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Suspension homotopy of 6–manifolds

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For a simply connected closed orientable manifold of dimension 6, we compute its homotopy decomposition after double suspension. This allows us to determine its K– and KO–groups easily. Moreover, in a special case we refine the decomposition to show the rigidity property of the manifold after double suspension.

55P15, 55P40, 57R19; 55N15, 55P10

1 Introduction

Let M be a closed orientable smooth manifold of dimension n. Numerous investigations have been made in geometric topology into the diffeomorphism or homeomorphism type of M in various cases. For instance, in the general case, Wall [30; 32] studied $(s-1)$–connected $2s$–manifolds and $(s-1)$–connected $(2s+1)$–manifolds. For concrete cases with specified dimension n, Barden [1] classified simply connected 5–manifolds, and Wall [31], Jupp [18] and Zhubr [35; 36] classified simply connected 6–manifolds. More recently, Kreck and Su [20] classified certain nonsimply connected 5–manifolds, while Crowley and Nordström [11] and Kreck [19] studied the classification problem of various kinds of 7–manifolds.

In the literature mentioned, the homotopy classification of M was usually carried out as a byproduct in terms of a system of invariants. However, it is almost impossible to extract nontrivial homotopy information of M directly from the classification. On the other hand, unstable homotopy theory is a powerful tool for studying the homotopy properties of manifolds. There have been several interesting investigations recently in this direction. For instance, Beben and Theriault [6] studied the loop decompositions of $(s-1)$–connected $2s$–manifolds, while Beben and Wu [8] and Huang and Theriault [17] studied the loop decompositions of the $(s-1)$–connected $(2s+1)$–manifolds. The homotopy groups of these manifolds were also investigated by Samik Basu and Somnath

Basu [2; 3] from a different point of view. Moreover, a theoretical method of loop decomposition was developed by Beben and Theriault [7], which is quite useful for studying the homotopy of manifolds. Additionally, the homotopy type of the suspension of a connected 4–manifold was determined by So and Theriault [28].

We study the homotopy of simply connected 6–manifolds. Let M be a simply connected closed orientable 6–manifold. By Poincaré duality and the universal coefficient theorem we have

$$H_*(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\oplus d} \oplus T & \text{if } * = 2, \\ \mathbb{Z}^{\oplus 2m} \oplus T & \text{if } * = 3, \\ \mathbb{Z}^{\oplus d} & \text{if } * = 4, \\ \mathbb{Z} & \text{if } * = 0, 6, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $m, d \geq 0$, and T is a finitely generated abelian torsion group. Our first main theorem concerns the double suspension splitting of M. Let ΣX denote the suspension of any CW–complex X. Let $P^n(T)$ be the Moore space such that the reduced cohomology $\widetilde{H}^*(P^n(T);\mathbb{Z})$ is isomorphic to T if $* = n$ and 0 otherwise; see Neisendorfer [25].

Theorem 1.1 *Suppose M is a simply connected closed orientable 6–manifold with homology of the form (1). Suppose that T has no 2 or 3–torsion. Then there is an integer c with $0 \leq c \leq d$ determined by the cohomology ring of M such that:*

- *If $c = 0$,*
$$\Sigma^2 M \simeq \Sigma W_0 \vee \bigvee_{j=1}^{d-1} (S^4 \vee S^6) \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$
where $W_0 \simeq (S^3 \vee S^5) \cup e^7$.
- *If $c = d$,*
$$\Sigma^2 M \simeq \Sigma W_d \vee \bigvee_{i=1}^{d-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$
where $W_d \simeq \Sigma \mathbb{C}P^2 \cup e^7$.
- *If $1 \leq c \leq d-1$,*
$$\Sigma^2 M \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6) \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$
where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$.

Theorem 1.1 classifies the homotopy type of $\Sigma^2 M$ up to an indeterminate term ΣW_c for $0 \leq c \leq d$. The Steenrod square $Sq^2 \colon H^2(M;\mathbb{Z}/2\mathbb{Z}) \to H^4(M;\mathbb{Z}/2\mathbb{Z})$ determines c. Since we need only the suspension of W_c, the attaching map of the top cell of W_c can be chosen so that it does not have a Whitehead product as a component. In general, the suspension of this attaching map depends on the manifold M itself; for instance, see Theorem 1.3 below.

Nevertheless, Theorem 1.1 is still useful, for instance, to calculate the K–group or the KO–group of M in Corollary 1.2. In particular, when M is a Calabi–Yau threefold, it partially reproduces the result of Doran and Morgan [12, Corollary 1.10] about its K–group by a different method, and provides a new computation of its KO–group. Moreover, there are many examples of simply connected Calabi–Yau threefolds. For instance, based on Kreuzer and Skarke [21], Batyrev and Kreuzer [4] showed that there are exactly 473 800 760 families of simply connected Calabi–Yau threefolds corresponding to 4–dimensional reflexive polytopes.

Corollary 1.2 *Let M be a manifold satisfying the conditions of Theorem 1.1. Then the reduced K–group and KO–group of M are*

$$\widetilde{K}(M) \cong \mathbb{Z}^{\oplus 2d+1} \oplus T \quad \text{and} \quad \widetilde{KO}(M) \cong \bigoplus_d (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).$$

If we specialize to the case when $d = 1$, we can obtain a complete description of M after double suspension, based on the work of Yamaguchi [34] (also summarized and corrected by Baues [5]). In particular, Yamaguchi's work [34] implies that a generator $x \in H^2(M;\mathbb{Z}) \cong \mathbb{Z}$ has the property that $x^2 = ky$ for some $k \in \mathbb{Z}$ and a generator $y \in H^4(M;\mathbb{Z})$ of infinite order. Let $\eta_i^3 = \eta_{i+2} \circ \eta_{i+1} \circ \eta_i \in \pi_{i+3}(S^i)$ (see Toda [29]), where $\eta_i \in \pi_{i+1}(S^i)$ is the Hopf element. Let V_3 be the manifold that is the total space of the sphere bundle of the oriented $\mathbb{R}^3$–bundle over S^4 determined by its first Pontryagin class $p_1 = 12s_4$, where $s_4 \in H^4(S^4;\mathbb{Z})$ is a generator.

