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On the cohomology ring of symplectic fillings

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We consider symplectic cohomology twisted by sphere bundles, which can be viewed as an analogue of symplectic cohomology with local systems. Using the associated Gysin exact sequence, we prove the uniqueness of part of the ring structure on cohomology of fillings for those asymptotically dynamically convex manifolds with vanishing property considered by Zhou (Int. Math. Res. Not. 2020 (2020) 9717–9729 and J. Topol. 14 (2021) 112–182). In particular, for any simply connected $4n+1$–dimensional flexibly fillable contact manifold Y, we show that the real cohomology $H^*(W)$ is unique as a ring for any Liouville filling W of Y as long as $c_1(W)=0$. Uniqueness of real homotopy type of Liouville fillings is also obtained for a class of flexibly fillable contact manifolds.

53D40; 57R17

1 Introduction

It is conjectured that Liouville fillings of certain contact manifolds are unique. The first result along this line is that Liouville fillings of the standard contact 3–sphere are unique; see Gromov [13]. The dimension 4 case is special because of the intersection theory of J–holomorphic curves. For higher-dimensional cases only weaker assertions can be made so far. Eliashberg, Floer and McDuff [21] proved that any symplectically aspherical filling of the standard contact sphere of dimension ≥ 5 is diffeomorphic to a ball. Oancea and Viterbo [22] showed that $H_*(Y;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ is surjective for a simply connected subcritically fillable contact manifold Y and any symplectically aspherical W. Barth, Geiges and Zehmisch [4] generalized the Eliashberg–Floer–McDuff theorem to the subcritically fillable case assuming Y is simply connected and of dimension ≥ 5. Roughly speaking, the method used to obtain the above results is finding a "homological foliation", which is hinted at by the splitting result of Cieliebak [8, Theorem 14.16] for subcritical domains.

On the other hand, contact manifolds considered above are asymptotically dynamically convex (ADC) in the sense of Lazarev [18], which is a much larger class of contact manifolds. Those contact manifolds admit only the trivial (contact DGA) augmentation due to degree reasons, hence one may expect that the filling is rigid in some sense. In [29; 30; 31], we studied fillings of such manifolds using Floer theories. Roughly speaking, a contact manifold is ADC if and only if the symplectic field theory (SFT) grading is positive (asymptotically). By the neck-stretching argument, such a condition is sufficient to prove invariance of many structures in [30; 31]. However, in many cases, the SFT gradings are greater than some positive integer k, which provides more room in the neck-stretching argument. The goal of this paper is trying to make use of this extra room and getting more information. In particular, we will study the ring structure of symplectic fillings. Throughout this paper, the default coefficient is $\mathbb{R}$. Our main theorem is the following, where we call a Liouville filling W of Y *topologically simple* if and only if $c_1(W)=0$ and $\pi_1(Y)\to\pi_1(W)$ is injective:

Theorem 1.1 *Let Y be a k–ADC contact manifold* (Definition 2.2) *with a topologically simple Liouville filling W_1 and $SH^*(W_1)=0$. Then for any topologically simple Liouville filling W_2, there is a linear isomorphism $\phi\colon H^*(W_1)\to H^*(W_2)$ preserving grading such that $\phi(\alpha\wedge\beta)=\phi(\alpha)\wedge\phi(\beta)$ for all $\alpha\in H^{2m}(W_1)$ with $2m\le k+1$.*

The main example where Theorem 1.1 can be applied is a flexibly fillable contact manifold Y^{2n-1}, which is $(n-3)$–ADC by Lazarev [18]. In particular, combining with [30, Corollary B], we have:

Corollary 1.2 *Let Y be a simply connected $4n+1$–dimensional flexibly fillable contact manifold with $c_1(Y)=0$. Then the real cohomology ring of Liouville fillings of Y with vanishing first Chern class is unique.*

Remark 1.3 By [30, Corollary B], manifolds considered in Theorem 1.1 have the property that $H^*(W;\mathbb{Z})\to H^*(Y;\mathbb{Z})$ is independent of topologically simple Liouville fillings. Therefore on the degree region where the restriction map is injective, we can infer the ring structure of the filling from the boundary. However, such a method cannot yield information when the product lands in a degree region where the restriction is not injective, for example, the middle degree for flexibly fillable manifolds. The method used in this paper asserts the uniqueness of product structure in those ambiguous regions.

There are also non-Weinstein examples to which Theorem 1.1 can be applied; see Section 2. If Y is subcritically fillable, then the π_1–injective condition can be dropped because Reeb orbits can be assumed to be contractible [18]. However, this case is covered by both [30, Corollary B] and [4, Theorem 1.2] along with the universal coefficient theorem.

In some cases, the knowledge of the cohomology ring is enough to determine the real homotopy type. In particular:

Corollary 1.4 *Let M be the product of $\mathbb{CP}^n$, $\mathbb{HP}^n$, S^{2n} and at most one copy of S^{2n+1} for $n \geq 1$, and let Y denote the contact boundary of the flexible cotangent bundle of M. Then the real homotopy type of a Liouville filling of Y is unique, as long as the Liouville filling has vanishing first Chern class.*

The method used in this paper is very different from the method used in [4; 20; 22], where they studied the moduli spaces of J–holomorphic curves in a partial compactification of W. The essential property needed for the partial compactification is that W splits as $V \times \mathbb{C}$ with V Weinstein. However, for many flexible critical Weinstein domains, such a splitting does not exist even in the topology category, eg flexible cotangent bundles T^*S^{2n} cannot be written as a complex line bundle over a manifold for $n > 1$.[1] Our method is based on symplectic cohomology and uses the index property of the contact boundary, hence we need to assume $c_1 = 0$. The strategy of the proof is to represent the cup product as a multiplication with an Euler class of a sphere bundle. Therefore we consider symplectic cohomology twisted by sphere bundles, which leads to Gysin exact sequences. The Gysin exact sequence associated to a k–sphere bundle uses moduli spaces of dimension up to k. We show that the Gysin exact sequence for a k–sphere bundle on the positive symplectic cohomology is independent of the filling by a neck-stretching argument, if the boundary is k–ADC. Then we can relate it to the regular Gysin sequence of the filling by the vanishing result in [29].

Remark 1.5 The reason for restricting to real coefficients is twofold. Firstly, it is not true that every class in $H^{2k}(M;\mathbb{Z})$ can be represented as the Euler class of an oriented vector bundle (see Walschap [25]) unless multiplied by a large integer (see Guijarro, Schick and Walschap [14]), which only depends on the degree and dimension. Secondly, the Gysin exact sequence is derived from the Morse–Bott framework developed in Zhou [28], which is defined over $\mathbb{R}$. In our case, one can get Gysin exact sequences in

[1] Assume otherwise. Then T^*S^{2n} can be written as a complex line bundle over some manifold V with boundary, and since $H^2(V;\mathbb{Z}) = 0$ when $n > 1$, the complex line bundle is necessarily trivial. Therefore $T^*S^{2n} = V \times \mathbb{D}$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. On the other hand, $H^*(T^*S^{2n};\mathbb{Z}) \to H^*(\partial T^*S^{2n};\mathbb{Z})$ is not injective (in degree $2n$) but $H^*(V \times \mathbb{D};\mathbb{Z}) \to H^*(\partial(V \times \mathbb{D});\mathbb{Z})$ is always injective, hence we arrive at a contradiction.

$\mathbb{Z}$–coefficient, since our moduli spaces do not have isotropy or weight. For example, one can generalize the Morse–Bott construction in Hutchings and Nelson [17] to sphere bundles to prove a $\mathbb{Z}$–coefficient Gysin exact sequence.

Remark 1.6 By symplectic cohomology we mean the symplectic cohomology generated by *contractible* orbits. The role of topological simplicity of the filling is to guarantee that the symplectic cohomology of the filling is canonically graded by $\mathbb{Z}$ using any trivialization of $\det \xi$ on Y. From the SFT perspective, it is related to the fact that the augmentation from the filling is (canonically) graded by $\mathbb{Z}$. Since the ADC condition only asserts unique contact DGA augmentation with a $\mathbb{Z}$ grading, we can only hope for uniqueness for topologically simple fillings using the ADC condition; see also [30, Remark 3.6].

Organization of the paper

Section 2 reviews the contact geometry background and provides a list of examples where Theorem 1.1 applies. In Section 3, we define the symplectic cohomology of sphere bundles and prove the independence result when the boundary is k–ADC. We finish the proof of Theorem 1.1, its corollaries and applications in Section 4.

