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On periodic groups of homeomorphisms of the 2–dimensional sphere

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We prove that every finitely generated group of homeomorphisms of the 2–dimensional sphere all of whose elements have a finite order which is a power of 2 and is such that there exists a uniform bound for the orders of the group elements is finite. We prove a similar result for groups of area-preserving homeomorphisms without the hypothesis that the orders of group elements are powers of 2 provided there is an element of even order.

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1 Introduction

Despite some remaining open questions, there is a very complete understanding of group actions on 1–manifolds (see Ghys [3] and Navas [13]). However, when passing to the 2–dimensional setting, many natural and fundamental questions remain unsolved. One of the most striking ones is related to the Burnside problem.

Recall that Burnside (see [1]) proved that every finitely generated linear group all of whose elements have finite order and such that there exists a uniform bound for the orders of the group elements is actually finite. This result has been extended to some other contexts, but fails in general, as is shown by classical examples due to Golod (see [4]). Later, Ol'shanskii (see [14]), Ivanov (see [8]), and Lysenok (see [12]) exhibited many other examples of infinite, finitely generated groups all of whose elements have a finite order which is bounded by a uniform constant. The case of groups of homeomorphisms is particularly interesting. The following question seems to be folklore: Does there exist an infinite, finitely generated group of homeomorphisms of the 2–dimensional sphere all of whose elements have finite order? Some progress on this question has been made by Guelman and Liousse [5; 6] (provided there is a finite orbit for the action and — in some cases — that all maps involved are of class C^1), and Hurtado [7] (provided the action is by C^∞ volume-preserving diffeomorphisms and there is a uniform bound for the orders of the group elements). The main result

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of this paper yields a new positive result for actions by homeomorphisms under some hypothesis on the orders of group elements. For short, in what follows, we will call *periodic* a group in which all elements have finite order, we will say that such a group is a 2–*group* if the orders of group elements are powers of 2. Also, we will say that a periodic group has *uniformly bounded order* if there exists a uniform bound for the orders of its group elements. Our main result is the following:

Theorem A *Let G be a finitely generated 2–group of homeomorphisms of the 2–dimensional sphere. Suppose that G has uniformly bounded order. Then G is finite.*

We note that the composition of two orientation-reversing homeomorphisms preserves orientation. We deduce that the subgroup of orientation-preserving homeomorphisms has at most index 2 in the group G above. Moreover, Schreier's lemma states that any finite-index subgroup in a finitely generated group is finitely generated. Hence, in order to prove Theorem A, it is enough to show that a finitely generated 2–group of *orientation-preserving* homeomorphisms of the 2–dimensional sphere is finite provided there is a uniform bound for the orders of the group elements. As a first step to proving this, we will show the next result, which is interesting by itself.

Theorem B *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G acts with a global fixed point. Then G is finite and cyclic.*

The second step in the proof is the following:

Theorem C *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has a finite orbit. Then G is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.*

As a by product of our methods, we obtain the following result for groups of area-preserving homeomorphisms:

Theorem D *Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has uniformly bounded order and contains an element of even order. Then G is finite.*

As above, in order to prove Theorem D, it is enough to show an analog of Theorem C in this setting.

Theorem E *Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has a finite orbit. Then G is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.*

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2 Preliminary results

2.1 Local rotation set

In this section, we introduce the notion of local rotation introduced by F Le Roux (see [11]). Since local dynamics (more precisely, the dynamics around a fixed point) does not fit into a compact framework, we consider only rotation numbers of "good orbits". This means that, in order to get a definition of a rotation set which is invariant under conjugacy, we consider only recurrent points close to the fixed point.

Let h be a homeomorphism of the plane $\mathbb{R}^2$ that preserves the orientation and fixes the vector $\mathbf{0} := (0,0) \in \mathbb{R}^2$. We will denote by $\widetilde{\mathbb{A}} = \mathbb{R} \times (0,+\infty)$ the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Let $\widetilde{\pi}\colon \widetilde{\mathbb{A}} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be the corresponding universal covering map and $p_1\colon \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ the projection on the first coordinate. Let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$. We say that the *rotation number (around* $\mathbf{0}$*)* of a h–recurrent point $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ under $\widetilde{h}$ is well defined and equal to $\rho_{\mathbf{0}}(\widetilde{h}, x) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ if for every sequence of integers $(n_k)_{k\in\mathbb{N}}$ which goes to $+\infty$ such that $(h^{n_k}(x))_{k\in\mathbb{N}}$ converges to x, the sequence $(\rho_{n_k}(\widetilde{h}, x))_{k\in\mathbb{N}}$, defined as

$$\rho_{n_k}(\widetilde{h}, x) := \frac{1}{n_k}(p_1(\widetilde{h}^{n_k}(\widetilde{x})) - p_1(\widetilde{x})),$$

where $\widetilde{x}$ is a point in $\widetilde{\pi}^{-1}(x)$, converges to $\rho_{\mathbf{0}}(\widetilde{h}, x)$. Notice that this definition does not depend on the choice of $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

