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Weighted sheaves and homology of Artin groups

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We expand the theory of weighted sheaves over posets, and use it to study the local homology of Artin groups. First, we use such theory to relate the homology of classical braid groups with the homology of certain independence complexes of graphs. Then, in the context of discrete Morse theory on weighted sheaves, we introduce a particular class of acyclic matchings. Explicit formulas for the homology of the corresponding Morse complexes are given, in terms of the ranks of the associated incidence matrices. We use such method to perform explicit computations for the new affine case $\widetilde{C}_n$, as well as for the cases A_n, B_n and $\widetilde{A}_n$ (which were already done before by different methods).

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1 Introduction

The topological theory of Artin groups started in the 1960s with classical braid groups (see Fox and Neuwirth [24] and Arnold [2; 3]), and was broadened to general Artin groups in the 1970s; see Brieskorn [7], Brieskorn and Saito [8], Cohen [16], Deligne [21], Fuks [26], Gorjunov [27] and Vaĭnšteĭn [44]. It has received much attention in connection with problems in singularity theory, homotopy theory, hyperplane arrangements and combinatorics (see Paris [37] for some references), up to recent applications of some special cases to conjectures about hyperbolic groups; see Agol [1].

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Thanks to a good topological model for the $K(\pi,1)$ space of Artin groups (see Salvetti [38; 39]), a combinatorial free resolution for these groups was found by De Concini and Salvetti [20], and several (co)homology computations were carried out by Callegaro, De Concini, Moroni, Procesi, Salvetti and Stumbo [19; 18; 14; 9; 10; 11; 12; 13] (equivalent resolutions also appeared in Squier [42], Ozornova [34] and Paolini [35]). The computational methods used in these papers are essentially based on filtering the algebraic complex and using the associated spectral sequence.

In Salvetti and Villa [41] a more combinatorial method of calculation was introduced, based on the application of discrete Morse theory to a particular class of sheaves over posets (called *weighted sheaves*). It is interesting to notice that similar ideas were considered later in Curry, Ghrist and Nanda [17], even if there the authors were mainly interested in computational aspects.

In this paper we expand the theory of weighted sheaves. First, we observe a surprising relationship between some twisted (co)homology of the classical braid groups and the (co)homology of certain *independence complexes*, which in their simplest form have already been studied in a combinatorial context (see for example Babson and Kozlov [4], Engström [22] and Kozlov [30]). This relationship is obtained in an indirect way, after localization. The exact formula we are going to prove is the following (it was announced in Salvetti [40] without proof).

Theorem 3.4 $H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \dfrac{R}{(\varphi_d)}\right).$

Here on the left we have the homology with local coefficients of the classical braid group and on the right we have the homology of some independence complex with trivial coefficients (we indicate by φ_d the d^{th} cyclotomic polynomial).

As said before, our method is based on the notions of weighted sheaves over posets and of *weighted matchings*, which are interesting objects by themselves. In our situation, the d–weight $v_d(\sigma)$ of a simplex σ is the maximal power of the d^{th} cyclotomic polynomial which divides the Poincaré polynomial of the (finite) parabolic subgroup generated by the vertices of the simplex. We allow matchings between simplices of the same weight. Applying discrete Morse theory, one obtains a Morse complex which still computes the local homology.

Then, by applying these methods to other Coxeter graphs (in particular to finite and affine type cases), we come up with the notion of *precise matching*. In all the cases we consider, it is possible to produce acyclic weighted matchings such that the Morse complex turns out to have a very nice property: the boundary of a critical p–simplex σ has a nonzero coefficient along a critical $(p-1)$–simplex τ if and only if $v_d(\sigma) = v_d(\tau) + 1$. Then the homology of the Morse complex reduces to the computation of the ranks of the incidence matrices among critical simplices (Theorems 4.2 and 4.4). We show that the existence of precise matchings can also be interpreted in terms of second-page collapsing of a spectral sequence which is naturally associated to the weighted sheaf (Proposition 4.3).

As we discuss at the beginning of Section 5, the existence of precise matchings for a certain Artin group has strong implications for its homology. In particular, we want to highlight the following nontrivial result:

Theorem 5.2 *Let $\boldsymbol{G_W}$ be an Artin group (corresponding to some finitely generated Coxeter group $\boldsymbol{W}$) that admits a φ–precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\boldsymbol{X_W}; R)$ does not have φ^k–torsion for $k \geq 2$.*

Here $\boldsymbol{X_W}$ (introduced in Section 2.4) is a finite CW–complex which is a deformation retract of the *orbit space* associated to $\boldsymbol{W}$, and $\pi_1(\boldsymbol{X_W}) = \boldsymbol{G_W}$. When $\boldsymbol{W}$ is finite and in a few other cases, $\boldsymbol{X_W}$ is known to be a $K(\boldsymbol{G_W}, 1)$, whereas this property is only conjectured in general; see Brieskorn [7], Callegaro, Moroni and Salvetti [13], Charney and Davis [15], Deligne [21], Hendriks [28], Lek [31], Okonek [33], Paris [37] and Salvetti [38; 39].

Theorem 5.2 can be seen as a result about the homology of the infinite cyclic covering of $\boldsymbol{X_W}$ and its monodromy. It is known that the conclusion of this theorem holds for all Artin groups of finite type, due to geometric reasons (see Remark 5.3). Our construction provides a new combinatorial condition that applies to a wider class of Artin groups (including, among others, the affine groups of type $\widetilde{A}_n$ and $\widetilde{C}_n$).

The methods we develop allow explicit and complete computations, in general in a more direct way compared to other known methods. We use them to compute the twisted homology in the case of affine Artin groups of type $\widetilde{C}_n$, obtaining the following result:

Theorem 5.20 *Let $\boldsymbol{G_W}$ be an Artin group of type $\widetilde{C}_n$. Then the φ_d–primary component of $H_*(\boldsymbol{G_W}; R)$ is trivial for d odd, and for d even is as follows:*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \leq m \leq n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k\frac{d}{2} + r$.

We also carry out complete computations in the cases A_n, B_n and $\widetilde{A}_n$; these cases were already done before by different methods [25; 19; 18; 11], but we need them to perform computations of more complicated cases.

In a recent preprint [36], the first author succeeded in constructing precise matchings for all remaining Artin groups of finite and affine type. Therefore, precise matchings seem to be a suitable and effective tool for studying the homology of Artin groups.

The paper is structured as follows. In Section 2 we recall the most important definitions and results of [41]. In particular, we introduce the general framework of weighted sheaves over posets, and then explain how it can be used to compute the homology of Artin groups. In Section 3 we study the case of braid groups (ie Artin groups of type A_n), and relate their homology with that of certain independence complexes. In Section 4 we introduce precise matchings and derive a general formula for the homology of the Morse complex. In Section 5 we apply the theory developed in Section 4 to the case of Artin groups. We construct precise matchings for Artin groups of type A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$, and use such matchings to explicitly compute the homology.

2 Preliminaries

In this section we are going to recall some definitions and constructions from [41] (see also [32]).

2.1 Weighted sheaves over posets

Let $(P, \preceq)$ be a finite poset.

Definition 2.1 Define a *sheaf of rings* over P (or a *diagram of rings* over P) as a collection

$$\{A_x \mid x \in P\}$$

of commutative rings together with a collection of ring homomorphisms

$$\{\rho_{x,y}\colon A_y \to A_x \mid x \preceq y\}$$

satisfying

$$\rho_{x,x} = \mathrm{id}_{A_x},$$

$$x \preceq y \preceq z \implies \rho_{x,z} = \rho_{x,y}\rho_{y,z}.$$

In other words, a sheaf of rings over P is a functor from P^{op} to the category of commutative rings.

Fix a PID R; usually we take $R = \mathbb{Q}[q^{\pm 1}]$. The divisibility relation

$$p_1 \,|\, p_2 \iff (p_1) \supseteq (p_2)$$

gives R the structure of a small category and R/R_* the structure of a poset (where R_* is the group of units in R) with minimum element the class of the units and maximum element 0. Any functor $w\colon (P, \preceq) \to (R, |)$, which maps every $x \in P$ to some $w(x) \in R \setminus \{0\}$, defines a sheaf over P by the collection

$$\{R/(w(x)) \mid x \in P\}$$

and

$$\{i_{x,y}\colon R/(w(y)) \to R/(w(x)) \mid x \preceq y\},$$

where $i_{x,y}$ is induced by the identity of R.

Definition 2.2 Given a poset P, a PID R and a morphism $w\colon (P, \preceq) \to (R, |)$ as above, we call the triple (P, R, w) a *weighted sheaf* over P and the coefficients $w(x)$ the *weights* of the sheaf.

Remark 2.3 Consider the *poset topology* over P, where a basis for the open sets is given by the upper intervals $\mathcal{B} = \{P_{>p},\ p \in P\}$. Let (P, R, w) be a weighted sheaf. Then one can see w as a functor from $(\mathcal{B}, \supseteq)$ to $(R, |)$ and thus a weighted sheaf defines a sheaf in the usual sense.

From now on our poset will be a simplicial complex K defined over a finite set S, with the partial ordering

$$\sigma \preceq \tau \iff \sigma \subseteq \tau.$$

We adopt the convention that K contains the empty simplex $\varnothing$. A weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w(\sigma) \in R$, with

$$\sigma \preceq \tau \implies w(\sigma) \,|\, w(\tau).$$

Let $C^0_*(K; R)$ be the 1–shifted standard algebraic complex computing the simplicial homology of K, ie

$$C^0_k(K; R) = \bigoplus_{\substack{\sigma \in K \\ |\sigma| = k}} R e^0_\sigma,$$

where e^0_σ is a generator associated to a given orientation of σ ($C^0_0(K;R) = Re^0_\varnothing$). The boundary is given by

$$\partial^0(e^0_\sigma) = \sum_{|\tau|=k-1} [\sigma:\tau] e^0_\tau,$$

where $[\sigma:\tau]$ is the incidence number (which is equal to ± 1 if $\tau \prec \sigma$, and otherwise is equal to 0).

Definition 2.4 The *weighted complex* associated to the weighted sheaf (K, R, w) is the algebraic complex $L_* = L_*(K)$ defined by

$$L_k = \bigoplus_{|\sigma|=k} \frac{R}{(w(\sigma))} \overline{e}_\sigma,$$

with boundary $\partial\colon L_k \to L_{k-1}$ induced by ∂^0:

$$\partial(a_\sigma \overline{e}_\sigma) = \sum_{\tau \prec \sigma} [\sigma:\tau] i_{\tau,\sigma}(a_\sigma) \overline{e}_\tau.$$

There is a natural projection $\pi\colon C^0_*(K;R) \to L_*$ which maps a generator e_σ in C^0_* to the generator $\overline{e}_\sigma$ in L_*.

Remark 2.5 The diagram $\{R/(w(\sigma)) \mid \sigma \in K\}$ also defines a sheaf over the poset K in the sense of Remark 2.3. The sheaf cohomology associated to the open covering given by the upper segments coincides with the homology of the weighted sheaf.

2.2 Decomposition and filtration

Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf. For any irreducible $\varphi \in R$, we define the φ–primary component $\mathcal{S}_\varphi = (K, R, w_\varphi)$ of the weighted sheaf $\mathcal{S}$ by setting

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)},$$

where $v_\varphi(\sigma)$ is the maximal r such that φ^r divides $w(\sigma)$. Since R is a PID and $w(\sigma) \neq 0$, such a maximal value exists. Notice that $\mathcal{S}_\varphi$ is a weighted sheaf. The weighted complex $(L_*)_\varphi$ associated to $\mathcal{S}_\varphi$ is called the φ–primary component of the weighted complex L_*. In degree k one has

$$(L_k)_\varphi = \bigoplus_{|\sigma|=k} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \overline{e}_\sigma.$$

The complex $(L_*)_\varphi$ has a natural increasing filtration by the subcomplexes

$$F^s(L_*)_\varphi = \bigoplus_{v_\varphi(\sigma)\le s} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\overline{e}_\sigma.$$

This filtration is associated to an increasing filtration of the simplicial complex K,

$$K_{\varphi,s} = \{\sigma \in K \mid v_\varphi(\sigma) \le s\}.$$

Then $F^s(L_*)_\varphi$ is the weighted complex associated to the weighted sheaf

$$(K_{\varphi,s}, R, w_\varphi|_{K_{\varphi,s}}).$$

Theorem 2.6 [41] *Let (K,R,w) be a weighted sheaf, with associated weighted complex L_*. For any irreducible $\varphi \in R$, there exists a spectral sequence*

$$E^r_{p,q} \Rightarrow H_*((L_*)_\varphi)$$

that abuts to the homology of the φ–primary component of the associated algebraic complex L_*. Moreover, the E^1–term

$$E^1_{p,q} = H_{p+q}(F^p/F^{p-1}) \cong H_{p+q}(K_{\varphi,p}, K_{\varphi,p-1}; R/(\varphi^p))$$

is isomorphic to the relative homology with trivial coefficients *of the simplicial pair $(K_{\varphi,p}, K_{\varphi,p-1})$.*

2.3 Discrete Morse theory for weighted complexes

Given x and y in a poset $(P,\prec)$, the notation $x \lhd y$ means that y *covers* x, ie $x \prec y$ and there is no other $z \in P$ such that $x \prec z \prec y$.

Recall the basic facts of discrete Morse theory (see for example [23; 30]). A *matching* in a poset P is a set $\mathcal{M} \subseteq P \times P$ such that $(x,y) \in M$ implies $x \lhd y$, and each $x \in P$ belongs to at most one pair of $\mathcal{M}$. An *alternating path* is a sequence

$$y_0 \rhd x_1 \lhd y_1 \rhd x_2 \lhd y_2 \rhd \cdots \rhd x_m \lhd y_m\ (\rhd x_{m+1})$$

such that each pair $x_i \lhd y_i$ belongs to $\mathcal{M}$ and no pair $x_i \lhd y_{i-1}$ belongs to $\mathcal{M}$. A *cycle* is a closed alternating path with $y_0 = y_m$. An *acyclic matching* over P is a matching with no cycles. We are going to describe a variant of discrete Morse theory which is suitable for our situation.

Definition 2.7 A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching $\mathcal{M}$ on P such that

$$(x, y) \in \mathcal{M} \implies (w(x)) = (w(y))$$

(in [41] we required $w(x) = w(y)$, but the above generalization does not change the results).

The standard theory generalizes as follows. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a finite simplicial complex K and let $\mathcal{M}$ an acyclic weighted matching. A *critical simplex* of K is a simplex σ which does not belong to any pair of $\mathcal{M}$. The following definition is equivalent to [41, Definition 3.3]:

Definition 2.8 The *Morse complex* of $\mathcal{S}$ with respect to $\mathcal{M}$ is defined as the torsion complex

$$L_*^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} \frac{R}{(w(\sigma))} \overline{e}_\sigma$$

with boundary

$$\partial^{\mathcal{M}}(\overline{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau|=|\sigma|-1}} [\sigma : \tau]^{\mathcal{M}} \overline{e}_\tau,$$

where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is given by the sum over all alternating paths

$$\sigma \rhd \tau_1 \lhd \sigma_1 \rhd \tau_2 \lhd \sigma_2 \rhd \cdots \rhd \tau_m \lhd \sigma_m \rhd \tau$$

from σ to τ of the quantity

$$(-1)^m [\sigma : \tau_1][\sigma_1 : \tau_1][\sigma_1 : \tau_2][\sigma_2 : \tau_2] \cdots [\sigma_m : \tau_m][\sigma_m : \tau].$$

The boundary map $\partial^{\mathcal{M}}$ is extended by R–linearity, where some care should be taken since each $\overline{e}_\tau$ lives in a component with a possibly different torsion ($\partial^{\mathcal{M}}$ is the same as the boundary map of [41]).

Theorem 2.9 [41] *Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a simplicial complex K. Let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}$. Then there is an isomorphism*

$$H_*(L_*, \partial) \cong H_*(L_*^{\mathcal{M}}, \partial^{\mathcal{M}})$$

between the homology of the weighted complex L_* associated to $\mathcal{S}$ and the homology of the Morse complex $L_*^{\mathcal{M}}$ of $\mathcal{S}$.

