

AG  
T

*Algebraic & Geometric  
Topology*

Volume 23 (2023)

**Nonslice linear combinations of iterated torus knots**

ANTHONY CONWAY

MIN HOON KIM

WOJCIECH POLITARCZYK



# Nonslice linear combinations of iterated torus knots

ANTHONY CONWAY

MIN HOON KIM

WOJCIECH POLITARCZYK

In 1976, Rudolph asked whether algebraic knots are linearly independent in the knot concordance group. We use twisted Blanchfield pairings to answer this question in the affirmative for new large families of algebraic knots.

57K10

## 1 Introduction

A knot is *algebraic* if it arises as a link of an isolated singularity of a complex curve. Algebraic knots are special cases of iterated torus knots. In 1976, Rudolph [28] asked whether the set of algebraic knots is linearly independent in the knot concordance group. For ease of reference and because of the later literature on the subject, we refer to this question as a conjecture.

**Conjecture 1** (Rudolph’s conjecture [28]) *The set of algebraic knots is linearly independent in the smooth knot concordance group  $\mathcal{C}$ .*

This question has been of particular interest due to its relevance to the slice-ribbon conjecture: a result of Miyazaki shows that nontrivial linear combinations of iterated torus knots are not ribbon [26, Corollary 8.4]. In particular, if the slice-ribbon conjecture holds, then Rudolph’s conjecture holds. Baker [2] and Abe and Tagami [1] recently noticed that the slice-ribbon conjecture implies a statement stronger than Rudolph’s conjecture:

**Conjecture 2** (Abe and Tagami [1] and Baker [2]) *The set of prime fibred strongly quasipositive knots is linearly independent in the smooth knot concordance group  $\mathcal{C}$ .*

This paper exhibits new large families of knots for which Conjectures 1 and 2 hold.

## 1.1 Statement of the results

Evidence of Rudolph's conjecture was first provided in 1979 by Litherland, who proved that positive torus knots are linearly independent in  $\mathcal{C}$  [21]. In 2010, Hedden, Kirk and Livingston showed that, for an appropriate choice of positive integers  $\{q_n\}_{n=1}^{\infty}$ , the set  $\{T(2, q_n), T(2, 3; 2, q_n)\}_{n=1}^{\infty}$  is linearly independent in  $\mathcal{C}$ , where  $T(p, q)$  and  $T(p, q; r, s)$  denote the  $(p, q)$ -torus knot and the  $(r, s)$ -cable of  $T(p, q)$ , respectively, and  $p$  is coprime to  $qrs$ . It is known that an iterated torus knot  $T(p_1, q_1; \dots; p_k, q_k)$  is algebraic if and only if  $p_i, q_i > 0$  and  $q_{i+1} > q_i p_{i+1} p_i$  for each  $i$ . Our main result, which relies on metabelian twisted Blanchfield pairings — see Miller and Powell [25] and Borodzik, Conway and Politarczyk [3; 4; 5] — reads as follows:

**Theorem 1.1** Fix a prime power  $p$ . Let  $S_p$  be the set of iterated torus knots

$$T(p, q_1; p, q_2; \dots; p, q_\ell),$$

where the sequences  $(q_1, q_2, \dots, q_\ell)$  of positive integers satisfy

- (i) for  $i = 1, \dots, \ell$ , the integer  $q_i$  is coprime to  $p$ ;
- (ii)  $q_\ell$  is a prime;
- (iii) for  $i = 1, \dots, \ell - 1$ , the integer  $q_i$  is coprime to  $q_\ell$  when  $\ell > 1$ .

The set  $S_p$  is linearly independent in the topological knot concordance group  $\mathcal{C}^{\text{top}}$ .

As an immediate corollary of Theorem 1.1, we obtain the following:

**Corollary 1.2** For every prime power  $p$ , the subset  $S_p^{\text{alg}} \subset S_p$  of algebraic knots in  $S_p$  is linearly independent in  $\mathcal{C}^{\text{top}}$  and therefore satisfies Conjecture 1.

Since positively iterated torus knots are strongly quasipositive — see Hedden [11, Theorem 1.2; 13, Proposition 2.1] — Theorem 1.1 also gives infinite families of knots satisfying Conjecture 2.

**Corollary 1.3** For every prime power  $p$ , the set  $S_p$  satisfies Conjecture 2, and  $S_p \setminus S_p^{\text{alg}}$  is an infinite family of nonalgebraic knots satisfying Conjecture 2.

Abe and Tagami also conjecture that the set of L-space knots is linearly independent in  $\mathcal{C}$  [1, Conjecture 3.4]. For a knot  $K$  with Seifert genus  $g$ , the  $(p, q)$ -cable  $K_{p,q}$  is an L-space knot if and only if  $K$  is an L-space knot and  $(2g - 1)p \leq q$ ; see Hedden [12] and Hom [16]. Since torus knots are L-space knots, we also obtain the following result:

**Corollary 1.4** For every prime power  $p$ , the subset  $\mathcal{S}_p^L \subset \mathcal{S}_p$  of  $L$ -space knots in  $\mathcal{S}_p$  is linearly independent in  $\mathcal{C}^{\text{top}}$ , and this statement also holds for the infinite family  $\mathcal{S}_p^L \setminus \mathcal{S}_p^{\text{alg}}$  of nonalgebraic  $L$ -space knots.

Note however that not all our examples are  $L$ -spaces knots: since the cable of an iterated torus knot need not be an  $L$ -space knot, [Corollary 1.3](#) shows that the infinite set  $\mathcal{S}_p \setminus \mathcal{S}_p^L$  contains no  $L$ -spaces knots but is nevertheless linearly independent in  $\mathcal{C}^{\text{top}}$ .

### 1.2 Context and comparison with smooth techniques

Litherland used the Levine–Tristram signature to show that torus knots are linearly independent in  $\mathcal{C}$  [21]. This approach is insufficient to answer Rudolph’s conjecture, since Livingston and Melvin showed in [23] that the following linear combinations of iterated torus knots are algebraically slice:

$$(1) \quad J(p, q, q_1, q_2) := T(p, q; p, q_1) \# -T(p, q_1) \# -T(p, q; p, q_2) \# T(p, q_2).$$

Classical knot invariants can thus not obstruct  $J(p, q, q_1, q_2)$  from being slice.

Hedden, Kirk and Livingston managed to leverage the Casson–Gordon invariants to provide further evidence of Rudolph’s conjecture [14]. Indeed, they showed that, for an appropriate choice of  $\{q_n\}_{n=1}^\infty$ , the knots  $\{J(2, 3, q_{2n-1}, q_{2n})\}_{n=1}^\infty$  generate an infinite-rank subgroup in  $\mathcal{C}$ . This result is particularly notable since they observe that the  $s$ -invariant from Khovanov homology and the  $\tau$ -invariant from Heegaard Floer homology both vanish on  $J(2, 3, q_{2n-1}, q_{2n})$  [14, Proposition 8.2]. In fact, their argument (combined with [Proposition 5.3](#)) generalises to show that, if  $K$  is a linear combination of algebraically slice knots belonging to  $\mathcal{S}_p$ , then  $\tau(K) = 0$  and  $s(K) = 0$ .

Next, we observe that the Upsilon invariant  $\Upsilon_K : [0, 2] \rightarrow \mathbb{R}$  from Ozsváth, Stipsicz and Szabó’s knot Floer homology [27] is also insufficient to prove [Theorem 1.1](#). First note that, if  $q_1, q_2 > p(p - 1)(q - 1)$ , then  $T(p, q; p, q_i)$  is an  $L$ -space knot [12], and thus a result of Tange shows that  $\Upsilon_{T(p,q;p,q_i)}(t) = \Upsilon_{T(p,q)}(pt) + \Upsilon_{T(p,q_i)}(t)$  for all  $t \in [0, 2]$  [29, Theorem 3]. The additivity of  $\Upsilon$  then establishes that  $\Upsilon_{J(p,q,q_1,q_2)}(t) = 0$  for all  $t \in [0, 2]$  whenever  $q_1, q_2 > p(p - 1)(q - 1)$ .

### 1.3 Strategy and ingredients of the proof

The proof of [Theorem 1.1](#) relies on Casson–Gordon theory [6; 7] — see also Kirk and Livingston [18] — and more specifically on the metabelian Blanchfield pairings introduced by Miller and Powell [25] and further developed by the first and third authors with Maciej Borodzik [3; 4; 5]. Since these invariants are somewhat technical, the next

paragraphs describe some background and ideas that go into the proof of [Theorem 1.1](#). For notational simplicity, however, we restrict ourselves to a very particular case: we apply our strategy to the knot  $J(p, q, q_1, q_2)$  described in [\(1\)](#).

**The sliceness obstruction** Let  $p$  be a prime power, let  $\Sigma_p(J)$  be the  $p$ -fold branched cover of the knot  $J := J(p, q, q_1, q_2)$ , let  $\chi$  be a character on  $H_1(\Sigma_p(J))$ , and let  $M_J$  be the 0-framed surgery of  $J$ . Associated to these data, there is a nonsingular sesquilinear and Hermitian *metabelian Blanchfield pairing*

$$\mathrm{Bl}_{\alpha(p,\chi)}(J): H_1(M_J; \mathbb{C}[t^{\pm 1}]^p) \times H_1(M_J; \mathbb{C}[t^{\pm 1}]^p) \rightarrow \mathbb{C}(t)/\mathbb{C}[t^{\pm 1}].$$

Here  $H_1(M_J; \mathbb{C}[t^{\pm 1}]^p)$  denotes the homology of  $M_J$  twisted by a metabelian representation  $\alpha(p, \chi): \pi_1(M_J) \rightarrow \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$ , whose definition will be recalled in [Section 3](#). The precise definition of  $\mathrm{Bl}_{\alpha(p,\chi)}(J)$  is irrelevant in this paper: only its properties are required. Informally, however, the pairing  $\mathrm{Bl}_{\alpha(p,\chi)}(J)$  contains the information from both twisted polynomial invariants and twisted signature invariants. We now describe how  $\mathrm{Bl}_{\alpha(p,\chi)}(J)$  provides a sliceness obstruction.

Let  $\lambda_p(J)$  denote the  $\mathbb{Q}/\mathbb{Z}$ -valued linking form on  $H_1(\Sigma_p(J))$ . Miller and Powell show that if, for every  $\mathbb{Z}_p$ -invariant metaboliser  $G$  of  $\lambda_p(J)$ , there exists a prime power-order character  $\chi$  that vanishes on  $G$  and is such that  $\mathrm{Bl}_{\alpha(p,\chi)}(J)$  is not metabolic, then  $J$  is not slice [[25](#), [Theorem 6.10](#)]. In order to make this obstruction more concrete, we now recall some terminology on linking forms and their metabolisers.

**The Witt group of linking forms** We focus on linking forms over  $\mathbb{C}[t^{\pm 1}]$ , referring to [Section 4](#) for a discussion over more general rings. A *linking form* over  $\mathbb{C}[t^{\pm 1}]$  is a sesquilinear Hermitian pairing  $V \times V \rightarrow \mathbb{C}(t)/\mathbb{C}[t^{\pm 1}]$ , where  $V$  is a torsion  $\mathbb{C}[t^{\pm 1}]$ -module. A linking form  $(V, \lambda)$  is *metabolic* if there is a submodule  $L \subset V$  such that  $L = L^\perp$ ; such an  $L$  is called a *metaboliser*. The *Witt group of linking forms*, denoted by  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ , consists of the monoid of nonsingular linking forms modulo the submonoid of metabolic linking forms. We write  $\lambda_1 \sim \lambda_2$  if two linking forms agree in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ . The Miller–Powell obstruction to sliceness, therefore, consists of deciding whether a certain twisted Blanchfield pairing  $\mathrm{Bl}_{\alpha(p,\chi)}(J)$  is zero in the group  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ . As we will now describe, one of our main ideas is to transfer a problem of linear independence in  $\mathcal{C}^{\mathrm{top}}$  (namely Rudolph’s conjecture) into a problem of linear independence in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ .

**From linear independence in  $\mathcal{C}^{\mathrm{top}}$  to linear independence in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$**  Since the knot  $J = T(p, q; p, q_1) \# -T(p, q_1) \# -T(p, q; p, q_2) \# T(p, q_2)$  is a connected sum of four knots, both  $H_1(\Sigma_p(J))$  and  $\lambda_p(J)$  can be decomposed into four

direct summands:

$$\lambda_p(J) = \lambda_p(T(p, q_1)) \oplus -\lambda_p(T(p, q_1)) \oplus \lambda_p(T(p, q_2)) \oplus -\lambda_p(T(p, q_2)).$$

In particular, any character on  $H_1(\Sigma_p(J))$  can be written as  $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$ . For each given  $\mathbb{Z}_p$ -invariant metaboliser  $M$  of  $\lambda_p(J)$ , the “sliceness-obstructing character” that we will produce will be of the form  $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$ , where  $\theta$  denotes the trivial character. Using the definition of  $J$ , together with the direct sum decomposition of [4, Corollary 4.21], the Witt class of the metabelian Blanchfield pairing of  $J$  is given by

$$(2) \quad \text{Bl}_{\alpha(p,\chi)}(J) \sim \text{Bl}_{\alpha(p,\chi_1)}(T(p, q; p, q_1)) \oplus -\text{Bl}_{\alpha(p,\chi_2)}(T(p, q_1)) \\ \oplus -\text{Bl}_{\alpha(p,\theta)}(T(p, q; p, q_2)) \oplus \text{Bl}_{\alpha(p,\theta)}(T(p, q_2)).$$

This expression can be further decomposed by applying the satellite formula for the metabelian Blanchfield forms given in [4, Theorem 4.19]. Regardless of the final expression, the problem has been converted into a question of linear independence in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ . In Proposition 4.3, we describe a criterion for linear independence in terms of roots of the orders of the underlying modules (recall that the order of a module over  $\mathbb{C}[t^{\pm 1}]$  is a Laurent polynomial in  $\mathbb{C}[t^{\pm 1}]$ ; it is defined up to multiplication by units of  $\mathbb{C}[t^{\pm 1}]$ ). Here is a simplified version of this statement:

**Proposition 1.5** *If  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$  are two nonmetabolic linking forms over  $\mathbb{C}[t^{\pm 1}]$  such that  $\text{Ord}(V_1)$  and  $\text{Ord}(V_2)$  have distinct roots, then the Witt classes  $[V_1, \lambda_1]$  and  $[V_2, \lambda_2]$  are linearly independent in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ .*

**1.3.1 Computation of twisted Alexander polynomials** In order to apply Proposition 1.5, we must therefore understand the roots of the metabelian twisted Alexander polynomials of  $T(p, q)$  associated to characters on  $H_1(\Sigma_p(T(p, q)))$  (indeed, these twisted polynomials arise as orders of modules of the form  $H_1(M_{T(p,q)}; \mathbb{C}[t^{\pm 1}]^p)$ , the same modules on which twisted Blanchfield pairings are defined). This is carried out in Section 3 and relies on our explicit understanding of the  $p$ -fold cover  $E_p(T(p, q)) \rightarrow E(T(p, q))$  from Section 2; here  $E(K)$  denotes the exterior of a knot  $K$ . Since this computation of twisted polynomials might be of independent interest, we summarise it as follows.

**Proposition 1.6** (Lemma 3.1 and Corollary 3.4) *Let  $p, q > 0$  be two coprime integers, and set  $\xi_p = e^{2\pi i/p}$ . The abelian group of characters on  $H_1(\Sigma_p(T(p, q))) \cong \mathbb{Z}_q^{p-1}$  is isomorphic to*

$$\{\mathbf{a} := (a_1, \dots, a_p) \in \mathbb{Z}_q^p \mid a_1 + \dots + a_p = 0\}.$$

We write  $\chi_{\mathbf{a}}$  for the character associated to  $\mathbf{a}$ . The metabelian twisted Alexander polynomial of the 0–framed surgery  $M_{T(p,q)}$  associated to the character

$$\chi_{\mathbf{a}}: H_1(\Sigma_p(T(p,q))) \rightarrow \mathbb{Z}_q$$

is given by

$$\Delta_1^{\alpha(p,\chi_{\mathbf{a}})}(M_{T(p,q)}) = \frac{(-1)^{p-1}(1-t^q)^{p-1}}{(t\xi_q^{a_1}-1)(t\xi_q^{a_2}-1)\cdots(t\xi_q^{a_p}-1)(t-1)}.$$

**Main steps of the proof** We now return to the knot  $J = J(p, q, q_1, q_2)$  from (1). Obstructing  $J$  from being slice has three main steps. In fact, the proof of [Theorem 1.1](#) in its full generality, as described in [Section 5](#), follows more complicated versions of these same steps:

- (i) Firstly, we use the previously described ingredients to study the implications of  $\text{Bl}_{\alpha(p,\chi)}(J)$  being metabolic on the characters  $\chi_1$  and  $\chi_2$ ; here  $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$  with  $\theta$  the trivial character. This is the content of [Section 5.2.2](#).
- (ii) Secondly, we show that, for every metaboliser  $L$  of  $\lambda_p(T_{p,q_1}) \oplus -\lambda_p(T_{p,q_1})$ , it is possible to build characters  $\chi_1$  and  $\chi_2$  that violate these conditions and are such that  $\chi_1 \oplus \chi_2$  vanishes on  $L$ . This is the content of [Section 5.2.3](#).
- (iii) Finally, we combine these two steps to obstruct the sliceness of  $J$ : for every metaboliser  $G$  of  $\lambda_p(J)$ , we are able to build a character  $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$  that vanishes on  $G$  and such that  $\text{Bl}_{\alpha(p,\chi)}(K)$  is not metabolic. This is the content of [Section 5.2.4](#).

