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Operadic actions on long knots and 2–string links

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We realize the space of 2–string links $\mathcal{L}$ as a free algebra over a colored operad denoted by $\mathcal{SCL}$ (for "Swiss-cheese for links"). This result extends works of Burke and Koytcheff about the quotient of $\mathcal{L}$ by its center, and is compatible with Budney's freeness theorem for long knots. From an algebraic point of view, our main result refines Blaire, Burke and Koytcheff's theorem on the monoid of isotopy classes of string links. Topologically, it expresses the homotopy type of the isotopy class of a 2–string link in terms of the homotopy types of the classes of its prime factors.

57R40; 55P48, 55U40, 57M99

Introduction

Motivation and context

The study of knots and links is a vast subject that emerged in the late nineteenth century and saw several renewals in the past thirty years. It is subject to many different approaches, being at the crossroads of topology, geometry, algebra, combinatorics and physics. The central theme in classical knot theory is the study of the isotopy

classes of knots, ie the isotopy classes of embeddings in $\mathrm{Emb}(S^1, S^3)$. They are the set of components $\pi_0 \mathrm{Emb}(S^1, S^3)$. A common method is to try to chop the knots into simpler pieces. Two ways of performing such a decomposition have proved themselves particularly fruitful: the prime decomposition and the satellite decomposition. The former splits a knot as the connected sum of other knots called its prime factors. The connected sum is a binary operation on $\pi_0 \mathrm{Emb}(S^1, S^3)$ denoted by $\#$. It endows the isotopy classes with a unital commutative monoidal structure. Intuitively, the product $k_1 \# k_2$ is the knot obtained by cutting open k_1 and k_2 and closing them up into a single knot. This decomposition is fairly well understood thanks to a theorem of Schubert [28] stating that $\pi_0 \mathrm{Emb}(S^1, S^3)$ is freely generated as a monoid by the isotopy classes of the prime knots. There are infinitely many prime knots and they can be very different in nature. This is why it is often useful to further decompose the prime knots as satellites of simpler knots. The satellite construction also originates in Schubert's work. It consists of a wide family of operations one can carry out on several knots at a time. Its rigorous definition is a bit more involved but clearly dispensed for example in Cromwell's book [10]. Jaco, Shalen, Johannson and Thurston played a major role in the study of satellite knots, with the results in [19; 20; 31]. Although they are quite complicated, the satellite operations have the advantage of generating the whole space of knots from a fairly small and classifiable class of knots. Namely, Budney shows in [4] — refining a result of [31] — that every knot can be obtained via successive satellite operations on hyperbolic and torus knots.

This depicts a duality between the two methods of decomposition: one is simple but leads to potentially complicated primes, while the other is more elaborate but has easier irreducible pieces. A similar story can be told for links, ie for the isotopy classes of embeddings in $\mathrm{Emb}(S^1 \sqcup \cdots \sqcup S^1, S^3)$. Adapting the connected sum $\#$ to this setting takes some work: problems arise once the components of a link are cut open, as there is no canonical way to close them up. However, a step-by-step adaptation of the satellite construction works for links, and a decomposition theorem exists as well. Its building blocks are the hyperbolic and Seifert-fibered links.

Nowadays, it is more common to study not only the set of components

$$\pi_0 \mathrm{Emb}(S^1 \sqcup \cdots \sqcup S^1, S^3),$$

but the full homotopy type of the spaces of knots and links. To adapt the decomposition approach described above, one needs to define an analogue of the connected sum and satellite operations on the space level, ie directly on the embeddings and not between

isotopy classes. To rigorously carry out this task, one uses the space of long knots $\mathcal{K}$ and the space of string links $\mathcal{L}$. Coupled with the language of operads, these spaces enable one to extend the existing operations on π_0 to the space level. More precisely, the different types of operations one can carry out on knots and links can be encoded in the action of an operad on a space of embeddings. This new framework is due to Budney in [5; 6] for the case of long knots. Namely, Budney constructs in [5] an action of the little 2–cubes operad $\mathcal{C}_2$ on a space $\widehat{\mathcal{K}}$ homotopy equivalent to $\mathcal{K}$, in such a way that all the induced operations descend to the connected sum on π_0. He builds in [6] another action on $\widehat{\mathcal{K}}$ of a more intricate operad which he calls the splicing operad. These operations correspond to the satellite constructions in many ways. In the case of string links, Burke and Koytcheff build in [8] a complex operad called the infection operad. It is an adaptation of Budney's splicing operad and deals with the satellite operations between string links. The authors mentioned above not only prove the existence of such actions but also their freeness, refining the unique decomposition results on π_0. It remains to find an operadic encoding of the connected sum of links, if possible, leading to a free algebra.

Present work

The present paper aims to fill in this gap in the case of 2–stranded string links. Unlike $\pi_0\mathcal{K}$, the monoid of isotopy classes of 2–string links is neither free nor commutative. Indeed, as explained by Blair, Burke and Koytcheff in [2], $\pi_0\mathcal{L}$ contains invertible elements in the form of the pure braid group KB_2. Together with three copies of $\pi_0\mathcal{K}$, these invertible elements generate the center of the monoid. Burke and Koytcheff state a partial result in [8] by building a free action of the little 1–cubes operad $\mathcal{C}_1$ on a subspace of 2–string links that ignores the homotopy center. They mention as an open problem the extension of such a structure to the whole space of 2–string links. Our main result provides an answer to this particular question. For this purpose, we introduce a four-colored operad $\mathcal{SCL}$ (standing for "Swiss-cheese for links") with set of colors $S = \{o, \uparrow, \downarrow, \updownarrow\}$. An $\mathcal{SCL}$–algebra is in particular a family of spaces $(X, A_\uparrow, A_\downarrow, A_\updownarrow)$, where X is a $\mathcal{C}_1$–algebra and each A_s, $s \in \{\uparrow, \downarrow, \updownarrow\}$, is a $\mathcal{C}_2$–algebra acting on X. One can think of the A_s as independent parts of the homotopy center of X. As in the case of Budney's $\mathcal{C}_2$–action on long knots, we consider a space $\widehat{\mathcal{L}}$ homotopy equivalent to $\mathcal{L}$ to prove a first result which can be summarized as follows:

Theorem 3.9 *The family $(\widehat{\mathcal{L}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is an $\mathcal{SCL}$–algebra. In particular, the family $(\mathcal{L}, \mathcal{K}, \mathcal{K}, \mathcal{K})$ is homotopy equivalent to an explicit $\mathcal{SCL}$–algebra.*

The structure so obtained is compatible with Budney's action on long knots and Burke and Koytcheff's action on their subspace of noncentral string links. It provides a complete understanding of the connected sum of 2–string links. As expected, we also prove a freeness result, refining the structure theorem for the monoid of isotopy classes proved in [2]. In order to do this, we discard the invertible elements by splitting $\pi_0\widehat{\mathcal{L}}$ as a product $\pi_0\widehat{\mathcal{L}}^0 \times KB_2$ and prove:

Theorem 4.11 *The quadruplet of spaces $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is homotopy equivalent as an $\mathcal{SCL}$–algebra to the free $\mathcal{SCL}$–algebra generated by prime knots and links.*

In addition to these algebraic considerations, this result has a homotopical significance as it expresses the homotopy type of the isotopy class of a 2–string link as a function of the homotopy types of the classes of its prime factors. This reduces the computation of the homotopy type of the whole $\mathcal{L}$ to figuring out the homotopy types of the components of the primes. As mentioned above, the latter can be further decomposed thanks to Burke and Koytcheff's infection operad defined in [8].

Organization of the paper

We define in a first section the different spaces of embeddings at stake here: long knots, string links and their fattened versions $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{L}}$. The second section sets up the operadic framework we use. We recall in particular the notion of colored operad and discuss the resulting algebras. We introduce along the way the operad that will appear in our main result, the Swiss-cheese operad for links $\mathcal{SCL}$. The third section aims to define various operadic actions on the spaces of knots and links. We recall the constructions of Budney, Burke and Koytcheff's actions and unify them in a single action of $\mathcal{SCL}$ on 2–string links. Finally, the fourth section proves the freeness result sketched above using low-dimensional and homotopical tools.

Upcoming projects

Our main statements concern the space of string links on two strands. One might naturally wonder what happens in the k–stranded case for some $k > 2$. The conjecture that Theorems 3.9 and 4.11 have adaptations to arbitrary string links seems reasonable, since most of the techniques used in Section 4.1 naturally generalize to the k–stranded case. However, lots of difficulties arise, even at the level of isotopy classes. Theorem 4.11 generalizes Blair, Burke and Koytcheff's explicit model for the monoid $\pi_0\mathcal{L}$, but it

does not provide an alternative proof for it. Actually, Blair, Burke and Koytcheff's result is used in the very first line of the proof of our Theorem 4.11. Thus, if one wants to adapt Theorem 4.11 to the k–stranded case, some preliminary work on the monoid of isotopy classes of string links on k strands is necessary. A key point is the understanding of its center. The commutation between string links on k strands has already been characterized in [2], but there are other relations among prime links that remain to be understood. For example, the invertible k–stranded string links are the pure braids on k strands, which already admit a wide family of fairly complex relations amongst themselves. Moreover, these units are not central anymore, which makes it harder to study prime decompositions. A solution to these difficulties could be to accept a less explicit construction for the replacement of $\mathcal{SCL}$, maybe a definition by induction on k. The operad for k–stranded string links would rely on a large set of colors, and would restrict to the operad for $(k-1)$–stranded string links on some subcollections of colors. The existence and freeness of an action could then be easier to prove, but the difficulty is only shifted towards understanding these potentially massive operads.

Another interesting question concerns the Goodwillie–Weiss calculus, introduced to study embedding spaces in [12; 34]. In the context of knots and links, this theory gives rise to two towers of fibrations $\{T_k\widehat{\mathcal{K}}\}$ and $\{T_k\widehat{\mathcal{L}}\}$, converging to the so-called polynomial approximations $T_\infty\widehat{\mathcal{K}}$ and $T_\infty\widehat{\mathcal{L}}$, respectively. Unfortunately, the natural applications $\iota_{\widehat{\mathcal{K}}}\colon\widehat{\mathcal{K}}\to T_\infty\widehat{\mathcal{K}}$ and $\iota_{\widehat{\mathcal{L}}}\colon\widehat{\mathcal{L}}\to T_\infty\widehat{\mathcal{L}}$ are not weak equivalences, but they preserve a lot of homotopical information. In particular, we know from Budney, Conant, Koytcheff and Sinha [7] that the map $\widehat{\mathcal{K}}\to T_k\widehat{\mathcal{K}}$ is a finite type-$(k-1)$ knot invariant and it has been conjectured that $T_k\widehat{\mathcal{K}}$ is actually the universal finite type-$(k-1)$ invariant. This conjecture is already proved rationally in Volić's thesis [32]. Moreover, the polynomial approximations can be simplified and identified to homotopy totalizations using the multiplicative Kontsevich operad K_3 obtained as a compactification of configurations of points in $\mathbb{R}^3$. Briefly speaking, one has the identifications

$$\widehat{\mathcal{K}}\xrightarrow{\iota_{\widehat{\mathcal{K}}}}T_\infty\widehat{\mathcal{K}}\xleftarrow{\mu_{\widehat{\mathcal{K}}}}\mathrm{hoTot}(K_3\circ\mathrm{SO}(3)),\quad \widehat{\mathcal{L}}\xrightarrow{\iota_{\widehat{\mathcal{L}}}}T_\infty\widehat{\mathcal{L}}\xleftarrow{\mu_{\widehat{\mathcal{L}}}}\mathrm{hoTot}(K_3^2\circ\mathrm{SO}(3)),$$

where $K_3^2(k)=K_3(2k)$ is a shifted version of the Kontsevich operad. The applications $\mu_{\widehat{\mathcal{K}}}$ and $\mu_{\widehat{\mathcal{L}}}$ have been proved to be weak equivalences by Sinha in [29] and Munson and Volić in [25], respectively. We know that the spaces $\widehat{\mathcal{K}}$, $T_\infty\widehat{\mathcal{K}}$ and $\mathrm{hoTot}(K_3\circ\mathrm{SO}(3))$ are $\mathcal{C}_2$–algebras by [5], [3] and [11], respectively. However, it is still unknown if $\iota_{\widehat{\mathcal{K}}}$ and $\mu_{\widehat{\mathcal{K}}}$ are $\mathcal{C}_2$–algebra maps. All these questions can be extended to the colored case. From the present work, the family $(\widehat{\mathcal{L}},\widehat{\mathcal{K}},\widehat{\mathcal{K}},\widehat{\mathcal{K}})$ is equipped with an explicit $\mathcal{SCL}$–algebra

structure. We believe that similar structures exist for the families

$$\big(T_\infty\widehat{\mathcal{L}}, T_\infty\widehat{\mathcal{K}}, T_\infty\widehat{\mathcal{K}}, T_\infty\widehat{\mathcal{K}}\big),$$

$$\big(\mathrm{hoTot}(K_3^2\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3))\big),$$

and that the zigzag of morphisms induced by $\iota_{\widehat{\mathcal{K}}}$, $\mu_{\widehat{\mathcal{K}}}$, $\iota_{\widehat{\mathcal{L}}}$ and $\mu_{\widehat{\mathcal{L}}}$ between the corresponding families are morphisms of $\mathcal{SCL}$–algebras.

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General framework and notation

We set up here the global framework we work in as well as some notation that might not be completely standard.

Topological spaces

By spaces, we understand compactly generated Hausdorff spaces. They form a full subcategory of topological spaces that we denote by **Top** by slight abuse of notation. Many useful properties of **Top** have been introduced by Steenrod in [30]. The standard Quillen model structure has then been adapted for it by Hovey in [18]. It is a convenient category in the sense that the natural curryfication map

$$\mathbf{Top}(X\times Y, Z)\cong \mathbf{Top}(X, \mathbf{Top}(Y, Z))$$

is a homeomorphism for any three spaces X, Y and Z in **Top**. The need to restrict ourselves to such a subcategory arises from the following fact: when defining an action of an algebraic structure that it also a topological space A on a space X, one can ask for the continuity of either $A\times X\to X$ or $A\to \mathbf{Top}(X, X)$. The homeomorphism above gives the equivalence between these two approaches and enables one to go back and forth between both frameworks. This will be useful when dealing with operadic actions.

Operations on maps

Let $f: A \to X$ and $g: B \to Y$ be maps between spaces. We use the following notation:

- $f \amalg g$ is the map between coproducts $A \amalg B \to X \amalg Y$.
- $f \oplus g$ is the map $A \amalg B \to X$ when $X = Y$.
- $f \times g$ is the map between products $A \times B \to X \times Y$.
- (f, g) is the map $A \to X \times Y$ when $A = B$.
- $A^{\times n}$ is the product of n copies of A and $f^{\times n}$ is the map $A^{\times n} \to B^{\times n}$.
- $B^{\amalg n}$ is the coproduct of n copies of B and $f^{\amalg n}$ is the map $A^{\amalg n} \to B^{\amalg n}$.

Smooth manifolds

When discussing manifolds, we think of usual (possibly bordered) C^∞ manifolds. We write $I = [0, 1]$ for the unit interval, $J = [-1, 1]$ for the 1–dimensional unit disk and $J^k = J^{\times k}$ for the k–dimensional unit cube. The set of C^∞ maps between two manifolds M and N is denoted by $C^\infty(M, N)$ and topologized with the usual C^∞–topology described in [17]. The space of embeddings, immersions, submersions or diffeomorphisms between manifolds are topologized as subspaces of the latter. This turns diffeomorphism groups into topological groups and makes every composition map continuous.

1 Embedding spaces

This section aims to review the construction of various spaces of embeddings, namely spaces of knots and 2–stranded links. We start by recalling the definition of the usual space of knots and introduce three variations: the long knots $\mathcal{K}$, the framed long knots $\mathrm{EC}(1, D^2)$ and the fat long knots $\widehat{\mathcal{K}}$. These spaces are meant to ease algebraic and homotopical manipulations. We also discuss the classical monoid structure on the space of knots, its interactions with these spaces and finally adapt these constructions to 2–stranded links.

1.1 Knot spaces

The first instance of a space of knots arises as the space of embeddings $\mathrm{Emb}(S^1, S^3)$. Its components $\pi_0 \mathrm{Emb}(S^1, S^3)$ are the isotopy classes of knots in the 3–sphere and are the central object of study in knot theory. The class of the standardly embedded

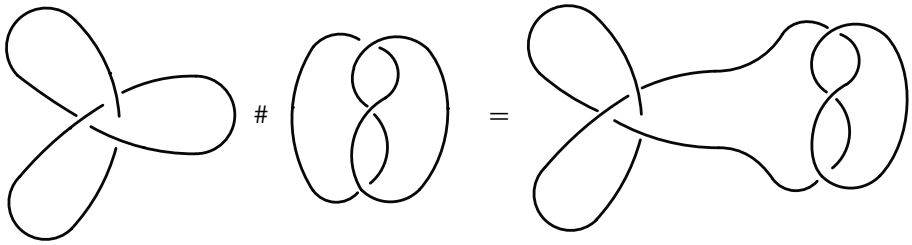

Figure 1: Illustration of the connected sum of two knots.

circle $S^1 \hookrightarrow \mathbb{R}^3 \subset S^3$ is called the trivial knot or unknot. Given two (isotopy classes of) knots k_1 and k_2, one can define the connected sum $k_1 \# k_2$ in various ways, as done for instance in [10]. Intuitively, $k_1 \# k_2$ is obtained by cutting open k_1 and k_2 and closing them back into a single knot. An example is provided in Figure 1. This operation turns out to be associative, commutative and unital with the unknot as unit. This turns $\pi_0 \operatorname{Emb}(S^1, S^3)$ into a commutative monoid. The nontrivial elements k which admit no nontrivial factorization $k = k_1 \# k_2$ are called prime. They are in a sense the most elementary knots. However, there are infinitely many of them and a further decomposition developed in [4] suggests that they form a fairly wide class of knots. The monoid structure on $\pi_0 \operatorname{Emb}(S^1, S^3)$ is completely understood thanks to a theorem of Schubert:

Theorem 1.1 (Schubert [28]) *The monoid $\pi_0 \operatorname{Emb}(S^1, S^3)$ is the free commutative monoid generated by prime knots.*

We now introduce long knots. They are a mild variation of usual knots for which the connected sum is easier to deal with. Let $\iota : \mathbb{R} \to \mathbb{R}^3$, with $\iota(t) = (t, 0, 0)$ be the standard embedding of the real line in $\mathbb{R}^3$.

Definition 1.2 A *long knot* is an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$ that agrees with the standard embedding ι outside of $J = [-1, 1]$ and maps the interior of J in the interior of $J \times D^2 \subset \mathbb{R}^3$. The space of long knots is denoted by $\mathcal{K}$. One can alternatively think of a long knot as a proper embedding $J \hookrightarrow J \times D^2$ whose values and derivatives at ∂J match those of ι.