The *local rotation set (around the fixed point* $\mathbf{0}$*) of* $\widetilde{h}$, which we denote by $\rho_{\mathbf{0}}(\widetilde{h})$, is the set of all rotation numbers of recurrent points of h.

We have the following properties (see [11] for more details):

(1) The rotation numbers of a recurrent point and, consequently, the local rotation set, are invariant under (local) oriented topological conjugacy. More precisely, if φ is a homeomorphism of $\mathbb{R}^2$ that preserves the orientation and fixes $\mathbf{0} \in \mathbb{R}^2$ and $\widetilde{\varphi}$ is a lift of φ to $\widetilde{\mathbb{A}}$, then

$$\rho_{\mathbf{0}}(\widetilde{\varphi}^{-1}\widetilde{h}\widetilde{\varphi}) = \rho_{\mathbf{0}}(\widetilde{h}).$$

(2) For every $p, q \in \mathbb{Z}$, we have $\rho_{\mathbf{0}}(\widetilde{h}^q + (p,0)) = q\rho_{\mathbf{0}}(\widetilde{h}) + p$. A similar formula holds for the rotation number of a recurrent point.

2.2 Periodic, orientation-preserving homeomorphisms of the 2–dimensional sphere

We say that an orientation-preserving homeomorphism g of the 2–dimensional sphere is *periodic* if its order is finite, that is, if there exists an integer q such that $g^q = \mathrm{Id}$. We recall that Keréjártó proved that every periodic, orientation-preserving homeomorphism of the 2–dimensional sphere is conjugate to a rotation (see [9; 2]). Formally, we have the following proposition:

Proposition 2.1 *Let g be a periodic, orientation-preserving homeomorphism of the 2–dimensional sphere. Then there exist an orientation-preserving homeomorphism h of the 2–dimensional sphere which has the same fixed points as g and a rotation R such that $hgh^{-1} = R$. In particular, if g is nontrivial, then it has exactly two fixed points, and every point that is not fixed is periodic. Additionally, the periods of all nonfixed, periodic points are the same.*

2.3 Local rotation set for periodic homeomorphisms

In our setting, let g be a periodic homeomorphism that preserves the orientation of $\mathbb{S}^2$ and fixes a point $z \in \mathbb{S}^2$. Considering a chart ϕ centered at z, we have that $h = \phi g \phi^{-1}$ is a periodic homeomorphism of the plane $\mathbb{R}^2$ that preserves the orientation and fixes the vector $\phi(z) = \mathbf{0} \in \mathbb{R}^2$. Let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$ the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, which, as before, we identify with $\mathbb{R} \times (0, +\infty)$. Suppose that h has order q, that is, q is the smallest positive integer such that $h^q = \mathrm{Id}$. We write $\mathrm{ord}(h) = q$.

Then there exists an integer p such that, for every $\widetilde{x} \in \widetilde{\mathbb{A}}$,

$$\widetilde{h}^q(\widetilde{x}) = \widetilde{x} + (p, 0).$$

It is not hard to prove that p and q are coprime. Also, every point is recurrent for h and has a rotation number around $\mathbf{0}$ equal to p/q. Because of the invariance under conjugacy by $\rho_{\mathbf{0}}(\widetilde{h})$ (property (1) above), this number does not depend on the choice of the chart ϕ. Hence, by property (2) above, we can associate to our periodic homeomorphism g a unique "local rotation number" around z, defined as

$$\rho_{\mathrm{loc},z}(g) = \frac{p}{q} \pmod 1 \in \mathbb{T}^1.$$

Remark 1 Clearly, if $\rho_{\mathrm{loc},z}(g) = 0$, then g is the identity.