**Remark 1.7** When  $p = 2$ , Hedden, Kirk and Livingston also use an obstruction based on the Casson–Gordon setup to show that, for an appropriate choice of positive integers  $\{q_n\}_{n=1}^{\infty}$ , the set  $\{T(2, q_n), T(2, 3; 2, q_n)\}_{n=1}^{\infty}$  is linearly independent in  $\mathcal{C}^{\text{top}}$  [14]. Our work differs from theirs in two main points:

- While [14] uses a blend of discriminants and signatures to prove its linear independence result, we use metabelian Blanchfield pairings. In a nutshell, the Blanchfield pairing encapsulates both the discriminant and (most of) the signature invariants allowing us to both streamline and generalise several of the arguments from [14].
- The result of [14] is proved without having to study invariant metabolisers; see also [5, Section 6]. This is a feature of iterated torus knots  $T(p, Q)$  with  $p = 2$  and fails when  $p > 2$ .

Passing from our outline to obstruct the sliceness of  $J(p, q, q_1, q_2)$  to the proof of [Theorem 1.1](#) requires additional steps. As often in Casson–Gordon theory, the main technical difficulty to overcome concerns the metabolisers of the linking form of the knot in question. Regarding these metabolisers, our strategy can be summarised as follows:

- (i) Given a metaboliser, we isolate certain technical conditions which guarantee that a character violates the sliceness obstruction. This is the content of [Lemma 5.8](#).
- (ii) We distinguish a certain family of metabolisers, called *graph metabolisers*, see [Section 4.2](#).
- (iii) The construction of the required character, for any fixed nongraph metaboliser is not overly challenging; see Cases 1 and 2 in the proof of [Lemma 5.8](#).
- (iv) Dealing with graph metabolisers requires more work. In Case 3, we show that either there exists a character satisfying the conditions from [Lemma 5.8](#), or the knot in question contains a slice summand  $K \# -K$ , for some knot  $K$ . Consequently, once we cancel all the summands of the form  $K \# -K$ , we are able to construct the desired obstructing character for any graph metaboliser, and finish the proof.

## 1.4 Assumptions and outlook

We conclude this introduction by commenting on the various technical assumptions that appear in [Theorem 1.1](#).

- (i) The assumption that the integers  $q_i$  are coprime to  $q_\ell$  is used in [Proposition 5.7](#) to ensure that certain Witt classes are linearly independent in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ . This hypothesis has its roots in the notion of *p-independence* introduced in [[14](#), Definition 6.2].
- (ii) We assume that  $p$  is a prime power in order to use Casson–Gordon theory [[6](#); [7](#)].
- (iii) We require that the  $q_i$  be positive mostly because of our interest in Rudolph’s conjecture: algebraic knots are iterated torus knots with *positive* cabling parameters.
- (iv) We use that  $q_\ell$  is prime in order to obtain the decomposition in ([21](#)) and to ensure that  $\mathbb{F}_{q_\ell}$  is a field.

Summarising, our assumptions are made for technical reasons: we have so far not encountered linear combinations of (algebraically slice) iterated torus knots whose sliceness is not obstructed by some Casson–Gordon invariants. Furthermore, this paper



does not fully use the techniques developed in [3; 4; 5] to compute the Casson–Gordon Witt class. Therefore, it would be interesting to study how far these methods can be pushed to investigate Rudolph’s conjecture.

## Organisation

This paper is organised as follows. In Section 2, we collect several results on the algebraic topology of the exterior of the torus knot  $T(p, q)$ . In Section 3, we use these results to compute Alexander polynomials of  $T(p, q)$  twisted by metabelian representations. In Section 4, we review some facts about linking forms. Finally in Section 5, we prove Theorem 1.1.

## Conventions

Manifolds are assumed to be compact and oriented. Throughout the paper, the  $p$ -fold branched cover of a knot is denoted by  $\Sigma_p(K)$ , and  $\lambda_p(K)$  denotes the linking form on  $H_1(\Sigma_p(K))$ .

## Acknowledgements

Conway thanks the MPIM for its financial support and hospitality. Kim was partly supported by the POSCO TJ Park Science Fellowship and NRF grants 2019R1A3B2067839 and 2021R1C1C1012939. Politarczyk was supported by the National Science Center grant 2016/22/E/ST1/00040. We wish to thank the CIRM in Luminy for providing excellent conditions where the bulk of this work was carried out.

## 2 Branched covers of torus knots

The aim of this section is to describe the  $\mathbb{Z}[\mathbb{Z}_p]$ -module structure of  $H_1(\Sigma_p(T(p, q)))$  induced from the  $\mathbb{Z}_p$ -covering action on  $\Sigma_p(T(p, q))$  when  $q$  is prime. Let  $E(T(p, q))$  be the complement of the torus knot  $T(p, q)$ , and let  $E_p(T(p, q))$  be its  $p$ -fold cyclic cover. In Section 2.1, to set up some notation, we recall the decomposition of  $E(T(p, q))$  coming from the standard genus 1 Heegaard splitting of  $S^3$ , as described in [10, Example 1.24]. In Section 2.2, this decomposition of  $E(T(p, q))$  is used to decompose  $E_p(T(p, q))$ ; after that,  $H_1(\Sigma_p(T(p, q)))$  can be computed via a Mayer–Vietoris sequence argument since  $\Sigma_p(T(p, q))$  is a union of  $E_p(T(p, q))$  with a solid torus glued along the torus boundary.

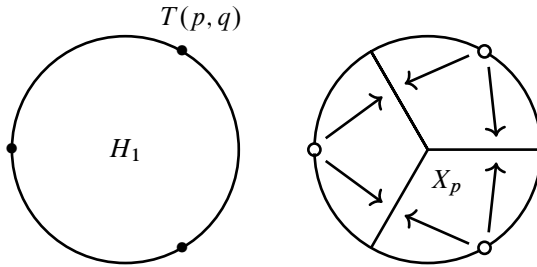


Figure 1: Left: the intersection  $T(p, q) \cap (\{x\} \times D^2)$ . Right: the complement  $H_1 \setminus T(p, q)$  deformation retracts onto a 2-complex  $X_p$ .

### 2.1 The homotopy type of $E(T(p, q))$

The goal of this subsection is to describe the homotopy type of  $E(T(p, q))$ , as well as describe explicit generators for  $\pi_1(E(T(p, q)))$ . To achieve this, we follow closely [10, Example 1.24].

Consider the standard decomposition  $S^3 = S^1 \times D^2 \cup D^2 \times S^1$  and denote  $S^1 \times D^2$  and  $D^2 \times S^1$  by  $H_1$  and  $H_2$ , respectively,  $H_1 \subset \mathbb{R}^3$  being the solid torus. We parametrise the  $(p, q)$ -torus knot  $T(p, q)$  on the torus  $H_1 \cap H_2$  as follows:

$$(3) \quad T(p, q) = \{(e^{2\pi i p t}, e^{2\pi i q t}) \mid t \in [0, 1]\} \subset S^1 \times S^1 = H_1 \cap H_2.$$

Using this description of  $T(p, q)$ , for each  $x \in S^1$ , we see that  $T(p, q)$  intersects  $\{x\} \times D^2 \subset H_1$  in  $p$  equidistributed points of  $\{x\} \times \partial D^2$ ; see Figure 1 for  $p = 3$ .

As depicted in Figure 1, right, the complement  $H_1 \setminus T(p, q)$  deformation retracts onto a 2-complex  $X_p \subset H_1$  which is the mapping cylinder of the degree  $p$  map  $f_p: S^1 \rightarrow c_1$ , where  $c_1$  is the core circle of  $H_1$ . The same argument shows that  $H_2 \setminus T(p, q)$  deformation retracts onto the mapping cylinder  $X_q$  of the degree  $q$  map  $f_q: S^1 \rightarrow c_2$ , where  $c_2$  is the core circle of  $H_2$ . By perturbing  $X_p$  near  $H_1 \cap H_2$ , we can arrange that  $X_p$  and  $X_q$  match up on  $H_1 \cap H_2$ . Next, let  $X_{p,q}$  be the union of  $X_p$  and  $X_q$ . Note that  $X_{p,q}$  is homeomorphic to the double mapping cylinder of the maps  $f_p$  and  $f_q$ , defined by

$$X_{p,q} := S^1 \times [0, 1] \cup c_1 \cup c_2 / \sim,$$

where  $(z, 0) \sim f_p(z)$  and  $(z, 1) \sim f_q(z)$  for all  $z \in S^1$  (see Figure 2). By van Kampen's theorem,

$$\pi_1(X_{p,q}) \cong \langle c_1, c_2 \mid c_1^p = c_2^q \rangle.$$

Summarising, we have the following proposition, which is implicit in Hatcher:

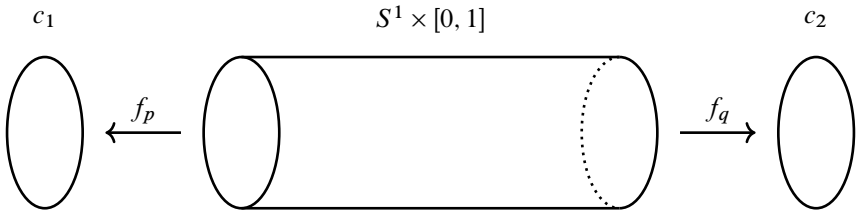


Figure 2: The double mapping cylinder  $X_{p,q}$  obtained by gluing  $S^1 \times [0, 1]$  with the circles  $c_1$  and  $c_2$  by the degree  $p$  and  $q$  maps  $f_p$  and  $f_q$ .

**Proposition 2.1** [10, Example 1.24] *There is a deformation retraction  $E(T(p, q)) \rightarrow X_{p,q}$  sending  $H_1 \setminus T(p, q)$  and  $H_2 \setminus T(p, q)$  to  $X_p$  and  $X_q$ , respectively. In particular,  $\pi_1(E(T(p, q))) \cong \langle c_1, c_2 \mid c_1^p = c_2^q \rangle$ , where  $c_i$  is the core circle of  $H_i$  for  $i = 1, 2$ .*

### 2.2 The computation of $H_1(\Sigma_p(T(p, q)))$ as a $\mathbb{Z}[\mathbb{Z}_p]$ -module

In this subsection, we describe the  $\mathbb{Z}[\mathbb{Z}_p]$ -module structure of  $H_1(\Sigma_p(T(p, q)))$ . To do so, we first study the  $p$ -fold cyclic covering map  $\pi : E_p(T(p, q)) \rightarrow E(T(p, q))$ , then we compute  $\pi_1(E_p(T(p, q)))$ , and finally we describe  $H_1(\Sigma_p(T(p, q)))$ .

We first use Section 2.1 to describe a deformation retract of  $E_p(T(p, q))$ . Using (3), we see that the torus knot  $T(p, q)$  links respectively  $q$  and  $p$  times the core circles  $c_1$  and  $c_2$ . Consequently,  $c_1$  and  $c_2$  are homologous to  $q\mu$  and  $p\mu$  in  $H_1(E(T(p, q)))$ , where  $\mu = c_1^k c_2^l$  is a meridian of  $T(p, q)$  and  $pk + ql = 1$ . Use  $(X_{p,q})_p$  to denote the preimage  $\pi^{-1}(X_{p,q})$ , and observe that, by Proposition 2.1,  $E_p(T(p, q))$  deformation retracts onto  $(X_{p,q})_p$ .

To describe  $\pi_1(E_p(T(p, q)))$  we study the homotopy type of  $(X_{p,q})_p$ . The (restricted) covering map  $\pi : (X_{p,q})_p \rightarrow X_{p,q}$  corresponds to the homomorphism  $\pi_1(X_{p,q}) \rightarrow \mathbb{Z}_p$  sending  $c_1$  to  $q \in \mathbb{Z}_p$  and  $c_2$  to  $0 \in \mathbb{Z}_p$ . We use  $\pi_* : \pi_1((X_{p,q})_p) \rightarrow \pi_1(X_{p,q})$  to denote the induced map. Let  $a$  be the preimage  $\pi^{-1}(c_1)$  and let  $b_0, \dots, b_{p-1}$  be the components of the preimage  $\pi^{-1}(c_2)$ ; we choose the indices of the  $b_i$  so that

$$(4) \quad \pi_*(b_i) = \mu^i c_2 \mu^{-i} \quad \text{for } i = 0, \dots, p-1.$$

Since  $\pi$  is a covering map, the induced map  $\pi_* : \pi_1((X_{p,q})_p) \rightarrow \pi_1(X_{p,q})$  is injective. For this reason, we shall often identify  $b_i$  with  $\mu^i c_2 \mu^{-i}$ . Since  $X_{p,q}$  is a double mapping cylinder, so is  $(X_{p,q})_p$ . More precisely, as illustrated in Figure 3,

$$(X_{p,q})_p = \bigcup_{i=0}^{p-1} S^1_i \times [0, 1] \cup a \cup b_0 \cup \dots \cup b_{p-1} / \sim,$$

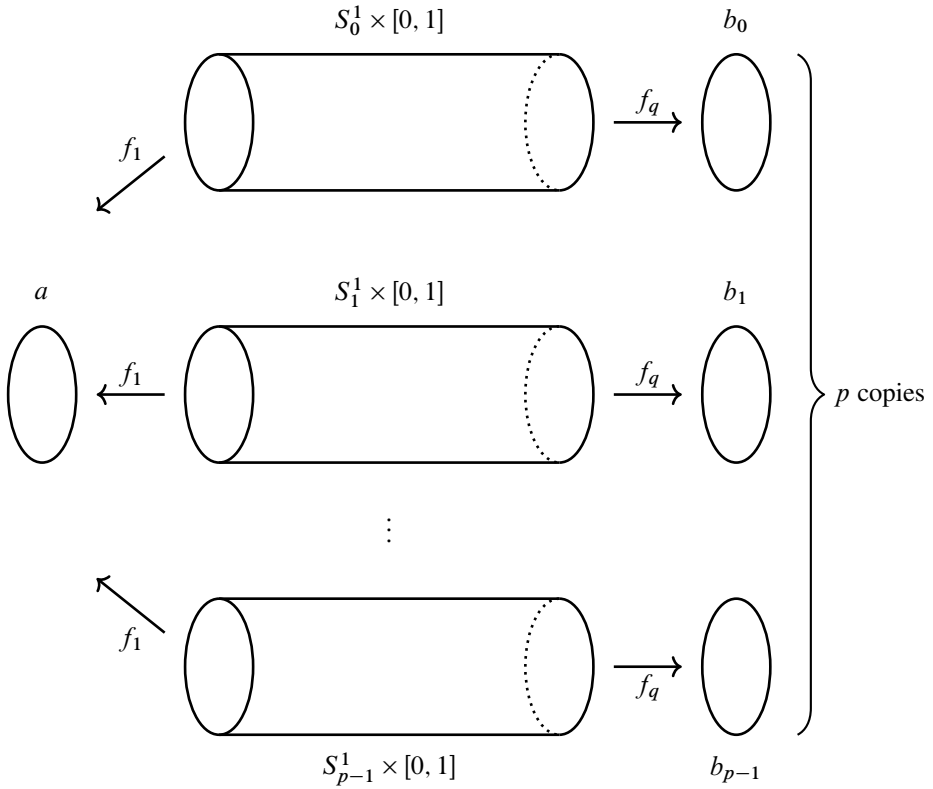


Figure 3: The  $p$ -fold cyclic cover  $(X_{p,q})_p$  of  $X_{p,q}$  is also a double mapping cylinder, where  $f_1$  and  $f_q$  denote the degree 1 and the degree  $q$  maps, respectively.

where each  $S_i^1 \times \{0\}$  is identified with the circle  $a$  by the identity map, and  $S_i^1 \times \{1\}$  is identified with the circle  $b_i$  by the degree  $q$  map. By van Kampen’s theorem, we deduce that

$$\pi_1((X_{p,q})_p) \cong \langle b_0, b_1, \dots, b_{p-1} \mid b_i^q = b_j^q \text{ for } 0 \leq i \neq j \leq p-1 \rangle.$$

Since  $E_p(T(p, q))$  deformation retracts onto  $(X_{p,q})_p$ , we obtain the following proposition:

**Proposition 2.2** *Let  $\pi : E_p(T(p, q)) \rightarrow E(T(p, q))$  be the  $p$ -fold cyclic covering and let  $b_0, b_1, \dots, b_{p-1}$  be the homotopy classes of the components of  $\pi^{-1}(c_2)$  such that  $\pi_*(b_i) = \mu^i c_2 \mu^{-i}$ . Then*

$$\pi_1(E_p(T(p, q))) = \langle b_0, b_1, \dots, b_{p-1} \mid b_i^q = b_j^q \text{ for } 0 \leq i \neq j \leq p-1 \rangle.$$

Next, we use this description of  $\pi_1(E_p(T(p, q)))$  to obtain generators of the finite abelian group  $H_1(\Sigma_p(T(p, q))) = TH_1(E_p(T(p, q)))$ . First, note that [Proposition 2.2](#) shows that  $H_1(E_p(T(p, q))) \cong \mathbb{Z} \oplus \mathbb{Z}_q^{p-1}$  has generators  $b_0, b_1, \dots, b_{p-1}$  and relations  $qb_i = qb_j$  for each  $i$  and  $j$ . In the remainder of this section, we describe a set of generators that will be more convenient for the twisted Alexander polynomial computations of [Section 3](#).

