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Proper 2–equivalences between infinite ended finitely presented groups

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Recall that two finitely presented groups G and H are "proper 2–equivalent" if they can be realized by finite 2–dimensional CW–complexes whose universal covers are proper 2–equivalent as (strongly) locally finite CW–complexes. This purely topological relation is coarser than the quasi-isometry relation, and those groups which are 1–ended and semistable at infinity are classified, up to proper 2–equivalence, by their fundamental pro-group. We show that if G and H are proper 2–equivalent and semistable at each end, then any two finite graph of groups decompositions of G and H with finite edge groups and finitely presented vertex groups with at most one end must have the same set of proper 2–equivalence classes of (infinite) nonsimply connected at infinity vertex groups (without multiplicities). Moreover, those simply connected at infinity vertex groups in such a decomposition (if any) are all proper 2–equivalent to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Thus, under the semistability hypothesis, this answers a question concerning the classification of infinite ended finitely presented groups up to proper 2–equivalence, and shows again the behavior of proper 2–equivalences versus quasi-isometries, in which the geometry of the group is taken into account.

57M07; 57M10

1 Introduction

In [5], Gromov outlined a program to understand and try to classify all finitely generated groups geometrically via the notion of quasi-isometry, regarded as metric spaces. Since then, those properties of finitely generated groups which are invariant under quasi-isometries have been of great interest and widely studied. On the other hand, the study of asymptotic invariants of a topological nature for finitely generated groups has also led to an interesting research area; see Geoghegan [4] for a good source on this subject.

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In this realm, in Cárdenas, Lasheras, Quintero and Roy [2] a topological equivalence relation was introduced within the class of finitely presented groups attending to their asymptotic topology rather than their asymptotic geometry. More precisely, two finitely presented groups G and H are said to be *proper 2–equivalent* if there exist (equivalently, for all) finite 2–dimensional CW–complexes X and Y, with $\pi_1(X) \cong G$ and $\pi_1(Y) \cong H$, such that their universal covers $\widetilde{X}$ and $\widetilde{Y}$ are proper 2–equivalent (as locally finite CW–complexes); in fact, the required proper 2–equivalence can be replaced by a proper homotopy equivalence after wedging with 2–spheres. It is worth pointing out that this equivalence relation is coarser than the quasi-isometry relation; quasi-isometric finitely presented groups are also related in this wider and "geometry forgetful" sense, by [4, Theorem 18.2.11].

It has been shown that two finite graph of groups decompositions with finite edge groups and finitely presented vertex groups with at most one end yield proper 2–equivalent groups if they have the same set of proper 2–equivalence classes of vertex groups; see [2, Theorem 3.9]. On the other hand, in contrast to the situation under the quasi-isometry relation (see Papasoglu and Whyte [14, Theorem 0.4]), the converse does not hold in general. The proper 2–equivalence class of a finitely presented group does not determine in general the set of proper 2–equivalence classes of vertex groups in such a decomposition of the group; see [2]. We establish a partial converse to [2, Theorem 3.9] under the semistability hypothesis:

Theorem 1.1 *Let G and H be two proper 2–equivalent finitely presented groups which are semistable at each end, and let $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ be finite graph of groups decompositions of G and H with finite edge groups and finitely presented vertex groups with at most one end. Then, $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ have the same set of proper 2–equivalence classes of (infinite) nonsimply connected at infinity vertex groups (without multiplicities). Moreover, those simply connected at infinity vertex groups in $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ (if any) are all proper 2–equivalent to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.*

Observe that, by [14, Theorem 0.4], if G and H are in fact quasi-isometric then $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ have the same set of proper 2–equivalence classes of vertex groups (without multiplicities), regardless of the semistability hypothesis.

Remark 1.2 The statement of Theorem 1.1 is the best possible, as the example in [2, Section 6] shows that $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and $H = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ are proper 2–equivalent, but the vertex groups in the given graph of groups decompositions are all either finite in the case of G or simply connected at infinity in the case of H.