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2 Asymptotically dynamically convex manifolds

Let α be a contact form of (Y^{2n-1}, ξ) and $D > 0$. We use $\mathcal{P}^{<D}(Y, \alpha)$ to denote the set of contractible Reeb orbits of α with period smaller than D. Letting α_1 and α_2 be two contact forms of (Y, ξ), we write $\alpha_1 \geq \alpha_2$ if $\alpha_1 = f\alpha_2$ for $f \geq 1$. For a nondegenerate Reeb orbit γ, the degree is defined to be $\mu_{CZ}(\gamma) + n - 3$, which is canonically defined in $\mathbb{Z}$ if $c_1(\xi) = 0$ and γ is contractible.

Definition 2.1 [18, Definition 3.6] A contact manifold (Y, ξ) with $c_1(\xi) = 0$ is asymptotically dynamically convex (ADC) if there exists a nonincreasing sequence of contact forms $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ for ξ and positive numbers $D_1 < D_2 < D_3 < \cdots$ going to infinity such that all elements of $\mathcal{P}^{<D_i}(Y, \alpha_i)$ are nondegenerate and have positive degree.

One important consequence of ADC is that the positive symplectic cohomology is independent of the filling W whenever $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective [18, Proposition 3.8]. Moreover, many Floer theoretic properties of the filling are independent of fillings [30; 31]. Roughly speaking, ADC guarantees the 0–dimensional moduli spaces used in the definition of the positive symplectic cohomology are completely contained in the cylindrical end of the completion $\widehat{W}$, hence are independent of the filling. We consider sphere bundles over (positive) symplectic cohomology. The information for k–sphere bundles is encoded in moduli spaces with dimension up to k. In particular, the associated Gysin exact sequence depends on moduli spaces with dimension up to k. Therefore we need more positivity in the degree of Reeb orbits, so the following finer dynamical convexity is needed:

Definition 2.2 A contact manifold (Y, ξ) with $c_1(\xi) = 0$ is k–ADC if there exists a nonincreasing sequence of contact forms $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ for ξ and positive numbers $D_1 < D_2 < D_3 < \cdots$ going to infinity such that all elements of $\mathcal{P}^{<D_i}(Y, \alpha_i)$ are nondegenerate and have degree greater than k.

Similarly, we say a Liouville domain (W, λ) with $c_1(W) = 0$ is k–ADC if and only if there exist positive functions $f_1 \geq f_2 \geq \cdots$ and positive numbers $D_1 < D_2 < D_3 < \cdots$ going to infinity such that all contractible (in W) orbits of $(\partial W, f_i \lambda)$ of period up to D_i are nondegenerate and have degree greater than k.

In particular, $(k+1)$–ADC implies k–ADC, and 0–ADC is the usual ADC condition in [18]. The basic example of a k–ADC manifold is the standard contact sphere S^{2n-1}, which is $(2n-3)$–ADC. From this basic example, the following propositions yield many k–ADC manifolds.

Proposition 2.3 [18, Theorems 3.15, 3.17, and 3.18] *Let Y be a $(2n-3-k)$–ADC contact manifold. Then the attachment of an index $k \neq 2$ subcritical or flexible handle to Y^{2n-1} preserves the $(2n-3-k)$–ADC property. When $k = 2$, the same holds if the conditions in* [18, Theorem 3.17] *are met.*

Let V be a manifold with boundary. We define the Morse dimension $\dim_M V$ to be the minimal value of the maximal index of an exhausting Morse function on V.

Proposition 2.4 [30, Theorem 6.3] *Let V^{2n} be a Liouville domain with $c_1(V)=0$. Then $\partial(V\times\mathbb{C})$ is $(2n-1-\dim_M V)$–ADC.*

Proposition 2.5 [30, Theorem 6.19] *Let V and W be p– and q–ADC domains, respectively. Then $\partial(V\times W)$ is k–ADC, where*

$$k=\min\{p+q+4,\ p+\dim W-\dim_M W, q+\dim V-\dim_M V\}.$$

Example 2.6 Using the above three propositions, we can apply Theorem 1.1 to classes of contact manifolds:

(i) By Proposition 2.3, for $n\geq 3$, any $2n-1$–dimensional flexibly fillable contact manifold Y with $c_1(Y)=0$ is $(n-3)$–ADC.

(ii) By Proposition 2.4, if V is the $2n$–dimensional Liouville but not Weinstein domain constructed in [19], then $\partial(V\times\mathbb{C}^k)$ is $(2k-2)$–ADC.

(iii) By Proposition 2.5, products of any k–ADC domain for $k>0$ with an example from the above two classes are m–ADC for a suitable $m>0$. We can also attach a flexible handle afterwards.

In general, there are many more k–ADC contact manifolds of interest, eg cotangent bundles and links of terminal singularities. For certain cotangent bundles, symplectic cohomology is zero with an appropriate local system [2]. In general, symplectic cohomology in these cases is not zero, hence they are beyond our scope.

3 Symplectic cohomology and fiber bundles

In this section, we review some basic properties of symplectic cohomology associated to a Liouville domain [9; 23; 24]. Then we introduce the symplectic cohomology of sphere bundles and the associated Gysin exact sequences using the abstract Morse–Bott framework developed in [28].

3.1 Symplectic cohomology

3.1.1 Floer cochain complexes To a Liouville filling (W,λ) of the contact manifold (Y,ξ), one can associate the completion $(\widehat{W},d\hat{\lambda})=(W\cup_Y[1,\infty)_r\times Y,d\hat{\lambda})$, where $\hat{\lambda}=\lambda$ on W and $\hat{\lambda}=r(\lambda|_Y)$ on $[1,\infty)_r\times Y$. Let $H\colon S^1\times\widehat{W}\to\mathbb{R}$ be a Hamiltonian. Our convention for the Hamiltonian vector field is

$$\omega(\,\cdot\,,X_H)=dH.$$

Then symplectic cohomology is defined as the "Morse cohomology" of the symplectic action functional

$$\mathcal{A}_H(x) := -\int x^*\hat{\lambda} + \int H \circ x(t)\, dt, \tag{1}$$

for a Hamiltonian $H = r^2$ for $r \gg 0$ [23; 24]. Equivalently, one may define symplectic cohomology as the direct limit of the Hamiltonian Floer cohomology of $H = Dr$ for $r \gg 0$ as D goes to infinity. For simplicity, we will use the former construction and a special class of Hamiltonian in this paper. Let α be a nondegenerate contact form of the contact manifold (Y, ξ) and R_α its associated Reeb vector field. Then we define

$$\mathcal{S}(Y,\alpha) := \left\{ \int_\gamma \alpha \;\middle|\; \gamma \text{ is the periodic orbit of } R_\alpha \right\}.$$

Following [7], we can choose a smooth family of time-dependent Hamiltonians H_R for $R \in [0,1]$ as a careful perturbation of an autonomous Hamiltonian, such that the following hold:

(i) $H_R|_W$ is time independent C^2–small Morse for $R \neq 0$, and $H_0|_W = 0$.

(ii) There exists a sequence of nonempty open intervals $(a_0, b_0), (a_1, b_1), \dots$ with a_i and b_i converging to infinity and $a_0 = 1$ such that $H_R|_{Y\times(a_i,b_i)} = f_{i,R}(r)$ with $f''_{i,R} > 0$ and $f'_{i,R} \notin \mathcal{S}(Y,\alpha)$, and $\lim_i \min f'_{i,R} = \infty$.

(iii) H_R outside $r = b_0$ does not depend on R.

(iv) For $R \neq 0$, the periodic orbits of X_{H_R} are nondegenerate, and are either critical points of $H_R|_W$ or nonconstant orbits in $\partial W \times [b_i, a_{i+1}]$.

(v) There exist $0 < D_0 < D_1 < \dots \to \infty$ such that all periodic orbits of X_{H_R} of action greater than $-D_i$ are contained in $W^i := \{r < a_i\}$.

(vi) $\partial_R H_R \leq 0$.

We use $\mathcal{C}(H_R)$ to denote the set of critical points of H_R on W and $\mathcal{P}^*(H)$ to denote the set of nonconstant *contractible* orbits of X_{H_R} outside W, which does not depend on R.