**Remark 2.3** While the meridian  $\mu$  of  $T(p, q)$  does not lift to  $E_p(T(p, q))$ , a loop representing  $\mu^p$  does. Since the projection-induced map  $\pi_*: \pi_1(E_p(T(p, q))) \rightarrow \pi_1(E(T(p, q)))$  is injective, we slightly abuse notation and also write  $\mu^p$  for the homotopy class of this lift in  $\pi_1(E_p(T(p, q)))$ .

We will make no notational distinction between elements in  $\pi_1(E_p(T(p, q)))$  and elements in  $H_1(E_p(T(p, q)))$ , despite switching from multiplicative to additive notation. In some rare instances, we will also use the multiplicative notation in homology. Keeping this in mind, for  $i = 0, \dots, p - 1$ , we consider  $\mu^{-p}b_i$  in  $\pi_1(E_p(T(p, q)))$  and  $x_i := b_i - \mu^p$  in  $H_1(E_p(T(p, q)))$ . The next proposition describes the homology group  $H_1(\Sigma_p(T(p, q)))$  as a  $\mathbb{Z}[\mathbb{Z}_p]$ -module.

**Proposition 2.4** *The abelian group  $H_1(\Sigma_p(T(p, q))) \cong \mathbb{Z}_q^{p-1}$  is generated by the  $x_i = b_i - \mu^p$ , and these elements satisfy the following relations:*

- (i)  $x_0 + x_1 + \dots + x_{p-1} = 0$ .
- (ii)  $x_i = t^i x_0$  for  $i = 0, \dots, p - 1$ , where  $t$  denotes the covering transformation of  $\Sigma_p(T(p, q))$ .

*In particular, there exists an isomorphism of  $\mathbb{Z}[\mathbb{Z}_p]$ -modules*

$$H_1(\Sigma_p(T(p, q))) \cong \mathbb{Z}_q[t]/(1 + t + t^2 + \dots + t^{p-1}).$$

**Proof** The proof has four steps. Firstly, we establish a criterion for an element in  $H_1(\Sigma_p(T(p, q)))$  to be torsion; secondly, we prove that the  $x_i$  are torsion; thirdly, we show that the  $x_i$  generate  $TH_1\Sigma_p(T(p, q))$  as an abelian group; fourthly and finally we prove that the  $x_i$  satisfy the two identities stated in the lemma.

We assert that an element  $x = \sum_{i=0}^{p-1} a_i b_i$  in  $H_1(E_p(T(p, q)))$  is torsion if and only if  $\sum_{i=0}^{p-1} a_i = 0$ . The map  $\pi_*: H_1(E_p(T(p, q))) \rightarrow H_1(E(T(p, q)))$  maps  $TH_1(E_p(T(p, q)))$  to zero and maps the infinite cyclic summand isomorphically

onto  $p\mathbb{Z} \cong \mathbb{Z}\langle c_2 \rangle$ .<sup>1</sup> In particular, a class  $x \in H_1(E_p(T(p, q)))$  is torsion if and only if  $\pi_*(x) = 0$ . On the other hand, using Proposition 2.2, we deduce that  $\pi$  induces the following map on homology, concluding the proof of the assertion:

$$\pi_*: H_1(E_p(T(p, q))) \rightarrow p\mathbb{Z} \subset \mathbb{Z} = H_1(E(T(p, q))), \quad \sum_{i=0}^{p-q} a_i b_i \mapsto \sum_{i=0}^{p-1} a_i.$$

We move on to the second step: we prove that the homology classes  $x_0, \dots, x_{p-1}$  are torsion. Using the criterion, we must show that  $\pi_*(x_i) = 0$  for each  $i$ . Since  $\pi_*(b_i) = 1$ , this reduces to showing that  $\pi_*(\mu^p) = 1$ . We start by computing the abelianisation of  $\mu^p$ . Since  $\mu = c_1^k c_2^l$ , we notice that, in  $\pi_1(E_p(T(p, q)))$ ,

$$(5) \quad \mu^p = (c_1^k c_2^l c_1^{-k}) \cdot (c_1^{2k} c_2^l c_1^{-2k}) \cdots (c_1^{(p-1)k} c_2^l c_1^{-(p-1)k}) c_1^{pk} c_2^l.$$

In order to compute the abelianisation of this expression, we claim that, for any  $0 \leq s \leq p-1$  and any  $k$ , the equation  $\mu^s c_2 \mu^{-s} = c_1^{ks} c_2 c_1^{-ks}$  holds in  $H_1(E_p(T(p, q))) = \pi_1(E_p(T(p, q)))^{\text{ab}}$ . This claim is a consequence of the following direct computation in  $\pi_1(E_p(T(p, q)))$ :

$$\mu^s c_2 \mu^{-s} = \left( \prod_{i=1}^{s-1} c_1^{ki} c_2^l c_1^{-ki} \right) \cdot (c_1^{ks} c_2 c_1^{-ks}) \cdot \left( \prod_{i=1}^{s-1} c_1^{ki} c_2^{-l} c_1^{-ki} \right).$$

Using consecutively (5), the equation  $\mu^s c_2 \mu^{-s} = c_1^{ks} c_2 c_1^{-ks}$  that we just established, and the identification  $b_i = \mu^i c_2 \mu^{-i}$  from (4) (as well as the presentation in Proposition 2.1 and  $qk + pl = 1$ ), we obtain the sequence of equalities, in  $H_1(E_p(T(p, q)))$ ,

$$(6) \quad \begin{aligned} \mu^p &= (c_1^k c_2^l c_1^{-k}) \cdot (c_1^{2k} c_2^l c_1^{-2k}) \cdots (c_1^{(p-1)k} c_2^l c_1^{-(p-1)k}) c_1^{pk} c_2^l \\ &= (\mu c_2^l \mu^{-1}) (\mu^2 c_2^l \mu^{-2}) \cdots (\mu^{(p-1)k} c_2^l \mu^{-(p-1)k}) c_1^{pk} c_2^l \\ &= l(b_0 + b_1 + \cdots + b_{p-1}) + qk b_0. \end{aligned}$$

As  $\pi_*(b_i) = 1$  for each  $i$ , this implies that  $\pi_*(\mu^p) = 1$ . It follows that  $\pi_*(x_i) = \pi_*(b_i) - \pi_*(\mu^p) = 0$ , and therefore each of the  $x_i$  is torsion. This concludes the second step of the proof.

Thirdly, we show that every element of  $TH_1(E_p(T(p, q)))$  can be written as a linear combination of the  $x_i$  for  $i = 0, 1, \dots, p-1$ : given  $x = \sum_{i=0}^{p-1} a_i b_i$ , adding and

<sup>1</sup>For any knot  $K$  and prime power  $n$ , one has the decomposition  $H_1(E_n(K)) = TH_1(E_n(K)) \oplus \mathbb{Z}$ , where the  $\mathbb{Z}$  summand is generated by a lift of the  $n$ -fold power of the meridian.

subtracting  $\mu^p$ , using  $\sum_{i=0}^{p-1} a_i = 0$  (which holds thanks to the first step) and the definition of  $x_i$ , we obtain

$$x = \sum_{i=0}^{p-1} a_i b_i = \sum_{i=0}^{p-1} a_i \mu^p + \sum_{i=0}^{p-1} a_i (b_i - \mu^p) = \sum_{i=0}^{p-1} a_i x_i.$$

Fourthly and finally, we establish the relations  $x_0 + x_1 + \cdots + x_{p-1} = 0$  and  $x_i = t^i x_0$ . The latter relation is clear (since  $b_i = t^i b_0$  and  $t\mu^p = \mu^p$ ) and so we focus on the former. Using consecutively (6), the relation  $qb_i = qb_j$ , and the fact that  $pl + qk = 1$ , we notice that, in  $H_1(E_p(T(p, q)))$ ,

$$\begin{aligned} p\mu^p &= pl(b_0 + b_1 + \cdots + b_{p-1}) + pqkb_0 \\ &= pl(b_0 + b_1 + \cdots + b_{p-1}) + qk(b_0 + b_1 + \cdots + b_{p-1}) \\ &= (b_0 + b_1 + \cdots + b_{p-1}). \end{aligned}$$

The conclusion now promptly follows from the definition of the  $x_i$ , establishing the proposition.  $\square$

Assume that  $q$  is a prime. In this case  $H_1(\Sigma_p(T(p, q)))$  becomes an  $\mathbb{F}_q$ -vector space. The covering action  $t$  is then an  $\mathbb{F}_q$ -linear endomorphism of  $V_{p,q}$ .

### 3 Twisted polynomials of torus knots

In this section, we compute the Alexander polynomial of the 0-framed surgery  $M_{T(p,q)}$  twisted by a metabelian representation  $\alpha_{T(p,q)}(p, \chi): \pi_1(M_{T(p,q)}) \rightarrow \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$  that frequently appears in Casson–Gordon theory [15]. In Section 3.1, we recall the definition of  $\alpha_K(p, \chi)$  for a general knot  $K$ . In Section 3.2, we study this representation in the case of torus knots. Finally, in Section 3.3, we compute the relevant twisted Alexander polynomials.

#### 3.1 The metabelian representation $\alpha_K(p, \chi)$

In this subsection, given a knot  $K$  and a positive integer  $p$ , we recall the definition of the representation  $\alpha_K(p, \chi): \pi_1(M_K) \rightarrow \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$  from [15], where  $\chi: H_1(\Sigma_p(K)) \rightarrow \mathbb{Z}_m$  is a character. In what follows,  $E_K$  denotes the exterior of  $K$  and  $M_K$  denotes its 0-framed surgery. Finally, we use  $\xi_m := e^{2\pi i/m}$  to denote the  $m^{\mathrm{th}}$  primitive root of unity.

We use

$$H_1(E(K); \mathbb{Z}[t_K^{\pm 1}]) \cong \pi_1(E(K))^{(1)} / \pi_1(E(K))^{(2)}$$

to denote the Alexander module of  $K$ . In what follows, we shall frequently identify  $H_1(\Sigma_p(K))$  with  $H_1(E(K); \mathbb{Z}[t_K^{\pm 1}]) / (t_K^p - 1)$ , as for instance in [8, Corollary 2.4]. Consider the composition of canonical projections

$$(7) \quad q_K: \pi_1(M_K)^{(1)} \rightarrow H_1(E(K); \mathbb{Z}[t_K^{\pm 1}]) \rightarrow H_1(\Sigma_p(K)).$$

Use  $\phi_K: \pi_1(E(K)) \rightarrow H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} = \langle t_K \rangle$  to denote the abelianisation homomorphism, and fix an element  $\mu_K$  in  $\pi_1(E(K))$  such that  $\phi_K(\mu_K) = t_K$ . Note that, for every  $g \in \pi_1(E(K))$ , we have  $\phi_K(\mu_K^{-\phi_K(g)} g) = 1$ . Since  $\phi_K$  is the abelianisation map, we deduce that  $\mu_K^{-\phi_K(g)} g$  belongs to  $\pi_1(E(K))^{(1)}$ . Combining this notation, we consider the representation

$$\alpha_K(p, \chi): \pi_1(E(K)) \rightarrow \text{GL}_p(\mathbb{C}[t^{\pm 1}])$$

given by

$$(8) \quad \alpha_K(p, \chi)(g) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ t & 0 & \cdots & 0 \end{pmatrix}^{\phi_K(g)} \cdot \begin{pmatrix} \xi_m^{\chi(q_K(\mu_K^{-\phi_K(g)} g))} & 0 & \cdots & 0 \\ 0 & \xi_m^{\chi(t_K \cdot q_K(\mu_K^{-\phi_K(g)} g))} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_m^{\chi(t_K^{p-1} \cdot q_K(\mu_K^{-\phi_K(g)} g))} \end{pmatrix} \\ =: A_p(t)^{\phi_K(g)} D_g.$$

Note that  $\alpha(p, \chi)$  can equally well be defined on  $\pi_1(M_K)$  instead of  $\pi_1(E(K))$ : the definition can be adapted verbatim, and we use the same notation

$$\alpha_K(p, \chi): \pi_1(M_K) \rightarrow \text{GL}_p(\mathbb{C}[t^{\pm 1}]).$$

A closely related observation is that  $\alpha(p, \chi)$  is a metabelian representation and therefore vanishes on the longitude of  $K$ ; this also explains why  $\alpha_K(p, \chi)$  descends to  $\pi_1(M_K)$ .

### 3.2 An explicit description of $\alpha_{T(p,q)}(p, \chi)$

We use the presentation of  $\pi_1(E(T(p, q)))$  from Proposition 2.1 to describe the representation  $\alpha_{T(p,q)}(p, \chi)$ . Throughout this subsection, we set  $K := T(p, q)$  in order to avoid cumbersome notation such as  $q_{T(p,q)}$ .

We recall the definition of the generators  $x_0, \dots, x_{p-1}$  of  $H_1(\Sigma_p(K)) \cong \mathbb{Z}_q^{p-1}$  described in Proposition 2.4, referring to Section 2 for further details. Using the notation of that section, we set  $x_i = b_i - \mu^p$ , where  $\mu$  is a meridian of  $K$ . For  $i = 0, \dots, p - 1$ ,



thinking of  $x_i$  as the abelianisation of  $\mu^{-p}b_i$ , and using Proposition 2.2 to identify  $b_i$  with  $\mu^i c_2 \mu^{-i}$ ,

$$(9) \quad t_K^i q_K(\mu^{-p} c_2) = q_K(\mu^{-p} \mu^i c_2 \mu^{-i}) = q_K(\mu^{-p} b_i) = x_i.$$

Recall furthermore that Proposition 2.4 also established the relations  $x_0 + \dots + x_{p-1} = 0$  as well as  $t_K x_i = x_{i+1}$  for  $i = 0, \dots, p-1$ . The next result follows immediately from these considerations.

**Lemma 3.1** Fix two coprime integers  $p, q > 0$  and set  $K := T(p, q)$ . The abelian group of characters on  $H_1(\Sigma_p(K))$  is isomorphic to

$$\{\mathbf{a} := (a_1, \dots, a_p) \in \mathbb{Z}_q^p \mid a_1 + \dots + a_p = 0\}.$$

The isomorphism maps a character  $\chi$  to  $(\chi(x_0), \dots, \chi(x_{p-1}))$ , and we write  $\chi_{\mathbf{a}}$  for the character associated to  $\mathbf{a}$ .

Recall that Proposition 2.1 described a two-generator, one-relation presentation for the knot group  $\pi_1(E(K))$ ; the generators were denoted by  $c_1$  and  $c_2$  and the unique relator was  $c_1^p c_2^{-q}$ . The next proposition describes the image of these generators under  $\alpha(p, \chi) := \alpha_K(p, \chi)$ . This will be useful in Proposition 3.3 when we compute the twisted Alexander polynomial of  $E(K)$ .

**Proposition 3.2** Fix two coprime integers  $p, q > 0$  and set  $K := T(p, q)$ . For a character  $\chi = \chi_{\mathbf{a}}: H_1(\Sigma_p(K)) \rightarrow \mathbb{Z}_q$ , the representation  $\alpha(p, \chi)$  is conjugated to a representation  $\alpha'(p, \chi)$  such that

$$\alpha'(p, \chi)(c_2) = t \cdot \text{diag}(\xi_q^{a_1}, \dots, \xi_q^{a_p}), \quad \alpha'(p, \chi)(c_1) = A_p(t)^q.$$

**Proof** We first compute  $\alpha(p, \chi)(c_2)$ . We know that  $\phi_K(c_2) = p$  and  $A_p(t)^p = t \cdot \text{id}$ . In order to compute the diagonal matrix which appears in the definition of  $\alpha(p, \chi)(c_2)$  (recall (8)), we use (9) and Lemma 3.1 to obtain  $\chi(t_K^{i-1} q_K(\mu^{-p} c_2)) = \chi(x_{i-1}) = a_i$ . The first assertion follows:

$$\alpha(p, \chi)(c_2) = t \cdot \text{diag}(\xi_q^{\chi(q_K(\mu^{-p} c_2))}, \dots, \xi_q^{\chi(t_K^{p-1} q_K(\mu^{-p} c_2))}) = t \cdot \text{diag}(\xi_q^{a_1}, \dots, \xi_q^{a_p}).$$

Next, we study the conjugacy class of  $\alpha(p, \chi)(c_1)$ : we must find an invertible matrix  $X$  such that

$$(10) \quad X\alpha(p, \chi)(c_1)X^{-1} = A_p(t)^q,$$

$$(11) \quad X\alpha(p, \chi)(c_2)X^{-1} = t \cdot \text{diag}(\xi_q^{a_1}, \dots, \xi_q^{a_p}).$$

For  $v \in H_1(\Sigma_p(K))$ , we define  $\tilde{\alpha}(v) := \text{diag}(\xi_q^{\chi(v)}, \xi_q^{\chi(t_K v)}, \dots, \xi_q^{\chi(t_K^{p-1} v)})$ . Observe that, if we set  $X := \tilde{\alpha}(z)$ , then (11) is satisfied for any  $z \in H_1(\Sigma_p(K))$ ; indeed,

$\alpha(p, \chi)(c_2)$  commutes with  $X$  since both are diagonal. Therefore, we just have to establish the existence of a  $z \in H_1(\Sigma_p(K))$  such that (10) is satisfied for  $X = \tilde{\alpha}(z)$ .