2 Some preliminaries

For the most part we will be working within the category of locally finite CW–complexes and proper maps. We recall that a *proper* map is a map with the property that the inverse image of every compact subset is compact. Thus, two locally finite CW–complexes are said to be proper homotopy equivalent if they are homotopy equivalent and all homotopies involved are proper. On the other hand, a proper cellular map $f: X \to Y$ between finite-dimensional locally finite CW–complexes is a proper n–equivalence if there is another proper cellular map $g: Y \to X$ such that the restrictions $g \circ f|_{X^{n-1}}$ and $f \circ g|_{Y^{n-1}}$ are proper homotopic to the inclusion maps $X^{n-1} \subseteq X$ and $Y^{n-1} \subseteq Y$. Observe that if two finite-dimensional locally finite CW–complexes are proper homotopy equivalent then they are proper n–equivalent, for all n. Also, one can easily check that if two finite-dimensional locally finite CW–complexes are proper n–equivalent then so are their n–skeleta, by the proper cellular approximation theorem [4, Theorem 10.1.14].

Given a noncompact (strongly) locally finite CW–complex Y, a *proper ray* in Y is a proper map $\omega: [0, \infty) \to Y$. We say that two proper rays ω and ω' *define the same end* if their restrictions to the natural numbers $\omega|_{\mathbb{N}}$ and $\omega'|_{\mathbb{N}}$ are properly homotopic. This equivalence relation gives rise to the notion of the *end determined by* ω as the corresponding equivalence class, as well as the space of ends $\mathcal{E}(Y)$ of Y as a compact totally disconnected metrizable space; see [4, Section 13.4]. The CW–complex Y is *semistable* at the end determined by ω if any other proper ray defining the same end is in fact properly homotopic to ω. Equivalently, Y is semistable if the fundamental pro-group pro-$\pi_1(Y, \omega)$ is pro-isomorphic to a tower of groups with surjective bonding homomorphisms; see [4, Proposition 16.1.2]. Recall that pro-$\pi_1(Y, \omega)$ is represented by the inverse sequence (tower) of groups

$$\pi_1(Y, \omega(0)) \xleftarrow{\phi_1} \pi_1(Y - C_1, \omega(t_1)) \xleftarrow{\phi_2} \pi_1(Y - C_2, \omega(t_2)) \leftarrow \cdots,$$

where $C_1 \subset C_2 \subset \cdots \subset Y$ with $\omega([t_i, \infty)) \subset Y - C_i$ is a filtration of Y by compact subspaces, and the bonding homomorphisms ϕ_i are induced by the inclusions and basepoint-change isomorphisms (which are defined using subpaths of ω). One can show the independence with respect to the filtration. Also, properly homotopic base rays yield pro-isomorphic fundamental pro-groups. We refer to [4; 9] for more details and the basics of the pro-category of towers of groups.

Given a CW–complex X with $\pi_1(X) \cong G$ we will denote by $\widetilde{X}$ the universal cover of X, constructed as prescribed in [4, Section 3.2], so that G acts freely on the CW–complex $\widetilde{X}$ via a cell-permuting left action with $G \backslash \widetilde{X} = X$. The number of ends of an (infinite) finitely generated group G represents the number of ends of the (strongly) locally finite CW–complex $\widetilde{X}^1$, for some (equivalently any) CW–complex X with $\pi_1(X) \cong G$ and with finite 1–skeleton, which is either 1, 2 or ∞ (finite groups have 0 ends [4; 15]). If G is finitely presented, then G is *semistable at each end* (or *at infinity*, if G is 1–ended) if the (strongly) locally finite CW–complex $\widetilde{X}^2$ is so, for some (equivalently any) CW–complex X with $\pi_1(X) \cong G$ and with finite 2–skeleton. In fact, we will refer to the fundamental pro-group of $\widetilde{X}^2$ (at each end) as the fundamental pro-group of G (at each end), and will say that G is *simply connected at each end* (or *at infinity*, if G is 1–ended) if it has pro-trivial fundamental pro-group at each end. Observe that any finite-dimensional locally finite CW–complex is strongly locally finite; see [4, Proposition 10.1.12].