Remark 3.1 A few remarks regarding our choice of Hamiltonian are in order.

(i) We do not define symplectic cohomology of sphere bundles as an invariant, but rather use one model to infer topological information. Therefore we choose to work with one specific Hamiltonian.

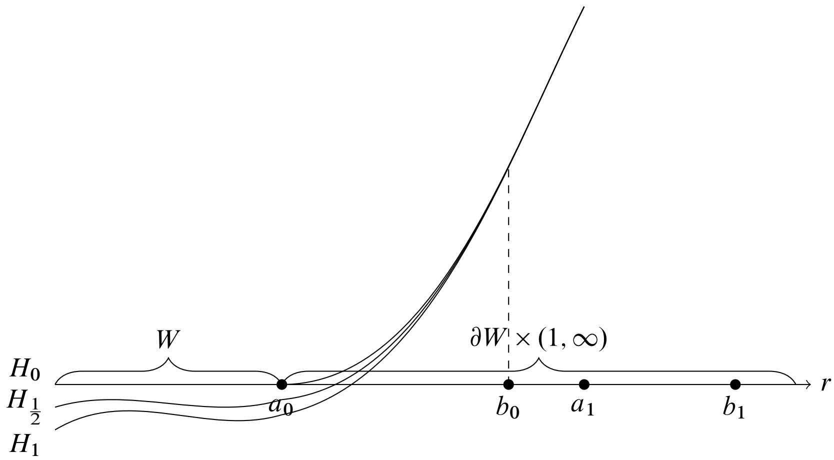

Figure 1: Graphs of H_R.

(ii) The requirement of H_R on interval (a_i, b_i) is for the purpose of the integrated maximum principle [1; 9]. In particular, with an admissible almost complex structure in Definition 3.2, any Floer cylinder asymptotic to orbits in W^i will be completely contained in W^i.

(iii) Ideally, we would like to work with H_0, where the neck-stretching argument will be cleaner. H_0 can be viewed as a "Morse–Bott" situation, which is used in [30]. We will use the nondegenerate Hamiltonian H_R for $R > 0$ to approximate H_0, because the relevant polyfolds are easier to construct and partially exist in the literature; see Remark 3.5.

(iv) The requirement $\partial_R H_R \leq 0$ ensures that the continuation map from H_{R_+} to H_{R_-} respects the action filtration for $R_+ > R_-$. The independence of H_R outside $r = b_0$ simplifies the continuation map for the positive symplectic cohomology to the identity map for different R.

For an admissible Hamiltonian H_R, there are infinitely many periodic orbits and they are not bounded in the r–coordinate. To guarantee the compactness of moduli spaces, we need to use the following almost complex structure so that the integrated maximum principle [1] can be applied.

Definition 3.2 An S^1–dependent almost complex structure J_t is admissible if the following hold:

(i) J_t is compatible with $d\hat{\lambda}$ on $\widehat{W}$.

(ii) J_t is cylindrical convex on $\partial W \times (a_i, b_i)$, that is, $\hat{\lambda} \circ J_t = dr$.

(iii) J_t is only required to be S^1–independent on W. We will often abbreviate J_t by J for simplicity.

The set of admissible almost complex structures is denoted by $\mathcal{J}(W)$.

Let $x, y \in \mathcal{C}(H_R) \cup \mathcal{P}^*(H)$ for $R > 0$ and J be an admissible almost complex structure. We use $\mathcal{M}_{x,y,H_R}$ to denote the compactified moduli space of solutions to the following equation modulo the $\mathbb{R}$–translation:

$$\partial_s u + J(\partial_t u - X_{H_R}) = 0, \quad \lim_{s \to \infty} u = x, \quad \lim_{s \to -\infty} u = y. \tag{2}$$

We will suppress H_R when there is no confusion. Then we have the following regularity result:

Proposition 3.3 *For any $R > 0$, there exists a subset $\mathcal{J}^R(W) \subset \mathcal{J}(W)$ of second Baire category such that the following hold*:

(i) *For all $x, y \in \mathcal{C}(H_R) \cup \mathcal{P}^*(H)$ the manifold $\mathcal{M}_{x,y}$ is compact and smooth with boundary and corners.*

(ii) $\partial \mathcal{M}_{x,z} = \bigcup_y \mathcal{M}_{x,y} \times \mathcal{M}_{y,z}$.

(iii) *$\mathcal{M}_{x,z}$ can be oriented so that the induced orientation of $\partial \mathcal{M}_{x,z}$ on $\mathcal{M}_{x,y} \times \mathcal{M}_{y,z}$ is given by the product orientation twisted by $(-1)^{\dim \mathcal{M}_{x,y}}$.*

(iv) *If $x \in \mathcal{C}(H_R)$ and $y \in \mathcal{P}^*(H)$, then $\mathcal{M}_{x,y} = \varnothing$.*

This proposition is folklore, although it is usually stated and proven for moduli spaces $\mathcal{M}_{x,y}$ with virtual dimensional smaller than or equal to 1. Since $\widehat{W}$ is exact and J can depend on $t \in S^1$, we have transversality for unbroken Floer trajectories. A more classical treatment to prove the first two claims is constructing compatible gluing maps for families of Floer trajectories. In the case of Lagrangian Floer theory, such a construction can be found in [3]. In the case of Morse theory, a more elementary approach can be used to give the compactified moduli spaces structures of manifolds with boundary and corners; see [27]. Another method is adopting the polyfold theory developed in [16]. In view of this, we make the following assumption.

Assumption 3.4 For any admissible almost complex structure J, there exists an M-polyfold construction for the symplectic cohomology moduli spaces. More precisely, for every $x, y \in \mathcal{C}(H_R) \cup \mathcal{P}^*(H)$, there exists a strong tame M-polyfold bundle $\mathcal{E}_{x,y} \to \mathcal{B}_{x,y}$ along with an oriented proper sc-Fredholm section $s_{x,y} \colon \mathcal{B}_{x,y} \to \mathcal{E}_{x,y}$ such that the following hold:

(i) $s_{x,y}^{-1}(0) = \mathcal{M}_{x,y}$, where $\mathcal{M}_{x,y}$ is the compact moduli space using J.

(ii) Classical transversality implies that $s_{x,y}$ is transverse and in general position.

(iii) The boundary of $\mathcal{B}_{x,z}$ is the union of products $\mathcal{B}_{x,y} \times \mathcal{B}_{y,z}$, over which the bundle and section have the same splitting.

Remark 3.5 Giving a detailed proof of Assumption 3.4 is not our goal. Symplectic cohomology is a special case of Hamiltonian Floer cohomology, whose polyfold construction was sketched in [26]. An alternative approach is using the full SFT polyfolds [12] as in [11]. In those constructions, the linearization in the polyfold and the linearization of the Floer equation modulo an $\mathbb{R}$–translation are the same. Then we have that classical transversality implies polyfold transversality, ie Assumption 3.4(ii) holds. We only use Assumption 3.4 to prove Proposition 3.3. In particular, we will not use any polyfold perturbation scheme but only the existence of polyfolds.

Proof of Proposition 3.3 To obtain the compactness of moduli spaces, in addition to including Floer breakings, we also need to rule out the possibility of a curve escaping to infinity. To this end, since we choose J to be cylindrical convex on $\partial W \times (a_i, b_i)$ where $H_R = f_{i,R}(r)$, we can apply the integrated maximum principle of Abouzaid and Seidel [1] to any $r \in (a_i, b_i)$; see also [9, Lemma 2.2] for the specific version of the integrated maximum principle we need here. We pick an admissible almost complex structure such that moduli spaces of unbroken Floer trajectories of any virtual dimension are cut out transversely. By Assumption 3.4, we have the M-polyfolds description of compactified moduli spaces as zero sets of sc-Fredholm sections. By Assumption 3.4(ii), those sc-Fredholm sections are cut out transversely. Then the M-polyfold implicit function theorem [16, Theorem 3.15][2] endows the compactified moduli spaces smooth structures of manifolds with boundary and corners. It is worth noting that we only need the existence of M-polyfolds with sc-Fredholm sections without evoking any abstract perturbation scheme. In particular, the first two claims hold. The claim on orientations follows from [28, Section 5.1.1]. If $\mathcal{M}_{x,y} \neq \varnothing$, then for energy reasons we have $\mathcal{A}_{H_R}(y) - \mathcal{A}_{H_R}(x) \geq 0$. Then the last claim follows from property (v) of H_R. □

The Hamiltonian Floer cochain complex is defined by counting the zero-dimensional moduli spaces $\mathcal{M}_{x,y}$. However, since we need to consider sphere bundles over the

[2] This theorem is stated for sections in good position. To obtain a decomposition of the boundary in the form of Proposition 3.3(ii) for sections in general position, one also needs [16, Theorem 4.3].

moduli spaces later, which is naturally a Morse–Bott situation, we need to introduce the Morse–Bott framework developed in [28]. To this purpose, we recall the concept of a *flow category*, which was first introduced in [10].