First, for any  $x \in H_1(\Sigma_p(K))$ , a computation shows that

$$\tilde{\alpha}(x)A_p(t)^q\tilde{\alpha}(x)^{-1} = A_p(t)^q\tilde{\alpha}((t_K^{-q} - 1)x).$$

Define  $y := q_K(\mu^{-q}c_1)$ , so that  $\alpha(p, \chi)(c_1) = A_p(t)^q\tilde{\alpha}(y)$ . Consequently, if we set  $X := \tilde{\alpha}(z)$  (for any  $z \in H_1(\Sigma_p(K))$ ), use the definition of  $y$ , the fact that  $\tilde{\alpha}(y)$  and  $X$  commute (both are diagonal), and the aforementioned identity, then we obtain

$$\begin{aligned} X\alpha(p, \chi)(c_1)X^{-1} &= XA_p(t)^q\tilde{\alpha}(y)X^{-1} = XA_p(t)^qX^{-1}\tilde{\alpha}(y) \\ &= A_p(t)^q\tilde{\alpha}((t_K^{-q} - 1)z + y). \end{aligned}$$

Therefore, if we choose  $z := -(t_K^{-q} - 1)^{-1}y$ , then (10) holds. For this to make sense, however, we must argue that  $t_K^{-q} - 1$  is an automorphism of  $H_1(\Sigma_p(K))$ . This is indeed the case: as  $t_K - 1$  is an automorphism of  $H_1(\Sigma_p(K))$ , the inverse is given by  $(t_K^{-1} - 1)^{-1}(1 + t_K^{-q} + t_K^{-2q} + \dots + t_K^{-(k-1)q})$ , where  $qk \equiv 1 \pmod p$ . Such a  $k$  exists because  $p$  and  $q$  are coprime. We have therefore found  $X$  such that (10) and (11) hold, and this concludes the proof of the proposition.  $\square$

### 3.3 The computation of the twisted polynomial

In this subsection, we compute the twisted Alexander polynomial of the 0–framed surgery  $M_{T(p,q)}$  with respect to  $\alpha(p, \chi)$ .

Recall that, given a space  $X$  and a representation  $\beta: \pi_1(X) \rightarrow \text{GL}_p(\mathbb{C}[t^{\pm 1}])$ , the twisted Alexander polynomial  $\Delta_1^\beta(X)$  is defined as the order of the twisted Alexander module  $H_1(X; \mathbb{C}[t^{\pm 1}]_\beta^p)$ . More generally, we write  $\Delta_i^\beta(X)$  for the order of the  $\mathbb{C}[t^{\pm 1}]$ –module  $H_i(X; \mathbb{C}[t^{\pm 1}]_\beta^p)$ . Recall that the  $\Delta_i^\beta(X)$  are defined up to multiplication by units of  $\mathbb{C}[t^{\pm 1}]$ .

The next proposition describes  $\Delta_1^{\alpha(p,\chi)}(E(T(p, q)))$ , where  $E(T(p, q))$  denotes the exterior of  $T(p, q)$ .

**Proposition 3.3** *Let  $p, q > 0$  be coprime integers. For  $\chi = \chi_a: H_1(\Sigma_p(T(p, q))) \rightarrow \mathbb{Z}_q$ , the metabelian twisted Alexander polynomial of  $E(T(p, q))$  is given by*

$$\Delta_1^{\alpha(p,\chi)}(E(T(p, q))) = \frac{(1 - t^q)^{p-1}}{(t\xi_q^{a_1} - 1)(t\xi_q^{a_2} - 1) \cdots (t\xi_q^{a_p} - 1)}.$$

**Proof** We use  $\tau^{\alpha(p,\chi)}(E(K))$  to denote the Reidemeister torsion of a knot exterior  $E(K)$  twisted by  $\alpha(p, \chi) := \alpha_K(p, \chi)$ . We refer to [9] for more on the subject, but

simply note that  $\tau^{\alpha(p,\chi)}(E(K))$  is defined since the chain complex  $C_*(E(K); \mathbb{C}(t)^P)$  of left  $\mathbb{C}(t)$ -modules is acyclic [6, Corollary after Lemma 4]. Since  $E(K)$  has torus boundary, by [9, Proposition 2(5)], the twisted Reidemeister torsion and twisted Alexander polynomial are related by

$$\tau^{\alpha(p,\chi)}(E(K)) = \frac{\Delta_1^{\alpha(p,\chi)}(E(K))}{\Delta_0^{\alpha(p,\chi)}(E(K))}.$$

Since  $\Delta_0^{\alpha(p,\chi)}(E(K)) = 1$  for every knot  $K$  [4, Lemma 4.1], we are reduced to computing  $\tau^{\alpha(p,\chi)}(E(T(p, q)))$ . By [19, Theorem A], this torsion invariant can be expressed via Fox calculus. In our case, using the presentation of  $\pi_1(E(T(p, q)))$  resulting from Proposition 2.1, we obtain

$$(12) \quad \Delta_1^{\alpha(p,\chi)}(E(T(p, q))) = \tau^{\alpha(p,\chi)}(E(T(p, q))) = \frac{\det(\alpha(p, \chi)(\partial(c_1^P c_2^{-q})/\partial c_1))}{\det(\alpha(p, \chi)(c_2) - \text{id})}.$$

Since this expression does not depend on the conjugacy class of  $\alpha(p, \chi)$ , we can work with the representation  $\alpha'(p, \chi)$  described in Proposition 3.2. Using the first item of Proposition 3.2, the denominator of (12) is given by the formula

$$(13) \quad \det(\alpha(p, \chi)(c_2) - \text{id}) = \det(\text{diag}(t \xi_q^{a_1} - 1, t \xi_q^{a_2} - 1, \dots, t \xi_q^{a_p} - 1)) \\ = \prod_{i=1}^p (t \xi_q^{a_i} - 1).$$

We will now compute the numerator of (12) and show that it equals  $(1 - t^q)^{p-1}$ . Recall from (8) that, for  $g \in \pi_1(E(K))$ , the metabelian representation  $\alpha_K(p, \chi)$  is given by  $\alpha_K(p, \chi)(g) = A_p(t)^{\phi_K(g)} D_g$ . An inductive argument involving the properties of the Fox derivative shows that

$$\frac{\partial(c_1^P c_2^{-q})}{\partial c_1} = \frac{\partial c_1^P}{\partial c_1} = 1 + c_1 + c_1^2 + \dots + c_1^{p-1} =: g.$$

We will now apply  $\alpha(p, \chi)$  to  $g$ . We recall from Proposition 3.2 that  $\alpha'(p, \chi)(c_1) = A_p(t)^q$ , and we now work over  $\mathbb{C}[t^{\pm 1/p}]$ . Indeed, as observed in [15, page 935], in this ring, the matrix  $A_p(t)$  is conjugated to the diagonal matrix

$$B_p(t) := \text{diag}(t^{1/p}, \xi_p t^{1/p}, \xi_p^2 t^{1/p}, \dots, \xi_p^{p-1} t^{1/p}).$$

Since (12) only depends on the conjugacy class of the representation  $\alpha(p, \chi)$ , we can work with  $B_p(t)$  instead of  $A_p(t)$ . We use  $\sim$  to denote the conjugacy relation. Since  $B_p(t)$  is diagonal, its powers are easy to compute and, as a consequence, we

obtain

$$\begin{aligned} \alpha'(p, \chi)(g) &\sim \text{id} + B_p(t)^q + B_p(t)^{2q} + \dots + B_p(t)^{(p-1)q} \\ &= \text{diag} \left( \frac{1-t^q}{1-t^{q/p}}, \frac{1-t^q}{1-\xi_p^q t^{q/p}}, \frac{1-t^q}{1-\xi_p^{2q} t^{q/p}}, \dots, \frac{1-t^q}{1-\xi_p^{q(p-1)} t^{q/p}} \right). \end{aligned}$$

Taking the determinant of this expression, we deduce that

$$(14) \quad \det \left( \alpha(p, \chi) \left( \frac{\partial(c_1^p c_2^{-q})}{\partial c_1} \right) \right) = \prod_{j=0}^{p-1} \frac{1-t^q}{1-\xi_p^{jq} t^{q/p}} = \frac{(1-t^q)^p}{1-t^q} = (1-t^q)^{p-1}.$$

Plugging (13) and (14) into (12) concludes the proof of the proposition. □

Using Proposition 3.3, we can compute the twisted polynomial of the 0–framed surgery  $M_{T(p,q)}$ .

**Corollary 3.4** *Let  $p, q > 0$  be coprime integers. For  $\chi = \chi_a : H_1(\Sigma_p(T(p, q))) \rightarrow \mathbb{Z}_q$ , the metabelian twisted Alexander polynomial of  $M_{T(p,q)}$  is given by*

$$\Delta_1^{\alpha(p,\chi)}(M_{T(p,q)}) = \frac{(-1)^{p-1} (1-t^q)^{p-1}}{(t\xi_q^{a_1} - 1)(t\xi_q^{a_2} - 1) \dots (t\xi_q^{a_p} - 1)(t-1)}.$$

**Proof** By Proposition 3.3, we need only show that  $(-1)^{p-1} (t-1) \Delta_1^{\alpha(p,\chi)}(M_K) = \Delta_1^{\alpha(p,\chi)}(E(K))$  for every knot  $K$ , where  $\alpha(p, \chi) := \alpha_K(p, \chi)$ . Using the equality  $\Delta_1^{\alpha(p,\chi)}(E(K)) = \tau^{\alpha(p,\chi)}(E(K))$  that was obtained in the proof of Proposition 3.3, Lemma 3 of [9], as well as [9, Propositions 2(8) and 5] and the fact that  $\Delta_0^{\alpha(p,\chi)}(M_K) = 1$  (by [4, Lemma 4.1]), we obtain the sequence of equalities

$$\begin{aligned} \Delta_1^{\alpha(p,\chi)}(E(K)) &= \tau^{\alpha(p,\chi)}(E(K)) = \det(\alpha(p, \chi)(\mu_K) - \text{id}) \tau^{\alpha(p,\chi)}(M_K) \\ &= \det(\alpha(p, \chi)(\mu_K) - \text{id}) \frac{\Delta_1^{\alpha(p,\chi)}(M_K)}{\Delta_0^{\alpha(p,\chi)}(M_K) \Delta_2^{\alpha(p,\chi)}(M_K)} \\ &= \det(\alpha(p, \chi)(\mu_K) - \text{id}) \frac{\Delta_1^{\alpha(p,\chi)}(M_K)}{\Delta_0^{\alpha(p,\chi)}(M_K) \overline{\Delta_0^{\alpha(p,\chi)}(M_K)}} \\ &= \det(\alpha(p, \chi)(\mu_K) - \text{id}) \Delta_1^{\alpha(p,\chi)}(M_K). \end{aligned}$$

It thus remains to show that  $\det(\alpha(p, \chi)(\mu_K) - \text{id}) = (-1)^{p-1} (t-1)$ ; this follows from the definition of  $\alpha(p, \chi)$  (recall (8)) since  $\alpha(p, \chi)(\mu_K) = A_p(t)$ . □

### 4 Linking forms and their metabolisers

This section collects some facts about linking forms and their metabolisers. This will be useful in Section 5 since both the metabelian Blanchfield pairing and  $\lambda_p(T(p, q))$

are linking forms. In Section 4.1, we recall some basics on linking forms and their Witt groups. In Section 4.2, we prove a result on metabolisers of linking forms of the type  $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$ .

## 4.1 The Witt group of linking forms

Let  $R$  be a PID with involution and let  $Q$  denote its field of fractions. This subsection is concerned with linking forms. Firstly, we recall the definition of the Witt group  $W(Q, R)$  of linking forms. Secondly, we collect some facts about  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$  that are used in Section 5 below.

A *linking form* over  $R$  is a pair  $(V, \lambda)$ , where  $V$  is a torsion  $R$ -module and  $\lambda: V \times V \rightarrow Q/R$  is a sesquilinear and Hermitian pairing. The set of linking forms over  $R$  forms a monoid under the direct sum. A linking form  $(V, \lambda)$  is *nonsingular* if its *adjoint*  $\lambda^*: V \rightarrow V^*$ ,  $x \mapsto \lambda(x, -)$  is an isomorphism. In the sequel, our linking forms will be either over  $\mathbb{Z}$  or  $\mathbb{C}[t^{\pm 1}]$ . From now on, we also assume that all linking forms are nonsingular. Given a linking form  $(V, \lambda)$  over  $R$ , a submodule  $L \subset V$  is *isotropic* if  $L \subset L^\perp$  and is a *metaboliser* if  $L = L^\perp$ . A linking form is *metabolic* if it admits a metaboliser. The set of metabolic linking forms over  $R$  forms a submonoid of the monoid of linking forms over  $R$ .

**Definition 4.1** The *Witt group of linking forms*, denoted by  $W(Q, R)$ , consists of the monoid of linking forms modulo the submonoid of metabolic linking forms. Two linking forms  $(V, \lambda)$  and  $(V', \lambda')$  are called *Witt equivalent* if they represent the same element in  $W(Q, R)$ .

The Witt group of linking forms is known to be an abelian group under direct sum, where the inverse of the class  $[(V, \lambda)]$  is represented by  $(V, -\lambda)$ . Next, we collect some facts on  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$  that will be used in Section 5 below.

**Remark 4.2** The Witt group  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$  is known to be free abelian and is detected by the signature jumps  $\delta\sigma_{(V, \lambda)}$  [3, Sections 4 and 5]. In particular, a linking form  $(V, \lambda)$  over  $\mathbb{C}[t^{\pm 1}]$  is metabolic if and only if all its signature jumps vanish [3, Theorem 5.3]. Reformulating,  $[V, \lambda] = 0$  in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$  if and only if  $\delta\sigma_{(V, \lambda)}(\omega) = 0$  for all  $\omega \in S^1$ . We refer to [3, Sections 4 and 5] for further details regarding signatures of linking forms but note that a linking form  $(V, \lambda)$  will have a trivial jump at  $\omega \in S^1$  if the order  $\text{Ord}(T)$  of the  $\mathbb{C}[t^{\pm 1}]$ -module  $T$  does not have a root at  $\omega$ .

In particular, Remark 4.2 implies the following result about linear independence in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ :

**Proposition 4.3** *If  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$  are two linking forms over  $\mathbb{C}[t^{\pm 1}]$  such that  $\text{Ord}(V_1)$  and  $\text{Ord}(V_2)$  have distinct roots, then the following assertions hold:*

- (i) *If  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$  are not metabolic, then the Witt classes  $[V_1, \lambda_1]$  and  $[V_2, \lambda_2]$  are linearly independent in  $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$ .*
- (ii) *If  $(V_1, \lambda_1) \oplus (V_2, \lambda_2)$  is metabolic, then  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$  are both metabolic.*

**Proof** We only prove the first assertion as the second assertion follows immediately. Assume that  $n_1[V_1, \lambda_1] + n_2[V_2, \lambda_2] = 0$  for some integers  $n_1$  and  $n_2$ . Remark 4.2 implies that all the signature jumps of  $n_1\lambda_1 \oplus n_2\lambda_2$  must vanish. Since  $\lambda_1$  is not metabolic, Remark 4.2 also implies that  $\lambda_1$  admits a nontrivial signature jump at some  $\omega_1 \in S^1$ . As a consequence of these two assertions, we infer that  $n_1\lambda_1$  and  $n_2\lambda_2$  must have a nontrivial signature jump at  $\omega_1$ . Since  $\text{Ord}(V_1)$  and  $\text{Ord}(V_2)$  have distinct roots, we deduce that  $n_1 = 0$ . The same reasoning shows that  $n_2 = 0$ , thus establishing the linear independence of  $[V_1, \lambda_1]$  and  $[V_2, \lambda_2]$  and establishing the proposition.  $\square$

## 4.2 Graph metabolisers

Given linking forms  $(V_1, \lambda_1), (V_2, \lambda_2)$ , we prove a result on metabolisers of linking forms of the type  $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$ . More precisely, Proposition 4.4 provides a criterion for when such a metaboliser must be a graph. This result will be used in Section 5 when we study metabolisers of  $\lambda_p(T(p, q))^N \oplus -\lambda_p(T(p, q))^N$ .