Given $z \in \mathbb{S}^2$, we will denote by $\mathrm{Homeo}^+(\mathbb{S}^2; z)$ the group of all homeomorphisms of $\mathbb{S}^2$ that preserve the orientation and fix z. By the discussion above, the "local rotation number map" is well defined for a periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. We have the following properties:

Lemma 2.2 *Let g be an element of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$.

(1) *The local rotation set around z is invariant under (local) oriented topological conjugacy. More precisely, if φ belongs to* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$, *then*

$$\rho_{\mathrm{loc},z}(\varphi^{-1} g \varphi) = \rho_{\mathrm{loc},z}(g).$$

(2) *For every $q \in \mathbb{Z}$, we have $\rho_{\mathrm{loc},z}(g^q) = q\rho_{\mathrm{loc},z}(g)$.*

The first nontrivial observation concerning the local rotation set is the following:

Proposition 2.3 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *The local rotation number map defined on G_0 is a group homomorphism into $\mathbb{T}^1$ if and only if G_0 is abelian.*

Proof Let f and g be two elements in G_0. We recall that $[f, g] := fgf^{-1}g^{-1}$ denotes the commutator of f and g. If $\rho_{\mathrm{loc},z}$ is a group homomorphism, then $\rho_{\mathrm{loc},z}([f, g])$ is null, which implies that $[f, g] = \mathrm{Id}$ (see Remark 1). As f and g are arbitrary, G_0 is abelian.

Conversely, assume that G_0 is abelian, and let f and g be two elements in G_0. Consider a chart ϕ centered at z, and let $h_1 := \phi f \phi^{-1}$ and $h_2 := \phi g \phi^{-1}$ be the conjugate homeomorphisms. Both h_1 and h_2 are periodic homeomorphisms of the plane $\mathbb{R}^2$ that preserve the orientation and fix the vector $\phi(z) = \mathbf{0} \in \mathbb{R}^2$. Since

h_1 and h_2 commute, we can consider commuting lifts $\widetilde{h}_1$ and $\widetilde{h}_2$ of h_1 and h_2, respectively. Suppose that $\rho_{\mathbf{0}}(\widetilde{h}_1) = p'/q'$ and $\rho_{\mathbf{0}}(\widetilde{h}_2) = p/q$. Then, for every $\widetilde{x}$ and $\widetilde{x}'$ in $\widetilde{\mathbb{A}}$, we have

$$\widetilde{h}_2^q(\widetilde{x}) = \widetilde{x} + (p,0) \quad \text{and} \quad \widetilde{h}_1^{q'}(\widetilde{x}') = \widetilde{x}' + (p',0).$$

Thus,

$$(\widetilde{h}_1\widetilde{h}_2)^{q'q}(\widetilde{x}) = \widetilde{h}_1^{q'q}(\widetilde{h}_2^{q'q}(\widetilde{x})) = \widetilde{h}_2^{q'q}(\widetilde{x}) + (qp',0) = \widetilde{x} + (q'p,0) + (qp',0).$$

Therefore,

$$\rho_{\mathrm{loc},z}(fg) = \frac{q'p+qp'}{q'q} = \frac{p}{q} + \frac{p'}{q'} = \rho_{\mathrm{loc},z}(f) + \rho_{\mathrm{loc},z}(g).$$

This shows that $\rho_{\mathrm{loc},z}$ is a group homomorphism. □

2.4 Consequences for abelian, periodic subgroups of orientation-preserving homeomorphisms of $\mathbb{S}^2$ that fix a point

From Proposition 2.3, we know that the "local rotation number map" is an injective group homomorphism for abelian, periodic subgroups of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$ (see Remark 1). So $\rho_{\mathrm{loc},z}$ gives an isomorphism with its image, a periodic subgroup of $\mathbb{R}/\mathbb{Z}$.

We deduce the following results:

Lemma 2.4 *Let A be an abelian, periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. Let a and b be two elements of A with $\mathrm{ord}(a) = \mathrm{ord}(b)$. Then there exists an integer $i \in \{1, \dots, \mathrm{ord}(a) - 1\}$ such that $b = a^i$. In particular, there exists at most one element of order 2 in A.*

Lemma 2.5 *If A is a finite, abelian subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$, then A is cyclic.*

3 Burnside problem for 2–groups of homeomorphisms of $\mathbb{S}^2$: particular cases

In order to prove Theorem A, it is enough to prove that every finitely generated 2–group G of *orientation-preserving* homeomorphisms of the 2–dimensional sphere is finite. This is the purpose of the following two subsections. Our proof consists in first considering the case where G has a global fixed point and later the case where G has a finite orbit. Finally, we settle the general case, and for this we prove that the group G contains only a finite number of involutions.