Theorem 1.3 *Suppose M is a simply connected closed orientable 6–manifold with homology of the form* (1) *such that $d = 1$. Let $x \in H^2(M;\mathbb{Z}) \cong \mathbb{Z}$ be a generator. Let $k \in \mathbb{Z}$ be such that $x^2 = ky$ for some generator $y \in H^4(M;\mathbb{Z})$ of infinite order. Suppose T has no 2– or 3–torsion. Then*:

- *If k is odd*, *then M is Spin. Moreover*,

  $$\Sigma^2 M \simeq \Sigma^2 \mathbb{C}P^3 \vee \textstyle\bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T)$$

  when $k \equiv \pm 1 \bmod 6$, *while*

  $$\Sigma^2 M \simeq \Sigma^2 V_3 \vee \textstyle\bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T)$$

  when $k \equiv 3 \bmod 6$.

- *If k is even and M is non-Spin,*

$$\Sigma^2 M \simeq S^4 \vee \Sigma^4\mathbb{C}P^2 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T).$$

- *If k is even and M is Spin,*

$$\Sigma^2 M \simeq (S^4 \cup_{\lambda\eta_4^3} e^8) \vee S^6 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

where $\lambda \in \mathbb{Z}/2$ is determined by M.

It should be remarked that there is no indeterminacy in the term $(S^4 \cup_{\lambda\eta_4^3} e^8)$ in the last decomposition of $\Sigma^2 M$. Indeed, the stable cube element $\eta_n^3 \in \pi_{n+3}(S^n)$ (for $n \geq 2$) is detected by the secondary operation $\mathbb{T}$ (see Harper [13, Exercise 4.2.5]), and in our case the homotopy decomposition has to preserve the module structure induced by the cohomology operations. Moreover, it is clear that $k \bmod 2$ and the spin condition of M are determined by the Steenrod square Sq^2. We will also see that $\Sigma\mathbb{C}P^3$ and ΣV_3 can be distinguished by the Steenrod power $\mathcal{P}^1\colon H^3(\Sigma M;\mathbb{Z}/3\mathbb{Z}) \to H^7(\Sigma M;\mathbb{Z}/3\mathbb{Z})$. Hence, we obtain the following rigidity result for manifolds of the type in Theorem 1.3 after double suspension:

Corollary 1.4 *Let M and M' be two manifolds of the type in Theorem 1.3. Then $\Sigma^2 M \simeq \Sigma^2 M'$ if and only if $H^*(\Sigma^2 M;\mathbb{Z}) \cong H^*(\Sigma^2 M';\mathbb{Z})$ as abelian groups, and $H^*(\Sigma^2 M;\mathbb{Z}/p\mathbb{Z}) \cong H^*(\Sigma^2 M';\mathbb{Z}/p\mathbb{Z})$ as $\mathbb{Z}/2\mathbb{Z}\{Sq^2, \mathbb{T}\}$–modules when $p = 2$, and as $\mathbb{Z}/3\mathbb{Z}\{\mathcal{P}^1\}$–modules when $p = 3$.*

The paper is organized as follows. In Section 2 we reduce the decomposition problem of 6–manifolds to that of those whose third Betti numbers are zero. In Section 3, we give a detailed procedure to decompose 6–manifolds after double suspension by the homology decomposition method. Sections 4 and 5 are devoted to proving Theorems 1.1 and 1.3, respectively. In Section 6, we compute some homotopy groups of odd primary Moore spaces used in Section 3.

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2 Reducing to the case when $b_3(M) = 0$

The following well known splitting theorem for 6–manifolds was proved by Wall [31] in the smooth category, while Jupp [18] pointed out that the theorem holds in the topological category by the same argument.

Theorem 2.1 [31, Theorem 1] *Suppose M is a simply connected closed orientable 6–manifold with third Betti number $b_3(M) = 2m$. Then there exists a 6–manifold M_1 such that*

$$M \cong M_1 \# \mathop{\#}_{m}(S^3 \times S^3) \quad \text{and} \quad H_3(M_1;\mathbb{Q}) = 0.$$

Corollary 2.2 *Let M and M_1 be manifolds as in Theorem 2.1. Then*

$$\Sigma M \simeq \Sigma M_1 \vee \bigvee_{i=1}^{m}(S^4 \vee S^4).$$

Proof Let M_1' and M' be the 5–skeletons of M_1 and M, respectively. It is known that $S^3 \vee S^3$ is the 5–skeleton of $S^3 \times S^3$ and $\Sigma(S^3 \times S^3) \simeq \Sigma(S^3 \vee S^3) \vee S^7$. In particular, there is a homotopy retraction $r\colon \Sigma(S^3 \times S^3) \to \Sigma(S^3 \vee S^3)$. For the connected sum $M_1 \# (S^3 \times S^3)$, there are the obvious pinch maps $q_1\colon M_1 \# (S^3 \times S^3) \to M_1$ and $q_2\colon M_1 \# (S^3 \times S^3) \to S^3 \times S^3$. Consider the composition

$$\begin{aligned}\phi\colon \Sigma(M_1 \# (S^3 \times S^3)) &\xrightarrow{\mu'} \Sigma(M_1 \# (S^3 \times S^3)) \vee \Sigma(M_1 \# (S^3 \times S^3)) \\ &\xrightarrow{Eq_1 \vee (r \circ Eq_2)} \Sigma M_1 \vee \Sigma(S^3 \vee S^3),\end{aligned}$$

where μ' is the standard comultiplication for a suspension, and E denotes the suspension of a map. It is easy to see that ϕ induces an isomorphism on homology and so is a homotopy equivalence by Whitehead's theorem. Since

$$M' \simeq M_1' \vee \bigvee_{i=1}^{m}(S^3 \vee S^3),$$

by repeating the above argument, we obtain the decomposition in the statement of the corollary. □

In Theorem 2.1 the connected summand M_1 satisfies $b_3(M_1) = 0$, so by Corollary 2.2 it suffices to consider such 6–manifolds in the sequel.

3 Homology decomposition of M after suitable suspensions

Let M be a simply connected closed orientable 6–manifold with $b_3(M)=0$. By Poincaré duality and the universal coefficient theorem, we have

$$H_*(M;\mathbb{Z})=\begin{cases}\mathbb{Z}^{\oplus d}\oplus T & \text{if } *=2,\\ T & \text{if } *=3,\\ \mathbb{Z}^{\oplus d} & \text{if } *=4,\\ \mathbb{Z} & \text{if } *=0,6,\\ 0 & \text{otherwise,}\end{cases} \tag{2}$$

where $d\geq 0$, and T is a finitely generated abelian torsion group. We may write

$$T=\bigoplus_{k=1}^{\ell}\mathbb{Z}/p_k^{r_k}\mathbb{Z}, \tag{3}$$

where each p_k is a prime and $r_k\geq 1$.

