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Extension DGAs and topological Hochschild homology

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We study differential graded algebras (DGAs) that arise from ring spectra through the extension of scalars functor. Namely, we study DGAs whose corresponding Eilenberg–Mac Lane ring spectrum is equivalent to $H\mathbb{Z} \wedge E$ for some ring spectrum E. We call these DGAs extension DGAs. We also define and study this notion for E_∞ DGAs.

The topological Hochschild homology (THH) spectrum of an extension DGA splits in a convenient way. We show that formal DGAs with nice homology rings are extension, and therefore their THH groups can be obtained from their Hochschild homology groups in many cases of interest. We also provide interesting examples of DGAs that are not extension.

In the second part, we study properties of extension DGAs. We show that, in various cases, topological equivalences and quasi-isomorphisms agree for extension DGAs. From this, we obtain that dg Morita equivalences and Morita equivalences also agree in these cases.

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1 Introduction

In [27], Stanley shows that the homotopy category of differential graded algebras is equivalent to the homotopy category of $H\mathbb{Z}$–algebras. Later, Shipley [26] improves this equivalence to a zigzag of Quillen equivalences between the model categories of DGAs and $H\mathbb{Z}$–algebras. This opens up a new opportunity to study DGAs, ie to study DGAs using ring spectra.

Dugger and Shipley [9] use this zigzag of Quillen equivalences to define new equivalences between DGAs called topological equivalences; see Definition 1.10 below. They show nontrivial examples of topologically equivalent DGAs and they use topological equivalences to develop a Morita theory for DGAs. In [2], the author uses topological equivalences to obtain classification results for DGAs. Moreover, topological equivalences for E_∞ DGAs are studied by the author in [1].

In this work, we follow this philosophy in a different way. We study what we call extension DGAs which are the DGAs that are obtained from ring spectra through the extension of scalars functor from $\mathbb{S}$–algebras to $H\mathbb{Z}$–algebras, ie the functor $H\mathbb{Z} \wedge -$. More generally, we work in R–DGAs for a discrete commutative ring R. There is a zigzag of Quillen equivalences between R–DGAs and HR–algebras [26]. Composing the corresponding derived functors, one obtains a functor from the category of R–DGAs to the category of HR–algebras. For a given R–DGA X, we let HX denote the HR–algebra corresponding to X under this composite functor. We often abuse notation and denote a cofibrant and fibrant replacement of HX also by HX.

Definition 1.1 An R–DGA X is *R–extension* if the HR–algebra corresponding to X is weakly equivalent to $HR \wedge E$ for some cofibrant $\mathbb{S}$–algebra E. For $R = \mathbb{Z}$, we omit $\mathbb{Z}$ and write extension instead of $\mathbb{Z}$–extension.

To define R–extension E_∞ R–DGAs, we use the zigzag of Quillen equivalences between E_∞ R–DGAs and commutative HR–algebras constructed by Richter and Shipley in [19]. As before, composing the corresponding derived functors, one obtains a functor from the category of E_∞ R–DGAs to the category of commutative HR–algebras. For a given E_∞ R–DGA X, the corresponding commutative HR–algebra, which we denote by $H_{E_\infty}X$, is obtained by applying this composite functor to X. Again, we often abuse notation and denote a cofibrant and fibrant replacement of $H_{E_\infty}X$ also by $H_{E_\infty}X$.

Definition 1.2 An E_∞ R–DGA X is *R–extension* if the commutative HR–algebra corresponding to X is weakly equivalent to $HR \wedge E$ for some cofibrant commutative $\mathbb{S}$–algebra E. For $R = \mathbb{Z}$, we omit $\mathbb{Z}$ and write extension instead of $\mathbb{Z}$–extension.

See Appendix A for a discussion on the compatibility of the two definitions above.

Although we only study the extension problems coming from the definitions above, it is also interesting to consider the following general extension problem. Let $\varphi\colon A \to B$ be a map of commutative $\mathbb{S}$–algebras and let X be a B–algebra. We say X is φ–extension if it is weakly equivalent to $B \wedge_A E$ for some cofibrant A–algebra E. For the map $\mathbb{S} \to HR$, this corresponds to the extension problem coming from Definition 1.1.

Let k be a perfect field of characteristic p and let $W(k)$ denote the Witt ring of k. The extension problem corresponding to the canonical map $\phi\colon HW(k) \to Hk$ is analogous to a classical lifting problem for schemes; see for instance Grothendieck [11, Section 6] and Serre [24]. One of the motivations for the classical Witt-lifting problem is to understand the crystalline cohomology of smooth algebraic varieties over $\mathbb{F}_p$ through the de Rham cohomology of their lifts to $W(\mathbb{F}_p)$ whenever such a lift exists; see Berthelot [5, V.2.3.2]. Following this philosophy, Petrov and Vologodsky [18] recently showed that if an Hk–algebra (ie a k–DGA) X is ϕ–extension, ie $X \simeq Hk \wedge_{HW(k)} E$ for some cofibrant $HW(k)$–algebra (ie a $W(k)$–DGA) E, then the p–completed periodic topological cyclic homology of X agrees with the p–completed periodic cyclic homology of E when $p > 2$. This boils down the computation of a topological homology theory to the computation of an algebraic homology theory.

Similarly, the extension property we study in this work boils down the computation of the topological Hochschild homology of an extension DGA to a Hochschild homology computation. Namely, for an R–extension DGA X (as in Definition 1.1), we have the following splitting at the level of topological Hochschild homology. This splitting is possibly well known to the experts in the field; see Schwänzl, Vogt and Waldhausen [20, Theorem 1] for an instance of this splitting when X is the Eilenberg–Mac Lane spectrum of a discrete ring. In the proposition below, $\mathrm{HH}^R(-)$ denotes $\mathrm{THH}^{HR}(-)$.

Proposition 1.3 *If X is an R–extension R–DGA, then there is an equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

If X is an R–extension E_∞ R–DGA, then the equivalence above is an equivalence of commutative $\mathbb{S}$–algebras.

For a map $\varphi\colon A \to B$ of commutative $\mathbb{S}$–algebras, there is a similar splitting of $\mathrm{THH}^A(X)$ whenever X is a φ–extension B–algebra; see Proposition B.1.

The splitting in Proposition 1.3 simplifies THH calculations significantly in many situations. Indeed, it is an important stepping stone in many THH calculations in the literature, particularly for the case where X is a discrete ring, ie a DGA whose homology is concentrated in degree 0. For example, Larsen and Lindenstrauss [16] show that this splitting exists at the level of homotopy groups for various discrete rings of characteristic p. Furthermore, Hesselholt and Madsen [12, Theorem 7.1] prove such a splitting for discrete rings that have a nice basis with respect to the ground ring R. In the following theorem, we generalize this result to connective formal DGAs. Note that a connective DGA is a DGA whose negative homology is trivial.

Theorem 1.4 *Let X be a connective formal R–DGA whose homology has a homogeneous basis as an R–module containing the multiplicative unit such that the multiplication of two basis elements is either zero or a basis element. In this situation, X is R–extension. As a result, we have the equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

Section 5 is devoted to the proof of this theorem. Furthermore, for a given R–DGA that satisfies the hypothesis of the theorem above, we provide an explicit description of the corresponding HR–algebra; see Proposition 5.8. The author and Moulinos [3, 4.8 and 6.1] show that for such HR–algebras, one often obtains nontrivial splittings at the level of topological negative cyclic homology and topological periodic homology. Using these splittings, the author and Moulinos compute the algebraic K–theory of $\mathrm{THH}(H\mathbb{F}_p)$, ie the algebraic K–theory of the formal DGA with homology $\mathbb{F}_p[x_2]$. In a future work, the author plans to compute the algebraic K–theory groups of various formal DGAs by using Proposition 5.8 and the splittings provided in [3].

Remark 1.5 Another way to state the hypothesis of Theorem 1.4 is the following. Let M be a monoid in the category of graded pointed sets. From M, one obtains a graded R–algebra $R\langle M\rangle$ whose underlying R–module is the free R–module over the graded set M_- obtained by removing the based point from M. The multiplication on $R\langle M\rangle$ is given by the multiplication on M where the based point of M is considered as the zero element in $R\langle M\rangle$. A graded R–algebra of the form $R\langle M\rangle$ is called a graded monoid R–algebra. With this definition, a connective formal R–DGA satisfies the hypothesis of Theorem 1.4 if and only if its homology is a graded monoid R–algebra.

Remark 1.6 We mention a few examples of graded rings that satisfy the hypothesis of the theorem above as homology of X. The polynomial algebra over R with a nonnegatively graded set S of generators $R[S]$ satisfies the hypothesis if all the elements of S are in even degrees. The basis of $R[S]$ is given by the monomials in S and the unit $1 \in R$. Similarly, many examples of quotients of polynomial rings with even degree generators also satisfy this hypothesis; for example $R[x]/(x^2)$, $R[x,y]/(y^2)$ and $R[x,y]/(x^2y, y^3)$ with even $|x|$ and $|y|$. However, there are rings that do not satisfy this hypothesis. For example, for $R = \mathbb{Z}$, the exterior algebra on two generators $\Lambda[x,y] \cong \Lambda[x] \otimes \Lambda[y]$ with odd $|x|$ and $|y|$ has a basis given by $\{x, y, xy\}$, but $yx = -xy$ and therefore yx is not one of the basis elements. Indeed, $\Lambda[x,y]$ has no basis that satisfies this hypothesis.

We prove the following nonextension results.

Theorem 1.7 *Let Y be an E_∞ DGA. For all primes p, if Y is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA then Y is not an extension E_∞ DGA.*

Theorem 1.8 *Let X be a DGA. If X is quasi-isomorphic to an $\mathbb{F}_2$–DGA then X is not an extension DGA.*

Remark 1.9 These theorems should be compared with the two commutative $H\mathbb{Z}$–algebras X and Y obtained from $H\mathbb{Z} \wedge H\mathbb{F}_p$ through the structure maps

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge H\mathbb{F}_p \quad \text{and} \quad H\mathbb{Z} \cong \mathbb{S} \wedge H\mathbb{Z} \to \mathbb{S} \wedge H\mathbb{F}_p \to H\mathbb{Z} \wedge H\mathbb{F}_p,$$

respectively. The E_∞ DGA corresponding to X is an extension E_∞ DGA and the E_∞ DGA corresponding to Y is an E_∞ $\mathbb{F}_p$–DGA. Although these two E_∞ DGAs are E_∞ topologically equivalent, they are not quasi-isomorphic due to [1, Theorem 5.3]. For the associative case with $p = 2$, the distinction between the two DGAs corresponding to X and Y is due to [9, Example 5.6].

In the results above, we work with (E_∞) DGAs in mixed characteristic, ie we work in (E_∞) $\mathbb{Z}$–DGAs. A natural question to ask is if there are examples of E_∞ k–DGAs that are not k–extension for a field k. In Example 1.12 below, we show that there are E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension.

Now we discuss topological equivalences of DGAs and the properties of extension DGAs regarding topological equivalences.

Definition 1.10 Two DGAs X and Y are *topologically equivalent* if the corresponding $H\mathbb{Z}$–algebras HX and HY are weakly equivalent as $\mathbb{S}$–algebras.

The definition of E_∞ topological equivalences is as follows.

Definition 1.11 Two E_∞ DGAs X and Y are E_∞ *topologically equivalent* if the corresponding *commutative* $H\mathbb{Z}$–algebras $H_{E_\infty}X$ and $H_{E_\infty}Y$ are weakly equivalent as *commutative* $\mathbb{S}$–algebras.

It follows from these definitions that quasi-isomorphic (E_∞) DGAs are (E_∞) topologically equivalent. However, there are examples of nontrivially topologically equivalent DGAs, ie DGAs that are topologically equivalent but not quasi-isomorphic [9]. Furthermore, examples of nontrivially E_∞ topologically equivalent E_∞ DGAs are constructed by the author in [1].

Example 1.12 This is an example of E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension. In [1, Example 5.1], the author constructs nontrivially E_∞ topologically equivalent E_∞ $\mathbb{F}_p$–DGAs that we call X and Y, ie X and Y are E_∞ topologically equivalent but they are not quasi-isomorphic. Although these E_∞ $\mathbb{F}_p$–DGAs are E_∞ topologically equivalent, their Dyer–Lashof operations are different.

For $p = 2$, the homology rings of these E_∞ $\mathbb{F}_p$–DGAs are given by

$$\mathbb{F}_2[x]/(x^4)$$

for both X and Y where $|x| = 1$. On the homology of X, the first Dyer–Lashof operation is trivial, ie $\mathrm{Q}^1 x = 0$. On the other hand, we have $\mathrm{Q}^1 x = x^3$ on the homology of Y. Using these properties we show (for all primes) that these E_∞ $\mathbb{F}_p$–DGAs are not $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGAs. See Section 3B for a proof of this fact.

