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Equivariant complex bundles, fixed points and equivariant unitary bordism

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We study the fixed points of the universal G–equivariant complex vector bundle of rank n and obtain a decomposition formula in terms of twisted equivariant universal complex vector bundles of smaller rank. We use this decomposition to describe the fixed points of the complex equivariant K–theory spectrum and the equivariant unitary bordism groups for adjacent families of subgroups.

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1 Introduction

In this article decomposition formulas for equivariant K–theory and geometric equivariant bordism of stably almost complex manifolds are obtained under suitable hypotheses. The underlying main technical idea behind such decompositions is a splitting formula for equivariant complex vector bundles first obtained by Gómez and Uribe [10] for the particular case of finite groups. We generalize this splitting formula for the general case of compact Lie groups and apply it to obtain the decompositions of equivariant K–theory and equivariant unitary bordism mentioned above.

More precisely, suppose that G is a compact Lie group that fits in a short exact sequence of compact Lie groups $1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$. Let X be a compact G–space such that A acts trivially on X. In the first part of this article we study G–equivariant complex vector bundles $p\colon E \to X$. Since A acts trivially on X, the fibers of E can be seen as A–representations. By decomposing E into A–isotypical pieces we obtain a splitting of E as an A–equivariant vector bundle in the form $\bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E) \cong E$. Here $\mathbb{V}_\tau$ denotes the trivial A–vector bundle $\pi_1\colon X \times V_\tau \to X$ associated to an irreducible representation $\tau\colon A \to U(V_\tau)$ and $\mathrm{Irr}(A)$ denotes the set of isomorphism classes of complex irreducible A–representations. This splitting is one of A–vector bundles and not one of G–vector bundles since in general the bundles $\mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ may not possess the structure of a G–vector bundle

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(see Example 3.5). A key technical observation of this work is that, up to isomorphism, the direct sum $\bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ can be rearranged using the different orbits of the action of Q on $\mathrm{Irr}(A)$ to obtain a decomposition of E in terms of G–vector bundles. This way a splitting of E as a G–equivariant vector bundle is obtained in Theorem 3.6. This result plays a key role in this paper.

Given an irreducible representation $\rho\colon A \to U(V_\rho)$ we can obtain in a natural way a central extension of the form $1 \to \mathbb{S}^1 \to \widetilde{Q}_\rho \to Q_\rho \to 1$, where $Q_\rho = G_\rho/A$ and $G_\rho = \{g \in G \mid g\cdot\rho \cong \rho\}$ (see Sections 2 and 3 for definitions). It turns out that each of the pieces in the splitting formula given in Theorem 3.6 can be used to define a twisted form of an equivariant K–theory, and as a consequence the following result is obtained:

Corollary 3.7 *Let G be a compact Lie group and X a G–space on which the normal subgroup A acts trivially. Then there is a natural isomorphism*

$$K^*_G(X) \cong \bigoplus_{\rho\in G\backslash \mathrm{Irr}(A)} {}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X),$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations and $Q_\rho = G_\rho/A$.

In the above formula, ${}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X)$ denotes a twisted form of Q_ρ–equivariant K–theory. This result generalizes a similar decomposition obtained in [10] for the particular case of finite groups.

On the other hand, the decomposition obtained in Theorem 3.6 can be carried out at the level of the universal G–equivariant complex bundle of rank n, denoted by $\gamma_G U(n) \to B_G U(n)$. Here $B_G U(n)$ is the classifying space of G–equivariant rank n complex vector bundles. Applying this decomposition to the restriction of $\gamma_G U(n)$ to $B_G U(n)^A$, we obtain an N_A/A–equivariant homotopy equivalence with a product of classifying spaces parametrized by the orbits of the action of the normalizer N_A on the set of nontrivial irreducible representations of A. This result is also one of the main results of this article and is summarized in Theorem 4.1.

The second part of this article adds to the understanding of the geometric equivariant bordism groups of stably almost complex manifolds with boundary, whenever the isotropy groups of the interior of the manifold differ by one conjugacy class of subgroups from the isotropies of the boundary. The equivariant version of the bordism theories was developed by Conner and Floyd in their monumental work [5; 6] and the unitary equivariant bordism theory was developed by Stong [21] among others. A compact G–equivariant manifold is unitary if the tangent bundle may be stabilized with trivial real bundles, thus becoming isomorphic to a G–equivariant complex vector bundle. The bordism group of unitary G–equivariant manifolds is denoted by Ω_*^G and the product of manifolds makes Ω_*^G into a ring and moreover an Ω_*–module. The calculation of the Ω_*–module structure of Ω_*^G has been elusive and very little is known whenever G is not abelian. Whenever G is abelian it is known that Ω_*^G is zero in odd degrees and a free Ω_*–module in even degrees, (see Comezaña [4, Theorem 5.3] and Ossa [18, Theorem 1]), and the question remains open whether this is also the case whenever G is not abelian.

The main calculational tool to understand Ω_*^G is to restrict attention to unitary manifolds on which the isotropy groups at each point lie on a prescribed family of subgroups of G. For a pair of families $(\mathcal{F}, \mathcal{F}')$ of subgroups of G with $\mathcal{F}' \subset \mathcal{F}$, denote by $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ the bordism classes of unitary G–manifolds with boundary $(M, \partial M)$ such that the isotropy groups of the points in M lie on $\mathcal{F}$ and the isotropy groups of the points of the boundary ∂M lie on $\mathcal{F}'$. Whenever the families differ by the set of groups conjugate to a fixed group A they are called adjacent. Whenever A is normal in G and the pair of families $(\mathcal{F}, \mathcal{F}')$ is adjacent differing by A, the bordism class of a manifold $(M, \partial M)$ in $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ is equivalent to the bordism class of the disk bundle of the tubular neighborhood of the fixed-point set M^A in M. Therefore, we may keep the information of the normal bundle by a map from M^A to the classifying space of G–equivariant complex vector bundles over trivial A–spaces. Hence, the unitary G–equivariant bordism groups for adjacent families can be written in terms of nonequivariant unitary bordism groups of a product of certain classifying spaces. As a consequence of Theorem 3.6 the following decomposition of G–equivariant bordisms is obtained. This theorem is the last main result in this article and is a new result for compact Lie groups that are not abelian.

Theorem 5.6 *Suppose that G is a compact Lie group and let A be a closed normal subgroup of G. If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of G differing by A, then*

$$\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0 \le 2k \le n - \dim(G/A)} \Omega_{n-2k}^{G/A}\{\{1\}\}\Big(X^A \times \coprod_{\overline{P} \in \overline{\mathcal{P}}(k,A)} B_{G/A}U(\overline{P})\Big),$$

where $\{1\}$ is the family of subgroups of G/A which only contains the trivial group.

In the above theorem $B_{G/A}U(\overline{P})$ denotes a product of classifying spaces for different twisted equivariant vector bundles (see Section 5 for the precise definition).

In the last section we use the previous theorem to determine the Ω_*–module structure of $\Omega_*^{D_{2p}}$, where D_{2p} is the dihedral group of order $2p$ with p an odd prime. We show that $\Omega_*^{D_{2p}}$ is a free Ω_*–module in even degrees and zero in odd degrees.

This paper is organized as follows: In Section 2 we review some preliminaries related to central extensions and twisted equivariant K–theory. In Section 3 we prove Theorem 3.6 and obtain Corollary 3.7 as a consequence. In Section 4 we calculate the homotopy type of the fixed-points space $B_G U(n)^A$ for a closed subgroup A of G. Section 5 is dedicated to studying geometric G–equivariant bordism and Theorem 5.6 is proved there. Finally, in Section 6 some applications are considered.

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2 Preliminaries

2.1 Central extensions and representations

Suppose that we have an exact sequence of compact Lie groups

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

and let $\rho\colon A \to U(V_\rho)$ be a complex, finite-dimensional, irreducible representation of A. Since A is normal in G, the group G acts on the left on the set $\mathrm{Hom}(A, U(V_\rho))$ of homomorphisms from A to $U(V_\rho)$ by the equation

$$(g \cdot \chi)(a) := \chi(g^{-1} a g)$$

for $\chi \in \mathrm{Hom}(A, U(V_\rho))$ and $a \in A$. Also, the unitary group $U(V_\rho)$ acts on the right on $\mathrm{Hom}(A, U(V_\rho))$ by conjugation by the equation

$$(\chi \cdot M)(a) := M^{-1} \chi(a) M$$

for $\chi \in \mathrm{Hom}(A, U(V_\rho))$ and $M \in U(V_\rho)$. Note further that this left G–action on $\mathrm{Hom}(A, U(V_\rho))$ commutes with the right $U(V_\rho)$–action.

In this section we are going to show that if the representation ρ is such that $g \cdot \rho \cong \rho$ for every $g \in G$, then we can associate to ρ a central extension of G by $\mathbb{S}^1$ and that this central extension can be thought as an obstruction for the existence of an extension $\widetilde{\rho}\colon G \to U(V_\rho)$ of the representation ρ. For this notice that the projective unitary group $\mathrm{PU}(V_\rho) := U(V_\rho)/Z(U(V_\rho)) = U(V_\rho)/\mathbb{S}^1$ can be identified with the inner automorphisms of $U(V_\rho)$ via the map $p(M) = \mathrm{Ad}_M$, where $\mathrm{Ad}_M(N) = MNM^{-1}$ for $M \in U(V_\rho)$.

Lemma 2.1 *Suppose that for all $g \in G$ the irreducible representation $g \cdot \rho$ is isomorphic to ρ. Then there is a unique homomorphism $f\colon G \to \mathrm{PU}(V_\rho)$ making the following diagram commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\iota} & G \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle f} \\ U(V_\rho) & \xrightarrow{p} & \mathrm{PU}(V_\rho) \end{array}$$

Proof Suppose that $g \in G$. Note that the representation $g \cdot \rho\colon A \to U(V_\rho)$, defined by $(g \cdot \rho)(a) = \rho(g^{-1}ag)$ for $a \in A$ and $g \in G$, also has V_ρ for its underlying vector space. By Schur's lemma we know that $\mathrm{Hom}_{U(V_\rho)}(g \cdot \rho, \rho) \cong \mathbb{C}$, thus there is only one inner automorphism $f(g^{-1}) \in \mathrm{Inn}(U(V_\rho))$ of $U(V_\rho)$ such that $g \cdot \rho = f(g^{-1}) \circ \rho$. Whenever $g \in A$, we have that $g \cdot \rho = \mathrm{Ad}_{\rho(g)^{-1}} \circ \rho$ and therefore we set $f(g) = \mathrm{Ad}_{\rho(g)}$ whenever $g \in A$.