Instead of using a skeletal decomposition, we may apply a *homology decomposition* to study the cell structure of M. For any finitely generated abelian group A, let $P^n(A)$ be the Moore space such that $\widetilde{H}^*(P^n(A);\mathbb{Z})\cong A$ if $*=n$ and 0 otherwise [25]. The information on the homotopy groups of $P^n(T)$ used in this section will be proved in Section 6.

Theorem 3.1 [14, Theorem 4H.3] *Let X be a simply connected CW–complex. Write $H_i=H_i(X;\mathbb{Z})$. Then there is a sequence of complexes $X_{(i)}$ (for $i\geq 2$) such that:*

- *$H_j(X_{(i)};\mathbb{Z})\cong H_j(X;\mathbb{Z})$ for $j\leq i$ and $H_j(X_{(i)};\mathbb{Z})=0$ for $j>i$.*
- *$X_{(2)}=P^3(H_2)$, and $X_{(i)}$ is defined by a homotopy cofibration*

$$P^i(H_i)\xrightarrow{f_{i-1}}X_{(i-1)}\xrightarrow{\iota_{i-1}}X_{(i)},$$

where f_{i-1} induces a trivial homomorphism

$$f_{i-1*}\colon H_{i-1}(P^i(H_i);\mathbb{Z})\to H_{i-1}(X_{(i-1)};\mathbb{Z}).$$

- $X\simeq \operatorname{hocolim}\{X_{(2)}\xrightarrow{\iota_2}\cdots\xrightarrow{\iota_{i-2}}X_{(i-1)}\xrightarrow{\iota_{i-1}}X_{(i)}\xrightarrow{\iota_i}\cdots\}$.

From this theorem, it is clear that the homology decomposition is compatible with the suspension functor. That is, for X in Theorem 3.1 the sequence of the triples $(\Sigma X_{(i)},Ef_i,E\iota_i)$ is a homology decomposition of ΣX.

3.1 Structure of $M_{(3)}$

By Theorem 3.1 and (2), there is a homotopy cofibration

$$P^3(T) \xrightarrow{f_2} M_{(2)} \xrightarrow{\iota_2} M_{(3)}, \quad \text{where } M_{(2)} \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T). \tag{4}$$

Notice that from (3), we have $P^n(T) \simeq \bigvee_{k=1}^{\ell} P^n(p_k^{r_k})$ by [23] or [25].

Lemma 3.2 *The map* f_2 *in* (4) *is nullhomotopic*, *and hence*

$$M_{(3)} \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T).$$

Proof Since $P^3(T) \simeq \bigvee_{k=1}^{\ell} P^3(p_k^{r_k})$, there is the embedding $j : \bigvee_{i=1}^{\ell} S^2 \to P^3(T)$ of the bottom cells. Consider the commutative diagram

$$\begin{array}{ccccc} \pi_2\big(\bigvee_{i=1}^{\ell} S^2\big) & \xrightarrow{j_*} & \pi_2(P^3(T)) & \xrightarrow{f_{2*}} & \pi_2(M_{(2)}) \\ \cong \big\downarrow \text{hur} & & \cong \big\downarrow \text{hur} & & \cong \big\downarrow \text{hur} \\ H_2\big(\bigvee_{i=1}^{\ell} S^2;\mathbb{Z}\big) & \xrightarrow{j_*} & H_2(P^3(T);\mathbb{Z}) & \xrightarrow{f_{2*}=0} & H_2(M_{(2)};\mathbb{Z}) \end{array}$$

where the Hurewicz homomorphisms hur are isomorphisms by the Hurewicz theorem, $f_{2*} = 0$ on homology by Theorem 3.1, and both maps j_* are epimorphisms. In particular, $f_{2*} \circ j_*$ is trivial on homotopy groups, and hence $f_2 \circ j$ is nullhomotopic. Thus, combined with (4), we have the diagram of homotopy cofibrations

$$\begin{array}{ccccc} \bigvee_{i=1}^{\ell} S^2 & \longrightarrow & * & \longrightarrow & \bigvee_{i=1}^{\ell} S^3 \\ \big\downarrow \bigvee_{i=1}^{\ell} p_k^{r_k} & & \big\downarrow & & \big\downarrow c \\ \bigvee_{i=1}^{\ell} S^2 & \xrightarrow{0} & M_{(2)} & \longrightarrow & M_{(2)} \vee \bigvee_{i=1}^{\ell} S^3 \\ \big\downarrow j & & \big\| & & \big\downarrow \\ P^3(T) & \xrightarrow{f_2} & M_{(2)} & \xrightarrow{\iota_2} & M_{(3)} \end{array} \tag{5}$$

where $p_k^{r_k} : S^n \to S^n$ is a map of degree $p_k^{r_k}$, and c is the induced map.

We claim that c can be chosen to be $i_2 \circ \big(\bigvee_{i=1}^{\ell} p_k^{r_k}\big)$ in (5), where

$$i_2 : \bigvee_{i=1}^{\ell} S^3 \to M_{(2)} \vee \bigvee_{i=1}^{\ell} S^3$$

is the injection onto the sphere summands. Indeed, we may replace (5) by a strictly commutative diagram up to homeomorphism. Start from the upper left square of (5). It becomes strictly commutative once we replace the 0 map by the constant map. On the one hand, extending the new square one step to the right by taking the mapping cone, we obtain $\tilde{c}\colon \bigvee_{i=1}^{\ell} S^3 \to M_{(2)} \vee \bigvee_{i=1}^{\ell} S^3$, which is exactly equal to $i_2 \circ \bigl(\bigvee_{i=1}^{\ell} p_k^{r_k}\bigr)$, and the upper right square automatically commutes. On the other hand, by the homotopy extension property of a cofibration, we may replace $f_2\colon P^3(T) \to M_{(2)}$ by a map $\tilde{f}_2$ such that $\tilde{f}_2 \simeq f_2$ and the lower left square strictly commutes. It follows that the mapping cone of $\tilde{f}_2$ is homotopy equivalent to $M_{(3)}$ and is homeomorphic to the mapping cone of $i_2 \circ (\bigvee_{i=1}^{\ell} p_k^{r_k})$. Hence we can choose $c = i_2 \circ (\bigvee_{i=1}^{\ell} p_k^{r_k})$ in (5).

Now from (5) it follows that

$$M_{(3)} \simeq M_{(2)} \vee P^4(T) \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T), \qquad \square$$

The following corollary follows from Lemma 3.2 and will be used in Lemma 3.6.