By [1, Theorem 1.6], E_∞ topological equivalences between E_∞ $\mathbb{F}_p$–DGAs with trivial first homology preserve Dyer–Lashof operations. We prove a stronger result for $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGAs.

Theorem 1.13 *Let X be an $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGA with $H_1 X = 0$ and let Y be an E_∞ $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are E_∞ topologically equivalent.*

In the following results, we show various situations where topological equivalences and quasi-isomorphisms agree.

Theorem 1.14 *Let Y be an $\mathbb{F}_p$–DGA and let X be an $\mathbb{F}_p$–extension $\mathbb{F}_p$–DGA. For odd p, assume that the homology of X is trivial in degrees $2p^r - 2$ for $r \geq 1$ and $2p^s - 1$ for $s \geq 0$. For $p = 2$, assume that the homology of X is trivial in degree $2^r - 1$ for $r \geq 1$. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

For the corollary below, note that a coconnective DGA is a DGA with trivial homology in positive degrees.

Corollary 1.15 *Let X be a coconnective extension $\mathbb{F}_p$–DGA and let Y be an $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Theorem 1.16 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$ and let X be an R–DGA whose corresponding HR–algebra is equivalent to $HR \wedge Z$ for some cofibrant $\mathbb{S}$–algebra Z whose underlying spectrum is equivalent to a coproduct of (de)suspensions of the sphere spectrum. Also, let Y be an R–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Our main interest for this theorem is due to its corollary stated below. This follows by Proposition 5.8 which implies that an R–DGA that satisfies the hypothesis of Theorem 1.4 also satisfies the hypothesis of the theorem above.

Corollary 1.17 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$, let Y be an R–DGA and let X be as in Theorem 1.4. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Two DGAs X and Y are said to be *Morita equivalent* if the model categories of X–modules and Y–modules are Quillen equivalent. There is a stronger notion of Morita equivalence for DGAs called *dg Morita equivalences* defined by Keller [14, Section 3.8]. Due to [9, 7.7], two DGAs X and Y are dg Morita equivalent if and only if the model categories of X–modules and Y–modules are *additively* Quillen equivalent; see Dugger and Shipley [8] for the definition of additive Quillen equivalences. This is a strictly stronger notion of Morita equivalence since there are examples of DGAs that are Morita equivalent but not dg Morita equivalent [9, Section 8]. However, in the situations where topological equivalences and quasi-isomorphisms agree, these two notions of Morita equivalences also agree [9, Proposition 7.7 and Theorem 7.2]. We obtain the following corollary to Theorems 1.14 and 1.16.

Corollary 1.18 *Assume that X and Y are as in Theorem 1.14 or Theorem 1.16. Then X and Y are Morita equivalent if and only if they are dg Morita equivalent.*

Organization In Section 2, we describe the dual Steenrod algebra and the Dyer–Lashof operations on it. In Section 3, we prove Theorems 1.13, 1.14 and 1.16. Section 4 is devoted to the proof of Theorems 1.7 and 1.8. In Section 5, we prove Theorem 1.4. This section is independent from Sections 2, 3 and 4, and it contains explicit descriptions of the $H\mathbb{Z}$–algebras corresponding to the formal DGAs as in Theorem 1.4, which is of independent interest. We leave the proof of Theorem 1.4 to the end because it uses different tools than the rest of the proofs in this work. Appendix A is devoted to a discussion on the compatibility of Definitions 1.1 and 1.2.

Terminology We work in the setting of symmetric spectra in simplicial sets; see Hovey, Shipley and Smith [13]. For commutative ring spectra, we use the positive $\mathbb{S}$–model structure developed by Shipley in [25]. When we work in the setting of associative ring spectra, we use the stable model structure of [13]. Throughout this work, R denotes a general discrete commutative ring except in Section 3C where R denotes a quotient of $\mathbb{Z}$. When we say (E_∞) DGA, we mean (E_∞) $\mathbb{Z}$–DGA.

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2 The dual Steenrod algebra

Here, we recall the ring structure and the Dyer–Lashof operations on the dual Steenrod algebra. Using the standard notation, we denote the dual Steenrod algebra by $\mathcal{A}_*$. We have $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) \cong \mathcal{A}_*$. Milnor shows that the dual Steenrod algebra is a free graded commutative $\mathbb{F}_p$–algebra [17].

For $p=2$, $\mathcal{A}_*$ is given by

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r \mid r \geq 1] = \mathbb{F}_2[\zeta_r \mid r \geq 1],$$

where $|\xi_r| = |\zeta_r| = 2^r - 1$. Let χ denote the action of the transpose map of the smash product on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$. We have $\chi(\xi_r) = \zeta_r$.

For an odd prime p, the dual Steenrod algebra is described by

$$\mathcal{A}_* = \mathbb{F}_p[\xi_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\tau_s \mid s \geq 0) = \mathbb{F}_p[\zeta_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\bar{\tau}_s \mid s \geq 0),$$

where $|\xi_r| = |\zeta_r| = 2(p^r - 1)$ and $|\tau_s| = |\bar{\tau}_s| = 2p^s - 1$. In this case, we have $\chi(\xi_r) = \zeta_r$ and $\chi(\tau_r) = \bar{\tau}_r$.

Dyer–Lashof operations are power operations that act on the homotopy rings of H_∞ $H\mathbb{F}_p$–algebras [7]. By forgetting structure, commutative $H\mathbb{F}_p$–algebras are examples of H_∞ $H\mathbb{F}_p$–algebras and therefore Dyer–Lashof operations are also defined on the homotopy ring of commutative $H\mathbb{F}_p$–algebras, and maps of commutative $H\mathbb{F}_p$–algebras preserve these operations. For $p=2$, there is a Dyer–Lashof operation denoted by Q^s for ever integer s where Q^s increases the degree by s. For odd p, there are

Dyer–Lashof operations denoted by βQ^s and Q^s for every integer s that increase the degree by $2s(p-1)-1$ and $2s(p-1)$, respectively. See [7, III.1.1] for further properties of these operations.

With the unit map

$$H\mathbb{F}_p \cong H\mathbb{F}_p \wedge \mathbb{S} \to H\mathbb{F}_p \wedge H\mathbb{F}_p,$$

$H\mathbb{F}_p \wedge H\mathbb{F}_p$ is a commutative $H\mathbb{F}_p$–algebra and therefore Dyer–Lashof operations are defined on the dual Steenrod algebra. These operations are first studied in [7, III.2]. Steinberger shows that the degree one element τ_0 for odd p and ξ_1 for $p=2$ generates the dual Steenrod algebra as an algebra with Dyer–Lashof operations, ie as an algebra over the Dyer–Lashof algebra. In particular for $p=2$, we have

$$Q^{2^s-2}\xi_1 = \zeta_s \quad \text{for } s > 1.$$

For odd p, we have

$$Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s \bar{\tau}_s, \quad \beta Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s \zeta_s$$

for $s \geq 1$.

3 Proof of the results on topological equivalences and the nonextension example

In this section, we prove Theorems 1.13, 1.14 and 1.16 which provide comparison results on (E_∞) topological equivalences and quasi-isomorphisms of (E_∞) DGAs for various cases. At the end, we prove Proposition 3.2 which justifies the last claim in Example 1.12. This provides examples of E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension.

These results are obtained using similar arguments. Therefore, we suggest the reader to go through their proof in the order presented in this section.

3A Proof of Theorems 1.13 and 1.14

In the proof of Theorems 1.13 and 1.14 and also in the proof of Theorem 1.16 and Proposition 3.2, we show that for various R–extension (E_∞) R–DGAs, (E_∞) topological equivalences and quasi-isomorphisms agree.

For this, we use the same technique to produce a quasi-isomorphism, ie an HR–algebra equivalence, out of a given topological equivalence, ie an $\mathbb{S}$–algebra equivalence. We start by describing this technique.

Let us focus on the E_∞ case. Assume that we are given commutative HR–algebras Y and $HR \wedge Z$, where Z denotes a cofibrant commutative $\mathbb{S}$–algebra and assume that we are given a weak equivalence

$$\varphi\colon HR \wedge Z \xrightarrow{\sim} Y$$

of commutative $\mathbb{S}$–algebras. Using φ, we produce a map of commutative HR–algebras through the composite

$$\psi\colon HR \wedge Z \cong HR \wedge \mathbb{S} \wedge Z \xrightarrow{i} HR \wedge HR \wedge Z \xrightarrow[\simeq]{HR\wedge\varphi} HR \wedge Y \xrightarrow{m} Y.$$

Here, i is the canonical map induced by the unit map $\mathbb{S} \to HR$ of HR and m is the commutative HR–algebra structure map of Y. Except Y, we provide the objects in the composite above with the commutative HR–algebra structure coming from the first HR factor. The maps i and $HR \wedge \varphi$ are maps of commutative HR–algebras as they are obtained using the functor $HR \wedge -$ from the category of commutative $\mathbb{S}$–algebras to the category of commutative HR–algebras. Furthermore, we assume that HR is cofibrant as a commutative $\mathbb{S}$–algebra in the positive $\mathbb{S}$–model structure of [25]. This implies that HR is cofibrant as an $\mathbb{S}$–module [25, 4.1] in the model structure of [25], ie HR is $\mathbb{S}$–cofibrant in the terminology of [13, 5.3.6]. Therefore, $HR \wedge \varphi$ is a weak equivalence [13, 5.3.10]. Note that m is the left adjoint of the identity map of Y under the adjunction between the categories of commutative $\mathbb{S}$–algebras and commutative HR–algebras whose left adjoint is given by the extension of scalars functor $HR \wedge -$ and whose right adjoint is given by the restriction of scalars functor. In particular, this shows that m is also a map of commutative HR–algebras. We deduce that ψ is a map of commutative HR–algebras as it is given by a composite of such maps. Compared to the commutative case, the definition of the map ψ is slightly more complicated in the associative case as we consider various cofibrant replacements. The results we prove in this section are obtained by showing that ψ is an equivalence under the given hypothesis.

We start with the proof of Theorem 1.13. We provide a restatement of this theorem below.

Theorem 1.13 *Let X be an $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGA with $H_1 X = 0$ and let Y be an E_∞ $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are E_∞ topologically equivalent.*

In what follows, we denote the category of commutative E–algebras by $E\text{–}c\mathcal{A}lg$ and the category of associative E–algebras by $E\text{–}\mathcal{A}lg$ for a given commutative ring spectrum E.

Proof Since quasi-isomorphic E_∞ DGAs are always E_∞ topologically equivalent, we only need to show that if X and Y are E_∞ topologically equivalent then they are quasi-isomorphic as E_∞ $\mathbb{F}_p$–DGAs.

Let $H\mathbb{F}_p$ denote a cofibrant model of $H\mathbb{F}_p$ in $\mathbb{S}$–$c\mathcal{A}lg$. The category of commutative $H\mathbb{F}_p$–algebra spectra is the same as the category of commutative $\mathbb{S}$–algebra spectra under $H\mathbb{F}_p$. Therefore we have a model structure on $H\mathbb{F}_p$–$c\mathcal{A}lg$ where the cofibrations, fibrations and weak equivalences are precisely the maps that forget to cofibrations, fibrations and weak equivalences in $\mathbb{S}$–$c\mathcal{A}lg$. We let Y also denote the commutative $H\mathbb{F}_p$–algebra corresponding to the E_∞ DGA Y. Therefore $\pi_1(Y)=0$. Taking a fibrant replacement, we assume Y is fibrant both in $H\mathbb{F}_p$–$c\mathcal{A}lg$ and in $\mathbb{S}$–$c\mathcal{A}lg$. Furthermore, we let $H\mathbb{F}_p \wedge Z$ denote the commutative $H\mathbb{F}_p$–algebra corresponding to the extension E_∞ $\mathbb{F}_p$–DGA X, where Z is a cofibrant object in $\mathbb{S}$–$c\mathcal{A}lg$. This ensures that $H\mathbb{F}_p \wedge Z$ is cofibrant in $H\mathbb{F}_p$–$c\mathcal{A}lg$. Therefore the composite $\mathbb{S} \rightarrowtail H\mathbb{F}_p \rightarrowtail H\mathbb{F}_p \wedge Z$ is also a cofibration in $\mathbb{S}$–$c\mathcal{A}lg$; this shows that $H\mathbb{F}_p \wedge Z$ is also cofibrant in $\mathbb{S}$–$c\mathcal{A}lg$. To prove Theorem 1.13, we need to show that $H\mathbb{F}_p \wedge Z$ and Y are weakly equivalent in $H\mathbb{F}_p$–$c\mathcal{A}lg$.