Definition 3.6 [28, Definition 2.9] A flow category is a small category $\mathcal{C}$ with:

(i) The object space $\mathrm{Obj}_{\mathcal{C}} = \bigsqcup_{i\in\mathbb{Z}} C_i$ is a disjoint union of closed manifolds C_i. The morphism space $\mathrm{Mor}_{\mathcal{C}} = \mathcal{M}$ is a manifold with boundary and corners. The source and target maps $s,t\colon \mathcal{M}\to C$ are smooth.

(ii) Let $\mathcal{M}_{i,j}$ denote $(s\times t)^{-1}(C_i\times C_j)$. Then $\mathcal{M}_{i,i} = C_i$, corresponding to the identity morphisms, and s and t restricted to $\mathcal{M}_{i,i}$ are identities. $\mathcal{M}_{i,j}=\varnothing$ for $j<i$, and $\mathcal{M}_{i,j}$ is a compact manifold with boundary and corners for $j>i$.

(iii) Let $s_{i,j}$ and $t_{i,j}$ denote $s|_{\mathcal{M}_{i,j}}$ and $t|_{\mathcal{M}_{i,j}}$. For every strictly increasing sequence $i_0<i_1<\cdots<i_k$,

$$\begin{aligned} t_{i_0,i_1}\times s_{i_1,i_2}\times t_{i_1,i_2}\times\cdots\times s_{i_{k-1},i_k}\colon \mathcal{M}_{i_0,i_1}\times\mathcal{M}_{i_1,i_2}\times\cdots\times\mathcal{M}_{i_{k-1},i_k}& \\ \to C_{i_1}\times C_{i_1}\times C_{i_2}\times C_{i_2}\times\cdots\times C_{i_{k-1}}\times C_{i_{k-1}}& \end{aligned}$$

is transverse to the submanifold $\Delta_{i_1}\times\cdots\times\Delta_{i_{k-1}}$, where Δ_{i_j} is the diagonal in $C_{i_j}\times C_{i_j}$. Therefore the fiber product

$$\begin{aligned} \mathcal{M}_{i_0,i_1}\times_{i_1}\mathcal{M}_{i_1,i_2}\times_{i_2}\cdots\times_{i_{k-1}}\mathcal{M}_{i_{k-1},i_k}& \\ := (t_{i_0,i_1}\times s_{i_1,i_2}\times t_{i_1,i_2}\times\cdots\times s_{i_{k-1},i_k})^{-1}(\Delta_{i_1}\times\Delta_{i_2}\times\cdots\times\Delta_{i_{k-1}})& \\ \subset \mathcal{M}_{i_0,i_1}\times\mathcal{M}_{i_1,i_2}\times\cdots\times\mathcal{M}_{i_{k-1},i_k}& \end{aligned}$$

is a submanifold.

(iv) The composition $m\colon \mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}\to\mathcal{M}_{i,k}$ is a smooth map such that

$$m\colon \bigsqcup_{i<j<k}\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}\to\partial\mathcal{M}_{i,k}$$

is a diffeomorphism up to zero-measure, ie m is a diffeomorphism from a full measure open subset to a full measure open subset.

In the case of Floer theory considered here, the object space is the set of critical points and the morphism space is the union of all compactified moduli spaces of Floer trajectories in addition to the identity morphisms. The source and target maps are evaluation maps at two ends and the composition is the concatenation of trajectories. The fiber product transversality is tautological, as both source and target maps map to 0–dimensional manifolds. If we label periodic orbits $\mathcal{C}(H_R) \cup \mathcal{P}^*(H)$ by integers so that $\mathcal{A}_{H_R}(x_i) \le \mathcal{A}_{H_R}(x_j)$ if and only if $i \le j$, then we have $\mathcal{M}_{x_i,x_j} = \varnothing$ if $i > j$. Moreover, we can require that x_i is a critical point of $H_R|_W$ if and only if $i \ge 0$. With such labels, Proposition 3.3 gives flow categories $\mathcal{C}^{R,J}$, $\mathcal{C}_0^{R,J}$ and $\mathcal{C}_+^{R,J}$:

$$\begin{aligned}
&\mathrm{Obj}(\mathcal{C}^{R,J}) := \{x_i\}, &&\mathrm{Mor}(\mathcal{C}^{R,J}) := \{\mathcal{M}_{i,j} := \mathcal{M}_{x_i,x_j}\};\\
&\mathrm{Obj}(\mathcal{C}_0^{R,J}) := \{x_i\}_{i\ge 0}, &&\mathrm{Mor}(\mathcal{C}_0^{R,J}) := \{\mathcal{M}_{i,j} := \mathcal{M}_{x_i,x_j}\}_{i,j\ge 0};\\
&\mathrm{Obj}(\mathcal{C}_+^{R,J}) := \{x_i\}_{i<0}, &&\mathrm{Mor}(\mathcal{C}_+^{R,J}) := \{\mathcal{M}_{i,j} := \mathcal{M}_{x_i,x_j}\}_{i,j<0}.
\end{aligned}$$

Moreover, $\mathcal{C}_0^{R,J}$ is a subflow category of $\mathcal{C}^{R,J}$ with quotient flow category $\mathcal{C}_+^{R,J}$ in the sense of [28, Proposition 3.38]. By considering only periodic orbits of action greater than $-D_i$, ie those contained in W^i, we have two subflow categories, $\mathcal{C}_{\le i}^{R,J} \subset \mathcal{C}^{R,J}$ and $\mathcal{C}_{+,\le i}^{R,J} \subset \mathcal{C}_+^{R,J}$. In particular $\mathcal{C}_{\le 0}^{R,J} = \mathcal{C}_0^{R,J}$. The orientation property of Proposition 3.3 implies that $\mathcal{C}^{R,J}$, $\mathcal{C}_0^{R,J}$ and $\mathcal{C}_+^{R,J}$, and the truncated versions $\mathcal{C}_{\le i}^{R,J}$ and $\mathcal{C}_{+,\le i}^{R,J}$ are *oriented flow categories* [28, Definition 2.15]. The main theorem of [28] is that for every oriented flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$, one can associate to it a cochain complex $C^*(\mathcal{C})$ over $\mathbb{R}$ generated by $H^*(C_i;\mathbb{R})$, whose homotopy type is well defined. The one feature of the construction in [28] that we will use is the following.

Proposition 3.7 [28, Corollary 3.13] *Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category. Assume $\dim C_i \le k$ for all i. Then the cochain complex $C^*(\mathcal{C})$ only depends on C_i and those $\mathcal{M}_{i,j}$ with $\dim \mathcal{M}_{i,j} \le 2k$.*

Remark 3.8 Roughly speaking, the part of the differential D from $H^*(C_i)$ to $H^*(C_{i+k})$ is defined by the composition $t_* \circ s^*$ through $C_i \xleftarrow{s} \mathcal{M}_{i,i+k} \xrightarrow{t} C_{i+k}$. However, since $\mathcal{M}_{i,i+k}$ is not closed, $t_* \circ s^*$ is not well defined on cohomology. In fact, after choosing representatives of $H^*(C_i)$ in $\Omega^*(C_i)$ (eg harmonic forms), the differential D for a Morse–Bott flow category is given by $t_* \circ s^*$ on $\mathcal{M}_{i,i+k}$, plus many correction terms from possible breakings of $\mathcal{M}_{i,i+k}$. Thus

$$\begin{aligned}
(3)\quad \int_{C_{i+k}} D\alpha\wedge\gamma = \pm&\int_{\mathcal{M}_{i,i+k}} s^*\alpha\wedge t^*\gamma\\
&+\lim_{n\to\infty}\sum_{0<j<k}\pm\int_{\mathcal{M}_{i,i+j}\times\mathcal{M}_{i+j,i+k}} s^*\alpha\wedge(t\times s)^* f_{i+j}^n\wedge t^*\gamma+\cdots,
\end{aligned}$$

where α and γ are the chosen differential form representatives of elements in $H^*(C_i)$ and $H^*(C_{i+k})$, and f_{i+j}^n is a $\dim C_{i+j}-1$–form on $C_{i+j} \times C_{i+j}$. The suppressed terms are integrations on products $\mathcal{M}_{i,*}\times\cdots\times\mathcal{M}_{*,i+k}$ with more f_*^n inserted; see [28] for details. It is clear that Proposition 3.7 follows from (3). Although (3) only depends on $\mathcal{M}_{i,j}$ with $\dim \mathcal{M}_{i,j} \leq 2k$, the proof that $D^2 = 0$ requires the existence of higher-dimensional ($\dim \leq 4k+1$) moduli spaces.