Corollary 3.3 *The homotopy cofiber of the obvious inclusion*

$$j\colon P^3(T) \vee P^4(T) \to M_{(3)} \to M$$

is a Poincaré duality complex V with cell structure

$$V = \bigvee_{i=1}^{d} S^2 \cup e^4_{(1)} \cup e^4_{(2)} \cdots \cup e^4_{(d)} \cup e^6.$$

Moreover, by [31, Theorem 8] V is homotopy equivalent to a closed smooth manifold.

3.2 Structure of $M_{(5)}$

By Theorem 3.1 and Lemma 3.2, there is a homotopy cofibration

(6)
$$\bigvee_{i=1}^{d} S^3 \xrightarrow{f_3} M_{(3)} \xrightarrow{\iota_3} M_{(4)} = M_{(5)}, \quad \text{where } M_{(3)} \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T).$$

We need to study the map

$$f_3\colon \bigvee_{i=1}^{d} S^3 \to \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T).$$

Let $i_3\colon P^4(T) \to \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T)$ be the inclusion. Define the complex Y by the homotopy cofibration

$$P^4(T) \xrightarrow{\iota_3 \circ i_3} M_{(4)} = M_{(5)} \to Y.$$

Lemma 3.4 *The map f_3 in* (6) *factors as*

$$f_3\colon \bigvee_{i=1}^d S^3 \xrightarrow{f_3'} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{i_1\vee i_2} \bigvee_{i=1}^d S^2 \vee P^3(T) \vee P^4(T)$$

for some f_3', where i_1 and i_2 are inclusions. Moreover, there is a homotopy cofibration

$$\bigvee_{i=1}^d S^3 \xrightarrow{f_3'} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{\iota_3'} Y, \tag{7}$$

and

$$M_{(5)} \simeq Y \vee P^4(T).$$

Proof First, there is the diagram of homotopy cofibrations

$$\begin{array}{ccccc}
* & \longrightarrow & P^4(T) & = & P^4(T) \\
\downarrow & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle \iota_3\circ i_3} \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f_3} & M_{(3)} & \xrightarrow{\iota_3} & M_{(5)} \\
\| & & \downarrow{\scriptstyle q_{1,2}} & & \downarrow \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f_3'} & \bigvee_{i=1}^d S^2 \vee P^3(T) & \xrightarrow{\iota_3'} & Y
\end{array}$$

where $q_{1,2}$ is the obvious projection, ι_3' is induced from ι_3, and $f_3' := q_{1,2}\circ f_3$. The diagram immediately implies that (7) is a homotopy cofibration.

Let $q_3\colon \bigvee_{i=1}^d S^2 \vee P^3(T)\vee P^4(T) \to P^4(T)$ be the canonical projection. By Theorem 3.1, $f_{3*}\colon H_3\big(\bigvee_{i=1}^d S^3;\mathbb{Z}\big) \to H_3\big(\bigvee_{i=1}^d S^2\vee P^3(T)\vee P^4(T);\mathbb{Z}\big)$ is trivial. In particular, $q_{3*}\circ f_{3*}=0$. Then, by the Hurewicz Theorem, $q_3\circ f_3$ is nullhomotopic. Further, by the Hilton–Milnor Theorem (see §XI.6 of [33], for instance), $\pi_3(S^2\vee P^3(T)\vee P^4(T)) \cong \pi_3(S^2\vee P^3(T))\oplus\pi_3(P^4(T))$, and hence

$$\Big[\bigvee_{i=1}^d S^3, S^2\vee P^3(T)\vee P^4(T)\Big] \cong \Big[\bigvee_{i=1}^d S^3, S^2\vee P^3(T)\Big]\oplus\Big[\bigvee_{i=1}^d S^3, P^4(T)\Big].$$

Under this isomorphism, the homotopy class of f_3 corresponds to $[f_3']+[q_3\circ f_3]$. However since we already showed that $[q_3\circ f_3]=0$, we have $f_3\simeq (i_1\vee i_2)\circ f_3'$, so $M_{(5)}\simeq Y\vee P^4(T)$ as required. □

3.3 Structure of $\Sigma M_{(5)}$

From this point, we may need extra conditions on the torsion group T. First recall that we already showed that by Lemma 3.4 there is a homotopy cofibration

$$\bigvee_{i=1}^d S^3 \xrightarrow{f_3'} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{\iota_3'} Y, \quad \text{where } M_{(5)}\simeq Y\vee P^4(T). \tag{8}$$

Let

$$q_1\colon \bigvee_{i=1}^d S^2 \vee P^3(T) \to \bigvee_{i=1}^d S^2$$

be the canonical projection, let $f_3'' := q_1 \circ f_3' = q_1 \circ q_{1,2} \circ f_3$, and also recall that $P^n(T) \simeq \bigvee_{k=1}^{\ell} P^n(p_k^{r_k})$. Let us suppose $p_k \geq 3$ for each k from now on.

Lemma 3.5 *Suppose T has no 2–torsion. Then there is a homotopy equivalence*

$$\Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T),$$

where X is the homotopy cofiber of the map $f_3''\colon \bigvee_{i=1}^d S^3 \to \bigvee_{i=1}^d S^2$.

Proof There is the diagram of homotopy cofibrations

$$\begin{array}{ccccc}
* & \longrightarrow & P^3(T) & = & P^3(T) \\
\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle \iota_3' \circ i_2} \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f_3'} & \bigvee_{i=1}^d S^2 \vee P^3(T) & \xrightarrow{\iota_3'} & Y \\
\| & & \downarrow{\scriptstyle q_1} & & \downarrow \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f_3''} & \bigvee_{i=1}^d S^2 & \longrightarrow & X
\end{array}$$

where i_2 is the canonical inclusion. Since $\pi_4(P^4(p^r)) = 0$ for odd p by Lemma 6.3, we have $\Sigma Y \simeq \Sigma X \vee P^4(T)$. The lemma then follows from (8). □

3.4 Structure of $\Sigma^2 M$

Recall, when T has no 2–torsion, by Lemma 3.5 there is a homotopy cofibration

(9) $$S^5 \xrightarrow{f_5} M_{(5)} \xrightarrow{\iota_5} M, \quad \text{where } \Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T).$$

Further, by Corollary 3.3 we have the homotopy cofibration

$$S^5 \to X \to V,$$

where X is defined in Lemma 3.5 without restriction on T, and V is a closed smooth manifold. We now further suppose T has no 3–torsion.