3 Proof of Theorem 1.1

We need the following result, which can be seen as a generalization of [12, Lemma 5]:

Proposition 3.1 *Let G be a finitely presented group which is semistable at each end, and assume G splits as an amalgamated product $G_0 *_F G_1$ (resp. an HNN-extension $H*_F$) over a finite group F. Then each factor G_0 and G_1 (resp. the base group H) is either finite or else it is also semistable at each end. Moreover, if the fundamental pro-group of G at a certain end is nontrivial then it is pro-isomorphic to the fundamental pro-group of one of the factors (resp. the base group) at one of its ends.*

The proof of the first part of this proposition mimics that of [7, Lemma 3.2]. Nonetheless, we include a detailed proof for the sake of completeness. It is worth mentioning that independently and essentially simultaneously, Mihalik has recently shown a similar result; see [11, Theorem 3.3].

Proof Let G_0 and G_1 be finitely presented groups and F be a finite group with presentation $\langle e_1, \dots, e_n; r_1, \dots, r_m \rangle$. Consider monomorphisms $\varphi_i \colon F \to G_i$ for $i = 0, 1$, and denote by $G = G_0 *_F G_1 = \langle G_0, G_1; \varphi_0(e_i) = \varphi_1(e_i), 1 \le i \le n \rangle$ the corresponding amalgamated product. Let K_0 and K_1 be finite 2–dimensional CW–complexes with $\pi_1(K_i) \cong G_i$, and let $f_i \colon \bigvee_{i=1}^n S^1 \to K_i$ $(i = 0, 1)$ be cellular maps such that $\operatorname{Im} f_{i_*} \subseteq \pi_1(K_i)$ corresponds to the subgroup $\operatorname{Im} \varphi_i \subseteq G_i$. Let L' be the standard 2–complex associated to the given presentation of F, with 1–cells $e_1, \dots, e_n$. Namely, L' is a wedge of circles each of which is directed and labeled by one of the generators, and a 2–cell attached to the 1–skeleton according to each of the defining relations. Consider the adjunction spaces $L = \left(\bigvee_{i=1}^{n} e_i\right) \times I \cup_{\left(\bigvee_{i=1}^{n} e_i\right)\times\{\frac{1}{2}\}} L' \times \left\{\frac{1}{2}\right\}$ (homotopy equivalent to L') and $X = L \cup_{f_0\times\{0\}\cup f_1\times\{1\}} (K_0 \sqcup K_1)$. By van Kampen's theorem, X is a finite 2–dimensional CW–complex with $\pi_1(X) \cong G_0 *_F G_1$. Let $\widetilde{X}$ be the universal cover of X with covering map $p\colon \widetilde{X} \to X$. Observe that $p^{-1}(K_i)$ consists of a disjoint union of copies of the universal cover $\widetilde{K}_i$ of K_i, since the inclusion $K_i \hookrightarrow X$ induces a monomorphism $G_i \hookrightarrow G_0 *_F G_1$ between the fundamental groups, for $i = 0, 1$; see [8]. Also, $p^{-1}\left(L' \times \left\{\frac{1}{2}\right\}\right)$ consists of a disjoint union of copies of the universal cover $\widetilde{L}'$ of L', as the inclusion $L' \times \left\{\frac{1}{2}\right\} \hookrightarrow X$ induces a monomorphism $F \hookrightarrow G_0 *_F G_1$. Let J be a connected component of $p^{-1}\left(\left(\bigvee_{i=1}^{n} e_i\right) \times \left\{\frac{1}{2}\right\}\right) \subset p^{-1}\left(L' \times \left\{\frac{1}{2}\right\}\right)$. Observe that J is a copy of the Cayley graph of F, and consider the finite 2–dimensional CW–complex $Y' = (J \times I) \cup_{J\times\{\frac{1}{2}\}} \widetilde{L}' \times \left\{\frac{1}{2}\right\}$. Thus, the universal cover $\widetilde{X}$ can be regarded as a push-out obtained from $\bigcup_{g\in G} g\widetilde{K}_0$, $\bigcup_{g\in G} g\widetilde{K}_1$ and $\bigcup_{g\in G} gY'$ (compare with the description given in [13], and see also [15, Section 3]) where:

(a) $\bigcup_{g\in G} g\widetilde{K}_0$ and $\bigcup_{g\in G} g\widetilde{K}_1$ are unions of copies of the "vertex spaces" $\widetilde{K}_0$ and $\widetilde{K}_1$, respectively, such that $g\widetilde{K}_0 \equiv g'\widetilde{K}_0$ if and only if $g^{-1}g' \in G_0$ (and are disjoint otherwise) and $g\widetilde{K}_1 \equiv g'\widetilde{K}_1$ if and only if $g^{-1}g' \in G_1$ (and are disjoint otherwise).