3.1 The case where the group has a global fixed point

In this section we will prove Theorem B, that is, every finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere that has a global fixed point is finite and cyclic. The idea of the proof is as follows: Notice that a (nontrivial) 2–group G_0 always contains involutions, that is, elements of order 2. The key step consists in proving that, in our case, there is a unique involution in G_0. This implies that such an involution must belong to the center of G_0, that is, it commutes with each element of G_0. Since we are assuming that G_0 is a 2–group, we can deduce that G_0 is abelian using the following property (see Proposition 3.6 below): if f and g^2 in G_0 commute, then f and g commute. Finally, using that G_0 is finitely generated, we can conclude that G_0 is finite, and hence cyclic by Lemma 2.5.

We start with a lemma that follows from classical properties of the local rotation set around a fixed point (Lemma 2.2).

Lemma 3.1 *Let g be a finite-order element in* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. Suppose that g is conjugate* (by an element in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$) *to its inverse. Then* $g^2 = \mathrm{Id}$.

Proof By hypothesis, there exists $\varphi \in \mathrm{Homeo}^+(\mathbb{S}^2; z)$ such that $\varphi^{-1} g \varphi = g^{-1}$. By Lemma 2.2,

$$\rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}(g^{-1}) = -\rho_{\mathrm{loc},z}(g).$$

This implies that $0 = 2\rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}(g^2)$. Since g has finite order, it must satisfy $g^2 = \mathrm{Id}$. □

Proposition 3.2 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. Let σ and σ' be two elements of order 2 in G_0. Then σ and σ' commute.*

Proof We know that, in any group, $\sigma\sigma'$ is conjugate (by σ) to $\sigma'\sigma$. Indeed,

$$\sigma'\sigma = (\sigma^{-1}\sigma)\sigma'\sigma = \sigma^{-1}(\sigma\sigma')\sigma.$$

Since σ and σ' have order 2, we have that $(\sigma\sigma')^{-1} = \sigma'\sigma$. By Lemma 3.1, it follows that $\sigma\sigma'$ has order 2. Since σ, σ', and $\sigma\sigma'$ have order 2, we deduce (using an argument due to Burnside) that

$$\mathrm{Id} = (\sigma\sigma')^2 = \sigma\sigma'\sigma\sigma' = \sigma\sigma'\sigma^{-1}\sigma'^{-1} := [\sigma, \sigma'].$$

This implies that σ and σ' commute. □

We deduce the following properties:

Proposition 3.3 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. Then G_0 has at most one element of order 2.*

Proof Suppose that σ and σ' are two elements of order 2 in G_0. By the previous proposition, the group generated by σ and σ' is an abelian, periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. By Lemma 2.4, we deduce that $\sigma = \sigma'$. □

Corollary 3.4 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. If $\sigma \in G_0$ has order 2, then σ belongs to the center of G_0.*

Proof Let g be an element of G_0. Since $g\sigma g^{-1}$ has order 2, by Proposition 3.3 we deduce that $g\sigma g^{-1} = \sigma$. This implies that g and σ commute. □

Lemma 3.5 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. Let f and g be two elements in G_0. Suppose that f and g^2 commute. Then, for every $i \in \{0, \dots, \mathrm{ord}(f) - 1\}$, the element $[f^i, g]$, the commutator of f^i and g, satisfies $[f^i, g]^2 = \mathrm{Id}$. Moreover, $[g, f^i]^2 = \mathrm{Id}$.*

Proof Since f and g^2 commute, we have that f^i and g^2 commute, that is, $f^i g^2 = g^2 f^i$. Hence,

$$g^{-1} f^{-i} = g f^{-i} g^{-2}.$$

Consequently,

$$[g, f^i] = g f^i g^{-1} f^{-i} = g f^i (g f^{-i} g^{-2}) = g(f^i g f^{-i} g^{-1}) g^{-1} = g[f^i, g] g^{-1}.$$

Since $[f^i, g]^{-1} = [g, f^i]$, it follows from Lemma 3.1 that $[f^i, g]^2 = \mathrm{Id}$. Finally, notice that $[g, f^i] = [f^i, g]^{-1}$. Hence, we deduce that $[g, f^i]^2 = \mathrm{Id}$. □