Lemma 3.6 *Suppose T has no 2– or 3–torsion. Then*

$$\Sigma^2 M \simeq \Sigma^2 V \vee P^6(T) \vee P^5(T).$$

Proof By the Hilton–Milnor theorem, we may write the suspension of f_5 as

$$Ef_5 = g_5^{(1)} + g_5^{(2)} + g_5^{(3)} + \theta\colon S^6 \to \Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T),$$

for some θ, where $E\theta = 0$, $g_5^{(i)} = q_i \circ Ef_5$, and q_i is the canonical projection of $\Sigma X \vee P^5(T) \vee P^4(T)$ onto its i^{th} summand. Then by Lemma 6.4 $g_5^{(2)} = 0$, and $Eg_5^{(3)} = 0$ by Lemma 6.6. It follows that $E^2 f_5 = Eg_5^{(1)}$. Furthermore, there is the diagram of homotopy cofibrations

$$\begin{array}{ccccc}
* & \longrightarrow & P^4(T) \vee P^3(T) & = & P^4(T) \vee P^3(T) \\
\downarrow & & \downarrow{\scriptstyle j_5} & & \downarrow{\scriptstyle j} \\
S^5 & \xrightarrow{f_5} & M_{(5)} & \xrightarrow{\iota_5} & M \\
\| & & \downarrow{\scriptstyle \pi_5} & & \downarrow{\scriptstyle \pi} \\
S^5 & \longrightarrow & X & \longrightarrow & V
\end{array}$$

where the homotopy cofibration in the last column is defined in Corollary 3.3 by using Lemma 3.2, and similarly the homotopy cofibration in the middle column can be also defined by using Lemma 3.2. Then it is clear that $g_5^{(1)} \simeq E(\pi_5 \circ f_5)$ and the lemma follows. □

4 Proof of Theorem 1.1 and Corollary 1.2

In Lemma 3.6 we established the double suspension splitting of M when $b_3(M) = 0$ and are now left to consider the homotopy type of V after suspension. Recall that V is a Poincaré duality complex of dimension 6, and its 5–skeleton $V_5 = X$ is the homotopy cofiber of the map $f_3''\colon \bigvee_{i=1}^d S^3 \to \bigvee_{i=1}^d S^2$ by Lemma 3.5. The following lemma, as a special case of [16, Lemma 6.1], determines the suspension homotopy type of X.

Lemma 4.1 [16, Lemma 6.1] *There is a homotopy equivalence*

$$\Sigma X \simeq \bigvee_{i=1}^c \Sigma \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c} (S^3 \vee S^5)$$

for some $0 \le c \le d$.

We may apply the method in [16, Section 3] to decompose $\Sigma^2 V$, in the same way that we used it to prove [16, Lemmas 6.4 and 6.6].

Lemma 4.2 *Suppose ΣX decomposes as in Lemma 4.1.*

- *If $c = 0$,*
$$\Sigma^2 V \simeq \Sigma W_0 \vee \bigvee_{j=1}^{d-1} (S^4 \vee S^6),$$
where $W_0 \simeq (S^3 \vee S^5) \cup e^7$.
- *If $c = d$,*
$$\Sigma^2 V \simeq \Sigma W_d \vee \bigvee_{i=1}^{d-1} \Sigma^2 \mathbb{C}P^2,$$
where $W_d \simeq \Sigma \mathbb{C}P^2 \cup e^7$.
- *If $1 \le c \le d-1$,*
$$\Sigma^2 V \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6),$$
where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$.

Proof Since the proof is similar to those of [16, Lemmas 6.4 and 6.6], we only sketch it. The interested reader can find the details in [16, Section 3]. Using Lemma 4.1, let $g\colon S^6 \to \Sigma X \simeq \bigvee_{i=1}^{c} \Sigma \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c} (S^3 \vee S^5)$ be the attaching map of the top cell of ΣV. To apply the method in [16, Section 3], we only need information about the homotopy groups $\pi_6(\Sigma \mathbb{C}P^2) \cong \mathbb{Z}/6\mathbb{Z}$ by [22, Proposition 8.2(i)], $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$ and $\pi_6(S^5) \cong \mathbb{Z}/2$, which are all finite cyclic groups. Then we can represent the attaching map Eg of the top cell of $\Sigma^2 V$ by a matrix B, and apply [16, Lemma 3.1] to transform B to a simpler matrix C. The new matrix representation C of the attaching map, corresponding to a base change of ΣX through a self homotopy equivalence, will give the desired decomposition. □

Proof of Theorem 1.1 First, by Theorem 2.1 and Corollary 2.2, we have
$$\Sigma M \simeq \Sigma M_1 \vee \bigvee_{i=1}^{m} (S^4 \vee S^4),$$
where M_1 is a closed 6–manifold with homology of the form (2). In particular $b_3(M_1) = 0$. Hence, by Lemmas 3.6 and 4.2, we have that if $1 \le c \le d-1$ then
$$\Sigma^2 M \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6) \vee P^6(T) \vee P^5(T) \vee \bigvee_{i=1}^{2m} S^5,$$
where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$. The decompositions for the cases when $c = 0$ or $c = d$ can be obtained similarly. Finally, notice that c records the number of the nontrivial Steenrod square $Sq^2\colon H^2(\Sigma^2 M; \mathbb{Z}/2\mathbb{Z}) \to H^4(\Sigma^2 M; \mathbb{Z}/2\mathbb{Z})$, which is preserved by the decomposition and the suspension operator. Since Sq^2 is the cup square on the elements of $H^2(M; \mathbb{Z}/2\mathbb{Z})$, this completes the proof of Theorem 1.1. □

$i \bmod 2$	0	1
$\widetilde{K}^{-i}(S^0)$	$\mathbb{Z}$	0

$j \bmod 8$	0	1	2	3	4	5	6	7
$\widetilde{KO}^{-j}(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0

Table 1: $\widetilde{K}^{-i}(S^0)$ and $\widetilde{KO}^{-j}(S^0)$.

To prove Corollary 1.2 we need Bott periodicity, which is described by Table 1, from which we can easily calculate the following, where only $\widetilde{KO}^2(P^5(T)) = 0$ requires that T has no 2–torsion:

Lemma 4.3 *Let W_c be the complex defined in Lemma 4.2 for $0 \le c \le d$.*

- $\widetilde{K}(P^5(T)) \cong T$ *and* $\widetilde{K}(P^6(T)) = 0$.
- $\widetilde{KO}^2(P^5(T)) = \widetilde{KO}^2(P^6(T)) = 0$.
- $\widetilde{KO}^1(\Sigma\mathbb{C}P^2) \cong \widetilde{KO}^1(S^3 \vee S^5) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
- $\widetilde{KO}^1(\Sigma^2\mathbb{C}P^2) \cong \widetilde{KO}^1(S^4 \vee S^6) = 0$.
- $\widetilde{KO}^1(W_0) \cong \widetilde{KO}^1(W_d) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *and* $\widetilde{KO}^1(W_c) \cong \bigoplus_2(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$.