With these conditions on the embeddings, it is natural to define a binary stacking operation between long knots as follows. Let $L, R : \mathbb{R}^3 \to \mathbb{R}^3$ be the maps sending (x, y, z) to $\left(\frac{1}{2}(x-1), y, z\right)$ and $\left(\frac{1}{2}(x+1), y, z\right)$, respectively. Given k_1 and k_2 in $\mathcal{K}$, we

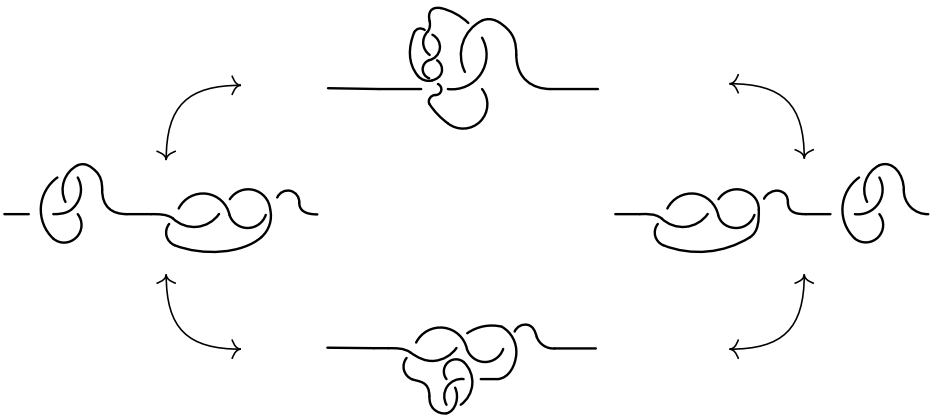

Figure 2: Illustration of the commutativity in the monoid $\pi_0\mathcal{K}$.

define the concatenation of k_1 and k_2 as the long knot that restricts to $t \mapsto L \circ k_1(2t+1)$ on $[-1, 0]$ and to $t \mapsto R \circ k_2(2t-1)$ on $[0, 1]$. This operation and its commutativity up to homotopy are illustrated in Figure 2. We still denote this operation by # as it is the analogue of the connected sum in the following sense. Each long knot is linear outside of J and can therefore be extended to an embedding $S^1 \hookrightarrow S^3$ by compactifying the domain and codomain. This specifies an inclusion $\mathcal{K} \hookrightarrow \mathrm{Emb}(S^1, S^3)$ which turns out to be a bijection on π_0. It is easy to verify that the concatenation of two knots is sent to their connected sum. The isotopy classes $\pi_0\mathcal{K}$ therefore inherit a monoid structure. When $\mathcal{P}$ denotes the collection of long knots which are prime, Schubert's theorem applies and gives:

Theorem 1.3 *The monoid $\pi_0\mathcal{K}$ is the free commutative monoid on the basis $\pi_0\mathcal{P}$.*

We now define framed long knots. The latter are meant to approximate the long knots defined above while being composable. The idea is to thicken the strand $\mathbb{R}$ into a long tube $\mathbb{R} \times D^2$. This definition is due to Budney and originates in [5]. We define:

Definition 1.4 (Budney [5]) A framed long knot is an embedding $\mathbb{R} \times D^2 \to \mathbb{R} \times D^2$ that restricts to the identity outside of $J \times D^2$. The space of framed long knots is denoted by $\mathrm{EC}(1, D^2)$. When $\mathrm{supp}(f)$ denotes the support of an embedding $f : \mathbb{R} \times D^2 \hookrightarrow \mathbb{R} \times D^2$, ie the closure of $\{(t, x) \in \mathbb{R} \times D^2 \mid f(t, x) \neq (t, x)\}$, the condition for f to lie in $\mathrm{EC}(1, D^2)$ can be reformulated as $\mathrm{supp}(f) \subset J \times D^2$.

Note that this space is still equipped with a stacking operation # defined just as in the case of long knots. It also still descends to an associative, commutative unital pairing on

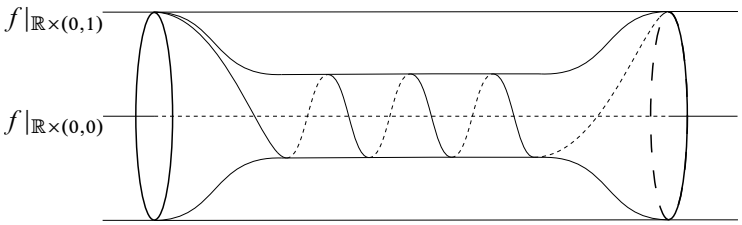

Figure 3: A framed long knot f with framing number 4.

isotopy classes. There is a restriction map $\mathrm{EC}(1, D^2) \to \mathcal{K}$, defined by $f \mapsto f|_{\mathbb{R}\times(0,0)}$, that preserves the concatenation. But, it does not induce a bijection on π_0. To see this, consider the diffeomorphism r of $J \times D^2$ that progressively performs a full turn rotation on the disk factor. One can parametrize $r : (t, x) \mapsto (t, e^{i\pi(t+1)}x)$. This diffeomorphism can be isotoped about $\partial J \times D^2$ to be extended into an element of $\mathrm{EC}(1, D^2)$. Now, for any long knot k and f an extension in $\mathrm{EC}(1, D^2)$, each composite $f \circ r^{\circ n}$ also extends k but no two are isotopic. This produces infinitely many components in the fiber over k, which shows that the restriction map does not induce a bijection on π_0. As of here, this twisting phenomenon is an unwanted byproduct of the thickening process. We will get rid of it when defining $\widehat{\mathcal{K}}$ as an unframed subspace of $\mathrm{EC}(1, D^2)$. To do so, we need to quantify the framing of a knot, which we do via the framing number.

Definition 1.5 We define the *framing number* $\omega(f)$ of a framed long knot f of $\mathrm{EC}(1, D^2)$ as the linking number $\mathrm{lk}(f|_{\mathbb{R}\times(0,0)}, f|_{\mathbb{R}\times(0,1)})$. Strictly speaking, the linking number is only defined between disjoint closed curves. We deal with this problem by identifying the curves above with their extension by compactification $S^1 \to S^3$ and isotoping $f|_{\mathbb{R}\times(0,1)}$ about the point at infinity to keep the disjointness.

Intuitively, the linking number counts the number of times a closed curve winds around another. Here, we think of $\omega(f)$ as the number of times a curve on the surface of the knot wraps around the core $f|_{\mathbb{R}\times(0,0)}$. This provides a way to quantify the framing of the elements of $\mathrm{EC}(1, D^2)$ and we have $\omega(r^{\circ n}) = n$. Another example is provided in Figure 3. Since the linking number can be computed by counting crossings in diagrams, it is easy to see that ω is additive with respect to the stacking product. It is also isotopy invariant and therefore descends to a morphism of monoids $\pi_0\mathrm{EC}(1, D^2) \to \mathbb{Z}$.

We are finally able to define $\widehat{\mathcal{K}}$, our preferred model to approximate $\mathcal{K}$. We simply set:

Definition 1.6 (Budney [5]) The space of *fat long knots* is the subspace $\widehat{\mathcal{K}} = \omega^{-1}(0)$.

Fat long knots are stable under # thanks to the additivity of ω. The primes in $\widehat{\mathcal{K}}$ are denoted by $\widehat{\mathcal{P}}$. There still is a restriction map $\widehat{\mathcal{K}} \to \mathcal{K}$ which preserves this structure. $\widehat{\mathcal{K}}$ is a good approximation for $\mathcal{K}$ in the sense that:

Proposition 1.7 (Budney [5]) *The restriction map $\widehat{\mathcal{K}} \to \mathcal{K}$ is a homotopy equivalence.*

In particular, we get an isomorphism on π_0 which enables us to transfer Schubert's theorem to fat long knots:

Corollary 1.8 *The monoid $\pi_0\widehat{\mathcal{K}}$ is the free commutative monoid on the basis $\pi_0\widehat{\mathcal{P}}$.*

1.2 Link spaces

We now adapt these constructions for 2–links. Most of this generalization work has already been carried out by Burke and Koytcheff in [8]. The space of usual 2–links arises as $\mathrm{Emb}(S^1 \sqcup S^1, S^3)$. There is no canonical version of a connected sum operation here, as there is no preferred strand in each link for one to merge. As in the case of long knots, the space of string links $\mathcal{L}$ deals with this problem by setting a framework where a stacking operation is naturally defined. Let $\iota_2 : \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}^3$ be the embedding of two copies of the real line in $\mathbb{R}^3$ mapping the first copy as $t \mapsto \left(t, 0, \frac{1}{2}\right)$ and the other one as $t \mapsto \left(t, 0, \frac{-1}{2}\right)$. We refer to ι_2 as the standard embedding for links with two strands. We then define:

Definition 1.9 A 2–*string link* is an embedding $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}^3$ that agrees with ι_2 outside of $J \sqcup J$ and maps the interior of $J \sqcup J$ in the interior of $J \times D^2 \subset \mathbb{R}^3$. The space of 2–string links is denoted by $\mathcal{L}$. One can alternatively think of a 2–string link as a proper embedding $J \sqcup J \hookrightarrow J \times D^2$ whose values and derivatives at $\partial J \sqcup \partial J$ match those of ι_2.

A binary stacking operation # can now be defined on $\mathcal{L}$ as in the case of long knots. It turns $\pi_0\mathcal{L}$ into a monoid with unit ι_2. A 2–string link is said to be prime if it is not invertible but cannot be factored without an invertible element. There is an injection $\mathcal{L} \hookrightarrow \mathrm{Emb}(S^1 \sqcup S^1, S^3)$ obtained by closing a truncated link $f|_{J \sqcup J}$ with two fixed smooth curves from $\left(-1, \frac{\pm 1}{2}\right)$ to $\left(1, \frac{\pm 1}{2}\right)$ as illustrated in Figure 4. But, this inclusion does not induce a bijection on isotopy classes. Indeed, there are pairs of nonisotopic 2–string links that yield isotopic links once closed as shown in Figure 4. Therefore, studying string links slightly differs from usual link theory.

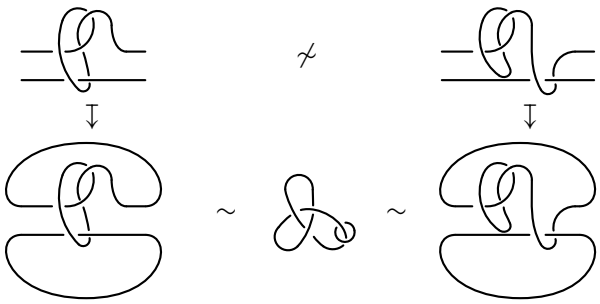

Figure 4: The two top string links are not isotopic (closing the first one up with two vertical lines results in a trivial knot, doing so with the second one yields a trefoil knot). The corresponding links are however isotopic.

Let us spend some time to investigate $\pi_0\mathcal{L}$. We first identify the invertible elements. There is a canonical map from the pure braid group on two strands to $\pi_0\mathcal{L}$ sending a pure braid to its isotopy class as a string link. It is a morphism of monoids that only maps to units in $\pi_0\mathcal{L}$ since braids form a group. This association is easily shown to be injective: the linking number map $\mathrm{lk}\colon \mathcal{L} \to \mathrm{Emb}(S^1 \sqcup S^1, S^3) \to \mathbb{Z}$ descends to a left inverse when one identifies the pure braids on two strands with the integers in the natural way. This provides a whole collection of invertible elements and it turns out that every unit in $\pi_0\mathcal{L}$ is of this form. Observe as well that these invertible links commute with every other link: an isotopy exhibiting this relation is suggested by Figure 5. Let now $\mathcal{L}^0$ be the preimage of 0 through the linking number map lk. The injection of the braid group provides a section in the short exact sequence

$$\pi_0\mathcal{L}^0 \to \pi_0\mathcal{L} \to \mathbb{Z}.$$

Thus $\pi_0\mathcal{L}$ splits as $\pi_0\mathcal{L}^0 \times \mathbb{Z}$ and we can focus on the first factor.

The monoid $\pi_0\mathcal{L}^0$ is not commutative but it contains several copies of $\pi_0\mathcal{K}$ in its center. Indeed, consider the injective morphism $\varphi^{\uparrow}\colon \pi_0\mathcal{K} \to \pi_0\mathcal{L}^0$ mapping a long knot k to the string link whose upper strand is knotted according to k and does not interact

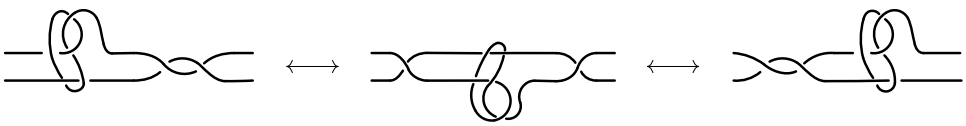

Figure 5: Illustration of the commutation between a braid and an arbitrary string link.

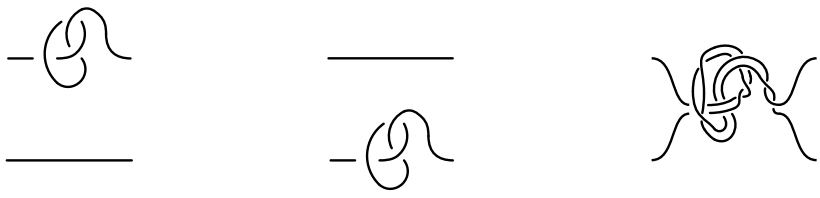

Figure 6: Illustration of the morphisms φ^s, $s \in \{\uparrow, \downarrow, \updownarrow\}$.

with the unknotted lower strand. An illustration of $\varphi^{\uparrow}$ is given in Figure 6. The image of $\varphi^{\uparrow}$ is a copy of $\pi_0\mathcal{K}$ lying in $\pi_0\mathcal{L}^0$, and one can build a similar morphism $\varphi^{\downarrow}$ by switching the roles of the strands. A third copy of the knot monoid can be found as follows. Consider the morphism $\varphi^{\updownarrow}$ that sends a knot k to the link whose strands are parallel and unlinked but knotted according to k. This $\varphi^{\updownarrow}$ maps to a third copy of $\pi_0\mathcal{K}$ in $\pi_0\mathcal{L}^0$.

The images of any two of these morphisms intersect only in the component of ι_2 so that $\pi_0\mathcal{K}^{\times 3}$ actually lives in $\pi_0\mathcal{L}^0$. Every string link in the image of a φ^s, for $s \in \{\uparrow, \downarrow, \updownarrow\}$, commutes with any other link. The structure theorem for 2–string links proved by Blair, Burke and Koytcheff in [2] actually shows that the center of $\pi_0\mathcal{L}^0$ is generated by the images of the maps φ^s, alongside the fact that the remaining links are freely generated by some prime elements. More precisely, when $\mathcal{Q}$ denotes the prime 2–string links in $\mathcal{L}$ that do not belong to the image of a φ^s and when $\mathcal{Q}^0 = \mathcal{Q} \cap \mathcal{L}^0$, one has the following result:

Theorem 1.10 (Blair, Burke and Koytcheff [2]) *The monoid $\pi_0\mathcal{L}^0$ is isomorphic to the product of $\pi_0\mathcal{K}^{\times 3}$ and the free (noncommutative) monoid on the basis $\pi_0\mathcal{Q}^0$. Moreover, an isomorphism is induced by the inclusion $\pi_0\mathcal{Q}^0 \hookrightarrow \pi_0\mathcal{L}^0$ and the maps φ^s.*

The string links generated by the images of $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ are called split. Such a string link can also be characterized by the existence of a properly embedded disk separating the two strands in the complement. The string links generated by the image of $\varphi^{\updownarrow}$ and invertible elements are called double cables. They can alternatively be defined as the links whose strands are parallel. Note that Theorem 1.10 above tells us that the center of $\pi_0\mathcal{L}$ precisely consists of the split links, double cables and their products, and that any other string link only commutes with central elements.

We now introduce framed 2–string links. They are the 2–stranded analogue of framed long knots as they arise by thickening the two strands. Let $\iota\colon D^2 \amalg D^2 \hookrightarrow D^2$ be

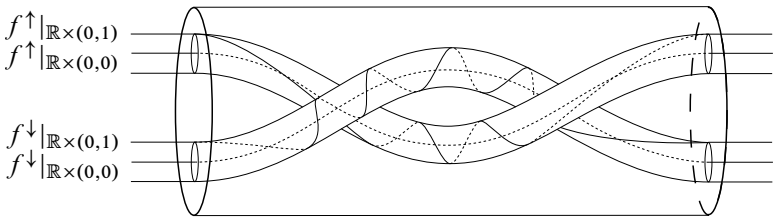

Figure 7: A framed 2–string link f with framing number $(3,-5)$.

the embedding that rescales the disks to make their radii $\frac{1}{8}$ and translates them so that they are centered at $\left(0,\frac{\pm 1}{2}\right)$. We refer to $\mathrm{id}_{\mathbb{R}}\times\iota$ as the standard embedding $(\mathbb{R}\times D^2)\sqcup(\mathbb{R}\times D^2)\hookrightarrow\mathbb{R}\times D^2$.

Definition 1.11 A *framed* 2*–string link* is an embedding

$$(\mathbb{R}\times D^2)\sqcup(\mathbb{R}\times D^2)\hookrightarrow\mathbb{R}\times D^2$$

that restricts to the standard embedding outside of $(J\times D^2)\sqcup(J\times D^2)$ and maps the interior of $(J\times D^2)\sqcup(J\times D^2)$ in the interior of $J\times D^2$. The space of framed 2–string links is denoted by $\mathrm{EC}_\iota(1,D^2)$. When $\mathrm{supp}_\iota(f)$ denotes the closure of $\{(t,x)\in(\mathbb{R}\times D^2)\sqcup(\mathbb{R}\times D^2)\mid f(t,x)\neq(t,\iota(x))\}$ for any embedding f, the condition for f to lie in $\mathrm{EC}_\iota(1,D^2)$ can be reformulated as $\mathrm{supp}_\iota(f)\subset(J\times D^2)\sqcup(J\times D^2)$ and $f\left(\mathrm{int}((J\times D^2)\sqcup(J\times D^2))\right)\subset J\times D^2$.

Framed 2–string links again dispose of a binary concatenation operation # and a restriction map $\mathrm{EC}_\iota(1,D^2)\to\mathcal{L}$ preserving it. This endows the isotopy classes $\pi_0\mathrm{EC}_\iota(1,D^2)$ with a monoid structure with the standard embedding as unit. An obstruction for the restriction map to be a homotopy equivalence is again the framing of each strand. We define the framing number ω of an element of $\mathrm{EC}_\iota(1,D^2)$ as in Definition 1.5, except that it now consists of a pair of integers, one for each strand.