We call a flow category Morse if and only if $\dim C_i = 0$ for all i, and Morse–Bott otherwise. In the Morse case considered in Proposition 3.3, since f_i^n has degree -1 ($f_i^n = 0$), the cochain complex associated to $\mathcal{C}^{R,J}$ is the usual Floer cochain complex generated by $\mathcal{C}(H_R) \cup \mathcal{P}^*(H)$ with differential solely contributed by zero-dimensional moduli spaces

$$\mathrm{D}x_i := \sum_j \Big(\int_{\mathcal{M}_{x_i,x_j}} 1 \Big) x_j,$$

that is, we count those moduli spaces $\mathcal{M}_{x_i,x_j}$ of dimension 0. Similarly, we have cochain complexes $C^*(\mathcal{C}_0^{R,J})$ and $C^*(\mathcal{C}_+^{R,J})$, and a tautological short exact sequence of cochain complexes

$$0 \to C^*(\mathcal{C}_0^{R,J}) \to C^*(\mathcal{C}^{R,J}) \to C^*(\mathcal{C}_+^{R,J}) \to 0, \tag{4}$$

as well as the truncated versions. Moreover, we have

$$C^*(\mathcal{C}^{R,J}) = \varinjlim_i C^*(\mathcal{C}_{\leq i}^{R,J}) \quad \text{and} \quad C^*(\mathcal{C}_+^{R,J}) = \varinjlim_i C^*(\mathcal{C}_{+,\leq i}^{R,J}).$$

Since J is time-independent on W, the cochain complex $C^*(\mathcal{C}_0^{R,J})$ is the Morse cochain complex of W for the Morse–Smale pair $(H_R, g := \omega(\cdot, J\,\cdot))$. Hence we have $H^*(C^*(\mathcal{C}_0^{R,J})) = H^*(W)$. Moreover, $H^*(C^*(\mathcal{C}^{R,J}))$ is the symplectic cohomology $SH^*(W)$, and $H^*(C^*(\mathcal{C}_+^{R,J}))$ is the positive symplectic cohomology $SH_+^*(W)$; see [9; 23; 24] for a more detailed discussion on those invariants. Then (4) gives rise to the tautological long exact sequence

$$\cdots \to H^*(W) \to SH^*(W) \to SH_+^*(W) \to H^{*+1}(W) \to \cdots.$$

Remark 3.9 Since we only consider contractible orbits in domains with vanishing first Chern class, the Conley–Zehnder index is well defined in $\mathbb{Z}$ independent of all choices. Our grading convention follows [23]: $|x_i| := n - \mu_{CZ}(x_i)$, where μ_{CZ} is the Conley–Zehnder index. Such convention implies that if x_i is a critical point of the C^2–small Morse function $H_R|_W$, then $|x_i|$ equals the Morse index. The convention here differs from [24] by n.

3.2 Continuation maps

We will only consider a special class of continuation maps, namely homotopies of almost complex structures and homotopies of Hamiltonians between H_R for different R.

Let $\rho(s)$ be a smooth nondecreasing function such that $\rho(s) = 0$ for $s \ll 0$, and $\rho(s) = 1$ for $s \gg 0$. Given $0 < R_- \le R_+ \le 1$, we have a homotopy of Hamiltonians $H_{R_+,R_-} := H_{\rho(s)R_+ + (1-\rho(s))R_-} : \mathbb{R}_s \times S^1 \times \widehat{W} \to \mathbb{R}$. Then we have the following properties for H_{R_+,R_-}:

(i) $H_{R_+,R_-} = H_{R_-}$ for $s \ll 0$ and $H_{R_+,R_-} = H_{R_+}$ for $s \gg 0$.

(ii) $\partial_s H_{R_+,R_-} \le 0$.

(iii) H_{R_+,R_-} outside $r = b_0$ does not depend on s.

Then for $x \in \mathcal{C}(H_{R_+}) \cup \mathcal{P}^*(H)$ and $y \in \mathcal{C}(H_{R_-}) \cup \mathcal{P}^*(H)$, let J_s be a homotopy of admissible almost complex structures. We use $\mathcal{H}_{x,y}$ to denote the compactified moduli space of solutions to

$$\partial_s u + J_s(u - X_{H_{R_+,R_-}}) = 0, \quad \lim_{s\to\infty} u = x, \quad \lim_{s\to-\infty} u = y.$$

Then for generic choice of J_s, $\mathcal{H}_{x,y}$ is a manifold with boundary and corners by an analogue of Proposition 3.3. They give rise to a flow morphism in the following sense.

Definition 3.10 [28, Definition 3.18] An oriented flow morphism $\mathfrak{H}: \mathcal{C} \Rightarrow \mathcal{D}$ between oriented flow categories $\mathcal{C} := \{C_i, \mathcal{M}^C_{i,j}\}$ and $\mathcal{D} := \{D_i, \mathcal{M}^D_{i,j}\}$ is a family of compact oriented manifolds with boundary and corners $\{\mathcal{H}_{i,j}\}_{i,j\in\mathbb{Z}}$ such that the following hold:

(i) There exists $N \in \mathbb{Z}$ such that when $i - j > N$, we have $\mathcal{H}_{i,j} = \varnothing$.

(ii) There are two smooth maps $s: \mathcal{H}_{i,j} \to C_i$ and $t: \mathcal{H}_{i,j} \to D_j$.

(iii) For every $i_0 < i_1 < \cdots < i_k$ and $j_0 < \cdots < j_{m-1} < j_m$, the fiber product $\mathcal{M}^C_{i_0,i_1} \times_{i_1} \cdots \times_{i_k} \mathcal{H}_{i_k,j_0} \times_{j_0} \cdots \times_{j_{m-1}} \mathcal{M}^D_{j_{m-1},j_m}$ is cut out transversely.

(iv) There are smooth maps $m_L: \mathcal{M}^C_{i,j} \times_j \mathcal{H}_{j,k} \to \mathcal{H}_{i,k}$ and $m_R: \mathcal{H}_{i,j} \times_j \mathcal{M}^D_{j,k} \to \mathcal{H}_{i,k}$ such that

$$\begin{aligned} s \circ m_L(a,b) &= s^C(a), \quad & t \circ m_L(a,b) &= t(b), \\ s \circ m_R(a,b) &= s(a), \quad & t \circ m_R(a,b) &= t^D(b), \end{aligned}$$

where map s^C is the source map for flow category $\mathcal{C}$ and map t^D is the target map for flow category $\mathcal{D}$.

(v) The map $m_L \cup m_R: \bigcup_j (\mathcal{M}^C_{i,j} \times_j \mathcal{H}_{j,k} \cup \mathcal{H}_{i,j} \times_j \mathcal{M}^D_{j,k}) \to \partial\mathcal{H}_{i,k}$ is a diffeomorphism up to zero measure.

(vi) The orientations of the $\mathcal{H}_{i,j}$ are compatible with orientations of C_i, D_i, $\mathcal{M}^C_{i,j}$ and $\mathcal{M}^D_{i,j}$ in the sense of [28, Definition 3.18(6)].