Because $H\mathbb{F}_p \wedge Z$ and Y are obtained from E_∞ topologically equivalent E_∞ DGAs, they are equivalent as commutative $\mathbb{S}$–algebras. Furthermore $H\mathbb{F}_p \wedge Z$ is cofibrant and Y is fibrant; therefore there is a weak equivalence $\varphi\colon H\mathbb{F}_p \wedge Z \xrightarrow{\simeq} Y$ of commutative $\mathbb{S}$–algebras. We consider the composite map

$$(1)\quad \psi\colon H\mathbb{F}_p \wedge Z \cong H\mathbb{F}_p \wedge \mathbb{S} \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow[\simeq]{H\mathbb{F}_p\wedge\varphi} H\mathbb{F}_p \wedge Y \xrightarrow{m} Y,$$

where the first map is induced by the unit map $u_{H\mathbb{F}_p}\colon \mathbb{S} \to H\mathbb{F}_p$ of $H\mathbb{F}_p$ and the last map is the $H\mathbb{F}_p$ structure map of Y. If we consider all the objects in this composite except Y to have the $H\mathbb{F}_p$ structure coming from the first smash factor, then all objects involved are commutative $H\mathbb{F}_p$–algebras and the maps involved are maps of commutative $H\mathbb{F}_p$–algebras. Note that i and $H\mathbb{F}_p \wedge \varphi$ are maps of commutative $H\mathbb{F}_p$–algebras as they are obtained via the functor $H\mathbb{F}_p \wedge -\colon \mathbb{S}\text{–}c\mathcal{A}lg \to H\mathbb{F}_p\text{–}c\mathcal{A}lg$. The last map m is a map of commutative $H\mathbb{F}_p$–algebras because it is the left adjoint of the identity map of Y under the usual adjunction between $\mathbb{S}$–$c\mathcal{A}lg$ and $H\mathbb{F}_p$–$c\mathcal{A}lg$. Since all the maps in the composite above are maps of commutative $H\mathbb{F}_p$–algebras, we deduce that ψ is a map of commutative $H\mathbb{F}_p$–algebras.

What remains is to show that ψ is a weak equivalence. For this, we take the homotopy groups of the composite defining ψ and show that it is an isomorphism. Firstly, we have a splitting

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z)$$

in $H\mathbb{F}_p$–$c\mathcal{A}lg$ where we consider the object on the right-hand side of the equality with the $H\mathbb{F}_p$ structure given by the first smash factor instead of the canonical one given by the smash product $\wedge_{H\mathbb{F}_p}$. Because the homotopy of $H\mathbb{F}_p$ is a field, we have $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z) \cong \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$; see [10, IV.4.1]. With this identification, we obtain that the composite map induced in homotopy by the composite defining ψ is given by

$$\psi_*\colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p*}Y \xrightarrow{m_*} Y_*. \tag{2}$$

Note that although we identify the domain of $\pi_*(H\mathbb{F}_p \wedge \varphi)$ as a tensor product, we do not claim that $\pi_*(H\mathbb{F}_p \wedge \varphi)$ splits as a tensor product of two maps.

Below, we state three claims. Afterwards, we assume these claims and prove that ψ_* is an isomorphism by showing $\psi_* = \varphi_*$, ie we prove the theorem assuming the claims below. After that, we provide a proof of the three claims listed below.

Claim 1 *The composite $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)$ maps every element of the form $a \otimes_{\mathbb{F}_p} x$ with $|a| > 0$ to zero in Y_*.*

Claim 2 *We have $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = \varphi_*(x)$ for every $x \in \pi_*(H\mathbb{F}_p \wedge Z)$.*

Claim 3 *We have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i$ for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$ and $x_i \in \pi_*(H\mathbb{F}_p \wedge Z)$.*

Now we show that ψ is a weak equivalence by assuming the claims above. We have

$$\begin{aligned}\psi_*(x) &= m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi) \circ i_*(x)\\ &= m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i)\\ &= \varphi_*(x)\end{aligned}$$

for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$. Here, the first equality follows by the definition of ψ_*, the second equality follows by Claim 3 and the third follows by Claims 1 and 2. This proves that ψ_* is an isomorphism since φ_* is an isomorphism. Therefore, we deduce that ψ is a weak equivalence as desired. What is left to prove is the three claims stated above.

Proof of Claim 1 The map $\mathbb{S} \to Z$ induces a map

$$(H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \to (HF_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z).$$

This map is in $H\mathbb{F}_p\text{–}c\mathcal{A}lg$, therefore the induced map in homotopy preserves the Dyer–Lashof operations. The induced map in homotopy is given by the inclusion $\mathcal{A}_* \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$ and this shows that Dyer–Lashof operations on this subset of $\mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$ are given by the action of the Dyer–Lashof operations on the dual Steenrod algebra, ie $\mathrm{Q}^s(a \otimes_{\mathbb{F}_p} 1) = (\mathrm{Q}^s a) \otimes_{\mathbb{F}_p} 1$. Let p be an odd prime. Since $\pi_1(Y)$ is trivial, $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(\tau_0 \otimes_{\mathbb{F}_p} 1) = 0$. Because the dual Steenrod algebra is generated with the Dyer–Lashof operations by τ_0, this shows that $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(a \otimes_{\mathbb{F}_p} 1) = 0$ for all $a \in \mathcal{A}_*$ with $|a| > 0$. Since all maps involved are ring maps and $a \otimes_{\mathbb{F}_p} x = (a \otimes_{\mathbb{F}_p} 1)(1 \otimes_{\mathbb{F}_p} x)$, this finishes the proof of our claim $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(a \otimes_{\mathbb{F}_p} x) = 0$ whenever $|a| > 0$. Note that for $p = 2$, one uses ξ_1 instead of τ_0. □

Proof of Claim 2 We consider the commutative diagram

$$\begin{array}{ccccccc} (\mathbb{S} \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z) & \xrightarrow{\cong} & \mathbb{S} \wedge H\mathbb{F}_p \wedge Z & \xrightarrow{\mathbb{S}\wedge\varphi} & \mathbb{S} \wedge Y & & \\ \downarrow h & & \downarrow & & \downarrow h_Y & \searrow^{\mathrm{id}} & \\ (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z) & \xrightarrow{\cong} & H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z & \xrightarrow{H\mathbb{F}_p\wedge\varphi} & H\mathbb{F}_p \wedge Y & \xrightarrow{m} & Y \end{array} \tag{3}$$

Because Y is in $H\mathbb{F}_p\text{–}c\mathcal{A}lg$, we have $m \circ h_Y = \mathrm{id}$. We also have

$$m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi) \circ h_*(x).$$

Carrying x through the top row and then composing with $m \circ h_Y$, we obtain the equality $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = \varphi_*(x)$ in our claim. □

Proof of Claim 3 The composite of the maps below is the identity

$$H\mathbb{F}_p \wedge Z \cong H\mathbb{F}_p \wedge \mathbb{S} \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow{m_{H\mathbb{F}_p} \wedge \mathrm{id}} H\mathbb{F}_p \wedge Z,$$

where $m_{H\mathbb{F}_p}$ is the multiplication map of $H\mathbb{F}_p$. With the identification

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z),$$

we obtain the composite in homotopy

$$\pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})} \pi_*(H\mathbb{F}_p \wedge Z), \tag{4}$$

where $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})$ is given by the augmentation $\mathcal{A}_* \to \mathbb{F}_p$. This description of $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})$ and the fact that $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id}) \circ i_* = \mathrm{id}$ proves our claim. □

This completes the proof of Theorem 1.13. □

Remark 3.1 The proof of Theorem 1.13 is showing slightly more. For a given cofibrant Z in $\mathbb{S}$–$c\mathcal{A}lg$ and a fibrant Y in $H\mathbb{F}_p$–$c\mathcal{A}lg$ with $\pi_1 Y = 0$ and an equivalence $H\mathbb{F}_p \wedge Z \xrightarrow{\sim} Y$ of $\mathbb{S}$–algebras, the map $H\mathbb{F}_p \wedge Z \to Y$ in $H\mathbb{F}_p$–$c\mathcal{A}lg$ given by the structure map of Y on $H\mathbb{F}_p$ and the map $\mathbb{S} \wedge Z \to H\mathbb{F}_p \wedge Z \xrightarrow{\sim} Y$ on Z is also a weak equivalence. Note that to construct this map, we use the fact that $H\mathbb{F}_p \wedge Z$ is a coproduct of $H\mathbb{F}_p$ and Z in $\mathbb{S}$–$c\mathcal{A}lg$.

The proof of Theorem 1.14 (restated below) is similar to the proof of Theorem 1.13. Therefore, in the proof of Theorem 1.14, we assume familiarity with the proof of Theorem 1.13.

Theorem 1.14 *Let Y be an $\mathbb{F}_p$–DGA and let X be an $\mathbb{F}_p$–extension $\mathbb{F}_p$–DGA. For odd p, assume that the homology of X is trivial in degrees $2p^r - 2$ for $r \geq 1$ and $2p^s - 1$ for $s \geq 0$. For $p = 2$, assume that the homology of X is trivial in degree $2^r - 1$ for $r \geq 1$. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof Here, we work in the setting of associative algebras. In this case, we need to be more careful with cofibrant replacements since the forgetful functor from $H\mathbb{F}_p$–$\mathcal{A}lg$ to $\mathbb{S}$–$\mathcal{A}lg$ does not necessarily preserve cofibrant objects. Let $H\mathbb{F}_p$ be cofibrant in $\mathbb{S}$–$c\mathcal{A}lg$ (with the model structure of [25]) as before and let Z be cofibrant in $\mathbb{S}$–$\mathcal{A}lg$ such that $H\mathbb{F}_p \wedge Z$ is an $H\mathbb{F}_p$–algebra that corresponds to X. By abuse of notation, let Y be a fibrant $H\mathbb{F}_p$–algebra corresponding to Y. Let $T \xrightarrow{\sim} H\mathbb{F}_p \wedge Z$ be a cofibrant replacement of $H\mathbb{F}_p \wedge Z$ in $\mathbb{S}$–$\mathcal{A}lg$. We have the lift

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & T \\ \downarrow & \overset{f}{\nearrow} & \downarrow \wr \\ Z & \longrightarrow & H\mathbb{F}_p \wedge Z \end{array} \tag{5}$$

in $\mathbb{S}$–$\mathcal{A}lg$ where the bottom map is given by the map $Z \cong \mathbb{S} \wedge Z \to H\mathbb{F}_p \wedge Z$. Since T and Y are obtained from topologically equivalent DGAs, they are equivalent in $\mathbb{S}$–$\mathcal{A}lg$. Also because T is cofibrant and Y is fibrant, we have a weak equivalence $\varphi\colon T \xrightarrow{\sim} Y$ of $\mathbb{S}$–algebras. We obtain the composite map of $H\mathbb{F}_p$–algebras

$$\psi\colon H\mathbb{F}_p \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge T \xrightarrow[\simeq]{H\mathbb{F}_p \wedge \varphi} H\mathbb{F}_p \wedge Y \xrightarrow{m} Y,$$

where $i = H\mathbb{F}_p \wedge f$ and m is the $H\mathbb{F}_p$ structure map of Y. The map m is a map of $H\mathbb{F}_p$–algebras because it is the left adjoint of the identity map of Y under the usual adjunction between $H\mathbb{F}_p$–$\mathcal{A}lg$ and $\mathbb{S}$–$\mathcal{A}lg$. Note that we denote $H\mathbb{F}_p \wedge f$ by i because the map i in the composite above should be compared to the map i in (1).

Again, what remains is to show that ψ_* is an isomorphism. Note that the functor $H\mathbb{F}_p \wedge -$ preserves weak equivalences [13, 5.3.10]. Identifying homotopy groups of T with homotopy groups of $H\mathbb{F}_p \wedge Z$ through the trivial fibration above, and similarly identifying the homotopy groups of $H\mathbb{F}_p \wedge T$ with those of $H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z$, we obtain a description of ψ_* similar to the one in (2),

$$\psi_* \colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p*}Y \xrightarrow{m_*} Y_*.$$

It is sufficient to show that the claims in the proof of Theorem 1.13 also hold in this case. Claim 1 follows by the hypothesis that $\pi_* Y$ is trivial at the degrees where the algebra generators of the dual Steenrod algebra are. Claim 2 follows similarly. For Claim 3, consider the sequence of maps

$$H\mathbb{F}_p \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge T \xrightarrow{\sim} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow{m_{H\mathbb{F}_p} \wedge \mathrm{id}} H\mathbb{F}_p \wedge Z,$$

where $m_{H\mathbb{F}_p}$ is the multiplication map of $H\mathbb{F}_p$. Due to diagram (5), the composite above is the identity map. Taking homotopy groups of the composite above and omitting the equivalence in the middle, one obtains (4). The rest of the proof of Claim 3 follows as before. □

3B Example 1.12

Here, we show that the E_∞ $\mathbb{F}_p$–DGAs provided in Example 1.12 are not $\mathbb{F}_p$–extension.