Remark 2.10 If we forget about the weights, the matching $\mathcal{M}$ is in particular an acyclic matching for the simplicial complex K. Therefore, the algebraic complex $C^0_*(K;A)$, which computes the (shifted and reduced) simplicial homology of K with coefficients in some ring A, admits a "classical" algebraic Morse complex

$$C^0_*(K;A)^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} A\overline{e}_\sigma$$

with boundary map

$$\delta^{\mathcal{M}}(\overline{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau|=|\sigma|-1}} [\sigma:\tau]^{\mathcal{M}} \overline{e}_\tau.$$

Remark 2.11 Set $C^0_* = C^0_*(K;R)$ and let $\pi\colon C^0_* \to L_*$ be the natural projection. The composition with the projection $L_* \to (L_*)_\varphi$, for some fixed irreducible element $\varphi \in R$, yields a natural projection $\pi_\varphi\colon C^0_* \to (L_*)_\varphi$. Similarly at the level of Morse complexes we have the projection $\overline{\pi}_\varphi\colon (C^0_*)^{\mathcal{M}} \to (L_*)^{\mathcal{M}}_\varphi$ which sends a generator $\overline{e}_\sigma$ in $(C^0_*)^{\mathcal{M}}$ to the generator $\overline{e}_\sigma$ in $(L_*)^{\mathcal{M}}_\varphi$. Then, by construction, the maps induced in homology by the projections π_φ and $\overline{\pi}_\varphi$ commute with the Morse isomorphisms of [30; 41]:

$$\begin{array}{ccc} H_*(C^0_*) & \xrightarrow{(\pi_\varphi)_*} & H_*((L_*)_\varphi) \\ \cong\downarrow & & \downarrow\cong \\ H_*((C^0_*)^{\mathcal{M}}) & \xrightarrow{(\overline{\pi}_\varphi)_*} & H_*((L_*)^{\mathcal{M}}_\varphi) \end{array}$$

Let $\varphi \in R$ be an irreducible element and let $\mathcal{M}$ be a weighted acyclic matching on $\mathcal{S}_\varphi$. Then the filtration on the weighted complex induces a filtration on the Morse complex,

$$F^p(L_*)^{\mathcal{M}}_\varphi = \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma)\le p}} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\overline{e}_\sigma.$$

Consider also the quotient complex

$$\mathcal{F}^p(L_*)^{\mathcal{M}}_\varphi = F^p(L_*)^{\mathcal{M}}_\varphi / F^{p-1}(L_*)^{\mathcal{M}}_\varphi \cong \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma)=p}} \frac{R}{(\varphi^p)}\overline{e}_\sigma.$$

Theorem 2.12 [41] *Let $\mathcal{S}$ and φ be as above and let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}_\varphi$. Then the E^1–page of the spectral sequence of Theorem 2.6 is identified with*

$$E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p(L_*)^{\mathcal{M}}_\varphi),$$

where $(L_*)_\varphi^{\mathcal{M}}$ *is the Morse complex of* $\mathcal{S}_\varphi$. *The differential*

$$d^1_{p,q}\colon E^1_{p,q} \to E^1_{p-1,q}$$

is induced by the boundary of the Morse complex, and thus it is also computed by using alternating paths.

2.4 Weighted sheaves for Artin groups

We use the constructions given above in the context of Artin groups. Let $(\boldsymbol{W}, S)$ be a Coxeter system, with $|S| = n$ finite, and let Γ be the corresponding Coxeter graph (with S as vertex set). Recall that $\boldsymbol{W}$ can always be realized as a group of reflections in some $\mathbb{R}^n$ (for example through the *Tits representation* — see [6; 43] and also [37]), so that it has the following naturally associated objects:

(i) a hyperplane arrangement in $\mathbb{R}^n$

$$\mathcal{A} = \{H \mid H \text{ is the fixed-point set of some reflection in } \boldsymbol{W}\};$$

(ii) configuration spaces

$$\boldsymbol{Y} = (\mathrm{int}(U) + i\mathbb{R}^n) \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}},$$
$$\boldsymbol{Y_W} = \boldsymbol{Y}/\boldsymbol{W},$$

where $U = \boldsymbol{W} \cdot \overline{C}_0$ is the *Tits cone* (here C_0 is a fixed chamber of the arrangement, and $\overline{C}_0$ is its topological closure);

(iii) a simplicial complex (defined over the finite set S)

$$K = K_{\boldsymbol{W}} = \{\sigma \subseteq S \mid \text{the parabolic subgroup } \boldsymbol{W}_\sigma \text{ generated by } \sigma \text{ is finite}\}.$$

One can define the *Artin group* $\boldsymbol{G_W}$ of type $\boldsymbol{W}$ as the fundamental group $\pi_1(\boldsymbol{Y_W})$. It has a presentation

$$\boldsymbol{G_W} = \langle g_s,\ s \in S \mid g_s g_{s'} g_s g_{s'} \cdots = g_{s'} g_s g_{s'} g_s \cdots \rangle$$

(both the products have $m(s,s')$ factors, where $m(s,s')$ is the order of ss' in $\boldsymbol{W}$). Recall also the following results [39, Theorems 1.4 and 1.8]:

(i) The orbit space $\boldsymbol{Y_W}$ deformation retracts onto a *finite* CW–complex $\boldsymbol{X_W}$ given by a union

$$Q = \bigcup_{\sigma \in K_{\boldsymbol{W}}} Q_\sigma$$

of convex polyhedra with explicit identifications of their faces.

(ii) Consider the action of the Artin group $\boldsymbol{G_W}$ on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by

$$g_s \mapsto [\text{multiplication by } -q] \quad \text{for all } s \in S.$$

Then the homology $H_*(\boldsymbol{X_W}; R)$ is computed by the algebraic complex

$$C_k = \bigoplus_{\substack{\sigma \in K_{\boldsymbol{W}} \\ |\sigma| = k}} R e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \lhd \sigma} [\sigma : \tau] \frac{\boldsymbol{W}_\sigma(q)}{\boldsymbol{W}_\tau(q)} e_\tau,$$

where $\boldsymbol{W}_\sigma(q) = \sum_{w \in \boldsymbol{W}_\sigma} q^{l(w)}$ is the Poincaré polynomial of $\boldsymbol{W}_\sigma$.

Remark 2.13 The R–module structure on $H_*(\boldsymbol{X_W}; R)$ is given by the transformation μ_q induced by q–multiplication. If the order of μ_q is n, then the homology groups decompose into cyclic factors which are either free, of the form R^k, or torsion, of the form $R/(\varphi_d)$, where φ_d is the d^{th} cyclotomic polynomial with $d \mid n$. Therefore, we are interested in localizing to cyclotomic polynomials.

Theorem 2.14 [41] *To a Coxeter system (W, S) we can associate a weighted sheaf (K, R, w) over the simplicial complex $K = K_{\boldsymbol{W}}$, by setting $w(\sigma) = \boldsymbol{W}_\sigma(q)$. Also, for any cyclotomic polynomial $\varphi = \varphi_d$, the map $w_\varphi(\sigma)$ — which gives the maximal power of φ which divides $\boldsymbol{W}_\sigma(q)$ — defines a weighted sheaf (K, R, w_φ) over $K = K_{\boldsymbol{W}}$.*

The homology of the associated weighted complex is strictly related to the homology of $\boldsymbol{X_W}$. Specifically, set $C_*^0 = C_*^0(K; R)$ and consider the diagonal map

$$\Delta\colon C_* \to C_*^0, \quad e_\sigma \mapsto \boldsymbol{W}_\sigma(q) e_\sigma^0.$$

By the formula for the boundary map it follows that Δ is an injective chain complex homomorphism, so there is an exact sequence of complexes

$$0 \to C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \to 0,$$

where

$$L_k = \bigoplus_{\substack{\sigma \in K_{\boldsymbol{W}} \\ |\sigma| = k}} \frac{R}{(\boldsymbol{W}_\sigma(q))} \bar{e}_\sigma$$

is the quotient complex. Passing to the associated long exact sequence we get

(1) $$\cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \to H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \to H_{k-1}(C_*) \xrightarrow{\Delta_*} \cdots.$$

Then the homology of L_* can be used to compute the homology of C_*.

The orbit space $\boldsymbol{Y_W}$ (and thus the CW–complex $\boldsymbol{X_W} \simeq \boldsymbol{Y_W}$) is conjectured to be a classifying space for the Artin group $\boldsymbol{G_W}$ [21; 7; 31; 37]. This conjecture was proved for Artin groups of finite type [21], for affine Artin groups of type $\widetilde{A}_n$, $\widetilde{B}_n$ and $\widetilde{C}_n$ [33; 13] and for some other families of Artin groups [28; 15]. Whenever the conjecture holds (in particular, this is true for all the cases we consider in this paper), the homology $H_*(\boldsymbol{X_W}; R)$ coincides with the twisted homology $H_*(\boldsymbol{G_W}; R)$ of the Artin group $\boldsymbol{G_W}$.

3 Homology of braid groups and independence complexes

In this section we are going to show how the twisted homology of braid groups (ie Artin groups of type A_n) is related to the homology of suitable independence complexes. Recall that a Coxeter graph of type A_n is a linear graph with n vertices, usually labeled $1, 2, \dots, n$ (see Figure 1), and that the corresponding Artin group is the braid group on $n+1$ strands (which we denote by Br_{n+1}). With a slight abuse of notation, by A_n we will sometimes indicate the graph itself.

Figure 1: A Coxeter graph of type A_n

The homology of Br_{n+1}, with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ described in the previous section, has been computed in [25; 18]. See also [10] for coefficients in $\mathbb{Z}[q^{\pm 1}]$ (but we will not address this case).

Theorem 3.1 [25; 18] *The φ_d–primary component of the twisted homology of a braid group is given by*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} = \begin{cases} R/(\varphi_d) & \text{if } n \equiv 0 \text{ or } -1 \pmod d, \\ 0 & \text{otherwise,} \end{cases}$$

where the nonvanishing term is in degree $(d-2)k$ if $n = dk$ or $n = dk - 1$.

Recall that, if G is a graph with vertex set VG, an *independent set* of G is a subset of VG consisting of pairwise nonadjacent vertices. Also, the *independence complex* $\mathrm{Ind}(G)$ of G is the abstract simplicial complex with VG as set of vertices and whose simplices are all the nonempty independent sets of G. Thus, $\mathrm{Ind}(G)$ is the clique complex of the complement graph of G. In contrast with the simplicial complex K introduced in Section 2, the simplicial complex $\mathrm{Ind}(G)$ does not contain the empty simplex, and the dimension of a simplex $\sigma \in \mathrm{Ind}(G)$ is given by $|\sigma|-1$. The homotopy type of $\mathrm{Ind}(A_n)$ has been computed in [30] by means of discrete Morse theory, and the result is the following:

Proposition 3.2 [30, Proposition 11.16] *We have*

$$\mathrm{Ind}(A_n) \simeq \begin{cases} S^{k-1} & \text{if } n = 3k \text{ or } n = 3k-1, \\ \{\mathrm{pt}\} & \text{if } n = 3k+1. \end{cases}$$

By comparing this with Theorem 3.1 we obtain the following relation between the homology of the independence complex of A_n and the φ_3–primary component of the twisted homology of Br_{n+1}:

Corollary 3.3 *We have*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_3} \cong \widetilde{H}_{*-2}\left(\mathrm{Ind}(A_{n-3}); \frac{R}{(\varphi_3)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients. □

In general, following [40], define the r*–independence complex* of a graph G as

$$\mathrm{Ind}_r(G) = \{\text{full subgraphs } G' \subseteq G \text{ such that each connected component of } G' \text{ has at most } r \text{ vertices}\}.$$

So $\mathrm{Ind}_r(G)$ is an abstract simplicial complex on the set of vertices VG of the graph G, which coincides with $\mathrm{Ind}(G)$ for $r=1$. The case $r=0$ also makes sense: $\mathrm{Ind}_0(G)=\varnothing$ for any graph G. We are going to prove the following generalization of Corollary 3.3:

Theorem 3.4 *We have*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients.

From the expression of the φ_d–primary component of the local homology of the braid group (Theorem 3.1) we obtain the following consequence:

Corollary 3.5 *We have*

$$\widetilde{H}_*(\mathrm{Ind}_{d-2}(A_n)) \cong \begin{cases} \widetilde{H}_*(S^{dk-2k-1}) & \text{for } n = dk \text{ or } n = dk-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the knowledge of the twisted homology of the braid group gives the homology (with trivial coefficients) of $\mathrm{Ind}_d(A_n)$. Conversely, the knowledge of the homology of $\mathrm{Ind}_d(A_n)$ gives the twisted homology of the braid group.

The Poincaré polynomial of a Coxeter group of type A_k is given by

$$\boldsymbol{W}_{A_k}(q) = [k+1]_q!, \quad \text{where } [k]_q = \frac{q^k-1}{q-1} = \prod_{\substack{d \mid k \\ d \geq 2}} \varphi_d$$

(see for example [5]). Consider now a simplex $\sigma \subseteq S \cong \{1, \dots, n\}$. Denote by $\Gamma(\sigma)$ the subgraph of A_n induced by σ. Denote by $\Gamma_1(\sigma)$, $\Gamma_2(\sigma)$, ..., $\Gamma_m(\sigma)$ the connected components of $\Gamma(\sigma)$ and by n_1, n_2, $\dots n_m$ their cardinalities (see Figure 2). The i^{th} connected component is a Coxeter graph of type A_{n_i}, so the entire Coxeter graph induced by σ has Poincaré polynomial

$$\begin{aligned} \boldsymbol{W}_\sigma(q) &= [n_1+1]_q! \cdot [n_2+1]_q! \cdots [n_m+1]_q! \\ &= \prod_{i=1}^{m} \prod_{d \geq 2} \varphi_d^{\lfloor (n_i+1)/d \rfloor} \\ &= \prod_{d \geq 2} \varphi_d^{\sum_{i=1}^m \lfloor (n_i+1)/d \rfloor}. \end{aligned}$$

Then we are interested in the homology of the weighted complex $(L_*)_\varphi$ associated to the weighted sheaf (K, R, w_φ), where $\varphi = \varphi_d$ is a cyclotomic polynomial (with $d \geq 2$) and

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)}, \quad v_\varphi(\sigma) = \sum_{i=1}^{m} \left\lfloor \frac{n_i+1}{d} \right\rfloor.$$

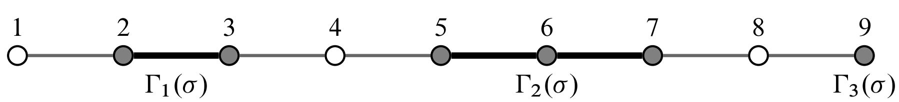


Figure 2: An example with $n = 9$ and $\sigma = \{2, 3, 5, 6, 7, 9\} \in \mathrm{Ind}_3(A_9)$. In this case, $|\Gamma_1(\sigma)| = 2$, $|\Gamma_2(\sigma)| = 3$ and $|\Gamma_3(\sigma)| = 1$.

Notice that only the connected components with at least $d-1$ vertices contribute to the φ–weight. Therefore, the weighted complex $(L_*)_\varphi$ is generated by the subgraphs having at least one component with $\geq d-1$ vertices.

Theorem 3.6 *There is a weighted acyclic matching $\mathcal{M}$ on K such that the set of critical simplices is given by*

$$\begin{aligned} Cr(\mathcal{M}) &= \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1},\ |\Gamma_i| \leq d-2\} \cup \mathrm{Ind}_{d-2}(A_n) \\ &= \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\} \cup \mathrm{Ind}_{d-2}(A_n), \end{aligned}$$

where A_{d-1} is the linear graph on the vertices $n-d+2, \dots, n$.

Proof First notice that removing the d^{th} vertex from an A_k component leaves the φ–weight unchanged:

$$v_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor = 1 + \left\lfloor \frac{k+1-d}{d} \right\rfloor = v_\varphi(A_{d-1} \sqcup A_{k-d})$$

(see Figure 3).

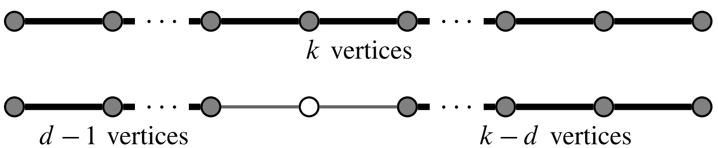


Figure 3: The φ–weight of an A_k component remains the same if we remove the d^{th} vertex, splitting A_k into $A_{d-1} \sqcup A_{k-d}$.

Let $K_0 = \mathrm{Ind}_{d-2}(A_n) \subseteq K$ (this is the set of the simplices $\sigma \in K$ such that all the connected components of the induced subgraph $\Gamma(\sigma)$ have cardinality $< d-1$). Let us define on $K' = K \setminus K_0$ the following matching $\mathcal{M}$. For $\sigma \in K'$, with $\Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, set

$$i(\sigma) = \min\{i \mid |\Gamma_i| \geq d-1\}.$$

Then match σ with the simplex τ obtained by adding or removing from σ the d^{th} vertex of the component $\Gamma_{i(\sigma)}$. By the remark at the beginning of the proof, $\mathcal{M}$ is a weighted matching. We prove that it is acyclic. In fact, suppose that an alternating path contains some subpath

$$\tau \lhd \sigma \rhd \tau' \lhd \sigma'$$

with $\tau \neq \tau'$. Then either $i(\tau) < i(\tau')$ or (set $j = i(\tau) = i(\tau')$)

$$\Gamma_j(\tau) = \{a, a+1, \dots, a+d-2\}, \quad \Gamma_j(\tau') = \{a+1, \dots, a+d-1\}$$

for some $a \leq n-d+1$. In this case the first vertex of $\Gamma_j(\tau')$ is greater than the first vertex of $\Gamma_j(\tau)$. Therefore, an alternating path in K' cannot be closed.

The set of critical elements of $\mathcal{M}$ in K' is given by

$$\mathrm{Cr}(\mathcal{M}) = \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1},\ |\Gamma_i| \leq d-2\},$$

where A_{d-1} is as in the statement of the theorem. □

Proof of Theorem 3.4 Consider the matching $\mathcal{M}$ of Theorem 3.6. Since $K_0 = \mathrm{Ind}_{d-2}(A_n)$ does not contribute to the weighted complex $(L_*)_\varphi$, we concentrate on the poset $K' = K \setminus K_0$. Notice that there are no nontrivial alternating paths between critical elements of K'. Therefore, the boundaries between simplices in

$$\mathrm{Cr}'(\mathcal{M}) = \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\}$$

are the same as the boundaries of $\mathrm{Ind}_{d-2}(A_{n-d})$. In the long exact sequence (1) the algebraic complex C^0_* has vanishing homology in all degrees (because K is the full simplicial complex on n vertices), so we have an isomorphism $H_{*+1}(L_*) \cong H_*(C_*)$. Therefore,

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong H_*(C_*)_{\varphi_d} \cong H_{*+1}(L_*)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right).$$

In the last isomorphism there is a $(d-1)$–shift in degree due to the loss of $d-1$ vertices when passing from $\mathrm{Cr}'(\mathcal{M})$ to $\mathrm{Ind}_{d-2}(A_{n-d})$; there is then a further 1–shift in degree due to the fact that in L_* a simplex σ has dimension $|\sigma|$, and in $\mathrm{Ind}_{d-2}(A_{n-d})$ it has dimension $|\sigma|-1$; finally, it is necessary to pass to reduced homology because the empty simplex is missing in $\mathrm{Ind}_{d-2}(A_{n-d})$. □

For the sake of completeness we also determine the homotopy type of the r–independence complex $\mathrm{Ind}_{d-2}(A_n)$ (obtaining again Corollary 3.5 as a consequence). This is a straightforward generalization of the case $d=3$, proved in [30, Proposition 11.16].