Given linking forms  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$ , a *morphism* of linking forms is an  $R$ -linear homomorphism  $f: V_1 \rightarrow V_2$  such that  $\lambda_2(f(x), f(y)) = \lambda_1(x, y)$  for all  $x, y \in V_1$ . If the forms are nonsingular, then a morphism is necessarily injective. An *isometry* of linking forms is a bijective morphism of linking forms. The graph

$$\Gamma_f = \{(v, f(v)) \in V_1 \oplus V_2 \mid v \in V_1\}$$

of a morphism  $f: (V_1, \lambda_1) \rightarrow (V_2, \lambda_2)$  is an isotropic submodule of  $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$ . If  $f$  is an isometry, then  $\Gamma_f$  is in fact a metaboliser of  $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$ . The next proposition provides an assumption under which the converse also holds.

**Proposition 4.4** *Let  $(V_1, \lambda_1)$  and  $(V_2, \lambda_2)$  be linking forms over  $R$  and let  $L \subset V_1 \oplus V_2$  be a metaboliser of  $\lambda_1 \oplus -\lambda_2$ . The following assertions hold:*

(i) If  $L \cap (V_1 \oplus 0) = 0 = L \cap (0 \oplus V_2)$ , then  $L$  is the graph of an isometry  $f : V_1 \rightarrow V_2$ ,

$$L = \{(v, f(v)) \in V_1 \oplus V_2 \mid v \in V_1\}.$$

(ii) If we additionally work over  $R = \mathbb{Z}$ , suppose that  $V_1$  and  $V_2$  are equipped with an isometric  $\mathbb{Z}_p$ -action, and  $L$  is a  $\mathbb{Z}_p$ -invariant metaboliser, then the isometry  $f$  is  $\mathbb{Z}_p$ -equivariant.

**Proof** We prove the first assertion. The isometry  $f$  will be defined by using the canonical projections  $\text{pr}_i : V_1 \oplus V_2 \rightarrow V_i$  for  $i = 1, 2$ . Since  $L \cap (V_1 \oplus 0) = 0 = L \cap (0 \oplus V_2)$ , it follows that  $\text{pr}_i|_L$  is injective, for  $i = 1, 2$ . Set  $W_i := \text{pr}_i(L)$ , for  $i = 1, 2$ , and define  $f$  as the composition

$$f : W_1 \xrightarrow[\cong]{\text{pr}_1^{-1}} L \xrightarrow[\cong]{\text{pr}_2} W_2.$$

Since  $f$  is an isomorphism of  $R$ -modules, it remains to check that it is a morphism of linking forms. First, however, we use the definition of  $f$  to observe that

$$(15) \quad L = \{(v, f(v)) \in V_1 \oplus V_2 \mid v \in W_1\} \subset V_1 \oplus V_2.$$

The fact that  $f$  is a morphism now follows from the fact that  $L$  is isotropic: for any  $v, w \in W_1$ , the pairs  $(v, f(v)), (w, f(w))$  belong to  $L$ , and therefore

$$0 = (\lambda_1 \oplus -\lambda_2)((v, f(v)), (w, f(w))) = \lambda_1(v, w) - \lambda_2(f(v), f(w)).$$

Looking at (15), it only remains to show that  $V_1 = W_1$  and  $V_2 = W_2$ . Since  $f$  is an isomorphism, we have  $\text{ord}(W_1) = \text{ord}(W_2)$  and therefore (15) implies that  $\text{ord}(L)^2 = \text{ord}(W_1) \text{ord}(W_2)$ . Since  $L$  is a metaboliser, we deduce that

$$(16) \quad \text{ord}(V_1) \text{ord}(V_2) = \text{ord}(L)^2 = \text{ord}(W_1) \text{ord}(W_2).$$

By way of contradiction, assume that  $\text{ord}(W_1)$  divides  $\text{ord}(V_1)$  with  $\text{ord}(W_1) \neq \text{ord}(V_1)$ ; we write  $\text{ord}(W_1) \nmid \text{ord}(V_1)$ . A glance at (16) shows that  $\text{ord}(V_2) \nmid \text{ord}(W_2)$ , contradicting the inclusion  $W_2 \subset V_2$ . We conclude that  $\text{ord}(W_i) = \text{ord}(V_i)$  and consequently  $W_i = V_i$  for  $i = 1, 2$ . This concludes the proof of the first assertion.

We prove the second assertion. Use  $t$  to denote a generator of  $\mathbb{Z}_p$ . As the metaboliser  $L$  is  $\mathbb{Z}_p$ -invariant, if  $(v, f(v)) \in L$ , then  $(tv, tf(v)) \in L$  for any  $v \in V_1$ . Moreover, as  $(tv, f(tv)) \in L$  and  $L \cap (0 \oplus V_2) = 0$ , it follows that  $(tv, f(tv)) = (tv, tf(v))$ . We have therefore established that  $f(tv) = tf(v)$  for any  $v \in V_1$ , and thus  $f$  is  $\mathbb{Z}_p$ -equivariant, as desired. □

## 5 Nonslice linear combinations of iterated torus knots

This section aims to prove [Theorem 1.1](#), whose statement we now recall. For an integer  $p \geq 2$  and a sequence  $Q = (q_1, q_2, \dots, q_\ell)$  of integers that are relatively prime to  $p$ , we write iterated torus knots as  $T(p, Q) := T(p, q_1; p, q_2; \dots; p, q_\ell)$ . Our main result reads as follows:

**Theorem 1.1** *Fix a prime power  $p$ . Let  $S_p$  be the set of iterated torus knots*

$$T(p, q_1; p, q_2; \dots; p, q_\ell),$$

where the sequences  $(q_1, q_2, \dots, q_\ell)$  of positive integers satisfy

- (i) for  $i = 1, \dots, \ell$ , the integer  $q_i$  is coprime to  $p$ ;
- (ii)  $q_\ell$  is a prime;
- (iii) for  $i = 1, \dots, \ell - 1$ , the integer  $q_i$  is coprime to  $q_\ell$  when  $\ell > 1$ .

The set  $S_p$  is linearly independent in the topological knot concordance group  $\mathcal{C}^{\text{top}}$ .

To prove [Theorem 1.1](#), we must obstruct the sliceness of linear combinations of knots belonging to  $S_p$ . The first step, which is carried out in [Section 5.1](#), is to determine which of these linear combinations are algebraically slice. In [Section 5.2](#), we use metabelian twisted Blanchfield pairings to obstruct the sliceness of such algebraically slice linear combinations.

### 5.1 Algebraically slice linear combinations of algebraic knots

Fix an integer  $p \geq 2$ . For  $i = 1, \dots, k$ , fix sequences  $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$  of  $\ell_i$  positive integers each of which is coprime to  $p$ , and let  $n_1, \dots, n_k \in \mathbb{Z}$ . The goal of this subsection is to determine when the following knot is algebraically slice:

$$(17) \quad K = n_1 T(p, Q_1) \# n_2 T(p, Q_2) \# \dots \# n_k T(p, Q_k).$$

In order to provide a convenient criterion, we define the  $s$ -level of  $K$  to be the knot

$$\mathcal{K}_s(K) := n_1 T(p, q_{1,\ell_1-s}) \# n_2 T(p, q_{2,\ell_2-s}) \# \dots \# n_k T(p, q_{k,\ell_k-s}).$$

Here, it is understood that  $T(p, q_{i,\ell_i-s})$  is the unknot  $U$  if  $\ell_i - s < 1$ . As an example of this notation, we see that, if  $Q = (q_1, \dots, q_\ell)$ , then  $\mathcal{K}_s(T(p, Q)) = T(p, q_{\ell-s})$  for  $0 \leq s \leq \ell - 1$  and  $\mathcal{K}_s(T(p, Q)) = U$  for  $s \geq \ell$ . In particular, the cabling formula for the classical Blanchfield form implies that

$$(18) \quad \text{Bl}(T(p, Q)) \cong \bigoplus_{s \geq 0} \text{Bl}(\mathcal{K}_s(T(p, Q)))(t^{p^s}).$$



Indeed, for a knot  $L$ , the cabling formula reads as [24]

$$\text{Bl}(L_{p,q})(t) = \text{Bl}(T(p, q))(t) \oplus \text{Bl}(L)(t^p).$$

Next, we move on to a slightly more involved example.

**Example 5.1** The  $s$ -levels of

$$J := T(p, q_1; p, q_2) \# T(p, q_3) \# -T(p, q_1; p, q_3) \# -T(p, q_2)$$

are given by

$$\mathcal{K}_s(J) = \begin{cases} T(p, q_2) \# T(p, q_3) \# -T(p, q_3) \# -T(p, q_2) & \text{if } s = 0, \\ T(p, q_1) \# -T(p, q_1) & \text{if } s = 1, \\ U & \text{if } s \geq 2. \end{cases}$$

For  $s = 1$ , we used that  $\mathcal{K}_1(J) = T(p, q_1) \# U \# -T(p, q_1) \# -U$  is  $T(p, q_1) \# -T(p, q_1)$ . In particular, the formula displayed in (18) also holds for  $J$ . As we shall use in Proposition 5.3 below, it holds for the linear combination of (17).

For later use, we note that the 0-level of  $K$  is the most important to us: the first homology of its  $p$ -fold branched cover equals that of  $K$ .

**Remark 5.2** Since  $H_1(\Sigma_p(J_{p,q})) = H_1(\Sigma_p(T(p, q)))$  for any knot  $J$ , we deduce

$$H_1(\Sigma_p(K)) = H_1(\Sigma_p(\mathcal{K}_0(K))) = \bigoplus_{i=1}^k H_1(\Sigma_p(T(p, q_{i,\ell_i}))).$$

The analogous decomposition holds for the linking form  $\lambda_p(K)$  [22, Lemma 4].

The next proposition uses  $s$ -levels to exhibit a criterion for the algebraic sliceness of  $K$ .

**Proposition 5.3** Fix an integer  $p \geq 2$  and choose sequences of positive integers  $Q_i = (q_{i,1}, \dots, q_{i,\ell_i})$  that are relatively prime to  $p$  for  $i = 1, 2, \dots, k$ . The following statements are equivalent:

- (i) The knot  $K = n_1 T(p, Q_1) \# \dots \# n_k T(p, Q_k)$  is algebraically slice.
- (ii) Each  $\mathcal{K}_s(K)$  is slice.

**Proof** We first assert that the polynomials  $\Delta_{\mathcal{K}_s(K)}(t^{p^s})$  and  $\Delta_{\mathcal{K}_u(K)}(t^{p^u})$  have distinct roots if  $s \neq u$ . For a positive integer  $m$ , we set  $\xi_m := e^{2\pi i/m}$ . The roots of  $\Delta_{T(p,q)}(t)$  occur at those  $\xi_{pq}^a$  where the integer  $1 \leq a \leq pq$  is such that neither  $p$  nor  $q$  divides  $a$ ,

ie  $(\xi_{pq}^a)^p \neq 1$  and  $(\xi_{pq}^a)^q \neq 1$ . Consequently, the roots of  $\Delta_{T(p,q)}(t^{P^s})$  occur at  $\xi_{p^{s+1}q}^a$  such that  $1 \leq a \leq p^{s+1}q$  and neither  $p$  nor  $q$  divides  $a$ .

We argue that, if  $s \neq u$ , then  $\Delta_{T(p,q_1)}(t^{P^s})$  and  $\Delta_{T(p,q_2)}(t^{P^u})$  have distinct roots. Assume to the contrary that they have a common root. This root must be of the form  $\xi_{p^{s+1}q_1}^a = \xi_{p^{u+1}q_2}^b$ , where  $q_1$  and  $p$  (resp.  $q_2$  and  $p$ ) do not divide  $a$  (resp.  $b$ ). Without loss of generality, assume that  $s < u$ , so that  $1 = (\xi_{p^{s+1}q_1}^a)^{p^{s+1}q_1} = (\xi_{p^{u+1}q_2}^b)^{p^{s+1}q_1} = \xi_{p^{u-s}q_2}^{bq_1}$ . This implies that  $p^{u-s}q_2$  divides  $bq_1$ . However, by assumption,  $p$  divides neither  $q_1$  nor  $b$ , yielding the desired contradiction.

Next, recall from the definition of the  $s$ -level that

$$\mathcal{K}_s(K) := n_1 T(p, q_1, \ell_{1-s}) \# n_2 T(p, q_2, \ell_{2-s}) \# \dots \# n_k T(p, q_k, \ell_{k-s}).$$

Thus, if  $s \neq u$ , then  $\Delta_{\mathcal{K}_s(K)}(t^{P^s})$  and  $\Delta_{\mathcal{K}_u(K)}(t^{P^u})$  have distinct roots. This proves the assertion.

Assume that  $K$  is algebraically slice. By the cabling formula for the Blanchfield pairing (see [Example 5.1](#)),

$$(19) \quad \text{Bl}(K)(t) \cong \bigoplus_{s \geq 0} \text{Bl}(\mathcal{K}_s(K))(t^{P^s})$$

is metabolic. By the assertion and [Proposition 4.3](#), we deduce that each  $\text{Bl}(\mathcal{K}_s(K))(t^{P^s})$  is metabolic. It follows that the jump function of each  $\text{Bl}(\mathcal{K}_s(K))(t^{P^s})$  is trivial, which is simply a reparametrisation of the jump function of  $\text{Bl}(\mathcal{K}_s(K))(t)$ , where the parameter  $t \in S^1$  is changed to  $t^{P^r}$ . Hence,  $\mathcal{K}_s(K)$  is a connected sum of torus knots such that the jump function of  $\sigma_\omega(\mathcal{K}_s)$  is trivial. Since Litherland showed in [\[21, Lemma 1\]](#) that the jump functions of  $\sigma_\omega(T(p, q))$  are linearly independent,  $\mathcal{K}_s(K)$  is slice, as desired.

Assume that each  $\mathcal{K}_s(K)$  is slice. As a linking form over  $\mathbb{Z}[t^{\pm 1}]$ ,  $\text{Bl}(\mathcal{K}_s(K))$  is metabolic. Combining this with the decomposition displayed in [\(19\)](#), we deduce that  $\text{Bl}(K)$  is metabolic, as a linking form over  $\mathbb{Z}[t^{\pm 1}]$ . This is equivalent to  $K$  being algebraically slice [\[17\]](#), completing the proof of [Proposition 5.3](#). □

When  $K$  is algebraically slice, we obtain a convenient description of the 0-level of  $K$ .

**Corollary 5.4** *Suppose that  $K$ ,  $p$ ,  $\ell_i$  and  $Q_i$  for  $i = 1, \dots, k$  are as in [Proposition 5.3](#). If  $K$  is algebraically slice, then  $k$  is even and, after renumbering if necessary, the 0-level of  $K$  is*

$$\mathcal{K}_0(K) = \#_{j=1}^{k/2} m_j (T(p, q_j, \ell_j) \# -T(p, q_j, \ell_j)).$$

**Proof** By [Proposition 5.3](#),  $\mathcal{K}_0(K)$  is a slice linear combination of torus knots. Since torus knots are linearly independent in the knot concordance group, the conclusion follows.  $\square$

## 5.2 Linearly independent families of iterated torus knots

Fix a prime power  $p$ . The goal of this section is to prove [Theorem 1.1](#).

For  $i = 1, \dots, k$ , we choose sequences  $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$  of positive integers, where  $q_{i,\ell_i}$  is prime for all  $i$ , and the integer  $q_{i,j}$  is coprime to  $p$  and to  $q_{i,\ell_i}$  for all  $j$ . We also let  $n_1, \dots, n_k \in \mathbb{Z}$  be integers. We will use metabelian Blanchfield pairings [[25](#); [3](#); [4](#); [5](#)] to obstruct the sliceness of the knot

$$K = n_1 T(p, Q_1) \# n_2 T(p, Q_2) \# \dots \# n_k T(p, Q_k).$$

The sliceness obstruction that we will use, due to Miller and Powell [[25](#), Theorem 6.10], reads as follows. If, for every  $\mathbb{Z}_p$ -invariant metaboliser  $G$  of  $\lambda_p(K)$ , there exists a prime power-order character  $\chi$  that vanishes on  $G$  and is such that  $\text{Bl}_{\alpha(p,\chi)}(K)$  is not metabolic, then  $K$  is not slice. Here, we use  $\alpha(p, \chi) := \alpha_K(p, \chi)$  to denote the metabelian representation that was described in [Section 3.1](#).

**Remark 5.5** The metabelian Blanchfield pairing is a linking form

$$\text{Bl}_{\alpha(p,\chi)}(K) : H_1(M_K; \mathbb{C}[t^{\pm 1}]_{\alpha(p,\chi)}^p) \times H_1(M_K; \mathbb{C}[t^{\pm 1}]_{\alpha(p,\chi)}^p) \rightarrow \mathbb{C}(t)/\mathbb{C}[t^{\pm 1}],$$

where  $H_1(M_K; \mathbb{C}[t^{\pm 1}]_{\alpha(p,\chi)}^p)$  denotes the homology of the 0-framed surgery of  $K$  twisted by  $\alpha(p, \chi)$ . The precise definition of  $\text{Bl}_{\alpha(p,\chi)}(K)$  is not needed in this paper (the interested reader can find it in [[25](#); [3](#); [4](#); [5](#)]). All we need is the behaviour of  $\text{Bl}_{\alpha(p,\chi)}(K)$  under satellite operations, and this will be recalled as the argument proceeds.