For $h, g \in G$ we know that $(hg \cdot \rho) = h \cdot (g \cdot \rho)$, thus implying that $f((hg)^{-1}) \circ \rho = h \cdot (f(g^{-1}) \circ \rho) = f(g^{-1}) \circ f(h^{-1}) \circ \rho$ and therefore $f((hg)^{-1}) = f(g^{-1}) \circ f(h^{-1})$. Hence, f is a homomorphism and by definition it is unique. □

Suppose now that we have an irreducible representation $\rho\colon A \to U(V_\rho)$ such that $g \cdot \rho \cong \rho$ for every $g \in G$. Let $f\colon G \to \mathrm{PU}(V_\rho)$ be the homomorphism constructed in the previous lemma, so that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & G \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle f} \\ U(V_\rho) & \xrightarrow{p} & \mathrm{PU}(V_\rho) \end{array}$$

Recall that the natural projection map

$$1 \to \mathbb{S}^1 \to U(V_\rho) \xrightarrow{p} \mathrm{PU}(V_\rho) \to 1$$

defines a central extension of $\mathrm{PU}(V_\rho)$ by $\mathbb{S}^1$. Define the Lie group $\widetilde{G}_\rho := f^* U(V_\rho)$ as the pullback of $U(V_\rho)$ under the homomorphism f, so that we obtain a central extension of Lie groups

$$1 \to \mathbb{S}^1 \to \widetilde{G}_\rho \xrightarrow{\tau_\rho} G \to 1.$$

If we denote by $\widetilde{f}\colon \widetilde{G}_\rho \to U(V_\rho)$ the induced homomorphism, we obtain the following commutative diagram in the category of Lie groups:

$$\begin{array}{ccccc} & & \mathbb{S}^1 & & \mathbb{S}^1 \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{\tilde{\iota}} & \widetilde{G}_\rho & \xrightarrow{\widetilde{f}} & U(V_\rho) \\ \downarrow{=} & & \downarrow & & \downarrow \\ A & \xrightarrow{\iota} & G & \xrightarrow{f} & \mathrm{PU}(V_\rho) \end{array} \tag{1}$$

In the above diagram the vertical sequences are $\mathbb{S}^1$–central extensions and the homomorphism $\tilde{\iota}\colon A \to \widetilde{G}_\rho$ is the unique homomorphism such that $\rho = \widetilde{f} \circ \tilde{\iota}$. Since A is normal in G and $\widetilde{G}_\rho$ is a central extension of G, we have that $\tilde{\iota}(A)$ is also normal in $\widetilde{G}_\rho$. Therefore, the quotient $\widetilde{G}_\rho/\tilde{\iota}(A)$ is a Lie group and we denote it by

$$\widetilde{Q}_\rho := \widetilde{G}_\rho/\tilde{\iota}(A)$$

since it depends only on ρ, and it fits into the diagram

$$\begin{array}{ccccc} & & \mathbb{S}^1 & & \mathbb{S}^1 \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{\tilde{\iota}} & \widetilde{G}_\rho & \xrightarrow{\tilde{\pi}} & \widetilde{Q}_\rho \\ \| & & \downarrow & & \downarrow \\ A & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q \end{array} \tag{2}$$

where the horizontal sequences are exact, the vertical are $\mathbb{S}^1$–central extensions and the square on the right-hand side is a pullback square.

Proposition 2.2 *Consider the short exact sequence $1 \to A \to G \to Q \to 1$ of compact Lie groups and $\rho\colon A \to U(V_\rho)$ an irreducible representation of A such that its isomorphism class is invariant under the G–action, namely that $(g \cdot \rho) \cong \rho$ for all $g \in G$. Then the representation ρ may be extended to an irreducible representation $\widetilde{\rho}\colon G \to U(V_\rho)$ if and only if the $\mathbb{S}^1$–central extension $\widetilde{Q}_\rho$ is trivial, that is, $\widetilde{Q}_\rho$ is isomorphic to $Q \times \mathbb{S}^1$ as Lie groups.*

Proof If $\widetilde{Q}_\rho$ is trivial as an $\mathbb{S}^1$–central extension, then $\widetilde{G}_\rho$ must also be trivial as an $\mathbb{S}^1$–central extension, that is, $\widetilde{G}_\rho \cong G \times \mathbb{S}^1$. Therefore, there is a homomorphism $\sigma\colon G \to \widetilde{G}_\rho$ compatible with the quotient homomorphism $\widetilde{G}_\rho \to G$ whose composition $\widetilde{\rho} := \widetilde{f} \circ \sigma\colon G \to U(V_\rho)$ is the desired extension of ρ.

Conversely, if $\widetilde{\rho}\colon G \to U(V_\rho)$ extends the homomorphism ρ then $\widetilde{\rho}$ defines a homomorphism $\sigma\colon G \to \widetilde{G}_\rho$ compatible with the quotient homomorphism $\widetilde{G}_\rho \to G$, thus making $\widetilde{G}$ a trivial $\mathbb{S}^1$–central extension. It follows that $\widetilde{Q}_\rho$ is also trivial as an $\mathbb{S}^1$–central extension. □

Remark 2.3 Recall that isomorphism classes of $\mathbb{S}^1$–central extensions of Q are in one-to-one correspondence with elements in $H^3(BQ, \mathbb{Z})$ (see [2, Proposition 6.3]). By the previous proposition we may say that the obstruction for the existence of the extension $\widetilde{\rho}\colon G \to U(V_\rho)$ of the irreducible representation $\rho\colon A \to U(V_\rho)$ is the cohomology class $[\widetilde{Q}_\rho] \in H^3(BQ, \mathbb{Z})$ which encodes the information of the $\mathbb{S}^1$–central extension $\mathbb{S}^1 \to \widetilde{Q}_\rho \to Q$. In [10, Section 1] the obstruction of the existence of extensions of representations $\rho\colon A \to U(V_\rho)$ was studied for the case of finite groups and the obstruction was explicitly described in terms of cocycles.

2.2 Twisted equivariant K–theory

Next we recall the definition of twisted equivariant K–theory that we will use throughout this article. For this suppose that Q is a compact Lie group and let

$$1 \to \mathbb{S}^1 \to \widetilde{Q} \xrightarrow{\tau} Q \to 1$$

be an $\mathbb{S}^1$–central extension of Q. Let X be a Q–space and endow it with the action of $\widetilde{Q}$ induced by the Q–action. Consider the set of isomorphism classes of $\widetilde{Q}$–vector bundles $p\colon E \to X$ on which the elements $z \in \mathbb{S}^1$ act by the scalar multiplication of z^{-1}; denote this set by ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. The set ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$ is a semigroup under direct sum of vector bundles and we define the twisted equivariant K–group ${}^{\widetilde{Q}}K^0_Q(X)$ as the Grothendieck construction applied to ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. For $n > 0$ the twisted groups

${}^{\widetilde{Q}}K^n_Q(X)$ are defined as ${}^{\widetilde{Q}}\widetilde{K}^0_Q(\Sigma^n X_+)$. We call the groups ${}^{\widetilde{Q}}K^*_Q(X)$ the $\widetilde{Q}$–twisted Q–equivariant K–theory groups of X. Notice that ${}^{\widetilde{Q}}K^*_Q(X)$ is naturally a module over $R(Q)$. The twisted groups ${}^{\widetilde{Q}}K^0_Q(X)$ can alternatively be defined as follows. The action of $\mathbb{S}^1$ on X obtained by restricting the $\widetilde{Q}$–action is trivial. Therefore, we obtain a natural map $K^0_{\widetilde{Q}}(X) \to K^0_{\mathbb{S}^1}(X) \cong K^0(X) \otimes R(\mathbb{S}^1)$. Composing this with the restriction map $K^0(X) \otimes R(\mathbb{S}^1) \to R(\mathbb{S}^1)$ we obtain a natural map $K^0_{\widetilde{Q}}(X) \to R(\mathbb{S}^1)$ and ${}^{\widetilde{Q}}K^0_Q(X)$ can also be defined as the inverse image of the subgroup generated by the $\mathbb{S}^1$–representations on which a scalar z acts by multiplication of z^{-1}. This description can also be used to define ${}^{\widetilde{Q}}K^n_Q(X)$. The cohomology class that classifies the twist is the image of the class $[\widetilde{Q}] \in H^3(BQ, \mathbb{Z})$ that corresponds to the central extension $\widetilde{Q}$ under the canonical map $H^3_Q(*, \mathbb{Z}) = H^3(BQ, \mathbb{Z}) \to H^3_Q(X, \mathbb{Z})$.

Remark 2.4 In the literature it is more common to encounter a different (but equivalent) definition of this twisted form of equivariant K–theory (see [1, Definition 7.1] for example). Suppose that we are given a central extension $1 \to \mathbb{S}^1 \to \widetilde{Q} \xrightarrow{\tau} Q \to 1$. We can also consider the set ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$ of isomorphism classes of $\widetilde{Q}$–vector bundles over $p\colon E \to X$ on which an element $z \in \mathbb{S}^1$ acts by the scalar multiplication of z. The set ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$ is also a semigroup under direct sum of vector bundles and we can also define a twisted form of K–theory, which we will denote by ${}^{\widetilde{Q}^+}K^0_Q(X)$, as the Grothendieck construction applied to ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$. For $n > 0$ the twisted groups ${}^{\widetilde{Q}^+}K^n_Q(X)$ can be defined in a similar way as above. These two twisted forms of equivariant K–theory are naturally isomorphic, as we show next. By definition it suffices to prove the case $n = 0$. Let $p\colon E \to X$ be a $\widetilde{Q}$–vector bundle such that a central element $z \in \mathbb{S}^1$ acts by the scalar multiplication of z^{-1}, so that $[E] \in {}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. Let $\mathrm{Hom}(E, \underline{\mathbb{C}})$ be the vector bundle dual to E, where $\underline{\mathbb{C}}$ denotes the trivial $\widetilde{Q}$–vector bundle $\pi_1\colon X \times \mathbb{C} \to X$. If $\phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_x$ and $q \in \widetilde{Q}$, then the action of q on ϕ is the element $q \cdot \phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_{qx}$ defined by $(q \cdot \phi)(v) = \phi(q^{-1} \cdot v)$ for every $v \in E_{qx}$. With this action $\mathrm{Hom}(E, \underline{\mathbb{C}})$ is a $\widetilde{Q}$–vector bundle. If $z \in \mathbb{S}^1$ is a central element and $\phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_x$, then as the action of z in E is given by scalar multiplication of z^{-1}, we have

$$(z \cdot \phi)(v) = \phi(z^{-1} \cdot v) = \phi(zv) = z\phi(v)$$

for every $v \in E_x$. This shows that $\mathrm{Hom}(E, \underline{\mathbb{C}})$ is a $\widetilde{Q}$–equivariant vector bundle on which the central factor $\mathbb{S}^1$ acts by multiplication of scalars on the fibers, so that $[\mathrm{Hom}(E, \underline{\mathbb{C}})] \in {}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$. The assignment

$${}^{\widetilde{Q}}\mathrm{Vec}_Q(X) \xrightarrow{\cong} {}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X), \quad [E] \mapsto [\mathrm{Hom}(E, \underline{\mathbb{C}})],$$

is an isomorphism of semigroups. After applying the Grothendieck construction we obtain an isomorphism ${}^{\widetilde{Q}}K^0_Q(X) \xrightarrow{\cong} {}^{\widetilde{Q}^+}K^0_Q(X)$. Throughout this article we will work with the twisted form of equivariant K–theory constructed using vector bundles on which the elements of the central factor $\mathbb{S}^1$ act by multiplication of their inverse; these are the bundles that appear naturally in our work.

3 Equivariant K–theory with prescribed fibers

The goal of this section is to generalize the decomposition of G–equivariant K–theory obtained in [10, Theorem 3.2] to the case of compact Lie groups.