Definition 1.12 Let f be a framed 2–string link with strands $f^{\uparrow}$ and $f^{\downarrow}$. We define the upper framing number $\omega_{\uparrow}(f)$ as the linking number $\mathrm{lk}(f^{\uparrow}|_{\mathbb{R}\times(0,0)},f^{\uparrow}|_{\mathbb{R}\times(0,1)})$. The lower framing number $\omega_{\downarrow}(f)$ is defined the same way and the whole framing number $\omega(f)$ is the pair of integers $(\omega_{\uparrow}(f),\omega_{\downarrow}(f))$.

The framing number is isotopy invariant and additive with respect to the concatenation for the same reasons as before. This makes it descend to a monoid morphism $\pi_0\mathrm{EC}_\iota(1,D^2)\to\mathbb{Z}^{\times 2}$. We are now able to get rid of this twisting phenomenon by defining:

Definition 1.13 The space of *fat 2–string links* $\widehat{\mathcal{L}}$ is the subspace

$$\omega^{-1}(0,0) \subset \mathrm{EC}_\iota(1, D^2).$$

Fat 2–string links are stable under concatenation. We denote by $\widehat{\mathcal{L}}^0$, $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}^0$ the elements of $\widehat{\mathcal{L}}$ whose restrictions to $(\mathbb{R}\times(0,0)) \amalg (\mathbb{R}\times(0,0))$ lie in $\mathcal{L}^0$, $\mathcal{Q}$ and $\mathcal{Q}^0$, respectively. $\widehat{\mathcal{L}}$ is a good approximation for $\mathcal{L}$ in the sense that:

Proposition 1.14 (Burke and Koytcheff [8]) *The restriction map $\widehat{\mathcal{L}} \to \mathcal{L}$ is a homotopy equivalence.*

In particular, the monoid $\pi_0\widehat{\mathcal{L}}$ is completely understood and has the same structure as $\pi_0\mathcal{L}$, made explicit in Theorem 1.10.

The remainder of this paper is dedicated to the elaboration of an algebraic structure on the space level of long knots and string links. The stacking products are examples of binary operations on the space level that relate to the monoids $\pi_0\mathcal{K}$ and $\pi_0\mathcal{L}$. We aim to find a refinement of these operations into a more subtle structure, in order to generalize Theorems 1.8 and 1.10 to the space level. These structures will be formalized as operadic actions and the isomorphisms described in the theorems above will generalize as equivariant homotopy equivalences. Budney's Theorem 11 in [5] precisely answers this problem in the case of knots, and Burke and Koytcheff's Theorem 6.8 in [8] partially deals with the case of 2–string links. In the following sections, we recall the work presented in these two papers and treat the case of 2–string links in a wider manner.

2 Operads

The purpose of this section is to recall the definition of (colored) operads, set up some notation and introduce the two operads of prime interest in this paper: the little cubes operad $\mathcal{C}_n$ and a 4–colored version of the Swiss-cheese operad that we call $\mathcal{SCL}$ for "Swiss-cheese for links". We also discuss the different types of algebras these objects encode and review free models for these structures.

2.1 Colored operads and their algebras

We start by reviewing the notion of colored operad. Let S be a set of colors. We denote by $S^\star$ the collection of tuples of elements of S. In other words, $S^\star$ is the union

$\coprod_{n\geq 0} S^{\times n}$. Each $S^{\times n}$ is naturally a right Σ_n–space. The length of a tuple $\underline{t} \in S^\star$ is denoted by $|\underline{t}|$ and, for every $s \in S$, $|\underline{t}|_s$ is the number of times s appears in $\underline{t}$. Given two tuples of colors $\underline{t}$, $\underline{u}$ and an integer $i \leq |\underline{t}|$, we denote by $\underline{t} \circ_i \underline{u}$ the $(|\underline{t}|+|\underline{u}|-1)$–tuple

$$\underline{t} \circ_i \underline{u} = (t_1, \dots, t_{i-1}, u_1, \dots, u_{|\underline{u}|}, t_{i+1}, \dots, t_{|\underline{t}|}).$$

Note that $|\underline{t} \circ_i \underline{u}|_s = |\underline{t}|_s + |\underline{u}|_s$ whenever $s \neq t_i$ and $|\underline{t}|_s + |\underline{u}|_s - 1$ for $s = t_i$. We write s^n for the tuple $(s, \dots, s) \in S^{\times n}$ for every $s \in S$ and $n \geq 0$. We are now in the right framework to define:

Definition 2.1 A *colored operad* $\mathcal{O}$ over the colors S consists of the following combined data:

(i) for every $\underline{t} \in S^\star$ and $s \in S$, a space $\mathcal{O}(\underline{t}; s)$;

(ii) for every $\underline{t}, \underline{u} \in S^\star$ and $i \leq |\underline{t}|$, operadic compositions

$$\circ_i : \mathcal{O}(\underline{t}; s) \times \mathcal{O}(\underline{u}; t_i) \to \mathcal{O}(\underline{t} \circ_i \underline{u}; s)$$

satisfying the usual associativity, symmetric and unital conditions — the latter are thoroughly detailed in [35];

(iii) for every permutation $\sigma \in \Sigma_n$, $\underline{t}$ an n–tuple of colors and s a color, a map

$$\sigma^* : \mathcal{O}(\underline{t}; s) \to \mathcal{O}(\underline{t}\sigma; s)$$

such that $\tau^* \circ \sigma^* = (\sigma \circ \tau)^*$ and $\mathrm{id}^* = \mathrm{id}$;

(iv) for every color s, a unit $1_s \in \mathcal{O}(s; s)$.

The elements of $\mathcal{O}(\underline{t}; s)$ are called operations with inputs $\underline{t}$ and output s. The units 1_s are sometimes referred to as identities. We also often write $a\sigma$ for $\sigma^*(a)$ and $a \circ_i b$ for $\circ_i(a, b)$. A morphism of colored operads $f : \mathcal{O} \to \mathcal{P}$ is a collection of maps $f_{(\underline{t};s)} : \mathcal{O}(\underline{t}; s) \to \mathcal{P}(\underline{t}; s)$ preserving operadic compositions, symmetric actions and units.

Example 2.2 The prototypical examples of colored operads are the endomorphism operads. Let $X = (X_s)_{s\in S}$ be an S–tuple of spaces. Given a vector of colors $\underline{t}$, we write $X^{\times \underline{t}}$ for the product $\prod_i X_{t_i}$. Now, for any output color s, we set $\mathcal{E}_X(\underline{t}; s)$ to be the mapping space $\mathbf{Top}(X^{\times \underline{t}}, X_s)$. An element $\sigma \in \Sigma_{|\underline{t}|}$ can act on an $f \in \mathcal{E}_X(\underline{t}; s)$ by permuting its entries, resulting in an element of $\mathcal{E}_X(\underline{t}\sigma; s)$. Also, when $\underline{u}$ is another vector of colors and g lies in $\mathcal{E}_X(\underline{u}; t_i)$, one can inject g into the i^{th} entry of f to get the composite $f \circ_i g \in \mathcal{E}_X(\underline{t} \circ_i \underline{u}; s)$. This specifies the data of a colored operad $\mathcal{E}_X$ on the colors S.

Definition 2.3 A colored operad $\mathcal{O}$ over a single color s is called an *operad*. In this case, we write $\mathcal{O}(n)$ for $\mathcal{O}(s^n; s)$ and call it the space in arity n. The unit 1_s is simply denoted by 1. The operadic compositions are now maps $\mathcal{O}(n)\times\mathcal{O}(m)\to\mathcal{O}(n+m-1)$ and the symmetric structure turns each $\mathcal{O}(n)$ into a right Σ_n–space.

Operads are useful to specify categories of algebraic objects. This is formalized via operadic actions which we define now.

Definition 2.4 Let $\mathcal{O}$ be a colored operad over the set of colors S and $X=(X_s)_{s\in S}$ an S–tuple of spaces. We say that X is an *$\mathcal{O}$–algebra* if it comes with a morphism of operads $\kappa:\mathcal{O}\to\mathcal{E}_X$. In other words, an *$\mathcal{O}$–algebra structure* on X is a collection of maps

$$\kappa_{(\underline{t};s)}:\mathcal{O}(\underline{t};s)\to\mathbf{Top}(X^{\times\underline{t}},X_s)$$

preserving the operadic compositions, identities and symmetric actions described in [35]. They may also be thought of as maps $\mathcal{O}(\underline{t};s)\times X^{\times\underline{t}}\to X_s$, and we shall use each framework when it is more convenient. A *morphism of $\mathcal{O}$–algebras* $f:X\to Y$ is a collection of maps $f_s:X_s\to Y_s$ preserving the operadic actions.

Example 2.5 Consider the operad obtained with the one point space in every arity and trivial symmetric actions and operadic compositions. An action of this operad on a space X is the data of a single map $X^{\times n}\to X$ for every nonnegative n. One readily checks that the required relations listed in [35] correspond to the associativity and commutativity of $X^{\times 2}\to X$, as well as the fact that the element specified by the zeroth map $X^{\times 0}\to X$ acts as a unit. In other words, an action of this operad on X is a commutative topological monoid structure on X. This justifies the terminology $\mathcal{C}om$ for this operad.

Example 2.6 Consider the operad whose space in arity n is the discrete symmetric group Σ_n as an evident right Σ_n–space with the following operadic composition. For every $\sigma\in\Sigma_n$, $\tau\in\Sigma_m$ and $i\leq n$, $\sigma\circ_i\tau$ permutes $\{1,\dots,n+m-1\}$ according to σ while treating $\{i,\dots,i+m-1\}$ as a single block, then shuffles the latter internally according to τ. An action of this operad on a space X is a data of a map $X^{\times n}\to X$ for every ordering of $\{1,\dots,n\}$. One readily checks that the required relations correspond to the associativity of $X^{\times 2}\to X$ and the fact that the element $X^{\times 0}\to X$ acts as a unit. In other words, the algebras over this operad are the not necessarily commutative topological monoids. This operad is called the associative operad and is denoted by $\mathcal{A}s$.

2.2 Free algebras

Before we introduce the two operads that will act on the spaces of knots and links, we take some time to discuss free algebras. When $\mathcal{O}$ is a colored operad over the set of colors S, the $\mathcal{O}$–algebras and their morphisms form a category denoted by $\mathcal{O}$–**Alg**. There is a forgetful functor $\mathcal{U}\colon \mathcal{O}\text{–}\mathbf{Alg} \to \mathbf{Top}^{\times S}$ mapping an $\mathcal{O}$–algebra to its underlying S–tuple of spaces. By free $\mathcal{O}$–algebra, we understand the left adjoint $\mathcal{O}[\_]$ to the forgetful functor $\mathcal{U}$. In other words, the free $\mathcal{O}$–algebra generated by $X = (X_s)_{s\in S}$ should lead to a bijection

$$\mathcal{O}\text{–}\mathbf{Alg}(\mathcal{O}[X], Y) \cong \mathbf{Top}^{\times S}(X, \mathcal{U}(Y))$$

for every $\mathcal{O}$–algebra Y. A well-known model for $\mathcal{O}[X]$ is obtained as follows. For every vector $\boldsymbol{x} = (x_1, \dots, x_n) \in X^{\times \underline{t}}$ and permutation σ, we will write $\sigma\boldsymbol{x}$ for $(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) \in X^{\times \underline{t}\sigma^{-1}}$. We set

$$\mathcal{O}[X]_s = \coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}} \big/ \sim,$$

where $\sim$ identifies each $(a, \boldsymbol{x})$ with $(a\sigma, \sigma^{-1}\boldsymbol{x})$ for every permutation σ. The action of $\mathcal{O}$ is obtained by composing in the $\mathcal{O}(\underline{t}; s)$ factor. The desired bijection above is then a formal verification. When $\mathcal{O}$ is an uncolored operad, $\sim$ corresponds to the Σ_n–orbits and we can simplify

$$\mathcal{O}[X] = \coprod_{n} \mathcal{O}(n) \times_{\Sigma_n} X^{\times n}.$$

We conclude this subsection with a quick observation about free algebras. When $\mathcal{O}$ is a colored operad with set of colors S, the components $\pi_0\mathcal{O}$ naturally inherit an operad structure. Similarly, if $X = (X_s)_{s\in S}$ is an $\mathcal{O}$–algebra, then the components $\pi_0 X = (\pi_0 X_s)_{s\in S}$ inherit a $\pi_0\mathcal{O}$–algebra structure. Finally, the following result will come in handy when proving that some actions yield free algebras in Section 4.

Proposition 2.7 *Let X be an S–tuple of spaces. Then $\pi_0(\mathcal{O}[X])$ is a model for the free algebra $\pi_0\mathcal{O}[\pi_0 X]$.*

Proof We have the two models

$$\pi_0(\mathcal{O}[X]_s) = \pi_0\Big(\coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}} \big/ \sim\Big), \quad \pi_0\mathcal{O}[\pi_0 X]_s = \coprod_{\underline{t}} \pi_0\mathcal{O}(\underline{t}; s) \times \pi_0 X^{\times \underline{t}} \big/ \sim.$$

Since π_0 commutes with products and coproducts, the right-hand side is equal to the quotient of $\pi_0\big(\coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}}\big)$ by the relations $[a, \boldsymbol{x}] \sim [a\sigma, \sigma^{-1}\boldsymbol{x}]$. The left-hand

side also matches this description so both spaces are the same. It is a tautological verification to see that the action of $\pi_0\mathcal{O}$ is the same under these identifications. □

2.3 The little cubes operad

We introduce the little cubes operad $\mathcal{C}_n$ and quickly discuss the algebras it encodes. It is an uncolored operad originated in [24] in order to understand iterated loop spaces. Our treatment is very similar to the one of Budney in [5].

Definition 2.8 The real functions of the form $x \mapsto ax + b$ for some positive a are said to be affine increasing. A *little n–cube* is an application $L\colon J^n \to J^n$ of the form $L = l^1 \times \cdots \times l^n$ for some affine increasing functions l^i. The space of k *overlapping little n–cubes* $\mathcal{C}_n^\infty(k)$ is the set of configurations of k little n–cubes $J^n \amalg \cdots \amalg J^n \to J^n$. We set $\mathcal{C}_n^\infty(0)$ to be the one point space. Given an element $L \in \mathcal{C}_n^\infty(k)$, we write its decomposition in little n–cubes $L = \bigoplus_i L^i$. Each L^i decomposes uniquely in affine increasing functions $l^{i,1} \times \cdots \times l^{i,n}$, so writing $l^{i,j} : x \mapsto a_{i,j}x + b_{i,j}$ gives rise to an injection $\mathcal{C}_n^\infty(k) \hookrightarrow \mathbb{R}^{2nk} : L \mapsto (a_{1,1}, b_{1,1}, \cdots, a_{k,n}, b_{k,n})$. This is used to transfer a topology on $\mathcal{C}_n^\infty(k)$. Considering the C^∞–topology actually has the same outcome.

An element of $\mathcal{C}_n^\infty(k)$ is represented by a drawing of the images of its little n–cubes. We now equip the family of spaces $\mathcal{C}_n^\infty(k)$ with an operadic structure. Thereafter, we define the little n–cubes operad as a suboperad by adding a disjointness conditions on the cubes.

Definition 2.9 The *overlapping little n–cubes operad* $\mathcal{C}_n^\infty$ is the operad specified as follows.

(i) The space in arity k is the set of configurations of little n–cubes $\mathcal{C}_n^\infty(k)$.

(ii) For every positive integer k, l and $i \leq k$, the operadic composition is given by

$$\circ_i : \mathcal{C}_n^\infty(k) \times \mathcal{C}_n^\infty(l) \to \mathcal{C}_n^\infty(l+k-1),$$
$$(L, P) \mapsto L^1 \oplus \cdots \oplus L^{i-1} \oplus L^i \circ P^1 \oplus \cdots \oplus L^i \circ P^l \oplus L^{i+1} \oplus \cdots \oplus L^k.$$

Composing $L \in \mathcal{C}_n^\infty(k)$ with the one point in $\mathcal{C}_n^\infty(0)$ discards the i^{th} cube of L.

(iii) The action of $\sigma \in \Sigma_l$ on $L \in \mathcal{C}_n^\infty(l)$ permutes the little n–cubes of L, ie

$$L\sigma = \bigoplus_i L^{\sigma(i)}.$$

(iv) The unit is the identity little n–cube $\mathrm{id}_{J^n} \in \mathcal{C}_n^\infty(1)$.

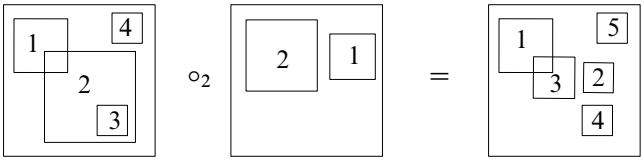

Figure 8: Illustration of the operadic composition $\circ_2 : \mathcal{C}_2^\infty(4) \times \mathcal{C}_2^\infty(2) \to \mathcal{C}_2^\infty(5)$.

Definition 2.10 Two little n–cubes are said to be *almost disjoint* if the interiors of their images are disjoint. The space of k little n–cubes $\mathcal{C}_n(k)$ is the subspace of $\mathcal{C}_n^\infty(k)$ consisting of pairwise almost disjoint little n–cubes. We still set $\mathcal{C}_n(0)$ to be the one point space. Operadic compositions in the overlapping little cubes operad preserve the property of being almost disjoint so the subspaces $\mathcal{C}_n(k)$ forms a suboperad $\mathcal{C}_n$ called the little n–cubes operad.

Remark 2.11 Budney defines in [6] an operad $\mathcal{C}_n'$ which he calls "the operad of overlapping little n–cubes". The resemblance with our terminology for $\mathcal{C}_n^\infty$ is only coincidental. The two objects are not equivalent (Budney's $\mathcal{C}_n'$ is equivalent to $\mathcal{C}_{n+1}$, while $\mathcal{C}_n^\infty$ has contractible underlying spaces). The operad $\mathcal{C}_n'$ does not appear in this article, so there should be no confusion.

We conclude this subsection by taking a look at the operad $\pi_0\mathcal{C}_n$. Given k little n–cubes $L \in \mathcal{C}_n(k)$, restricting L to the center of each cube leads to an injective map $\{1,\dots,k\} \hookrightarrow J^n$, ie an element of the configuration space $\mathrm{conf}_k(J^n)$. Conversely, given k points $\underline{x}$ in $\mathrm{conf}_k(J^n)$, one gets an element of $\mathcal{C}_n(k)$ by considering the identical cubes centered at $\underline{x}$ whose size is the maximal size that keeps them almost disjoint. Intuitive straight line homotopies show that these two maps are homotopy inverses, so that the homotopy type of $\mathcal{C}_n(k)$ is the one of $\mathrm{conf}_k(J^n)$. In particular, when $n > 1$, each $\mathcal{C}_n(k)$ is path connected, so $\pi_0\mathcal{C}_n = \mathcal{C}om$ from Example 2.5. The free algebras over this operad are the free commutative monoids. In dimension 1, the isotopy classes of $\mathcal{C}_1(k)$ are indexed by the orderings of $\{1,\dots,n\}$, so $\pi_0\mathcal{C}_1 = \mathcal{A}s$ from Example 2.6. The free $\mathcal{A}s$–algebras are the free monoids.