Proposition 3.6 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$*. Let f and g be two elements in G_0. If f and g^2 commute, then f and g commute.*

Proof By the previous lemma, we have that $[f, g]^2 = \mathrm{Id}$ and so $g[f, g]^2 = g$, that is,

$$\begin{aligned} g &= g(fgf^{-1}g^{-1})(fgf^{-1}g^{-1}) \\ &= (gf)(gf^{-1}g^{-1}f)g(f^{-1}g^{-1}) \\ &= (gf)[g, f^{-1}]g(gf)^{-1}. \end{aligned}$$

Applying the local rotation map and using its invariance under (local) topological conjugacy, we obtain

$$\rho_{\mathrm{loc},z}([g, f^{-1}]g) = \rho_{\mathrm{loc},z}(g).$$

Since $[g, f^{-1}]$ commutes with g (Corollary 3.4), the local rotation map restricted to the group generated by $[g, f^{-1}]$ and g is a group homomorphism (Proposition 2.3). Thus,

$$\rho_{\mathrm{loc},z}([g, f^{-1}]) + \rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}([g, f^{-1}]g) = \rho_{\mathrm{loc},z}(g).$$

Therefore, $\rho_{\mathrm{loc},z}([g, f^{-1}]) = 0$, and so $[g, f^{-1}] = \mathrm{Id}$. This contradiction proves that f and g commute. □

End of the proof of Theorem B Let G_0 be a finitely generated 2–group contained in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. Let f and g be two elements in G_0. Since G_0 is a 2–group, g has order 2^{p+1} for a certain integer $p \geq 0$, and then g^{2^p} has order 2. It follows from Corollary 3.4 that g^{2^p} and f commute. Applying the previous proposition, we obtain that $g^{2^{p-1}}$ and f commute. Iterating this argument, we get that f and g commute. Therefore, G_0 is an abelian, finitely generated group, and so it is finite. Finally, we deduce from Lemma 2.5 that G_0 is cyclic. □

3.2 The case where the group has a finite orbit

In this section, we prove Theorem C, that is, every finitely generated 2–group G of orientation-preserving homeomorphisms of the 2–dimensional sphere which has a finite orbit is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.

Proof of Theorem C Let z_0 be a point with finite G–orbit. We write $\mathcal{O}_G(z_0) = \{z_0, z_1, \ldots, z_n\}$, where, for every $i \in \{0, \ldots, n\}$, $z_i = g_i(z_0)$ for some $g_i \in G$. We denote by $\mathrm{Sta}_G(z_0)$, the stabilizer in G of z_0, that is, the set

$$\mathrm{Sta}_G(z_0) := \{g \in G : g(z_0) = z_0\}.$$

We first have, by Theorem B, that $\mathrm{Sta}_G(z_0)$ is a finite cyclic group. Finally, we conclude that G is finite, by proving that $G = \bigcup_{i=0}^{n} g_i(\mathrm{Sta}_G(z_0))$. Indeed, if $g \in G$, since $g(z_0) \in \mathcal{O}_G(z_0)$ there exists an integer $i \in \{0, \ldots, n\}$ such that $g(z_0) = g_i(z_0)$. Hence, $g_i^{-1} g \in \mathrm{Sta}_G(z_0)$ and then $g \in g_i(\mathrm{Sta}_G(z_0))$. This proves that G is finite. Now suppose that G has a finite orbit of cardinality 2. We will prove, in this case, G is either a cyclic or a dihedral finite group. Let z be a point with G–orbit of cardinality 2.

We write $\mathcal{O}_G(z) = \{z, z'\}$. We will consider the subgroup G_0 of homeomorphisms that fix both z and z'.

Lemma 3.7 *The group G_0 is an index-2, normal subgroup of G. In particular, G_0 is a finitely generated 2–group contained in* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$.