(b) $\bigcup_{g\in G} gY'$ is a union of copies of the "edge space" Y' such that the subcomplex corresponding to $J \times \{i\}$ inside gY' for $i = 0, 1$ is glued to $g\widetilde{K}_i$ via a lift $\tilde{f}_{i.g}\colon J\times\{i\} \to g\widetilde{K}_i$ of the map f_i. Furthermore $\tilde{f}_{i.g} \equiv \tilde{f}_{i.g'}$ if $g^{-1}g' \in \operatorname{Im}\varphi_i \cong F$ for $i = 0, 1$ (and their images are disjoint otherwise).

Next, fix a copy gY' and consider each map $\tilde{f}_{i,g}\colon J \times \{i\} \to g\widetilde{K}_i$ for $i = 0, 1$. Observe that this map is nullhomotopic (as $J \times \{i\}$ is a finite connected 1–dimensional complex and $g\widetilde{K}_i$ is simply connected), so we can replace it (without altering the homotopy type of the entire construction) by a constant map $h_{i,g}\colon J \times \{i\} \to g\widetilde{K}_i$ whose image is a vertex inside $\operatorname{Im} \tilde{f}_{i,g}$. We do the same for any other copy $g'Y'$ and maps $\tilde{f}_{i,g'}$ for $i = 0, 1$ via homotopies which may be taken as translates (within $\widetilde{X}$) of those for gY' and $\tilde{f}_{i,g}$, $i = 0, 1$. Since the G–action on $\widetilde{X}$ is properly discontinuous, the collection of all these homotopies together with the gluing lemma [1, Lemma I.4.9] yields a proper homotopy equivalence between $\widetilde{X}$ and a new push-out $\widehat{X}$ in the proper category, where $\widehat{X}$ can be seen as the 2–dimensional CW–complex obtained from a collection of copies of $Y = \Sigma J \cup_{J\times\{\frac{1}{2}\}} \widetilde{L}' \times \left\{\frac{1}{2}\right\}$ (here "Σ" stands for "suspension") and the collection of copies $g\widetilde{K}_0$ and $g\widetilde{K}_1$ of $\widetilde{K}_0$ and $\widetilde{K}_1$ glued together appropriately through the suspension vertices of the copies of $\Sigma J \subset Y$ (via the image of the new maps $h_{i,g}$)

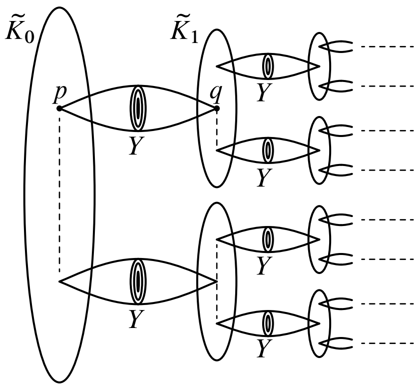

Figure 1

in such a way that the copies of Y inside $\widehat{X}$ are in a bijective correspondence with the copies gY' in the construction of $\widetilde{X}$. Observe that the universal cover $\widetilde{X}$ is modeled after the Bass–Serre tree for $G = G_0 *_F G_1$, but the group G is no longer acting on $\widehat{X}$. Nevertheless, $\widehat{X}$ still keeps that same tree-like structure. See Figure 1 for a pictorial description of $\widehat{X}$ in the infinite ended case.