To start, assume that G is a compact Lie group and let A be a normal subgroup of G, so that we have an extension of compact Lie groups

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1,$$

where $Q = G/A$. Let G act on a compact space X in such a way that A acts trivially on X. Assume that $p\colon E \to X$ is a G–equivariant complex vector bundle. We can give E a Hermitian metric that is invariant under the action of G; in particular, this metric is A–invariant. If we see $p\colon E \to X$ as an A–vector bundle then, as the action of A on X is trivial, by [20, Proposition 2.2] we have a natural isomorphism of A–vector bundles

$$\beta\colon \bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E) \xrightarrow{\cong} E, \quad v \otimes f \mapsto f(v).$$

In the above equation $\mathrm{Irr}(A)$ denotes the set of isomorphism classes of complex irreducible A–representations and, if $\tau\colon A \to U(V_\tau)$ is an irreducible A–representation, then $\mathbb{V}_\tau$ denotes the trivial A–vector bundle $\pi_1\colon X \times V_\tau \to X$. The decomposition of the vector bundle E provided above is a decomposition as an A–equivariant bundle and not as a G–equivariant bundle. Furthermore, the summands $\mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ that appear in this decomposition do not in general have the structure of a G–vector bundle in such a way that the map β is G–equivariant. Following [10], we have the next definition.

Definition 3.1 Suppose that $\rho\colon A \to U(V_\rho)$ is a complex irreducible representation and that $1 \to A \to G \to Q \to 1$ is a short exact sequence of compact Lie groups. A (G,ρ)–equivariant vector bundle over X is a G–vector bundle $p\colon E \to X$ such that the map

$$\beta\colon \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E, \quad v \otimes f \mapsto f(v),$$

is an isomorphism of A–vector bundles.

A (G,ρ)–equivariant vector bundle is a G–equivariant vector bundle $p\colon E\to X$ such that for every $x\in X$ the A–representation E_x is isomorphic to a direct sum of copies of the representation ρ. Notice that if $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle then for every $g\in G$ we have $g\cdot\rho\cong\rho$, so that (G,ρ)–equivariant vector bundles can only exist when this happens. We can define a direct summand of the equivariant K–theory using (G,ρ)–equivariant vector bundles. For this let $\mathrm{Vec}_{G,\rho}(X)$ denote the set of isomorphism classes of (G,ρ)–equivariant vector bundles, where two (G,ρ)–equivariant vector bundles are isomorphic if they are isomorphic as G–vector bundles. Notice that if E_1 and E_2 are two (G,ρ)–equivariant vector bundles then so is the direct sum $E_1\oplus E_2$. Therefore, $\mathrm{Vec}_{G,\rho}(X)$ is a semigroup.

Definition 3.2 Assume that G acts on a compact space X in such a way that A acts trivially on X. We define $K^0_{G,\rho}(X)$, the (G,ρ)–equivariant K–theory of X, as the Grothendieck construction applied to $\mathrm{Vec}_{G,\rho}(X)$. For $n>0$ the group $K^n_{G,\rho}(X)$ is defined as $\widetilde{K}^0_{G,\rho}(\Sigma^n X_+)$, where as usual X_+ denotes the space X with an added basepoint.

Following [10] we relate the (G,ρ)–equivariant K–theory of X with a suitable twisted form of equivariant K–theory as defined in Section 2.2. For this suppose that $\rho\colon A\to U(V_\rho)$ is an irreducible representation and let $f\colon G\to \mathrm{PU}(V_\rho)$ be the homomorphism associated to ρ as constructed in Lemma 2.1. Consider $\widetilde{G}_\rho=f^*U(V_\rho)$ and $\widetilde{Q}_\rho:=\widetilde{G}_\rho/\tilde{\iota}(A)$, so that we have a commutative diagram of central extensions as in diagram (2).

Theorem 3.3 *Let X be a G–space such that A acts trivially on X. Assume that $g\cdot\rho\cong\rho$ for every $g\in G$. If $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle, then $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ has the structure of a $\widetilde{Q}_\rho$–vector bundle on which the elements of the central factor $\mathbb{S}^1$ act by multiplication by their inverse. Moreover, the assignment*

$$[E]\mapsto[\mathrm{Hom}_A(\mathbb{V}_\rho,E)]$$

defines a natural one-to-one correspondence between isomorphism classes of (G,ρ)–equivariant vector bundles over X and isomorphism classes of $\widetilde{Q}_\rho$–equivariant vector bundles over X for which the elements of the central $\mathbb{S}^1$ act by multiplication of their inverse.

Proof Suppose $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle. Then $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ is a complex vector bundle over X. Next we give $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ an action of $\widetilde{G}_\rho$ on which $\tilde{\iota}(A)$ acts trivially.

Take $\phi \in \mathrm{Hom}_A(\mathbb{V}_\rho, E)_x$ and $\widetilde{g} \in \widetilde{G}_\rho$, and define $\widetilde{g} \bullet \phi \in \mathrm{Hom}_A(\mathbb{V}_\rho, E)_{g \cdot x}$ by

$$(\widetilde{g} \bullet \phi)(v) = g\phi(\widetilde{f}(\widetilde{g})^{-1} v),$$

where $\widetilde{g}$ projects to g in G and $\widetilde{f}\colon \widetilde{G}_\rho \to U(V_\rho)$ is the homomorphism defined in diagram (1). The action is a composition of continuous maps and therefore it is continuous. It is straightforward to check that it is a homomorphism.

Now let us take $a \in A$ and consider the action of $\tilde{\iota}(a)$ on ϕ. In this case we have

$$(\tilde{\iota}(a) \bullet \phi)(v) = a\phi\big(\widetilde{f}(\tilde{\iota}(a))^{-1} v\big) = a\phi(\rho(a)^{-1} v) = \phi(v),$$

which implies that $\tilde{\iota}(a) \bullet \phi = \phi$. Hence, the action of $\tilde{\iota}(A)$ on $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is trivial and therefore there is an induced action of $\widetilde{Q}_\rho = \widetilde{G}_\rho / \tilde{\iota}(A)$ on $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ compatible with the action of Q on X.

Now, if $\lambda \in \ker(\widetilde{G}_\rho \to G)$, the action becomes

$$(\lambda \bullet \phi)(v) = \phi(\widetilde{f}(\lambda)^{-1} v) = \lambda^{-1}\phi(v),$$

which implies that $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a $\widetilde{Q}_\rho$–equivariant bundle where the elements of $\mathbb{S}^1$ act by multiplication by their inverse.

Now let us take a $\widetilde{Q}_\rho$–equivariant bundle $F \to X$ where the elements of $\mathbb{S}^1$ act by multiplication by their inverse. Consider the vector bundle $\mathbb{V}_\rho \otimes F$ and define a $\widetilde{G}_\rho$–action in the following way: for $\widetilde{g} \in \widetilde{G}_\rho$ and $v \otimes e \in \mathbb{V}_\rho \otimes F$, let the action be

$$\widetilde{g} \cdot (v \otimes e) := (\widetilde{f}(\widetilde{g})v) \otimes (\widetilde{\pi}(\widetilde{g}) \cdot e),$$

where $\widetilde{\pi}\colon \widetilde{G}_\rho \to \widetilde{Q}_\rho$ is the homomorphism induced by $\pi\colon G \to Q$. This is clearly a continuous $\widetilde{G}_\rho$–action and for $\lambda \in \ker(\widetilde{G}_\rho \to G)$ we have that

$$\lambda \cdot (v \otimes e) := (\widetilde{f}(\lambda)v) \otimes (\widetilde{\pi}(\lambda) \cdot e) = \lambda v \otimes \lambda \cdot e = \lambda v \otimes \lambda^{-1} e = v \otimes e,$$

which implies that the action factors through $G = \widetilde{G}_\rho / \mathbb{S}^1$.

Let us see now what the action looks like once restricted to A. Take $a \in A$ and consider the element $\tilde{\iota}(a) \in \widetilde{G}_\rho$. The action of a on $v \otimes e$ becomes

$$\tilde{\iota}(a) \cdot (v \otimes e) = \big(\widetilde{f}(\tilde{\iota}(a))v\big) \otimes \big(\widetilde{\pi}(\tilde{\iota}(a)) \cdot e\big) = \rho(a)v \otimes e,$$

which implies that the action of A on the fibers of $\mathbb{V}_\rho \otimes F$ is determined by the representation ρ. Hence, $\mathbb{V}_\rho \otimes F$ is a (G, ρ)–equivariant vector bundle.

If E is a (G,ρ)–equivariant vector bundle then $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is also a (G,ρ)–equivariant vector bundle and the canonical map

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E, \quad v \otimes \phi \mapsto \phi(v),$$

is by definition an isomorphism of vector bundles. Note that the map is moreover G–equivariant: for $\widetilde{g} \in \widetilde{G}_\rho$ which projects to $g \in G$ we have that

$$g \cdot (v \otimes \phi) = \widetilde{g} \cdot (v \otimes \phi) = (\widetilde{f}(\widetilde{g})v) \otimes (\widetilde{g} \bullet \phi),$$

whose evaluation becomes

$$(\widetilde{g} \bullet \phi)(\widetilde{f}(\widetilde{g})v) = g\phi(\widetilde{f}(\widetilde{g})^{-1}\widetilde{f}(\widetilde{g})v) = g\phi(v),$$

thus implying that the canonical evaluation map is a G–equivariant isomorphism.

Finally, if F is a $\widetilde{Q}_\rho$–equivariant bundle where the elements of $\mathbb{S}^1$ act by multiplication by their inverse, we may consider the canonical isomorphism of vector bundles

$$F \to \mathrm{Hom}_A(\mathbb{V}_\rho, \mathbb{V}_\rho \otimes F), \quad e \mapsto \phi_e \colon v \mapsto v \otimes e.$$

It is straightforward to check that it is moreover an isomorphism of $\widetilde{Q}_\rho$–equivariant vector bundles.

We conclude that the inverse map of the assignment $[E] \mapsto [\mathrm{Hom}_A(\mathbb{V}_\rho, E)]$ is precisely the map defined by the assignment $[F] \mapsto [\mathbb{V}_\rho \otimes F]$. The theorem follows. □

Theorem 3.3 provides the following identification of the (G,ρ)–equivariant K–groups of Definition 3.1 with the twisted groups ${}^{\widetilde{Q}_\rho}K^*_Q(X)$ defined in Section 2.2.

Corollary 3.4 *Let G be a compact Lie group and X be a compact G–space such that the normal subgroup A of G acts trivially on X. Assume furthermore that $\rho \colon A \to U(V_\rho)$ is an irreducible representation whose isomorphism class is fixed by G, that is, $g \cdot \rho \cong \rho$ for every $g \in G$. Then the homomorphism*

$$K^*_{G,\rho}(X) \xrightarrow{\cong} {}^{\widetilde{Q}_\rho}K^*_Q(X), \quad [E] \mapsto [\mathrm{Hom}_A(\mathbb{V}_\rho, E)],$$

is a natural isomorphism of $R(Q)$–modules. The inverse map is $F \mapsto \mathbb{V}_\rho \otimes F$.

Whenever the isomorphism class of the A–representation ρ is not fixed by the whole group G we need to be more careful. Define

$$G_\rho := \{g \in G \mid g \cdot \rho \cong \rho\} \quad \text{and} \quad Q_\rho := G_\rho/A$$

and call $\widetilde{G}_\rho$ and $\widetilde{Q}_\rho$ the $\mathbb{S}^1$–central extensions which measure the obstruction for the extension of ρ to G_ρ constructed in Proposition 2.2.