2.4 The operad $\mathcal{SCL}$

We go through the construction of the Swiss-cheese operad for links $\mathcal{SCL}$. It is a 4–colored operad that is also defined in terms of little cubes. This terminology is motivated by the fact that $\mathcal{SCL}$ restricts to the 2–colored Swiss-cheese operad on several pairs of colors. We then conclude by investigating the operad of components $\pi_0\mathcal{SCL}$.

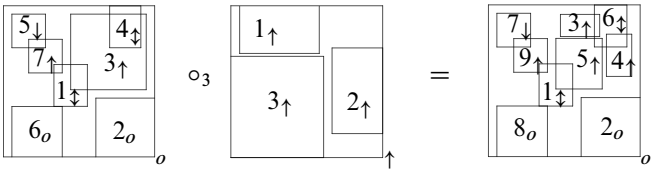

Figure 9: Illustration of the operadic composition in $\mathcal{SCL}$, more precisely of $\circ_3 : \mathcal{SCL}(\updownarrow, o, \uparrow, \updownarrow, \downarrow, o, \uparrow; o) \times \mathcal{SCL}(\uparrow, \uparrow, \uparrow; \uparrow) \to \mathcal{SCL}(\updownarrow, o, \uparrow, \uparrow, \uparrow, \updownarrow, \downarrow, o, \uparrow; o)$.

Definition 2.12 A little n–cube $L = l^1 \times \cdots \times l^n$ is said to *meet the lower face of the unit cube* if $l^n(-1) = -1$. Visually, this happens when the image of L intersects $J^{n-1} \times -1 \subset J^n$. The configurations of k almost disjoint little n–cubes meeting the lower face of J^n is denoted by $\mathcal{C}_n^\circ(k)$ and these spaces form an operad $\mathcal{C}_n^\circ$ just as in the case of $\mathcal{C}_n$.

Consider the set of four colors $S = \{o, \uparrow, \downarrow, \updownarrow\}$. The notation o is meant to remind one of the "open" color in the Swiss-cheese operad as it will play a similar role. The other symbols call up to the upper and lower strands of a string link.

Definition 2.13 The Swiss-cheese operad for links $\mathcal{SCL}$ is specified as follows.

(i) For $s \in \{\uparrow, \downarrow, \updownarrow\}$, the only inputs $\underline{t}$ that do not lead to an empty $\mathcal{SCL}(\underline{t}; s)$ are the monochromatic ones such that $\underline{t} = s^n$. In that case, we set $\mathcal{SCL}(s^n; s) = \mathcal{C}_2(n)$. When $s = o$, we set $\mathcal{SCL}(\underline{t}; o)$ to be those $L \in \mathcal{C}_2^\infty(|\underline{t}|)$ such that each L^i with $t_i = o$ meets the lower face of J^2 while being almost disjoint from any other cube, the L^i with $t_i = \uparrow$ are almost disjoint from each other and the same holds for $t_i = \downarrow$ and $\updownarrow$.

(ii) The operadic compositions, symmetric actions and units are inherited from $\mathcal{C}_2^\infty$.

An element of $\mathcal{SCL}(\underline{t}; s)$ is represented by a drawing of the images of its little n–cubes. We decorate the numbering of each little cube with its associated color to distinguish the cubes that simply happen to meet the lower face of J^n from those that have to. The output color also appears as an index of the whole drawing. Figure 9 gives an example. It is immediate from its definition that $\mathcal{SCL}$ restricts to the (cubic) Swiss-cheese operad on the pairs of colors $\{o, \uparrow\}$, $\{o, \downarrow\}$ and $\{o, \updownarrow\}$. It also clearly restricts to the little cubes operad $\mathcal{C}_2$ on $\uparrow$, $\downarrow$ and $\updownarrow$. Originally, the 2–dimensional Swiss-cheese operad is a 2–colored operad on the set of colors $\{o, c\}$. In some sense, the color o encodes a homotopy associative algebra and the color c describes part of its center. In the

case of 2–string links, the center of $\pi_0\mathcal{L}^0$ decomposes as $\pi_0\mathcal{K}^{\times 3}$. To encode this extra information, we split the color c into three independent colors $\{\uparrow, \downarrow, \updownarrow\}$.

We now investigate $\pi_0\mathcal{SCL}$ and its algebras. Observe that for every k and $n > 1$, the projection map $\mathcal{C}_n^\circ(k) \to \mathcal{C}_{n-1}(k)$ is a homotopy equivalence. A homotopy inverse is obtained by inflating $(n-1)$–cubes into n–cubes of some fixed height. This reasoning can readily be adapted to show that $\mathcal{SCL}(\underline{t}; o)$ is homotopy equivalent to the product

$$\mathcal{C}_1(|\underline{t}|_o) \times \mathcal{C}_2(|\underline{t}|_\uparrow) \times \mathcal{C}_2(|\underline{t}|_\downarrow) \times \mathcal{C}_2(|\underline{t}|_\updownarrow).$$

In particular, $\pi_0\mathcal{SCL}(\underline{t}; s)$ is either

- empty if $s \neq o$ and $\underline{t} \neq s^n$,
- a single point if $s \neq o$ and $\underline{t} = s^n$,
- the discrete space $\Sigma_{|\underline{t}|_o}$ when $s = o$.

Therefore, an action of $\pi_0\mathcal{SCL}$ on $X = (X_o, X_\uparrow, X_\downarrow, X_\updownarrow)$ is the data of a monoid structure on each space, such that the X_s are commutative for $s \neq o$ and act on X_o with compatible actions. With this description, it is easy to see with the universal property of free objects that the free such quadruplet on the basis (A, B, C, D) is given by

$$\pi_0\mathcal{SCL}[A, B, C, D] = \big(\mathcal{A}s[A] \times \mathcal{C}om[B] \times \mathcal{C}om[C] \times \mathcal{C}om[D], \mathcal{C}om[B], \mathcal{C}om[C], \mathcal{C}om[D]\big),$$

where the last three monoids act on the first one on their respective factor.

3 Operadic actions

We gather here the objects introduced in the previous sections and endow the spaces $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{L}}$ with operadic actions. In the first subsection, the fat long knots $\widehat{\mathcal{K}}$ are equipped with a $\mathcal{C}_2$–algebra structure originally exhibited by Budney in [5]. In the case of fat 2–string links, there is a $\mathcal{C}_1$–algebra structure that follows from Burke and Koytcheff's work in [8]. We recall its construction in the second subsection and extend it to an action of $\mathcal{SCL}$ in a third one.

3.1 Budney's action on fat long knots

We start with the little 2–cubes action on $\widehat{\mathcal{K}}$ originated in [5]. Following Budney's work, we first define an action of the affine increasing automorphisms of $\mathbb{R}$ on the self-embeddings of $\mathbb{R} \times D^2$, then proceed to extend it to the 2–dimensional little cubes operad $\mathcal{C}_2$. We denote by CAut_1 the group of real affine increasing functions. A

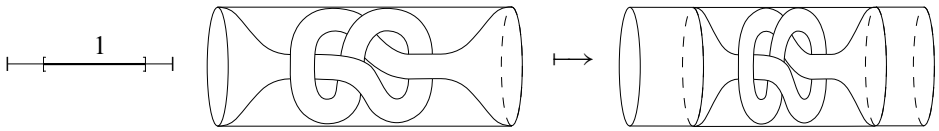

Figure 10: Illustration of the action of $\mathcal{C}_1(1)$ on $\mathrm{EC}(1, D^2)$.

little 1–cube is identified with its natural extension to the real line so that $\mathcal{C}_1(1)$ lives in CAut_1. We topologize CAut_1 as we topologized $\mathcal{C}_1(1)$, which coincides with the C^∞–topology and turns it into a topological group.

Proposition 3.1 (Budney [5]) *The topological group* CAut_1 *acts on the space of embeddings* $\mathrm{Emb}(\mathbb{R}\times D^2, \mathbb{R}\times D^2)$ *via*

$$\mathrm{CAut}_1 \times \mathrm{Emb}(\mathbb{R}\times D^2, \mathbb{R}\times D^2) \to \mathrm{Emb}(\mathbb{R}\times D^2, \mathbb{R}\times D^2),$$
$$(L, f) \mapsto (L\times \mathrm{id}_{D^2}) \circ f \circ (L^{-1}\times \mathrm{id}_{D^2}).$$

Moreover, this restricts to an action of $\mathcal{C}_1(1)$ *on* $\mathrm{EC}(1, D^2)$*, which we write as* $(L, f) \mapsto Lf$*.*

Proof That this map defines a valid action of a topological group is immediate. To prove the statement about the restriction, we just need to check that Lf restricts to the identity outside of $J\times D^2$, provided $L\in\mathcal{C}_1(1)$ and $f\in\mathrm{EC}(1, D^2)$. For any $t\notin J$, $L^{-1}(t)$ does not lie in J because $L(J)\subset J$. So, for every $x\in D^2$, $(L^{-1}\times\mathrm{id}_{D^2})(t, x)$ is outside of $J\times D^2$, where f restricts to the identity. Thus $Lf(t, x) = (t, x)$, which proves the second part of the proposition. □

Definition 3.2 (Budney [5]) We define two operations and a partial order on little 2–cubes.

(i) Given a little 2–cube $L = l^1\times l^2$, we write L_π for the little 1–cube l^1. When $L\in\mathcal{C}_2(k)$, L_π denotes $\bigoplus_i L^i_\pi$. These little 1–cubes may overlap so L_π lies in $\mathcal{C}_1^\infty(k)$ but not necessarily in $\mathcal{C}_1(k)$.

(ii) For every $L = l^1\times l^2\in\mathcal{C}_2(1)$, let L_t denote the number $l^2(-1)$. Again, if $L\in\mathcal{C}_2(k)$, then L_t is the k–tuple of reals $(L^1_t, \dots, L^k_t)\in J^k$.

(iii) We define a partial order on the little cubes L^i of an element $L\in\mathcal{C}_2(k)$. This binary relation is the order generated by setting $L^i < L^j$ if and only if $L^i_t < L^j_t$ and the interiors of L^i_π and L^j_π intersect. Then, a permutation $\sigma\in\Sigma_k$ is said to *order* L if the mapping $i\mapsto L^{\sigma(i)}$ is nondecreasing.

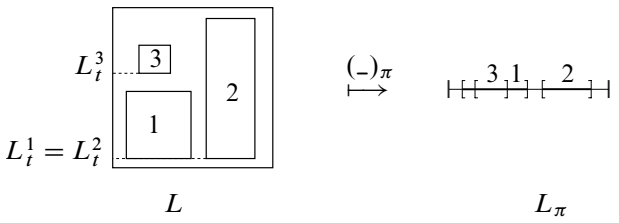

Figure 11: Illustration of the operations $(\_)_\pi$ and $(\_)_t$. The permutations id, (12) and (23) order L.

We are now ready to define an action of $\mathcal{C}_2$ on $\mathrm{EC}(1, D^2)$, ie a map of operads $\kappa : \mathcal{C}_2 \to \mathcal{E}_{\mathrm{EC}(1,D^2)}$. For every element $L \in \mathcal{C}_2(k)$ and permutation $\sigma \in \Sigma_k$ that orders L, we set

$$\begin{aligned} \kappa_k(L) : \mathrm{EC}(1, D^2)^{\times k} &\to \mathrm{EC}(1, D^2), \\ \boldsymbol{f} = (f_i)_i &\mapsto (L_\pi^{\sigma(1)} f_{\sigma(1)}) \circ \cdots \circ (L_\pi^{\sigma(k)} f_{\sigma(k)}). \end{aligned}$$

One can be assured that $\kappa_k(L)$ does not depend on σ as follows. Two choices for σ differ by a sequence of transpositions (ab) such that L^a and L^b are incomparable, ie such that L_π^a and L_π^b are almost disjoint. Then, $\mathrm{supp}(L_\pi^a f_a)$ and $\mathrm{supp}(L_\pi^b f_b)$ are almost disjoint as well so both orders of composition yield the same outcome. For the continuity, consider for every $\tau \in \Sigma_k$ the map

$$\begin{aligned} \kappa_k^\tau : \mathcal{C}_2(j) \times \mathrm{EC}(1, D^2)^{\times k} &\to \mathrm{EC}(1, D^2), \\ (L, \boldsymbol{f}) &\mapsto (L_\pi^{\tau(1)} f_{\tau(1)}) \circ \cdots \circ (L_\pi^{\tau(k)} f_{\tau(k)}). \end{aligned}$$

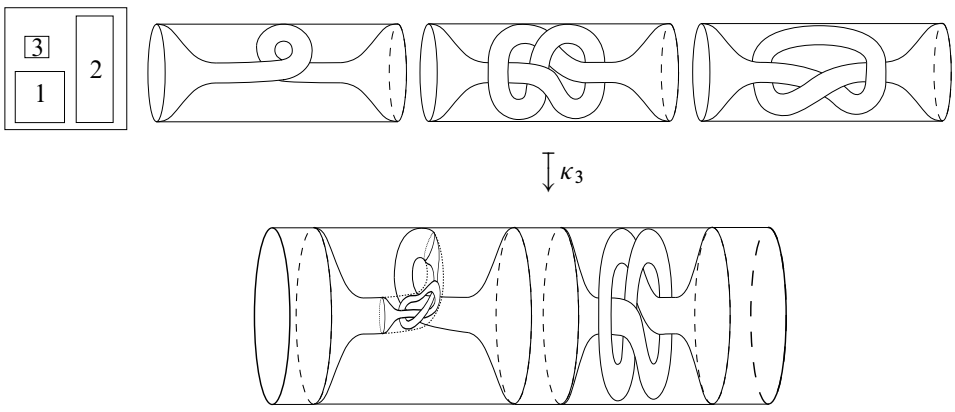

Figure 12: Illustration of Budney's action on fat long knots.

Each κ_k^τ is continuous and coincides with κ_k on

$$F_\tau = \{L \in \mathcal{C}_2(k) \mid \tau \text{ orders } L\} \times \mathrm{EC}(1, D^2)^{\times k}.$$

The sets F_τ are closed and cover $\mathcal{C}_2(k) \times \mathrm{EC}(1, D^2)^{\times k}$ so κ_k is continuous. In arity 0, we set κ_0 to be the map sending the single point in $\mathcal{C}_2(0) \times \mathrm{EC}(1, D^2)^{\times 0}$ to $\mathrm{id}_{\mathbb{R}\times D^2}$.

Theorem 3.3 (Budney [5]) *The operations κ turn $\mathrm{EC}(1, D^2)$ into a $\mathcal{C}_2$–algebra.*

Even though a proof of this result is readily available in [5], we provide one here as the methods and ideas at stake will be reused before the end of this section.

Proof The operation κ_1 clearly maps the basepoint $\mathrm{id}_{J^2} \in \mathcal{C}_2(1)$ to the identity. We need to check the compatibility of κ with the symmetric group actions and the operadic compositions. We start with the symmetric structure. Recall that a permutation τ acting on the right of $L \in \mathcal{C}_2(k)$ yields $\bigoplus_i L^{\tau(i)}$. It also acts on the left of

$$\boldsymbol{f} = (f_i)_i \in \mathrm{EC}(1, D^2)^{\times k}$$

to give $\tau \boldsymbol{f} = (f_{\tau^{-1}(i)})_i$. Thus, if σ is a permutation that orders $L\tau$, then $\tau \circ \sigma$ orders L. This proves the needed equality

$$\kappa_k(L\tau, \boldsymbol{f}) = (L_\pi^{\tau\circ\sigma(1)} f_{\sigma(1)}) \circ \cdots \circ (L_\pi^{\tau\circ\sigma(k)} f_{\sigma(k)}) = \kappa_k(L, \tau \boldsymbol{f}).$$

We are left to prove that κ preserves operadic compositions. Given little 2–cubes $L \in \mathcal{C}_2(k)$, $P \in \mathcal{C}_2(l)$ and an integer $i \leq k$, we need to show that

$$\kappa_{k+l-1}(L \circ_i P) = \kappa_k(L) \circ_i \kappa_l(P).$$

Let σ and τ be permutations that respectively order L and P. Unravelling the definition of κ shows that the desired equality boils down to checking that $\sigma \circ_i \tau$ orders $L \circ_i P$. Recall the definitions of $\sigma \circ_i \tau$ and $L \circ_i P$:

- $\sigma \circ_i \tau$ shuffles the interval $\{i, \dots, i + l - 1\}$ according to τ, then permutes $\{1, \dots, k + l - 1\}$ according to σ while treating the shuffled interval as a single block.
- $L \circ_i P$ is obtained from L by replacing L^i with $\bigoplus_r L^i \circ P^r$.

If L^i and L^j are incomparable, $L^i \circ P^r$ and L^j also are, so the result follows. □

This recently developed structure on the space of framed long knots generalizes the stacking operation in the following sense: acting with two side-by-side rectangles of

width 1 on two knots results in their concatenation. In particular, the $\mathcal{C}om$–algebra structure on $\pi_0\mathrm{EC}(1, D^2)$ induced from $\mathcal{C}_2 \to \mathcal{E}_{\mathrm{EC}(1,D^2)}$ is the monoid structure described in Section 1.1.

Theorem 3.4 (Budney [5]) *The fat long knots $\widehat{\mathcal{K}}$ form a sub-$\mathcal{C}_2$–algebra of* $\mathrm{EC}(1, D^2)$.

Proof As mentioned in Section 1.1, the framing number ω descends to a morphism of monoids $\pi_0\mathrm{EC}(1, D^2) \to \mathbb{Z}$. Recall also that $\pi_0\mathcal{C}_2$ is the commutative operad $\mathcal{C}om$. The integers $\mathbb{Z}$ form an abelian group and thus a commutative monoid. They can therefore be seen as a $\mathcal{C}_2$–algebra via the structure map $\mathcal{C}_2 \to \mathcal{C}om \to \mathcal{E}_{\mathbb{Z}}$. In this framework, the framing number ω is a $\mathcal{C}_2$–algebra morphism, hence the result. □

We conclude this subsection with a quick discussion about κ. Let L be an element of $\mathcal{C}_2(k)$. The heights of the little cubes of L only appear in the formula of $\kappa_k(L)$ to dictate a composition order. This is done via an ordering permutation, which we defined as an element $\sigma \in \Sigma_k$ such that the mapping $i \mapsto L^{\sigma(i)}$ is nondecreasing. Here, one can replace the word "decreasing" with "increasing" and define another action with the same formula. We refer to it as *Budney's reverse action*. There is no substantial difference between these two versions of κ, nor is there a reason to prefer one or the other. We still introduce the two of them now, as they will both play a role in the next subsections. Informally, the need for a reversed action arises because knots yielding split links must be tied at the beginning of a composition, while knots yielding cables must be tied at the end.