Proof of Corollary 1.2 We only compute the $\widetilde{KO}$–group of M when $1 \le c \le d-1$, as the other cases can be computed similarly. By Theorem 1.1 and Lemma 4.3, we have

$$\begin{aligned}
\widetilde{KO}(M) \cong \widetilde{KO}^2(\Sigma^2 M) \cong \widetilde{KO}^2(\Sigma W_c) \oplus \bigoplus_{i=1}^{c-1} \widetilde{KO}^2(\Sigma^2\mathbb{C}P^2) \oplus \bigoplus_{j=1}^{d-c-1} \widetilde{KO}^2(S^4 \vee S^6) \\
\oplus \bigoplus_{j=1}^{2m} \widetilde{KO}^2(S^5) \oplus \widetilde{KO}^2(P^6(T)) \oplus \widetilde{KO}^2(P^5(T)) \\
\cong \bigoplus_2(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{c-1}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{d-c-1}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \\
= \bigoplus_d(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).
\end{aligned}$$

□

5 The case when $H^2(M;\mathbb{Z}) \cong \mathbb{Z}$

By Lemma 3.6, we may consider the torsion free case first. In [34], Yamaguchi classified the homotopy types of CW–complexes of the form $V \simeq S^2 \cup e^4 \cup e^6$. Specializing to the case when V is a manifold, we can summarize the necessary result in the following theorem (see [5, Section 1]):

Theorem 5.1 [34, Corollary 4.6, Lemmas 2.6 and 4.3] *Let $V \simeq S^2 \cup_{k\eta_2} e^4 \cup_b e^6$ be a closed smooth manifold, where $k\eta_2$ with $k \in \mathbb{Z}$ and b are the attaching maps of the cells e^4 and e^6, respectively. Then the top attaching map b is determined by a generator $b_W \in \pi_5(S^2 \cup_{k\eta_2} e^4)$ of infinite order, the second Stiefel–Whitney class $\omega_2(V) \in H^2(V;\mathbb{Z}/2)$ of V, and an indeterminacy term $b' \in \mathbb{Z}/2$ which depends on the following three cases:*

- *If k is odd, then V is Spin, the homotopy type of V is uniquely determined by k, and $b = b_W$.*
- *If k is even and V is non-Spin, then the homotopy type of V is uniquely determined by k, and $b = b_W + \tilde{\eta}_4$ with $\tilde{\eta}_4$ representing the generator of a $\mathbb{Z}/2$ summand determined by $\omega_2(V)$.*
- *If k is even and V is Spin, then V has precisely two homotopy types depending on the value of $b' \in \mathbb{Z}/2$, and $b = b_W + b'$.*

Remark 5.2 In Theorem 5.1, b_W, as a generator of the $\mathbb{Z}$–summand of $\pi_5(S^2 \cup_{k\eta_2} e^4)$, is indeed a relative Whitehead product when $k \neq 0$ by [34, Lemma 2.6]. It is possible that the suspension Eb_W is not nullhomotopic. The class $\tilde{\eta}_4$ is derived from the homotopy class of

$$S^5 \xrightarrow{b} S^2 \cup_{k\eta_2} e^4 \xrightarrow{q} S^4,$$

where q is the quotient map onto the 4–cell of $S^2 \cup_{k\eta_2} e^4$ (see [5, Section 1]). The class b' is from a class of $\pi_5(S^2) \cong \mathbb{Z}/2\{\eta_2^3\}$ by [34, Lemma 2.6] or [5, Section 1]. Also, as pointed out in Mathematical Reviews [26], the original theorem of [34] was misstated, but is corrected here and in [5, Section 1] as well.

Thanks to Theorem 5.1 and Remark 5.2, we can describe the suspension homotopy type of V. Recall that $\pi_6(\Sigma\mathbb{C}P^2) \cong \mathbb{Z}/6\mathbb{Z}\{E\pi_2\}$ [22, Theorem 8.2(i)], where $\pi_2\colon S^5 \to \mathbb{C}P^2$ is the Hopf map with the cofibre $\mathbb{C}P^3$, and E is the suspension of a map.

Proposition 5.3 *Let $V \simeq S^2 \cup_{k\eta_2} e^4 \cup_b e^6$ be a closed smooth manifold.*

- *If k is odd, then V is Spin and*

  $$\Sigma V \simeq \Sigma\mathbb{C}P^2 \cup_{k' E\pi_2} e^7,$$

  where $k' = 1$ or 3 is such that $k' \equiv \pm k \bmod 6$.

- *If k is even and V is non-Spin then*

$$\Sigma V \simeq S^3 \vee \Sigma^3\mathbb{C}P^2.$$

- *If k is even and V is Spin then*

$$\Sigma V \simeq (S^3 \cup_{b'\eta_3^3} e^7) \vee S^5,$$

where $b' \in \mathbb{Z}/2$ is from Theorem 5.1.

Proof When k is even the decompositions follow immediately from Theorem 5.1 and Remark 5.2. When k is odd, V is Spin and $\Sigma V \simeq \Sigma\mathbb{C}P^2 \cup_{Eb_W} e^7$ by Theorem 5.1. Also notice that $\Sigma\mathbb{C}P^2 \cup_{Eb_W} e^7 \simeq \Sigma\mathbb{C}P^2 \cup_{-Eb_W} e^7$. Hence, to prove the statement in the proposition it suffices to show that the suspension map

$$E\colon \pi_5(S^2 \cup_{k\eta_2} e^4) \to \pi_6(\Sigma\mathbb{C}P^2)$$

sends the generator b_W to $kE\pi_2$ up to sign.