We choose the copy of Y in $\widehat{X}$ (which we again refer to as Y) corresponding to the copy Y' for $\widetilde{X}$ (ie $g \equiv 1$), and consider the copies $\widetilde{K}_0$ and $\widetilde{K}_1$ in $\widehat{X}$ of the universal cover of K_0 and K_1, which intersect Y at vertices $p \in \widetilde{K}_0$ and $q \in \widetilde{K}_1$, taken as basepoints. We consider filtrations by compact subsets $C_{i,1} \subset C_{i,2} \subset \cdots \subset \widetilde{K}_i$ with $p \in C_{0,1}$ and $q \in C_{1,1}$, and such that no vertex of $\widetilde{K}_i$ is in the boundary of any of the $C_{i,j}$. These $C_{i,j}$ may be taken as finite subcomplexes in some barycentric subdivision of $\widetilde{K}_i$. We proceed to build a filtration by compact subsets $C_1 \subset C_2 \subset \cdots \subset \widehat{X}$ inductively. The subset C_1 consists of $Y \cup C_{0,1} \cup C_{1,1}$. Assume C_n is constructed. Then, C_{n+1} is the union of $C_n \cup C_{0,n+1} \cup C_{1,n+1}$, the copies of Y which intersect C_n, and the translates of $C_{0,n+1}$ and $C_{1,n+1}$ on all those copies $g\widetilde{K}_0$ and $g\widetilde{K}_1$ which may intersect these copies of Y at a vertex (so that the corresponding basepoint on $\widetilde{K}_i$ is sent to the corresponding intersection vertex in $g\widetilde{K}_i$).

Observe that the fundamental pro-group of $G = G_0 *_F G_1$ at each end is represented by

$$\text{pro-}\pi_1(\widetilde{X}) \cong \text{pro-}\pi_1(\widehat{X}) \equiv \{\{1\} \leftarrow \pi_1(\widehat{X} - C_1) \leftarrow \pi_1(\widehat{X} - C_2) \leftarrow \cdots\},$$

where the basepoints are taken on any base ray determining the given end. We will show that if $\widetilde{K}_0$ is noncompact then $\widetilde{K}_0$ (and hence the group G_0) is semistable at each end; the proof for G_1 is analogous. In fact, one can define continuous retractions

$h_j \colon \widehat{X} - C_j \to \widetilde{K}_0 - C_{0,j}$ as follows. Fix $p_j \in \widetilde{K}_0 - C_{0,j}$ (along a given base ray). If $z \in \widehat{X} - C_j$ is in $\widetilde{K}_0 - C_{0,j}$ then we set $h_j(z) = z$. Otherwise, define $h_j(z)$ in the following way. Let γ be a path in $\widehat{X}$ from z to p, and let w be the vertex at which γ meets $\widetilde{K}_0$ for the first time. If $w \notin C_{0,j}$ then we define $h_j(z) = w$, otherwise, we set $h_j(z) = p_j$. One can check that each h_j is a continuous retraction, by the choice of the compact subsets $C_{0,j} \subset \widetilde{K}_0$. Observe that h_j is not a proper map, as it maps noncompact subsets of each copy of the universal cover of K_i (except for $\widetilde{K}_0$) to a point; in fact, if a copy of Y is being attached to a vertex w of $\widetilde{K}_0 - C_{0,j}$, then the connected component of $\widehat{X} - \{w\}$ whose closure contains that copy of Y is sent by h_j to the single point w. However, as a loop in $\widehat{X}$ is a product (modulo change of basepoints) of loops each of which lives inside some copy of Y, $g\widetilde{K}_0$ or $g\widetilde{K}_1$, one can check that these maps h_j lead us to commutative diagrams

$$\begin{tikzcd}
\pi_1(\widetilde{K}_0 - C_{0,j}) \arrow[d] & \pi_1(\widetilde{K}_0 - C_{0,j+1}) \arrow[l] \arrow[d] \\
\pi_1(\widehat{X} - C_j) \arrow[d, "(h_j)_*"'] & \pi_1(\widehat{X} - C_{j+1}) \arrow[l] \arrow[d, "(h_{j+1})_*"] \\
\pi_1(\widetilde{K}_0 - C_{0,j}) & \pi_1(\widetilde{K}_0 - C_{0,j+1}) \arrow[l]
\end{tikzcd}$$

where the unmarked arrows are induced by the inclusions, and the composition of any two consecutive vertical arrows is the corresponding identity homomorphism. Therefore the homomorphisms $\{(h_j)_*\}_{j\geq 1}$ are all surjective, and the conclusion of the first part of the proposition follows as G is semistable at each end and hence pro-$\pi_1(\widehat{X})$ ($\cong$ pro-$\pi_1(\widetilde{X})$) is pro-isomorphic to a tower whose bonding maps are also surjective homomorphisms; see Section 2. In fact, given the filtrations above, one can easily check that a repeated use of van Kampen's theorem yields that each vertical inclusion-induced homomorphism (in the diagram above) is an isomorphism (and hence so is each $(h_j)_*$), by construction of $\widehat{X}$.