Proposition 3.2 *Let X and Y be as in Example 1.12. As E_∞ $\mathbb{F}_p$–DGAs, X and Y are not $\mathbb{F}_p$–extension.*

Proof Recall that in Example 1.12, we provide examples of E_∞ $\mathbb{F}_p$–DGAs that are E_∞ topologically equivalent but not quasi-isomorphic. We prove that X is not an extension E_∞ $\mathbb{F}_p$–DGA. In order to show Y is not extension, it suffices to exchange the roles of X and Y in the proof below.

We assume that X is an extension E_∞ $\mathbb{F}_p$–DGA and obtain a contradiction by showing that X and Y are quasi-isomorphic under this assumption. This is similar to the proof of Theorem 1.13, which we assume familiarity with. Following the constructions there, we obtain a map of commutative $H\mathbb{F}_p$–algebras

$$\psi \colon H\mathbb{F}_p \wedge Z \cong H\mathbb{F}_p \wedge \mathbb{S} \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow[\simeq]{H\mathbb{F}_p \wedge \varphi} H\mathbb{F}_p \wedge Y \xrightarrow{m} Y$$

as in (1), where $H\mathbb{F}_p \wedge Z$ denotes a commutative $H\mathbb{F}_p$–algebra corresponding to X and Y denotes a commutative $H\mathbb{F}_p$–algebra corresponding to the E_∞ $\mathbb{F}_p$–DGA Y by abusing notation. This is a map of commutative $H\mathbb{F}_p$–algebras as before. Therefore, it is sufficient to show that ψ_* is an isomorphism.

As in (2), ψ_* is given by

$$\psi_*\colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p*}Y \xrightarrow{m_*} Y_*.$$

By Claim 3 in the proof of Theorem 1.14, for every $x \in \pi_*(H\mathbb{F}_p \wedge Z)$ we have

$$i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i \tag{6}$$

for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$ and $x_i \in \pi_*(H\mathbb{F}_p \wedge Z)$.

For $p = 2$, $\pi_*(H\mathbb{F}_p \wedge Z) \cong \mathbb{F}_2[x]/(x^4)$ with $|x| = 1$. By degree reasons, we either have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x$ or $i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \xi_1 \otimes_{\mathbb{F}_p} 1$. Since $(1 \otimes_{\mathbb{F}_p} x + \xi_1 \otimes_{\mathbb{F}_p} 1)^4 \neq 0$ but $x^4 = 0$, the second option is not possible. Therefore we have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x$. Since i is a map of ring spectra, i_* is multiplicative so $i_*(x^l) = 1 \otimes_{\mathbb{F}_p} x^l$ for every l. By Claim 2 in the proof of Theorem 1.13, this shows that ψ_* is an isomorphism. This provides a contradiction as X and Y are not quasi-isomorphic as E_∞ $\mathbb{F}_2$–DGAs.

For odd p, we have

$$\pi_*Y \cong \pi_*(H\mathbb{F}_p \wedge Z) \cong \Lambda_{\mathbb{F}_p}[x, y]$$

with $|x| = 1$ and $|y| = 2p-2$. By (6) above, either

$$i_*(y) = 1 \otimes_{\mathbb{F}_p} y \quad \text{or} \quad i_*(y) = c\xi_1 \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} y$$

for some unit $c \in \mathbb{F}_p$. However, $y^2 = 0$ but $(c\xi_1 \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} y)^2 \neq 0$ so only the first option is possible. This shows that $\psi_*(y) = y$ due to Claim 2 in the proof of Theorem 1.13. The $2p-2$ Postnikov sections of Y and $H\mathbb{F}_p \wedge Z$ agrees with that of $H\mathbb{F}_p \wedge H\mathbb{F}_p$ in commutative $H\mathbb{F}_p$–algebras; see [1, Example 5.1]. Using this together with the fact that $\beta\mathrm{Q}^1\tau_0 = -\zeta_1$ in the dual Steenrod algebra, $\beta\mathrm{Q}^1 x = y$ up to a unit both in $\pi_*(H\mathbb{F}_p \wedge Z)$ and in π_*Y. Because ψ is a map of commutative $H\mathbb{F}_p$–algebras, ψ_* preserves Dyer–Lashof operations. Since $\psi_*(y) = y$, we obtain that $\psi_*(x) = x$ up to a unit of $\mathbb{F}_p$. Because ψ_* is a ring map, we deduce that ψ_* is indeed an isomorphism. Therefore ψ is a weak equivalence of commutative $H\mathbb{F}_p$–algebras between the commutative $H\mathbb{F}_p$–algebras corresponding to the E_∞ $\mathbb{F}_p$–DGAs X and Y. This contradicts the fact that X and Y are not quasi-isomorphic as E_∞ $\mathbb{F}_p$–DGAs and finishes our proof. □

3C Proof of Theorem 1.16

Theorem 1.16 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$ and let X be an R–DGA whose corresponding HR–algebra is equivalent to $HR \wedge Z$ for some cofibrant $\mathbb{S}$–algebra Z whose underlying spectrum is equivalent to a coproduct of (de)suspensions of the sphere spectrum. Also, let Y be an R–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof Let HR be cofibrant as a commutative $\mathbb{S}$–algebra in Shipley's convenient model structure. This guarantees that $HR \wedge -$ preserves weak equivalences [13, 5.3.10]. Since $HR \wedge -$ preserves weak equivalences, we can further assume Z to be fibrant.

Let Y be an R–DGA. Since quasi-isomorphic R–DGAs are always topologically equivalent, we only need to show that X and Y are quasi-isomorphic if they are topologically equivalent. Abusing notation, we also let Y denote a fibrant HR–algebra corresponding to the R–DGA Y. We assume that X and Y are topologically equivalent, ie $HR \wedge Z$ and Y are equivalent as $\mathbb{S}$–algebras. Using this, we are going to show that there is a weak equivalence

$$\psi\colon HR \wedge Z \xrightarrow{\sim} Y$$

of HR–algebras.

Let $g\colon T \xrightarrow{\sim} HR \wedge Z$ be a cofibrant replacement of $HR \wedge Z$ in $\mathbb{S}$–algebras. As in diagram (5), there exists a map $f: Z \to T$ such that

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & T \\ \downarrow & \overset{f}{\nearrow} & \simeq \downarrow g \\ Z & \xrightarrow[h_Z]{} & HR \wedge Z \end{array} \tag{7}$$

commutes. Here, h_Z denotes the canonical map

$$h_Z\colon Z \cong \mathbb{S} \wedge Z \to HR \wedge Z.$$

Since X and Y are topologically equivalent, T and Y are equivalent as $\mathbb{S}$–algebras. Furthermore, T is cofibrant and Y is fibrant; therefore we have a weak equivalence

$$\varphi\colon T \xrightarrow{\sim} Y$$

of $\mathbb{S}$–algebras.

We obtain the composite map

$$\psi\colon HR \wedge Z \xrightarrow{HR \wedge f} HR \wedge T \xrightarrow{HR \wedge \varphi} HR \wedge Y \xrightarrow{m} Y$$

of HR–algebras where m denotes the HR–module structure map of Y. Note that the last map above is a map of HR–algebras as it is the left adjoint of the identity map of Y under the usual adjunction between the categories of HR–algebras and $\mathbb{S}$–algebras. Since ψ is a map of HR–algebras, it is sufficient to show that ψ induces an isomorphism in homotopy.

We have the commuting diagram

$$\begin{array}{ccccc} \mathbb{S} \wedge T & \xrightarrow{\mathbb{S}\wedge\varphi} & \mathbb{S} \wedge Y & & \\ \downarrow{\scriptstyle h_T} & & \downarrow & \overset{\text{id}}{\searrow} & \\ HR \wedge T & \xrightarrow{HR\wedge\varphi} & HR \wedge Y & \xrightarrow{m} & Y \end{array}$$

where the vertical maps are the canonical maps induced by the unit map $u_R\colon \mathbb{S} \to HR$. This shows that the composite map starting from $T \cong \mathbb{S} \wedge T$ and ending in Y is given by φ and therefore is a weak equivalence. In particular, $\pi_*(m \circ (HR \wedge \varphi))$ is an isomorphism when it is restricted to the image of the Hurewicz map of T

$$\pi_* h_T\colon \pi_*(\mathbb{S} \wedge T) \to \pi_*(HR \wedge T).$$

Therefore, in order to prove that ψ_* is an isomorphism, it is sufficient to show that the map

$$\pi_*(HR \wedge f)\colon \pi_*(HR \wedge Z) \to \pi_*(HR \wedge T)$$

is injective and its image agrees with the image of $\pi_* h_T$. For this, it is sufficient to prove that the corresponding statements are true after composing with the isomorphism

$$\pi_*(HR \wedge g)\colon \pi_*(HR \wedge T) \xrightarrow{\cong} \pi_*(HR \wedge HR \wedge Z).$$

In other words, it is sufficient to show that

$$\pi_*(HR \wedge g) \circ \pi_*(HR \wedge f)$$

is injective and the image of this map agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$. Due to (7), $g \circ f = h_Z$. Therefore, it is sufficient to show that $\pi_*(HR \wedge h_Z)$ is injective in homotopy and its image agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$.

The composite

$$HR \wedge Z \xrightarrow{HR\wedge h_Z} HR \wedge HR \wedge Z \xrightarrow{m\wedge \text{id}} HR \wedge Z$$

is the identity map, where m denotes the multiplication map of HR and id denotes the identity map of Z. From this, we deduce that $\pi_*(HR \wedge h_Z)$ is injective in homotopy, as desired. What remains to prove is that the image of $\pi_*(HR \wedge h_Z)$ agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$.

Due to the commuting diagram

$$\begin{array}{ccc} \mathbb{S} \wedge T & \xrightarrow[\simeq]{g} & \mathbb{S} \wedge HR \wedge Z \\ \downarrow h_T & & \downarrow h_{HR \wedge Z} \\ HR \wedge T & \xrightarrow[\simeq]{HR \wedge g} & HR \wedge HR \wedge Z \end{array}$$

the image of the map $\pi_*(HR \wedge g) \circ \pi_* h_T$ is given by the image of the Hurewicz map

$$\pi_*(h_{HR \wedge Z}) \colon \pi_*(\mathbb{S} \wedge HR \wedge Z) \to \pi_*(HR \wedge HR \wedge Z)$$

of $HR \wedge Z$. Note that $h_{HR \wedge Z}$ is induced by the unit map of HR as usual. Therefore, it is sufficient to show that the image of $\pi_*(HR \wedge h_Z)$ agrees with the image of $\pi_*(h_{HR \wedge Z})$.

The map $HR \wedge h_Z$ is the canonical map

$$HR \wedge Z \cong HR \wedge \mathbb{S} \wedge Z \to HR \wedge HR \wedge Z.$$

This is the same as the composite

$$HR \wedge Z \cong \mathbb{S} \wedge HR \wedge Z \xrightarrow{h_{HR \wedge Z}} HR \wedge HR \wedge Z \xrightarrow{\tau \wedge \mathrm{id}} HR \wedge HR \wedge Z, \tag{8}$$

where τ is the transposition map of the monoidal structure. Since the map $h_{HR \wedge Z}$ in the middle of the composite in (8) induces $\pi_*(h_{HR \wedge Z})$, it is sufficient to show that $\pi_*(\tau \wedge \mathrm{id})$ is the identity map on the image of $\pi_*(h_{HR \wedge Z})$.

By hypothesis, the underlying spectrum of Z is a wedge of suspensions of the sphere spectrum. Let

$$E = \bigvee_{a \in A} \Sigma^{|a|} \mathbb{S}$$

be weakly equivalent to Z as a spectrum where A is a graded set. Since E is cofibrant and Z is fibrant, there is a weak equivalence of spectra $E \xrightarrow{\sim} Z$.