3.2 Burke and Koytcheff's actions on fat 2–string links

This subsection is a first step towards an adaptation of Budney's work to 2–string links. Namely, we build an action of $\mathcal{C}_1$ on $\mathrm{EC}_\iota(1, D^2)$ and $\widehat{\mathcal{L}}$. This structure has already been exhibited by Burke and Koytcheff [8, Theorem 6.8], with $\mathcal{C}_1$ appearing as a suboperad of a way bigger object called the infection operad. As before, we start with an action of CAut_1 on the embeddings $(\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \hookrightarrow \mathbb{R} \times D^2$, then proceed to extend it to $\mathcal{C}_1$.

Proposition 3.5 *The topological group* CAut_1 *acts on* $\mathrm{Emb}((\mathbb{R} \times D^2)^{\amalg 2}, \mathbb{R} \times D^2)$ *via*

$$\mathrm{CAut}_1 \times \mathrm{Emb}((\mathbb{R} \times D^2)^{\amalg 2}, \mathbb{R} \times D^2) \to \mathrm{Emb}((\mathbb{R} \times D^2)^{\amalg 2}, \mathbb{R} \times D^2),$$
$$(L, l) \mapsto (L \times \mathrm{id}_{D^2}) \circ l \circ (L^{-1} \times \mathrm{id}_{D^2})^{\amalg 2}.$$

Moreover, this restricts to an action of $\mathcal{C}_1(1)$ on $\mathrm{EC}_\iota(1, D^2)$, which we write as $(L, l) \mapsto Ll$.

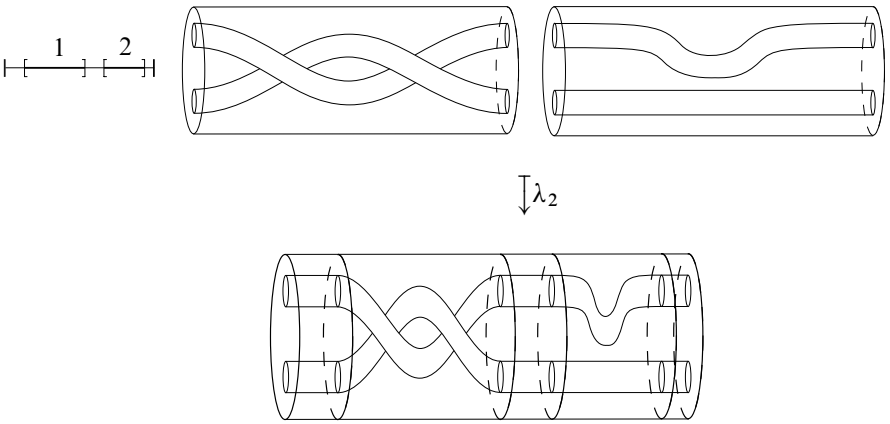

Figure 13: Illustration of Burke and Koytcheff's action on fat 2–string links.

Proof The proof of this is very similar to the proof of Proposition 3.1: the fact that the formula above specifies a valid action of a topological group is still clear and the restriction statement is proved just as in the case of framed long knots. □

To distinguish Budney's action from the one we build now, we denote the structure map by λ. The space $\mathcal{C}_1(0)$ in arity 0 still consists of a single point that λ_0 maps to $\mathrm{id}_{\mathbb{R}} \times \iota$ from Definition 1.11. For any positive integer k and $L \in \mathcal{C}_1(k)$, we set $\lambda_k(L)$ to the map that concatenates k framed string links according to the configuration of intervals L. That is, for every $\boldsymbol{f} = (f_i)_i \in \mathrm{EC}_\iota(1, D^2)^{\times k}$,

$$\lambda_k(L)(\boldsymbol{f}) \colon (\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \to \mathbb{R} \times D^2,$$
$$(t, x) \mapsto \begin{cases} L^i f_i(t, x) & \text{when } t \in L^i(J), \\ (t, \iota(x)) & \text{elsewhere.} \end{cases}$$

The embeddings patch in a differentiable way because the little cubes are almost disjoint. The outcome lies in $\mathrm{EC}_\iota(1, D^2)$ because $\mathrm{supp}_\iota(L^i f_i) = (L^i \times \mathrm{id}_{D^2})^{\amalg 2}(\mathrm{supp}_\iota(f_i))$.

Theorem 3.6 (Burke and Koytcheff [8]) *The operations λ turn $\mathrm{EC}_\iota(1, D^2)$ into a $\mathcal{C}_1$–algebra.*

Proof It is clear that $\lambda_1(\mathrm{id}_J)$ is the identity on $\mathrm{EC}_\iota(1, D^2)$. We check the compatibility with the symmetric actions. Let τ be a permutation, $L \in \mathcal{C}_1(k)$ and $\boldsymbol{f} \in \mathrm{EC}_\iota(1, D^2)^{\times k}$. To prove the desired $\lambda_k(L\tau, \boldsymbol{f}) = \lambda_k(L, \tau \boldsymbol{f})$, we show that these maps agree on the images of every little cube of L. This is enough as they clearly restrict to $\mathrm{id}_{\mathbb{R}} \times \iota$

outside of these intervals. For every $i \leq k$, the left-hand side of the equation restricts to $(L\tau)^i f_i$ on $(L\tau)^i(J^k)$. The right-hand side restricts to $L^{\tau(i)} f_{\tau^{-1}\circ\tau(i)} = (L\tau)^i f_i$ so we are done. The associative compatibility is verified the same way. □

As in the case of knots, the stacking operation arises as a special case of this recently developed action. More precisely, acting with two side-by-side intervals of width 1 on two string links results in their concatenation. Therefore, the $\mathcal{A}s$–algebra structure on $\pi_0\mathrm{EC}_\iota(1, D^2)$ induced from $\lambda\colon \mathcal{C}_1 \to \mathcal{E}_{\mathrm{EC}_\iota(1,D^2)}$ is the monoid structure on $\pi_0\mathrm{EC}_\iota(1, D^2)$ discussed in Section 1.2. Moreover, as in the case of knots, we can restrict ourselves to unframed embeddings:

Theorem 3.7 *The fat 2–string links $\widehat{\mathcal{L}}$ form a sub-$\mathcal{C}_1$–algebra of* $\mathrm{EC}_\iota(1, D^2)$.

Proof Just as in Theorem 3.4, $\mathbb{Z}^{\times 2}$ is a group that we can think of as an $\mathcal{A}s$–algebra and therefore a $\mathcal{C}_1$–algebra. This turns the framing number ω into a morphism of $\mathcal{C}_1$–algebras, hence the result. □

3.3 The action of $\mathcal{SCL}$ on fat 2–string links

This section aims to merge the two actions defined above into a single $\mathcal{SCL}$–algebra structure on the spaces of fat long knots and fat 2–string links. More precisely, we build an action of $\mathcal{SCL}$ on the quadruplet of spaces $X = (X_o, X_\uparrow, X_\downarrow, X_\updownarrow)$, where $X_o = \mathrm{EC}_\iota(1, D^2)$ and $X_s = \mathrm{EC}(1, D^2)$ for every $s \in \{\uparrow, \downarrow, \updownarrow\}$. We start with a lemma to ease the construction.

Lemma 3.8 *There is a map*

$$\mathrm{EC}_\iota(1, D^2) \times \mathrm{EC}(1, D^2)^{\times 3} \to \mathrm{EC}_\iota(1, D^2), \quad (l, k_\uparrow, k_\downarrow, k_\updownarrow) \mapsto k_\updownarrow \circ l \circ [k_\uparrow \amalg k_\downarrow].$$

Proof The continuity of this application immediately follows from the continuity of the composition in the C^∞–topology. The purpose of this lemma is actually to check that $k_\updownarrow \circ l \circ [k_\uparrow \amalg k_\downarrow]$ indeed lives in $\mathrm{EC}_\iota(1, D^2)$. This follows from the inclusions $\mathrm{supp}(k_s) \subset J \times D^2$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$, and $l\big(\mathrm{int}((J \times D^2) \amalg (J \times D^2))\big) \subset J \times D^2$. □

This map is in some way a combination of the morphisms φ^s from Section 1.2. Indeed, if one restricts this application to the subspace $\{\mathrm{id}_\mathbb{R} \times \iota\} \times \mathrm{EC}(1, D^2) \times \{\mathrm{id}_{\mathbb{R}\times D^2}\}^{\times 2}$, the formula becomes $k_\uparrow \mapsto (\mathrm{id}_\mathbb{R} \times \iota) \circ [k_\uparrow \amalg \mathrm{id}_{\mathbb{R}\times D^2}]$, which is a fattened version of $\varphi^\uparrow$. We denote it by $\hat{\varphi}^\uparrow$. The same goes for $\downarrow$. In the case of $\updownarrow$, one is left with $k_\updownarrow \mapsto k_\updownarrow \circ (\mathrm{id}_\mathbb{R} \times \iota)$. This map sends a long knot $k_\updownarrow$ to the string link whose strands

are parallel and knotted according to $k_{\updownarrow}$. In other words, it is again a fattened version of $\varphi^{\updownarrow}$, which we denote by $\hat{\varphi}^{\updownarrow}$.

We are now ready to define the morphism $\mu\colon \mathcal{SCL} \to \mathcal{E}_X$ for the new action. As the values of $\mu_{(\underline{t};s)}$ heavily depend on $(\underline{t};s)$, defining μ takes several steps.

We start by specifying the values of μ in monochromatic cases. The operad $\mathcal{SCL}$ restricts to the little cubes operad $\mathcal{C}_2$ on the colors $s \in \{\uparrow, \downarrow\}$. We set μ to Budney's action on these colors. In other words, $\mu_{(s^k;s)} = \kappa_k$ from Theorem 3.3. When $s = \updownarrow$, we similarly set $\mu_{(\updownarrow^k;\updownarrow)}$ to Budney's reverse action, which we will still denote by κ by slight abuse of notation. On the color o, $\mathcal{SCL}$ restricts to the operad $\mathcal{C}_2^{\circ}$. There is a morphism $(\_)_\pi\colon \mathcal{C}_2^{\circ} \to \mathcal{C}_1$ and we set μ to the composite $\lambda \circ (\_)_\pi$ on this suboperad.

For every $s \in \{\uparrow, \downarrow, \updownarrow\}$, the only input colors $\underline{t}$ that do not lead to an empty $\mathcal{SCL}(\underline{t};s)$ are the monochromatic ones such that $\underline{t} = s^k$. Thus, we are left to specify $\mu_{(\underline{t};s)}$ when $s = o$ and the inputs are mixed. To this end, we introduce color sorting functions. Let $\underline{t} \in S^\star$. Consider four injective (not necessarily increasing) maps

$$\alpha_s\colon [|\underline{t}|_s] = \{1, \dots, |\underline{t}|_s\} \to [|\underline{t}|] = \{1, \dots, |\underline{t}|\}, \quad s \in S,$$

whose disjoint images cover $[|\underline{t}|]$ and such that $t_{\alpha_s(i)} = s$ for every $i \in [|\underline{t}|_s]$. These maps regroup inputs of the same color and are said to sort the colors of $\underline{t}$. Observe that each α_s lifts to a map $\mathcal{SCL}(\underline{t};o) \to \mathcal{C}_2(|\underline{t}|_s)$ that discards the little cubes whose colors are different from s,

$$\alpha_s\colon \mathcal{SCL}(\underline{t};s) \to \mathcal{C}_2(|\underline{t}|_s), \quad L \mapsto \bigoplus_i L^{\alpha_s(i)}.$$

Discarding embeddings also yields a map

$$\alpha_s\colon X^{\times \underline{t}} \to X_s^{\times |\underline{t}|_s}, \quad \boldsymbol{f} \mapsto \alpha_s \boldsymbol{f} = (f_{\alpha_s(1)}, \dots, f_{\alpha_s(|\underline{t}|_s)}).$$

The behavior of these lifts with respect to the symmetric structures on $\mathcal{SCL}$ and $X^{\times \underline{t}}$ is captured by the following relations: for every $\sigma \in \Sigma_{|\underline{t}|}$, $\tau \in \Sigma_{|\underline{t}|_s}$, $L \in \mathcal{SCL}(\underline{t};o)$ and $\boldsymbol{f} \in X^{\times \underline{t}}$,

$$(\sigma \circ \alpha_s)L = \bigoplus_i L^{\sigma\circ\alpha_s(i)} = \bigoplus_i (L\sigma)^{\alpha_s(i)} = \alpha_s(L\sigma), \quad (\sigma \circ \alpha_s)\boldsymbol{f} = \alpha_s(\sigma^{-1}\boldsymbol{f}),$$

$$(\alpha_s \circ \tau)L = \bigoplus_i L^{\alpha_s\circ\tau(i)} = \bigoplus_i (\alpha_s L)^{\tau(i)} = (\alpha_s L)\tau, \quad (\alpha_s \circ \tau)\boldsymbol{f} = \tau^{-1}(\alpha_s \boldsymbol{f}).$$

We can finally combine the previous actions and define $\mu_{(\underline{t};o)}(L)$ as the map

$$\mu_{(\underline{t};o)}(L)\colon X^{\times \underline{t}} \to \mathrm{EC}_\iota(1, D^2),$$
$$\boldsymbol{f} \mapsto \kappa(\alpha_\updownarrow L, \alpha_\updownarrow \boldsymbol{f}) \circ \lambda(\alpha_o L_\pi, \alpha_o \boldsymbol{f}) \circ \big[\kappa(\alpha_\uparrow L, \alpha_\uparrow \boldsymbol{f}) \amalg \kappa(\alpha_\downarrow L, \alpha_\downarrow \boldsymbol{f})\big],$$

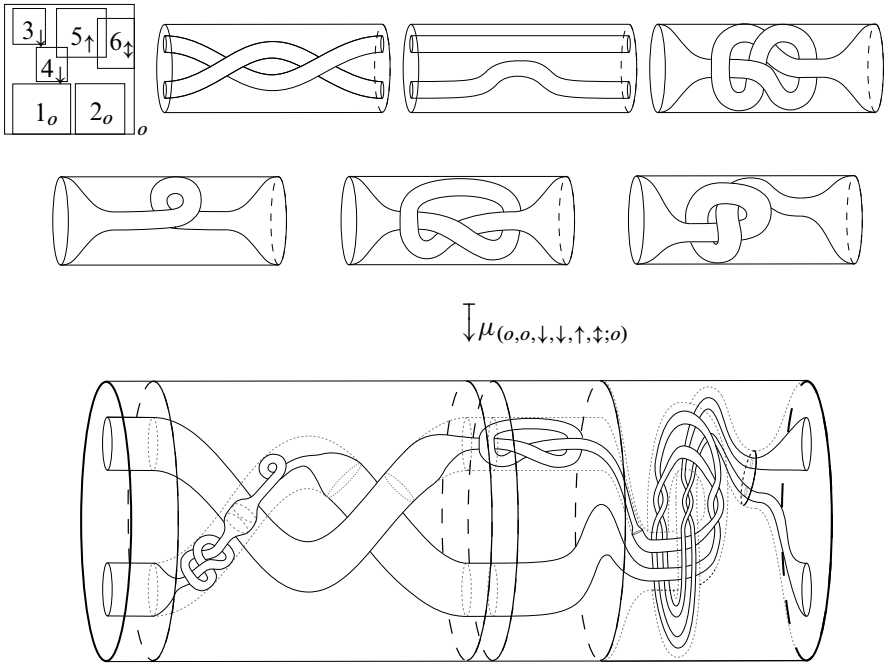

Figure 14: Illustration of the action of $\mathcal{SCL}$ on fat 2–string links and fat long knots.

where the κ on the left-hand side refers to Budney's reverse action and the other two to Budney's regular action.

The continuity of μ is immediate and its values do not depend on the color sorting functions: if one chooses to replace α_o with ${\alpha_o}'$, there is a permutation τ such that ${\alpha_o}' = \alpha_o \circ \tau$. Then, the relations above give

$$\begin{aligned}\lambda({\alpha_o}' L_\pi, {\alpha_o}' \boldsymbol{f}) &= \lambda((\alpha_o \circ \tau) L_\pi, (\alpha_o \circ \tau)\boldsymbol{f}) \\ &= \lambda((\alpha_o L_\pi)\tau, \tau^{-1}(\alpha_o \boldsymbol{f})) \\ &= \lambda(\alpha_o L_\pi, \alpha_o \boldsymbol{f}).\end{aligned}$$

The same argument with κ shows that the remaining α_s can be replaced as well.

Theorem 3.9 *The operations μ turn the quadruplet X into an $\mathcal{SCL}$–algebra.*

Proof First of all, it is clear that every $\mu_{(s;s)}$ sends id_{J^2} to the identity. The symmetric compatibility is also quickly verified: when functions $(\alpha_s)_s$ sort the colors of $\underline{t}$, the composites $(\sigma^{-1} \circ \alpha_s)_s$ sort the colors of $\underline{t}\sigma$ and the needed equality follows.

We are left to check the compatibility with the operadic composition. Let $L \in \mathcal{SCL}(\underline{t}; s)$ and $P \in \mathcal{SCL}(\underline{u}; t_i)$ for some i. We need to show that $\mu(L \circ_i P) = \mu(L) \circ_i \mu(P)$. The validity of Budney, Burke and Koytcheff's actions (Theorems 3.3 and 3.6) implies the result when $\underline{t}$ and $\underline{u}$ are monochromatic of the same color. Since there is no operation with output color $s \in \{\uparrow, \downarrow, \updownarrow\}$ and input colors $\underline{t} \neq s^k$, we may assume that $s = o$ and $\underline{t} \neq o^k$. When evaluated in embeddings $\boldsymbol{f} \in X^{\times \underline{t} \circ_i \underline{u}}$, the desired equality reads

$$\begin{aligned}\mu_{(\underline{t}\circ_i \underline{u}; o)}&(L \circ_i P, \boldsymbol{f}) \\ &= \mu_{(\underline{t};o)}\big(L, f_1, \dots, f_{i-1}, \mu_{(\underline{u};t_i)}(P, f_i, \dots, f_{i+|\underline{u}|-1}), f_{i+|\underline{u}|}, \dots, f_{|\underline{t}\circ_i \underline{u}|}\big).\end{aligned}$$

We split cases and unravel the definition of μ on both sides of this equation.