Proposition 3.7 *We have*

$$\mathrm{Ind}_{d-2}(A_n) \simeq \begin{cases} S^{dk-2k-1} & \text{if } n = dk \text{ or } n = dk-1, \\ \{\mathrm{pt}\} & \text{otherwise.} \end{cases}$$

Proof We denote here for brevity $K_0 = \mathrm{Ind}_{d-2}(A_n)$. Let $n = qd + r$ be the euclidean division of n by d.

Let $P = \{c_d > c_{2d} > c_{3d} > \cdots > c_{qd} > c_*\}$ be a linearly ordered set, where c_* is the minimum element. Define a map $f\colon K_0 \to P$ as follows. If $\sigma \in K_0$ does not contain any multiples of d, then set $f(\sigma) = c_*$. Otherwise, set $f(\sigma) = c_{jd}$ if $jd \in \sigma$ but $j'd \notin \sigma$ for $j' < j$. Clearly f is a poset map (if we remove a vertex from σ, j increases or remains the same).

In $f^{-1}(c_d)$ consider the matching $\mathcal{M}_1 = \{(\sigma \setminus \{1\}, \sigma \cup \{1\}) \mid \sigma \in f^{-1}(c_d)\}$, which is justified by

$$\sigma \in f^{-1}(c_d) \implies \{2, \ldots, d-1\} \not\subseteq \sigma.$$

Similarly, in $f^{-1}(c_{jd})$ for $j \le q$, consider the matching

$$\mathcal{M}_j = \bigl\{(\sigma \setminus \{(j-1)d+1\},\ \sigma \cup \{(j-1)d+1\}) \mid \sigma \in f^{-1}(c_{jd})\bigr\},$$

justified by

$$\sigma \in f^{-1}(c_{jd}) \implies \{(j-1)d+2, \ldots, jd-1\} \not\subseteq \sigma.$$

Each $\mathcal{M}_j$ is acyclic in $f^{-1}(c_{jd})$, thus

$$\mathcal{M} = \bigcup_{j=1}^{q} \mathcal{M}_j$$

is an acyclic matching on K_0 by the patchwork theorem (see [29, Lemma 4.2; 30, Theorem 11.10] for example). Since each $\mathcal{M}_j$ is a perfect matching on $f^{-1}(c_{jd})$ for $j = 1, \ldots, q$ and $K_1 = f^{-1}(c_*)$ is a subcomplex of K_0, it follows that K_0 deformation retracts onto K_1.

Notice now that K_1 is the join of q copies of $\mathrm{Ind}_{d-2}(A_{d-1}) \cong S^{d-3}$ and one copy of $\mathrm{Ind}_{d-2}(A_r)$. For $r = 0$ we get

$$K_0 \cong (S^{d-3})^{*q} \cong S^{q(d-3)+q-1} = S^{qd-2q-1}$$

and for $r = d-1$ we get

$$K_0 \cong (S^{d-3})^{*(q+1)} \cong S^{(q+1)(d-3)+q} = S^{(q+1)d-2(q+1)-1},$$

which give the first case in the corollary. If r is not 0 or $d-1$ then $\mathrm{Ind}_{d-2}(A_r)$ is a full simplex Δ^{r-1} on r vertices, and therefore K_0 is contractible. □

4 Precise matchings

In this section we are going to introduce the notion of *precise matching* on a weighted sheaf. The motivation comes from the study of the twisted homology of some families of Artin groups, as we will show in the subsequent sections. For now we go back to the general framework of weighted sheaves over simplicial complexes (later we will specialize in the case of Artin groups).

Assume from now on that the PID R contains some field $\mathbb{K}$. Our main case of interest is $\mathbb{K} = \mathbb{Q}$ and $R = \mathbb{Q}[q^{\pm 1}]$. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over the finite simplicial complex K, with associated weighted complex L_*. Given a fixed irreducible element φ of R, let $\mathcal{S}_\varphi$ be the φ–primary component of $\mathcal{S}$ and let $(L_*)_\varphi$ be its associated weighted complex. Let $\mathcal{M}$ be a weighted acyclic matching for $\mathcal{S}_\varphi$.

Let $G^{\mathcal{M}}$ be the incidence graph of the corresponding Morse complex: the vertices of $G^{\mathcal{M}}$ are the critical simplices of K, and there is an (oriented) edge $\sigma \to \tau$ whenever $[\sigma : \tau]^{\mathcal{M}}$ is not 0 in R (or equivalently in $\mathbb{K}$), where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is the incidence number between σ and τ in the Morse complex of K. In other words, there is an edge $\sigma \to \tau$ if $[\sigma : \tau]^{\mathcal{M}}$ is not a multiple of $\operatorname{char} \mathbb{K} = \operatorname{char} R$. When $\mathbb{K} = \mathbb{Q}$, this simply means that $[\sigma : \tau]^{\mathcal{M}} \neq 0$.

Let $\mathcal{I}$ be the set of connected components of $G^{\mathcal{M}}$ (computed ignoring the orientation of the edges). Recall that $v_\varphi(\sigma)$ is the maximal $k \in \mathbb{N}$ such that φ^k divides $w(\sigma)$.

Definition 4.1 The matching $\mathcal{M}$ is *φ–precise* (or simply *precise*) if, for any edge $\sigma \to \tau$ of $G^{\mathcal{M}}$, we have that $v_\varphi(\sigma) = v_\varphi(\tau) + 1$.

In other words, $\mathcal{M}$ is precise if, for any two simplices σ and τ lying in the same connected component $i \in \mathcal{I}$, the following relation holds:

$$v_\varphi(\sigma) - v_\varphi(\tau) = |\sigma| - |\tau|.$$

Equivalently the quantity $|\sigma| - v_\varphi(\sigma)$, as a function of σ, is constant within a fixed connected component of $G^{\mathcal{M}}$. This definition is motivated by the fact that precise matchings exist in many cases of interest (we will see this in Section 5), and that the homology of the Morse complex is much simpler to compute (and takes a particularly nice form) when the matching is precise. The name "precise" has been chosen because for a generic matching one only has $v_\varphi(\sigma) \geq v_\varphi(\tau)$ (when $\sigma \to \tau$ is an edge of $G^{\mathcal{M}}$), and we require $v_\varphi(\sigma)$ to be precisely $v_\varphi(\tau) + 1$.

Assume from now on in this section that $\mathcal{M}$ is a φ–precise matching. To simplify the notation, set $(A_*, \partial) = ((L_*)^{\mathcal{M}}_{\varphi}, \partial^{\mathcal{M}})$ and $(V_*, \delta) = (C^0_*(K, \mathbb{K})^{\mathcal{M}}, \delta^{\mathcal{M}})$. Our aim is to derive a formula for the homology of the Morse complex A_*. Since the differential δ vanishes between simplices in different connected components of $G^{\mathcal{M}}$, the complex (V_*, δ) splits as

$$(V_*, \delta) = \bigoplus_{i \in \mathcal{I}} (V^i_*, \delta^i),$$

where

$$V^i_* = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \mathbb{K} \bar{e}_\sigma$$

and the boundary map $\delta^i \colon V^i_* \to V^i_*$ is the restriction of δ to V^i_*. The differential ∂ of A_* is induced by δ, and thus it also vanishes between simplices in different connected components. Therefore, we have an analogous splitting for (A_*, ∂),

$$(A_*, \partial) = \bigoplus_{i \in \mathcal{I}} (A^i_*, \partial^i),$$

where

$$A^i_* = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \frac{R}{(w_\varphi(\sigma))} \bar{e}_\sigma$$

and the boundary map $\partial^i \colon A^i_* \to A^i_*$ is simply the restriction of ∂ to A^i_*.

Fix now a connected component $i \in \mathcal{I}$. Since $\mathcal{M}$ is precise, there exists some $k \in \mathbb{Z}$ (which depends on i) such that $v_\varphi(\sigma) = |\sigma| + k$ for all $\sigma \in i$. Therefore, in degree m we have

$$A^i_m = \bigoplus_{\sigma \in \mathrm{Cr}^i_m} \frac{R}{(\varphi^{m+k})} \bar{e}_\sigma = \Bigg(\bigoplus_{\sigma \in \mathrm{Cr}^i_m} \mathbb{K} \bar{e}_\sigma \Bigg) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} = V^i_m \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})},$$

where Cr^i_m is the set of critical simplices σ such that $\sigma \in i$ and $|\sigma| = m$. By construction the boundary $\partial^i_m \colon A^i_m \to A^i_{m-1}$ factors accordingly:

$$\partial^i_m = \delta^i_m \otimes_{\mathbb{K}} \pi_m,$$

where

$$\pi_m \colon \frac{R}{(\varphi^{m+k})} \to \frac{R}{(\varphi^{m+k-1})}$$

is the projection induced by the identity $R \to R$. Since $\operatorname{im} \delta^i_{m+1} \subseteq \ker \delta^i_m$, each V^i_m splits (as a vector space over $\mathbb{K}$) as a direct sum of linear subspaces

$$V^i_m = W^i_{m,1} \oplus W^i_{m,2} \oplus W^i_{m,3},$$

where $W^i_{m,1} = \operatorname{im} \delta^i_{m+1}$ and $W^i_{m,1} \oplus W^i_{m,2} = \ker \delta^i_m$. Then

$$\begin{aligned}
\ker(\partial^i_m) &= \ker(\delta^i_m \otimes_{\mathbb{K}} \pi_m) \\
&= \left(\left(W^i_{m,1} \oplus W^i_{m,2} \right) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W^i_{m,3} \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right); \\
\operatorname{im}(\partial^i_{m+1}) &= \operatorname{im}(\delta^i_{m+1} \otimes_{\mathbb{K}} \pi_{m+1}) = W^i_{m,1} \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}.
\end{aligned}$$

Therefore, the homology of (A^i_*, δ^i) is given, as an R–module, by

$$\begin{aligned}
H_m(A^i_*) &= \frac{\ker(\partial^i_m)}{\operatorname{im}(\partial^i_{m+1})} \\
&= \left(W^i_{m,2} \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W^i_{m,3} \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right) \\
&\cong \left(H_m(V^i_*, \delta^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta^i_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).
\end{aligned}$$

In the last isomorphism we used the fact that $\dim W^i_{m,3} = \dim V^i_m - \dim \ker \delta^i_m = \operatorname{rk} \delta^i_m$.

Recall that the previous formula holds for a fixed connected component $i \in \mathcal{I}$, and k depends on i. Since we now need to take the direct sum over the connected components, let k_i be the value of k for the component i.

Theorem 4.2 *The homology of $(L_*)_\varphi$ is given, as an R–module, by*

$$H_m((L_*)_\varphi, \partial) \cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).$$

Proof By Theorem 2.9, $H_m((L_*)_\varphi, \partial) \cong H_m((L_*)^{\mathcal{M}}_\varphi, \partial^{\mathcal{M}})$. Using what we have done in this section, we have that

$$\begin{aligned}
H_m((L_*)^{\mathcal{M}}_\varphi, \partial^{\mathcal{M}}) &= H_m(A_*, \partial) \\
&= \bigoplus_{i \in \mathcal{I}} H_m(A^i_*, \partial) \\
&\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\left(\bigoplus_{i \in \mathcal{I}} \mathbb{K}^{\operatorname{rk} \delta^i_m} \right) \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right) \\
&\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).
\end{aligned}$$ □

Let us see how the existence of a precise matching can be interpreted in terms of the spectral sequence associated to the weighted sheaf (see Theorem 2.6).

Proposition 4.3 *If a φ–precise matching $\mathcal{M}$ exists, then the spectral sequence $E^r_{p,q}$ associated to the weighted sheaf $\mathcal{S}_\varphi$ collapses at the E^2–page.*

Proof By Theorem 2.12, the E^1–page can be computed through the Morse complex of our matching $\mathcal{M}$:

$$E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p(L_*)^{\mathcal{M}}_\varphi),$$

and the differential $d^1_{p,q}$ is induced by the boundary of the Morse complex. The spectral sequence then splits as a direct sum over the connected components of $G^{\mathcal{M}}$,

$$E^r_{p,q} = \bigoplus_{i \in \mathcal{I}} E^{r,i}_{p,q},$$

where $E^{1,i}_{p,q} \cong H_{p+q}(\mathcal{F}^p A^i_*)$. Since the matching is precise, for $m \neq p - k_i$ we have

$$\mathcal{F}^p A^i_m = F^p A^i_m / F^{p-1} A^i_m = 0.$$

This means that the page $E^{0,i}_{p,q} \cong E^{1,i}_{p,q}$ is nontrivial only in the row $q = -k_i$, and the entire spectral sequence $E^r_{p,q}$ collapses at the E^2–page. □

What we have done so far in this section assumed φ to be some fixed irreducible element of the PID R. In order to recover the full homology of L_* we need to let φ vary among all equivalence classes of irreducible elements of R modulo the units. Suppose from now on we have a φ–precise matching $\mathcal{M}_\varphi$ on $\mathcal{S}_\varphi$ for each φ. The following result follows immediately from Theorem 4.2, provided that we add a "φ" subscript (or superscript) to all the quantities that depend on the matching $\mathcal{M}_\varphi$.

Theorem 4.4 *The homology of L_* is given, as an R–module, by*

$$H_m(L_*, \partial) \cong \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta^\varphi_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \qquad □$$

For later applications, we finally need to study how the isomorphism of Theorem 4.4 behaves with respect to the projection $\pi \colon C^0_* \to L_*$ of Remark 2.11. This projection is the direct sum over φ of the projections

$$\pi_\varphi \colon C^0_* \to (L_*)_\varphi.$$

Instead of studying the induced map $(\pi_\varphi)_*\colon H_m(C^0_*) \to H_m((L_*)_\varphi)$, we study the map

$$(\overline{\pi}_\varphi)_*\colon H_m((C^0_*)^{\mathcal{M}}) \to H_m((L_*)^{\mathcal{M}}_\varphi)$$

between the Morse complexes (here $\mathcal{M} = \mathcal{M}_\varphi$ is a precise matching which depends on φ). For $i \in \mathcal{I}_\varphi$, let $\pi_i\colon (C^0_*)^{\mathcal{M}} \to V^{\varphi,i}_* \otimes_{\mathbb{K}} R \subseteq (C^0_*)^{\mathcal{M}}$ be the projection on the subcomplex corresponding to the connected component i, and let $(\pi_i)_*$ be the map induced in homology. Let $[c] \in H_m((C^0_*)^{\mathcal{M}})$, for some cycle $c \in \ker \delta^{\mathcal{M}} \subseteq (C^0_m)^{\mathcal{M}}$. Applying the map $(\overline{\pi}_\varphi)_*\colon H_m((C^0_*)^{\mathcal{M}}) \to H_m((L_*)^{\mathcal{M}}_\varphi)$ we obtain

$$(\overline{\pi}_\varphi)_*([c]) = (\overline{\pi}_\varphi)_*\Bigg(\sum_{i \in \mathcal{I}_\varphi} (\pi_i)_*([c])\Bigg) = \sum_{i \in \mathcal{I}_\varphi} (\overline{\pi}_\varphi)_*((\pi_i)_*([c])).$$

Applying the isomorphism of Theorem 4.2, this element is sent to

$$\sum_{i \in \mathcal{I}_\varphi} ((\pi_i)_*([c])) \otimes_{\mathbb{K}} [1] \in \bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

We are going to use these computations to prove the following result, which describes the kernel and the cokernel of π_*:

Proposition 4.5 *The cokernel of $\pi_*\colon H_m(C^0_*) \to H_m(L_*)$ is given by*

$$\operatorname{coker} \pi_* \cong \bigoplus_\varphi \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta^\varphi_m}.$$

In addition, the kernel of π_* is a free R–module isomorphic to $H_m(C^0_*)$.

Proof Throughout the proof, consider the following R–modules identified one with each other, without explicitly mentioning the isomorphisms between them:

$$\begin{aligned} H_m(L_*) &\cong \bigoplus_\varphi H_m((L_*)_\varphi) \\ &\cong \bigoplus_\varphi H_m((L_*)^{\mathcal{M}}_\varphi) \\ &\cong \bigoplus_\varphi \Bigg(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \Bigg) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta^\varphi_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \end{aligned}$$

Recall that the matching $\mathcal{M}$ depends on φ, although we write $\mathcal{M}$ instead of $\mathcal{M}_\varphi$ in order to make the notation more readable. Also recall that the isomorphisms $H_m((L_*)_\varphi) \cong H_m((L_*)^{\mathcal{M}}_\varphi)$ occur in the commutative diagram of Remark 2.11.