Proof It is easy to check that G_0 is normal in G. Moreover, notice that if σ and σ' are in $G \setminus G_0$, then $\sigma\sigma'$ is in G_0. Hence, G_0 has index 2 in G. Moreover, Schreier's lemma states that any finite-index subgroup in a finitely generated group is finitely generated. Hence, as G is a finitely generated 2–group, we deduce that G_0 is a finitely generated 2–group contained in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. □

Lemma 3.8 *Every $g \in G \setminus G_0$ has order 2.*

Proof If $g \in G \setminus G_0$, then $g(z) = z'$ and $g(z') = z$. We deduce that $g^2(z) = z$. As the local rotation number of g is a singleton, we deduce that g has order 2 (see Proposition 2.1). □

End of the proof of Theorem C By Theorem B, we know that G_0 is a finite cyclic group. If $G = G_0$, then G is finite and cyclic. Otherwise, let g_0 in G be a generator of G_0, and let $g \in G \setminus G_0$. Consider Γ the subgroup of G generated by g_0 and g. We claim that $\Gamma = G$. Indeed, if g' is any element in $G \setminus G_0$, then $gg' \in G_0$, hence $g' \in \Gamma$. Moreover, as gg_0 and g do not belong to G_0, we have that gg_0 and g have order 2 (by the previous lemma). Hence, we have $gg_0gg_0 = \mathrm{Id} = g^2$, which yields $gg_0g^{-1} = g_0^{-1}$. It follows that G is a dihedral group. □

4 Burnside problem for 2–groups of homeomorphisms of the 2–dimensional sphere

In this section, we prove Theorem A, that is, every finitely generated 2–group G of homeomorphisms of the 2–dimensional sphere for which there is a uniform bound for the orders of the group elements is finite. Recall that a nontrivial 2–group always contains involutions, that is, elements of order 2. Let $\mathrm{Inv}(G) := \{g \in G \setminus \mathrm{Id} : g^2 = \mathrm{Id}\}$, and let $Z(\sigma)$ be the centralizer of σ in G, that is, $Z(\sigma) = \{g \in G : g\sigma = \sigma g\}$. In order to prove Theorem A, we start by proving, using Theorem C, that, for every $\nu \in \mathrm{Inv}(G)$, the set $Z(\nu) \cap \mathrm{Inv}(G)$ is finite (following the proof of Theorem A, this is the only part where we use the existence of a uniform bound for the orders of the group elements).

Then we will prove that the set $\mathrm{Inv}(G)$ is finite. Since each $g \in G \setminus \{\mathrm{Id}\}$ has exactly two fixed points (by Proposition 2.1), we obtain that the union of fixed points of the involutions is also finite. Moreover, this set is nonempty and G–invariant, and has finite cardinality. We deduce that G is finite by Theorem C. For $\sigma \in G$, let us denote by $\mathrm{Fix}(\sigma)$ the set of all fixed points of σ.

Proposition 4.1 *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. Suppose that G has uniformly bounded order. Then the following assertions hold:*

(1) *If $\nu \in \mathrm{Inv}(G)$, then the set $Z(\nu) \cap \mathrm{Inv}(G)$ is finite.*

(2) *The set $\mathrm{Inv}(G)$ is finite.*

Proof Let us prove (1). Suppose that there is an infinite sequence $\nu, \nu_1, \dots, \nu_n, \dots$ contained in $Z(\nu) \cap \mathrm{Inv}(G)$. Fix an integer $n \geq 1$. The group G_n generated by $\nu,\ \nu_1, \dots, \nu_n$ is finitely generated, periodic, and preserves the set of fixed points of ν (because each ν_i commutes with ν). Then, by Theorem C, the group G_n is finite and either cyclic or dihedral. Moreover,

$$\{\mathrm{Id}\} \subset G_0 \subset \cdots \subset G_n \cdots .$$

Since we are assuming that elements in G have uniformly bounded order, this sequence must stabilize at some integer n_0. That is, for every integer $n \geq n_0$, one has $G_n = G_{n_0}$. This proves that $Z(\nu) \cap \mathrm{Inv}(G)$ is finite.

Let us prove (2). Suppose that there exists an infinite sequence of involutions σ_1, σ_2, $\dots, \sigma_n, \dots$ contained in G. For every integer n, the group generated by σ_1 and σ_n is either cyclic or dihedral, because $\sigma_1\sigma_n$ has finite order. Therefore, it contains an involution ν_n that commutes with σ_1 and σ_n ($\nu_n = \sigma_1$ in the cyclic case, and $\nu_n = (\sigma_1\sigma_n)^{\mathrm{ord}(\sigma_1\sigma_n)/2}$ in the dihedral case). Since $Z(\sigma_1) \cap \mathrm{Inv}(G)$ is finite (by (1)), we can suppose (by passing to a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$) that $\nu_n = \nu$ for every integer n. This implies that ν commutes with all σ_n, and hence the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is contained in $Z(\nu) \cap \mathrm{Inv}(G)$. But this last set is finite by (1). This contradiction proves that the set $\mathrm{Inv}(G)$ is finite. □