In the case of an HNN-extension $H*_F = \langle H, t; t^{-1}\psi_0(e_i)t = \psi_1(e_i), 1 \leq i \leq n\rangle$ with monomorphisms $\psi_i \colon F \to H$ for $i = 0, 1$, let K be a finite 2–dimensional CW–complex with $\pi_1(K) \cong H$ and $f_i \colon \bigvee_{i=1}^n S^1 \to K$ for $i = 0, 1$ be cellular maps such that $\operatorname{Im} f_{i*} \subseteq \pi_1(K)$ corresponds to the subgroup $\operatorname{Im}\psi_i \subseteq H$. Let L be the 2–dimensional CW–complex constructed as above and consider the adjunction space $X = L \cup_{f_0\times\{0\}\cup f_1\times\{1\}} K$, with $\pi_1(X) \cong H*_F$. Then the proof is similar to the one given above for the amalgamated product.

For the second part of the proposition, observe that the fundamental pro-group of G at a certain end can also be thought of as the fundamental pro-group of $\widehat{X}$ at the corresponding end, as the universal cover $\widetilde{X}$ is proper homotopy equivalent to $\widehat{X}$. Moreover, if G is semistable at each end then so is the 2–dimensional CW–complex $\widehat{X}$. Consider a base ray $\omega\colon [0,\infty) \to \widehat{X}$ with $\omega(t_n) \in \widehat{X} - C_n$ for $n \geq 1$. We are to study pro-$\pi_1(\widehat{X},\omega)$. For this, we will distinguish the following two cases:

Case 1 There is a copy $g\widetilde{K}_i \subset \widehat{X}$ for $i = 0, 1$ such that for all $t \geq 0$ there exists $t' \geq t$ with $\omega(t') \in g\widetilde{K}_i$. In particular, the end determined by ω can be represented by a sequence of points within the copy $g\widetilde{K}_i \subset \widehat{X}$. In this case, we can consider a reparametrization ω' of ω such that $\omega'(k) \in g\widetilde{K}_i$ for all $k \in \mathbb{N}$ (and yielding pro-isomorphic fundamental pro-groups). On the other hand, we may also consider a proper ray $\omega''\colon [0,\infty) \to g\widetilde{K}_i$ satisfying $\omega''(k) = \omega'(k)$ for all $k \in \mathbb{N}$, and hence defining the same end of $\widehat{X}$ as ω' (and ω). By the semistability hypothesis, we have a pro-isomorphism pro-$\pi_1(\widehat{X},\omega') \cong$ pro-$\pi_1(\widehat{X},\omega'')$; in fact, it is not hard to show that two such proper rays ω' and ω'' are always properly homotopic within $\widehat{X}$; compare with [10, Lemma 4]. Finally, taking similar filtrations as in the first part of the proof, one can easily check that a repeated use of van Kampen's theorem yields a pro-isomorphism pro-$\pi_1(\widehat{X},\omega'') \cong$ pro-$\pi_1(g\widetilde{K}_i,\omega'')$, by construction of $\widehat{X}$. Thus, this relates the fundamental pro-group of G at a certain end (determined by ω) to the fundamental pro-group of one of the factors G_i at one of its ends. In the case of an HNN-extension the argument is similar.

Case 2 For every copy $g\widetilde{K}_i \subset \widehat{X}$ of $\widetilde{K}_i$ for $i = 0, 1$ there exists $t_{i,j} \geq 0$ such that $\omega([t_{i,j},\infty)) \cap g\widetilde{K}_i = \varnothing$ (ie ω eventually leaves any copy $g\widetilde{K}_i$ inside $\widehat{X}$). In this case, by construction, one can easily check that for each $n \geq 1$ there exists $\phi(n) \geq n$ (sufficiently large) so that every loop in the component of $\widehat{X} - C_{\phi(n)}$ which contains $\omega(t_{\phi(n)})$ is homotopic within the component of $\widehat{X} - C_n$ which contains $\omega(t_n)$ to a constant map (with image one of the suspension vertices of some copy of $Y \subset \widehat{X}$), and hence the inclusion-induced homomorphism $\pi_1(\widehat{X} - C_{\phi(n)}, \omega(t_{\phi(n)})) \to \pi_1(\widehat{X} - C_n, \omega(t_n))$ is trivial. Thus, the fundamental pro-group pro-$\pi_1(\widehat{X},\omega)$, which is also represented by the tower (see [4, Section 16.2])