This equivalence induces the vertical maps in the commuting diagram of $\mathbb{S}$–modules,

$$\begin{array}{ccccc} HR \wedge E & \xrightarrow{h_{HR \wedge E}} & HR \wedge HR \wedge E & \xrightarrow{\tau \wedge \mathrm{id}} & HR \wedge HR \wedge E \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ HR \wedge Z & \xrightarrow{h_{HR \wedge Z}} & HR \wedge HR \wedge Z & \xrightarrow{\tau \wedge \mathrm{id}} & HR \wedge HR \wedge Z \end{array}$$

where $h_{HR\wedge E}$ denotes the canonical map that induces the Hurewicz map of $HR \wedge E$ in homotopy. In order to show that $\pi_*(\tau \wedge \mathrm{id})$ (of the bottom row) is the identity map on the image of $\pi_*(h_{HR\wedge Z})$, it is sufficient to show that $\pi_*(\tau \wedge \mathrm{id})$ (of the top row) is given by the identity map on the image of $\pi_*(h_{HR\wedge E})$. For this, it is sufficient to show that the composite of the maps on the top row is given by $\pi_*(h_{HR\wedge E})$ in homotopy.

Note that the canonical R–module basis elements of

$$\pi_*(HR \wedge E) = \pi_*\Bigl(HR \wedge \Bigl(\bigvee_{a\in A} \Sigma^{|a|}\mathbb{S}\Bigr)\Bigr) \cong \bigoplus_{a\in A} \Sigma^{|a|} R$$

are also abelian group generators because $R = \mathbb{Z}/(m)$ for some integer m. Therefore, it is sufficient to show that

$$\pi_*(\tau \wedge \mathrm{id}) \circ \pi_*(h_{HR\wedge E})(x) = \pi_*(h_{HR\wedge E})(x)$$

for every canonical basis element x. Such an x is represented by a map

$$u_{HR} \wedge i_a \colon \mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \to HR \wedge \Bigl(\bigvee_{a\in A} \Sigma^{|a|}\mathbb{S}\Bigr) = HR \wedge E,$$

where i_a is the inclusion of the cofactor corresponding to an $a \in A$.

In other words, it is sufficient to show that the composite

$$\mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \xrightarrow{u_{HR}\wedge i_a} HR \wedge E \xrightarrow{h_{HR\wedge E}} HR \wedge HR \wedge E \xrightarrow{\tau\wedge \mathrm{id}} HR \wedge HR \wedge E$$

agrees with the composite

$$\mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \xrightarrow{u_{HR}\wedge i_a} HR \wedge E \xrightarrow{h_{HR\wedge E}} HR \wedge HR \wedge E.$$

To see this, note that the composite maps above are of the form $\upsilon \wedge i_a$ and $\nu \wedge i_a$, respectively, where υ and ν are $\mathbb{S}$–algebra maps from $\mathbb{S}$ to $HR \wedge HR$. Since $\mathbb{S}$ is the initial object in the category of $\mathbb{S}$–algebras, we deduce that $\upsilon = \nu$. Therefore, the two composites above agree, as claimed. □

4 E–infinity $\mathbb{F}_p$–DGAs are not extension

This section is devoted to the proof of Theorems 1.7 and 1.8. We restate these theorems below. Recall that when we say extension (E_∞) DGA, we mean $\mathbb{Z}$–extension (E_∞) $\mathbb{Z}$–DGA.

Theorem 1.7 *Let Y be an E_∞ DGA. For all primes p, if Y is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA then Y is not an extension E_∞ DGA.*

Theorem 1.8 *Let X be a DGA. If X is quasi-isomorphic to an $\mathbb{F}_2$–DGA then X is not an extension DGA.*

In the proof of these theorems, we use the ring structure and the Dyer–Lashof operations on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{Z}) = H\mathbb{F}_{p*}H\mathbb{Z}$. For odd p, the ring structure is given by

$$H\mathbb{F}_{p*}H\mathbb{Z} \cong \mathbb{F}_p[\zeta_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\bar{\tau}_s \mid s \geq 1),$$

where the degrees of the generators are the same as those of the dual Steenrod algebra. Note that $H\mathbb{F}_{p*}H\mathbb{Z}$ has the same generators as the dual Steenrod algebra except that $H\mathbb{F}_{p*}H\mathbb{Z}$ does not contain the degree 1 generator τ_0. Indeed, the map

$$H\mathbb{F}_{p*}H\mathbb{Z} \to H\mathbb{F}_{p*}H\mathbb{F}_p = \mathcal{A}_*$$

induced by $H\mathbb{Z} \to H\mathbb{F}_p$ is the canonical inclusion [21, II.10.26]. This inclusion is induced by a map of commutative $H\mathbb{F}_p$–algebras and therefore it preserves the Dyer–Lashof operations. Therefore through this map, the Dyer–Lashof operations on the dual Steenrod algebra determine the Dyer–Lashof operations on $H\mathbb{F}_{p*}H\mathbb{Z}$; see [7, III.2].

For $p = 2$, we have

$$H\mathbb{F}_{2*}H\mathbb{Z} = \mathbb{F}_2[\zeta_1^2] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\zeta_r \mid r \geq 2],$$

where $|\zeta_i| = 2^i - 1$ for $i \geq 2$ and $|\zeta_1^2| = 2$. Again, the canonical map

$$H\mathbb{F}_{2*}H\mathbb{Z} \to H\mathbb{F}_{2*}H\mathbb{F}_2 = \mathcal{A}_*$$

is the canonical inclusion and this determines the Dyer–Lashof operations on $H\mathbb{F}_{2*}H\mathbb{Z}$.

For the rest of this section, we assume that $H\mathbb{Z}$ is cofibrant as a commutative $\mathbb{S}$–algebra and $H\mathbb{F}_p$ is cofibrant as a commutative $H\mathbb{Z}$–algebra in the model structure developed in [25]. Since the category of commutative $H\mathbb{Z}$–algebras is the same as the category of commutative $\mathbb{S}$–algebras under $H\mathbb{Z}$, cofibrations of commutative $H\mathbb{Z}$–algebras forget to cofibrations of commutative $\mathbb{S}$–algebras. Therefore, $H\mathbb{Z} \to H\mathbb{F}_p$ is also a cofibration of commutative $\mathbb{S}$–algebras. This ensures that $H\mathbb{F}_p$ is also cofibrant as a commutative $\mathbb{S}$–algebra and therefore the functor $H\mathbb{F}_p \wedge -$ preserves all weak equivalences [13, 5.3.10].

We start by proving the following lemma. This lemma is obvious if one assumes that for a map of discrete commutative rings $R \to R'$, the Quillen equivalences of [19; 26] are compatible with the restriction of scalars functors from (E_∞) R'–DGAs to (E_∞) R–DGAs and from (commutative) HR'–algebras to (commutative) HR–algebras. However, there is no such compatibility result available in the literature and proving it is beyond the scope of this work.

Lemma 4.1 *Let X be an E_∞ DGA that is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA. Then there is a map of commutative $H\mathbb{Z}$–algebras*

$$H\mathbb{F}_p \to H_{E_\infty}X,$$

where $H_{E_\infty}X$ denotes a fibrant commutative $H\mathbb{Z}$–algebra corresponding to the E_∞ DGA X.

If X is a DGA that is quasi-isomorphic to an $\mathbb{F}_p$–DGA, then there is a map of $H\mathbb{Z}$–algebras

$$c(H\mathbb{F}_p) \to HX,$$

where HX denotes a fibrant $H\mathbb{Z}$–algebra corresponding to the DGA X. Here, $c(H\mathbb{F}_p)$ denotes a cofibrant replacement of $H\mathbb{F}_p$ in $H\mathbb{Z}$–algebras.

Proof We only prove the E_∞ case; the associative case follows in a similar manner. Assume that we are using a unital E_∞ operad, ie an operad given by the monoidal unit $\mathbb{F}_p$ in operadic degree zero. The Barratt–Eccles operad is an example of a unital E_∞–operad [4]. In this situation, $\mathbb{F}_p$ is the free E_∞ $\mathbb{F}_p$–DGA generated by the trivial $\mathbb{F}_p$–chain complex 0. Therefore, $\mathbb{F}_p$ is the initial object in E_∞ $\mathbb{F}_p$–DGAs. This, together with the fact that X is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA implies that there is a map $\mathbb{F}_p \to X$ in the homotopy category of E_∞ DGAs.

The equivalence of categories between the homotopy categories of commutative $H\mathbb{Z}$–algebras and E_∞ DGAs implies that there is also a map $H\mathbb{F}_p \to H_{E_\infty}X$ in the homotopy category of commutative $H\mathbb{Z}$–algebras. Since $H_{E_\infty}X$ is fibrant in commutative $H\mathbb{Z}$–algebras and $H\mathbb{F}_p$ is cofibrant in commutative $H\mathbb{Z}$–algebras due to our standing assumptions, there is a map $H\mathbb{F}_p \to H_{E_\infty}X$ of commutative $H\mathbb{Z}$–algebras as desired. □

The following starts with the proof of Theorem 1.7, and at the end we mention how this also shows Theorem 1.8.

Proof of Theorems 1.7 and 1.8 Assume to the contrary that there is an extension E_∞ DGA X that is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA. It follows by Lemma 4.1 that there is a map $H\mathbb{F}_p \to H_{E_\infty}X$ of commutative $H\mathbb{Z}$–algebras where $H_{E_\infty}X$ denotes a fibrant commutative $H\mathbb{Z}$–algebra corresponding to the E_∞ DGA X. In particular, the $H\mathbb{Z}$–structure map $H\mathbb{Z} \to H_{E_\infty}X$ of $H_{E_\infty}X$ factors as

$$H\mathbb{Z} \xrightarrow{\varphi_{H\mathbb{F}_p}} H\mathbb{F}_p \to H_{E_\infty}X,$$

where $\varphi_{H\mathbb{F}_p}$ denotes the canonical map.

Since X is a $\mathbb{Z}$–extension E_∞ DGA, there is a cofibrant commutative $\mathbb{S}$–algebra Y such that $H\mathbb{Z} \wedge Y$ is weakly equivalent to $H_{E_\infty} X$ in commutative $H\mathbb{Z}$–algebras.

Note that $H\mathbb{Z} \wedge Y$ is cofibrant as a commutative $H\mathbb{Z}$–algebra; this is the case because $H\mathbb{Z} \wedge -$ is a left Quillen functor from commutative $\mathbb{S}$–algebras to commutative $H\mathbb{Z}$–algebras and therefore it preserves cofibrant objects.

Since $H_{E_\infty} X$ is fibrant and $H\mathbb{Z} \wedge Y$ is cofibrant, there is a weak equivalence of commutative $H\mathbb{Z}$–algebras $\psi \colon H\mathbb{Z} \wedge Y \xrightarrow{\sim} H_{E_\infty} X$. Because ψ is a map of commutative $H\mathbb{Z}$–algebras, we obtain a commutative diagram

$$\begin{array}{ccccc} & & H\mathbb{Z} & & \\ & \swarrow^{\varphi_{H\mathbb{Z}\wedge Y}} & & \searrow^{\varphi_{H\mathbb{F}_p}} & \\ & & & & H\mathbb{F}_p \\ & & & & \searrow \\ H\mathbb{Z} \wedge Y & & \xrightarrow[\simeq]{\psi} & & H_{E_\infty} X \end{array}$$

where the composite on the right from $H\mathbb{Z}$ to $H_{E_\infty} X$ is the composite given above. The map $\varphi_{H\mathbb{Z}\wedge Y}$ is the $H\mathbb{Z}$–structure map of $H\mathbb{Z} \wedge Y$ which is given by

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge Y.$$

Applying the homology functor $H\mathbb{F}_{p*}$ to this diagram and inverting $H\mathbb{F}_{p*}\psi$, we obtain

$$\begin{array}{ccccc} & & H\mathbb{F}_{p*}H\mathbb{Z} & & \\ & \swarrow^{H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}} & & \searrow^{H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}} & \\ & & & & H\mathbb{F}_{p*}H\mathbb{F}_p \\ & & & & \searrow \\ H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y & & \xleftarrow[(H\mathbb{F}_{p*}\psi)^{-1}]{\cong} & & H\mathbb{F}_{p*}H_{E_\infty} X \end{array}$$

By the Künneth spectral sequence in [10, IV.4.1],

$$H\mathbb{F}_{p*}(H\mathbb{Z} \wedge Y) \cong H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$$

and the morphism on the left is given by

$$H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}(a) = a \otimes_{\mathbb{F}_p} 1. \tag{9}$$

Since the diagram above commutes, $H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}$ factors as

$$H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y} \colon H\mathbb{F}_{p*}H\mathbb{Z} \xrightarrow{H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}} H\mathbb{F}_{p*}H\mathbb{F}_p \xrightarrow{f} H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y, \tag{10}$$

where the second map f is the composite in the triangle above starting from $H\mathbb{F}_{p*}H\mathbb{F}_p$ and ending in the bottom left corner. Both maps in the composite above are ring maps that preserve the Dyer–Lashof operations.