For this purpose, start with the diagram of homotopy cofibrations

$$\begin{array}{ccccc} S^3 & \xrightarrow{k\eta_2} & S^2 & \longrightarrow & S^2 \cup_{k\eta_2} e^4 \\ \downarrow k & & \| & & \downarrow r \\ S^3 & \xrightarrow{\eta_2} & S^2 & \longrightarrow & \mathbb{C}P^2 \\ \downarrow & & \downarrow & & \downarrow \\ P^4(k) & \longrightarrow & * & \longrightarrow & P^5(k) \end{array} \tag{10}$$

which defines the map r. Then there is the diagram of homotopy fibrations

$$\begin{array}{ccccccc} S^1 & \longrightarrow & Z & \longrightarrow & S^2 \cup_{k\eta_2} e^4 & \xrightarrow{f_x} & K(\mathbb{Z},2) \\ \| & & \downarrow \tilde{r} & & \downarrow r & & \| \\ S^1 & \longrightarrow & S^5 & \xrightarrow{\pi_2} & \mathbb{C}P^2 & \xrightarrow{f_c} & K(\mathbb{Z},2) \end{array} \tag{11}$$

where f_c and f_x represent the generators $c \in H^2(\mathbb{C}P^2;\mathbb{Z})$ and $x \in H^2(S^2 \cup_{k\eta_2} e^4;\mathbb{Z})$, respectively, and Z is the homotopy fibre of f_x mapping to S^5 by the induced map $\tilde{r}$. By analyzing the Serre spectral sequences of the homotopy fibrations in (11), it can be shown that $Z \simeq P^4(k) \cup e^5$ and $\tilde{r}^*\colon H^5(S^5;\mathbb{Z}) \to H^5(Z;\mathbb{Z})$ is of degree k. Since by Lemma 6.3 $\pi_4(P^4(k)) = 0$ when k is odd, we see that $Z \simeq P^4(k) \vee S^5$, and then $\tilde{r}_*$ is of degree k on homology. Moreover, by the naturality of the Hurewicz homomorphism and Lemma 6.4, it is easy to see that $\tilde{r}_*\colon \pi_5(Z)\cong\mathbb{Z}\to\pi_5(S^5)$ is of degree k. It follows that $r_*\colon \pi_5(S^2\cup_{k\eta_2}e^4)\cong\mathbb{Z}\to\pi_5(\mathbb{C}P^2)\cong\mathbb{Z}$ is of degree k by (11).

Now the naturality of suspension map induces the commutative diagram

$$\begin{array}{ccc} \pi_5(S^2\cup_{k\eta_2}e^4) & \xrightarrow{r_*} & \pi_5(\mathbb{C}P^2) \\ \downarrow E & & \downarrow E \\ \pi_6(\Sigma\mathbb{C}P^2) & \xrightarrow{Er_*} & \pi_6(\Sigma\mathbb{C}P^2) \end{array} \tag{12}$$

where $E\colon \pi_5(\mathbb{C}P^2)\cong\mathbb{Z}\to\pi_6(\Sigma\mathbb{C}P^2)\cong\mathbb{Z}/6\mathbb{Z}$ is surjective by [22, Theorem 8.2(i)]. We have shown that r_* in (12) is of degree k. On the other hand, from the last column of (10) we have the homotopy cofibration

$$\Sigma\mathbb{C}P^2\xrightarrow{Er}\Sigma\mathbb{C}P^2\to P^6(k).$$

Applying the Blakers–Massey theorem [9], we obtain the exact sequence

$$\pi_6(\Sigma\mathbb{C}P^2)\xrightarrow{Er_*}\pi_6(\Sigma\mathbb{C}P^2)\to\pi_6(P^6(k)). \tag{13}$$

Since $\pi_6(P^6(k))=0$ by Lemma 6.3 and $\pi_6(\Sigma\mathbb{C}P^2)\cong\mathbb{Z}/6\mathbb{Z}\{E\pi_2\}$, we see that Er_* is an isomorphism from (13). Then by (12), $E\colon \pi_5(S^2\cup_{k\eta_2}e^4)\to\pi_6(\Sigma\mathbb{C}P^2)$ sends the generator b_W to $kE\pi_2$ up to sign. This proves the statement in the case when k is odd, and we have completed the proof of the proposition. □

Now we can prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3 First, by Theorem 2.1, Corollary 2.2 and Lemma 3.6, we have

$$\begin{aligned}\Sigma^2M&\simeq\Sigma^2M_1\vee\textstyle\bigvee_{i=1}^m(S^5\vee S^5)\\ &\simeq\Sigma^2V\vee P^6(T)\vee P^5(T)\vee\textstyle\bigvee_{i=1}^m(S^5\vee S^5),\end{aligned}$$

where M_1 is a closed 6–manifold with homology of the form (2) such that $b_3(M_1)=0$ and $d=1$. Moreover, by Corollary 3.3 and the assumption on the ring structure of $H^*(M;\mathbb{Z})$, we have $V\simeq S^2\cup_{k\eta_2}e^4\cup_b e^6$ for some attaching map b. Let $\lambda=b'$ in Theorem 5.1. The theorem for the two cases when k is even then follows immediately from Proposition 5.3. For the case when k is odd, recall that there is the fibre bundle [15, Section 1.1]

$$S^2\to\mathbb{C}P^3\xrightarrow{\sigma}S^4,$$

with its first Pontryagin class $p_1 = 4s_4$ where $s_4 \in H^4(S^4;\mathbb{Z})$ is a generator. Taking the pullback of this bundle with the self-map of S^4 of degree 3, we obtain the 6–manifold V_3 with bundle projection σ_3 onto S^4 in the following diagram of S^2–bundles:

$$\begin{array}{ccccc} S^2 & \longrightarrow & V_3 & \xrightarrow{\sigma_3} & S^4 \\ \| & & \downarrow & & \downarrow{\scriptstyle 3} \\ S^2 & \longrightarrow & \mathbb{C}P^3 & \xrightarrow{\sigma} & S^4 \end{array}$$

From this diagram it is easy to see that the first Pontryagin class of σ_3 is $12s_4$ as required, and $x^2 = 3y$, where by abuse of notation $x, y \in H^*(V_3;\mathbb{Z})$ are two generators such that $\deg(x) = 2$ and $\deg(y) = 4$. Hence by Proposition 5.3, $\Sigma V \simeq \Sigma\mathbb{C}P^3$ when $k \equiv \pm 1 \bmod 6$ and $\Sigma V \simeq \Sigma V_3$ when $k \equiv 3 \bmod 6$, and then the two decompositions when k is odd in the theorem follow. □

Proof of Corollary 1.4 As discussed before Corollary 1.4, the number $k \bmod 2$ and the spin condition of M are determined by the Steenrod square Sq^2. Since the attaching maps of the top cells of $\Sigma\mathbb{C}P^3$ and ΣV_3 are $E\pi_2$ of order 6 and $3E\pi_2$ of order 2, respectively, by Proposition 5.3, after localization at 3 we can consider the Steenrod power $\mathcal{P}^1 : H^3(\Sigma M;\mathbb{Z}/3\mathbb{Z}) \to H^7(\Sigma M;\mathbb{Z}/3\mathbb{Z})$. Then since $\Sigma V_3 \simeq_{(3)} S^3 \vee S^5 \vee S^7$, $\mathcal{P}^1$ acts trivially on its cohomology. In contrast, $\Sigma\mathbb{C}P^3 \simeq_{(3)} S^3 \cup_{\alpha_1} e^7 \vee S^5$ with α_1 an element detected by $\mathcal{P}^1$ [13, Section 1.5.5]. Hence, $\Sigma\mathbb{C}P^3$ and ΣV_3 can be distinguished by $\mathcal{P}^1$. Moreover, the stable cube element $\eta_n^3 \in \pi_{n+3}(S^n)$ (for $n \geq 2$) is detected by the secondary operation $\mathbb{T}$ [13, Exercise 4.2.5]. Therefore the two cases of $S^4 \cup_{\lambda\eta_4^3} e^8$ depending on $\lambda \in \mathbb{Z}$ can be distinguished by $\mathbb{T}$. From the above discussions on cohomology operations, we can prove the corollary easily by the decompositions in Theorem 1.3. □