$$\{1\} \leftarrow \pi_1(\widehat{X}-C_1,\omega(t_1)) \leftarrow \pi_1(\widehat{X}-C_{\phi(1)},\omega(t_{\phi(1)})) \leftarrow \pi_1(\widehat{X}-C_{\phi^2(1)},\omega(t_{\phi^2(1)})) \cdots,$$

is pro-isomorphic to the trivial tower.

Therefore, if the fundamental pro-group of G at a certain end is nontrivial, then Case 1 above must hold and the conclusion follows. □

Observe that any finitely presented group G with more than one end can be decomposed as the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ whose edge groups are finite and whose vertex groups are finitely presented groups with at most one end, by Stallings' structure theorem [16] and Dunwoody's accessibility theorem for finitely presented groups [3]. Thus, as the fundamental group of a graph of groups with $n+1$ edges can be built out of graphs with fewer edges (by amalgamated products or HNN-extensions), an inductive argument gives us:

Corollary 3.2 *Let G be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ with finite edge groups and finitely presented vertex groups with at most one end. If G is semistable at each end then each vertex group in $(\mathcal{G}, \Gamma)$ is either finite or else it is semistable at infinity. Moreover, if the fundamental pro-group of G at a certain end is nontrivial then it is pro-isomorphic to the fundamental pro-group of one of the vertex groups in $(\mathcal{G}, \Gamma)$.*

Finally, the following result is also crucial for the proof of Theorem 1.1:

Proposition 3.3 [2, Proposition 2.9] *Let G and H be two finitely presented groups which are 1–ended and semistable at infinity. Then G and H are proper 2–equivalent if and only if they have pro-isomorphic fundamental pro-groups.*

Proof of Theorem 1.1 Assume that G and H are expressed as the fundamental group of some finite graphs of groups $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ respectively, with finite edge groups and finitely presented vertex groups with at most one end. In particular, they can be expressed as a combination of amalgamated products and HNN-extensions of the corresponding vertex groups over the corresponding (finite) edge groups. Let G_i and H_j denote the vertex groups in the graphs of groups $(\mathcal{G}, \Gamma)$ and $(\mathcal{H}, \Gamma')$ respectively. Following an argument similar to that in the proof of Proposition 3.1 one can get two finite graphs of 2–dimensional CW–complexes $(\mathcal{X}, \Gamma)$ and $(\mathcal{Y}, \Gamma')$ with vertex spaces X_i and Y_j having $\pi_1(X_i) \cong G_i$ and $\pi_1(Y_j) \cong H_j$, and such that their associated total complexes $\mathbb{X}$ and $\mathbb{Y}$ (obtained as the corresponding adjunction spaces [4, Section 6.2]) are finite 2–dimensional CW–complexes satisfying $\pi_1(\mathbb{X}) \cong G$ and $\pi_1(\mathbb{Y}) \cong H$.

Again, as in the proof of Proposition 3.1, one can make the universal covers $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ proper homotopy equivalent to 2–dimensional CW–complexes $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{Y}}$ each one obtained from a collection of copies of the universal covers of the corresponding vertex spaces, and a collection of copies of the suspension of the Cayley graphs of the edge spaces together with a copy of the (finite) universal cover of the corresponding edge space attached at the middle level, all glued together appropriately so that all copies of the universal covers of the vertex spaces are mutually disjoint, and if a copy of the universal cover of a vertex space intersects a copy of one of the suspensions above corresponding to an edge space it does so in exactly one of the suspension vertices. By [2, Theorem 3.5], there exists a proper 2–equivalence between the universal covers $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ (as G and H are proper 2–equivalent), which in turn yields a proper 2–equivalence $f:\widehat{\mathbb{X}}\to\widehat{\mathbb{Y}}$.