Now consider a G–vector bundle E over the compact G–space X on which A acts trivially. We know that as a A–vector bundle we have the isomorphism

$$\bigoplus_{\rho \in \mathrm{Irr}(A)} \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \xrightarrow{\cong} E \tag{3}$$

given by the evaluation, where ρ runs over the set of isomorphism classes of irreducible A–representations. Each of the vector bundles $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a G_ρ–vector bundle. However, it is not possible in general to provide each factor $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ with the structure of a G–vector bundle in such a way that the isomorphism (3) is an isomorphism of G–equivariant vector bundles as the G–action intertwines the vector bundles associated to irreducible A–representations which are related by the action of G. To illustrate this issue we explore the following example:

Example 3.5 Suppose that $G = D_8$ is the dihedral group generated by the elements a and b with relations $a^4 = b^2 = 1$ and $bab = a^{-1}$, and let $A = \langle a \rangle = \mathbb{Z}/4$, so that we have a short exact sequence

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

with $Q = G/A = \{1, \tau\} = \mathbb{Z}/2$, where $\tau = [b] \in Q$. Let $\rho\colon A \to \mathbb{S}^1$ be the representation given by $\rho(a) = e^{2\pi i/4} = i$. In this example we have $\mathrm{Irr}(\mathbb{Z}/4) = \{\mathbf{1}, \rho, \rho^2, \rho^3\}$ and the action of Q on $\mathrm{Irr}(\mathbb{Z}/4)$ is such that τ permutes the isomorphism classes of the representations ρ and ρ^3. Let $V_\rho := \mathbb{C}$ equipped with the A–representation ρ and $V_{\rho^3} := \mathbb{C}$ equipped with the A–representation ρ^3. Consider the balanced product $E := D_8 \times_A V_\rho$ seen as a D_8–equivariant vector bundle over $D_8/A = \{*, *'\}$. Note that, as a D_8–equivariant vector bundle, E is isomorphic to the bundle $V_\rho \sqcup V_{\rho^3}$, where the action of b maps V_ρ to V_{ρ^3} and vice versa using the explicit isomorphisms with $\mathbb{C}$. Here we see V_ρ as a bundle over $\{*\}$ and V_{ρ^3} as a bundle over $\{*'\}$. In this case we have

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \cong V_\rho \sqcup \{*'\},$$
$$\mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E) \cong \{*\} \sqcup V_{\rho^3}.$$

Hence,

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \oplus \mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E) \cong V_\rho \sqcup \{*'\} \oplus \{*\} \sqcup V_{\rho^3} \cong E$$

as D_8–equivariant vector bundles. However, the factors

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \quad \text{and} \quad \mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E)$$

do not possess a structure of a D_8–vector bundle that is compatible with the above isomorphism.

Suppose now that $p\colon E \to X$ is a G–vector bundle over the compact G–space X on which A acts trivially. As our next step we show that the factors in the decomposition described in formula (3) can be arranged in a suitable way to obtain a decomposition of E as a G–vector bundle. Choosing representatives $\{g_i\}_i$ for each class in G/G_ρ, we know that the image of the evaluation map

$$\bigoplus_i \mathbb{V}_{g_i\cdot\rho} \otimes \mathrm{Hom}_A(\mathbb{V}_{g_i\cdot\rho}, E) \to E$$

becomes a G–equivariant vector bundle. Notice that G/G_ρ is finite since G_ρ contains the connected component of the identity of G and G is compact; this follows from the fact that G is acting on the discrete set $\mathrm{Irr}(A)$. Now, in order to define the bundle above in a coordinate-free fashion, we need to promote the G_ρ–equivariant bundle $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ over X to a G–equivariant bundle over the same space X. This construction was called *multiplicative induction* in [3, Section 4] and here we will recall its properties.

Let G be a compact Lie group and H a closed subgroup. The restriction functor r^G_H from G–spaces to H–spaces which restricts the action to H has a left adjoint which maps a H–space Y to the G–space $G \times_H Y$, thus giving a homeomorphism

$$\mathrm{map}(G \times_H Y, X)^G \cong \mathrm{map}(Y, r^G_H X)^H$$

for any G–space X. This left adjoint is additive, but in general it is not multiplicative. A right adjoint for the restriction functor r^G_H can be defined on an H–space Y as the G–space of H–equivariant maps from G to Y,

$$m^G_H(Y) := \mathrm{map}(G, Y)^H,$$

where G is considered as an H–space via left multiplication. The G–action on $m^G_H(Y)$ is given by $(g\cdot f)(k) := f(kg)$. A map of H–spaces $\phi\colon Y_1 \to Y_2$ induces a map of G–spaces $m^G_H(\phi)\colon m^G_H(Y_1) \to m^G_H(Y_2)$ by composition. In this case there is a homeomorphism

$$\mathrm{map}(r^G_H(X), Y)^H \cong \mathrm{map}(X, m^G_H(Y))^G$$

for any G–space X and any H–space Y. The maps are defined by

$$\begin{aligned}\mathrm{map}(X, m^G_H(Y))^G &\to \mathrm{map}(r^G_H(X), Y)^H, & F &\mapsto (x \mapsto F(x)(1_G)),\\ \mathrm{map}(r^G_H(X), Y)^H &\to \mathrm{map}(X, m^G_H(Y))^G, & f &\mapsto m^G_H(f)\circ p^G_H,\end{aligned}$$

where $p^G_H\colon X \to m^G_H(r^G_H(X))$ is defined by the equation

$$(p^G_H(x))(g) = gx$$

and is the unit of the adjunction.

The functor m^G_H is called multiplicative because

$$m^G_H(Y_1 \times Y_2) \cong m^G_H(Y_1) \times m^G_H(Y_2)$$

for any H–spaces Y_1 and Y_2.

Note also that the space $m^G_H(Y)$ is homeomorphic to the space $\Gamma(G \times_H Y, G/H)$ of sections of the projection map $G \times_H Y \to G/H$, endowed with the G–action given by $(g \cdot \sigma)(kH) = g\sigma(g^{-1}kH)$, where σ is any section. In the case that G/H is finite the space $m^G_H(Y)$ is homeomorphic to the product of $|G : H|$ copies of Y.

Let us now consider the G–equivariant bundle

$$m^G_{G_\rho}(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)) \to m^G_{G_\rho}(X)$$

and construct the pullback bundle

$$(p^G_{G_\rho})^*\big(m^G_{G_\rho}(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big) \to X.$$

The G–equivariant bundle $(p^G_{G_\rho})^*\big(m^G_{G_\rho}(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big)$ is endowed with a natural G–equivariant map to E, defined by the restriction of the natural map

$$\mathrm{map}(G, \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))^{G_\rho} \to E, \quad \phi \mapsto \mathrm{ev}(\phi(1_G)),$$

induced by the evaluation map ev: $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E$, $\mathrm{ev}(v \otimes f) = f(v)$. Therefore, we have constructed the desired G–vector bundle over X.

Theorem 3.6 *Let G be a compact Lie group and E a G–equivariant complex vector bundle over the compact G–space X. If the action on X by the normal subgroup A of G is trivial, then the following decomposition formula is an isomorphism of G–equivariant bundles:*

$$\bigoplus_{\rho \in G\backslash \mathrm{Irr}(A)} (p^G_{G_\rho})^*\big(m^G_{G_\rho}(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big) \xrightarrow{\cong} E,$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations.

Note that when the Lie group G is connected, then $G_\rho = G$ for all ρ and the decomposition simplifies to the isomorphism

$$\bigoplus_{\rho\in \mathrm{Irr}(A)} \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \xrightarrow{\cong} E$$

of G–equivariant bundles. This is also the case, for example, when G is abelian.

The decomposition formula of the equivariant vector bundle E induces an isomorphism in K–theory as follows:

Corollary 3.7 *Let G be a compact Lie group and X a G–space on which the normal subgroup A acts trivially. Then there is a natural isomorphism*

$$K^*_G(X) \cong \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} {}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X),$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations and $Q_\rho = G_\rho/A$.

Proof The isomorphism follows from the isomorphism

$$\bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} K^*_{G_\rho,\rho}(X) \xrightarrow{\cong} K^*_G(X), \qquad \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} E_\rho \mapsto \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} (p^G_{G_\rho})^*(m^G_{G_\rho} E_\rho),$$

and Corollary 3.4. □

4 The decomposition at the level of classifying spaces

In this section we will write the results of the previous section at the level of the classifying space of G–equivariant complex vector bundles. This will show us how the spectrum of G–equivariant K–theory decomposes at the fixed-point set of each subgroup.

The universal bundles for twisted equivariant K–theory associated to central extensions of the group G are constructed as follows. Consider a central extension

$$1 \to \mathbb{S}^1 \to \widetilde{G} \to G \to 1$$

of the compact Lie group G by $\mathbb{S}^1$. Let $\widetilde{\boldsymbol{C}}^\infty$ denote the direct sum of countably many copies of all irreducible $\widetilde{G}$–representations on which elements in $\mathbb{S}^1 = \ker(\widetilde{G} \to G)$ act by scalar multiplication of their inverse. Let ${}^{\widetilde{G}}B_G U(n)$ denote the Grassmannian of n–dimensional complex subspaces of $\widetilde{\boldsymbol{C}}^\infty$ and denote by ${}^{\widetilde{G}}\gamma_G U(n)$ the universal n–plane bundle over ${}^{\widetilde{G}}B_G U(n)$. The complex vector bundle

$$\mathbb{C}^n \to {}^{\widetilde{G}}\gamma_G U(n) \to {}^{\widetilde{G}}B_G U(n) \tag{4}$$

is the universal $\widetilde{G}$–twisted G–equivariant complex vector bundle of rank n, and therefore for a finite G–CW complex X we have

$${}^{\widetilde{G}}\mathrm{Vec}^n_G(X) \cong [X, {}^{\widetilde{G}}B_G U(n)]_G.$$

Note that since $\mathbb{S}^1$ acts by multiplication of the inverse of scalars, its action on the Grassmannian of complex n–planes is trivial, and therefore the $\widetilde{G}$–action on ${}^{\widetilde{G}}B_G U(n)$ reduces to a G–action. If $V \subset \widetilde{\boldsymbol{C}}^\infty$ is a finite-dimensional complex $\widetilde{G}$–subrepresentation, then there is a map

$${}^{\widetilde{G}}\gamma_G U(n) \oplus \mathbb{V} \to {}^{\widetilde{G}}\gamma_G U(n + |V|)$$

which induces a map $\iota_V\colon {}^{\widetilde{G}}B_G U(n) \to {}^{\widetilde{G}}B_G U(n + |V|)$ at the level of the classifying spaces. The colimit

$${}^{\widetilde{G}}B_G U := \varinjlim_{V \subset \widetilde{\boldsymbol{C}}^\infty} \bigsqcup_{n \geq 0} {}^{\widetilde{G}}B_G U(n) \tag{5}$$

is the classifying space for reduced $\widetilde{G}$–twisted G–equivariant complex K–theory,

$${}^{\widetilde{G}}\widetilde{K}^0_G(X) \cong [X, {}^{\widetilde{G}}B_G U]_G$$

for X a finite G–CW complex.

Whenever the extension is trivial $\widetilde{G} \cong \mathbb{S}^1 \times G$, the spaces ${}^{\mathbb{S}^1\times G}B_G U(n)$ classify G–equivariant $U(n)$–principal bundles, and therefore we may write $B_G U(n) := {}^{\mathbb{S}^1\times G}B_G U(n)$ and $B_G U := {}^{\mathbb{S}^1\times G}B_G U$, thus having that, for X a compact G–space,

$$\widetilde{K}^0_G(X) \cong [X, B_G U]_G.$$

Furthermore, the vector bundles $\gamma_G U(n) := {}^{\mathbb{S}^1\times G}\gamma_G U(n)$ are the universal G–equivariant complex vector bundles of rank n.