Assume first that $t_i = \uparrow$. This forces $\underline{u} = \uparrow^{|\underline{u}|}$. Let $(\alpha_s)_s$ sort the colors of $\underline{t}$. We may ask for $\alpha_\uparrow(|\underline{t}|_\uparrow) = i$. We sort the colors $s \neq \uparrow$ in $\underline{t} \circ_i \underline{u}$ with functions $(\gamma_s)_s$ mapping j to $\alpha_s(j)$ if $\alpha_s(j) < i$ or to $\alpha_s(j) + |\underline{u}|$ if $\alpha_s(j) > i$. The reason for this choice is $\gamma_s(L \circ_i P) = \alpha_s L$. For the remaining $\gamma_\uparrow$, we use the same construction on $[|\underline{t}|_\uparrow - 1]$ and extend it to $[|\underline{t} \circ_i \underline{u}|_\uparrow]$ via the increasing map onto the interval $\{i + j,\ j < |\underline{u}|\}$. The equality reduces to

$$K_\updownarrow \circ \Lambda \circ [K_\uparrow^L \amalg K_\downarrow] = K_\updownarrow \circ \Lambda \circ [K_\uparrow^R \amalg K_\downarrow],$$

where

$$\begin{aligned}\Lambda &= \lambda(\alpha_o L_\pi, \gamma_o \boldsymbol{f}),\\ K_\uparrow^L &= \kappa(\gamma_\uparrow(L \circ_i P), \gamma_\uparrow \boldsymbol{f}),\\ K_\updownarrow &= \kappa(\alpha_\updownarrow L, \gamma_\updownarrow \boldsymbol{f}),\\ K_\uparrow^R &= \kappa(\alpha_\uparrow L, f_{\alpha_\uparrow(1)}, \dots, f_{\alpha_\uparrow(|\underline{t}|_\uparrow - 1)}, \kappa(P, f_i, \dots, f_{|\underline{u}|+i-1})),\\ K_\downarrow &= \kappa(\alpha_\downarrow L, \gamma_\downarrow \boldsymbol{f}).\end{aligned}$$

Furthermore, $\gamma_\uparrow(L \circ_i P) = \alpha_\uparrow L \circ_{|\underline{t}|_\uparrow} P$, so the validity of Budney's action (Theorem 3.3) completes the proof in this case. The same manipulations treat the cases $t_i = \downarrow$ and $\updownarrow$.

We are left to treat the case $s = o$ and $t_i = o$. Let $(\alpha_s)_s$ and $(\beta_s)_s$ be color sorting functions for $\underline{t}$ and $\underline{u}$, respectively. We may again ask for $\alpha_o(|\underline{t}|_o) = i$. Let $(\gamma_s)_s$ be the color sorting functions for $\underline{t} \circ_i \underline{u}$ one naturally builds from $(\alpha_s)_s$ and $(\beta_s)_s$. More precisely, γ_s agrees with α_s on $\alpha_s{}^{-1}(\{l \mid l < i\})$, with $\alpha_s + |\underline{u}|$ on $\alpha_s{}^{-1}(\{l \mid l > i\})$ and maps the remaining interval to the inputs s in $\{i + l \mid l < |\underline{u}|\}$ according to β_s. These choices are the ones giving $\gamma_s(L \circ_i P) = \alpha_s L \oplus (L^i \circ \beta_s P)$ for every s in $\{\uparrow, \downarrow, \updownarrow\}$ and $\gamma_o(L \circ_i P) = \alpha_o L \circ_{|\underline{t}|_o} \beta_o P$. The left-hand side of the equality reads

$K^L_{\updownarrow} \circ \Lambda^L \circ [K^L_{\uparrow} \amalg K^L_{\downarrow}]$, where

$$\Lambda^L = \lambda(\alpha_o L \circ_{|\underline{t}|_o} \beta_o P, \gamma_o \boldsymbol{f}),$$
$$K^L_s = \kappa(\alpha_s L \oplus (L^i \circ \beta_s P), \gamma_s \boldsymbol{f}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

On the other hand, the right-hand side of the equality is $K^R_{\updownarrow} \circ \Lambda^R \circ [K^R_{\uparrow} \amalg K^R_{\downarrow}]$, where

$$\Lambda^R = \lambda(\alpha_o L, f_{\gamma_o(1)}, \dots, f_{\gamma_o(|\underline{t}|_o - 1)}, \mu_{(\underline{u};o)}(P, f_i, \dots, f_{i+|\underline{u}|-1})),$$
$$K^R_s = \kappa(\alpha_s L, f_{\gamma_s(1)}, \dots, f_{\gamma_s(|\underline{t}|_s)}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

But $\mu_{(\underline{u};o)}(P, f_i, \dots, f_{i+|\underline{u}|-1})$ is itself of the form $K_{\updownarrow}{}' \circ \Lambda' \circ [K_{\uparrow}{}' \amalg K_{\downarrow}{}']$, where

$$\Lambda' = \lambda(\beta_o P, f_{\gamma_o(|\underline{t}|_o)}, \dots, f_{\gamma_o(|\underline{t} \circ_i \underline{u}|_o)}),$$
$$K_s{}' = \kappa(\beta_o P, f_{\gamma_s(|\underline{t}|_s + 1)}, \dots, f_{\gamma_s(|\underline{t} \circ_i \underline{u}|_s)}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

It is easy to check from the definition of λ and Theorem 3.6 that

$$\Lambda^R = L^i K_{\updownarrow}{}' \circ \Lambda^L \circ [L^i K_{\uparrow}{}' \amalg L^i K_{\downarrow}{}'].$$

We get the following new expression for the whole right-hand side of the equation:

$$(K^R_{\updownarrow} \circ L^i K_{\updownarrow}{}') \circ \Lambda^L \circ [L^i K_{\uparrow}{}' \circ K^R_{\uparrow} \amalg L^i K_{\downarrow}{}' \circ K^R_{\downarrow}].$$

Thus we are left to identify K factors. We previously computed

$$K^L_{\uparrow} = \kappa(\alpha_{\uparrow} L \oplus (L^i \circ \beta_{\uparrow} P), \gamma_{\uparrow} \boldsymbol{f}).$$

Recall that when evaluating κ, one chooses a permutation that orders $\alpha_{\uparrow} L \oplus (L^i \circ \beta_{\uparrow} P)$ and composes the embeddings accordingly. Here, L^i is a little 2–cube that meets the lower face of the unit cube. In other words, $L^i_t = -1$ and cannot get any lower. Thus, the factors $(L^i \circ \beta_{\uparrow} P)^j f_{\gamma_{\uparrow}(|\underline{t}|_{\uparrow} + j)}$ can be placed in first position when computing $K^L_{\uparrow}$. This ultimately shows that

$$K^L_{\uparrow} = L^i K_{\uparrow}{}' \circ K^R_{\uparrow}.$$

One deals with $\downarrow$ the exact same way. For $\updownarrow$, the same phenomenon with Budney's reverse action shows that the factors $(L^i \circ \beta_{\updownarrow} P)^j f_{\gamma_{\updownarrow}(|\underline{t}|_{\updownarrow} + j)}$ can be placed in last position when computing $K^L_{\updownarrow}$, which again shows the desired

$$K^L_{\updownarrow} = K^R_{\updownarrow} \circ L^i K_{\updownarrow}{}'. \qquad \square$$

Once again, the concatenation comes as a special case with side-by-side cubes of equal width. Budney's action on knots can be recovered and one can also turn a knot into a double cable or a split link using identity cubes in $\mathcal{SCL}(s; o)$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$. More precisely, $\mu_{(s;o)}(\mathrm{id}_{J^2}) = \hat{\varphi}^s$. This shows that the $\pi_0 \mathcal{SCL}$–algebra structure on

the quadruplet $\pi_0 X$ is the data of the usual monoids $\pi_0\mathrm{EC}(1, D^2)$ and $\pi_0\mathrm{EC}_\iota(1, D^2)$, together with the three distinct independent actions of $\pi_0\mathrm{EC}(1, D^2)$ on $\pi_0\mathrm{EC}_\iota(1, D^2)$ given by the $\hat{\varphi}^s$, $s \in \{\uparrow, \downarrow, \updownarrow\}$. Finally, the spaces of unframed knots and links are still stable:

Theorem 3.10 *The quadruplet $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ forms a sub-$\mathcal{SCL}$–algebra of X.*

Proof Consider the two monoids $\mathbb{Z}$ and $\mathbb{Z}^{\times 2}$. The first one acts on the second one in three different ways: on the first factor of $\mathbb{Z}^{\times 2}$, on the second factor or diagonally. The data of these three actions is precisely that of a $\pi_0\mathcal{SCL}$–algebra structure on the quadruplet $(\mathbb{Z}^{\times 2}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$. One can think of this structure as an $\mathcal{SCL}$–algebra structure. Thanks to the additive properties of the linking number with respect to the concatenation of curves, one easily checks that the framing number ω turns into a morphism of $\mathcal{SCL}$–algebras. The result follows. □

This action of $\mathcal{SCL}$ on $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ combines all the structure we have met on long knots and 2–string links so far. Moreover, the isotopies exhibiting the commutativity relations discussed in Section 1.2 can all be obtained with paths in $\mathcal{SCL}$ from a configuration of cubes to another. The next section aims to show that this correspondence actually follows from a deeper result: a homotopy equivalence between $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ and a free algebra over $\mathcal{SCL}$.

Remark 3.11 It is possible to extend Theorem 3.9 to manifolds other than the 2–dimensional disk D^2. More precisely, when M is a manifold of dimension n, we can consider the space $\mathrm{EC}(k, M)$ consisting of the embeddings from $\mathbb{R}^k \times M$ to itself that restrict to the identity outside of $J^k \times M$. The notation "EC" comes from the terminology "embedding" and "cubical". This space has been intensively studied by Budney and proved to be an algebra over the $(k+1)$–dimensional little cubes operad $\mathcal{C}_{k+1}$ in [5].

Similarly, for any fixed embedding $\iota\colon M \amalg M \hookrightarrow M$, one can define $\mathrm{EC}_\iota(k, M)$, the space consisting of embeddings from $\mathbb{R}^k \times M^{\amalg 2}$ to $\mathbb{R}^k \times M$ that restrict to $\mathrm{id}_{\mathbb{R}^k} \times \iota$ outside of $J^k \times M^{\amalg 2}$ and map the interior of $J^k \times M^{\amalg 2}$ to the interior of $J^k \times M$. Burke and Koytcheff mentioned these spaces in [2], alongside their work on the special case corresponding to framed string links.

In order to understand the structure on the quadruplet of spaces

$$X_M = (\mathrm{EC}_\iota(k, M), \mathrm{EC}(k, M), \mathrm{EC}(k, M), \mathrm{EC}(k, M)),$$

we need a higher-dimensional version of the Swiss-cheese operad for links. Roughly speaking, one can define $\mathcal{SCL}_k$ the same way we did $\mathcal{SCL}$, except that the operad $\mathcal{C}_k$ is used in the construction, instead of the 2–dimensional little cubes operad $\mathcal{C}_2$. We can then extend Budney's action on $\mathrm{EC}(k, M)$ in order to get an $\mathcal{SCL}_{k+1}$–algebraic structure on the quadruplet X_M. The precise formula for this action is the same as the one introduced before Theorem 3.9, and checking that is specifies a valid $\mathcal{SCL}_{k+1}$–algebra structure boils down to the verifications already carried out in the proof above.

4 Freeness results

We prove here that the operadic actions constructed in Section 3 lead to free algebras over different operads. More precisely, we first introduce the main result of Budney in [5], which states that $\widehat{\mathcal{K}}$ is homotopy equivalent as a $\mathcal{C}_2$–algebra to $\mathcal{C}_2[\widehat{\mathcal{P}}]$. A second theorem proved by Burke and Koytcheff in [8] provides an analogous statement about the action of $\mathcal{C}_1$ on a subspace of $\widehat{\mathcal{L}}$. We then combine these results to prove the main theorem of this paper, Theorem 4.11, stating that $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is homotopy equivalent to a free $\mathcal{SCL}$–algebra. These three theorems are proved with very similar methods, most of them coming from 3–manifold topology and homotopy theory. The first subsection recalls the concepts we need from these fields, and the following three are each dedicated to a freeness theorem. The proofs of the results of Budney, Burke and Koytcheff are only quickly outlined, since thorough treatments are available in [5] and [8]. We still dispense sketches of proofs as the arguments they involve will be useful for Theorem 4.11.

4.1 Notions of 3–dimensional topology

We introduce some basic concepts of 3–manifold theory. The instances of 3–manifolds we will encounter mostly lie in $\mathbb{R}^3$, so they inherit very nice features. Furthermore, they are compact, orientable, connected and irreducible. It is very common when studying 3–manifolds to deal with embedded surfaces: we denote by S^2 the 2–sphere, D^2 the disk, A the annulus, T^2 the torus and $(T^2)^{\#2}$ the genus 2 oriented surface. We denote by P_n the n–punctured disk, whose boundary splits as an external component $\partial_{\mathrm{ext}} P_n$ and n internal components $\partial_{\mathrm{int}} P_n$. As for common 3–manifolds, we note $B = J \times D^2$ the cylinder, homeomorphic to a 3–ball D^3, $H_n = P_n \times I$ the n–handlebody and $C_f \subset B$ the complement of a fat long knot or a fat 2–string link f. The boundary of C_f is a torus when f is a fat long knot, and a 2–torus when f is a fat 2–string

link. A recurring procedure in the upcoming proofs is the cutting of C_f along essential surfaces. We define the latter now.

Definition 4.1 Let S be a (not necessarily connected) orientable surface embedded in an orientable 3–manifold M properly (ie $S \cap \partial M = \partial S$). A disk $D \subset M$ with $D \cap S = \partial D$ is said to be a *compressing disk for* S if its boundary does not bound a disk in S. A surface that admits a compressing disk is said to be *compressible*, and a surface different from S^2 or D^2 admitting no compressing disk is said to be *incompressible*.

Definition 4.2 Let S be a bordered surface properly embedded in a 3–manifold M. A *∂–compressing disk for S* is a disk $D \subset M$ whose boundary consists of two arcs α and β with $\alpha \subset S$ and $\beta \subset \partial M$, whose interior is disjoint from S and ∂M, such that there is no arc γ in ∂S such that $\gamma \cup \alpha$ bounds a disk in S. A surface that admits a ∂–compressing disk is said to be *∂–compressible*. Otherwise, it is *∂–incompressible*.

Definition 4.3 A properly embedded surface S in a 3–manifold M is said to be *∂–parallel* if it can be isotoped to a piece of ∂M.

Definition 4.4 A properly embedded orientable surface S in a 3–manifold M is *essential* if one of the following holds:

(i) S is a sphere and does not bound a ball.

(ii) S is a disk whose boundary does not bound a disk in ∂M.

(iii) S is not a sphere nor a disk, it is incompressible, ∂–incompressible and not ∂–parallel.

Spaces of embeddings of incompressible surfaces have been extensively studied by Hatcher in [14]. He describes in his paper how the homotopy type of such a space depends on S. This result will be used repeatedly so we formulate a precise version here.

Theorem 4.5 (Hatcher [14]) *Let M be an orientable compact connected irreducible 3–manifold and $S \hookrightarrow M$ an essential orientable compact surface in M. Let* $\mathrm{Emb}(S, M, \partial S)$ *be the space of embeddings of S in M whose values at ∂S are fixed. Then the component* $\mathrm{Emb}(S, M, \partial S)_S$ *of $S \hookrightarrow M$ in* $\mathrm{Emb}(S, M, \partial S)$ *is weakly contractible unless S is closed and the fiber of a bundle structure on M, or if S is a torus. In these exceptional cases $\pi_i \mathrm{Emb}(S, M) = 0$ for all $i > 1$. In the bundle case, the inclusion of the subspace consisting of embeddings with image a fiber induces an isomorphism on π_1. When S is a torus but not the fiber of a bundle structure, the inclusion* $\mathrm{Diff}(S) \hookrightarrow \mathrm{Emb}(S, M)$ *obtained by precomposing $S \hookrightarrow M$ induces an isomorphism on π_1.*

Another tool of 3–manifold theory that will come in handy is the JSJ–decomposition. It provides a way to cut an irreducible manifold into simpler ones. The cuts are performed along essential tori, but if one keeps on cutting a manifold until no such torus is available, the obtained decomposition might not be unique. A manifold that admits no essential torus is said to be *atoroidal*. In order to get a unique decomposition, one must agree not to cut the pieces that are Seifert-fibered. The latter are manifolds consisting of disjoint parallel circles forming a particular fibering. The precise definition of this fibering is looser than the notion of fiber bundle with fiber S^1. It is specified for example in [15]. The decomposition theorem we use is the following:

Theorem 4.6 (Jaco, Shalen and Johannson [19; 20]) *Every orientable, compact, irreducible 3–manifold M contains a collection of embedded, incompressible tori T so that if one removes an open tubular neighborhood of T from M, the outcome is a disjoint union of Seifert-fibered and atoroidal manifolds. Moreover, a minimal collection of such tori is unique up to isotopy.*

The minimal collection of tori T from Theorem 4.6 (or sometimes its isotopy class) is called the JSJ–decomposition of M. In the case where ∂M consists of a single component, the piece of the cut M containing ∂M is called the root of the decomposition. The tori of T bounding the root are referred to as the base-level tori of T.

Our main concern while studying an orientable compact 3–manifold M will actually be the homotopy type of the group of its boundary-fixing diffeomorphisms $\mathrm{Diff}(M, \partial M)$. More precisely, we are interested of the subgroup $\mathrm{Diff}_\mathrm{d}(M, \partial M)$ consisting of the diffeomorphisms whose derivatives at ∂M agree with those of the identity. This extra condition is relevant for our work because it enables one to postcompose a fat long knot by an element of $\mathrm{Diff}_\mathrm{d}(B, \partial B)$ and still end up with a fat long knot. The main ingredient we use to prove the three upcoming freeness theorems is the following proposition:

Proposition 4.7 *Let M be an orientable compact connected irreducible 3–manifold and S an essential surface which cuts M into pieces M_i, such that the component $\mathrm{Emb}(S, M, \partial S)_S$ is weakly contractible and stable under the postcomposition action of $\mathrm{Diff}(M, \partial M)$. Then the inclusion*

$$\prod_i \mathrm{Diff}_\mathrm{d}(M_i, \partial M_i) \hookrightarrow \mathrm{Diff}_\mathrm{d}(M, \partial M)$$

is a weak homotopy equivalence.

Proof Thanks to the stability condition, we have a well-defined postcomposition map $\mathrm{Diff}(M,\partial M)\to \mathrm{Emb}(S,M,\partial S)_S$. Restriction maps such as this one have been shown to be locally trivial, and in particular fibrations. This problem, as well as the local triviality of the restriction map $\mathrm{Emb}(M,N)\to\mathrm{Emb}(M',N)$ for a submanifold $M'\subset M$, has been treated in several articles: in [26] for the case of closed manifolds and in [9] for the case of bordered manifolds. A more recent exposition is provided in the third section of [21]: the precise result we use here is formulated as Corollary 3.7 in [21]. The fiber over S of this map is the subgroup of diffeomorphisms fixing S, ie

$$\mathrm{Diff}(M,\partial M\cup S)\hookrightarrow \mathrm{Diff}(M,\partial M)\longrightarrow \mathrm{Emb}(S,M,\partial S)_S.$$

The base space is weakly contractible so the inclusion of the fiber is a weak homotopy equivalence. We are left to add the derivative condition on the diffeomorphisms.