We want to show that the image of $\pi_*\colon H_m(C^0_*) \to H_m(L_*)$ is given by

$$\operatorname{im}\pi_* = \bigoplus_{\varphi}\Bigl(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}\Bigr) \subseteq H_m(L_*).$$

Let $\psi_\varphi\colon H_*(C^0_*) \to H_*((L_*)^{\mathcal{M}}_\varphi)$ be the map defined as the composition

$$H_*(C^0_*) \xrightarrow{\cong} H_*((C^0_*)^{\mathcal{M}}) \xrightarrow{(\bar\pi_\varphi)_*} H_*((L_*)^{\mathcal{M}}_\varphi).$$

By commutativity of the diagram of Remark 2.11, the image of $\pi_* = \bigoplus_\varphi (\pi_\varphi)_*$ is the same as the image of

$$\bigoplus_\varphi \psi_\varphi\colon H_*(C^0_*) \to \bigoplus_\varphi H_*((L_*)^{\mathcal{M}}_\varphi).$$

We have already proved that, for any $[c] \in H_m((C^0_*)^{\mathcal{M}})$,

$$(\bar\pi_\varphi)_*([c]) = \sum_{i\in\mathcal{I}_\varphi}((\pi_i)_*([c])) \otimes_R [1] \in \bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})},$$

which means in particular that

$$\operatorname{im}(\bar\pi_\varphi)_* \subseteq \bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

Therefore, we immediately have the inclusion

$$\operatorname{im}(\pi_*) \subseteq \sum_\varphi \operatorname{im}\psi_\varphi = \sum_\varphi \operatorname{im}(\bar\pi_\varphi)_* \subseteq \bigoplus_\varphi\Bigl(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}\Bigr).$$

To prove the opposite inclusion, we show that any element of the form

$$[c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}$$

is in the image of π_* (for any fixed φ and i). To do so, choose $\alpha \in R$ such that $\alpha \equiv 1 \pmod{\varphi^{m+k_{\varphi,i}}}$ and $\alpha \equiv 0 \pmod{\eta^{m+k_{\eta,j}}}$ for any irreducible element $\eta \neq \varphi$ which divides some weight $w(\sigma)$ and for any connected component $j \in \mathcal{I}_\eta$ (there is only a finite number of such η up to multiplication by units, because K is finite). The element $c \otimes_{\mathbb{K}} \alpha$ is a cycle in $V_m^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq C^0_m(K,\mathbb{K})^{\mathcal{M}} \otimes_{\mathbb{K}} R \cong (C^0_m)^{\mathcal{M}}$. Then, if $[\tilde c]$ is the preimage of $[c \otimes_{\mathbb{K}} \alpha]$ under the isomorphism $H_*(C^0_*) \xrightarrow{\cong} H_*((C^0_*)^{\mathcal{M}})$, we have that

$$\psi_\varphi([\widetilde{c}]) = (\overline{\pi}_\varphi)_*([c \otimes_{\mathbb{K}} \alpha]) = [c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})},$$
$$\psi_\eta([\widetilde{c}]) = (\overline{\pi}_\eta)_*([c' \otimes_{\mathbb{K}} \alpha]) = 0 \quad \text{for any } \eta \neq \varphi,$$

where $[c']$ is the image of $[c]$ under the isomorphism

$$H_*((C_*^0)^{\mathcal{M}_\varphi}) \xrightarrow{\cong} H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}_\eta}).$$

Therefore, $[c] \otimes_{\mathbb{K}} [1]$ is in the image of π_*. We have thus proved that

$$\operatorname{im} \pi_* = \bigoplus_\varphi \Biggl(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \Biggr).$$

Then the cokernel of π_* can be easily computed:

$$\operatorname{coker} \pi_* = \frac{H_m(L_*)}{\operatorname{im} \pi_*} = \bigoplus_\varphi \Biggl(\mathbb{K}^{\operatorname{rk} \delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \Biggr) = \bigoplus_\varphi \Biggl(\frac{R}{(\varphi)} \Biggr)^{\oplus \operatorname{rk} \delta_m^\varphi}.$$

The R–module $H_m(C_*^0) \cong H_m(C_*^0(K;\mathbb{K})) \otimes_{\mathbb{K}} R$ is free and finitely generated, because $H_m(C_*^0(K;\mathbb{K}))$ is a finite-dimensional vector space over $\mathbb{K}$ (recall that K is a finite simplicial complex). The kernel of π_* is a submodule of $H_m(C_*^0)$, so it is itself a free R–module with lower or equal rank. Let $[c_1], \dots, [c_k]$ be an R–base of $H_m(C_*^0)$. Consider the nonzero ideal

$$I = \bigcap_\varphi \bigcap_{i \in \mathcal{I}_\varphi} (\varphi^{m+k_i}) \subseteq R,$$

where φ varies among the (finitely many) irreducible elements which divide some weight $w(\sigma)$ (for $\sigma \in K$). Fix any nonzero element $\alpha \in I$. Then the elements $\alpha[c_1], \dots, \alpha[c_k]$ generate a free submodule of $\ker \pi_*$ of rank $k = \operatorname{rk} H_m(C_*^0)$. Therefore, $\ker \pi_*$ and $H_m(C_*^0)$ have the same rank, so they are isomorphic as R–modules. □

5 Precise matchings for Artin groups

Consider now the case of Artin groups, as in Section 2.4. For a Coxeter system (W, S) we have constructed a weighted sheaf $\mathcal{S} = (K, R, w)$ with

$$K = \{\sigma \subset S \mid \text{the parabolic subgroup } W_\sigma \text{ is finite}\}$$

and $R = \mathbb{Q}[q^{\pm 1}]$. The associated weighted complex L_* fits into the short exact sequence

$$0 \to C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \to 0,$$

which gives rise to the long exact sequence (1):

$$\cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \to H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \to H_{k-1}(C_*) \xrightarrow{\Delta_*} \cdots .$$

In order to compute $H_*(C_*) = H_*(\boldsymbol{X_W}; R)$, we split this long exact sequence into the short exact sequences

$$0 \to \operatorname{coker} \pi_* \to H_m(C_*) \xrightarrow{\Delta_*} \ker \pi_* \to 0,$$

where on the left we have the cokernel of $\pi_*\colon H_{m+1}(C_*^0) \to H_{m+1}(L_*)$ and on the right we have the kernel of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$. Since $\ker \pi_*$ is a free R–module, these short exact sequences split:

$$H_m(C_*) \cong \operatorname{coker} \pi_* \oplus \ker \pi_*.$$

Recall that the only irreducible elements of R that occur in the factorization of the weights are the cyclotomic polynomials φ_d for $d \geq 2$. As in Section 4, suppose from now on that we have constructed a φ–precise matching $\mathcal{M}_\varphi$ for each cyclotomic polynomial $\varphi = \varphi_d$ (with $d \geq 2$). Then we have an explicit description of $\operatorname{coker} \pi_*$ and $\ker \pi_*$ thanks to Proposition 4.5, and we obtain the following result:

Theorem 5.1 *Under the above hypothesis, the homology of $\boldsymbol{X_W}$ with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ is given by*

$$H_m(\boldsymbol{X_W}; R) \cong \left(\bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_{m+1}^{\varphi}} \right) \oplus H_m(C_*^0). \qquad \square$$

In particular, the term $H_m(C_*^0)$ gives the free part of the homology, and the other direct summands give the torsion part. The torsion part actually takes a very particular form, and we are going to highlight this in the following result:

Theorem 5.2 *Let $\boldsymbol{G_W}$ be an Artin group that admits a φ–precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\boldsymbol{X_W}; R)$ does not have φ^k–torsion for $k \geq 2$.* $\square$

We are particularly interested in Artin groups of finite and affine type. When G_W is an Artin group of finite type with n generators, K is the full simplicial complex on $S \cong \{1, \dots, n\}$ and therefore C^0_* has trivial homology in every dimension. Thus, the formula of Theorem 5.1 reduces to

$$H_m(X_W; R) \cong \bigoplus_{\varphi} \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta^{\varphi}_{m+1}}.$$

When G_W is an Artin group of affine type with $n+1$ generators, K is obtained from the full simplicial complex on $S \cong \{0, 1, \dots, n\}$ by removing the single top-dimensional simplex. Then we have

$$H_m(X_W; R) \cong \begin{cases} R & \text{for } m = n, \\ \bigoplus_{\varphi} (R/(\varphi))^{\oplus \operatorname{rk} \delta^{\varphi}_{m+1}} & \text{for } m < n. \end{cases}$$

Remark 5.3 When G_W is an Artin group of finite type, the corresponding reflection arrangement of hyperplanes $\mathcal{A}$ is finite. In this case it is well known that there is an R–module isomorphism between the twisted homology $H_*(X_W; R)$ and the homology with constant coefficients $H_*(F; \mathbb{Q})$ of the Milnor fiber F of $\mathcal{A}$ [9]. The q–multiplication on the homology of X_W corresponds to the action of the monodromy operator on the homology of F. If $N = |\mathcal{A}|$, the square of the defining polynomial of the arrangement is W–invariant, thus the order of the monodromy of the Milnor fibration divides $2N$. It follows that the polynomial $q^{2N} - 1$ must annihilate the homology. Since $q^{2N} - 1$ is square-free in characteristic 0, the homology cannot have φ^k–torsion for $k \geq 2$. So the conclusion of Theorem 5.2 is not surprising in the case of Artin groups of finite type.

In the rest of this paper we are going to construct precise matchings for Artin groups of type A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$. For each of these cases we are then going to: describe the critical simplices with respect to the constructed matching; find all alternating paths and incidence numbers between critical simplices; determine the ranks $\operatorname{rk} \delta^{\varphi}_*$ and use Theorem 5.1 to compute the homology $H_*(X_W; R)$. The final results are stated in Theorem 3.1 (for A_n), Theorem 5.13 (for B_n), Theorem 5.16 (for $\widetilde{A}_n$) and Theorem 5.20 (for $\widetilde{C}_n$).

Let us first introduce some notation. We will have $S = \{1, 2, \dots, n\}$ for the finite cases (A_n and B_n), and $S = \{0, 1, 2, \dots, n\}$ for the affine cases ($\widetilde{A}_n$ and $\widetilde{C}_n$). A simplex $\sigma \subseteq S$ will be also represented as a string of bits $\epsilon_i \in \{0, 1\}$ (for $i \in S$), where $\epsilon_i = 1$

if $i \in S$ and $\epsilon_i = 0$ if $i \notin S$. For example, if $S = \{1, 2, 3, 4\}$, the string representation of $\sigma = \{1, 2, 4\}$ is 1101. Also, for $\sigma \subseteq S$ and $v \in S$, let

$$\sigma \veebar v = \begin{cases} \sigma \cup \{v\} & \text{if } v \notin \sigma, \\ \sigma \setminus \{v\} & \text{if } v \in \sigma; \end{cases}$$

it can be regarded as the bitwise xor between the string representation of σ and the string with $\epsilon_v = 1$ and $\epsilon_i = 0$ for $i \neq v$.

5.1 Case A_n

Many properties of the homology $H_*(X_W; R)$ in the case A_n have been thoroughly discussed in Section 3. Using precise matchings we are going to obtain a new proof of the formula for the homology of braid groups (Theorem 3.1).

Let $S = \{1, 2, \ldots, n\}$ be the set of vertices of the Coxeter graph of type A_n, as in Figure 1 on page 3954. In this case K is the full simplicial complex on S. Fix now an integer $d \geq 2$ and set $\varphi = \varphi_d$. The φ–weight of a simplex $\sigma \in K$ has been computed in Section 3 and is as follows:

$$v_\varphi(\sigma) = \sum_{i=1}^{m} \omega_\varphi(A_{n_i}),$$

where n_i is the size of the i^{th} (linear) connected component of the subgraph induced by σ and $\omega_\varphi(A_k)$ stands for the φ–weight of a connected component of type A_k, given by

$$\omega_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor.$$

Fix also an integer f with $0 \leq f \leq d-1$. Let $K^A_{n,f} \subseteq K$ be

$$K^A_{n,f} = \{\sigma \in K \mid 1, 2, \ldots, f \in \sigma\}.$$

Notice that $K^A_{n,0} = K$, and $K^A_{n,f}$ is not a subcomplex of K for $f \geq 1$ (but it is still a subposet of K, so it makes sense to define a matching on it). We are going to construct a φ–precise matching on $K^A_{n,f}$. In particular, for $f = 0$, we will get a φ–precise matching for K. The precise matchings on $K^A_{n,f}$ for $f \geq 1$ will become useful when treating the cases B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$.

For a fixed d, the matching will be constructed recursively in n and f for $n \geq 0$ and $0 \leq f \leq d-1$. We will write $K_{n,f}$ for $K^A_{n,f}$ throughout Section 5.1. The matching is as follows:

(a) If $\{1,\dots,d-1\} \subseteq \sigma$ then match σ with $\sigma \veebar d$ (unless $n = d-1$, in which case σ is critical). Here σ is matched with a simplex, which also occurs in case (a). Notice that for $f = d-1$ case (a) always applies, thus in the subsequent cases we can assume $f \le d-2$.

(b) Otherwise, if $n = f$ then σ is critical.

(c) Otherwise, if $f+1 \in \sigma$ then match σ with $\sigma \setminus \{f+1\}$. Notice that $\sigma \setminus \{f+1\}$ occurs in case (d).

(d) Otherwise, if $\{f+2,\dots,d-1\} \not\subseteq \sigma$ then match σ with $\sigma \cup \{f+1\}$. Notice that $\sigma \cup \{f+1\}$ occurs in case (c).

(e) We are left with the simplices σ such that $\{1,\dots,f, f+2,\dots,d-1\} \subseteq \sigma$ and $f+1 \notin \sigma$. If we ignore the vertices $1,\dots,f+1$ we are left with the simplices on the vertex set $\{f+2,\dots,n\}$ which contain $f+2,\dots,d-1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1,\dots,n-f-1\}$ which contain $1,\dots,d-2-f$. Then construct the matching recursively as in $K_{n-f-1,\,d-2-f}$.

Example 5.4 For $n=5$, $d=3$ and $f=1$, $K_{n,f}$ contains $2^4=16$ simplices of which 14 are matched and 2 are critical. For instance, consider the simplex $\sigma = \{1,4,5\}$. Case (e) applies because $1 \in \sigma$ and $2 \notin \sigma$; the recursion requires us to consider the new simplex $\sigma' = \{2,3\} \in K_{3,0}$. Again case (e) applies because $1 \notin \sigma'$ and $2 \in \sigma'$; the recursion requires us to consider the new simplex $\sigma'' = \{1,2\} \in K_{2,1}$. Finally, case (a) applies because $\{1,2\} \subseteq \sigma''$, and σ'' is critical because $\sigma'' = \{1,2\}$. Therefore, $\sigma = \{1,4,5\} \in K_{5,1}$ is also critical. See Table 1 for an explicit description of the matching for $K_{5,1}$, $d=3$.

Remark 5.5 A peculiarity of this matching is that, if $\sigma' \to \tau'$ is in the matching, then $\sigma' = \tau' \cup \{v\}$ with $v \equiv f+1$ or $v \equiv 0 \pmod d$. This can be easily checked by induction.

Lemma 5.6 *The matching described above is an acyclic weighted matching on* $K_{n,f}$.

Proof We proceed in two parts:

Part 1 (the matching is acyclic) The proof is by induction on n, the case $n=0$ being trivial. For $n \le f$, in $K_{n,f}$ there are either 1 or 0 simplices. Assume from now on $n > f$. Let $P = \{p_{\mathrm{a}}, p_{\mathrm{e}}, p_{\mathrm{c,d}}\}$ be a three-element totally ordered poset, with the order

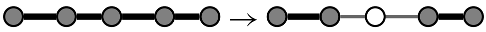
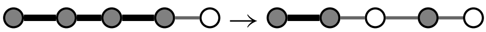
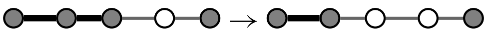
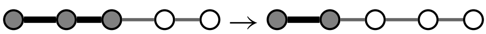
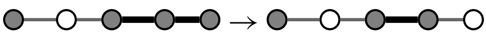
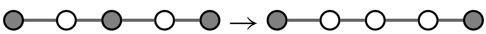
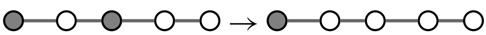
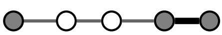
Simplices		$v_\varphi(\sigma)$	Step
		2	(a)
		1	(a)
		1	(a)
		1	(a)
		1	(e) $\rightsquigarrow$ (a)
		0	(e) $\rightsquigarrow$ (c)/(d)
		0	(e) $\rightsquigarrow$ (c)/(d)
	(critical)	1	(e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (a)
	(critical)	0	(e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (b)

Table 1: Matching in the case A_n with $n = 5,\ d = 3,\ f = 1$. The last columns indicates the case where simplices occur. When case (e) is reached, the arrow "$\rightsquigarrow$" indicates how the recursion continues (after $K_{5,1}$, the recursion involves $K_{3,0}$, $K_{2,1}$ and $K_{0,0}$).

given by $p_{\mathrm{a}} > p_{\mathrm{e}} > p_{\mathrm{c,d}}$. Consider the map $\eta\colon K_{n,f} \to P$ which sends $\sigma \in K_{n,f}$ to the p_{x} such that σ occurs in case (x) (here cases (c) and (d) are united). For instance, $\eta(\sigma) = p_{\mathrm{a}}$ if and only if $\{1, \dots, d-1\} \subseteq \sigma$. Notice that the map η is compatible with the matching, ie two matched simplices lie in the same fiber of η.

Let us prove that η is a poset map. Given two simplices $\sigma \geq \tau$, we want to prove that $\eta(\sigma) \geq \eta(\tau)$. If $\eta(\tau) = p_{\mathrm{a}}$ then $\{1, \dots, d-1\} \subseteq \tau \subseteq \sigma$, so $\eta(\sigma) = p_{\mathrm{a}}$ also. If $\eta(\tau) = p_{\mathrm{e}}$ then $\{1, \dots, f, f+2, \dots, d-1\} \subseteq \tau \subseteq \sigma$, thus $\eta(\sigma) \in \{p_{\mathrm{a}}, p_{\mathrm{e}}\}$. Finally, if $\eta(\tau) = p_{\mathrm{c,d}}$, there is nothing to prove.

