading on which d_F is nonzero, and let x be an element of $H_H(S)\langle 1\rangle$ in bigrading $(q, h_{\min})$ for some q with $d_F(x) \neq 0$. Define $d_F(x) = y$.

We know also that $H_H(S)\langle 1\rangle$ is bounded below in quantum grading and d_n changes the quantum grading by $2n$, so choose N sufficiently large that y cannot be in the image of d_N. Since y lies in horizontal grading $h_{\min}+2$ (in particular, not $h_{\min}$), it follows that $y \in \operatorname{Ker}(d_N)$ if and only if $y \in \operatorname{Im}(d_N)$. Thus, y is not in the kernel of d_N.

But then $d_N \circ d_F(x)$ is nonzero, while $d_F \circ d_N(x)$ must be zero because $d_N(x)$ lies in horizontal grading $h_{\min}-2$, and $d_F = 0$ for all horizontal gradings less than $h_{\min}$, which contradicts the fact that d_N and d_F anticommute. □

This theorem is proved with $\mathbb{Q}$ coefficients. However, HOMFLY-PT homology over $\mathbb{Z}$ of braid graphs is known to be torsion-free. This can be seen from the MOY relations (see [11; 18]), which we won't describe explicitly here, but the argument can be summarized as follows. There is a map $\iota\colon \{\text{singular braids}\} \to \mathbb{N}$, called the *complexity*, such that the HOMFLY-PT homology of any singular braid can be written in terms of the HOMFLY-PT homologies of singular braids with lower complexity. The fact that $H_H(S,\mathbb{Z})$ is $\mathbb{Z}$–torsion-free then follows from an induction argument.

Thus, HOMFLY-PT homology still satisfies the composition product formula over $\mathbb{Z}$,

$$H_H(S,\mathbb{Z})\langle 1\rangle \cong \bigoplus_Z H_H(S-Z,\mathbb{Z}).$$

In our computation of $H_*(C_F(S), d_0)$, we only canceled those differentials with coefficient ± 1, so

$$H_*(C_F(S), d_0, \mathbb{Z}) \cong \bigoplus_Z H_H(S-Z,\mathbb{Z}) \cong H_H(S,\mathbb{Z})\langle 1\rangle.$$

So, for the theorem to hold with $\mathbb{Z}$ coefficients, we need the filtered homology $H_*(C_F(S), d_0, \mathbb{Z})$ to be isomorphic to the unfiltered homology $H_*(C_F(S), d, \mathbb{Z})$, ie we need the induced higher differentials on $H_*(C_F(S), d_0, \mathbb{Z})$ to be zero. But any nonzero differential over $\mathbb{Z}$ would also be nonzero over $\mathbb{Q}$, contradicting Theorem 4.26. Thus, Theorem 4.26 is true with $\mathbb{Z}$ coefficients as well.

We relate this theorem to Manolescu's conjecture with the following corollary. Note that the original conjecture was over $\mathbb{Z}$ rather than $\mathbb{Q}$. By the previous argument, there is no $\mathbb{Z}$–torsion, so the two contexts are equivalent.

Corollary 4.27 *If S is a connected braid graph, then there is an isomorphism of* Tor *groups*

$$\mathrm{Tor}_R(R/L, R/N) \cong \mathrm{Tor}_R(R/L, R/Q)$$

as bigraded vector spaces. The bigrading on $\mathrm{Tor}_R(R/L, R/N)$ is given by (q,h), where q is the quantum grading coming from the polynomial ring and h is the homological grading, and the bigrading on $\mathrm{Tor}_R(R/L, R/Q)$ is $(q+h,h)$.

Proof The difference between our complex $C_F(S)$ and Manolescu's knot Floer complex for S is that we have added an additional unknotted component at infinity, placed the marked edge on that component and reduced that component (ie set that U_i equal to zero).

Manolescu instead placed the marked edge on the leftmost strand of the braid. Let $C_F^M(S)$ denote Manolescu's complex. By the argument at the beginning of Section 4, we know that this has the effect of doubling the homology. In particular, with respect to the (q,h) bigrading there is an isomorphism

$$H_F(S) \cong H_F^M(S) \otimes V,$$

where $V = \mathbb{Q}\{-1,-1\} \oplus \mathbb{Q}\{1,1\}$.

Similarly, the middle HOMFLY-PT homology can be viewed as the unreduced HOMFLY-PT homology of the $1-1$ tangle obtained by breaking an edge in the diagram. Let $C_H^M(S)$ denote the middle HOMFLY-PT homology of S. Then, with respect to the (q,h) grading, we have the isomorphism

$$H_H(S) \cong H_H^M(S)\{0,-1\} \oplus H_H^M(S)\{0,1\}.$$

But when we switch to the $(q+h,h)$ grading, this becomes

$$H_H(S)\langle 1\rangle \cong H_H^M(S)\langle 1\rangle \otimes V.$$

The previous theorem states that $H_F(S) \cong H_H(S)\langle 1\rangle$. Using the above arguments, this becomes

$$H_F^M(S) \otimes V \cong H_H^M(S)\langle 1\rangle \otimes V$$

as bigraded groups.

Since all of our theories are bounded in h–grading and bounded below in q–grading, the above isomorphism implies an isomorphism without tensoring with V,

$$H_F^M(S) \cong H_H^M(S)\langle 1\rangle.$$

But Manolescu showed that

$$H_F^M(S) \cong \mathrm{Tor}_R(R/L, R/N) \quad \text{and} \quad H_H^M(S) \cong \mathrm{Tor}_R(R/L, R/Q),$$

which proves the corollary. □

Another significant corollary of this result relates to the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. This is the page which was conjectured by Manolescu to give HOMFLY-PT homology. In [2], we showed that the graded Euler characteristic of the homology

$$E_2^f(D) = H_*\big(H_*(C_F(D), d_0^f), (d_1^f)^*\big)$$

is the HOMFLY-PT polynomial, where d_i^f denotes the component of the differential on $C_F(D)$ which increases the cube grading by i and preserves the basepoint filtration. In particular, we define the triple grading on this complex by the i, j and k gradings, where k denotes twice the cube grading, and

$$i = 2A - 2M - k, \quad j = 4A - 2M - k,$$

where M and A are the Maslov and Alexander gradings, respectively. With respect to this triple grading, we showed that

$$\sum_{i,j,k} (-1)^{(k-j)/2} \dim(E_2^f(D)^{i,j,k}) = P_H(aq, q, D).$$

But we have just seen that $H_*(C_F(D), d_0^f) \cong H_*(C_F(D), d_0)$, so there is a spectral sequence from $H_*\big(H_*(C_F(D), d_0^f), (d_1^f)^*\big)$ to the E_2 page $H_*\big(H_*(C_F(D), d_0), d_1^*\big)$. But all of these differentials have triple grading $\{0,0,2\}$, so they do not change the Euler characteristic. Thus we have shown the following:

Corollary 4.28 *Let $C_F(D)$ denote the oriented cube of resolutions complex for a braid diagram D, and let $E_2(D)$ denote the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. Then the graded Euler characteristic of $E_2(D)$ with the triple grading given above is the* HOMFLY-PT *polynomial $P_H(aq, q, D)$.*

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