End of the proof of Theorem A Assume G is nontrivial. Applying Proposition 4.1, we obtain that the set $\mathrm{Inv}(G)$ is finite. Since each $g \in G \setminus \{\mathrm{Id}\}$ has exactly two fixed points (by Proposition 2.1), we obtain that the set

$$F := \bigcup_{\sigma \in \mathrm{Inv}(G)} \mathrm{Fix}(\sigma)$$

is also finite. As $\mathrm{Fix}(g\sigma g^{-1}) = g(\mathrm{Fix}(\sigma))$ and $g\sigma g^{-1}$ is an involution, the set F is nonempty and G–invariant, and has finite cardinality. We deduce that G is finite by Theorem C. □

5 Burnside problem for area-preserving homeomorphisms of the 2–dimensional sphere

In this section, we prove Theorem D, that is, every finitely generated, periodic group of area-preserving homeomorphisms of the 2–dimensional sphere having uniformly bounded order and an element of even order — equivalently, of order two — is finite. As in the case of a 2–group, we start by proving Theorem E (which is the analog of Theorem C in the area-preserving setting). Then using Theorem E we deduce that Proposition 4.1 holds in the area-preserving case (in the case where the set $\mathrm{Inv}(G)$ is nonempty). We then finish the proof of Theorem D in the same way as that of Theorem A. In order to prove Theorem E, we first introduce the rotation set for a homeomorphism of the open annulus.

5.1 Rotation set for a homeomorphism of the open annulus

Let $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ be the open annulus and $\widetilde{\mathbb{A}} := \mathbb{R} \times \mathbb{R}$ its universal covering. We denote by $\widetilde{\pi}\colon \widetilde{\mathbb{A}} \to \mathbb{A}$ the corresponding universal covering map and $p_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the projection on the first coordinate. By the two-point compactification, one can identify $\mathbb{A}$ to the punctured sphere $\mathbb{S}^2 \setminus \{N, S\}$, where N and S are two distinct points of $\mathbb{S}^2$ (the north and south poles). The Lebesgue measure on $\mathbb{S}^2$ induces a probability measure on $\mathbb{A}$, which we still call the Lebesgue measure and denote by Leb.

Let h be a homeomorphism of $\mathbb{A}$ that is isotopic to the identity, and let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$. Following [10], we say that the *rotation number* of an h–recurrent point $x \in \mathbb{A}$ under $\widetilde{h}$ is well defined and equal to $\rho(\widetilde{h}, x) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ if, for every sequence of integers $(n_k)_{k\in\mathbb{N}}$ which goes to $+\infty$ such that $(h^{n_k}(x))_{k\in\mathbb{N}}$ converges to x, the sequence $(\rho_{n_k}(\widetilde{h}, x))_{k\in\mathbb{N}}$, defined as

$$\rho_{n_k}(\widetilde{h}, x) := \frac{1}{n_k}(p_1(\widetilde{h}^{n_k}(\widetilde{x})) - p_1(\widetilde{x})),$$

where $\widetilde{x}$ is a point in $\widetilde{\pi}^{-1}(x)$, converges to $\rho(\widetilde{h}, x)$. Again, this definition does not depend on the choice of $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

Assume that h preserves a probability measure μ on $\mathbb{A}$. We say that the *rotation number* of $\widetilde{h}$ (with respect to μ) is well defined and equal to $\rho(\widetilde{h}, \mu)$ if

(1) μ–almost every point $x \in \mathbb{A}$ has a rotation number $\rho(\widetilde{h}, x)$, and

(2) the function $x \mapsto \rho(\widetilde{h}, x)$ is μ–integrable, with

$$\rho(\widetilde{h}, \mu) := \int_{\mathbb{A}} \rho(\widetilde{h}, x)\, d\mu.$$

Notice that, by the Birkhoff ergodic theorem, we have

$$\rho(\widetilde{h}, \mu) := \int_{\mathbb{A}} \rho_1(\widetilde{h}, x)\, d\mu,$$

where $\rho_1(\widetilde{h}, x) = p_1(\widetilde{h}(\widetilde{x})) - p_1(\widetilde{x})$, with $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