Let p denote an odd prime; we discuss the case $p = 2$ at the end of this proof. We have $\beta Q^1\tau_0 = \zeta_1$ (up to a unit we are going to omit) in $H\mathbb{F}_{p*}H\mathbb{F}_p$. Note that $f(\zeta_1) = \zeta_1 \otimes_{\mathbb{F}_p} 1$. This follows by considering the composite in (10), equality (9) and by noting that $H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}$ is the canonical inclusion. Since f preserves Dyer–Lashof operations,

$$\beta Q^1 f(\tau_0) = f(\beta Q^1\tau_0) = f(\zeta_1) = \zeta_1 \otimes_{\mathbb{F}_p} 1.$$

We conclude that $\beta Q^1 f(\tau_0) = \zeta_1 \otimes_{\mathbb{F}_p} 1$ in $H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$.

We obtain a contradiction by showing that there is no z in $H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$ that satisfies $\beta Q^1 z = \zeta_1 \otimes_{\mathbb{F}_p} 1$, ie there is no candidate for $f(\tau_0)$. For an element of the form $1 \otimes_{\mathbb{F}_p} y \in H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$, we have that $\beta Q^1(1 \otimes_{\mathbb{F}_p} y) = 1 \otimes_{\mathbb{F}_p} \beta Q^1 y$ does not contain $\zeta_1 \otimes_{\mathbb{F}_p} 1$ as a summand. Now consider an element of the form $a \otimes_{\mathbb{F}_p} y \in H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$ with $|a| > 0$. By the Cartan formula and the fact that the Bockstein operation is a derivation, $\beta Q^1(a \otimes_{\mathbb{F}_p} y)$ is a sum of elements of the form $a' \otimes_{\mathbb{F}_p} y'$ where a' is obtained by applying a Dyer–Lashof operation to a. In particular, $|a'| > |a| \geq |\zeta_1|$; therefore $\beta Q^1(a \otimes_{\mathbb{F}_p} y)$ does not contain $\zeta_1 \otimes_{\mathbb{F}_p} 1$ as a summand either. We deduce that $\beta Q^1 z$ does not contain $\zeta_1 \otimes_{\mathbb{F}_p} 1$ as a summand for all $z \in H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$.

For $p = 2$, we do not need to use the Dyer–Lashof operations. In this case, we have $f(\zeta_1^2) = \zeta_1^2 \otimes_{\mathbb{F}_2} 1$ due to the composite in (10). We obtain that $f(\zeta_1)^2 = \zeta_1^2 \otimes_{\mathbb{F}_2} 1$. However, there is no element in $H\mathbb{F}_{2*}H\mathbb{Z} \otimes_{\mathbb{F}_2} H\mathbb{F}_{2*}Y$ that squares to $\zeta_1^2 \otimes_{\mathbb{F}_2} 1$. Since this does not use Dyer–Lashof operations, this argument at $p = 2$ also works for DGAs and $H\mathbb{Z}$–algebras and provides a proof of Theorem 1.8. □

5 Formal DGAs to $H\mathbb{Z}$–algebras

This section is devoted to the proof of Proposition 5.8 which provides an explicit description of the HR–algebra corresponding to a formal R–DGA whose homology satisfies the hypothesis of Theorem 1.4. This description provides Theorem 1.4. Recall that we also use Proposition 5.8 to obtain Corollary 1.17.

We work in several different monoidal categories in this section. When we work in the category of chain complexes or in the category of differential graded algebras, we denote the monoidal product by $\otimes$. For the categories of HR–modules and HR–algebras, we denote the smash product by $\wedge_{HR}$ as before. In all the other cases, we let $\wedge$ denote the monoidal product. In this section, HR denotes the Eilenberg–Mac Lane spectrum of a general discrete commutative ring as in [13, 1.2.5].

Let X be an R–DGA satisfying the hypothesis of Theorem 1.4. Recall from Remark 1.5 that there is a monoid M in graded pointed sets for which $H_*(X) \cong R\langle M\rangle$ as R–algebras where the underlying R–module of $R\langle M\rangle$ is the free graded R–module over the graded set M_- obtained by removing the base point of M. Furthermore, the multiplication on $R\langle M\rangle$ is the canonical one induced by that of M. For the rest of this section, let M denote a monoid in nonnegatively graded pointed sets.

5A A monoid object corresponding to M

Here, we construct a monoid in a general monoidal category by using M. Furthermore, we show that this construction is preserved by strong monoidal Quillen pairs.

We start by explaining a notation we use for the symmetric monoidal pointed model categories we consider in this section. For a cofibrant C, ΣC denotes the pushout of the diagram $\bar{*} \leftarrowtail C \rightarrowtail \bar{*}$, where $\bar{*}$ is obtained by a factorization $C \rightarrowtail \bar{*} \xrightarrow{\sim} \!\!\!\!\rightarrow *$ of the map $C \to *$ by a cofibration followed by a trivial fibration, and $*$ denotes the final object. For the unit $\mathbb{I}$ of the monoidal structure, $\Sigma^k\mathbb{I}$ denotes $(\Sigma\mathbb{I})^{\wedge k}$ for $k > 0$ and denotes $\mathbb{I}$ for $k = 0$.

Construction 5.1 Let $(\mathcal{C}, \wedge, \mathbb{I})$ denote a pointed cofibrantly generated closed symmetric monoidal model category whose unit $\mathbb{I}$ is cofibrant. Furthermore, assume that $\mathcal{C}$ satisfies the monoid axiom and the smallness axioms of [22]. This implies that the category of modules over a monoid in $\mathcal{C}$ carries an induced model structure where the weak equivalences and the fibrations are those created by the forgetful functor to $\mathcal{C}$ [22, 4.1]. For a given M as above, we construct a monoid structure on

$$\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I},$$

where $\vee$ denotes the coproduct in $\mathcal{C}$. The multiplication map

$$(11)\quad \Big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}\Big) \wedge \Big(\bigvee_{n\in M_-} \Sigma^{|n|}\mathbb{I}\Big) \cong \bigvee_{(m,n)\in M_-\times M_-} \Sigma^{|m|+|n|}\mathbb{I} \to \bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}$$

is given (on the cofactor corresponding to $(m,n) \in M \times M$) by the inclusion of the cofactor corresponding to $mn \in M$ if $mn \neq 0$ and given by the zero map if $mn = 0$.

Note that in a pointed model category, there is a unique zero map between every pair of objects which is defined to be the map that factors through the point object. One easily checks that the multiplication above is associative and unital.

If E is a commutative monoid in $\mathcal{C}$, then the category of E–modules is also a symmetric monoidal model category [22, 4.1]. We let $\bigvee_{m\in M_-} \Sigma^{|m|} E$ denote the monoid we obtain by applying the construction above in the category of E–modules. In particular, $\bigvee_{m\in M_-} \Sigma^{|m|} E$ is an E–algebra.

Using the construction above, we obtain an HR–algebra $\bigvee_{m\in M_-} \Sigma^{|m|} HR$. In order to prove Theorem 1.4, we go through the zigzag of Quillen equivalences between the model categories of R–DGAs and HR–algebras to show that the HR–algebra corresponding to the formal R–DGA with homology $R\langle M\rangle$ is given by $\bigvee_{m\in M_-} \Sigma^{|m|} HR$ [26]. We deduce that the formal R–DGA with homology $R\langle M\rangle$ is R–extension by showing that $\bigvee_{m\in M_-} \Sigma^{|m|} HR$ is weakly equivalent to $HR \wedge c\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\big)$ in HR–algebras where c denotes the cofibrant replacement functor in $\mathbb{S}$–algebras. For this, we start with the following lemmas.

Lemma 5.2 *Assume that $(\mathcal{C}, \wedge, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ are pointed and closed symmetric monoidal model categories with cofibrant units. Furthermore, let*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a Quillen pair, where F denotes the left adjoint. If there is a weak equivalence $\upsilon\colon F(\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \mathbb{I}_{\mathcal{D}}$, then there exists a weak equivalence

$$\varphi\colon F(\Sigma\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma\mathbb{I}_{\mathcal{D}}.$$

Proof By factoring the map $\mathbb{I}_{\mathcal{C}} \to *$ by a cofibration followed by a trivial fibration, we obtain a factorization $F(\mathbb{I}_{\mathcal{C}}) \rightarrowtail F(\bar{*}) \xrightarrow{\sim} F(*) \cong *$. Note that the isomorphism follows by the fact that F is a left adjoint functor between pointed categories. To see that the second map is a weak equivalence, note that $*$ is cofibrant in the pointed model category $\mathcal{C}$ and that F preserves all weak equivalences between cofibrant objects. Similarly, we have a factorization $\mathbb{I}_{\mathcal{D}} \rightarrowtail \bar{*} \overset{\sim}{\twoheadrightarrow} *$ consisting of a cofibration followed by a trivial fibration. We use the equivalence $\upsilon\colon F(\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \mathbb{I}_{\mathcal{D}}$ and the lift in the square

$$\begin{array}{ccccc}
F(\mathbb{I}_{\mathcal{C}}) & \xrightarrow[\sim]{\upsilon} & \mathbb{I}_{\mathcal{D}} & \longrightarrow & \bar{*} \\
\downarrow & & \overset{\sim}{\nearrow} & & \downarrow\sim \\
F(\bar{*}) & & \xrightarrow{\qquad\sim\qquad} & & F(*) \cong *
\end{array}$$

to obtain a weak equivalence of diagrams

$$(F(\bar{*}) \leftarrowtail F(\mathbb{I}_{\mathcal{C}}) \rightarrowtail F(\bar{*})) \xrightarrow{\sim} (\bar{*} \leftarrowtail \mathbb{I}_{\mathcal{D}} \rightarrowtail \bar{*}).$$

This in turn gives a map φ of the corresponding pushouts of these diagrams. This is a weak equivalence because these are diagrams consisting only of cofibrations between cofibrant objects; therefore their pushout is the homotopy pushout. Since the pushout of the diagram on the left-hand side is $F(\Sigma\mathbb{I}_{\mathcal{C}})$ and the pushout of the diagram on the right-hand side is $\Sigma\mathbb{I}_{\mathcal{D}}$, we obtain the weak equivalence

$$\varphi\colon F(\Sigma\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma\mathbb{I}_{\mathcal{D}}$$

we wanted to construct. □

Lemma 5.3 *Assume that $(\mathcal{C}, \wedge, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ are pointed and closed symmetric monoidal model categories with cofibrant units as in Construction 5.1. Furthermore, let*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a Quillen pair where the left adjoint F is a strong monoidal functor. In this situation, $Fc\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$ are weakly equivalent as monoids in $\mathcal{D}$, where c denotes the cofibrant replacement functor in the model category of monoids in $\mathcal{C}$ [22, 4.1].

Proof Since F is a strong monoidal functor, we have a natural isomorphism

$$F(X) \wedge F(Y) \cong F(X \wedge Y)$$

and an isomorphism $F(\mathbb{I}_{\mathcal{C}}) \cong \mathbb{I}_{\mathcal{D}}$. This isomorphism provides the weak equivalence υ in the hypothesis of Lemma 5.2. Thus, there is a weak equivalence $\varphi\colon F(\Sigma\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma\mathbb{I}_{\mathcal{D}}$.

Using φ, we produce a weak equivalence of monoids,

$$\Phi\colon F\Big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\Big) \cong \bigvee_{m\in M_-} F(\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}.$$

Here, Φ is the coproduct of maps given by the isomorphism $F(\mathbb{I}_{\mathcal{C}}) \cong \mathbb{I}_{\mathcal{D}}$ for $|m| = 0$ and the map

$$F(\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}) = F((\Sigma\mathbb{I}_{\mathcal{C}})^{\wedge|m|}) \cong F(\Sigma\mathbb{I}_{\mathcal{C}})^{\wedge|m|} \underset{\sim}{\xrightarrow{\varphi^{\wedge|m|}}} (\Sigma\mathbb{I}_{\mathcal{D}})^{\wedge|m|} = \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$$

for $|m| > 0$, where the first and the last equalities follow by our definition of $\Sigma^k -$ for $k > 0$ and the second isomorphism comes from the strong monoidal structure of F.

Also, note that $\varphi^{\wedge|m|}$ is a weak equivalence because it is a smash product of weak equivalences between cofibrant objects. Since Φ is a coproduct of weak equivalences between cofibrant objects, it is a weak equivalence by [28, Lemma 4.7]. It is clear that Φ is a map of monoids by the definition of the monoidal structure on both sides and from the fact that left adjoint functors between pointed categories preserve the zero maps. This shows that Φ is a weak equivalence of monoids between $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$.