Assume there is a vertex space X_{i_0} in $(\mathcal{X},\Gamma)$ such that pro-$\pi_1(\widetilde{X}_{i_0},\omega)$ is nontrivial (meaning not pro-isomorphic to the trivial tower) for some (equivalently any) base ray $\omega\colon [0,\infty)\to\widetilde{X}_{i_0}$. By construction, one can easily check that pro-$\pi_1(\widetilde{X}_{i_0},\omega)$ is pro-isomorphic to pro-$\pi_1(\widehat{\mathbb{X}},\omega)$, which in turn is pro-isomorphic to pro-$\pi_1(\widehat{\mathbb{Y}},f\circ\omega)$; see [4, Proposition 16.2.3]. By Corollary 3.2 (see also the proof of Proposition 3.1) there must be some vertex space Y_{j_0} in $(\mathcal{Y},\Gamma')$ such that pro-$\pi_1(\widehat{\mathbb{Y}},f\circ\omega)$ is pro-isomorphic to the fundamental pro-group of Y_{j_0}. Therefore, the corresponding vertex groups G_{i_0} and H_{j_0} are 1–ended and semistable at infinity (by Corollary 3.2) with pro-isomorphic fundamental pro-groups and hence proper 2–equivalent by Proposition 3.3. And conversely, for each vertex group in $(\mathcal{H},\Gamma')$ with nontrivial fundamental pro-group there is some vertex group in $(\mathcal{G},\Gamma)$ with pro-isomorphic fundamental pro-group, and hence in the same proper 2–equivalence class.

Finally, the second part of the theorem follows from the fact that any 1–ended and simply connected at infinity finitely presented group is proper 2–equivalent to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, by [6, Corollary 1.3] and [2, Theorem 5.1]. $\square$

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References

[1] **H-J Baues**, **A Quintero**, *Infinite homotopy theory*, K–Monographs in Mathematics 6, Kluwer, Dordrecht (2001) MR Zbl

[2] **M Cárdenas**, **F F Lasheras**, **A Quintero**, **R Roy**, *A topological equivalence relation for finitely presented groups*, J. Pure Appl. Algebra 224 (2020) art. id. 106300 MR Zbl

[3] **M J Dunwoody**, *The accessibility of finitely presented groups*, Invent. Math. 81 (1985) 449–457 MR Zbl

[4] **R Geoghegan**, *Topological methods in group theory*, Graduate Texts in Math. 243, Springer (2008) MR Zbl

[5] **M Gromov**, *Asymptotic invariants of infinite groups*, from "Geometric group theory, II" (G A Niblo, M A Roller, editors), Lond. Math. Soc. Lect. Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR Zbl

[6] **F F Lasheras**, *Ascending HNN-extensions and properly 3–realisable groups*, Bull. Austral. Math. Soc. 72 (2005) 187–196 MR Zbl

[7] **F F Lasheras**, **R Roy**, *Relating the Freiheitssatz to the asymptotic behavior of a group*, Rev. Mat. Iberoam. 29 (2013) 75–89 MR Zbl

[8] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. 89, Springer (1977) MR Zbl

[9] **S Mardešić**, **J Segal**, *Shape theory*, North-Holland Math. Library 26, North-Holland, Amsterdam (1982) MR Zbl

[10] **M L Mihalik**, *Semistability at ∞, ∞–ended groups and group cohomology*, Trans. Amer. Math. Soc. 303 (1987) 479–485 MR Zbl

[11] **M Mihalik**, *Semistability of graph products*, preprint (2020) arXiv 2004.11333

[12] **M L Mihalik**, **S T Tschantz**, *One relator groups are semistable at infinity*, Topology 31 (1992) 801–804 MR Zbl

[13] **M L Mihalik**, **S T Tschantz**, *Semistability of amalgamated products and HNN-extensions*, Mem. Amer. Math. Soc. 471, Amer. Math. Soc., Providence, RI (1992) MR Zbl

[14] **P Papasoglu**, **K Whyte**, *Quasi-isometries between groups with infinitely many ends*, Comment. Math. Helv. 77 (2002) 133–144 MR Zbl

[15] **P Scott**, **T Wall**, *Topological methods in group theory*, from "Homological group theory" (C T C Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 137–204 MR Zbl

[16] **J Stallings**, *Group theory and three-dimensional manifolds*, Yale Mathematical Monographs 4, Yale Univ. Press, New Haven, CT (1971) MR Zbl

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