Suppose now that A is a closed subgroup of the compact Lie group G. Consider the fixed-point set $B_G U(n)^A$ and the restriction $\gamma_G U(n)|_{B_G U(n)^A}$ of the universal vector bundle. Denote by N_A the normalizer of A in G and by $W_A = N_A/A$ the quotient. Therefore, $\gamma_G U(n)|_{B_G U(n)^A} \to B_G U(n)^A$ is an N_A–equivariant vector bundle such that A acts trivially on the base space. In this way we are in the situation of the previous section for the short exact sequence $1 \to A \to N_A \to W_A \to 1$. Take $\rho \in \mathrm{Irr}(A)$. By Theorem 3.3 the bundle $\mathrm{Hom}_A(\mathbb{V}_\rho, \gamma_G U(n)|_{B_G U(n)^A})$ is a $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundle, but since the space $B_G U(n)^A$ is not necessarily connected, it may not have constant rank. Therefore, in order to construct a universal N_A–equivariant complex bundle over spaces with trivial A–actions using universal $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundles, we need to work with bundles of all ranks. We claim the following result:

Theorem 4.1 *There is a W_A–equivariant homotopy equivalence*

$$\bigsqcup_{n=0}^{\infty} B_G U(n)^A \simeq \prod_{\rho \in W_A \backslash \mathrm{Irr}(A)} m^{W_A}_{(W_A)_\rho}\Big(\bigsqcup_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\Big).$$

The stable version is

$$B_G U^A \simeq \prod_{\rho \in W_A \backslash \mathrm{Irr}(A)} m^{W_A}_{(W_A)_\rho}({}^{(\widetilde{W}_A)_\rho} B U_{(W_A)_\rho} U)$$

as W_A–spaces.

Proof To start notice that by [17, Chapter V, Lemma 4.7 and Chapter VII, Theorem 2.4] it follows that $B_G U(n)^A$ classifies N_A–equivariant complex vector bundles of rank n over A–trivial N_A–spaces. Therefore, $\bigsqcup_{n=0}^{\infty} B_G U(n)^A$ classifies N_A–equivariant complex bundles (of any rank) over spaces with trivial A–actions. The theorem will follow by Theorem 3.6 since both sides classify N_A–equivariant complex bundles over spaces with trivial A–actions. Let us define the maps.

Since $\mathrm{Hom}_A(\mathbb{V}_\rho, \gamma_G U(n)|_{B_G U(n)^A})$ is a $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundle, there is a $(W_A)_\rho$–equivariant classifying map

$$f_\rho\colon B_G U(n)^A \to \bigsqcup_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)$$

which induces a W_A–equivariant map

$$m^{W_A}_{(W_A)_\rho}(f_\rho) \circ p^{W_A}_{(W_A)_\rho}\colon B_G U(n)^A \to m^{W_A}_{(W_A)_\rho}\Big(\bigsqcup_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\Big).$$

This constructs the map from left to right.

For the map from the right to the left, we know from Theorem 3.3 that

$$\coprod_{n_\rho=0}^{\infty} \mathbb{V}_\rho \otimes {}^{(\widetilde{W}_A)_\rho}\gamma_{W_A} U(n_\rho)$$

is an $(N_A)_\rho$–equivariant complex bundle and

$$m_{(N_A)_\rho}^{N_A}\left(\coprod_{n_\rho=0}^{\infty} \mathbb{V}_\rho \otimes {}^{(\widetilde{W}_A)_\rho}\gamma_{W_A} U(n_\rho)\right)$$

is an N_A–equivariant complex bundle. The product over $\rho \in W_A \backslash \operatorname{Irr}(A)$ is also an N_A–equivariant complex bundle and therefore there is a classifying map

$$\prod_{\rho \in W_A \backslash \operatorname{Irr}(A)} m_{(W_A)_\rho}^{W_A}\left(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\right) \to \coprod_{n=0}^{\infty} B_G U(n)^A.$$

The homotopy equivalence follows from Theorem 3.6. The homotopy equivalence of the stable version follows from Corollary 3.7. □

Remark 4.2 Distributing the product over union, we obtain a homeomorphism

$$\prod_{\rho \in W_A \backslash \operatorname{Irr}(A)} m_{(W_A)_\rho}^{W_A}\left(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\right) \cong \coprod_{n=0}^{\infty} \coprod_{\sum n_\rho |\rho| = n} \prod_{\rho \in \operatorname{Irr}(A)} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho),$$

where $|\rho|$ denotes the complex dimension of the representation ρ. We note that the expression on the right-hand side is not canonically a W_A–space, therefore we induce the W_A–action on the right-hand side from the one of the expression on the left.

Let $\mathcal{P}(n, A)$ be the set of arrays $P = (n_\rho)_{\rho \in \operatorname{Irr}(A)}$ such that

$$\sum_{\rho \in \operatorname{Irr}(A)} n_\rho |\rho| = n.$$

Restricting to G–equivariant complex bundles of rank n over G–spaces, one gets the W_A–homotopy equivalence

$$B_G U(n)^A \simeq \coprod_{P \in \mathcal{P}(n,A)} \prod_{\rho \in \operatorname{Irr}(A)} ({}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)),$$

where W_A acts on $\mathcal{P}(n, A)$ by permuting the arrays of numbers according to its action on $\operatorname{Irr}(A)$, and the isotropy subgroups $(W_A)_\rho$ act on the appropriate coordinate space ${}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)$.

Whenever G is abelian G acts trivially on $\mathrm{Irr}(A)$ and therefore we recover the homotopy equivalence of G/A–spaces

$$B_G U(n)^A \simeq \bigsqcup_{P\in\mathcal{P}(n,A)} \prod_{\rho\in\mathrm{Irr}(A)} B_{G/A} U(n_\rho)$$

that appeared in [11, Proposition 4.3]. Note that in this case $\sum_\rho n_\rho = n$ since all irreducible representations of A are 1–dimensional.

Whenever the normalizer of A is connected, the right-hand side simplifies as

$$B_G U(n)^A \simeq \bigsqcup_{P\in\mathcal{P}(n,A)} \prod_{\rho\in\mathrm{Irr}(A)} {}^{\widetilde{W}_{A\rho}} B_{W_A} U(n_\rho),$$

where $\widetilde{W}_{A\rho}$ is the $\mathbb{S}^1$–central extension of W_A that depends on ρ.

5 Equivariant unitary bordism

The decomposition of equivariant complex vector bundles on fixed-point sets carried out in the previous sections is a key ingredient in the calculation of equivariant unitary bordisms for families. Conner and Floyd in their monumental work on the study of the bordism groups [5; 6] introduced the use of families of subgroups in order to restrict the bordisms to manifolds whose isotropy groups lie in a prescribed family.

We will concentrate on the tangentially stably equivariant unitary bordism groups Ω_*^G, which will be called the geometric G–equivariant unitary bordisms. The explicit definition of these homology groups and their stable versions can be found in [12, Section 2; 11, Definition 3.1]. Let us recall the main ingredients. In all of what follows, G will be a compact Lie group.

Definition 5.1 Let M be a smooth G–manifold. A tangentially stably almost complex G–structure on M is a complex G–structure on $TM \oplus \underline{\mathbb{R}}^k$ for some $k \geq 0$, where $\underline{\mathbb{R}}^k$ denotes the trivial bundle $M \times \mathbb{R}^k$ over M with trivial G–action; that is, there exists a G–equivariant complex bundle ξ over M such that $TM \oplus \underline{\mathbb{R}}^k \cong \xi$ as G–equivariant real vector bundles. Two tangentially stably almost complex G–structures are identified if after stabilization with further G–trivial $\underline{\mathbb{C}}$ summands the structures become G–homotopic through complex G–structures.

With this definition, if H is a closed subgroup of G then the fixed points M^H also have a tangentially stably almost complex N_H–structure. Moreover, an N_H–tubular neighborhood around M^H in M possesses an N_H–complex structure by

[11, Proposition 3.2]; let us see why. The bundle $TM|_{M^H}$ contains $T(M^H)$ as a subbundle and

$$TM|_{M^H} = T(M^H) \oplus \nu(M^H, M),$$

where $\nu(M^H, M)$ is the normal bundle of M^H in M. Also $T(M^H) = (TM|_{M^H})^H$ is an N_H–equivariant real vector bundle.

Given the tangentially stably almost complex G–structure ξ, we have that $\xi|_{M^H}$ is a complex vector bundle over M^H with a complex vector subbundle ξ^H, and also

$$\xi^H \cong (TM \oplus \underline{\mathbb{R}}^k)^H = (TM|_{M^H})^H \oplus \underline{\mathbb{R}}^k = T(M^H) \oplus \underline{\mathbb{R}}^k.$$

Therefore, M^H has a tangentially stably almost complex N_H–structure.

Now, since

$$\xi|_{M^H} \cong TM|_{M^H} \oplus \underline{\mathbb{R}}^k = T(M^H) \oplus \nu(M^H, M) \oplus \underline{\mathbb{R}}^k,$$

the normal bundle $\nu(M^H, M)$ is isomorphic to the quotient bundle $\xi|_{M^H}/\xi^H$, which is an N_H–complex bundle on M^H, thus showing that the normal bundle of M^H on M possesses an N_H–complex structure.

Definition 5.2 For a cofibration of G–spaces $Y \to X$ the geometric G–equivariant unitary bordism groups $\Omega_n^G(X, Y)$ are defined as G–bordism classes of singular tangentially stably almost complex n–dimensional G–manifolds $(M^n, \partial M^n) \to (X, Y)$.

When G is trivial, a tangentially stably almost complex structure is the same as a normally stably almost complex structure and $\Omega_n^{\{1\}}(X, Y)$ is the usual unitary bordism of the pair (X, Y).

One way to study the equivariant bordism groups is through the study of the equivariant bordism groups of manifolds M whose isotropies lie in a fixed family of subgroups of G. This way of studying equivariant bordism groups was developed by Conner and Floyd [6, Section 5] and it is currently one of the most useful techniques for calculating the equivariant bordism groups.

A family of subgroups $\mathcal{F}$ of G is a set (possibly empty) consisting of subgroups of G which is closed under taking subgroups and under conjugation. Denote by $E\mathcal{F}$ the classifying space for the family $E\mathcal{F}$, a G–space which is terminal in the category of $\mathcal{F}$–numerable G–spaces [9, Section 1, Theorem 6.6], and which is characterized by the following properties on fixed-point sets: $E\mathcal{F}^H \simeq *$ if $H \in \mathcal{F}$ and $E\mathcal{F}^H = \varnothing$ if $H \notin \mathcal{F}$.

Given families of subgroups $\mathcal{F}' \subset \mathcal{F}$ of G the induced map $E\mathcal{F}' \to E\mathcal{F}$ can be constructed so that it is a G–cofibration.

Following tom Dieck [8, page 310], we can define equivariant unitary bordism groups for families $\Omega_*^G[\mathcal{F},\mathcal{F}']$ as follows. Given a G–space X, $\Omega_*^G[\mathcal{F},\mathcal{F}'](X)$ is defined as

$$\Omega_*^G[\mathcal{F},\mathcal{F}'](X) := \Omega_*^G(X \times E\mathcal{F}, X \times E\mathcal{F}').$$

Alternatively, we may define the geometric G–equivariant unitary bordism groups $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X,A)$ in a geometric way, as was done in [21, Section 2]. We recall the definition of the absolute unitary bordism groups $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$ for completeness.