The strategy behind the proof of [Theorem 1.1](#) is as follows:

- (i) Firstly, we study the characters on  $H_1(\Sigma_p(K))$ .
- (ii) Secondly, we study the consequences of  $\text{Bl}_{\alpha(p,\chi)}(K)$  being metabolic. This will impose substantial restrictions on  $\chi$ .
- (iii) Thirdly, we build characters that violate these restrictions.
- (iv) Finally, we combine these first three steps to conclude the proof.

The reader that wishes to see how these steps combine might glance at the end of the argument, after the conclusion of the proof of [Lemma 5.8](#); see [Section 5.2.4](#).

**5.2.1 Characters on  $H_1(\Sigma_p(K))$**  Assume that  $K$  is slice. The first step is to study the possible characters on the  $p$ -fold branched cover of  $K$ . Since  $K$  is algebraically slice, [Corollary 5.4](#) implies that  $k$  is even and, after renumbering if necessary, for some prime  $r$  (which is one of the  $q_{j,\ell_j}$ ) and some integers  $m_1, \dots, m_{k/2}$ , we can write

$$\mathcal{K}_0(K) = m_1(T(p, r) \# -T(p, r)) \# \#_{j=2}^{k/2} m_j(T(p, q_{j,\ell_j}) \# -T(p, q_{j,\ell_j})),$$

where  $q_{i,\ell_i} = r$  if and only if  $1 \leq i \leq 2m_1$ . It follows that, if we set  $M_j = m_1 + \dots + m_{j-1}$  for  $j = 2, \dots, \frac{1}{2}k$ , then, after further possible renumbering, the knot  $K$  can be rewritten as

$$(20) \quad K = \#_{i=1}^{m_1} (T(p, Q_{2i-1}) \# -T(p, Q_{2i})) \# \#_{j=2}^{k/2} \#_{i=1}^{m_j} (T(p, Q_{2M_j+2i-1}) \# -T(p, Q_{2M_j+2i})).$$

As [Remark 5.2](#) implies that  $H_1(\Sigma_p(K)) \cong H_1(\Sigma_p(\mathcal{K}_0(K)))$ , the description of  $\mathcal{K}_0(K)$ , the primary decomposition and the fact that the  $q_{i,\ell_i}$  are prime show that

$$(21) \quad H_1(\Sigma_p(K)) = H_1(\Sigma_p(T(p, r)))^{m_1} \oplus H_1(\Sigma_p(-T(p, r)))^{m_1} \oplus \bigoplus_{j=2}^{k/2} (H_1(\Sigma_p(T(p, q_{j,\ell_j})))^{m_j} \oplus H_1(\Sigma_p(-T(p, q_{j,\ell_j})))^{m_j}).$$

The linking form  $\lambda_p(K)$  on  $\Sigma_p(K)$  decomposes analogously.

From now on,  $\theta$  denotes the trivial character. Also, since  $H_1(\Sigma_p(T(p, r))) \cong \mathbb{Z}_r^{p-1}$ , we write characters  $H_1(\Sigma_p(T(p, r))) \rightarrow \mathbb{Z}_r$  as  $\chi_{\mathbf{a}}$ , where  $\mathbf{a} \in \mathbb{Z}_r^p$ . Since  $r$  is distinct from  $q_{i,\ell_i}$  for  $i > 2m_1$ , the decomposition of (21) implies that any character  $\chi: H_1(\Sigma_p(K)) \rightarrow \mathbb{Z}_r$  must be of the form

$$(22) \quad \chi = \bigoplus_{i=1}^{m_1} (\chi_{\mathbf{a}^i} \oplus \chi_{\mathbf{b}^i}) \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} \theta \oplus \theta,$$

where  $\{\mathbf{a}^j\}_{j=1}^{m_1}$  and  $\{\mathbf{b}^j\}_{j=1}^{m_1}$  are sequences of  $p$  elements in  $\mathbb{Z}_r$ .

**Remark 5.6** Recall that the Miller–Powell obstruction requires that, for every  $\mathbb{Z}_p$ -invariant metaboliser  $G$  of  $\lambda_p(K)$ , we construct a prime power-order character  $\chi$  that vanishes on  $G$  and is such that  $\text{Bl}_{\alpha(p,\chi)}(K)$  is not metabolic. The primary decomposition implies that every such metaboliser decomposes as a direct sum of metabolisers of the summands in (21).

Consequently, thanks to the form of the character in (22), it suffices to prove the following result: for every  $\mathbb{Z}_p$ -invariant metaboliser  $L$  of  $\lambda_p(T(p, r))^{m_1} \oplus -\lambda_p(T(p, r))^{m_1}$ , there is a prime power-order character  $\bigoplus_{i=1}^{m_1} (\chi_{a^i} \oplus \chi_{b^i})$  that vanishes on  $L$  and is such that  $\text{Bl}_{\alpha(p, \chi)}(K)$  is not metabolic, with  $\chi$  as in (22).

**5.2.2 The metabelian Blanchfield pairing of  $K$**  We now study the metabelian Blanchfield pairing of  $K$ . We first use satellite formulas to decompose it, and we then study the implications of it being metabolic. We use  $\alpha(p, \chi) := \alpha_K(p, \chi)$  to denote the metabelian representation that was described in Section 3.1. The behaviour of metabelian Blanchfield pairings under connected sums [4, Corollary 4.21] implies that  $\text{Bl}_{\alpha(p, \chi)}(K)$  is Witt equivalent to the linking form

$$\begin{aligned}
 (23) \quad & \text{Bl}_{\alpha(p, \chi)}(K) \\
 & \sim \bigoplus_{i=1}^{m_1} (\text{Bl}_{\alpha(p, \chi_{a^i})}(T(p, Q_{2i-1})) \oplus -\text{Bl}_{\alpha(p, \chi_{b^i})}(T(p, Q_{2i}))) \\
 & \quad \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} (\text{Bl}_{\alpha(p, \theta)}(T(p, Q_{2M_j+2i-1})) \oplus -\text{Bl}_{\alpha(p, \theta)}(T(p, Q_{2M_j+2i}))).
 \end{aligned}$$

For a sequence  $S = (q_1, \dots, q_k)$ , we use  $T(p, \widehat{S})$  to denote the iterated torus knot  $T(p, q_1; \dots; p, q_{k-1})$ . Next, we apply the satellite formula for the metabelian Blanchfield pairing [4, Theorem 4.19] to both expressions in (23). As we are working with  $p$ -fold covers and the sequences  $Q_{2i-1}$  and  $Q_{2i}$  (resp.  $Q_{2M_j+2i-1}$  and  $Q_{2M_j+2i}$ ) both have  $r$  (resp.  $q_j, \ell_j$ ) as the prime in last position, we claim

$$\begin{aligned}
 (24) \quad & \text{Bl}_{\alpha(p, \chi)}(K) \\
 & \sim \bigoplus_{i=1}^{m_1} (\text{Bl}_{\alpha(p, \chi_{a^i})}(T(p, r)) \oplus -\text{Bl}_{\alpha(p, \chi_{b^i})}(T(p, r))) \\
 & \quad \oplus \bigoplus_{i=1}^{m_1} \bigoplus_{u=1}^p (\text{Bl}(T(p, \widehat{Q}_{2i-1}))(\xi_r^{a^i} t) \oplus -\text{Bl}(T(p, \widehat{Q}_{2i}))(\xi_r^{b^i} t)) \\
 & \quad \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} (\text{Bl}_{\alpha(p, \theta)}(T(p, q_j, \ell_j)) \oplus -\text{Bl}_{\alpha(p, \theta)}(T(p, q_j, \ell_j))) \\
 & \quad \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} \bigoplus_{u=1}^p (\text{Bl}(T(p, \widehat{Q}_{2M_j+2i-1}))(t) \oplus -\text{Bl}(T(p, \widehat{Q}_{2M_j+2i}))(t)).
 \end{aligned}$$

The satellite formula of [4, Theorem 4.19] involves  $\text{Bl}(K)(\xi_{q_1}^{\chi(t_{\mathcal{Q}}^{i-1}q_{\mathcal{Q}}(\mu_{\mathcal{Q}}^{-w}\eta))}t)$ , where  $\mu_{\mathcal{Q}}$  denotes the meridian of the satellite knot  $\mathcal{Q} = P_{\eta}(K)$  with pattern  $P$ , companion  $K$  and infection curve  $\eta$ ; furthermore,  $q_{\mathcal{Q}}: \pi_1(M_{\mathcal{Q}}) \rightarrow H_1(\Sigma_p(\mathcal{Q}))$  denotes the map described in (7). Recalling the notation of Section 2, we see that, in our case,  $\eta$  coincides with the curve  $c_2$ , and  $\mu_{\mathcal{Q}} = \mu_{T(p,q)}$ . Thus, as explained in (9) for  $\chi = \chi_{\mathbf{a}}$ , we deduce that  $\chi(t_{\mathcal{Q}}^{u-1}q_{\mathcal{Q}}(\mu_{\mathcal{Q}}^{-w}\eta)) = \mathbf{a}_u$ , and this explains the second summand of (24). The decomposition in (24) is now justified, concluding the claim.

Next, we wish to apply the cabling formula  $\text{Bl}(J_{p,q})(t) = \text{Bl}(T(p,q))(t) \oplus \text{Bl}(J)(t^p)$  for the classical Blanchfield pairing. To make notation more manageable, however, for  $s \geq 1$ , coprime integers  $p$  and  $q$ , and  $\mathbf{a} \in \mathbb{Z}_r^p$ , we consider the linking form

$$\Lambda(p, q, \chi_{\mathbf{a}}, s) := \bigoplus_{u=0}^{p-1} \text{Bl}(T(p, q))(\xi_r^{p^{s-1}\mathbf{a}_u}t^{p^{s-1}}).$$

If the character  $\chi_{\mathbf{a}}$  is trivial, then we write  $\Lambda(p, q, s)$  instead of  $\Lambda(p, q, \theta, s)$ . These pairings appear as summands of the Blanchfield pairing of a cable. Indeed, using this notation and the aforementioned untwisted cabling formula, we deduce from (24) that

$$\begin{aligned} & \text{Bl}_{\alpha(p,\chi)}(K) \\ & \sim \bigoplus_{i=1}^{m_1} (\text{Bl}_{\alpha(p,\chi_{\mathbf{a}^i})}(T(p, r)) \oplus -\text{Bl}_{\alpha(p,\chi_{\mathbf{b}^i})}(T(p, r))) \dots \\ & \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} (\text{Bl}_{\alpha(p,\theta)}(T(p, q_j, \ell_j)) \oplus -\text{Bl}_{\alpha(p,\theta)}(T(p, q_j, \ell_j))) \\ & \oplus \bigoplus_{i=1}^{m_1} \bigoplus_{s \geq 1} (\Lambda(p, q_{2i-1}, \ell_{2i-1}-s, \chi_{\mathbf{a}^i}, s) \oplus -\Lambda(p, q_{2i}, \ell_{2i}-s, \chi_{\mathbf{b}^i}, s)) \\ & \oplus \bigoplus_{j=1}^{k/2} \bigoplus_{i=2}^{m_j} \bigoplus_{s \geq 1} (\Lambda(p, q_{2M_j+2i-1}, \ell_{2M_j+2i-1}-s, s) \oplus -\Lambda(p, q_{2M_j+2i}, \ell_{2M_j+2i}-s, s)) \\ & =: B_1^{\chi} \oplus B_2 \oplus B_3^{\chi} \oplus B_4. \end{aligned}$$

Now that we have decomposed  $\text{Bl}_{\alpha(p,\chi)}(K)$ , we study the consequences of it being metabolic.

**Claim 1** *If  $\text{Bl}_{\alpha(p,\chi)}(K)$  is metabolic, then  $B_1^{\chi}$  and  $B_3^{\chi} \oplus B_4$  are metabolic.*

**Proof** As  $\text{Bl}_{\alpha(p,\chi)}(K)$  and  $B_2$  are metabolic,  $B_1^{\chi} \oplus (B_3^{\chi} \oplus B_4)$  is metabolic. By Proposition 4.3, it suffices to prove that the orders of  $B_1^{\chi}$  and  $B_3^{\chi} \oplus B_4$  have distinct roots:

the roots of the twisted polynomial occur at prime powers of unity (by Proposition 3.3), while this is never the case for the classical Alexander polynomial [8, proof of Proposition 3.3(3)].<sup>2</sup> This proves Claim 1.  $\square$

In order to study the consequences of  $B_3^X \oplus B_4$  being metabolic, for  $s \geq 1$ , we set

$$B_3^X(s) := \bigoplus_{i=1}^{m_1} (\Lambda(p, q_{2i-1, \ell_{2i-1}-s}, \chi_{\mathbf{a}^i}, s) \oplus -\Lambda(p, q_{2i, \ell_{2i}-s}, \chi_{\mathbf{b}^i}, s)),$$

$$B_4(s) := \bigoplus_{j=1}^{k/2} \bigoplus_{i=2}^{m_j} (\Lambda(p, q_{2M_j+2i-1, \ell_{2M_j+2i-1}-s}, s) \oplus -\Lambda(p, q_{2M_j+2i, \ell_{2M_j+2i}-s}, s)).$$

Using these forms, we derive a further consequence of  $\text{Bl}_{\alpha(p, \chi)}(K)$  being metabolic.

**Claim 2** *If  $B_3^X \oplus B_4$  is metabolic, then  $B_3^X(s) \oplus B_4(s)$  is metabolic for each  $s$ .*

**Proof** By definition, we have the decompositions  $B_3^X = \bigoplus_{s \geq 1} B_3^X(s)$  and  $B_4 = \bigoplus_{s \geq 1} B_4(s)$ . For  $u \neq v$ , the order of  $B_3^X(u) \oplus B_4(u)$  and the order of  $B_3^X(v) \oplus B_4(v)$  have distinct roots. By Proposition 4.3, Claim 2 follows.  $\square$

Consequently, it is sufficient to study the linking forms  $B_3^X(s) \oplus B_4(s)$  for a fixed  $s \geq 1$ . To further decompose  $B_3^X(s) \oplus B_4(s)$ , we want to group these linking forms according to the torus knots that appear. We also need to be attentive to the fact that the torus knot  $T(p, q_i, \ell_i - s)$  is trivial when  $i \leq \ell_i$ . As a consequence, for  $s \geq 1$ , we consider the sets

$$\begin{aligned} \mathcal{I}_1(q, s) &:= \{1 \leq i \leq m_1 \mid \ell_{2i-1} > s, q_{2i-1, \ell_{2i-1}-s} = q\}, \\ \mathcal{I}_2(q, s) &:= \{1 \leq i \leq m_1 \mid \ell_{2i} > s, q_{2i, \ell_{2i}-s} = q\}, \\ (25) \quad \mathcal{I}_3(q, s) &:= \bigcup_{j=2}^{k/2} \{1 \leq i \leq m_j \mid \ell_{2M_j+2i-1} > s, q_{2M_j+2i-1, \ell_{2M_j+2i-1}-s} = q\}, \\ \mathcal{I}_4(q, s) &:= \bigcup_{j=2}^{k/2} \{1 \leq i \leq m_j \mid \ell_{2M_j+2i} > s, q_{2M_j+2i, \ell_{2M_j+2i}-s} = q\}. \end{aligned}$$

<sup>2</sup>Here is a topological proof of this fact: for a knot  $K$  and an integer  $q$ , the order of  $H_1(\Sigma_q(K))$  is  $\prod_{a=1}^{q-1} \Delta_K(\xi_q^a)$  [20, Corollary 9.8]; since  $q$  is a prime power,  $H_1(\Sigma_q(K))$  is a finite group, and thus none of the  $\Delta_K(\xi_q^a)$  can vanish.

Note that, for some  $q$ , the set  $\mathcal{I}_i(q, s)$  may well be empty. However, from now on, we will implicitly assume that we only consider  $q$  for which this is not the case. In order to study the consequences of  $B_3^\chi(s) \oplus B_4(s)$  being metabolic, we set

$$B_3^\chi(q, s) := \bigoplus_{k \in \mathcal{I}_1(q, s)} \Lambda(p, q, \chi_{\mathbf{a}^k}, s) \oplus - \bigoplus_{k \in \mathcal{I}_2(q, s)} \Lambda(p, q, \chi_{\mathbf{b}^k}, s),$$

$$B_4(q, s) := \bigoplus_{k \in \mathcal{I}_3(q, s)} \Lambda(p, q, s) \oplus - \bigoplus_{k \in \mathcal{I}_4(q, s)} \Lambda(p, q, s).$$

Note that  $B_4(q, s)$  is not automatically metabolic as the cardinality of  $\mathcal{I}_3(q, s)$  need not agree with that of  $\mathcal{I}_4(q, s)$ . Observe however that, if  $K$  is algebraically slice, [Proposition 5.3](#) implies that

$$(26) \quad \#\mathcal{I}_1(q, s) - \#\mathcal{I}_2(q, s) + \#\mathcal{I}_3(q, s) - \#\mathcal{I}_4(q, s) = 0.$$

Indeed, note that the sets  $\mathcal{I}_i(q, s)$  record where  $T(p, q)$  appears in the  $s$ -level of  $K$ . Using the  $B_i(q, s)$ , we now derive a further consequence of  $\text{Bl}_{\alpha(p, \chi)}(K)$  being metabolic.