Therefore, in order to finish the proof of the lemma, it is sufficient to show that the monoids $Fc\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ are weakly equivalent. Since c is the cofibrant replacement functor in the category of monoids, there is a weak equivalence of monoids

$$f: c\Big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\Big) \xrightarrow{\sim} \bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}.$$

By [22, Theorem 4.1], the source of f is cofibrant in $\mathcal{C}$. This means that f is a weak equivalence between cofibrant objects and therefore $F(f)$ is a weak equivalence. Furthermore, $F(f)$ is a weak equivalence of monoids because a strong monoidal functor preserves maps of monoids. Therefore, the monoids $Fc\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ are weakly equivalent as desired. □

5B From DGAs to $H\mathbb{Z}$–algebras

Here, we carry out our discussion for the case $R=\mathbb{Z}$. The case of general discrete commutative ring R follows similarly.

The DGA corresponding to an $H\mathbb{Z}$–algebra is obtained using the zigzag of monoidal Quillen equivalences of [26]

$$H\mathbb{Z}\text{-}\mathcal{M}od \underset{U}{\overset{Z}{\rightleftarrows}} \mathrm{Sp}^{\Sigma}(s\mathcal{A}B) \underset{\phi^*N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^{\Sigma}(\mathcal{C}h^+) \underset{R}{\overset{D}{\rightleftarrows}} \mathcal{C}h,$$

where the left adjoints are the top arrows and the pairs (Z,U) and (D,R) are both strong monoidal Quillen equivalences. The pair (L,ϕ^*N) is a weak monoidal Quillen equivalence. See [23, 3.6] for the definitions of strong monoidal Quillen equivalences and weak monoidal Quillen equivalences. We often use the fact that the model categories in the zigzag above are pointed.

Since each Quillen equivalence in the zigzag is a monoidal Quillen equivalence, there is an induced zigzag of Quillen equivalences of the corresponding model categories of monoids. This gives the induced derived functors $\mathbb{H}\colon \mathcal{D}GA \to H\mathbb{Z}\text{–}\mathcal{A}lg$ and

$\Theta\colon H\mathbb{Z}\text{–}\mathcal{A}lg \to \mathcal{D}GA$ in [26, Theorem 1.1]. We have

$$\Theta = Dc\phi^* NZc, \quad \mathbb{H} = UL^{\mathrm{mon}}cR,$$

where L^{mon} is the induced left adjoint at the level of monoids and c denotes the cofibrant replacement functors in the corresponding model category of monoids. See [23, Section 3.3] for a definition of the induced left adjoint at the level of monoids. Recall that for a given DGA X, we often write HX to denote $\mathbb{H}X$ or a cofibrant and/or fibrant replacement of $\mathbb{H}X$ as an $H\mathbb{Z}$–algebra.

In the lemmas below, $\mathbb{I}_1$ and $\mathbb{I}_2$ denote the monoidal units of $\mathrm{Sp}^{\Sigma}(s\mathcal{A}B)$ and $\mathrm{Sp}^{\Sigma}(\mathcal{C}h^+)$ respectively. Note that the units of the monoidal model categories in the zigzag above are all cofibrant [26, Definition 2.1 and Corollary 3.4]. By Construction 5.1, we have the monoids $\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_1$ and $\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_2$ in $\mathrm{Sp}^{\Sigma}(s\mathcal{A}B)$ and $\mathrm{Sp}^{\Sigma}(\mathcal{C}h^+)$, respectively.

Lemma 5.4 *In $\mathrm{Sp}^{\Sigma}(s\mathcal{A}B)$, $Zc\big(\bigvee_{m\in M_-} \Sigma^{|m|} H\mathbb{Z}\big)$ and $\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_1$ are weakly equivalent as monoids. In $\mathcal{C}h$, $Dc(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_2)$ and the formal DGA with homology $\mathbb{Z}\langle M\rangle$ are quasi-isomorphic as DGAs.*

Proof The first statement is a direct consequence of Lemma 5.3. We prove the second statement of the lemma. It again follows by Lemma 5.3 that $Dc\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_2\big)$ and $\bigoplus_{m\in M_-} \Sigma^{|m|}\mathbb{Z}$ are quasi-isomorphic as DGAs (ie weakly equivalent as monoids in $\mathcal{C}h$).

Therefore, it is sufficient to show that $\bigoplus_{m\in M_-} \Sigma^{|m|}\mathbb{Z}$ is quasi-isomorphic to the formal DGA with homology $\mathbb{Z}\langle M\rangle$. Let $\bar{0}$ denote the chain complex consisting of $\mathbb{Z}$ in degrees 0 and 1 and the trivial module in the rest of the degrees; its differentials are trivial except degree 1 where the differential is the identity. There is a factorization $\mathbb{Z} \rightarrowtail \bar{0} \xrightarrow{\sim} 0$ of the trivial map $\mathbb{Z} \to 0$ as a cofibration followed by a trivial fibration.

Let $\sigma\mathbb{Z}$ denote the chain complex consisting of $\mathbb{Z}$ in degree 1 and the trivial module in the rest of the degrees. This is the pushout of the diagram $\bar{0} \leftarrowtail \mathbb{Z} \to 0$.

Note that due to our conventions, $\Sigma\mathbb{Z}$ is the pushout of the diagram $\bar{0} \leftarrowtail \mathbb{Z} \to \bar{0}$. Since the category of chain complexes is left proper, there is a weak equivalence $\varphi\colon \Sigma\mathbb{Z} \xrightarrow{\sim} \sigma\mathbb{Z}$. Let $\sigma^n\mathbb{Z}$ denote $(\sigma\mathbb{Z})^{\otimes n}$. Following Construction 5.1, we obtain a formal DGA $\bigoplus_{m\in M_-} \sigma^{|m|}\mathbb{Z}$. Similar to the map Φ in the proof of Lemma 5.3, we obtain a quasi-isomorphism of DGAs

$$\Phi\colon \bigoplus_{m\in M_-} \Sigma^{|m|}\mathbb{Z} \xrightarrow{\sim} \bigoplus_{m\in M_-} \sigma^{|m|}\mathbb{Z}$$

given by the identity map for $|m| = 0$ and given by $\varphi^{|m|}$ for $|m| > 0$. This shows that $\bigoplus_{m \in M_-} \Sigma^{|m|}\mathbb{Z}$ and $\bigoplus_{m \in M_-} \sigma^{|m|}\mathbb{Z}$ are quasi-isomorphic as DGAs where the latter is the formal DGA with homology $\mathbb{Z}\langle M \rangle$. □

We state and prove the following two lemmas, which we use in the proof of Lemma 5.7.

Lemma 5.5 *The functor $\phi^* N$ preserves colimits.*

Proof The category of symmetric spectra in a closed symmetric monoidal model category $\mathcal{C}$ is the category of modules over a monoid in symmetric sequences in $\mathcal{C}$; see [26, Definition 2.7]. Since symmetric sequences in $\mathcal{C}$ is a diagram category in $\mathcal{C}$, the colimits in symmetric sequences are levelwise. Furthermore, the forgetful functor from modules over a monoid to the underlying closed monoidal category preserves colimits. Therefore colimits of symmetric spectra in $\mathcal{C}$ are also levelwise.

Here, N is the normalization functor $s\mathcal{AB} \to \mathcal{C}h^+$ of the Dold–Kan correspondence, an equivalence of categories, applied levelwise. Therefore it preserves colimits. Furthermore, ϕ^* is the restriction of scalars functor between the categories of modules over two monoids induced by a map of these monoids in symmetric sequences in $\mathcal{C}h^+$; see [26, page 358]. Therefore ϕ^* is the identity functor on the underlying symmetric sequences and therefore it also preserves colimits. □

Lemma 5.6 *For every cofibrant A in $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$ and every B in $\mathrm{Sp}^\Sigma(s\mathcal{AB})$, a map $L(A) \to B$ is a weak equivalence if and only if its adjoint $A \to \phi^* N(B)$ is a weak equivalence.*

Proof This follows from the fact that $\phi^* N$ preserves weak equivalences. Let $B \xrightarrow{\sim} fB$ be a fibrant replacement of B. The adjoint of the composite $L(A) \to B \xrightarrow{\sim} fB$ is given by the composite $A \to \phi^* N(B) \xrightarrow{\sim} \phi^* N(fB)$ whose first map is the adjoint of the map $L(A) \to B$. Because $(L, \phi^* N)$ is a Quillen equivalence, the first composite is a weak equivalence if and only if the second composite is a weak equivalence. The result follows by the two-out-of-three property of weak equivalences. □

The following lemma takes care of the middle step in the zigzag of Quillen equivalences between the model categories of $H\mathbb{Z}$–algebras and DGAs. Note that since $(L, \phi^* N)$ is a weak monoidal Quillen pair, $\phi^* N$ is a lax monoidal functor; see [23, Definition 3.3]. Therefore, $\phi^* N$ carries monoids to monoids. In particular, $\phi^* N\big(\bigvee_{m \in M_-} \Sigma^{|m|}\mathbb{I}_1\big)$ is a monoid.

Lemma 5.7 *In* $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$, $\phi^*N\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1\big)$ *and* $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2$ *are weakly equivalent as monoids.*

Proof By Lemma 5.5, ϕ^*N preserves coproducts. Therefore, there is an isomorphism

$$\phi^*N\Big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1\Big)\cong\bigvee_{m\in M_-}\phi^*N(\Sigma^{|m|}\mathbb{I}_1). \tag{12}$$

Similar to Construction 5.1, the object on the right-hand side above carries a canonical monoid structure given by the multiplication on M and the lax monoidal structure of ϕ^*N. Namely, the multiplication map

$$\bigvee_{m\in M_-}\phi^*N(\Sigma^{|m|}\mathbb{I}_1)\wedge\bigvee_{n\in M_-}\phi^*N(\Sigma^{|n|}\mathbb{I}_1)\to\bigvee_{m\in M_-}\phi^*N(\Sigma^{|m|}\mathbb{I}_1)$$

is given (on the cofactor corresponding to $(m,n)\in M_-\times M_-$) by the composite

$$\phi^*N(\Sigma^{|m|}\mathbb{I}_1)\wedge\phi^*N(\Sigma^{|n|}\mathbb{I}_1)\to\phi^*N(\Sigma^{|m|}\mathbb{I}_1\wedge\Sigma^{|n|}\mathbb{I}_1)=\phi^*N(\Sigma^{|mn|}\mathbb{I}_1)$$

followed by the inclusion of the cofactor corresponding to $mn\in M$ if $mn\neq 0$ and given by the zero map if $mn=0$. Note that the map above is the lax monoidal structure map of ϕ^*N and the equality above follows by our definition of Σ^k-. Furthermore, one checks using this definition that the isomorphism in (12) is an isomorphism of monoids. Therefore, in order to prove the lemma, it is sufficient to show that there is an isomorphism of monoids between $\bigvee_{m\in M_-}\phi^*N(\Sigma^{|m|}\mathbb{I}_1)$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2$.

There is a weak equivalence $L(\mathbb{I}_2)\xrightarrow{\sim}\mathbb{I}_1$ since (L,ϕ^*N) is a weak monoidal Quillen pair; see [23, 3.6]. Therefore, there is also a weak equivalence $\varphi\colon L(\Sigma\mathbb{I}_2)\xrightarrow{\sim}\Sigma\mathbb{I}_1$ by Lemma 5.2. Let

$$\psi\colon\Sigma\mathbb{I}_2\to\phi^*N(\Sigma\mathbb{I}_1)$$

be the adjoint of φ.

Let ψ^0 denote the unit $\mathbb{I}_2\to\phi^*N(\mathbb{I}_1)$ of the lax monoidal structure of ϕ^*N and let ψ^1 denote ψ. For $\ell>1$, let ψ^ℓ denote the composite

$$\psi^\ell\colon\Sigma^\ell\mathbb{I}_2=(\Sigma\mathbb{I}_2)^{\wedge\ell}\xrightarrow{\psi^{\wedge\ell}}(\phi^*N(\Sigma\mathbb{I}_1))^{\wedge\ell}\to\phi^*N((\Sigma\mathbb{I}_1)^{\wedge\ell})=\phi^*N(\Sigma^\ell\mathbb{I}_1),$$

where the equalities follow by our definition of Σ^ℓ and the second map is obtained by successive applications of the transformation $\phi^*N(-)\wedge\phi^*N(-)\to\phi^*N(-\wedge-)$ that is a part of the lax monoidal structure of ϕ^*N; see [23, 3.3].