5.2 Rotation set for periodic homeomorphisms of $\mathbb{S}^2$

In our setting, let g be a periodic, orientation-preserving homeomorphism of $\mathbb{S}^2$ that preserves the Lebesgue measure. We know that if g is nontrivial, then it fixes two distinct points N and S of $\mathbb{S}^2$. As in the local case, we can associate to our periodic homeomorphism g a unique "rotation number" on the open annulus $\mathbb{A}_{N,S} := \mathbb{S}^2 \setminus \{N, S\}$, defined as

$$\rho_{\mathbb{A}_{N,S}}(g) := \int_{\mathbb{A}_{N,S}} \rho_1(g, x)\, d\mathrm{Leb} \in \mathbb{T}^1.$$

Remark 2 If $\rho_{\mathbb{A}_{N,S}}(g) = 0$, then g is the identity.

Given two distinct points N and S of $\mathbb{S}^2$, we will denote by $\mathrm{Homeo}_0(\mathbb{A}_{N,S})$ the group of all homeomorphisms of $\mathbb{S}^2$ that preserve the orientation and fix both N and S. As in the local case we have the following result:

Proposition 5.1 *Let G_0 be a periodic subgroup of $\mathrm{Homeo}_0(\mathbb{A}_{N,S})$. The rotation number map defined on G_0 is a group homomorphism into $\mathbb{T}^1$ if and only if G_0 is abelian.*

5.3 Proof of Theorems D and E

We start by proving Theorem E.

Proof of Theorem E Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Let z be a point with G–orbit of cardinality 2. We write $\mathcal{O}_G(z) = \{z, z'\}$. We consider the subgroup G_0 of homeomorphisms that fix both z and z'. By Lemma 3.7, the group G_0 is an index-2, normal subgroup of G. In particular G_0 is a finitely generated periodic group contained in $\mathrm{Homeo}_0(\mathbb{A}_{z,z'})$ all of whose elements preserve the Lebesgue measure. Since the rotation number is a group homomorphism in the area-preserving case (see Lemma 5.2 below), we can invoke an analog of Proposition 5.1 to conclude that G_0 is abelian.

Lemma 5.2 *Let G_0 be a subgroup of $\mathrm{Homeo}_0(\mathbb{A}_{z,z'})$. Suppose each element of G_0 preserves the Lebesgue measure. Then the rotation map is a group homomorphism.*

Proof Let f and g be two elements of G_0. We have that

$$\begin{aligned}\rho_{\mathbb{A}_{z,z'}}(fg) &= \int_{\mathbb{A}_{z,z'}} \rho_1(fg,x)\,d\mathrm{Leb}(x)\\ &= \int_{\mathbb{A}_{z,z'}} \rho_1(f,g(x))\,d\mathrm{Leb}(x) + \int_{\mathbb{A}_{z,z'}} \rho_1(g,x)\,d\mathrm{Leb}(x)\\ &= \int_{\mathbb{A}_{z,z'}} \rho_1(f,y)\,d\mathrm{Leb}(y) + \int_{\mathbb{A}_{z,z'}} \rho_1(g,x)\,d\mathrm{Leb}(x)\\ &= \rho_{\mathbb{A}_{z,z'}}(f) + \rho_{\mathbb{A}_{z,z'}}(g).\end{aligned}$$

This shows that $g \mapsto \rho_{\mathbb{A}_{z,z'}}(g)$ is a group homomorphism. □

Since G_0 is finitely generated, periodic and abelian, we deduce that it is finite. Moreover, by an analog of Lemma 2.5 (using the rotation number instead of the local rotation set), we deduce that G_0 is cyclic. The proof finishes as the proof of Theorem C. □

Now we can prove Theorem D.

Proof of Theorem D The proof is a straightforward adaptation of the proof of Theorem A. Let G be a finitely generated periodic group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. Suppose that each element of G preserves the Lebesgue measure, that G has at least one element of even order, and that G has uniformly bounded order. Let $\mathrm{Inv}(G) := \{g \in G \setminus \mathrm{Id} : g^2 = \mathrm{Id}\}$. Notice that G always contains involutions. Indeed, if $g^{2p} = \mathrm{Id}$ for some integer p, then $g^p \in \mathrm{Inv}(G)$. Applying Proposition 4.1 (using Theorem E instead of Theorem C), we obtain that the set $\mathrm{Inv}(G)$ is finite. The proof follows as the proof of Theorem A. □

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