Since η is a poset map, our matching is acyclic if and only if it is acyclic on each fiber of η. The restriction of the matching to $\eta^{-1}(p_{\mathrm{a}})$ consists of edges $\sigma \to \tau$ with $\sigma = \tau \cup \{d\}$, so there is no alternating path of length ≥ 4 and the matching is acyclic. The restriction to $\eta^{-1}(p_{\mathrm{c,d}})$ is acyclic for the same reason (it is always the vertex $f+1$ which is added or removed). The restriction to $\eta^{-1}(p_{\mathrm{e}})$ is acyclic by induction.

Part 2 (the matching is weighted) The proof is by induction on n, the case $n=0$ being trivial. In case (a), $v_\varphi(\sigma) = v_\varphi(\sigma \veebar d)$ by what we have already said in Section 3 (see the proof of Theorem 3.6). In cases (c) and (d), both the simplices σ and $\sigma \veebar (f+1)$ do not contain $\{1,\dots,d-1\}$. Thus, the (possibly empty) connected component of 1 has size $\le d-2$ and does not contribute to the weight. Therefore, $v_\varphi(\sigma) = v_\varphi(\sigma \veebar (f+1))$. Finally, suppose that σ and τ are simplices that occur in case (e). Let $\hat\sigma, \hat\tau \in K_{n-f-1,d-2-f}$ be the simplices obtained ignoring the vertices $1,\dots,f+1$, as described above. Since $f \le d-2$ we have that

$$v_\varphi(\sigma) = v_\varphi(\tau) \quad \text{if and only if} \quad v_\varphi(\hat\sigma) = v_\varphi(\hat\tau),$$

so we are done by induction on n. □

The critical simplices of the matching on $K_{n,f}$ are quite simple to describe. If $f \le d-2$, $n > f$ and $n \equiv f$ or $-1 \pmod d$, there are two critical simplices, one of weight 1 and one of weight 0. If $n=f$ there is one critical simplex (in fact there is only one simplex in $K_{n,f}$). In all the other cases the matching has no critical simplices. We are going to prove this in the following theorem. See Table 2, and Figures 4 and 5, for an explicit description of the critical simplices. Notice in particular that, when there are two critical simplices, one is a face of the other in $K_{n,f}$.

Case		Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$
$n > f$ $f \le d-2$	$n = kd+f$ (Figure 4)	$(1^f 0 1^{d-2-f} 0)^{k-1} 1^f 0 1^{d-1}$	$n-2k+1$	1
		$(1^f 0 1^{d-2-f} 0)^{k} 1^f$	$n-2k$	0
	$n = kd-1$ (Figure 5)	$(1^f 0 1^{d-2-f} 0)^{k-1} 1^{d-1}$	$n-2k+2$	1
		$(1^f 0 1^{d-2-f} 0)^{k-1} 1^f 0 1^{d-2-f}$	$n-2k+1$	0
$n=f$		1^f	n	1 or 0

Table 2: Description of the critical simplices for A_n

Theorem 5.7 (critical simplices in case A_n) *The critical simplices for the matching on $K_{n,f}$ are those listed in Table 2. In particular, the matching is always precise, and has no critical simplices if $n \not\equiv f, -1 \pmod d$ or $n > f = d-1$. In addition, when there are two critical simplices there is only one alternating path between them (the trivial one).*

Proof We proceed in two parts:

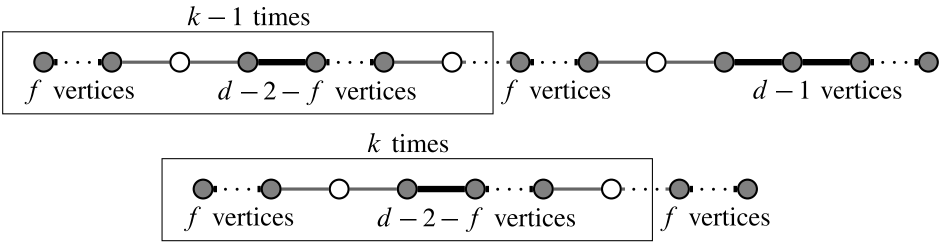


Figure 4: Critical simplices for $f \leq d-2$ and $n = kd + f$ $(k \geq 1)$

Part 1 (critical simplices) The proof is by induction on n, the case $n = 0$ being trivial (there is one critical simplex for $f = 0$, namely the empty simplex $\varnothing$, and no critical simplices for $f > 0$). Let $\sigma \in K_{n,f}$ be a simplex. Let us consider each of the five cases that can occur in the construction of the matching:

(a) We have $\{1, \dots, d-1\} \subseteq \sigma$. Then σ is critical if and only if $n = d-1$. If $n = d-1$ then σ is indeed listed in Table 2, as the first of the two critical simplices in the case $n = kd - 1$ (here $k = 1$). Conversely, the only simplex of Table 2 which contains $\{1, \dots, d-1\}$ is the first one of the case $n = kd - 1$ when $k = 1$.

(b) We have $n = f$ and $\sigma = \{1, \dots, f\}$. Then σ is critical, and it is indeed listed as the second of the two critical simplices in the case $n = kd + f$ (here $k = 0$).

(c) We have $\{1, \dots, f+1\} \subseteq \sigma$. Then σ is not critical. The only listed simplex which contains $\{1, \dots, f+1\}$ is the first one of the case $n = kd - 1$ for $k = 1$, but it is equal to $\{1, \dots, d-1\}$ and so it must be different from σ.

(d) We have $f+1 \notin \sigma$ and $\{f+2, \dots, d-1\} \not\subseteq \sigma$. Then σ is not critical. It is easy to check that all the listed simplices τ with $n > f$ and $f+1 \notin \tau$ satisfy $\{f+2, \dots, d-1\} \subseteq \tau$, so none of them can be equal to σ.

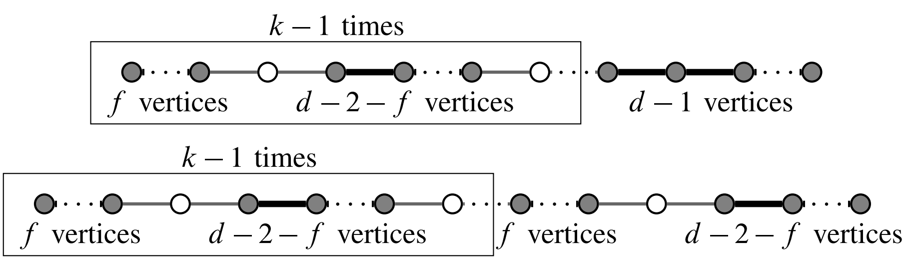


Figure 5: Critical simplices for $f \leq d-2$ and $n = kd - 1$ $(k \geq 1)$

(e) We have $\{1,\dots,f,f+2,\dots,d-1\}\subseteq\sigma$ and $f+1\notin\sigma$. Then σ is critical if and only if the simplex $\hat{\sigma}\in K_{n-f-1,d-2-f}$ is critical, where $\hat{\sigma}$ is constructed from σ by ignoring the first $f+1$ vertices. By induction, $\hat{\sigma}$ is critical if and only if it is listed in Table 2. By taking the simplices of Table 2 for $K_{n-f-1,d-2-f}$ and adding $1^f 0$ at the beginning, one obtains exactly the simplices of Table 2 for $K_{n,f}$ (but the two cases are exchanged). This concludes the induction argument.

For fixed n and f, the quantity $|\sigma|-v_\varphi(\sigma)$ is constant among the critical simplices σ. More precisely, it is equal to $n-2k$ for $n=kd+f$ (except for $f=d-1$) and to $n-2k+1$ for $n=kd-1$. Therefore, the matching is precise.

Part 2 (alternating paths) We want to prove that the only alternating path between critical simplices is the trivial one. Consider the case $n=kd+f$ for $k\geq 1$ (the case $n=kd-1$ is analogous). Let σ and τ be the two critical simplices (the ones of Figure 4). Notice that $\sigma=\tau\cup\{n-f\}$. To establish a contradiction suppose we have a nontrivial alternating path

$$\sigma\rhd\tau_1\lhd\sigma_1\rhd\tau_2\lhd\sigma_2\rhd\cdots\rhd\tau_m\lhd\sigma_m\rhd\tau,\qquad m\geq 1.$$

Since $\tau_1\neq\tau$, we must have $\sigma=\tau_1\cup\{v_1\}$ with $v_1\neq n-f$. Notice that $n-f$ is the only vertex $v\in\sigma$ which satisfies $v\equiv f+1$ or $v\equiv 0 \pmod d$. Therefore, by Remark 5.5, the vertex v_1 will never be recovered throughout the alternating path. This is a contradiction since $v_1\in\tau$. □

As a consequence we can compute the homology $H_*(\boldsymbol{G_W};R)$ for $\boldsymbol{G_W}$ of type A_n. This gives a proof of Theorem 3.1.

Proof of Theorem 3.1 (homology in case A_n) We simply need to apply the formula given by Theorem 5.1 using our precise matching on $K=K_{n,0}$. Since $f=0$, there are critical simplices for $n=kd$ or $n=kd-1$. The boundary δ^φ_{m+1} is nontrivial only for

$$\begin{cases} m=n-2k=k(d-2) & \text{if } n=kd,\\ m=n-2k+1=k(d-2) & \text{if } n=kd-1.\end{cases}$$

In both cases for $m=k(d-2)$ we have $\operatorname{rk}\delta^\varphi_{m+1}=1$, and all the other boundaries are trivial. Theorem 3.1 follows. □

Remark 5.8 A relation with independence complexes can be found for $K_{n,f}$ also when $1\leq f\leq d-2$. Indeed, choosing a suitable weighted matching (similar to the one of Theorem 3.6), the set of critical simplices of positive weight in $K_{n,f}$ is

$$\{\sigma \sqcup A_{d-1} \mid \sigma \in \mathrm{Ind}_{d-2}(A_{n-d}) \cap K_{n-d,f}\}.$$

5.2 Case B_n

Consider now a Coxeter graph of type B_n, as in Figure 6. In this case K is again the full simplex on vertices $\{1, 2, \dots, n\}$. The Poincaré polynomial of a Coxeter group of type B_k is given by

$$W_{B_k}(q) = [2k]_q!! = \prod_{i=1}^{n} [2i]_q = \prod_{\varphi} \varphi^{\omega_\varphi(B_k)},$$

where, for a given cyclotomic polynomial $\varphi = \varphi_d$, we have

$$\omega_\varphi(B_k) = \begin{cases} \lfloor k/d \rfloor & \text{if } d \text{ is odd,} \\ \lfloor (k/\frac{d}{2}) \rfloor & \text{if } d \text{ is even;} \end{cases}$$

see for example [5]. Then we can compute the φ–weight of any simplex $\sigma \in K$ by looking at the subgraph $\Gamma(\sigma)$ induced by σ. Let $\Gamma_1(\sigma), \dots, \Gamma_m(\sigma)$ be the connected components of $\Gamma(\sigma)$, with cardinality $n_1, \dots, n_m$, respectively, where $\Gamma_1(\sigma)$ is the (possibly empty) component that contains the vertex $1 \in S$. Then

$$v_\varphi(\sigma) = \omega_\varphi(B_{n_1}) + \sum_{i=2}^{m} \omega_\varphi(A_{n_i}).$$

The situation is quite different depending on the parity of d. If d is odd, the φ–weight of a B_{k+1} component is equal to the φ–weight of an A_k component:

$$\omega_\varphi(B_{k+1}) = \left\lfloor \frac{k+1}{d} \right\rfloor = \omega_\varphi(A_k) \quad (d \text{ odd}).$$

For this reason it is possible to construct a very simple matching on K: match any simplex $\sigma \in K$ with $\sigma \veebar 1$.

Figure 6: A Coxeter graph of type B_n

Lemma 5.9 *The matching constructed above for d odd is an acyclic weighted matching on K, with no critical simplices.*

Proof Clearly the matching is acyclic and there are no critical simplices. Let us prove that the matching is weighted. Let $\sigma \in K$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\sigma \veebar 1)$ is the leftmost connected component, which in one case is of type B_{k+1} and in the other case is of type A_k. Therefore, $v_\varphi(\sigma) = v_\varphi(\sigma \veebar 1)$. □

Suppose from now on that d is even. The simplicial complex K is partitioned as

$$K = \bigsqcup_{q \geq 0} K_q, \quad \text{where } K_q = \left\{ \sigma \in K \;\middle|\; \left\lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \right\rfloor = q \right\}.$$

Here $\Gamma_1(\sigma)$ is the (possibly empty) connected component of $\Gamma(\sigma)$ which contains the vertex 1. Notice that each K_q is a subposet of K, but not a subcomplex in general. For a given simplex $\sigma \in K_q$, let $|\Gamma_1(\sigma)| = q\frac{d}{2} + r$ with $0 \leq r < \frac{d}{2}$. The matching on K_q is as follows:

(a) If $r \geq 1$ $\left(\text{ie } q\frac{d}{2} + 1 \in \sigma\right)$ then match σ with $\sigma \setminus \left\{q\frac{d}{2} + 1\right\}$.

(b) If $r = 0$ $\left(\text{ie } q\frac{d}{2} + 1 \notin \sigma\right)$ and $\left\{q\frac{d}{2} + 2, \dots, (q+1)\frac{d}{2}\right\} \not\subseteq \sigma$, then match σ with $\sigma \cup \left\{q\frac{d}{2} + 1\right\}$ (unless $n = q\frac{d}{2}$, in which case σ is critical).

(c) We are left with the simplices σ for which neither (a) nor (b) apply, ie with $r = 0$ and $\left\{q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\right\} \subseteq \sigma$. Ignore the first $q\frac{d}{2}+1$ vertices and relabel the remaining ones from 1 to $n - q\frac{d}{2} - 1$, so that we are left exactly with the simplices of $K^A_{n-qd/2-1,\, d/2-1}$. Then construct the matching on $K^A_{n-qd/2-1,\, d/2-1}$ as in Section 5.1.

Putting together the matchings on each K_q, we obtain a matching on the full simplicial complex K.

Example 5.10 For $n = 4$ and $d = 4$, the simplicial complex K contains $2^4 = 16$ simplices of which 12 are matched and 4 are critical. For instance, consider $\sigma = \{1, 2\} \in K$. Then $q = 1$ and $r = 0$. Since $4 \notin \sigma$, case (b) occurs. Therefore, σ is matched with $\sigma \cup \{3\} = \{1, 2, 3\}$. See Table 3 for an explicit description of the matching in this case.

Lemma 5.11 *The matching constructed above for d even is an acyclic weighted matching on K.*

Proof We proceed in two parts:

Part 1 (the matching is acyclic) The map $K \to (\mathbb{N}, \leq)$ which sends σ to $q = \left\lfloor |\Gamma_1(\sigma)| / \frac{d}{2} \right\rfloor$ is a poset map compatible with the matching, and its fibers are exactly

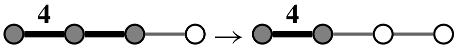
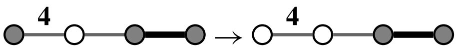
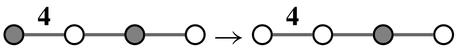
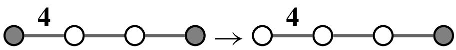
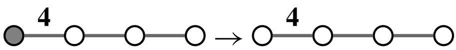
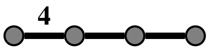
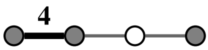
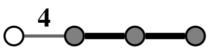
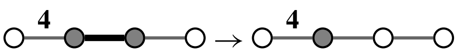
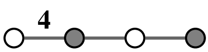
Simplices	$v_\varphi(\sigma)$	q	Step
	1	1	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
 (critical)	2	2	(b)
 (critical)	1	1	(c)
 (critical)	1	0	(c)
	0	0	(c)
 (critical)	0	0	(c)

Table 3: Matching in the case B_n with $n = 4$ and $d = 4$

the subsets K_q for $q \in \mathbb{N}$. Therefore, we only need to prove that the matching on each fiber K_q is acyclic.

Let $P = \{p_{\mathrm{c}}, p_{\mathrm{a,b}}\}$ be a two-element totally ordered poset with $p_{\mathrm{c}} > p_{\mathrm{a,b}}$. For a fixed $q \in \mathbb{N}$, consider the map $\eta\colon K_q \to P$ which sends σ to the p_{x} such that σ occurs in case (x) (here cases (a) and (b) are united). Clearly η is compatible with the matching. We want to prove that it is a poset map, and for this we only need to show that if $\eta(\tau) = p_{\mathrm{c}}$ and $\tau \leq \sigma$ then $\eta(\sigma) = p_{\mathrm{c}}$ also. We have that $\{1, \dots, q\frac{d}{2}, q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\} \subseteq \tau \subseteq \sigma$. The simplex σ cannot contain the vertex $q\frac{d}{2}+1$, because otherwise we would have $\sigma \in K_{q+1}$. Therefore, $\eta(\sigma) = p_{\mathrm{c}}$.

On the fiber $\eta^{-1}(p_{\mathrm{a,b}})$ the matching is acyclic because the same vertex $q\frac{d}{2}+1$ is always added or removed. On the fiber $\eta^{-1}(p_{\mathrm{c}})$ the matching is acyclic by Lemma 5.6. Therefore, the entire matching on K_q is acyclic.