An $(\mathcal{F},\mathcal{F}')$*–free geometric unitary bordism element* of X is an equivalence class of a triple $(M,\partial M,f)$, where M is an n–dimensional G–manifold endowed with tangentially stably almost complex G–structure which is moreover $\mathcal{F}$–free; this means that all isotropy groups $G_m = \{g \in G \mid gm = m\}$ for $m \in M$ belong to $\mathcal{F}$, ∂M is $\mathcal{F}'$–free and $f\colon M \to X$ is a G–equivariant map. Two triples $(M,\partial M,f)$ and $(M',\partial M',f')$ are equivalent if there exists a G–manifold V that is $\mathcal{F}$–free such that $\partial V = M \cup M' \cup V^+$, and $M \cap V^+ = \partial M$, $M' \cap V^+ = \partial M'$, $M \cap M' = \varnothing$, $V^+ \cap (M \cup M') = \partial V^+$ and V^+ is $\mathcal{F}'$–free, together with a G–equivariant map $F\colon V \to X$ that restricts to f on M and to f' on M'.

Definition 5.3 The set of equivalence classes of $(\mathcal{F},\mathcal{F}')$–free geometric unitary bordism elements of X, consisting of classes $(M,\partial M,f)$ where the dimension of M is n, and under the operation of disjoint union, forms an abelian group, denoted by $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$. We refer to these groups as the geometric unitary bordisms of X restricted to the pair of families $\mathcal{F}' \subset \mathcal{F}$. The equivalence class corresponding to the triple $(M,\partial M,f)$ will be denoted by $[M,\partial M,f]$.

Notice that if N is a stably almost complex closed manifold, we can define

$$[N]\cdot[M,\partial M,f] := [N \times M, N \times \partial M, f \circ \pi_M],$$

thus making $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$ a module over the unitary bordism ring Ω_*.

The covariant functor $\Omega_*^G\{\mathcal{F},\mathcal{F}'\}$ defines a G–equivariant homology theory; see [21, Proposition 2.1]. A natural transformation $\mu\colon \Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X) \to \Omega_n^G[\mathcal{F},\mathcal{F}'](X)$ can be defined as in [8, Satz 3] in the following way. Suppose that $(M,\partial M,f)$ is a representative of an element in $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$. Since the inclusion $E\mathcal{F}' \subset E\mathcal{F}$ is a G–cofibration, ∂M is $\mathcal{F}'$–free and M is $\mathcal{F}$–free, there is a G–equivariant map $k\colon M \to E\mathcal{F}$ such that $k(\partial M) \subset E\mathcal{F}'$. The G–equivariant map

$$(f,k)\colon M \to X \times E\mathcal{F}, \quad m \mapsto (f(m), k(m)),$$

maps ∂M into $X \times E\mathcal{F}'$ and therefore (f,k) becomes an element in $\Omega_n^G[\mathcal{F},\mathcal{F}'](X)$.

Proposition 5.4 *The natural transformation*

$$\mu\colon \Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X) \to \Omega_n^G[\mathcal{F},\mathcal{F}'](X), \quad [M,\partial M, f] \mapsto [(M,\partial M,(f,k))],$$

is an isomorphism.

The proof of this proposition follows the same lines as the one done by tom Dieck [8, Satz 3] in the case of an equivariant unoriented bordism. We will not reproduce the proof here.

The long exact sequence of the pair $(X \times E\mathcal{F}', X \times E\mathcal{F})$ becomes

$$\cdots \to \Omega_n^G\{\mathcal{F}'\}(X) \to \Omega_n^G\{\mathcal{F}\}(X) \to \Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X) \to \Omega_{n-1}^G\{\mathcal{F}'\}(X) \to \cdots,$$

where $\Omega_n^G\{\mathcal{F}\}(X) = \Omega_n^G\{\mathcal{F},\varnothing\}(X)$ is the bordism group of $\mathcal{F}$–free tangentially stably almost complex closed manifolds with an equivariant map to X. Note that for finite G, if $\mathcal{F} = \{e\}$ then $\Omega_n^G\{\{1\}\}(X)$ is the bordism group of tangentially stably complex closed manifolds with a free G–action and an equivariant map to X, which can be identified with the usual unitary bordism group $\Omega_n(EG \times_G X)$.

Similarly, for three families of representations $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$, we have the corresponding long exact sequence of a triple

$$\cdots \to \Omega_n^G\{\mathcal{F}_2,\mathcal{F}_1\}(X) \to \Omega_n^G\{\mathcal{F}_3,\mathcal{F}_1\}(X) \to \Omega_n^G\{\mathcal{F}_3,\mathcal{F}_2\}(X) \\ \to \Omega_{n-1}^G\{\mathcal{F}_2,\mathcal{F}_1\}(X) \to \cdots.$$

Following the same argument as in [6, Lemma 5.2] we can obtain the next lemma.

Lemma 5.5 *Let $(M^n,\partial M^n, f)$ be a $(\mathcal{F},\mathcal{F}')$–free geometric unitary bordism element of X and W^n a compact manifold with boundary regularly embedded in the interior of M^n and invariant under the G–action. If $G_m \in \mathcal{F}'$ for all $m \in M^n \backslash W^n$, then $[M^n,\partial M^n, f] = [W^n,\partial W_n, f|_{W^n}]$ in $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$.*

A pair of families $\mathcal{F}' \subset \mathcal{F}$ of subgroups of G is said to be an *adjacent pair of families of groups* if $\mathcal{F} \backslash \mathcal{F}' = (A)$, where A is a subgroup of G and (A) is the set of subgroups conjugate to A in G. We then say that $\mathcal{F}$ and $\mathcal{F}'$ differ by A. Notice that if A is a normal subgroup of G, then a pair of families $\mathcal{F}$ and $\mathcal{F}$ differ by A precisely if $\mathcal{F} = \mathcal{F}' \sqcup \{A\}$. Moreover, in this case, if M is a G–manifold such that $G_m \in \mathcal{F}$ for every $m \in M$, then the fixed-point set M^A has a free action of G/A.

Building on the notation of Theorem 4.1 and Remark 4.2, we denote by $\overline{\mathcal{P}}(n, A)$ the set of arrays $\overline{P} = (n_\rho)_{\rho \in \mathrm{Irr}(A),\, \rho \neq 1}$ of nonnegative integers, where the number n_1 associated to the trivial representation is not considered, such that

$$\sum_{\rho \in \mathrm{Irr}(A),\, \rho \neq 1} n_\rho |\rho| = n.$$

In the above equation n_ρ is a nonnegative integer and $|\rho|$ denotes the complex dimension of the representation ρ. Suppose now that A is a closed and normal subgroup of G. For any such partition $\overline{P}$ we define the space

$$B_{G/A} U(\overline{P}) := \prod_{\rho \in \mathrm{Irr}(A), \rho \neq 1} {}^{\widetilde{(G/A)}_\rho} B_{(G/A)_\rho} U(n_\rho)$$

with the G/A–action induced by the homeomorphism shown in Remark 4.2.

As an application of Theorem 4.1 we obtain the following decomposition formula for the geometric G–equivariant unitary equivariant bordism groups. This decomposition is well known for the case of a compact abelian Lie group but is new for the case of nonabelian groups. Since we follow the same line of argument as in the abelian case we only sketch part of its proof.

Theorem 5.6 *Suppose that G is a compact Lie group and let A be a closed normal subgroup of G. If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of G differing by A, then*

$$\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0 \leq 2k \leq n - \dim(G/A)} \Omega_{n-2k}^{G/A}\{\{1\}\}\Big(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k,A)} B_{G/A} U(\overline{P})\Big),$$

where $\{1\}$ is the family of subgroups of G/A which only contains the trivial group.

Proof We are going to define an isomorphism Φ between these groups. Suppose that $f\colon M \to X$ is an $(\mathcal{F}, \mathcal{F}')$–free geometric unitary bordism of X such that

$$[M, \partial M, f] \in \Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X).$$

Let $M^A = M^A_1 \cup \cdots \cup M^A_l$ be a decomposition of disjoint manifolds, where each M^A_j is an $n_1(j)$–dimensional manifold which is moreover connected. Using the G–equivariant tubular neighborhood theorem, we may find pairwise disjoint tubular neighborhoods U_j of M^A_j in M which are diffeomorphic to the G–manifolds $D(\nu_j)$ through the diffeomorphisms $\phi_j\colon U_j \xrightarrow{\cong} D(\nu_j)$, where $D(\nu_j)$ denotes the unit disk bundle of the G–equivariant normal bundle $\nu_j \to M^A_j$ of the inclusion $M^A_j \subset M$ for $j = 1, \dots, l$. By Lemma 5.5 we know that

$$[M, \partial M, f] = \sum_{j=1}^{l} [U_j, \partial U_j, f|_{U_j}] = \sum_{j=1}^{l} [D(\nu_j), S(\nu_j), f|_{U_j} \circ \phi_j^{-1}]$$

in $\Omega^G_n\{\mathcal{F}, \mathcal{F}'\}(X)$. The bundle $\nu_j \to M^A_j$ is a complex G–equivariant bundle with the property that the trivial A–representation does not appear on the fibers. Let $r_j = \operatorname{rank}_{\mathbb{C}}(\nu_j)$, so that $n = n^j_1 + 2r_j$ for each $j = 1, \dots, l$ with n^j_1 the dimension of M^A_j. By Theorem 4.1 we know that the bundle ν_j is classified by a G/A–equivariant map

$$\kappa_j\colon M^A_j \to \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}).$$

Now set $f_j\colon M^A_j \to X^A$ to be the restriction of f to M^A_j, and define the product map

$$(f_j, \kappa_j)\colon M^A_j \to X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}).$$

Notice that $0 \le \dim(G/A) \le n^j_1$ and thus the class $[M^A_j, \varnothing, (f_j, \kappa_j)]$ defines an element in $\Omega^{G/A}_{n-2r_j}\{\{1\}\}(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}))$ for $j = 1, \dots, l$. We define

$$\Phi([M, \partial M, f]) := \sum_{j=1}^{l} [M^A_j, \varnothing, (f_j, \kappa_j)].$$

We claim that map Φ is an isomorphism. To see that Φ is surjective, suppose that

$$\Big[Y, \varnothing, \varphi\colon Y \to X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})\Big] \in \Omega^{G/A}_{n-2k}\{\{1\}\}\Big(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})\Big).$$

Let $p\colon E \to Y$ be the G–equivariant complex vector bundle defined by the map $\pi_2 \circ \varphi\colon Y \to \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})$ as is given in Theorem 4.1. Since Y is a closed manifold with a free G/A–action and with a tangentially stably almost complex G/A–structure, the closed unit disk of the bundle $D(E)$ is an n–dimensional manifold endowed with a tangentially stably complex G–structure. Moreover, the boundary

$S(E)$ of $D(E)$ is $\mathcal{F}'$–free since the trivial A–representation does not appear on the fibers of E. Denoting by $\psi\colon D(E) \to X$ the composition of the maps

$$D(E) \to Y \to X^A \hookrightarrow X,$$

where the first is the projection on the base, the second is $\pi_1 \circ \varphi$ and the third is the inclusion, we see that the bordism class $[D(E),\, S(E),\, \psi\colon D(E) \to X]$ lives in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$ and, by construction,

$$\Phi([D(E),\, S(E),\, \psi\colon D(E) \to X]) = [Y, \varnothing, \varphi].$$

This proves the surjectivity of Φ. The injectivity of Φ can be proved in a similar way as in the case of a compact abelian Lie group using Theorem 4.1. □

To be able to extend the previous theorem to general subgroups that are not necessarily normal, we consider $G = N_A$ and extend N_A–bordisms to G–bordisms with the change of groups formula [11, Lemma 3.4]: For any subgroup H of a finite group G, we have an isomorphism

$$\begin{aligned} \Omega_*^H(X) &\xrightarrow{\cong} \Omega_*^G(G \times_H X), \\ [M,\, \partial M,\, f\colon M \to X] &\longmapsto [G \times_H M,\, \partial(G \times_H M),\, G \times_H f\colon G \times_H M \to G \times_H X]. \end{aligned}$$

Corollary 5.7 *If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of the finite group G differing by the subgroup A, then*

$$\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0 \le 2k \le n} \Omega_{n-2k}^{W_A}\{\{1\}\}\Big(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k,A)} B_{W_A} U(\overline{P})\Big),$$

where $\{1\}$ is the family of subgroups of W_A which only contains the trivial group.