Therefore $\{\mathcal{H}_{x,y}\}$ defines an oriented flow morphism $\mathfrak{H}^{R_+,R_-}$ from $\mathcal{C}^{R_+,J_+}$ to $\mathcal{C}^{R_-,J_-}$. By [28, Theorem 3.21], flow morphisms induce cochain maps between the cochain complexes of the flow categories according to a formula similar to (3). Hence in our situation, $\mathfrak{H}^{R_+,R_-}$ is the geometric data required to define the continuation map. In the Morse case, the cochain map is defined by counting zero-dimensional moduli spaces in $\{\mathcal{H}_{x,y}\}$, which is indeed the classical continuation map. Since we have $\partial_s H_{R_+,R_-} \le 0$, then if $\mathcal{H}_{x,y} \neq \varnothing$, we have $\mathcal{A}_{H_{R_-}}(y) - \mathcal{A}_{H_{R_+}}(x) \ge 0$. Therefore the flow morphism $\mathfrak{H}^{R_+,R_-}$ preserves the action filtration, and in particular, the filtration induced by W^i. Hence we have the flow morphisms

$$\mathfrak{H}_0^{R_+,R_-} : \mathcal{C}_0^{R_+,J_+} \Rightarrow \mathcal{C}_0^{R_-,J_-}, \quad \mathfrak{H}_+^{R_+,R_-} : \mathcal{C}_+^{R_+,J_+} \Rightarrow \mathcal{C}_+^{R_-,J_-},$$
$$\mathfrak{H}_{\le i}^{R_+,R_-} : \mathcal{C}_{\le i}^{R_+,J_+} \Rightarrow \mathcal{C}_{\le i}^{R_-,J_-}, \quad \mathfrak{H}_{+,\le i}^{R_+,R_-} : \mathcal{C}_{+,\le i}^{R_+,J_+} \Rightarrow \mathcal{C}_{+,\le i}^{R_-,J_-}.$$

3.3 Sphere bundles and Gysin exact sequences

For any oriented k–sphere bundle $\pi : E \to W$ with k odd, there is an associated Gysin exact sequence

$$\to H^i(W) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\pi_*} H^{i-k}(W) \xrightarrow{\wedge(-e)} H^{i+1}(W) \to \tag{5}$$

Here π_* is integration along the fiber using the convention in [5, Section 6] and e is the Euler class of π; the extra sign is for consistency with [28, Proposition 6.24]. In this subsection, we consider sphere bundles over symplectic cohomology and deduce the associated Gysin exact sequences. This construction can be viewed as a higher-dimensional analogue of Floer cohomology with local systems. Gysin exact sequences in Floer theory were first considered by Bourgeois and Oancea [6], where the exact sequence arises from an S^1–bundle in the construction of S^1–equivariant symplectic homology. Fiber bundles over Floer theory were considered by Barraud and Cornea [3], where they considered the path–loop fibration. The smooth fiber bundles we consider are technically easier to deal with. The construction in [28] works as long as the moduli spaces support integration [15]. We first recall the concept of sphere bundles over flow categories:

Definition 3.11 [28, Definition 6.17] Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}^C\}$ be an oriented flow category. An oriented k–sphere bundle over $\mathcal{C}$ is a flow category $\mathcal{E} = \{E_i, \mathcal{M}_{i,j}^E\}$ with functor $\pi : \mathcal{E} \to \mathcal{C}$ such that the following hold:

(i) π maps E_i to C_i and $\mathcal{M}_{i,j}^E$ to $\mathcal{M}_{i,j}^C$.

(ii) The maps $\pi : E_i \to C_i$ and $\pi : \mathcal{M}_{i,j}^E \to \mathcal{M}_{i,j}^C$ are oriented sphere bundles such that both bundle maps $s_{i,j}^E$ and $t_{i,j}^E$ preserve the orientation.

By [28, Proposition 6.18], an oriented k–sphere bundle $\mathcal{E}$ over an oriented flow category is an oriented flow category. The construction [28, Definition 3.8] assigns $\mathcal{E}$ to a cochain complex, and we have:

Proposition 3.12 [28, Theorem 6.19] *Let $\mathcal{E}$ be an oriented k–sphere bundle over an oriented flow category $\mathcal{C}$. Then we have a short exact sequence of cochain complexes*[3]

$$0 \to C^*(\mathcal{C}) \xrightarrow{\pi^*} C^*(\mathcal{E}) \xrightarrow{\pi_*} C^{*-k}(\mathcal{C}) \to 0.$$

It induces the Gysin exact sequence

$$\cdots \to H^*(\mathcal{C}) \to H^*(\mathcal{E}) \to H^{*-k}(\mathcal{C}) \to H^{*+1}(\mathcal{C}) \to \cdots . \tag{6}$$

Remark 3.13 Both π^* and π_* are induced by oriented flow morphisms, which are completely determined by $\mathcal{E}$. Here we give an explanation in the special case when $E \to C$ is an actual sphere bundle. The compact manifold C can be understood as a flow category whose object space is diffeomorphic to C and morphism space consists of only identity morphisms. Then E can be understood as a sphere bundle over the flow category C, π^* is given by the flow morphism $C \xleftarrow{s=\pi} E \xrightarrow{t=\mathrm{id}} E$ and π_* is given by the flow morphism $E \xleftarrow{s=\mathrm{id}} E \xrightarrow{t=\pi} C$. In particular, π^* is the composition $t_* \circ s^*$ from $C \xleftarrow{s=\pi} E \xrightarrow{t=\mathrm{id}} E$, which is indeed the pullback π^* on cohomology, and π_* is the composition $t_* \circ s^*$ from $E \xleftarrow{s=\mathrm{id}} E \xrightarrow{t=\pi} C$, which is the pushforward π_* on cohomology. In general, the underlying flow morphisms of π^* and π_* are induced from the identity flow morphism of $\mathcal{E}$ [28, Definition 3.23].

Remark 3.14 [28, Corollary 6.23] Assume $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ is a Morse flow category, ie $\dim C_i = 0$. Then $H^*(\mathcal{E})$ and the Gysin exact sequence only depend on $\mathcal{M}^E_{i,j}$ with $\dim \mathcal{M}^E_{i,j} \le 2k$. In particular, we only use moduli spaces $\mathcal{M}_{i,j}$ of dimension up to k. The nontriviality of higher-dimensional moduli spaces $\mathcal{M}_{i,j}$ is the foundation of the existence of interesting sphere bundles. Although the formula only requires $\mathcal{M}^E_{i,j}$ of dimension up to $2k$, we need a priori the existence of the full flow category to guarantee the existence of Gysin sequences.

Remark 3.15 The Gysin exact sequence considered in [28] works for any Morse–Bott flow category $\mathcal{C}$. In the case considered here ($\mathcal{C}$ is Morse) it is possible to generalize the construction in [17] to the S^k case to get a $\mathbb{Z}$–coefficient Gysin exact sequence.

[3] To be more precise, we have a short exact sequence using certain choices in the construction. However, in the special case that $\mathcal{C}$ is Morse, the minimal construction in [28, Theorem 3.10], ie the one in Remark 3.8, gives the short exact sequence.

We call a Gysin exact sequence (6) trivial when the Euler part $H^*(\mathcal{C}) \to H^{*+k+1}(\mathcal{C})$ is zero. In the case we consider, a sphere bundle over the Liouville domain will induce a sphere bundle over the symplectic flow category.

Proposition 3.16 *Let W be a Liouville domain and $J \in \mathcal{J}^R(W)$. Let $\pi\colon E \to W$ be an oriented k–sphere bundle and P_γ the parallel transport along path γ for a fixed connection on E. Then we have oriented k–sphere bundles $\mathcal{E}^{R,J}$, $\mathcal{E}_0^{R,J}$, $\mathcal{E}_+^{R,J}$, $\mathcal{E}_{\le i}^{R,J}$ and $\mathcal{E}_{+,\le i}^{R,J}$ over $\mathcal{C}^{R,J}$, $\mathcal{C}_0^{R,J}$, $\mathcal{C}_+^{R,J}$, $\mathcal{C}_{\le i}^{R,J}$ and $\mathcal{C}_{+,\le i}^{R,J}$, respectively.*

Proof If $\mathcal{C} = \{x_i, \mathcal{M}_{i,j}\}$, then we define $E_i := E_{x_i(0)} \simeq S^k$ and $\mathcal{M}_{i,j}^E := \mathcal{M}_{i,j} \times E_i$. The structure maps are

$$\begin{aligned}
s^E &\colon \mathcal{M}_{i,j} \times E_i \to E_i \quad \text{given by } (u,v) \mapsto v,\\
t^E &\colon \mathcal{M}_{i,j} \times E_i \to E_j \quad \text{given by } (u,v) \mapsto P_{u(-\cdot,0)}v,\\
m &\colon (\mathcal{M}_{i,j} \times E_i) \times_{E_j} (\mathcal{M}_{j,k} \times E_j) \to \mathcal{M}_{i,k} \times E_i\\
&\qquad\qquad \text{given by } (u_1, v, u_2, P_{u_1(-\cdot,0)}v) \mapsto (u_1, u_2, v).
\end{aligned}$$