Part 2 (the matching is weighted) Let $\sigma \in K_q$ be a simplex which occurs either in case (a) or case (b). We want to show that $v_\varphi(\sigma) = v_\varphi\big(\sigma \veebar \big(q\frac{d}{2}+1\big)\big)$. Suppose

Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
$\sigma_q = 1^{qd/2}(01^{d/2-1})^{k-q-2}01^{d-1}$	$n-k+q+1$	$q+1$	$0 \le q \le k-2$
$\sigma'_q = 1^{qd/2}(01^{d/2-1})^{k-q}$	$n-k+q$	q	$0 \le q \le k$

Table 4: Description of the critical simplices for B_n, where d is even and $n = k\frac{d}{2}$

without loss of generality that σ occurs in case (a), ie $r \ge 1$ and $q\frac{d}{2}+1 \in \sigma$. Let $\tau = \sigma \setminus \{q\frac{d}{2}+1\}$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\tau)$ is that $\Gamma(\sigma)$ has a $B_{qd/2+r}$ component, whereas $\Gamma(\tau)$ has a $B_{qd/2}$ component and an A_{r-1} component instead. Since $1 \le r < \frac{d}{2}$, we have that

$$\omega_\varphi(B_{q\frac{d}{2}+r}) = \omega_\varphi(B_{q\frac{d}{2}}) = q \quad \text{and} \quad \omega_\varphi(A_{r-1}) = 0$$

(the second equality holds because $r-1 \le \frac{d}{2}-1 \le d-2$). Therefore, $v_\varphi(\sigma) = v_\varphi(\tau)$.

For simplices occurring in case (c), the matching only involves changes in the connected components not containing the first vertex (ie connected components of type A_k). Therefore, Lemma 5.6 applies. □

Now we are going to describe the critical simplices. The matching has no critical simplices when n is not a multiple of $\frac{d}{2}$. On the other hand, if $n = k\frac{d}{2}$, we have two families of critical simplices: σ_q (for $0 \le q \le k-2$) and σ'_q (for $0 \le q \le k$). See Table 4 and Figure 7 for the definition of these simplices. For instance, in Example 5.10, the critical simplices are $\sigma_0 = \{2,3,4\}$, $\sigma'_2 = \{1,2,3,4\}$, $\sigma'_1 = \{1,2,4\}$ and $\sigma'_0 = \{1,3\}$. See also Table 5, where the critical simplices are listed by dimension.

Theorem 5.12 (critical simplices in case B_n) *The critical simplices for the matching on K are those defined in Table 4. In particular, the matching is always precise, and*

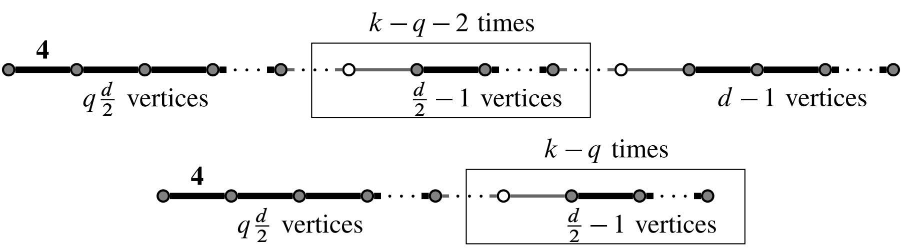


Figure 7: Critical simplices for B_n, where d is even and $n = k\frac{d}{2}$. Above is the simplex σ_q $(0 \le q \le k-2)$ and below is the simplex σ'_q $(0 \le q \le k)$.

$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	Simplices
n	k	σ'_k
$n-1$	$k-1$	$\sigma_{k-2}, \sigma'_{k-1}$
$n-2$	$k-2$	$\sigma_{k-3}, \sigma'_{k-2}$
$\vdots$	$\vdots$	$\vdots$
$n-k+1$	1	σ_0, σ'_1
$n-k$	0	σ'_0

Table 5: Critical simplices for B_n by dimension, where d is even and $n = k\frac{d}{2}$

has no critical simplices if d is odd or if d is even and $n \not\equiv 0 \pmod{\frac{d}{2}}$. In addition, if d is even and $n \equiv 0 \pmod{\frac{d}{2}}$, the incidence numbers between the critical simplices in the Morse complex are as follows:

$$\begin{aligned}
[\sigma_q : \sigma_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)d/2}, \\
[\sigma_q : \sigma'_q]^{\mathcal{M}} &= (-1)^{(k-1)(d/2-1)+q}, \\
[\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k(d/2-1)+q}, \\
[\sigma'_q : \sigma'_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)d/2}.
\end{aligned}$$

Proof We proceed in two parts:

Part 1 (critical simplices) For d odd there is nothing to prove. Suppose from now on that d is even. For a given simplex $\sigma \in K$, let us consider each of the three cases that can occur in the construction of the matching.

(a) We have $q\frac{d}{2}+1 \in \sigma$, and σ is not critical. Indeed, none of the simplices of Table 4 contains the vertex $q\frac{d}{2}+1$.

(b) We have $q\frac{d}{2}+1 \notin \sigma$ and $\{q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\} \not\subseteq \sigma$. In this case σ is critical if and only if $n = q\frac{d}{2}$. Indeed, the only simplex of this type in Table 4 is σ'_k (which occurs for $q = k$, ie $n = q\frac{d}{2}$).

(c) In the remaining case, we end up with the matching on $K^A_{n-qd/2-1,\, d/2-1}$. By Theorem 5.7, this matching admits critical simplices if and only if

$$n - q\frac{d}{2} - 1 \equiv \frac{d}{2} - 1 \pmod{d} \quad \text{or} \quad n - q\frac{d}{2} - 1 \equiv -1 \pmod{d},$$

ie for $n \equiv 0 \pmod{\frac{d}{2}}$. Notice that if $f = \frac{d}{2} - 1$ then $d - 2 - f = \frac{d}{2} - 1$ also. Therefore, again by Theorem 5.7, for $n \equiv 0 \pmod{\frac{d}{2}}$ the critical simplices are exactly the ones listed in Table 4.

For a fixed $n = k\frac{d}{2}$, the quantity $|\sigma| - v_\varphi(\sigma)$ is constant among the critical simplices and is equal to $n - k$. Thus, the matching is precise.

Part 2 (incidence numbers) We are going to show that there is exactly one alternating path for each of the four pairs. Notice that if $\sigma \to \tau$ is in the matching of K_q, then $\sigma = \tau \cup \{v\}$ with $v \equiv 1 \pmod{\frac{d}{2}}$ and $v \geq q\frac{d}{2} + 1$. In particular, if at a certain point of an alternating path one removes a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, then that vertex will never be added again. But all the critical simplices contain every vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, so any alternating path between critical simplices cannot ever drop a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$.

Let us now consider each of the four pairs.

$(\boldsymbol{\sigma_q, \sigma_{q-1}})$ We have that $\sigma_q = \sigma_{q-1} \cup \{v\}$ where $v = (q-1)\frac{d}{2} + 1$, so there is the trivial alternating path $\sigma_q \to \sigma_{q-1}$ which contributes to the incidence number $[\sigma_q : \sigma_{q-1}]^{\mathcal{M}}$ by

$$[\sigma_q : \sigma_{q-1}] = (-1)^{|\{w \in \sigma_q \mid w < v\}|} = (-1)^{(q-1)d/2}.$$

Suppose by contradiction that there exists some other (nontrivial) alternating path

$$\sigma_q \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-1}, \quad m \geq 1.$$

Let $\sigma_q = \tau_1 \cup \{u\}$. By the previous considerations, we must have $u \equiv 1 \pmod{\frac{d}{2}}$. If $u = (k-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma'_q$, but this is not possible since alternating paths stop at critical simplices. Similarly, if $u = (q-1)\frac{d}{2} + 1$ then we would stop at $\tau_1 = \sigma_{q-1}$. Therefore, we must have $u \leq (q-2)\frac{d}{2} + 1$. But then $\tau_1 \in K_{q'}$ with $q' \leq q - 2$, and by induction all the subsequent simplices in the alternating path must lie in

$$\bigcup_{q'' \leq q-2} K_{q''}.$$

In particular, $\sigma_{q-1} \in K_{q''}$ for some $q'' \leq q - 2$, but this is a contradiction since $\sigma_{q-1} \in K_{q-1}$.

$(\boldsymbol{\sigma_q, \sigma'_q})$ This case is similar to the previous one, except for the fact that $\sigma_q = \sigma'_q \cup \{v\}$ for $v = (k-1)\frac{d}{2} + 1$. So the only alternating path is the trivial one, which contributes to the incidence number by

$$[\sigma_q : \sigma'_q] = (-1)^{|\{w \in \sigma_q \mid w < v\}|} = (-1)^{qd/2+(k-q-1)(d/2-1)} = (-1)^{(k-1)(d/2-1)+q}.$$

$(\sigma'_q, \sigma'_{q-1})$ This case is also similar to the previous ones. Here we have $\sigma'_q = \sigma'_{q-1} \cup \{v\}$ with $v = (q-1)\frac{d}{2} + 1$, so the contribution to the incidence number due to the trivial alternating path is

$$[\sigma'_q : \sigma'_{q-1}] = (-1)^{|\{w \in \sigma'_q \mid w < v\}|} = (-1)^{(q-1)d/2}.$$

$(\sigma'_q, \sigma_{q-2})$ This case is more complicated because the only alternating path is nontrivial. Suppose we have an alternating path

$$\sigma'_q \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-2}, \quad m \geq 1$$

(m must be at least 1 because σ_{q-2} is not a face of σ'_q in K). Let $\sigma'_q = \tau_1 \cup \{v\}$, with $v \equiv 1 \pmod{\frac{d}{2}}$. If $v = (q-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma'_{q-1}$ and we must stop because σ'_{q-1} is already critical. If $v \leq (q-3)\frac{d}{2} + 1$ we fall into some $K_{q'}$ with $q' \leq q-3$ and it is not possible to reach $\sigma_{q-2} \in K_{q-2}$. Therefore, necessarily $v = (q-2)\frac{d}{2} + 1$. Then τ_1 is matched with

$$\rho_1 = \tau_1 \cup \left\{q\frac{d}{2} + 1\right\}.$$

Then again, $\tau_2 = \rho_1 \setminus \{v_2\}$ and the only possibility for v_2 (in order to have $\tau_2 \neq \tau_1$ and not to fall into some $K_{q'}$ with $q' \leq q-3$) is $v_2 = (q-1)\frac{d}{2} + 1$. Proceeding by induction we obtain that

$$\tau_i = \rho_{i-1} \setminus \left\{(q-3+i)\frac{d}{2} + 1\right\},$$
$$\rho_i = \tau_i \cup \left\{(q-1+i)\frac{d}{2} + 1\right\}.$$

The path stops at $\tau_{k-q+1} = \sigma_{q-2}$, and its length is $m = k-q$. See Table 6 for an example. The contribution of this path to the incidence number is

$$(-1)^{k-q}[\sigma'_q : \tau_1][\rho_1 : \tau_1] \cdots [\rho_{k-q} : \tau_{k-q}][\rho_{k-q} : \sigma_{q-2}],$$

where

$$[\rho_{i-1} : \tau_i] = (-1)^{(q-2)d/2+(i-1)(d/2-1)},$$
$$[\rho_i : \tau_i] = (-1)^{(q-2)d/2+(i-1)(d/2-1)+d-1}$$
$$= (-1)^{(q-2)d/2+(i-1)(d/2-1)+1}$$

(these formulas also hold for $\rho_0 = \sigma'_q$ and $\tau_{k-q+1} = \sigma_{q-2}$). Then

$$\begin{aligned}[\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k-q} \left(\prod_{i=1}^{k-q} [\rho_{i-1} : \tau_i][\rho_i : \tau_i] \right) [\rho_{k-q} : \sigma_{q-2}] \\ &= (-1)^{k-q} (-1)^{k-q} (-1)^{(q-2)d/2 + (k-q)(d/2-1)} \\ &= (-1)^{k(d/2-1)+q}. \end{aligned}$$

□

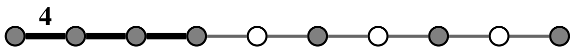
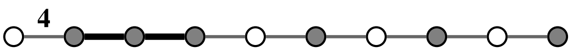
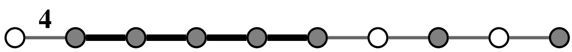
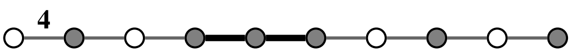
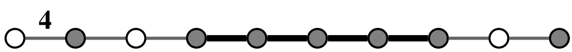
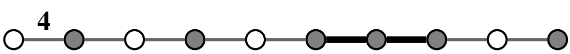
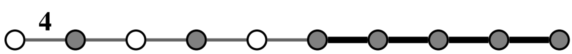
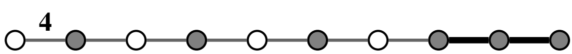
Simplices		$v_\varphi(\sigma)$
$\sigma'_2 =$		2
$\tau_1 =$		1
$\rho_1 =$		1
$\tau_2 =$		1
$\rho_2 =$		1
$\tau_3 =$		1
$\rho_3 =$		1
$\sigma_0 =$		1

Table 6: The only alternating path from σ'_q to σ_{q-2} for $n = 10$, $d = 4$, $k = 5$ and $q = 2$

Having a complete description of a precise matching on K, we can now compute the homology $H_*(\boldsymbol{G_W}; R)$ for $\boldsymbol{G_W}$ of type B_n. We recover the result of [19].

Theorem 5.13 (homology in case B_n [19]) *For an Artin group $\boldsymbol{G_W}$ of type B_n, we have*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} R/(\varphi_d) & \text{if } d \text{ is even, } n = k\frac{d}{2} \text{ and } n-k \le m \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof For a fixed $\varphi = \varphi_d$, we need to compute the boundary δ^{φ}_{m+1}. Assume that $n = k\frac{d}{2}$, otherwise there are no critical simplices. In top dimension ($m+1 = n$) we have $\operatorname{rk}\delta_n^{\varphi}=1$, because $[\sigma'_k:\sigma'_{k-1}]\neq 0$. For $m\leq n-k-1$ the boundary δ_{m+1}^{φ} vanishes, because there are no critical simplices in dimension $\leq n-k-1$. For $m=n-k$ we have $\operatorname{rk}\delta_{m+1}^{\varphi}=1$, because $[\sigma'_1:\sigma'_0]\neq 0$. Finally, for $n-k+1\leq m\leq n-2$, we have (set $q=m-n+k$)

$$\operatorname{rk}\delta_{m+1}^{\varphi}=\operatorname{rk}\begin{pmatrix}[\sigma_q:\sigma_{q-1}][\sigma'_{q+1}:\sigma_{q-1}]\\ [\sigma_q:\sigma'_q][\sigma'_{q+1}:\sigma'_q]\end{pmatrix}=\operatorname{rk}\begin{pmatrix}(-1)^{(q-1)d/2}(-1)^{k(d/2-1)+q+1}\\ (-1)^{(k-1)(d/2-1)+q}(-1)^{qd/2}\end{pmatrix}=1,$$

because

$$\det\begin{pmatrix}(-1)^{(q-1)d/2} & (-1)^{k(d/2-1)+q+1}\\ (-1)^{(k-1)(d/2-1)+q} & (-1)^{qd/2}\end{pmatrix}=(-1)^{d/2}-(-1)^{d/2}=0.$$

To summarize, we have $\operatorname{rk}\delta_{m+1}^{\varphi}=1$ for $n-k\leq m\leq n-1$, and $\operatorname{rk}\delta_{m+1}^{\varphi}=0$ otherwise. We conclude applying Theorem 5.1. □

5.3 Case $\widetilde{A}_n$

Consider now the case of affine Artin groups of type $\widetilde{A}_n$ (see Figure 8 for a picture of the corresponding Coxeter graph). Recall that, for affine Artin groups, the simplicial complex K consists of all the simplices $\sigma\subseteq S=\{0,\dots,n\}$ except for the full simplex $\sigma=\{0,\dots,n\}$. For any $\sigma\in K$, the induced subgraph $\Gamma(\sigma)$ consists only of connected components of type A_k. The weight $v_{\varphi}(\sigma)$ is then computed as in Section 5.1.

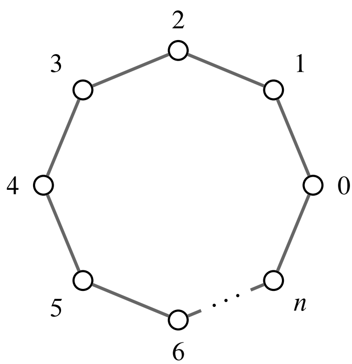


Figure 8: A Coxeter graph of type $\widetilde{A}_n$

If σ is a simplex of K then σ misses at least one vertex of $\{0,1,\dots,n\}$. Define h to be the first vertex missing from σ reading counterclockwise from 0. Then σ belongs to a unique

$$K_h=\{\sigma\in K\mid h\notin\sigma \text{ and } \{0,\dots,h-1\}\subseteq\sigma\}.$$

In other words, the subsets K_h form a partition of K,

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Let $h = qd + r$ with $0 \le r \le d-1$, and let $m = n - qd$. Consider the following clockwise relabeling of the vertices: $r-1$ becomes 1, $r-2$ becomes $2, \dots, 0$ becomes r, n becomes $r+1, \dots, h+2$ becomes $m-1$, $h+1$ becomes m (the vertices $r, r+1, \dots, h$ are forgotten). This relabeling induces a poset isomorphism

$$K_h \xrightarrow{\cong} K^A_{m,r}.$$

Then equip K_h with the pullback of the matching on $K^A_{m,r}$ defined in Section 5.1 (for $h = n$ the only simplex in K_h is critical).