**Claim 3** *If  $B_3^\chi(s) \oplus B_4(s)$  is metabolic, then  $B_3^\chi(q, s) \oplus B_4(q, s)$  is metabolic for each  $q$ .*

**Proof** We have decompositions  $B_3^\chi(s) = \bigoplus_{q \geq 1} B_3^\chi(q, s)$  and  $B_4(s) = \bigoplus_{q \geq 1} B_4(q, s)$ . Since all the  $q_i$  are positive, for  $u \neq v$ , the order of  $B_3^\chi(u, s) \oplus B_4(u, s)$  and the order of  $B_3^\chi(v, s) \oplus B_4(v, s)$  have distinct roots. By [Proposition 4.3](#), [Claim 3](#) follows.  $\square$

Summarising these claims, we have shown that, if the metabelian Blanchfield pairing  $\text{Bl}_{\alpha(p, \chi)}(K)$  is metabolic, then the linking forms  $B_3^\chi(q, s) \oplus B_4(q, s)$  are metabolic for all  $q$  and  $s$ . This concludes the second part of the proof.

**5.2.3 Building the characters that vanish on metabolisers** The third part consists in showing that, for every  $\mathbb{Z}_p$ -invariant metaboliser  $L$  of  $\lambda_p(T(p, r))^{m_1} \oplus -\lambda_p(T(p, r))^{m_1}$ , there are characters  $\chi_{\mathbf{a}} = \bigoplus_{i=1}^{m_1} \chi_{\mathbf{a}^i}$  and  $\chi_{\mathbf{b}} = \bigoplus_{i=1}^{m_1} \chi_{\mathbf{b}^i}$  such that  $\chi_{\mathbf{a}} \oplus \chi_{\mathbf{b}}$  vanishes on  $L$ , but for which the linking forms  $B_3^\chi(q, s) \oplus B_4(q, s)$  are not all metabolic, where  $\chi = \chi_{\mathbf{a}} \oplus \chi_{\mathbf{b}} \oplus \theta$  is as in [\(22\)](#).

The next proposition describes characters for which  $B_3^\chi(q, s) \oplus B_4(q, s)$  is not metabolic.



**Proposition 5.7** *Let  $q, s > 0$  be positive integers with  $q$  coprime to  $p$ . If a character  $\bigoplus_{i=1}^{m_1} \chi_{\mathbf{a}^i} \oplus \chi_{\mathbf{b}^i}$  satisfies either*

- (i)  $\chi_{\mathbf{b}^k} = \theta$  for every  $k \in \mathcal{I}_2(q, s)$  and  $\chi_{\mathbf{a}^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_1(q, s)$ , or
- (ii)  $\chi_{\mathbf{a}^k} = \theta$  for every  $k \in \mathcal{I}_1(q, s)$  and  $\chi_{\mathbf{b}^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_2(q, s)$ ,

*then the linking form  $B_3^\chi(q, s) \oplus B_4(q, s)$  is not metabolic.*

**Proof** We will only consider case (i). In order to give the proof in case (ii), just exchange the roles of  $\chi_{\mathbf{a}}$  and  $\chi_{\mathbf{b}}$ . Assume that  $\chi_{\mathbf{b}^k} = \theta$  for every  $k \in \mathcal{I}_2(q, s)$  and  $\chi_{\mathbf{a}^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_1(q, s)$ . Since  $K$  is algebraically slice, recall from (26) that

$$\#\mathcal{I}_1(q, s) - \#\mathcal{I}_2(q, s) + \#\mathcal{I}_3(q, s) - \#\mathcal{I}_4(q, s) = 0.$$

We thus define  $N := \#\mathcal{I}_1(q, s) - \#\mathcal{I}_2(q, s) - \#\mathcal{I}_3(q, s) + \#\mathcal{I}_4(q, s)$ , leading to the Witt equivalence

$$(27) \quad B_3^\chi(q, s) \oplus B_4(q, s) \sim \bigoplus_{k \in \mathcal{I}_1(q, s)} \Lambda(p, q, \chi_{\mathbf{a}^i}, s) \oplus - \bigoplus_{i=1}^{p \cdot N} \text{Bl}(T(p, q))(t^{p^{s-1}}).$$

We assert that the orders of the modules underlying the summands of the right-hand side of (27) have distinct roots. First, note that  $r$  is coprime to  $q$ : as  $k \in \mathcal{I}_1(q, s)$ , we know that  $q \in Q_i$  for some  $i < 2m_1$  and, since  $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i-1}, r)$  for  $i < 2m_1$ , this follows from the assumption of Theorem 1.1. It is known that  $\Delta_{T(p,q)}(\xi_r^{a_1} t)$  and  $\Delta_{T(p,q)}(\xi_r^{a_2} t)$  have distinct roots whenever  $a_1 \neq a_2$  and  $r$  and  $q$  are coprime [14, Theorem 7.1]. This establishes the assertion.

Thanks to the assertion, we may apply Proposition 4.3. Indeed, the fact that  $\chi_{\mathbf{a}^{k_0}} \neq \theta$  and Proposition 4.3 now guarantees that the linking form on the right-hand side of (27) is not metabolic. □

Before constructing the required characters, we introduce some terminology. We say that the knot  $K$  is *simplified* if there are no indices  $k_1 \in \mathcal{I}_1(q, s)$  and  $k_2 \in \mathcal{I}_2(q, s)$  such that  $Q_{2k_1-1} = Q_{2k_2}$ . If  $K$  is not simplified, then it contains a slice connected summand  $T(p, Q_{2k_1-1}) \# -T(p, Q_{2k_1-1})$ .

**Lemma 5.8** *Let  $p$  be a prime power. If the knot  $K$  is simplified, then, for any  $\mathbb{Z}_p$ -invariant metaboliser  $L \subset H_1(\Sigma_p(T(p, r)))^{m_1} \oplus H_1(\Sigma_p(T(p, r)))^{m_1}$ , there exist  $q, s$  and a character  $\chi_{\mathbf{a}} \oplus \chi_{\mathbf{b}} = \bigoplus_{i=1}^{m_1} \chi_{\mathbf{a}^i} \oplus \chi_{\mathbf{b}^i}$  vanishing on  $L$  such that either*

- (i)  $\chi_{\mathbf{b}^k} = \theta$  for every  $k \in \mathcal{I}_2(q, s)$  and  $\chi_{\mathbf{a}^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_1(q, s)$ , or
- (ii)  $\chi_{\mathbf{a}^k} = \theta$  for every  $k \in \mathcal{I}_1(q, s)$  and  $\chi_{\mathbf{b}^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_2(q, s)$ .

**Proof** Fix a metaboliser  $L \subset H_1(\Sigma_p(T(p, r)))^{m_1} \oplus H_1(\Sigma_p(T(p, r)))^{m_1}$  of

$$\lambda_p(T(p, r))^{m_1} \oplus -\lambda_p(T(p, r))^{m_1}.$$

For  $i = 1, 2$ , consider the projection  $\text{pr}_i : H_1(\Sigma_p(T(p, r)))^{m_1} \oplus H_1(\Sigma_p(T(p, r)))^{m_1} \rightarrow H_1(\Sigma_p(T(p, r)))^{m_1}$  onto the  $i^{\text{th}}$  factor. The proof is divided into three separate cases.

**Case 1** ( $\text{pr}_1(L)$  is a proper subspace of  $H_1(\Sigma_p(T(p, r)))^{m_1}$ ) In this case, we can define the characters  $\chi_a$  and  $\chi_b$  as  $\chi_b = \theta$  and

$$\chi_a : H_1(\Sigma_p(T(p, r)))^{m_1} \rightarrow H_1(\Sigma_p(T(p, r)))^{m_1} / \text{pr}_1(L) \xrightarrow{\text{nontrivial character}} \mathbb{Z}_r.$$

It is not difficult to see that  $\chi_a$  and  $\chi_b$  satisfy (i) and are such that  $\chi_a \oplus \chi_b$  vanishes on  $L$ .

**Case 2** ( $\text{pr}_2(L)$  is a proper subspace of  $H_1(\Sigma_p(T(p, r)))^{m_1}$ ) In this case, we exchange the roles of  $\chi_a$  and  $\chi_b$  and repeat the argument from the first case. This way, we obtain characters  $\chi_a$  and  $\chi_b$  that satisfy (ii) and are such that  $\chi_a \oplus \chi_b$  vanishes on  $L$ .

**Case 3** ( $\text{pr}_1(L) = H_1(\Sigma_p(T(p, r)))^{m_1}$  and  $\text{pr}_2(L) = H_1(\Sigma_p(T(p, r)))^{m_1}$ ) We wish to apply [Proposition 4.4](#) in order to prove that  $L$  is a graph. We verify the hypothesis of this proposition. Using the assumption of Case 3 and the definition of the projections, we have

$$\begin{aligned} 0 &= \ker(\text{pr}_1|_L) = L \cap (0 \oplus H_1(\Sigma_p(T(p, r)))^{m_1}), \\ 0 &= \ker(\text{pr}_2|_L) = L \cap (H_1(\Sigma_p(T(p, r)))^{m_1} \oplus 0). \end{aligned}$$

Consequently, by [Proposition 4.4](#),  $L$  is the graph of an isometry

$$g : (H_1(\Sigma_p(T(p, r)))^{m_1}, \lambda_p(T(p, r))^{m_1}) \rightarrow (H_1(\Sigma_p(T(p, r)))^{m_1}, \lambda_p(T(p, r))^{m_1}).$$

For each  $q, s$  and  $j = 1, 2$ , consider the subsets of  $H_1(\Sigma_p(T(p, r)))^{m_1}$

$$\begin{aligned} S_{\mathcal{I}_j(q, s)} &= \{(v_1, v_2, \dots, v_{m_1}) \in H_1(\Sigma_p(T(p, r)))^{m_1} : v_i = 0 \text{ for } i \notin \mathcal{I}_j(q, s)\} \\ &= \bigoplus_{k \in \mathcal{I}_j(q, s)} H_1(\Sigma_p(T(p, Q_k))), \end{aligned}$$

where  $\mathcal{I}_j(q, s)$  is as defined in [\(25\)](#).

Next, we use these sets and the isometry  $g$  to describe a sufficient criterion to obtain the characters  $\chi_a$  and  $\chi_b$  required by the statement of [Lemma 5.8](#).

**Claim 4** *If there exist  $q$  and  $s$  such that  $g(S_{\mathcal{I}_1(q,s)}) \neq S_{\mathcal{I}_2(q,s)}$ , then there are characters  $\chi_a$  and  $\chi_b$  satisfying either (i) or (ii) and such that  $\chi_a \oplus \chi_b$  vanishes on  $L$ .*

**Proof** If  $g(S_{\mathcal{I}_1(q,s)}) \setminus S_{\mathcal{I}_2(q,s)} \neq \emptyset$ , then choose  $v \in S_{\mathcal{I}_1(q,s)}$  such that  $g(v) \notin S_{\mathcal{I}_2(q,s)}$ . Since  $r$  is a prime,  $H_1(\Sigma_p(T(p,r)))^{m_1}$  is an  $\mathbb{F}_r$ -vector space and so we obtain a direct-sum decomposition  $H_1(\Sigma_p(T(p,r)))^{m_1} = \langle v \rangle \oplus W$  for some  $\mathbb{F}_r$ -vector space  $W$ . We can then define the characters as

$$\chi_a(v) = 1, \quad \chi_a|_W = \theta, \quad \chi_b(x) = -\chi_a(g^{-1}(x)).$$

Such choices of  $\chi_a$  and  $\chi_b$  satisfy condition (i). We verify that  $\chi_a \oplus \chi_b$  vanishes on  $L$ , where we recall that  $L$  is the graph of  $g$ . For an element  $(h, g(h)) \in L$  of this graph, one has  $(\chi_a \oplus \chi_b)(h, g(h)) = \chi_a(h) - \chi_a(g^{-1}(g(h))) = 0$ . This concludes the proof in this case.

If, on the other hand, we assume that  $S_{\mathcal{I}_2(q,s)} \setminus g(S_{\mathcal{I}_1(q,s)}) \neq \emptyset$ , the argument is nearly identical. Choose  $v \in S_{\mathcal{I}_2(q,s)} \setminus g(S_{\mathcal{I}_1(q,s)})$  and write once more  $H_1(\Sigma_p(T(p,r)))^{m_1} = \langle v \rangle \oplus W$  and define the required characters as

$$\chi_b(v) = 1, \quad \chi_b|_W = \theta, \quad \chi_a(x) = -\chi_b(g(x)).$$

These choices of  $\chi_a$  and  $\chi_b$  satisfy condition (ii) and  $\chi_a \oplus \chi_b$  vanishes on  $L$ .  $\square$

By [Claim 4](#), to prove [Lemma 5.8](#), it is enough to show that there always exist  $q$  and  $s$  such that  $g(S_{\mathcal{I}_1(q,s)}) \neq S_{\mathcal{I}_2(q,s)}$ . Assume by way of contradiction that  $g(S_{\mathcal{I}_1(q,s)}) = S_{\mathcal{I}_2(q,s)}$  for all  $q$  and  $s$ . We will show in [Claim 5](#) below that this assumption implies that  $K$  is not simplified. This is a contradiction since we assumed that  $K$  is simplified. This proves [Lemma 5.8](#) modulo [Claim 5](#).  $\square$

**Claim 5** *If  $g(S_{\mathcal{I}_1(q,s)}) = S_{\mathcal{I}_2(q,s)}$  for all  $q$  and  $s$ , then  $K$  is not simplified.*

**Proof** We will observe that, under the assumption of the claim,  $K$  contains a summand of the form  $T(p, Q_{2k_0-1})\#-T(p, Q_{2k_0-1})$  for some integer  $k_0$ . To be precise, choose  $1 \leq k_0 \leq m_1$  such that the length  $\ell_{2k_0-1}$  of the sequence of  $Q_{2k_0-1}$  is maximal among

all the  $\ell_{2k-1}$  for  $k = 1, \dots, m_1$ , and define<sup>3</sup>

$$X(k_0) = \{1 \leq k \leq m_1 \mid Q_{2k_0-1} = Q_{2k-1}\} = \bigcap_{s=1}^{\ell_{2k_0-1}} \mathcal{I}_1(q_{2k_0-1, \ell_{2k_0-1}-s}, s),$$

$$Y(k_0) = \{1 \leq k \leq m_1 \mid Q_{2k_0-1} = Q_{2k}\} = \bigcap_{s=1}^{\ell_{2k_0-1}} \mathcal{I}_2(q_{2k_0-1, \ell_{2k_0-1}-s}, s).$$

We will need the following properties of these sets:

- (a) Since  $k_0 \in X(k_0)$ ,  $X(k_0)$  is nonempty.
- (b) If  $j \in X(k_0)$ , then  $T(p, Q_{2j-1}) = T(p, Q_{2k_0-1})$ .
- (c) If  $j \in Y(k_0)$ , then  $T(p, Q_{2j}) = T(p, Q_{2k_0-1})$ .

It is enough to show that  $Y(k_0) \neq \emptyset$ . By (a)–(c), this would imply that  $K$  is not simplified since  $K$  contains a summand of the form  $T(p, Q_{2k_0-1}) \# -T(p, Q_{2k_0-1})$ .

To show that  $Y(k_0) \neq \emptyset$ , consider the subspaces of  $H_1(\Sigma_p(T(p, r)))^{m_1}$

$$S_{X(k_0)} := \{(v_1, v_2, \dots, v_{m_1}) \in H_1(\Sigma_p(T(p, r)))^{m_1} : v_i = 0 \text{ for } i \notin X(k_0)\},$$

$$= \bigoplus_{k \in X(k_0)} H_1(\Sigma_p(T(p, Q_{2k-1}))),$$

$$S_{Y(k_0)} := \{(v_1, v_2, \dots, v_{m_1}) \in H_1(\Sigma_p(T(p, r)))^{m_1} : v_i = 0 \text{ for } i \notin Y(k_0)\}$$

$$= \bigoplus_{k \in Y(k_0)} H_1(\Sigma_p(T(p, Q_{2k}))).$$

The advantage of writing  $X(k_0)$  and  $Y(k_0)$  as intersections of the  $\mathcal{I}_j(q_{k_0, \ell_{k_0}-s}, s)$  is that the action of  $g$  on  $S_{X(k_0)}$  can be described as

$$g(S_{X(k_0)}) = \bigcap_{s \geq 1} S_{g(\mathcal{I}_1(q_{2k_0-1, \ell_{2k_0-1}-s}, s))} = \bigcap_{s \geq 1} S_{\mathcal{I}_2(q_{2k_0-1, \ell_{2k_0-1}-s}, s)} = S_{Y(k_0)},$$

where the second equality follows from the assumption. As  $g$  is an  $\mathbb{F}_r$ -linear automorphism,  $\dim S_{X(k_0)} = \dim S_{Y(k_0)}$ . Since the  $\mathbb{F}_r$ -dimension of  $H_1(\Sigma_p(T(p, r)))$  is  $p - 1$ , we deduce that

$$(p - 1) \#X(k_0) = \dim S_{X(k_0)} = \dim S_{Y(k_0)} = (p - 1) \#Y(k_0).$$

It follows that  $\#X(k_0) = \#Y(k_0)$ . Since  $X(k_0) \neq \emptyset$  by (a), it follows that  $Y(k_0) \neq \emptyset$ . As we mentioned, this implies that  $K$  is not simplified by (a)–(c) and Claim 5 is proved. □

<sup>3</sup>Without the maximality assumption on  $\ell_{2k_0-1}$ , we would have had to replace the condition  $Q_{k_0} = Q_k$  by  $Q_{k_0} \subset Q_k$ .