Now we define a map of monoids

$$\Psi\colon \bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_2 \to \bigvee_{m\in M_-} \phi^* N(\Sigma^{|m|}\mathbb{I}_1)$$

as the coproduct of $\psi^{|m|}$ over $m \in M_-$. By the associativity and the unitality of the lax monoidal structure on $\phi^* N$ and by the fact that right adjoint functors preserve the zero maps between pointed categories, Ψ is a map of monoids; see [6, 6.4.1].

Finally, we need to show that Ψ is a weak equivalence. By Lemmas 5.5 and 5.6, it is sufficient to show that the adjoint of Ψ is a weak equivalence. Since both $\phi^* N$ and L preserve coproducts and since Ψ is a coproduct of maps $\psi^{|m|}$, the adjoint of Ψ is a coproduct of the adjoints of the maps $\psi^{|m|}$. Note that a coproduct of weak equivalences of cofibrant objects is again a weak equivalence by [28, 4.7]. Since the adjoint of ψ^ℓ is a map between cofibrant objects, it is sufficient to show that the adjoint of ψ^ℓ is a weak equivalence for each $\ell \geq 0$.

For the case $\ell = 0$, we have that the adjoint of ψ^0 is the weak equivalence $L(\mathbb{I}_2) \xrightarrow{\sim} \mathbb{I}_1$ mentioned above. For $\ell = 1$, the adjoint of ψ^1 is the map φ above which is also a weak equivalence.

We show the $\ell = 2$ case and the rest follow similarly. Let $m_{\phi^* N}$ denote the natural transformation

$$m_{\phi^* N}\colon \phi^* N(- \wedge -) \to \phi^* N(-) \wedge \phi^* N(-)$$

that is part of the lax monoidal structure of $\phi^* N$. We show that the adjoint to the composite defining ψ^2

$$\psi^2\colon \Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\psi\wedge\psi} \phi^* N(\Sigma\mathbb{I}_1) \wedge \phi^* N(\Sigma\mathbb{I}_1) \xrightarrow{m_{\phi^* N}} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1)$$

is the composite map

$$L(\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2) \underset{\simeq}{\xrightarrow{c_L}} L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2) \underset{\simeq}{\xrightarrow{\varphi\wedge\varphi}} \Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1. \tag{13}$$

The first map in this composite is the comonoidal map induced by the lax monoidal structure of $\phi^* N$ and this is a weak equivalence since $(L, \phi^* N)$ is a weak monoidal Quillen pair [26, 4.4]. Furthermore, the second map in the composite is a smash product of weak equivalences between cofibrant objects; therefore, it is also a weak equivalence. This shows that the composite is a weak equivalence.

To show that ψ^2 is the adjoint to this composite, first note that by the discussion on equation (3.4) in [23], the comonoidal map c_L is the adjoint of the composite map

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{m_{\phi^* N}} \phi^* N(L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2)),$$

where η denotes the unit of the adjunction $(L, \phi^* N)$. Considering the adjoint of the composite (13) as the adjoint of the first map c_L in the composite followed by $\phi^* N(\varphi \wedge \varphi)$, we obtain that the adjoint of (13) is given by the composite

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{m_{\phi^* N}} \phi^* N(L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2)) \xrightarrow{\phi^* N(\varphi\wedge\varphi)} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1).$$

By the naturality of $m_{\phi^* N}$, this composite is equal to the canonical composite

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{\phi^* N(\varphi)\wedge\phi^* N(\varphi)} \phi^* N(\Sigma\mathbb{I}_1) \wedge \phi^* N(\Sigma\mathbb{I}_1) \xrightarrow{m_{\phi^* N}} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1).$$

Note that the composition of the first two maps is the smash product of adjoints of φ which is $\psi \wedge \psi$. Therefore, this composite is precisely the composite that defines ψ^2 above. This shows that the adjoint of ψ^2 is the composite weak equivalence in (13). □

5C Proof of Theorem 1.4

We prove the following proposition which provides an explicit description of the HR–algebra corresponding to the formal R–DGA with homology $R\langle M\rangle$. After that, we use this description to prove Theorem 1.4.

Proposition 5.8 *The R–DGA corresponding to the HR–algebra $\bigvee_{m\in M_-} \Sigma^{|m|} HR$ is the formal R–DGA with homology $R\langle M\rangle$. Furthermore, there is an equivalence of HR–algebras*

$$\bigvee_{m\in M_-} \Sigma^{|m|} HR \simeq HR \wedge c\Big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\Big),$$

where c denotes the cofibrant replacement functor in $\mathbb{S}$–algebras.

Proof For the first statement, we discuss the case $R = \mathbb{Z}$, the proof for general R follows similarly. The first statement is a consequence of Lemmas 5.4 and 5.7.

Now we prove the second statement. Recall that $HR \wedge -$ is a symmetric monoidal functor between $\mathbb{S}$–modules and HR–modules. Therefore, the second statement is consequence of Lemma 5.3. □

Theorem 1.4 *Let X be a connective formal R–DGA whose homology has a homogeneous basis as an R–module containing the multiplicative unit such that the multiplication of two basis elements is either zero or a basis element. In this situation, X is R–extension. As a result, we have the equivalence of spectra,*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

Proof Recall from Remark 1.5 that the homology of X is $R\langle M\rangle$ for some monoid M in nonnegatively graded pointed sets. In other words, X is the formal R–DGA with homology $R\langle M\rangle$. Using Proposition 5.8, we deduce that the HR–algebra corresponding to X is $\bigvee_{m\in M_-} \Sigma^{|m|} HR$. By the equivalence given in Proposition 5.8, X is an R–extension R–DGA.

Since X is an R–extension R–DGA, the splitting for $\mathrm{THH}(X)$ is a consequence of Proposition 1.3. □

We are ready to prove the following corollaries of our results.

Corollary 1.17 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$, let Y be an R–DGA and let X be as in Theorem 1.4. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof By Remark 1.5, X is the formal R–DGA with homology $R\langle M\rangle$ for some monoid M in nonnegatively graded pointed sets. Using Proposition 5.8, we deduce that the HR–algebra corresponding to X is given by $HR \wedge c\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\big)$, where c denotes the cofibrant replacement functor in HR–algebras. In particular, $Z = c\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\big)$ is weakly equivalent as a spectrum to a wedge of suspensions of the sphere spectrum. We deduce that X satisfies the hypothesis of Theorem 1.16. This implies that X and Y are quasi-isomorphic if and only if they are topologically equivalent. □

Corollary 1.18 *Assume that X and Y are as in Theorem 1.14 or Theorem 1.16. Then X and Y are Morita equivalent if and only if they are dg Morita equivalent.*

Proof We need to show that the model categories of X–modules and Y–modules are additively Quillen equivalent if and only if they are Quillen equivalent [9, 7.7]. By definition, additively Quillen equivalent additive model categories are Quillen equivalent [8]. Therefore, we only need to prove one direction.

If the model categories of X–modules and Y–modules are Quillen equivalent then there exists a fibrant and cofibrant representative P of a compact generator of the homotopy category of Y–modules such that the endomorphism DGA $\mathrm{End}_{Y\text{–mod}}(P)$ of P is topologically equivalent to X [9, 7.2]. Since Y is an R–DGA, $\mathrm{End}_{Y\text{–mod}}(P)$ is also an R–DGA. It follows by Theorems 1.14 and 1.16 that $\mathrm{End}_{Y\text{–mod}}(P)$ quasi-isomorphic to X. By [9, 7.2], this implies that the model categories of X–modules and Y–modules are additively Quillen equivalent, as desired. □

Appendix A

Here, we provide a short discussion on the compatibility of Definitions 1.1 and 1.2.

If we choose our E_∞ operad to be the Barratt–Eccles operad, then every E_∞ R–DGA is at the same time an R–DGA; see [4, Section 1.1.1]. Let X be an R–extension E_∞ R–DGA and let $U(X)$ denote its underlying R–DGA. The canonical compatibility question asks if $U(X)$ is R–extension as an R–DGA. In other words, we want to know if every R–extension E_∞ R–DGA forgets to an R–extension R–DGA.

Let $H_{E_\infty}X$ denote the commutative HR–algebra corresponding to X and let $HU(X)$ denote the HR–algebra corresponding to $U(X)$. For the moment, assume that $H_{E_\infty}X$ is weakly equivalent to $HU(X)$ as an HR–algebra. Under this assumption, we conclude that $U(X)$ is R–extension. To see this, let $H_{E_\infty}X \simeq HR \wedge E$ for some cofibrant commutative $\mathbb{S}$–algebra E and let c denote the cofibrant replacement functor in $\mathbb{S}$–algebras. Since cofibrant (commutative) $\mathbb{S}$–algebras forget to cofibrant $\mathbb{S}$–modules [22; 25] and since the left Quillen functor $HR\wedge -$ preserves weak equivalences between cofibrant objects, we deduce that $HR \wedge E$ is equivalent to $HR \wedge cE$ in HR–algebras. Hence, $HU(X)$ is weakly equivalent to $HR \wedge cE$ and therefore $U(X)$ is R–extension, as desired.

However, it is not known whether $H_{E_\infty}X$ and $HU(X)$ are weakly equivalent in HR–algebras. In other words, it is not known if the zigzag of Quillen equivalences between HR–algebras and R–DGAs in [26] is compatible with the zigzag of Quillen equivalences between commutative HR–algebras and E_∞ R–DGAs in [19]. In conclusion, if we assume that these Quillen equivalences are compatible, then Definitions 1.1 and 1.2 are also compatible in the sense described above.

Appendix B

Here, we provide a proof of Proposition 1.3. Indeed, we prove the following more general statement.

Proposition B.1 *Let $\varphi\colon A \to B$ be a map of commutative $\mathbb{S}$–algebras and let X be a B–algebra. If X is φ–extension, ie if $X \simeq B \wedge_A E$ for some cofibrant A–algebra E, then there is the equivalence of spectra*

$$\mathrm{THH}^A(X) \simeq \mathrm{THH}^A(B) \wedge_B \mathrm{THH}^B(X).$$

Furthermore, if X is a commutative B–algebra that is weakly equivalent to $B \wedge_A E$ for some cofibrant commutative A–algebra E, then the equivalence above is an equivalence of commutative ring spectra.

Proof Let $X \simeq B \wedge_A E$ for some cofibrant A–algebra E. The equivalence in the proposition is given by the composite of the chain of equivalences

$$\begin{aligned}\mathrm{THH}^A(B \wedge_A E) &\simeq \mathrm{THH}^A(B) \wedge_A \mathrm{THH}^A(E)\\ &\simeq \mathrm{THH}^A(B) \wedge_B (B \wedge_A \mathrm{THH}^A(E))\\ &\simeq \mathrm{THH}^A(B) \wedge_B \mathrm{THH}^B(B \wedge_A E).\end{aligned} \tag{14}$$

The first equivalence follows by the fact that $\mathrm{THH}^A(-)$ is a monoidal functor and the last equivalence follows by the base change formula for topological Hochschild homology; see [15, Conventions]. The base change formula and the monoidality of $\mathrm{THH}^A(-)$ can be easily shown using the cyclic bar construction defining topological Hochschild homology [10, IX.2.1].

When E is a cofibrant commutative A–algebra, the equivalences given in (14) are those of commutative A–algebras. This is because $\mathrm{THH}^A(-)$ is a symmetric monoidal functor and the base change formula provides an equivalence of commutative A–algebras. □

The following is the special case of the proposition above corresponding to the map of commutative $\mathbb{S}$–algebras $\mathbb{S} \to HR$. Note that for an R–DGA X, we let $\mathrm{THH}(X)$ denote $\mathrm{THH}(HX)$ and $\mathrm{HH}^R(X)$ denote $\mathrm{THH}^{HR}(HX)$. For an E_∞ R–DGA X, we let $\mathrm{THH}(X)$ denote $\mathrm{THH}(H_{E_\infty}X)$ and $\mathrm{HH}^R(X)$ denote $\mathrm{THH}^{HR}(H_{E_\infty}X)$.

Proposition 1.3 *If X is an R–extension R–DGA, then there is an equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

If X is an R–extension E_∞ R–DGA, then the equivalence above is an equivalence of commutative $\mathbb{S}$–algebras.

Proof For an R–extension R–DGA X, we have that HX satisfies the first hypothesis of Proposition B.1 for the map of commutative $\mathbb{S}$–algebras $\varphi\colon \mathbb{S} \to HR$. This provides the equivalence in the proposition. Similarly, for an R–extension E_∞ R–DGA X, $H_{E_\infty}X$ satisfies the last hypothesis of Proposition B.1. This provides the second statement of the proposition. □

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