It is proved in Kupers' book on diffeomorphism groups [22] that the inclusion of the subgroup $\mathrm{Diff}_{\mathrm{d}}(N,\partial N)\hookrightarrow\mathrm{Diff}(N,\partial N)$ is a weak homotopy equivalence for every compact manifold N. This justifies the bottom left equivalence in the diagram of inclusions

$$\begin{array}{ccccc}
\mathrm{Diff}_{\mathrm{d}}(M,\partial M\cup S) & \hookrightarrow & \mathrm{Diff}(M,\partial M\cup S) & \xhookrightarrow{\simeq} & \mathrm{Homeo}(M,\partial M\cup S)\\
\| & & & & \| \\
\prod_i \mathrm{Diff}_{\mathrm{d}}(M_i,\partial M_i) & \xhookrightarrow{\simeq} & \prod_i \mathrm{Diff}(M_i,\partial M_i) & \xhookrightarrow{\simeq} & \prod_i \mathrm{Homeo}(M_i,\partial M_i)
\end{array}$$

while the right horizontal equivalences come from works of Cerf in [9]. The two-out-of-three rule assures us that the top left inclusion is a weak equivalence as well. This same rule in the diagram

$$\begin{array}{ccc}
\mathrm{Diff}_{\mathrm{d}}(M,\partial M\cup S) & \hookrightarrow & \mathrm{Diff}_{\mathrm{d}}(M,\partial M)\\
\downarrow\simeq & & \downarrow\simeq \\
\mathrm{Diff}(M,\partial M\cup S) & \xhookrightarrow{\simeq} & \mathrm{Diff}(M,\partial M)
\end{array}$$

concludes the proof. □

The need to study these diffeomorphism groups arises from the following classical result in modern knot theory:

Proposition 4.8 *Let f be a fat long knot or a fat 2–string link. Then the component $\widehat{\mathcal{K}}_f$ or $\widehat{\mathcal{L}}_f$ of f in $\widehat{\mathcal{K}}$ or $\widehat{\mathcal{L}}$ is a model for the classifying space of $\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f)$. Moreover, it is a $K(G,1)$.*

Proof We treat the case where f is a fat long knot, the other one is treated identically. When B is the solid cylinder and C_f the complement of f in B, we have the inclusion and restriction maps

$$\mathrm{Diff}_\mathrm{d}(C_f, \partial C_f) \hookrightarrow \mathrm{Diff}_\mathrm{d}(B, \partial B) \longrightarrow \widehat{\mathcal{K}}_f.$$

The application on the right-hand side precomposes a diffeomorphism by f and is a fibration thanks to [21, Corollary 3.7]. Now, $\mathrm{Diff}_\mathrm{d}(B, \partial B) \simeq \mathrm{Diff}(B, \partial B)$ is weakly contractible, as proved in [13]. Thus, $\mathrm{Diff}_\mathrm{d}(C_f, \partial C_f)$ acts properly and freely on a contractible space, and the quotient of this action can be identified to $\widehat{\mathcal{K}}_f$, so $\widehat{\mathcal{K}}_f$ is a model for $B\mathrm{Diff}_\mathrm{d}(C_f, \partial C_f)$. Moreover, spaces of diffeomorphisms of orientable Haken bordered 3–manifolds which preserve the boundary pointwise always have vanishing higher homotopy groups by [14, Theorem 2]. Thus, the long exact sequence in homotopy coming from the fibration above assures us that $\widehat{\mathcal{K}}_f$ is a $K(\pi_0\mathrm{Diff}_\mathrm{d}(C_f, \partial C_f), 1)$. □

4.2 Budney's freeness theorem

We now recall Budney's main theorem in [5], which is a freeness statement about the space of fat long knots as a $\mathcal{C}_2$–algebra. Recall from Section 1.1 that the prime fat long knots are denoted by $\widehat{\mathcal{P}} \subset \widehat{\mathcal{K}}$, and that they are the embeddings whose isotopy classes are prime elements in the monoid $\pi_0\widehat{\mathcal{K}}$. The result we present appears as Theorem 11 in Budney's paper, and we only dispense a sketch of proof here, as the complete proof is fairly long.

Theorem 4.9 (Budney [5]) *The restriction of the structure map*

$$\kappa : \mathcal{C}_2[\widehat{\mathcal{P}}] \to \widehat{\mathcal{K}}$$

from Theorem 3.3 is a homotopy equivalence.

Sketch of Proof Thanks to Corollary 1.8, Proposition 2.7 and the fact that applying π_0 to the structure map $\kappa : \mathcal{C}_2 \to \mathcal{E}_{\widehat{\mathcal{K}}}$ endows $\pi_0\widehat{\mathcal{K}}$ with its usual monoid structure, we are assured that κ induces a bijection on components. Thus, we are left to prove that it is a homotopy equivalence on each of these components.

On the component of the unknot, κ restricts to the map $\mathcal{C}_2(0) \times \widehat{\mathcal{P}}^{\times 0} \to \widehat{\mathcal{K}}_{\mathrm{id}_{\mathbb{R}\times D^2}}$. The complement of an unknot is a 1–handlebody H_1, and the diffeomorphism group $\mathrm{Diff}_\mathrm{d}(H_1, \partial H_1) \simeq \mathrm{Diff}(H_1, \partial H_1)$ is contractible. Proposition 4.8 therefore gives the contractibility of $\widehat{\mathcal{K}}_{\mathrm{id}_{\mathbb{R}\times D^2}} = B\mathrm{Diff}_\mathrm{d}(H_1, \partial H_1)$, so we have an equivalence in this case.

On the component of a prime knot $f \in \widehat{\mathcal{P}}$, κ restricts to the map $\mathcal{C}_2(1)\times\widehat{\mathcal{P}}_f \to \widehat{\mathcal{P}}_f = \widehat{\mathcal{K}}_f$. The homotopy retracting $\mathcal{C}_2(1)$ onto the identity little cube shows that this map is an equivalence as well.

Suppose now that f is a composite knot $f = f_1\#\cdots\# f_n$. Budney proved in [4] that the base-level tori T of the JSJ–decomposition of the complement of f split C_f into $n+1$ pieces: the complements of the prime factors C_{f_i} and a root homeomorphic to $S^1 \times P_n$. One would like to apply Proposition 4.7 in order to split $\mathrm{Diff}_\partial(C_f, \partial C_f)$ as a product of diffeomorphism groups involving each $\mathrm{Diff}_\partial(C_{f_i}, \partial C_{f_i})$. But, cutting along T is not possible here as the component $\mathrm{Emb}(T^2 \amalg \cdots \amalg T^2, C_f)_T$ is not contractible by Theorem 4.5. It is also not stable under the postcomposition action of $\mathrm{Diff}(C_f, \partial C_f)$. Indeed, when T_i is the torus bounding C_{f_i}, two tori T_i and T_j can be permuted by some diffeomorphism of C_f if and only if f_i and f_j are isotopic. Let Σ_f be the subgroup of Σ_n preserving the partition of $\{1,\dots,n\}$ given by $i \sim j$ if and only if f_i and f_j are isotopic. Then, by considering the components of $\mathrm{Emb}(T^2 \amalg \cdots \amalg T^2, C_f)$ where the image of the i^{th} torus is isotopic to a T_j with $i \sim j$, and by quotienting out these components by the parametrization of each torus, one gets a space homotopy equivalent to Σ_f. It is stable under the postcomposition action of $\mathrm{Diff}(C_f, \partial C_f)$ so one can use an argument similar to the proof of Proposition 4.7 to split $\mathrm{Diff}_\partial(C_f, \partial C_f)$ into a product involving each C_{f_i}. Namely, after some manipulations on the diffeomorphisms, Budney manages to fit $\mathrm{Diff}_\partial(C_f, \partial C_f)$ up to homotopy in a fibration

$$\mathrm{KDiff}(P_n, \partial P_n) \times \prod_i \mathrm{Diff}_\partial(C_{f_i}, \partial C_{f_i}) \to \mathrm{Diff}_\partial(C_f, \partial C_f) \to \Sigma_f,$$

for some subgroup $\mathrm{KDiff}(P_n, \partial P_n)$ homotopy equivalent to the pure braid group on n strands KB_n. Since $BKB_n = \mathrm{conf}_n(J^2) \simeq \mathcal{C}_2(n)$, applying the classifying space functor B leads to

$$\begin{aligned}
\widehat{\mathcal{K}}_f &\simeq B\mathrm{Diff}_\partial(C_f, \partial C_f) \\
&\simeq B\,\mathrm{KDiff}(P_n, \partial P_n) \times_{\Sigma_f} \prod_i B\mathrm{Diff}_\partial(C_{f_i}, \partial C_{f_i}) \\
&\simeq \mathcal{C}_2(n) \times_{\Sigma_f} \prod_i \widehat{\mathcal{K}}_{f_i} = \mathcal{C}_2[\widehat{\mathcal{P}}]_f.
\end{aligned}$$

At this stage, this equivalence is merely an abstract map. But the vast majority of the group morphisms we met are inclusion-based, so Budney manages to show that this equivalence coincides with κ via explicit models. Weak equivalences are finally promoted to strong ones via an application of Whitehead's theorem. □

This result can be thought of as a generalization of Schubert's theorem: the connected sum operation # is extended to an algebraic structure on the space level of $\widehat{\mathcal{K}}$, and the isomorphism $\pi_0\widehat{\mathcal{K}} = \mathcal{C}om[\pi_0\widehat{\mathcal{P}}]$ is extended to the $\mathcal{C}_2$–equivariant homotopy equivalence $\widehat{\mathcal{K}} \simeq \mathcal{C}_2[\widehat{\mathcal{P}}]$. Note however that Budney uses Corollary 1.8 in the very first sentence of the proof, so that this generalization is in no way an alternative argument for Schubert's result.

4.3 Burke and Koytcheff's freeness theorem

We now get to Burke and Koytcheff's result about fat 2–string links. Recall from Section 1.2 that $\widehat{\mathcal{Q}}^0 \subset \widehat{\mathcal{L}}^0$ denotes the fat 2–string links which are prime but not in the image of one of the maps $\hat{\varphi}^s$, $s \in \{\uparrow, \downarrow, \updownarrow\}$. The theorem we present here is [8, Theorem 6.8]. We again provide a quick sketch of proof, as the ideas involved will reappear in the proof of our main result, Theorem 4.11.

Theorem 4.10 (Burke and Koytcheff [8]) *Let $\widehat{\mathcal{S}}^0$ be the subspace of $\widehat{\mathcal{L}}^0$ consisting of the fat 2–string links whose prime factors lie in $\widehat{\mathcal{Q}}^0$. Then, the restriction of the structure map*

$$\lambda : \mathcal{C}_1[\widehat{\mathcal{Q}}^0] \to \widehat{\mathcal{S}}^0$$

from Theorem 3.6 is a homotopy equivalence.

Sketch of proof Thanks to Theorem 1.10, we are assured that λ induces a bijection on components. We are therefore again left to show that it is an equivalence on each of these components.

On the component of the trivial link, λ restricts to the map $\mathcal{C}_1(0) \times (\widehat{\mathcal{Q}}^0)^{\times 0} \to \widehat{\mathcal{L}}^0_{\mathrm{id}_{\mathbb{R}} \times \iota}$. The complement of $\mathrm{id}_{\mathbb{R}} \times \iota$ is a 2–handlebody H_2, and the diffeomorphism group $\mathrm{Diff}_\partial(H_2, \partial H_2) \simeq \mathrm{Diff}(H_2, \partial H_2)$ is contractible so we can conclude as in the case of knots.

Let now f be a nontrivial element of $\widehat{\mathcal{S}}^0$ and $f = f_1 \# \cdots \# f_n$ its decomposition in noncentral prime fat 2–string links. According to [2, Theorem 4.1], there are $n-1$ twice-punctured disks separating C_f into the complements of the f_i. Moreover, as shown in the fourth step of the proof of this same theorem, these disks are unique up to isotopy. This enables us to split $\mathrm{Diff}_\partial(C_f, \partial C_f)$ as the product $\prod_i \mathrm{Diff}_\partial(C_{f_i}, \partial C_{f_i})$ thanks to Proposition 4.7. Applying the classifying space functor yields the equivalences

$$\widehat{\mathcal{S}}^0_f = B\mathrm{Diff}_\partial(C_f, \partial C_f) \simeq \prod_i B\mathrm{Diff}_\partial(C_{f_i}, \partial C_{f_i}) \simeq \prod_i \widehat{\mathcal{L}}_{f_i}.$$

When we denote by $\mathcal{C}_1(n)_{(1,\dots,n)}$ the (contractible) component of $\mathcal{C}_1(n)$ where the intervals appear from left to right in the order $(1,\dots,n)$, we can go further and write

$$\widehat{\mathcal{S}}^0_f \simeq \prod_i \widehat{\mathcal{L}}_{f_i} \simeq \mathcal{C}_1(1)_{(1,\dots,n)} \times \prod_i \widehat{\mathcal{L}}_{f_i} = \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_f\,.$$

At this stage, the equivalence is still given by an abstract map but it is easy to keep track of the identifications and see that it coincides with λ. Weak equivalences are finally promoted to strong ones via an application of Whitehead's theorem. □

Again, this result is a generalization of Theorem 1.10 as its provides a free algebraic structure on the space level of $\widehat{\mathcal{S}}^0$, which descends to the usual free monoid on the basis $\widehat{\mathcal{Q}}^0$ on isotopy classes.

4.4 Fat long knots and 2–string links form a free $\mathcal{SCL}$–algebra

We combine in this last subsection the theorems of Budney, Burke and Koytcheff presented above to prove a freeness result for the whole space of fat 2–string links. Namely, we prove:

Theorem 4.11 *The restriction of the structure map*

$$\mu\colon \mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}] \to (\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$$

from Theorem 3.9 is a homotopy equivalence.

Just as in the preceding subsections, the proof mainly consists in reducing ourselves to each connected component and splitting the diffeomorphism group of the complement of a link along suitable surfaces. We state two technical lemmas to prepare this cutting process, then proceed to the proof of the theorem.

Lemma 4.12 *Let f be a fat 2–string link decomposing as*

$$f^o \# \hat{\varphi}^{\uparrow}(f^{\uparrow}) \# \hat{\varphi}^{\downarrow}(f^{\downarrow}) \# \hat{\varphi}^{\updownarrow}(f^{\updownarrow})$$

for some element f^o in $\widehat{\mathcal{S}}^0$ and fat long knots f^s, $s \in \{\uparrow, \downarrow, \updownarrow\}$. Then, the three vertical twice-punctured disks D cutting C_f into C_{f^o}, $C_{\hat{\varphi}^{\uparrow}(f^{\uparrow})}$, $C_{\hat{\varphi}^{\downarrow}(f^{\downarrow})}$ and $C_{\hat{\varphi}^{\updownarrow}(f^{\updownarrow})}$ are unique up to isotopy fixing the boundary. Moreover, the component $\mathrm{Emb}(D, C_f, \partial D)_D$ *is weakly contractible and stable under the postcomposition action of* $\mathrm{Diff}(C_f, \partial C_f)$.

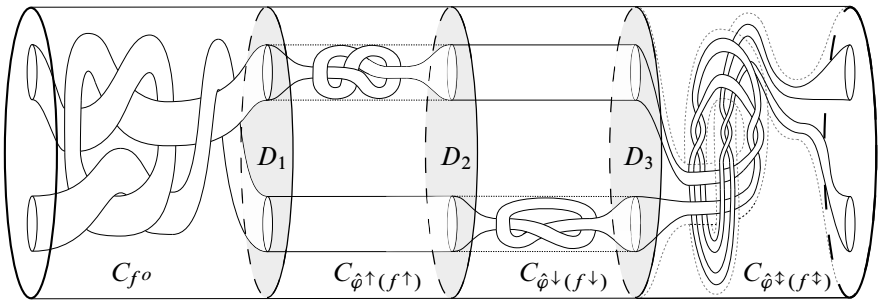

Figure 15: Illustration of the three disks $D = D_1 \sqcup D_2 \sqcup D_3$ in C_f.

Proof The uniqueness statement is proved in the steps 1 and 3 of Blair, Burke and Koytcheff's proof of their Theorem 4.1 in [2]. Applying a diffeomorphism of $\mathrm{Diff}(C, \partial C_f)$ to the punctured disks D does not change the fact that they split C_f into pieces homeomorphic to the complement of f^o and $\hat{\varphi}^s(f^s)$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$. Thus, the component $\mathrm{Emb}(D, C_f, \partial D)_D$ is stable under the postcomposition action of this diffeomorphism group. Its contractibility immediately follows from Hatcher's work on incompressible surfaces (Theorem 4.5). □

Lemma 4.13 *Let f be a fat long knot and $\hat{\varphi}^s(f)$ the central link obtained from f for $s \in \{\uparrow, \downarrow, \updownarrow\}$. Consider the annulus A_s in the complement $C_{\hat{\varphi}^s(f)}$ specified by*

(i) $A_\uparrow = (\mathrm{id}_\mathbb{R} \times \iota)((J \times \partial D^2) \sqcup \varnothing)$,

(ii) $A_\downarrow = (\mathrm{id}_\mathbb{R} \times \iota)(\varnothing \sqcup (J \times \partial D^2))$,

(iii) $A_\updownarrow = f(J \times \partial D^2)$.

Then the isotopy class of A_s is stable under the postcomposition action of

$$\mathrm{Diff}(C_{\hat{\varphi}^s(f)}, \partial C_{\hat{\varphi}^s(f)})$$

and the component $\mathrm{Emb}(A, C_{\hat{\varphi}^s(f)}, \partial A)_{A_s}$ is weakly contractible.

Proof We first treat the case where $s = \uparrow$. Consider a horizontal disk $E \subset C_{\hat{\varphi}^\uparrow(f)}$ separating the two strands of $\hat{\varphi}^\uparrow(f)$. Cutting along E yields two manifolds: an upper piece containing $A_\uparrow$, homeomorphic to C_f, and a lower one that is a 1–handlebody H_1. Any two disks in $C_{\hat{\varphi}^\uparrow(f)}$ sharing their boundary are isotopic because $C_{\hat{\varphi}^\uparrow(f)}$ is irreducible. Therefore, given a diffeomorphism g in $\mathrm{Diff}(C_{\hat{\varphi}^\uparrow(f)}, \partial C_{\hat{\varphi}^\uparrow(f)})$, there is an isotopy from $g(E)$ to E. It can be extended to a boundary preserving ambient isotopy and postcomposing g by the latter shows that we may assume that $g(E) = E$. In other

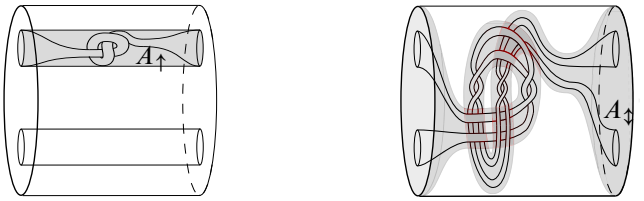

Figure 16: Illustration of the annuli $A_\uparrow$ and $A_\updownarrow$.

words, g preserves the cut and in particular $g(A_\uparrow)$ lies in the upper piece. But $A_\uparrow$ is boundary parallel after the cut, so it is unique up to boundary-fixing isotopy in its piece, which concludes the proof in this case. The weak contractibility statement immediately follows from Theorem 4.5. The case where $s = \downarrow$ is treated the exact same way.