It is direct to check that they form a category. The fiber product transversality follows since s^E and t^E are submersive. Because $E \to W$ is an oriented sphere bundle, we have that $E_i = E_{x_i(0)}$ is oriented and P_γ preserves the orientation. Hence $\mathcal{E}^{R,J} = \{E_i, \mathcal{M}_{i,j}^E\}$ is an oriented k–sphere bundle over $\mathcal{C}^{R,J}$. Similarly for other flow categories. □

Example 3.17 To further explain Remark 3.14, we can look at two flow categories: $\mathrm{Obj}_{\mathcal{C}_1}$ is set of two points $\{x_0, x_1\}$ with $\mathcal{M}_{0,1} = \varnothing$, and $\mathrm{Obj}_{\mathcal{C}_2} = \{x_0, x_1\}$ while $\mathcal{M}_{0,1} = S^1$. $\mathcal{C}_2$ can be viewed the flow category associated to the Morse theory of the height function on S^2. Then $\mathcal{C}_1$ does not admit any nontrivial S^n bundle; in particular, the associated Euler part is always trivial. Even though $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same cohomology of rank 2, $\mathcal{C}_2$ admits a nontrivial S^1 bundle $\mathcal{E}_2 = \{S_0^1, S_1^1, \mathcal{M}_{0,1}^E = S^1 \times S^1\}$, where $\mathcal{M}_{0,1}^E$ is viewed as an S^1 bundle over the second factor S^1, which is viewed as $\mathcal{M}_{0,1}$. The structural maps are defined as $s^E\colon (\theta, t) \mapsto \theta$ and $t^E\colon (\theta, t) \mapsto \theta + t$. One may check the induced Gysin exact sequence has nontrivial Euler part. Indeed, $\mathcal{E}_2$ is the S^1 bundle induced from the Hopf fibration over S^2 using an appropriate parallel transport. This example shows that higher-dimensional moduli spaces are foundations for interesting fibrations.

Similarly, there is a notion of oriented sphere bundles over flow morphisms. Given two oriented k–sphere bundles $\mathcal{E} \to \mathcal{C}$ and $\mathcal{F} \to \mathcal{D}$, let $\mathfrak{H}\colon \mathcal{C} \Rightarrow \mathcal{D}$ be an oriented flow morphism. Then a k–sphere bundle $\mathfrak{P}$ over $\mathfrak{H}$ is defined as follows:

(i) $\mathfrak{P} = \{\mathcal{P}_{i,j}\}$ is a flow morphism from $\mathcal{E}$ to $\mathcal{F}$.

(ii) $\pi : \mathcal{P}_{i,j} \to \mathcal{H}_{i,j}$ is a k–sphere bundle such that s^P and t^P are bundles maps covering s^H and t^H.

(iii) $\pi : \mathcal{P}_{i,j} \to \mathcal{H}_{i,j}$ is an oriented bundle, and s^P and t^P preserve the orientation.

Given a sphere bundle $E \to W$ with a parallel transport, let $\mathfrak{H}$, $\mathfrak{H}_0$, $\mathfrak{H}_+$, $\mathfrak{H}_{\le i}$ and $\mathfrak{H}_{+,\le i}$ be the flow morphisms constructed from H_{R_+,R_-}. Then by the same construction as in Proposition 3.16, there are induced oriented sphere bundles $\mathfrak{P}$, $\mathfrak{P}_0$, $\mathfrak{P}_+$, $\mathfrak{P}_{\le i}$ and $\mathfrak{P}_{+,\le i}$ over them. Moreover, the parallel transport at two ends can be different. In this case, we need to fix a smooth family of connections $\{\xi_s\}$ such that ξ_s is the connection for the negative end for $s \ll 0$ and ξ_s is the connection for the positive end for $s \gg 0$. Then given a Floer solution $u(s,t)$ in the flow morphism for continuation maps, the structure maps for the sphere bundle are defined using the parallel transport with respect to xi_s over $u(s,0)$.

By [28, Proposition 6.27], sphere bundles over flow morphisms induce morphisms of Gysin sequences. We define $\mathcal{J}^R_{\le i}$ to be the set of almost complex structures such that the flow category $\mathcal{C}^{R,J}_{\le i}$ is defined. Given a sequence of real numbers $1 > R_1 > R_2 > \cdots > 0$ and a sequences of almost complex structures J_i such that $J_i \in \mathcal{J}^{R_i}_{\le i}(W)$, if we fix any oriented S^k bundle $E \to W$ along with a connection, then Proposition 3.12 induces the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & \varinjlim_i H^{*+k}(\mathcal{E}_0^{R_i,J_i}) & \to & \varinjlim_i H^*(\mathcal{C}_0^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{C}_0^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{E}_0^{R_i,J_i}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & \varinjlim_i H^{*+k}(\mathcal{E}^{R_i,J_i}) & \to & \varinjlim_i H^*(\mathcal{C}^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{C}^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{E}^{R_i,J_i}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & \varinjlim_i H^{*+k}(\mathcal{E}_+^{R_i,J_i}) & \to & \varinjlim_i H^*(\mathcal{C}_+^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{C}_+^{R_i,J_i}) & \to & \varinjlim_i H^{*+k+1}(\mathcal{E}_+^{R_i,J_i}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

Note that

$$\varinjlim_i H^*(\mathcal{C}_0^{R_i,J_i}) = H^*(W), \quad \varinjlim_i H^*(\mathcal{C}^{R_i,J_i}) = SH^*(W),$$
$$\varinjlim_i H^*(\mathcal{C}_+^{R_i,J_i}) = SH^*_+(W).$$

We expect $\varinjlim_i H^*(\mathcal{E}^{R_i,J_i})$ and $\varinjlim_i H^*(\mathcal{E}_+^{R_i,J_i})$ are also well-defined objects, but this requires proving invariance under changing various defining data like H_R, R_i, J_i and the parallel transport P. In the Morse–Bott situation considered here, we need to use the flow-homotopy introduced in [28, Definition 3.29] to prove the invariance.

However, for the purpose of this paper, we do not need a well-defined Floer theory for the sphere bundle and are only interested in the Euler part. We will proceed with this version involving all specific choices. We will suppress the choice of parallel transport for simplicity, and only specify our choice when it matters.

Since the constant orbits part corresponds to the Morse theory on W, there the Gysin sequence should be the regular Gysin sequence.

Proposition 3.18 [28, Theorem 8.14] *The Gysin sequence*

$$\to \varinjlim_i H^{*+k}(\mathcal{E}_0^{R_i,J_i}) \to \varinjlim_i H^*(\mathcal{C}_0^{R_i,J_i}) \to \varinjlim_i H^{*+k+1}(\mathcal{C}_0^{R_i,J_i}) \\ \to \varinjlim_i H^{*+k+1}(\mathcal{E}_0^{R_i,J_i}) \to$$

is the classical Gysin exact sequence (5) *for* $\pi\colon E \to W$.

By the Gysin exact sequence for symplectic cohomology $SH^*(W)$, we have the following vanishing result:

Proposition 3.19 *If* $SH^*(W) = 0$ *and* E *is an oriented sphere bundle over the Liouville domain* W, *then* $\varinjlim_i H^*(\mathcal{E}^{R_i,J_i}) = 0$ *for any defining data.*

3.4 Naturality

In the neck-stretching argument, we need to compare moduli spaces of two fillings, hence naturality is important. Moreover, we can only get the moduli spaces appearing in the counting matched up for two fillings, ie moduli spaces of dimension up to k. But to apply Proposition 3.12 we need the full flow category. In particular, it is possible that the higher-dimensional moduli spaces are not cut out transversely in the neck-stretching. In the following, we discuss those aspects in a similar way to [30].

Definition 3.20 $\mathcal{J}^{R,\le k}(W) \subset \mathcal{J}(W)$ is the set of admissible almost complex structures such that moduli spaces of H_R up to dimension k are cut out transversely. $\mathcal{J}_+^{R,\le k}(W)$ stands for the positive version, and $\mathcal{J}_{\le i}^{R,\le k}(W)$ and $\mathcal{J}_{+,\le i}^{R,\le k}(W)$ are the truncated versions.