This concludes the third part of the proof.

**5.2.4 Conclusion of the proof of Theorem 1.1** Let  $K$  be a (nontrivial) linear combination of iterated torus knots of the form  $T(p, Q_i)$  for  $i = 1, \dots, k$ . Here, the  $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$  are sequences of  $\ell_i$  positive integers where  $q_{i,\ell_i}$  is prime for all  $i$  and the integer  $q_{i,j}$  is coprime to  $p$  and to  $q_{i,\ell_i}$  for all  $j$ . Assume that  $K$  is slice to obtain a contradiction. In particular  $K$  is algebraically slice and, as we saw in (20), we can therefore assume without loss generality that it is of the form

$$(28) \quad K = \#_{i=1}^{m_1} (T(p, Q_{2i-1}) \# -T(p, Q_{2i})) \\ \# \#_{j=2}^{k/2} \#_{i=1}^{m_j} (T(p, Q_{2M_j+2i-1}) \# -T(p, Q_{2M_j+2i})).$$

Here we arranged that  $q_{i,\ell_i} = r$  if and only if  $1 \leq i \leq 2m_1$ . Furthermore, we can assume that  $K$  is simplified by cancelling terms of the form  $J \# -J$  if any such term appears in (28). We can also assume that there is an index  $i$  such that  $\ell_i > 1$ : otherwise,  $K$  would be a linear combination of torus knots, which is impossible since the latter are linearly independent in  $C^{\text{top}}$  [21]. To prove that  $K$  is not slice, we saw that it is enough to show that, for every  $\mathbb{Z}_p$ -invariant metaboliser  $L$  of  $\lambda_p(T(p, r))^{m_1} \oplus -\lambda_p(T(p, r))^{m_1}$ , there is a character  $\chi_a \oplus \chi_b = \bigoplus_{k=1}^{m_1} (\chi_{a^k} \oplus \chi_{b^k})$  that vanishes on  $L$  such that  $\text{Bl}_{\alpha(p,\chi)}(K)$  is not metabolic, where  $\chi = \chi_a \oplus \chi_b \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} \theta \oplus \theta$ ; recall Remark 5.6. We then applied satellite formulas to show that  $\text{Bl}_{\alpha(p,\chi)}(K)$  decomposes (up to Witt equivalence) as

$$\text{Bl}_{\alpha(p,\chi)}(K) \sim B_1^\chi \oplus B_2 \oplus B_3^\chi \oplus B_4 = B_1^\chi \oplus B_2 \oplus \bigoplus_{q,s} B_3^\chi(q, s) \oplus \bigoplus_{q,s} B_4(q, s).$$

Claim 1 shows that, if  $\text{Bl}_{\alpha(p,\chi)}(K)$  is metabolic, then  $B_1^\chi$  and  $B_3^\chi \oplus B_4$  are metabolic. By Claims 2 and 3, it follows that  $B_3^\chi(q, s) \oplus B_3^\chi(q, s)$  must be metabolic for all  $q$  and  $s$  and all characters  $\chi_a \oplus \chi_b$ . On the other hand, as the knot  $K$  is simplified, Lemma 5.8 implies that, for any  $\mathbb{Z}_p$ -invariant metaboliser  $L \subset H_1(\Sigma_p(T(p, r)))^{m_1} \oplus H_1(\Sigma_p(T(p, r)))^{m_1}$ , there exist  $q, s$  and a character  $\chi_a \oplus \chi_b$  vanishing on  $L$  such that either

- (i)  $\chi_{b^k} = \theta$  for every  $k \in \mathcal{I}_2(q, s)$  and  $\chi_{a^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_1(q, s)$ , or
- (ii)  $\chi_{a^k} = \theta$  for every  $k \in \mathcal{I}_1(q, s)$  and  $\chi_{b^{k_0}} \neq \theta$  for some  $k_0 \in \mathcal{I}_2(q, s)$ .

Applying Proposition 5.7, we deduce that for such characters and such integers  $q$  and  $s$ , the linking form  $B_3^\chi(q, s) \oplus B_4(q, s)$  is not metabolic. This is the desired contradiction, and Theorem 1.1 is proved. □

## References

- [1] **T Abe, K Tagami**, *Fibered knots with the same 0–surgery and the slice-ribbon conjecture*, *Math. Res. Lett.* 23 (2016) 303–323 [MR](#) [Zbl](#)
- [2] **K L Baker**, *A note on the concordance of fibered knots*, *J. Topol.* 9 (2016) 1–4 [MR](#) [Zbl](#)
- [3] **M Borodzik, A Conway, W Politarczyk**, *Twisted Blanchfield pairings and twisted signatures, I: Algebraic background*, *Linear Algebra Appl.* 655 (2022) 236–290 [MR](#) [Zbl](#)
- [4] **M Borodzik, A Conway, W Politarczyk**, *Twisted Blanchfield pairings and twisted signatures, II: Relation to Casson–Gordon invariants*, preprint (2021) [arXiv 1809.08791v4](#)
- [5] **M Borodzik, A Conway, W Politarczyk**, *Twisted Blanchfield pairings and twisted signatures, III: Applications*, preprint (2021) [arXiv 2111.10631](#)
- [6] **A J Casson, C M Gordon**, *On slice knots in dimension three*, from “Algebraic and geometric topology, II” (R J Milgram, editor), *Proc. Sympos. Pure Math.* XXXII, Amer. Math. Soc., Providence, RI (1978) 39–53 [MR](#) [Zbl](#)
- [7] **A J Casson, C M Gordon**, *Cobordism of classical knots*, from “À la recherche de la topologie perdue” (L Guillou, A Marin, editors), *Progr. Math.* 62, Birkhäuser, Boston, MA (1986) 181–199 [MR](#) [Zbl](#)
- [8] **S Friedl**, *Eta invariants as sliceness obstructions and their relation to Casson–Gordon invariants*, *Algebr. Geom. Topol.* 4 (2004) 893–934 [MR](#) [Zbl](#)
- [9] **S Friedl, S Vidussi**, *A survey of twisted Alexander polynomials*, from “The mathematics of knots” (M Banagl, D Vogel, editors), *Contrib. Math. Comput. Sci.* 1, Springer (2011) 45–94 [MR](#) [Zbl](#)
- [10] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) [MR](#) [Zbl](#)
- [11] **M Hedden**, *Some remarks on cabling, contact structures, and complex curves*, from “Proceedings of Gökova Geometry–Topology Conference 2007” (S Akbulut, T Önder, R J Stern, editors), *GGT, Gökova* (2008) 49–59 [MR](#) [Zbl](#)
- [12] **M Hedden**, *On knot Floer homology and cabling, II*, *Int. Math. Res. Not.* 2009 (2009) 2248–2274 [MR](#) [Zbl](#)
- [13] **M Hedden**, *Notions of positivity and the Ozsváth–Szabó concordance invariant*, *J. Knot Theory Ramifications* 19 (2010) 617–629 [MR](#) [Zbl](#)
- [14] **M Hedden, P Kirk, C Livingston**, *Non-slice linear combinations of algebraic knots*, *J. Eur. Math. Soc.* 14 (2012) 1181–1208 [MR](#) [Zbl](#)
- [15] **C Herald, P Kirk, C Livingston**, *Metabelian representations, twisted Alexander polynomials, knot slicing, and mutation*, *Math. Z.* 265 (2010) 925–949 [MR](#) [Zbl](#)
- [16] **J Hom**, *A note on cabling and  $L$ -space surgeries*, *Algebr. Geom. Topol.* 11 (2011) 219–223 [MR](#) [Zbl](#)
- [17] **C Kearton**, *Cobordism of knots and Blanchfield duality*, *J. London Math. Soc.* 10 (1975) 406–408 [MR](#) [Zbl](#)

- [18] **P Kirk, C Livingston**, *Twisted Alexander invariants, Reidemeister torsion, and Casson–Gordon invariants*, *Topology* 38 (1999) 635–661 [MR](#) [Zbl](#)
- [19] **T Kitano**, *Twisted Alexander polynomial and Reidemeister torsion*, *Pacific J. Math.* 174 (1996) 431–442 [MR](#) [Zbl](#)
- [20] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Math. 175, Springer (1997) [MR](#) [Zbl](#)
- [21] **R A Litherland**, *Signatures of iterated torus knots*, from “Topology of low-dimensional manifolds” (R A Fenn, editor), *Lecture Notes in Math.* 722, Springer (1979) 71–84 [MR](#) [Zbl](#)
- [22] **R A Litherland**, *Cobordism of satellite knots*, from “Four-manifold theory” (C Gordon, R Kirby, editors), *Contemp. Math.* 35, Amer. Math. Soc., Providence, RI (1984) 327–362 [MR](#) [Zbl](#)
- [23] **C Livingston, P Melvin**, *Algebraic knots are algebraically dependent*, *Proc. Amer. Math. Soc.* 87 (1983) 179–180 [MR](#) [Zbl](#)
- [24] **C Livingston, P Melvin**, *Abelian invariants of satellite knots*, from “Geometry and topology” (J Alexander, J Harer, editors), *Lecture Notes in Math.* 1167, Springer (1985) 217–227 [MR](#) [Zbl](#)
- [25] **A Miller, M Powell**, *Symmetric chain complexes, twisted Blanchfield pairings and knot concordance*, *Algebr. Geom. Topol.* 18 (2018) 3425–3476 [MR](#) [Zbl](#)
- [26] **K Miyazaki**, *Nonsimple, ribbon fibered knots*, *Trans. Amer. Math. Soc.* 341 (1994) 1–44 [MR](#) [Zbl](#)
- [27] **PS Ozsváth, A I Stipsicz, Z Szabó**, *Concordance homomorphisms from knot Floer homology*, *Adv. Math.* 315 (2017) 366–426 [MR](#) [Zbl](#)
- [28] **L Rudolph**, *How independent are the knot-cobordism classes of links of plane curve singularities?*, *Notices Amer. Math. Soc.* 23 (1976) 410
- [29] **M Tange**, *Upsilon invariants of L-space cable knots*, *Topology Appl.* 324 (2023) art. id. 108335 [MR](#) [Zbl](#)

Max-Planck-Institut für Mathematik

Bonn, Germany

Current address: Massachusetts Institute of Technology

Cambridge, MA, United States

Department of Mathematics Education, Kyungpook National University

Daegu, South Korea

Institute of Mathematics, University of Warsaw

Warszawa, Poland

[anthonyconway@gmail.com](mailto:anthonyconway@gmail.com), [minhoonkim@knu.ac.kr](mailto:minhoonkim@knu.ac.kr),

[wpolitarczyk@mimuw.edu.pl](mailto:wpolitarczyk@mimuw.edu.pl)

Received: 29 November 2020

Revised: 13 August 2021

# ALGEBRAIC & GEOMETRIC TOPOLOGY

[msp.org/agt](http://msp.org/agt)

## EDITORS

### PRINCIPAL ACADEMIC EDITORS

John Etnyre  
[etnyre@math.gatech.edu](mailto:etnyre@math.gatech.edu)  
Georgia Institute of Technology

Kathryn Hess  
[kathryn.hess@epfl.ch](mailto:kathryn.hess@epfl.ch)  
École Polytechnique Fédérale de Lausanne

### BOARD OF EDITORS

Julie Bergner	University of Virginia <a href="mailto:jeb2md@eservices.virginia.edu">jeb2md@eservices.virginia.edu</a>	Robert Lipshitz	University of Oregon <a href="mailto:lipshitz@uoregon.edu">lipshitz@uoregon.edu</a>
Steven Boyer	Université du Québec à Montréal <a href="mailto:cohf@math.rochester.edu">cohf@math.rochester.edu</a>	Norihiko Minami	Nagoya Institute of Technology <a href="mailto:nori@nitech.ac.jp">nori@nitech.ac.jp</a>
Tara E. Brendle	University of Glasgow <a href="mailto:tara.brendle@glasgow.ac.uk">tara.brendle@glasgow.ac.uk</a>	Andrés Navas	Universidad de Santiago de Chile <a href="mailto:andres.navas@usach.cl">andres.navas@usach.cl</a>
Indira Chatterji	CNRS & Université Côte d'Azur (Nice) <a href="mailto:indira.chatterji@math.cnrs.fr">indira.chatterji@math.cnrs.fr</a>	Thomas Nikolaus	University of Münster <a href="mailto:nikolaus@uni-muenster.de">nikolaus@uni-muenster.de</a>
Alexander Dranishnikov	University of Florida <a href="mailto:dranish@math.ufl.edu">dranish@math.ufl.edu</a>	Robert Oliver	Université Paris 13 <a href="mailto:bobol@math.univ-paris13.fr">bobol@math.univ-paris13.fr</a>
Corneli Druţu	University of Oxford <a href="mailto:cornelia.drutu@maths.ox.ac.uk">cornelia.drutu@maths.ox.ac.uk</a>	Birgit Richter	Universität Hamburg <a href="mailto:birgit.richter@uni-hamburg.de">birgit.richter@uni-hamburg.de</a>
Tobias Ekholm	Uppsala University, Sweden <a href="mailto:tobias.ekholm@math.uu.se">tobias.ekholm@math.uu.se</a>	Jérôme Scherer	École Polytech. Féd. de Lausanne <a href="mailto:jerome.scherer@epfl.ch">jerome.scherer@epfl.ch</a>
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México <a href="mailto:mario@matem.unam.mx">mario@matem.unam.mx</a>	Zoltán Szabó	Princeton University <a href="mailto:szabo@math.princeton.edu">szabo@math.princeton.edu</a>
David Futер	Temple University <a href="mailto:dfuter@temple.edu">dfuter@temple.edu</a>	Ulrike Tillmann	Oxford University <a href="mailto:tillmann@maths.ox.ac.uk">tillmann@maths.ox.ac.uk</a>
John Greenlees	University of Warwick <a href="mailto:john.greenlees@warwick.ac.uk">john.greenlees@warwick.ac.uk</a>	Maggy Tomova	University of Iowa <a href="mailto:maggy-tomova@uiowa.edu">maggy-tomova@uiowa.edu</a>
Ian Hambleton	McMaster University <a href="mailto:ian@math.mcmaster.ca">ian@math.mcmaster.ca</a>	Nathalie Wahl	University of Copenhagen <a href="mailto:wahl@math.ku.dk">wahl@math.ku.dk</a>
Hans-Werner Henn	Université Louis Pasteur <a href="mailto:henn@math.u-strasbg.fr">henn@math.u-strasbg.fr</a>	Chris Wendl	Humboldt-Universität zu Berlin <a href="mailto:wendl@math.hu-berlin.de">wendl@math.hu-berlin.de</a>
Daniel Isaksen	Wayne State University <a href="mailto:isaksen@math.wayne.edu">isaksen@math.wayne.edu</a>	Daniel T. Wise	McGill University, Canada <a href="mailto:daniel.wise@mcgill.ca">daniel.wise@mcgill.ca</a>
Christine Lescop	Université Joseph Fourier <a href="mailto:lescop@ujf-grenoble.fr">lescop@ujf-grenoble.fr</a>		

---

See inside back cover or [msp.org/agt](http://msp.org/agt) for submission instructions.


The subscription price for 2023 is US \$650/year for the electronic version, and \$940/year (+ \$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

---

AGT peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers



# ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23

Issue 2 (pages 509–962)

2023

---

Parametrized higher category theory	509
JAY SHAH	
Floer theory of disjointly supported Hamiltonians on symplectically aspherical manifolds	645
YANIV GANOR and SHIRA TANNY	
Realization of graded monomial ideal rings modulo torsion	733
TSELEUNG SO and DONALD STANLEY	
Nonslice linear combinations of iterated torus knots	765
ANTHONY CONWAY, MIN HOON KIM and WOJCIECH POLITARCZYK	
Rectification of interleavings and a persistent Whitehead theorem	803
EDOARDO LANARI and LUIS SCOCCOLA	
Operadic actions on long knots and 2–string links	833
ETIENNE BATELIER and JULIEN DUCOULOMBIER	
A short proof that the $L^p$ –diameter of $\text{Diff}_0(S, \text{area})$ is infinite	883
MICHAŁ MARCINKOWSKI	
Extension DGAs and topological Hochschild homology	895
HALDUN ÖZGÜR BAYINDIR	
Bounded cohomology of classifying spaces for families of subgroups	933
KEVIN LI	