We now deal with the case where $s = \updownarrow$. The proof of the stability statement uses the JSJ–decomposition recalled in Theorem 4.6, especially the uniqueness part. The idea is to show that the JSJ–decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$ admits a single base-level torus $T_\updownarrow$, which must be unique up to isotopy, and that $A_\updownarrow$ is a suitable piece of it.

Let $A_\updownarrow{}'$ be the annulus $J \times \partial D^2 \subset \partial C_{\hat{\varphi}^\updownarrow(f)}$. The two annuli $A_\updownarrow$ and $A_\updownarrow{}'$ share their boundary so that their union is a torus in $C_{\hat{\varphi}^\updownarrow(f)}$. Let $T_\updownarrow$ be the torus obtained by pushing $A_\updownarrow \cup A_\updownarrow{}'$ in the interior of $C_{\hat{\varphi}^\updownarrow(f)}$. It is essential and cuts $C_{\hat{\varphi}^\updownarrow(f)}$ into two manifolds, one containing $\partial C_{\hat{\varphi}^\updownarrow(f)}$, $A_\updownarrow$ and $A_\updownarrow{}'$ that we denote by V, the other one homeomorphic to C_f. We now proceed to show that $T_\updownarrow$ is the base-level torus of the JSJ–decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$.

Claim 1 *Let P_2 be a twice-punctured disk and $\gamma_\updownarrow$ a curve in the interior of P_2 parallel to the external boundary circle. Then V is homeomorphic to a 2–handlebody $J \times P_2$ deprived of a solid torus that is a tubular neighborhood of $0 \times \gamma_\updownarrow$.*

Proof This unknotting process is similar to Budney's "untwisted reembedding" described in the beginning of his paper on knot complements [4]. Cutting V along $A_\updownarrow$ results in two pieces: an external one containing $T_\updownarrow$ and an internal one that is a (knotted) 2–handlebody H_2. The torus $T_\updownarrow$ is boundary parallel in the external part, so this piece is a fattened torus $T^2 \times I$. This shows that V is the manifold obtained by gluing a 2–handlebody H_2 and a fattened torus $T^2 \times I$ along specific annuli in their boundaries. This description also matches our new model for V so we are done. □

This new model makes a lot of considerations easier since it forgets all the complexity of the knotting of f. We call the two unknotted annuli $J \times \partial_{\text{int}} P_2$ the *strands* of ∂V.

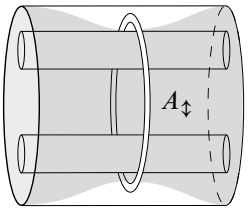

Figure 17: Illustration of the new model for V and $A_\updownarrow$.

Through this identification, $A_\updownarrow$ turns into the annulus bounding a neighborhood of the two strands not containing $T_\updownarrow$, as illustrated in Figure 17. The second annulus $A_\updownarrow{}'$ remains in the boundary as $J \times \partial D^2$.

Claim 2 *V is atoroidal.*

Proof In many cases including that of V, the definition of an atoroidal 3–manifold can be reformulated algebraically using the fundamental group. More precisely, one defines a peripheral subgroup of a 3–manifold M to be a subgroup of $\pi_1 M$ that lies in the image of the inclusion of a boundary component, then declares M to be atoroidal if every subgroup of $\pi_1 M$ isomorphic to $\mathbb{Z}^{\times 2}$ is conjugate to a peripheral subgroup. The equivalence between the two definitions is alluded for example in the beginning of [1], but the implication we are about to use follows from Corollary 5.5 in Waldhausen's article [33].

We start by computing $\pi_1 V$. The application of van Kampen's theorem summarized in Figure 18 gives

$$\pi_1 V = \langle \alpha_1, \alpha_2, \beta \mid \alpha_1 \beta \alpha_1^{-1} = \alpha_2 \beta \alpha_2^{-1} \rangle.$$

We can further write

$$\pi_1 V \cong \langle a, b, c \mid ab = ba \rangle = \mathbb{Z}^{\times 2} * \mathbb{Z}$$

via $\alpha_1 \mapsto c$, $\alpha_2 \mapsto ca^{-1}$ and $\beta \mapsto b$.

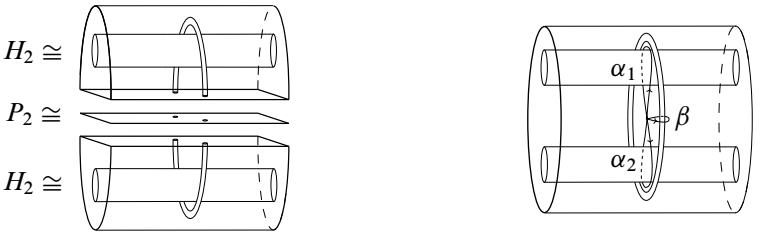

Figure 18: Applying van Kampen's theorem to compute $\pi_1 V$.

The inclusion of the toric boundary component $T_{\updownarrow} \subset V$ has image in π_1 the subgroup generated by a and b. It corresponds to the $\mathbb{Z}^{\times 2}$ factor under the isomorphism $\pi_1 V \cong \mathbb{Z}^{\times 2} * \mathbb{Z}$. We prove that any subgroup of $\mathbb{Z}^{\times 2} * \mathbb{Z}$ isomorphic to $\mathbb{Z}^{\times 2}$ is conjugate to a subgroup of the latter. Consider an arbitrary injection $\mathbb{Z}^{\times 2} \hookrightarrow \mathbb{Z}^{\times 2} * \mathbb{Z}$ and denote by x and y the images of $(0,1)$ and $(1,0)$. These x and y commute, they are nontrivial and they are not allowed to be powers of some third element. Now, applying Theorem 4.5 on page 209 of the book [23] on groups and presentations yields the following three possibilities for x and y:

(i) x or y may be trivial;

(ii) if neither x nor y is trivial, but x is in the conjugate of a factor, then y is in that same conjugate of a factor;

(iii) if neither x nor y is in a conjugate of a factor, then they are powers of a third element of $\mathbb{Z}^{\times 2} * \mathbb{Z}$.

Thanks to our observations, (i) and (iii) are ruled out and x and y must lie in the same conjugate of a factor of $\mathbb{Z}^{\times 2} * \mathbb{Z}$. The $\mathbb{Z}$ factor does not admit any subgroup isomorphic to $\mathbb{Z}^{\times 2}$, so we are done. □

At this point, we showed that $T_{\updownarrow}$ and the tori of the JSJ–decomposition of C_f cut $C_{\hat{\varphi}^{\updownarrow}(f)}$ into atoroidal and Seifert-fibered pieces. Finally, $T_{\updownarrow}$ cannot be removed to obtain a smaller decomposition because V can never be part of a Seifert-fibered manifold, having a boundary component homeomorphic to a 2–torus. This shows that $T_{\updownarrow}$ and the tori of the JSJ–decomposition of C_f form a minimal decomposition of $C_{\hat{\varphi}^{\updownarrow}(f)}$ into atoroidal and Seifert-fibered manifolds, which proves that this collection is the JSJ–decomposition of $C_{\hat{\varphi}^{\updownarrow}(f)}$. The root V is bounded by $\partial C_{\hat{\varphi}^{\updownarrow}(f)}$ and $T_{\updownarrow}$, so $T_{\updownarrow}$ is the only base-level torus. Its image is therefore unique up to isotopy.

Consider now $g \in \mathrm{Diff}(C_{\hat{\varphi}^{\updownarrow}(f)}, \partial C_{\hat{\varphi}^{\updownarrow}(f)})$. Thanks to the work above, there is an isotopy from $g(T_{\updownarrow})$ to $T_{\updownarrow}$. It can be extended to a boundary-fixing ambient isotopy, and postcomposing g by the latter shows that we may assume that $g(T_{\updownarrow}) = T_{\updownarrow}$. In other words, g preserves the cut along $T_{\updownarrow}$ and $g(A_{\updownarrow})$ lies in V. Let us now take a look at the (boundary-fixing) isotopy classes of annuli in V with boundary $\partial A_{\updownarrow}$. We prove that there are only two of them: the one of $A_{\updownarrow}$ and the one of ${A_{\updownarrow}}'$. Let $A \subset V$ be an arbitrary annulus with adequate boundary. We may assume that the interiors of A and ${A_{\updownarrow}}'$ are disjoint because ${A_{\updownarrow}}'$ is ∂–parallel. Now, $A \cup {A_{\updownarrow}}'$ is an embedded torus in V. Its image in π_1 is generated by $\alpha_2^{-1}\alpha_1 = a$ and some other element y that commutes with a. Using [23, Theorem 4.5] again, we see that y is either trivial or in

the $\mathbb{Z}^{\times 2}$ factor of $\mathbb{Z}^{\times 2} * \mathbb{Z} \cong \pi_1 V$. If y is trivial or a power of a, then $A \cup A_\updownarrow{}'$ bounds a solid torus and the two annuli are isotopic. If y is not just a power of a, then $A \cup A_\updownarrow{}'$ is incompressible and must parallel to $T_\updownarrow$ by Claim 2. In this situation, A is isotopic to $A_\updownarrow$. The two annuli cannot be permuted by a diffeomorphism of V because $A_\updownarrow{}'$ is ∂–parallel and $A_\updownarrow$ is not. This shows that $g|_V(A_\updownarrow)$ is isotopic to $A_\updownarrow$ via an isotopy that fixes the boundary in V. This concludes this second case. The weak contractibility of the component in the embedding space again follows from Theorem 4.5. □

We now implement all the tools at our disposal to complete the proof of Theorem 4.11.

Proof of Theorem 4.11 Thanks to Theorems 1.8 and 1.10 and Proposition 2.7, we are assured that μ induces a bijection on components. Thus, we are left to prove that it is a homotopy equivalence on each of these components.

On the component of the unlink, μ restricts to the map

$$\mathcal{SCL}(\varnothing; o) \times (\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}})^{\times \varnothing} \to \widehat{\mathcal{L}}_{\mathrm{id}_\mathbb{R} \times \iota}.$$

The space $\mathcal{SCL}(\varnothing; o)$ consists of a single point. The complement of the unlink is a 2–handlebody H_2, and the diffeomorphism group $\mathrm{Diff}_\partial(H_2, \partial H_2) \simeq \mathrm{Diff}(H_2, \partial H_2)$ is contractible. Proposition 4.8 then gives the contractibility of $\widehat{\mathcal{L}}_{\mathrm{id}_\mathbb{R} \times \iota} = B\mathrm{Diff}_\partial(H_2, \partial H_2)$, so we have an equivalence in this case.

On the component of a fat long knot or an element of $\widehat{\mathcal{S}}^0$, Theorems 4.9 and 4.10 imply the result because the action of $\mathcal{SCL}$ restricts to κ and λ on these components.

Now let f be a nontrivial fat 2–string link. The action map μ restricts on the component of f to the map $\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f \to \widehat{\mathcal{L}}_f$. Suppose f decomposes as the concatenation $f^o \# \hat{\varphi}^\uparrow(f^\uparrow) \# \hat{\varphi}^\downarrow(f^\downarrow) \# \hat{\varphi}^\updownarrow(f^\updownarrow)$ for some element f^o of $\widehat{\mathcal{S}}^0$ and some fat long knots f^s for $s \in \{\uparrow, \downarrow, \updownarrow\}$. We denote by $f^o = \#_{i \le |f|_o} f_i^o$ and $f^s = \#_{i \le |f|_s} f_i^s$ the prime decompositions of f^o and f^s. Then, one readily checks with the usual model presented in Section 2.2 that the component $\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f$ in the free algebra is given by

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)} \times_{(\Sigma_{f^\uparrow} \times \Sigma_{f^\downarrow} \times \Sigma_{f^\updownarrow})} \prod_{i \le |f|_o} \widehat{\mathcal{L}}_{f_i^o} \times \prod_s \prod_{i \le |f|_s} \widehat{\mathcal{K}}_{f_i^s},$$

where $\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)}$ is the component of

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)$$

where the cubes indexed by o appear from left to right in the order $(1, \dots, |f|_o)$ and where Σ_{f^s} is the subgroup of $\Sigma_{|f|_s}$ preserving the partition specified by $i \sim j$

if and only if f_i^s is isotopic to f_j^s. This group acts on the cubes indexed by s in $\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)}$ and permutes the entries in $\prod_{i\le|f|_s}\widehat{\mathcal{K}}_{f_i^s}$. The inclusion

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)} \hookrightarrow \mathcal{C}_2^\circ(|f|_o)_{(1,\dots,|f|_o)} \times \prod_s \mathcal{C}_2(|f|_s)$$

is a homotopy equivalence, so that we have, by rearranging terms in the product, the natural equivalences

$$\begin{aligned}\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f &\simeq \left[\mathcal{C}_2^\circ(|f|_o)_{(1,\dots,|f|_o)} \times \prod_i \widehat{\mathcal{L}}_{f_i^o}\right] \times \prod_s \left[\mathcal{C}_2(|f|_s) \times_{\Sigma_{f^s}} \prod_i \widehat{\mathcal{K}}_{f_i^s}\right] \\ &\simeq \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_{f^o} \times \prod_s \mathcal{C}_2[\widehat{\mathcal{P}}]_{f^s}.\end{aligned}$$

On the other hand, thanks to Proposition 4.8, we know that $\widehat{\mathcal{L}}_f$ is a model for the classifying space of $\mathrm{Diff}_\partial(C_f, \partial C_f)$. The three vertical twice-punctured disks D from Lemma 4.12 splitting C_f as C_{f^o}, $C_{\hat\varphi^\uparrow(f^\uparrow)}$, $C_{\hat\varphi^\downarrow(f^\downarrow)}$ and $C_{\hat\varphi^\updownarrow(f^\updownarrow)}$ are stable under the action of $\mathrm{Diff}(C_f, \partial C_f)$ and have their component in $\mathrm{Emb}(D, C_f, \partial D)$ weakly contractible. Therefore, Proposition 4.7 gives the inclusion-based weak equivalence

$$\mathrm{Diff}_\partial(C_{f^o}, \partial C_{f^o}) \times \prod_s \mathrm{Diff}_\partial(C_{\hat\varphi^s(f^s)}, \partial C_{\hat\varphi^s(f^s)}) \hookrightarrow \mathrm{Diff}_\partial(C_f, \partial C_f).$$

Now, Lemma 4.13 states that the three annuli A_s also satisfy the conditions of Proposition 4.7. They each split $C_{\hat\varphi^s(f^s)}$ into a 2–handlebody H_2 and a manifold diffeomorphic to C_{f^s}. Actually, this second piece is precisely the image of C_{f^s} under $\mathrm{id}_\mathbb{R} \times \iota$ when $s \in \{\uparrow, \downarrow\}$ and $C_{f^\updownarrow}$ itself when $s = \updownarrow$. The diffeomorphism group $\mathrm{Diff}_\partial(H_2, \partial H_2)$ is contractible so we get the further natural equivalences

$$\mathrm{Diff}_\partial(C_{f^s}, \partial C_{f^s}) \hookrightarrow \mathrm{Diff}_\partial(C_{f^s}, \partial C_{f^s}) \times \mathrm{Diff}_\partial(H_2, \partial H_2) \hookrightarrow \mathrm{Diff}_\partial(C_{\hat\varphi^s(f^s)}, \partial C_{\hat\varphi^s(f^s)}).$$

Composing these results with Proposition 4.8 yields the natural equivalences

$$\begin{aligned}\widehat{\mathcal{L}}_{f^o} \times \prod_s \widehat{\mathcal{K}}_{f^s} &\simeq B\mathrm{Diff}_\partial(C_{f^o}, \partial C_{f^o}) \times \prod_s B\mathrm{Diff}_\partial(C_{f^s}, \partial C_{f^s}) \\ &\simeq B\mathrm{Diff}_\partial(C_f, \partial C_f) \\ &\simeq \widehat{\mathcal{L}}_f.\end{aligned}$$

We are now able to use Budney, Burke and Koytcheff's freeness results (Theorems 4.9 and 4.10) and our previous discussion to get the equivalence

$$\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f \simeq \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_{f^o} \times \prod_s \mathcal{C}_2[\widehat{\mathcal{P}}]_{f^s} \simeq \widehat{\mathcal{L}}_{f^o} \times \prod_s \widehat{\mathcal{K}}_{f^s} \simeq \widehat{\mathcal{L}}_f.$$

We merely have an abstract equivalence at this stage. To show that it coincides with μ, we need to check the commutativity up to homotopy of the diagram

$$\begin{array}{ccc} B\mathrm{Diff}_{\mathrm{d}}(C_{fo},\partial C_{fo}) \times \prod_s B\mathrm{Diff}_{\mathrm{d}}(C_{fs},\partial C_{fs}) & \xrightarrow{\simeq} & B\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f) \\ \downarrow\simeq & & \downarrow\simeq \\ \widehat{\mathcal{L}}_{fo} \times \prod_s \widehat{\mathcal{K}}_{fs} & \xrightarrow{\quad\mu\quad} & \widehat{\mathcal{L}}_f \end{array}$$

but the spaces at stake are $K(G,1)$'s by Proposition 4.8, so it is enough to check the commutativity in π_1. This verification is very similar to the end of Budney's proof of Theorem 11 in [5]. An element of $\pi_1\widehat{\mathcal{L}}_f$ is (a homotopy class of) a based path in $\widehat{\mathcal{L}}_f$, ie an isotopy from f to f. The elements of $\pi_1 B\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f)$ can canonically be identified with $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f)$ in the long exact sequence of the fibration realizing $\widehat{\mathcal{L}}_f$ as $B\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f)$ in Proposition 4.8. In this framework, picking a class $\phi \in \pi_0\mathrm{Diff}_{\mathrm{d}}(C_{fo},\partial C_{fo})$ and chasing the diagram along the clockwise route turns it into an element of $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_f,\partial C_f)$ with its support lying between $-1\times D^2$ and $D_1 \subset C_f$, then converts it into an isotopy of f according to the construction in Proposition 4.8. Chasing ϕ along the counterclockwise route converts it into an isotopy of f^o in $\pi_1\widehat{\mathcal{L}}_{fo}$, then applies μ to it. The outcome is the same as each $\hat{\varphi}^s(f^s)$ is fixed all along this last isotopy. The same argument shows that picking a class in $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_{fs},\partial C_{fs})$ and chasing the diagram in either direction has the same effect. When evaluated in π_1, the upper-left product is a direct product, on which a factor-by-factor verification is thus sufficient to get the commutativity.

Finally, Hatcher and McCullough proved in [16] that the classifying spaces of the diffeomorphism groups at stake here have the homotopy types of (aspherical finite) CW–complexes (what we use here could also be deduced from Palais' earlier article [27]). Thus, Whitehead's theorem promotes the weak homotopy equivalence μ to a strong one. □

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