Proof We just need to note that $M^A \cap M^{gAg^{-1}} = \varnothing$ whenever g does not belong to N_A. Therefore, we can choose an N_A–equivariant tubular neighborhood U of M^A in M such that its G–orbit $G \cdot U$ is a G–equivariant tubular neighborhood of $G \cdot M^A$ and such that

$$G \times_{N_A} U \to G \cdot U, \quad [(g,u)] \mapsto gu,$$

is a G–equivariant diffeomorphism. Hence, we have an isomorphism

$$\begin{aligned} \Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(X) &\xrightarrow{\cong} \Omega_*^{N_A}\{\mathcal{F}|_{N_A}, \mathcal{F}'|_{N_A}\}(X), \\ [M,\, \partial M,\, f\colon M \to X] &\longmapsto [U,\, \partial U,\, f|_U\colon U \to X], \end{aligned}$$

which composed with the isomorphism of Theorem 5.6 for the group N_A and its normal subgroup A provides the desired result. □

Let us use the decomposition formula given in Corollary 5.7 of the equivariant bordism groups for adjacent families in the case of finite groups to give an alternative proof of Theorem 1.1 in [19]. Let $(\mathcal{F},\mathcal{F}')$ be an adjacent pair of families of subgroups of the finite group G differing by the subgroup A, and consider the restriction map from the G–bordisms to A–bordisms. Let $\mathcal{A}$ denote the family of all subgroups of A and let $\mathcal{P}$ denote the family of all subgroups of A besides A itself. The restriction of G–manifolds to A–manifolds gives a homomorphism

$$r_A^G\colon \Omega_n^G\{\mathcal{F},\mathcal{F}'\} \to \Omega_n^A\{\mathcal{A},\mathcal{P}\}^{W_A},$$

which lies in the W_A–invariants since the action of the inner automorphisms of G in $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}$ is trivial (see [5, Section 20]) and the restriction map is N_A–equivariant.

Applying the fixed-point construction done in Theorem 5.6 to both sides of the homomorphism above we obtain the following diagram with horizontal isomorphisms:

$$\begin{array}{ccc}
\Omega_n^G\{\mathcal{F},\mathcal{F}'\} & \xrightarrow{\cong} & \bigoplus_{0\le 2k\le n} \Omega_{n-2k}^{W_A}\{\{1\}\}\Big(\bigsqcup_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big) \\
\Big\downarrow r_A^G & & \Big\downarrow r_{\{1\}}^{W_A} \\
\Omega_n^A\{\mathcal{A},\mathcal{P}\}^{W_A} & \xrightarrow{\cong} & \bigoplus_{0\le 2k\le n} \Omega_{n-2k}\Big(\bigsqcup_{\overline{P}'\in\overline{\mathcal{P}}(k,A)} BU(\overline{P}')\Big)^{W_A}.
\end{array}$$

Note that $B_{W_A}U(\overline{P})$ is a model for $BU(\overline{P})$ and therefore we may take $BU(\overline{P}) := B_{W_A}U(\overline{P})$.

If we tensor with the ring Z_P of P–local integers, where P is the collection of primes which do not divide the order of the group, the right vertical map

$$\Omega_{n-2k}^{W_A}\{\{1\}\}\Big(\bigsqcup_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big)\otimes Z_P \cong \Omega_{n-2k}\Big(\bigsqcup_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big)^{W_A}\otimes Z_P$$

induces an isomorphism. Therefore, the restriction map

$$r_A^G\colon \Omega_n^G\{\mathcal{F},\mathcal{F}'\}\otimes Z_P \to \Omega_n^A\{\mathcal{A},\mathcal{P}\}^{W_A}\otimes Z_P$$

becomes an isomorphism (see [19, Proposition 3.1]). The spaces $B_{W_A}U(\bar{P})$ are products of $BU(j)$'s and therefore the bordism groups $\Omega_*(B_{W_A}U(\bar{P}))$ are zero in odd degrees and Ω_*–free in even degrees. Since

$$\Omega_*({}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho}U(n_\rho))$$

is $(W_A)_\rho$–invariant, the action of W_A on

$$\bigoplus_{\bar{P}\in\bar{\mathcal{P}}(k,A)} \Omega_{n-2k}(B_{W_A}U(\bar{P}))$$

permutes the generators and therefore the W_A invariants are also Ω_*–free. Hence, we conclude that the bordism groups $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}\otimes Z_P$ for adjacent families are $\Omega_*\otimes Z_P$–free in even degrees and zero in odd degrees. Therefore, the short exact sequences

$$0\to\Omega_*^G\{\mathcal{F}'\}\otimes Z_P\to\Omega_*^G\{\mathcal{F}\}\otimes Z_P\to\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\otimes Z_P\to 0$$

are all split for all pairs of families of subgroups of G, $\Omega_*^G\otimes Z_P$ is an $\Omega_*\otimes Z_P$–free module and there is a canonical isomorphism

$$\Omega_*^G\otimes Z_P\cong\bigoplus_{(A)}\Omega_*^A\{\mathcal{A},\mathcal{P}\}^{W_A}\otimes Z_P,$$

where (A) runs over the set of conjugacy classes of subgroups of G (see Theorem 1.1 of [19]).

6 Applications

In this section we use Corollary 5.7 to calculate the Ω_*–module structure of the equivariant unitary bordism groups of the dihedral groups of order $2p$, where p is an odd prime number.

Let $D_{2p}=\langle a,b\mid a^p=b^2=1,\ bab=a^{-1}\rangle$ denote the dihedral group of order $2p$. Notice that $\langle a\rangle\cong\mathbb{Z}/p$ is a normal subgroup of D_{2p} and we have an extension of groups

$$1\to\mathbb{Z}/p\xrightarrow{i}D_{2p}\xrightarrow{\pi}\mathbb{Z}/2\to 1.$$

If we write $\mathbb{Z}/2=\{1,\tau\}$, where $\tau^2=1$, then the map $j(\tau)=b$ defines a splitting of the previous short exact sequence and thus $D_{2p}\cong\mathbb{Z}/p\rtimes\mathbb{Z}/2$.

Let us recall what we know about the unitary bordism groups of free D_{2p}–actions. Denote by S_-^{2i-1} the sphere with the antipodal $\mathbb{Z}/2$–action and by $S_{\lambda^l}^{2k-1}$ the spheres with the action of $\mathbb{Z}/p$ given by multiplying by $\lambda = e^{2\pi l/p}$. By [13, Corollary 2.5] the bordism classes of the D_{2p}–free unitary manifolds defined by the balanced products $D_{2p} \times_{\mathbb{Z}/2} S^{2i-1}_{-}$ and $D_{2p} \times_{\mathbb{Z}/p} S^{4k-1}_{\lambda^l}$ with $\mathbb{Z}/2 \cong \langle b\rangle$ and $\mathbb{Z}/p \cong \langle a\rangle$ form a generating set of $\widetilde{\Omega}^{D_{2p}}_*\{\{1\}\}$ as an Ω_*–module. In [13, Theorem. 2.6] it is shown that the map

$$(6) \qquad i_* \oplus j_*\colon (\widetilde{\Omega}^{\mathbb{Z}/p}_*\{\{1\}\})^{\mathbb{Z}/2} \oplus \widetilde{\Omega}^{\mathbb{Z}/2}_*\{\{1\}\} \xrightarrow{\cong} \widetilde{\Omega}^{D_{2p}}_*\{\{1\}\}$$

induced by the balanced products is an isomorphism of Ω_*–modules, where

$$i_*\big(\tfrac{1}{2}([S^{4k-1}_{\lambda^l}] + [S^{4k-1}_{\lambda^{p-l}}])\big) = [D_{2p} \times_{\mathbb{Z}/p} S^{4k-1}_{\lambda^l}],\ j_*[S^{2i-1}_{-}] = [D_{2p} \times_{\mathbb{Z}/2} S^{2i-1}_{-}].$$

Here $\widetilde{\Omega}^G_*\{\{1\}\} := \widetilde{\Omega}_*(BG)$ denotes the reduced bordism groups of BG; that is, $\widetilde{\Omega}_*(BG)$ is the kernel of the augmentation map $\Omega_*(BG) \to \Omega_*$.

Using [15, Theorem 3] we see that since $H^n(BD_{2p};\mathbb{Z}) = 0$ for all $n \geq 1$ odd, the projective dimension of $\Omega_*(BD_{2p})$ over Ω_* is at most 1. Since $\Omega_*(BD_{2p})$ contains torsion elements we conclude that it is not an Ω_*–projective module and thus it has projective dimension 1 over Ω_*.

For what follows we will use the notation $\Omega^G_+\{\mathcal{F}\}(X) := \bigoplus_{n \text{ even}} \Omega^G_n\{\mathcal{F}\}(X)$ and $\Omega^G_-\{\mathcal{F}\}(X) := \bigoplus_{n \text{ odd}} \Omega^G_n\{\mathcal{F}\}(X)$ for any family $\mathcal{F}$ of subgroups of a finite group G. Therefore, $\Omega^G_*\{\mathcal{F}\}(X) = \Omega^G_+\{\mathcal{F}\}(X) \oplus \Omega^G_-\{\mathcal{F}\}(X)$, and similarly for pairs of families. With this notation we have that $\Omega^{D_{2p}}_+\{\{1\}\} \cong \Omega_+$ and $\Omega^{D_{2p}}_-\{\{1\}\}$ is all torsion. Moreover, we can identify Ω_* with Ω_+ as $\Omega_- = 0$.

The following theorem was originally proved in [16] and we offer here a simpler proof which makes use of the results of the previous sections.

Theorem 6.1 *The unitary bordism group $\Omega^{D_{2p}}_*$ is a free Ω_*–module on even-dimensional generators.*

Proof Consider the families $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ of subgroups of D_{2p} defined by

$$\mathcal{F}_0 := \{\{1\}\},$$
$$\mathcal{F}_1 := \{\{1\}, \langle a\rangle\},$$
$$\mathcal{F}_2 := \{\{1\}, \langle a\rangle, \langle b\rangle, \langle aba^{-1}\rangle, \dots, \langle a^{p-1}ba^{1-p}\rangle\} = \text{all} \setminus \{D_{2p}\},$$
$$\mathcal{F}_3 := \text{all}.$$

The proof of the theorem will be based on the following facts that will be proved later:

- The unitary bordism group $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module on even-dimensional generators.
- The unitary bordism group $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is a free Ω_*–module.
- The boundary map $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective.
- Both $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ and $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ have projective dimension 1 as modules over Ω_*.
- The boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective.