6 Some computations on homotopy groups of odd primary Moore spaces

In this section, we work out the homotopy groups of Moore spaces used in Section 3. Consider the Moore space $P^{2n+1}(p^r)$ with $n \geq 1$, $p \geq 3$ and $r \geq 1$. We have the homotopy fibration

$$F^{2n+1}\{p^r\} \to P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1}, \tag{14}$$

where q is the pinch map of the bottom cell. Cohen, Moore and Neisendorfer proved the following the famous decomposition theorem:

Theorem 6.1 [10; 24] *Let p be an odd prime. Then there is a p–local homotopy equivalence*

$$\Omega F^{2n+1}\{p^r\} \simeq_{(p)} S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\} \times \Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r),$$

where $S^i\{p^r\}$ is the homotopy fibre of the degree map $p^r : S^i \to S^i$, and $\bigvee_\alpha P^{n_\alpha}(p^r)$ is an infinite bouquet of mod p^r Moore spaces, with only finitely many Moore spaces in each dimension and the least value of n_α being $4n-1$.

We also need the following classical result:

Lemma 6.2 [25, Proposition 6.2.2] *Let p be an odd prime. Then there is a homotopy equivalence*

$$P^m(p^r) \wedge P^n(p^r) \simeq P^{m+n}(p^r) \vee P^{m+n-1}(p^r).$$

Lemma 6.3 [27; 28] *Let p be an odd prime. Then there are isomorphisms*

$$\pi_3(P^3(p^r)) = \mathbb{Z}/p^r\mathbb{Z} \quad \text{and} \quad \pi_n(P^n(p^r)) = 0$$

for $n \geq 4$.

Proof The cases when $n = 3$ and 4 were already proved in [28, Lemma 2.1] and [27, Lemma 3.3], respectively, while the remaining cases follow immediately from the Freudenthal suspension theorem. □

Lemma 6.4 *Let p be an odd prime. Then there is an isomorphism*

$$\pi_{n+1}(P^n(p^r)) = 0$$

for $n \geq 3$.

Proof The isomorphism $\pi_4(P^3(p^r)) = 0$ was shown in [27, Lemma 3.3]. Let us consider $\pi_5(P^4(p^r))$. By the classical EHP-sequence [33, Chapter XII, Theorem 2.2], there is an exact sequence

$$0 = \pi_4(P^3(p^r)) \to \pi_5(P^4(p^r)) \xrightarrow{H} \pi_5(P^4(p^r) \wedge P^3(p^r)) \xrightarrow{P} \pi_3(P^3(p^r)) \to \pi_4(P^4(p^r)) = 0.$$

By Lemma 6.2,

$$\pi_5(P^4(p^r) \wedge P^3(p^r)) \cong \pi_5(P^6(p^r) \vee P^7(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Hence, by Lemma 6.3 and the above exact sequence, P is an isomorphism, implying that $\pi_5(P^4(p^r)) = 0$. The remaining cases follow immediately from the Freudenthal suspension theorem, and this completes the proof of the lemma. □

In the remaining two lemmas, we exclude the case when $p = 3$.

Lemma 6.5 *Let $p \geq 5$. Then there is an isomorphism*

$$\pi_{n+2}(P^n(p^r)) = 0$$

for $n \geq 6$.

Proof By the Freudenthal suspension theorem, it suffices to show $\pi_9(P^7(p^r)) = 0$. To do this, let us compute $\pi_9(F^7\{p^r\})$ first. By Theorem 6.1,

$$\pi_9(F^7\{p^r\}) \cong \pi_8(\Omega F^7\{p^r\}) \cong \pi_8(S^5)_{(p)}.$$

Since $\pi_8(S^5) \cong \mathbb{Z}/24\mathbb{Z}$ and $p \geq 5$, we have $\pi_9(F^7\{p^r\}) = 0$. Now from the exact sequence of homotopy groups of the homotopy fibration (14) (for $n = 3$)

$$0 = \pi_9(F^7\{p^r\}) \to \pi_9(P^7(p^r)) \to \pi_9(S^7)_{(p)} = 0,$$

we see that $\pi_9(P^7(p^r)) = 0$. □

Lemma 6.6 *Let $p \geq 5$. The suspension morphism*

$$E\colon \pi_6(P^4(p^r)) \to \pi_7(P^5(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$$

is trivial.

Proof On the one hand there is the EHP-sequence of $P^4(p^r)$

$$\pi_6(P^4(p^r)) \xrightarrow{E} \pi_7(P^5(p^r)) \xrightarrow{H} \pi_7(P^5(p^r) \wedge P^4(p^r)) \to \pi_5(P^4(p^r)) = 0,$$

where $\pi_5(P^4(p^r)) = 0$ by Lemma 6.4, and

$$\pi_7(P^5(p^r) \wedge P^4(p^r)) \cong \pi_7(P^8(p^r) \vee P^9(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$$

by Lemma 6.2. It follows that

$$\pi_7(P^5(p^r))/\operatorname{Im}(E) \cong \mathbb{Z}/p^r\mathbb{Z}. \tag{15}$$

On the other hand there is the EHP-sequence of $P^5(p^r)$

$$\pi_9(P^6(P^r) \wedge P^5(p^r)) \xrightarrow{P} \pi_7(P^5(p^r)) \to \pi_8(P^6(p^r)) = 0,$$

where $\pi_8(P^6(p^r)) = 0$ by Lemma 6.5, and

$$\pi_9(P^6(P^r) \wedge P^5(p^r)) \cong \pi_9(P^{10}(P^r) \vee P^{11}(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$$

by Lemma 6.2. It follows that

$$\pi_7(P^5(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}/\operatorname{Ker}(P). \tag{16}$$

By (15) and (16), we see that $\pi_7(P^5(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}$, and $\operatorname{Im}(E) = \operatorname{Ker}(P) = 0$. □

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