Lemma 5.14 *The matching defined above is an acyclic weighted matching on K.*

Proof Consider the map $\eta\colon K \to (\mathbb{N}, \le)$ which sends $\sigma \in K$ to $\min\{h \in \mathbb{N} \mid h \notin \sigma\}$. Then η is a poset map with fibers $\eta^{-1}(h) = K_h$. The matching on each $K_h \cong K^A_{m,r}$ is acyclic by Lemma 5.6, therefore, the whole matching on K is acyclic.

Fix $h \in \{0, \dots, n\}$. The bijection $K_h \xrightarrow{\cong} K^A_{m,r}$ is such that, if $\sigma \mapsto \hat{\sigma}$, then $v_\varphi(\sigma) = v_\varphi(\hat{\sigma}) + q$. Indeed, $\Gamma(\sigma)$ is obtained from $\Gamma(\hat{\sigma})$ by adding qd vertices to one (possibly empty) connected component, and this increases the weight by q. Then, since the matching on $K^A_{m,r}$ is weighted, its pullback also is. □

Theorem 5.15 (critical simplices in case $\tilde{A}_n$) *The critical simplices for the matching on K are those listed in Table 7. The only nontrivial incidence numbers between critical cells are*

$$\begin{aligned} [\tau_q : \tau'_q]^{\mathcal{M}} &= \pm 1 \quad (\textit{for } n = kd + r \textit{ with } 0 \le r \le d-2), \\ [\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} &= \pm 1, \\ [\bar{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} &= \pm 1 \quad (\textit{for } n = kd - 1). \end{aligned}$$

In particular, the matching is precise.

Proof We proceed in three parts:

Part 1 (critical simplices) For $n \not\equiv -1 \pmod d$, the matching on K_h has critical simplices only for $h \equiv n \pmod d$. If we write $n = kd + r$ with $0 \le r \le d-2$, then

Case	Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
$n = kd + r$	τ_q	$n-2(k-q)+1$	$q+1$	$0 \le q \le k-1$
$\not\equiv -1 \pmod d$	τ'_q	$n-2(k-q)$	q	$0 \le q \le k$
	$\sigma_{q,r}$	$n-2(k-q)+2$	$q+1$	$0 \le q \le k-2, 0 \le r \le d-2$
$n = kd - 1$	$\sigma'_{q,r}$	$n-2(k-q)+1$	q	$0 \le q \le k-1, 0 \le r \le d-2$
	$\bar\sigma = 1^n 0$	n	k	

Simplices
$\tau_q = 1^{qd+r}01^{d-1}0(1^r01^{d-2-r}0)^{k-q-1}$
$\tau'_q = 1^{qd+r}0(1^r01^{d-2-r}0)^{k-q}$
$\sigma_{q,r} = 1^{qd+r}01^{d-1}0(1^{d-2-r}01^r0)^{k-q-2}1^{d-2-r}0$
$\sigma'_{q,r} = 1^{qd+r}0(1^{d-2-r}01^r0)^{k-q-1}1^{d-2-r}0$

Table 7: Description of the critical simplices for $\widetilde{A}_n$. Below, the binary string notation is used.

$K_{qd+r} \cong K^A_{n-qd,r}$ has two critical simplices for $0 \le q \le k-1$ and one critical simplex for $q = k$. These simplices are the ones listed in Table 7.

Suppose now that $n = kd - 1$. For any $h = qd + r$, with $0 \le r \le d-2$, the complex $K_{qd+r} \cong K^A_{n-qd,r}$ has two critical simplices because $n - qd \equiv -1 \pmod d$. Moreover, for $h = n$ the complex $K_n \cong K^A_{d-1,d-1}$ has one critical simplex. In the remaining cases the matching on K_h has no critical simplices. Again, the critical simplices are those listed in Table 7.

Part 2 (incidence numbers for $n = kd + r$) We want to find the incidence numbers between critical simplices of consecutive dimensions. We start with the case $n = kd + r$, with $0 \le r \le d-2$. First, let us look for alternating paths from τ_q to τ'_q (for $0 \le q \le k-1$). Set $h = qd + r$. Suppose we have one such path,

$$\tau_q \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau'_q.$$

If at some point a vertex $v \in \{0, \dots, h-1\}$ is removed, then the path falls into some $K_{h'}$ with $h' < h$ and can never return in K_h. Therefore, the path must be entirely contained in $K_h \cong K^A_{n-qd,r}$, and by Theorem 5.7 it must be the trivial path $\tau_q \rhd \tau'_q$. Thus, $[\tau_q : \tau'_q] = \pm 1$.

The other pairs of critical simplices in consecutive dimensions are (τ'_{q+1}, τ_q) for $0 \le q \le k-1$. There is a trivial path $\tau'_{q+1} \rhd \tau_q$ which consists in removing the vertex $h = qd + r$, and contributes to the incidence number by

$$[\tau'_{q+1} : \tau_q] = (-1)^h.$$

Suppose we have some other (nontrivial) alternating path

$$\tau'_{q+1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau_q, \quad m \ge 1. \tag{2}$$

Let $\rho_m = \tau_q \cup \{v\}$. If $v = h$ then $\rho_m = \tau'_{q+1}$, which is excluded. Then v must be one of the other vertices that do not belong to τ_q. They are of the form $v = n - sd$ (for $0 \le s \le k-q-1$) or $v = n-(d-1-r)-sd$ (for $0 \le s \le k-q-2$). If $v = n-(d-1-r)-sd$, then ρ_m is matched with $\rho_m \cup \{n-(s+1)d\}$. This is impossible because ρ_m must be matched with some $\zeta_m \lhd \rho_m$. Similarly, if $v = n-sd$ with $0 \le s \le k-q-2$, then ρ_m is matched with $\rho_m \cup \{n-(d-1-r)-sd\}$ and not with some $\zeta_m \lhd \rho_m$. The only remaining possibility is $v = n-sd$ with $s = k-q-1$, ie $v = n-(k-q-1)d = (q+1)d + r$. In this case ρ_m is matched with $\zeta_m = \rho_m \setminus \{qd+2r+1\}$. Going on with the same argument, we have exactly one way to continue the alternating path (from right to left in (2)), and we eventually end up with τ'_{q+1}. From left to right, the obtained alternating path is the following:

$$\begin{aligned}
\zeta_1 &= \tau'_{q+1} \setminus \{qd+2r+1\}, \\
\rho_1 &= \zeta_1 \cup \{n\}, \\
\zeta_2 &= \rho_1 \setminus \{qd+r\}, \\
\rho_2 &= \zeta_2 \cup \{n-(d-1-r)\}, \\
\zeta_3 &= \rho_2 \setminus \{n\}, \\
\rho_3 &= \zeta_3 \cup \{n-d\}, \\
\zeta_4 &= \rho_3 \setminus \{n-(d-1-r)\}, \\
&\ \vdots \\
\zeta_m &= \rho_{m-1} \setminus \{(q+1)d+2r+1\}, \\
\rho_m &= \zeta_m \cup \{qd+2r+1\}, \\
\tau_q &= \rho_m \setminus \{(q+1)d+r\}.
\end{aligned}$$

The length of the path is $m = 2(k-q)$. Apart from $qd+r$, which is the vertex in $\tau'_{q+1} \setminus \tau_q$, the other vertices are added and removed exactly once during the path. If a certain v is added in some ρ_i and removed in some ζ_j (the removal might possibly come before the addition), then

$$[\rho_i : \zeta_i][\rho_{j-1} : \zeta_j] = \begin{cases} 1 & \text{if } v = n, \\ -1 & \text{otherwise.} \end{cases}$$

This is true because, except for $v = n$, between the addition and the removal of v exactly one vertex u with $u < v$ has been added/removed. Namely, between the addition and the removal of a vertex $v = jd + 2r + 1$ (for $q \le j \le k-1$) the vertex $u = jd + r$ is added/removed, and between the addition of the removal of a vertex $v = jd + r$ (for $q + 1 \le j \le k - 1$) the vertex $u = (j-1)d + 2r + 1$ is added. Therefore, the alternating path (2) contributes to the incidence number by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1} = (-1)^{h+1}.$$

Finally, the incidence number is given by

$$[\tau'_{q+1} : \tau_q]^{\mathcal{M}} = (-1)^h + (-1)^{h+1} = 0.$$

Part 3 (incidence numbers for $n = kd - 1$) Consider a generic alternating path starting from $\overline{\sigma} = \{0, \dots, n-1\}$,

$$\overline{\sigma} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \zeta_{m+1}.$$

Let $\overline{\sigma} = \zeta_1 \cup \{v\}$. If $n - d + 1 \le v \le n - 1$ then $\zeta_1 = \sigma'_{k-1,r}$ for $r = n - 1 - v$. Therefore, we have trivial alternating paths from $\overline{\sigma}$ to any of the $\sigma'_{k-1,r}$. If $v \le n - d$ then $\zeta_1 \in K_h$ with $h \le n - d = (k-1)d - 1$. None of these K_h's contains critical simplices ζ_{m+1} with $|\zeta_{m+1}| = n - 1$, and the alternating path cannot return in any $K_{h'}$ with $h' \ge (k-1)d$. Thus, there are no other alternating paths from $\overline{\sigma}$ to critical simplices of K. Then the nontrivial incidence numbers involving $\overline{\sigma}$ are

$$[\overline{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} = \pm 1.$$

Consider now a generic alternating path from σ_{q,r_1} to σ'_{q,r_2},

$$\sigma_{q,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma'_{q,r_2}.$$

Let $\rho_m = \sigma'_{q,r_2} \cup \{v\}$. Adding v to σ'_{q,r_2} causes the creation of a connected component Γ_i with $|\Gamma_i| \equiv -1 \pmod{d}$. This means that ρ_m is matched with a simplex of higher dimension, or is not matched at all (this happens for $v = (q+1)d - 1$). Therefore, the alternating path must be trivial, and it occurs only for $r_1 = r_2$. Then the nontrivial incidence numbers of the form $[\sigma_{q,r_1} : \sigma'_{q,r_2}]^{\mathcal{M}}$ are

$$[\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} = \pm 1.$$

Finally, consider a generic alternating path from σ'_{q+1,r_1} to σ_{q,r_2},

$$\sigma'_{q+1,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma_{q,r_2}. \tag{3}$$

As before, we work backwards. Let $\rho_m = \sigma_{q,r_2} \cup \{v\}$. Apart from the choices $v = qd + r$ and $v = (q+1)d + r$, in all other cases ρ_m has a connected component of size $\equiv -1 \pmod{d}$ and this prevents the continuation of the alternating path. For $v = qd + r$ we obtain $\rho_m = \sigma'_{q+1,r_2}$, so we have a trivial alternating path from σ'_{q+1,r_2} to σ_{q,r_2}. For $v = (q+1)d + r$, iterating the same argument, we have exactly one way to continue the alternating path (from right to left in (3)) and we end up with σ'_{q+1,r_2}. From left to right, the path is as follows (set $r = r_2$):

$$\begin{aligned}
\zeta_1 &= \sigma'_{q+1,r} \setminus \{(q+1)d - 1\}, \\
\rho_1 &= \zeta_1 \cup \{n\}, \\
\zeta_2 &= \rho_1 \setminus \{qd + r\}, \\
\rho_2 &= \zeta_2 \cup \{n - (d-1-r)\}, \\
\zeta_3 &= \rho_2 \setminus \{n\}, \\
\rho_3 &= \zeta_3 \cup \{n - d\}, \\
\zeta_4 &= \rho_3 \setminus \{n - (d-1-r)\}, \\
&\vdots \\
\zeta_m &= \rho_{m-1} \setminus \{(q+2)d - 1\}, \\
\rho_m &= \zeta_m \cup \{(q+1)d - 1\}, \\
\sigma_{q,r} &= \rho_m \setminus \{(q+1)d + r\}.
\end{aligned}$$

The length of the path is $m = 2(k-q) - 1$. As happened in the case $n = kd + r$, the contribution of this alternating path to the incidence number is given by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1}.$$

The contribution of the trivial path $\sigma'_{q+1,r} \rhd \sigma_{q,r}$ is given by $(-1)^{qd+r}$. Therefore,

$$[\sigma'_{q+1,r} : \sigma_{q,r}]^{\mathcal{M}} = (-1)^{qd+r} + (-1)^{qd+r+1} = 0. \qquad \square$$

We are now able to recover the result of [12] about the homology $H_*(\boldsymbol{G_W}; R)$ when $\boldsymbol{G_W}$ is an Artin group of type $\widetilde{A}_n$.

Theorem 5.16 (homology in case $\widetilde{A}_n$ [12]) *For an Artin group $\boldsymbol{G_W}$ of type $\widetilde{A}_n$, we have*

$$H_m(\boldsymbol{G_W};R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus d-1} & \text{if } n=kd-1 \text{ and } m=n-2i+1 \ (1\le i\le k),\\ R/(\varphi_d) & \text{if } n=kd+r \text{ and } m=n-2i \ (1\le i\le k),\\ 0 & \text{otherwise,}\end{cases}$$

where $0\le r\le d-2$ in the second case.

Proof We apply Theorems 5.1 and 5.15. For $n = kd + r$ $(0 \le r \le d-2)$, the boundary map δ^{φ}_{m+1} has rank 1 when $m+1 = n-2(k-q)+1$ (for $0 \le q \le k-1$, due to τ_q); it has rank 0 otherwise. For $n = kd-1$, the boundary map δ^{φ}_{m+1} has rank $d-1$ if $m+1 = n-2(k-q)+2$ (for $0 \le q \le k-2$, due to the simplices $\sigma_{q,r}$) or if $m+1 = n$ (due to $\overline{\sigma}$); it has rank 0 otherwise. □

5.4 Case $\widetilde{C}_n$

The last case we consider is that of Artin groups of type $\widetilde{C}_n$. The corresponding Coxeter graph Γ is shown in Figure 9. As in the $\widetilde{A}_n$ case, the simplicial complex K consists of all the simplices $\sigma \subseteq S = \{0,\dots,n\}$ except for the full simplex $\sigma = \{0,\dots,n\}$. For any $\sigma \in K$, the subgraph $\Gamma(\sigma)$ of Γ splits as a union of connected components of type B_k (those containing the first or the last vertex) and of type A_k (the remaining ones).

Figure 9: A Coxeter graph of type $\widetilde{C}_n$

For each m with $1 \le m \le n$, let K^B_m be the full simplicial complex on $\{1,\dots,m\}$, endowed with the weight function of B_m (see Section 5.2). We are going to construct a precise matching for the case $\widetilde{C}_n$ using the precise matching on K^B_m.

For $h \in \{0,\dots,n\}$ set

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0,\dots,h-1\} \subseteq \sigma\}.$$

As in the $\widetilde{A}_n$ case, since every simplex $\sigma \in K$ misses at least one vertex, the subsets K_h form a partition of K,

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Ignore the vertices $0, \dots, h$ and relabel the remaining ones from right to left: n becomes 1, $n-1$ becomes 2, …, $h+1$ becomes $n-h$. This induces a poset isomorphism

$$K_h \xrightarrow{\cong} K^B_{n-h}.$$

Then equip K_h with the pullback of the matching on K^B_{n-h} constructed in Section 5.2 (for $h = n$ the single simplex in K_h is critical).

Remark 5.17 For d odd we are simply matching σ with $\sigma \veebar n$ for all σ except $\sigma = \{0, \dots, n-1\}$.

Lemma 5.18 *The matching defined above is an acyclic weighted matching on K.*

Proof As in the proof of Lemma 5.14, the subsets K_h are the fibers of a poset map and the matching is acyclic on each $K_h \cong K^B_{n-h}$.

If a simplex $\sigma \in K_h$ is sent to $\hat\sigma$ by the isomorphism $K_h \xrightarrow{\cong} K^B_{n-h}$, then $v_\varphi(\sigma) = v_\varphi(\hat\sigma) + \omega_\varphi(B_h)$. Since the matching on K^B_{n-h} is weighted, the matching on K_h also is. □

Theorem 5.19 (critical simplices in case $\widetilde{C}_n$) *The critical simplices for the matching on K are those listed in Table 8. In particular, the matching is precise. In addition the only nontrivial incidence numbers between critical simplices in the Morse complex are as follows (for d even and $n = k\frac{d}{2} + r$):*

$$\begin{aligned}
[\sigma_{q_1,q_2} : \sigma_{q_1,q_2-1}]^{\mathcal{M}} &= (-1)^{\alpha},\\
[\sigma_{q_1,q_2} : \sigma_{q_1-1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+1},\\
[\sigma_{q_1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+d/2+1},\\
[\sigma'_{q_1,q_2} : \sigma_{q_1,q_2-2}]^{\mathcal{M}} &= (-1)^{\beta+1},\\
[\sigma'_{q_1,q_2} : \sigma_{q_1-2,q_2}]^{\mathcal{M}} &= (-1)^{\beta},\\
[\sigma'_{q_1,q_2} : \sigma'_{q_1,q_2-1}]^{\mathcal{M}} &= (-1)^{\alpha+1},\\
[\sigma'_{q_1,q_2} : \sigma'_{q_1-1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+d/2},
\end{aligned}$$

where $\alpha = (k-q_2)\left(\frac{d}{2}-1\right) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$.

Proof As we have already said, for d odd there is exactly one critical simplex. Suppose from now on that d is even. Let $n = k\frac{d}{2} + r$, with $0 \le r \le \frac{d}{2} - 1$.