Using these facts we can prove the theorem as follows. Since $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is trivial, the long exact sequence associated to the families $\mathcal{F}_0\subset\mathcal{F}_1\subset\mathcal{F}_3$ becomes

$$0\to\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\beta}\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\xrightarrow{\delta}\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to 0.$$

Since $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective, the previous exact sequence yields the short exact sequence

$$0\to\operatorname{Im}(\beta)\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\xrightarrow{\delta}\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to 0.$$

We know that $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ has projective dimension 1 as a module over Ω_*. Since $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module, we conclude, by Schanuel's lemma, that $\operatorname{Im}(\beta)$ must be a projective Ω_*–module and hence free by [7, Proposition 3.2]. On the other hand, using the long exact sequence given above we obtain the short exact sequence

$$0\to\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\beta}\operatorname{Im}(\beta)\to 0.$$

As both $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ and $\operatorname{Im}(\beta)$ are free modules over Ω_* we conclude that $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module as well. Moreover, since the boundary map $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective, then $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is trivial. Hence, $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module on even-dimensional generators.

Now, the long exact sequence associated to the families $\mathcal{F}_0\subset\mathcal{F}_3$ becomes

$$0\to\Omega_+\to\Omega_+^{D_{2p}}\xrightarrow{\gamma}\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\partial}\Omega_-^{D_{2p}}\{\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\to 0$$

since $\Omega_+^{D_{2p}}\{\mathcal{F}_0\}\cong\Omega_+$ and $\Omega_-\{\mathcal{F}_3,\mathcal{F}_0\}=0$. We know that the boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective and thus we conclude that $\Omega_-^{D_{2p}}$ is zero. On the other hand, using the previous long exact sequence we obtain the short exact sequence

$$0\to\operatorname{Im}(\gamma)\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\partial}\Omega_-^{D_{2p}}\{\mathcal{F}_0\}\to 0.$$

In this short exact sequence we know that $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ has projective dimension 1 as an Ω_*–module and that $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module. By Schanuel's lemma we conclude that $\mathrm{Im}(\gamma)$ is a projective Ω_*–module and hence free by [7, Proposition 3.2]. Finally, using the short exact sequence

$$0 \to \Omega_+ \to \Omega_+^{D_{2p}} \xrightarrow{\gamma} \mathrm{Im}(\gamma) \to 0$$

we conclude that $\Omega_+^{D_{2p}}$ is a free Ω_*–module because Ω_+ and $\mathrm{Im}(\gamma)$ are free as well. The theorem follows. □

Let us now check each one of the facts listed above.

Lemma 6.2 *The unitary bordism group $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module on even-dimensional generators.*

Proof Consider the adjacent pairs of families $(\mathcal{F}_3,\mathcal{F}_2)$ and $(\mathcal{F}_2,\mathcal{F}_1)$ with $A = D_{2p}$ in the first case and $A = \langle b\rangle$ in the second. Since both D_{2p} and $\langle b\rangle$ are their own normalizers in D_{2p}, in both cases the group W_A is trivial. By Corollary 5.7 we know that both $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_2\}$ and $\Omega_*^{D_{2p}}\{\mathcal{F}_2,\mathcal{F}_1\}$ are isomorphic to unitary bordism groups of copies of $BU(k)$'s and therefore free Ω_*–modules on even-dimensional generators. The long exact sequence associated to the families $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ implies that $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is trivial and the short exact sequence

$$0 \to \Omega_+^{D_{2p}}\{\mathcal{F}_2,\mathcal{F}_1\} \to \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\} \to \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_2\} \to 0$$

implies that the middle term is also a free Ω_*–module. □

Lemma 6.3 *The unitary bordism group $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is a free Ω_*–module.*

Proof By Corollary 5.7 we know that

$$\Omega_*^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\} \cong \Omega_*^{\mathbb{Z}/2}\{\{1\}\}\Bigg(\coprod_{n_1,n_2,\dots,n_{p-1}\in\mathbb{N}} BU(n_1)\times\cdots\times BU(n_{p-1})\Bigg),$$

where the number n_l parametrizes the rank of the irreducible representation of $\mathbb{Z}/p$ given by multiplication of $e^{2\pi l/p}$. The action of $\mathbb{Z}/2$ interchanges the coordinates

$$BU(n_1)\times\cdots\times BU(n_{p-1}) \to BU(n_{p-1})\times\cdots\times BU(n_1),$$
$$(x_1,\dots,x_{p-1}) \mapsto (x_{p-1},\dots,x_1),$$

and it only has fixed points whenever $n_l = n_{p-l}$ for all $1 \leq l \leq \frac{1}{2}(p-1)$. Therefore,

$$\Omega_*^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\} \cong M_* \oplus N_*,$$

where M_* is isomorphic to a direct sum of unitary bordism groups of copies of $BU(k)$'s (thus a free Ω_*–module) and

$$N_* := \bigoplus_{n_1, n_2, \dots, n_{(p-1)/2} \in \mathbb{N}} \Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2),$$

where $X := \prod_{l_1}^{l=(p-1)/2} BU(n_l)$ and $\mathbb{Z}/2$ acts on X^2 by permutation of the coordinates. Next we study $\Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$. The second page of the Atiyah–Hirzebruch spectral sequence becomes

$$E^2_{s,t} \cong H_s(\mathbb{Z}/2, \Omega_t(X^2))$$

and therefore $E^2_{s,\mathrm{odd}} = 0$ and $E^2_{2k,t} = 0$ for $k > 0$. Whenever $s = 0$ we have that the groups $E^2_{0,*}$ are the $\mathbb{Z}/2$–coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$. By [14, Propositions 4.3.2 and 4.3.3] we know that $\Omega_*(BU(k))$ is a free Ω_*–module with basis

$$\{\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_s} \mid 1 \leq \alpha_{j_1} \leq \cdots \leq \alpha_{j_s},\ s \leq k\},$$

where the degree of $\alpha_{j_1}\cdots\alpha_{j_s}$ is $2(j_1 + \cdots + j_s)$, therefore it follows that the $\mathbb{Z}/2$–coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$ form a free Ω_*–module. Since the odd columns and the even rows are trivial, the vertical axis is a free Ω_*–module and the other components of the first quadrant are $\mathbb{Z}/2$–torsion, the spectral sequence collapses on the second page. This implies that $\Omega_+(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is isomorphic to the coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$, and therefore $\Omega_+(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is a free Ω_*–module. Hence, N_+ is a free Ω_*–module, and therefore $\Omega_+^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is a free Ω_*–module. □

Lemma 6.4 *The boundary map* $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ *is surjective.*

Proof Following the argument of the proof of the previous lemma it is enough to show that there are elements in $\Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\}$ whose boundary correspond in $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ to the generators of $\Omega_-(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ as Ω_*–module for $X = \prod_{l_1}^{l=(p-1)/2} BU(n_l)$.

The bordism group $\Omega_*(X)$ is generated as an Ω_*–module by unitary manifolds $M \to X$ (see [14, Propositions 4.3.2 and 4.3.3]) and therefore the trivial $\mathbb{Z}[\mathbb{Z}/2]$–submodule of $\Omega_*(X^2)$ is generated as an Ω_*–module by the manifolds $M^2 \to X^2$.

We claim that $\Omega_-(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is generated as an Ω_*–module by the unitary manifolds

$$S_-^{2i-1} \times_{\mathbb{Z}/2} M^2 \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2.$$

This follows from the following argument. Consider the maps

$$S_-^{2i-1} \times_{\mathbb{Z}/2} M^2 \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} M^2 \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2,$$

where the first one is induced by the inclusion $S_-^{2i-1} \to S_-^\infty = E\mathbb{Z}/2$ and the second is induced by the map $M^2 \to X^2$. If the dimension of M is n, the composition of the maps in homology

$$H_{2i+2n-1}(S_-^{2i-1} \times_{\mathbb{Z}/2} M^2) \to H_{2i+2n-1}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} M^2) \to H_{2i+2n-1}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$$

sends the volume form $[S_-^{2i-1} \times_{\mathbb{Z}/2} M^2]$ to the $\mathbb{Z}/2$–torsion class in the group $H_{2i+2n-1}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ which corresponds in $E^2_{2i-1,2n} \cong H_{2i-1}(\mathbb{Z}/2, H_{2n}(X^2))$ of the Serre spectral sequence to the class in $H_{2i-1}(\mathbb{Z}/2, \mathbb{Z}[M^2]) \cong \mathbb{Z}/2$. Therefore, the homology classes in $H_-(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ defined by the volume forms of the unitary manifolds $S_-^{2i-1} \times_{\mathbb{Z}/2} M^2$ generate the homology in odd degrees. This implies that the Thom homomorphism $\mu\colon \Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2) \to H_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is surjective and that the bordism spectral sequence collapses. We conclude that the unitary manifolds $S_-^{2i-1} \times_{\mathbb{Z}/2} M^2 \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2$ generate $\Omega_-(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ as an Ω_*–module.

Now consider the complex vector bundle $E \to M$ of rank $n_1 + \cdots + n_{(p-1)/2}$ that the map $M \to X = \prod_{l_1}^{l=(p-1)/2} BU(n_l)$ defines, with the appropriate induced action of $\mathbb{Z}/p = \langle a \rangle$ on the fibers. Take the manifold $S_-^{2i-1} \times D(E \times E)$, where $D(E \times E)$ is the disk bundle of $E^2 \to M^2$, and define the D_{2p}–action on it as follows: for $(x,y,z) \in S_-^{2i-1} \times D(E \times E)$, let $a \cdot (x,y,z) := (x, ay, a^{-1}z)$ and $b \cdot (x,y,z) := (-x, z, y)$. The class of the D_{2p}–manifold $S_-^{2i-1} \times D(E \times E)$ lies in $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ and corresponds to the class of $S_-^{2i-1} \times_{\mathbb{Z}/2} M^2 \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2$ in $\Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$. The class of the D_{2p}–manifold $D(\mathbb{C}_-^i) \times D(E \times E)$ lies in $\Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\}$ and its boundary its the class of $S_-^{2i-1} \times D(E \times E)$ in $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$. Hence, the boundary map $\Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is surjective and the lemma follows. □

Lemma 6.5 *As modules over Ω_*, both $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ and $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ have projective dimension 1.*

Proof Notice that $\Omega_*^{D_{2p}}\{\mathcal{F}_0\} = \Omega_*(BD_{2p})$ and thus it has projective dimension 1 over Ω_*. On the other hand, using the previous lemma we conclude that the Thom map corresponding to $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is surjective and thus Proposition 4 of [15] implies that $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ has projective dimension at most 1 as a module over Ω_*. The projective dimension of this module is 1 because it also contains torsion elements. □

Lemma 6.6 *The boundary map* $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_0\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ *is surjective.*

Proof By the isomorphism described in formula (6) we know that the bordism classes $[D_{2p} \times_{\mathbb{Z}/2} S_-^{2i-1}]$ and $[D_{2p} \times_{\mathbb{Z}/p} S_{\lambda^l}^{4k-1}]$ generate $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ as an Ω_*–module. Let $D(\mathbb{C}_-^i)$ and $D(\mathbb{C}_{\lambda^l}^{2k})$ denote the disks of the representations of $\mathbb{Z}/2$ and $\mathbb{Z}/p$, respectively, whose boundaries are S_-^{2i-1} and $S_{\lambda^l}^{4k-1}$. The manifolds $D_{2p} \times_{\mathbb{Z}/2} D(\mathbb{C}_-^i)$ and $D_{2p} \times_{\mathbb{Z}/p} D(\mathbb{C}_{\lambda^l}^{2k})$ are both $(\mathcal{F}_3, \mathcal{F}_0)$–free, and their boundaries are $D_{2p} \times_{\mathbb{Z}/2} S_-^{2i-1}$ and $D_{2p} \times_{\mathbb{Z}/p} S_{\lambda^l}^{4k-1}$, respectively. Therefore, the boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_0\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective and the lemma follows. □

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