Case	Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
d odd	$\overline{\sigma} = 1^n 0$	n	1 or 0	
d even	σ_{q_1,q_2}	$n-k+q_1+q_2+1$	q_1+q_2+1	$0 \le q_1+q_2 \le k-2$
	σ'_{q_1,q_2}	$n-k+q_1+q_2$	q_1+q_2	$0 \le q_1+q_2 \le k$

Simplices
$\sigma_{q_1,q_2} = 1^{q_1 d/2+r} 0 1^{d-1} 0 (1^{d/2-1} 0)^{k-q_1-q_2-2} 1^{q_2 d/2}$
$\sigma'_{q_1,q_2} = 1^{q_1 d/2+r} 0 (1^{d/2-1} 0)^{k-q_1-q_2} 1^{q_2 d/2}$

Table 8: Description of the critical simplices for $\widetilde{C}_n$. When d is even, set $n = k\frac{d}{2} + r$.

Part 1 (the critical simplices) In K_h there are critical simplices if and only if $n - h \equiv 0 \pmod{\frac{d}{2}}$, ie when $h = q_1\frac{d}{2} + r$ for some q_1 (with $0 \le q_1 \le k$). By Theorem 5.12 there are two families of critical simplices: σ_{q_1,q_2} (for $0 \le q_1 + q_2 \le k-2$) and σ'_{q_1,q_2} (for $0 \le q_1 + q_2 \le k$), as shown in Table 8 and Figure 10. In a fixed dimension $m = n - k + l$ ($0 \le l \le k$) there are $2l+1$ critical simplices if $l \le k-1$ and l critical simplices if $l = k$. See Table 9.

Part 2 (paths ending in σ_{q_1,q_2}) Consider a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ_{q_1,q_2},

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q_1,q_2}.$$

Let $\rho_m = \sigma_{q_1,q_2} \cup \{v\}$. If $v = q_1\frac{d}{2} + r$ then $\rho_m = \sigma'_{q_1+2,q_2}$ and the alternating path stops. If $v = n - q_2\frac{d}{2}$ then the alternating path stops at $\rho_m = \sigma_{q_1,q_2+1}$. Suppose

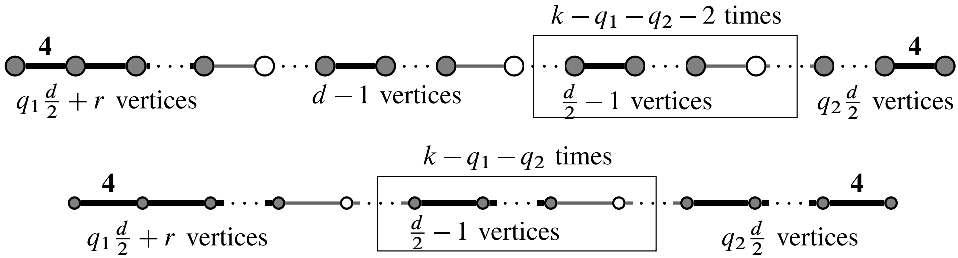


Figure 10: Critical simplices for $\widetilde{C}_n$, with d even and $n = k\frac{d}{2} + r$. The diagram for the simplex σ_{q_1,q_2} is at the top and the diagram for the simplex σ'_{q_1,q_2} is below it.

$q_1+q_2 \le k-3$, otherwise there are no more cases. If $q_1\frac{d}{2}+r+d < v < n-q_2\frac{d}{2}$, then $\Gamma(\rho_m)$ has at least one connected component of size $d-1$ and therefore ρ_m is matched with a simplex of higher dimension; thus the path stops without having reached a critical simplex. If $v = q_1\frac{d}{2}+r+d$ then ρ_m is matched with

$$\tau_m = \rho_m \setminus \left\{q_1\frac{d}{2}+r+\frac{d}{2}\right\} = 1^{q_1 d/2+r}01^{d/2-1}01^{d-1}0(1^{d/2-1}0)^{k-q_1-q_2-3}01^{q_2 d/2}$$

in the binary string notation. From here the path can continue in many ways. Let $\rho_{m-1} = \tau_m \cup \{w\}$.

- If $w = q_1\frac{d}{2}+r$, we end up with $\rho_{m-1} = \sigma_{q_1+1,q_2}$.
- If $w = q_1\frac{d}{2}+r+\frac{d}{2}$, we would be going back to ρ_m.
- If $w > q_1\frac{d}{2}+r+\frac{3}{2}d$, then $\Gamma(\rho_{m-1})$ has at least one connected component of size $d-1$ and therefore ρ_{m-1} is matched with a simplex of higher dimension.
- If $w = q_1\frac{d}{2}+r+\frac{3}{2}d$, then ρ_{m-1} is matched with
$$\begin{aligned}\tau_{m-1} &= \rho_{m-1} \setminus \left\{q_1\frac{d}{2}+r+d\right\}\\ &= 1^{q_1 d/2+r}01^{d/2-1}01^{d/2-1}01^{d-1}0(1^{d/2-1}0)^{k-q_1-q_2-4}01^{q_2 d/2}.\end{aligned}$$
By induction, repeating the same argument as above, this path can be continued in exactly one way and it eventually arrives at the critical simplex σ'_{q_1,q_2+2}. The path has length $m = k-q_1-q_2-2$ and is as follows:
$$\begin{aligned}\tau_1 &= \sigma'_{q_1,q_2+2} \setminus \left\{n-q_2\frac{d}{2}\right\},\\ \rho_1 &= \tau_1 \cup \left\{n-q_2\frac{d}{2}-d\right\},\\ \tau_2 &= \rho_1 \setminus \left\{n-q_2\frac{d}{2}-\frac{d}{2}\right\},\\ \rho_2 &= \tau_2 \cup \left\{n-q_2\frac{d}{2}-\frac{3}{2}d\right\},\\ \tau_3 &= \rho_2 \setminus \left\{n-q_2\frac{d}{2}-d\right\},\\ &\;\vdots\\ \rho_{m-1} &= \tau_{m-1} \cup \left\{q_1\frac{d}{2}+r+d\right\},\\ \tau_m &= \rho_{m-1} \setminus \left\{q_1\frac{d}{2}+r+\frac{3}{2}d\right\},\\ \rho_m &= \tau_m \cup \left\{q_1\frac{d}{2}+r+\frac{d}{2}\right\},\\ \sigma_{q_1,q_2} &= \rho_m \setminus \left\{q_1\frac{d}{2}+r+d\right\}.\end{aligned}$$

Simplices	
$\sigma_{l-1,0}$	$\sigma'_{l,0}$
$\sigma_{l-2,1}$	$\sigma'_{l-1,1}$
$\vdots$	$\vdots$
$\sigma_{1,l-2}$	$\sigma'_{2,l-2}$
$\sigma_{0,l-1}$	$\sigma'_{1,l-1}$
	$\sigma'_{0,l}$

Table 9: Critical simplices for $\widetilde{C}_n$ in dimension $m = n-k+l$ $(0 \le l \le k)$, where d is even and $n = k\frac{d}{2}+r$. For $l=k$ only the second column occurs.

Part 3 (paths ending in σ'_{q_1,q_2}) Consider now a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ'_{q_1,q_2},

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma'_{q_1,q_2}. \tag{4}$$

As usual, let $\rho_m = \sigma'_{q_1,q_2} \cup \{v\}$. For the same reasons as above, there are only three possibilities: $v = q_1\frac{d}{2}+r$, $v = q_1\frac{d}{2}+r+\frac{d}{2}$ and $v = n-q_2\frac{d}{2}$; in all the other cases, ρ_m is matched with a simplex of higher dimension. If $v = q_1\frac{d}{2}+r$ then the path ends (to the left in (4)) at $\rho_m = \sigma'_{q_1+1,q_2}$. If $v = q_1\frac{d}{2}+r+\frac{d}{2}$ then the path ends at σ_{q_1,q_2}. Finally, if $v = n-q_2\frac{d}{2}$ then the path ends at σ'_{q_1,q_2+1}.

Part 4 (incidence numbers) We have seven families of incidence numbers to compute, each coming from one of the alternating paths we have found:

From σ'_{q_1+2,q_2} to σ_{q_1,q_2} The alternating path is trivial and consists in removing the vertex $v = q_1\frac{d}{2}+r$, so

$$[\sigma'_{q_1+2,q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)^{|\{w \in \sigma_{q_1,q_2} \mid w<v\}|} = (-1)^{q_1 d/2+r},$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma_{q_1-2,q_2}]^{\mathcal{M}} = (-1)^{q_1 d/2+r} = (-1)^{\beta}.$$

From σ_{q_1,q_2+1} to σ_{q_1,q_2} Again the path is trivial, and it consists in removing $v = n-q_2\frac{d}{2}$. Therefore,

$$\begin{aligned}[\sigma_{q_1,q_2+1} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{|\{w \in \sigma_{q_1,q_2} \mid w<v\}|} \\ &= (-1)^{n-q_2 d/2-(k-q_1-q_2-1)}\end{aligned}$$

$$= (-1)^{kd/2+r-q_2d/2-k+q_1+q_2+1}$$
$$= (-1)^{(k-q_2)(d/2-1)+r+q_1+1},$$

which implies

$$[\sigma_{q_1,q_2} : \sigma_{q_1,\,q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)(d/2-1)+r+q_1+1} = (-1)^{\alpha}.$$

From $\sigma_{q_1+1,\,q_2}$ to σ_{q_1,q_2} The path consists in removing $q_1\frac{d}{2}+r$, adding $q_1\frac{d}{2}+r+\frac{d}{2}$ and removing $q_1\frac{d}{2}+r+d$. Therefore,

$$\begin{aligned}[\sigma_{q_1+1,\,q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)(-1)^{q_1d/2+r}(-1)^{q_1d/2+r+d/2-1}(-1)^{q_1d/2+r+d-1}\\ &= (-1)^{q_1d/2+r+d/2+1},\end{aligned}$$

which implies

$$[\sigma_{q_1,q_2} : \sigma_{q_1-1,\,q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)d/2+r+d/2+1} = (-1)^{\beta+1}.$$

From $\sigma'_{q_1,\,q_2+2}$ to σ_{q_1,q_2} The path is the one we explicitly wrote at the end of Part 2. Notice that from ρ_{i-1} to τ_i one removes the vertex $v_i = n-(q_2-i+1)\frac{d}{2}$, and from τ_i to ρ_i one adds the vertex $v'_i = n-(q_2-i-1)\frac{d}{2}$. Therefore,

$$[\rho_{i-1} : \tau_i][\rho_i : \tau_i] = (-1)^{|\{w\in\tau_i \mid v'_i<w<v_i\}|} = (-1)^{d-1} = -1.$$

Then we can compute the incidence number in the Morse complex:

$$\begin{aligned}[\sigma'_{q_1,\,q_2+2} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)^m\Bigl(\prod_{i=1}^{m}[\rho_{i-1} : \tau_i][\rho_i : \tau_i]\Bigr)[\sigma_{q_1,q_2} : \rho_m]\\ &= (-1)^m(-1)^m(-1)^{q_1d/2+r+(d-1)}\\ &= (-1)^{q_1d/2+r+1},\end{aligned}$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma_{q_1,\,q_2-2}]^{\mathcal{M}} = (-1)^{q_1d/2+r+1} = (-1)^{\beta+1}.$$

From $\sigma'_{q_1+1,\,q_2}$ to σ'_{q_1,q_2} The path is trivial and consists of removing $v = q_1\frac{d}{2}+r$. Then

$$[\sigma'_{q_1+1,\,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1d/2+r},$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma'_{q_1-1,\,q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)d/2+r} = (-1)^{\beta+d/2}.$$

From σ_{q_1,q_2} to σ'_{q_1,q_2} The path is trivial and consists in removing $v = q_1\frac{d}{2} + r + \frac{d}{2}$. Then

$$[\sigma_{q_1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1 d/2+r+d/2-1} = (-1)^{\beta+d/2+1}.$$

From σ'_{q_1,q_2+1} to σ'_{q_1,q_2} The path is trivial and consists in removing $v = n - q_2\frac{d}{2}$. Then

$$\begin{aligned}[\sigma'_{q_1,q_2+1} : \sigma'_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{n-q_2 d/2-(k-q_1-q_2)} \\ &= (-1)^{kd/2+r-q_2 d/2-k+q_1+q_2} \\ &= (-1)^{(k-q_2)(d/2-1)+r+q_1},\end{aligned}$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma'_{q_1,q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)(d/2-1)+r+q_1} = (-1)^{\alpha+1}.$$ □

We can finally compute the homology $H_*(\boldsymbol{G_W}; R)$ for Artin groups of type $\widetilde{C}_n$.

Theorem 5.20 (homology in case $\widetilde{C}_n$) *Let $\boldsymbol{G_W}$ be an Artin group of type $\widetilde{C}_n$. Then the φ_d–primary component of $H_*(\boldsymbol{G_W}; R)$ is trivial for d odd, and for d even is as follows:*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \le m \le n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k\frac{d}{2} + r$.

Proof In order to apply Theorem 5.1 we need to find the rank of the boundary maps δ^{φ}_{m+1} of the Morse complex. For d odd there is only one critical cell, thus all the boundaries vanish. Suppose from now on that d is even, and let $n = k\frac{d}{2} + r$. In order to have a nontrivial boundary δ^{φ}_{m+1} we must have at least one critical simplex both in dimension m and in dimension $m+1$, thus $m = n-k+l$ with $0 \le l \le k-1$ (see Table 9).

Case 1 $(l \le k-2)$ We are going to prove that, for $l \le k-2$, a basis for the image of $\delta = \delta^{\varphi}_{m+1}$ is given by

$$\mathcal{B} = \{\delta\sigma_{q_1,q_2} \mid q_1 + q_2 = l\}.$$

By Theorem 5.19 we obtain the following formula for $\delta\sigma_{q_1,q_2}$:

$$\delta\sigma_{q_1,q_2} = (-1)^{\alpha}\sigma_{q_1,q_2-1} + (-1)^{\beta+1}\sigma_{q_1-1,q_2} + (-1)^{\beta+d/2+1}\sigma'_{q_1,q_2},$$

where $\alpha = (k-q_2)\big(\frac{d}{2}-1\big)+q_1+r+\frac{d}{2}$ and $\beta = q_1\frac{d}{2}+r$. In this formula the σ_{q_1,q_2-1} (resp. σ_{q_1-1,q_2}) term vanishes if $q_2=0$ (resp. $q_1=0$). The term σ'_{q_1,q_2} appears in $\delta\sigma_{q_1,q_2}$ but not in any other element of $\mathcal{B}$, thus $\mathcal{B}$ is a linearly independent set. In addition, if $q_1+q_2=l+1$, we have that

$$\begin{aligned}\delta\sigma'_{q_1,q_2} &= (-1)^{\beta+1}\sigma_{q_1,q_2-2}+(-1)^{\beta}\sigma_{q_1-2,q_2}+(-1)^{\alpha+1}\sigma'_{q_1,q_2-1} \\ &\qquad\qquad +(-1)^{\beta+d/2}\sigma'_{q_1-1,q_2} \\ &= (-1)^{\alpha+\beta+d/2}\big((-1)^{\alpha+d/2+1}\sigma_{q_1,q_2-2}+(-1)^{\beta+1}\sigma_{q_1-1,q_2-1} \\ &\qquad\qquad +(-1)^{\beta+d/2+1}\sigma'_{q_1,q_2-1}\big) \\ &\quad +(-1)^{d/2+1}\big((-1)^{\alpha+1}\sigma_{q_1-1,q_2-1}+(-1)^{\beta+d/2+1}\sigma_{q_1-2,q_2} \\ &\qquad\qquad +(-1)^{\beta+1}\sigma'_{q_1-1,q_2}\big) \\ &= (-1)^{\alpha+\beta+d/2}\delta\sigma_{q_1,q_2-1}+(-1)^{d/2+1}\delta\sigma_{q_1-1,q_2}.\end{aligned}$$

Therefore, $\mathcal{B}$ generates the image of δ^{φ}_{m+1}. Thus, $\operatorname{rk}\delta^{\varphi}_{m+1}=|\mathcal{B}|=l+1$ for $l\le k-2$.

Case 2 $(l=k-1)$ For $l=k-1$, ie $m=n-1$, the situation is a bit different because there are no critical simplices of the form σ_{q_1,q_2} with $q_1+q_2=l$. However, we can still define

$$\epsilon_{q_1,q_2}=(-1)^{\alpha}\sigma_{q_1,q_2-1}+(-1)^{\beta+1}\sigma_{q_1-1,q_2}+(-1)^{\beta+d/2+1}\sigma'_{q_1,q_2}$$

for $q_1+q_2=l$, and

$$\mathcal{B}'=\{\epsilon_{q_1,q_2}\mid q_1+q_2=l\}.$$

The term σ'_{q_1,q_2} appears in ϵ_{q_1,q_2} but not in any other element of $\mathcal{B}'$, thus $\mathcal{B}'$ is a linearly independent set. As above we have that, for $q_1+q_2=l+1$,

$$\delta\sigma'_{q_1,q_2}=(-1)^{\alpha+\beta+d/2}\epsilon_{q_1,q_2-1}+(-1)^{d/2+1}\epsilon_{q_1-1,q_2}.$$

Then $\mathcal{B}'$ generates (and so it is a basis of) the image of δ^{φ}_{n}.

We have proved that, for $0\le l\le k-1$, the rank of δ^{φ}_{m+1} is equal to $l+1=m+k-n+1$. Then we conclude by applying Theorem 5.1. □

Concluding remarks

Future work will focus on other families of Artin groups. In particular, it seems that precise matchings can be constructed in all finite and affine cases (see [36]), possibly allowing explicit homology computations.

The methods developed in this paper are particularly powerful when the coefficients are over a PID, but Artin groups with roots of different lengths (eg B_n, $\widetilde{B}_n$ and $\widetilde{C}_n$) also admit natural representations over polynomial rings with more than one variable. We believe that some of our theory can be extended to such cases, and this can also be the aim of future work.

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