All above sets are of second Baire category. Moreover, as a consequence of compactness, $\mathcal{J}_{(+),\le i}^{R,\le k}$ is open and dense. The following is a standard result in Floer theory:

Proposition 3.21 *Let* $J_0 \in \mathcal{J}_{+,\le i}^{R_0,\le 0}(W)$ *and* $J_1 \in \mathcal{J}_{+,\le i+1}^{R_1,\le 0}(W)$ *for* $R_0 > R_1$. *Then* $H^*(\mathcal{C}_{+,\le i}^{R_0,J_0}) \to H^*(\mathcal{C}_{+,\le i+1}^{R_1,J_1})$, *the continuation map, is independent of the homotopy of almost complex structures.*

We also recall the following result from [30]:

Proposition 3.22 [30, Lemma 2.15] *Let $J_s, s \in [0,1]$ be a smooth path in $\mathcal{J}(W)$ and R_s be a nonincreasing function in $(0,1]$ such that $J_s \in \mathcal{J}^{R_s,\le 0}_{+,\le i}(W)$. Then the continuation map $C^*(\mathcal{C}^{R_0,J_0}_{+,\le i}) \to C^*(\mathcal{C}^{R_1,J_1}_{+,\le i})$ is homotopic to the identity.*[4]

Note that we assume H_R stays the same outside $r = b_0$ for any R, meaning the generators for positive symplectic cohomology stay the same. The same argument of [30, Lemma 2.15] can be applied here for positive symplectic cohomology, even though we assume $H = 0$ on W in [30, Lemma 2.15].

Although the full flow category requires transversality for all moduli spaces, the Gysin sequence is well defined for almost complex structure of low regularity:

Proposition 3.23 *Let $E \to W$ be a k–sphere bundle. Then the Euler part of the Gysin exact sequence*

$$\to \varinjlim_i H^{*+k}(\mathcal{E}^{R_i,J_i}_{+,\le i}) \to \varinjlim_i H^*(\mathcal{C}^{R_i,J_i}_{+,\le i}) \to \varinjlim_i H^{*+k+1}(\mathcal{C}^{R_i,J_i}_{+,\le i}) \to \varinjlim_i H^{*+k+1}(\mathcal{E}^{R_i,J_i}_{+,\le i}) \to$$

is well defined for $J_i \in \mathcal{J}^{R_i,\le k}_{+,\le i}(W)$.

Proof We first prove the truncated Gysin sequence

$$\to H^{*+k}(\mathcal{E}^{R_i,J_i}_{+,\le i}) \to H^*(\mathcal{C}^{R_i,J_i}_{+,\le i}) \to H^{*+k+1}(\mathcal{C}^{R_i,J_i}_{+,\le i}) \to H^{*+k+1}(\mathcal{E}^{R_i,J_i}_{+,\le i}) \to$$

is defined. Since we can find an open neighborhood $\mathcal{U} \subset \mathcal{J}^{R_i,\le k}_{+,\le i}(W)$ of J_i, we have a universal moduli space $\bigcup_{J \in \mathcal{U}} M_{x,y,J}$, where $M_{x,y,J}$ is the moduli space of *unbroken* Floer trajectories using J in (2) for the positive symplectic cohomology for $x, y \subset W^i$. The universal moduli space is a Banach manifold and its projection to $\mathcal{U}$ is regular. For each $J \in \mathcal{U} \cap \mathcal{J}^{R_i}_{+,\le i}(W)$, we have a flow category with sphere bundle. We use d_J to denote the differential on the cochain complex of the sphere bundle. Moreover, d_J is well defined by (3) for $J \in \mathcal{U}$, even though d_J^2 may not be zero a priori unless $J \in \mathcal{U} \cap \mathcal{J}^{R_i}_{+,\le i}(W)$ since the integration (3) only depends on the full measure set $M_{x,y,J}$. We have that d_J varies continuously[5] over $\mathcal{U}$. Since $\mathcal{U} \cap \mathcal{J}^{R_i}_{+,\le i}(W)$ is dense in $\mathcal{U}$, we have $d_J^2 = 0$ for every $J \in \mathcal{U}$. As a consequence, the Gysin sequence is

[4] Note that generators are the same for (H_{R_0}, J_0) and (H_{R_1}, J_1), hence the identity map makes sense.

[5] The compactification $\mathcal{M}_{x,y}$ also varies continuously for $J \in \mathcal{U} \subset \mathcal{J}^{R,\le k}_{+,\le i}$ when $\dim \mathcal{M}_{x,y} \le k$.

defined for every $J \in \mathcal{U}$, in particular for J_i. Then by a similar argument, by finding $J_{i,i+1} \in \mathcal{J}^{R_i,R_{i+1},\leq k}_{+,\leq i}(W)$ we have a commutative diagram of the truncated Gysin sequence. This yields a Gysin sequence of the direct limit. Since we only need the well-definedness of the Euler part, the continuation map $H^*(\mathcal{C}^{R_i,J_i}_{+,\leq i}) \to H^*(\mathcal{C}^{R_{i+1},J_{i+1}}_{+,\leq i+1})$ is independent of the choice of $J_{i,i+1}$ by Proposition 3.21. □

Corollary 3.24 *Let $E \to W$ be a k–sphere bundle. Assume J_s, $s \in [0,1]$ is a smooth path in $\mathcal{J}(W)$ and R_s is a nonincreasing smooth function taking values in $(0,1]$ such that $J_s \in \mathcal{J}^{R_s,\leq k}_{+,\leq i}(W)$. Then the Euler parts of the Gysin exact sequences are commutative*:

$$\begin{array}{ccc} H^*(\mathcal{C}^{R_0,J_0}_{+,\leq i}) & \longrightarrow & H^{*+k+1}(\mathcal{C}^{R_0,J_0}_{+,\leq i}) \\ \downarrow & & \downarrow \\ H^*(\mathcal{C}^{R_1,J_1}_{+,\leq i}) & \longrightarrow & H^{*+k+1}(\mathcal{C}^{R_1,J_1}_{+,\leq i}) \end{array} \tag{7}$$

Here the vertical arrows are the continuation maps, which are homotopic to the identity by Proposition 3.22.

Proof Assume in addition that $J_0 \in \mathcal{J}^{R_0}_{+,\leq i}$ and $J_1 \in \mathcal{J}^{R_1}_{+,\leq i}$. Then we can find a regular enough homotopy from J_0 to J_1 such that we have a flow morphism between the associated flow categories. The induced continuation map induces an isomorphism on the Euler parts of the Gysin exact sequences. By Proposition 3.22, the continuation map $H^*(\mathcal{C}^{R_0,J_0}_{+,\leq i}) \to H^*(\mathcal{C}^{R_1,J_1}_{+,\leq i})$ is the identity. Therefore the Euler parts of the Gysin sequences are the same for J_0 and J_1, since $\mathcal{J}^{R_*,\leq k}_{+,\leq i}(W)$ is open and contains $\mathcal{J}^{R_*}_{+,\leq i}(W)$ as a dense set. Then the argument in Proposition 3.23 shows that the Euler part varies continuously with respect to J. □

Proposition 3.25 *For $J_i \in \mathcal{J}^{R_i,\leq k}_{+,\leq i}(W)$ we have the well-defined commutative diagram*

$$\begin{array}{ccc} \varinjlim_i H^*(\mathcal{C}^{R_i,J_i}_{+,\leq i}) & \longrightarrow & \varinjlim_i H^{*+k+1}(\mathcal{C}^{R_i,J_i}_{+,\leq i}) \\ \downarrow & & \downarrow \\ H^{*+1}(W) & \xrightarrow{\wedge(-e(E))} & H^{*+k+2}(W) \end{array}$$

where the horizontal map is the Euler part, and the vertical map is the connecting map from the positive symplectic cohomology to the cohomology of the filling.

Proof Since $\mathcal{J}^{R_i,\leq k}_{+,\leq i}(W)$ is open, we can choose J_i' in a connected neighborhood of J_i in $\mathcal{J}^{R_i,\leq k}_{+,\leq i}(W)$ such that $J_i' \in \mathcal{J}^{R_i}_{\leq i}(W)$. Then by Corollary 3.24 we have the commutative diagram

$$\begin{array}{ccc}
\varinjlim_i H^*(\mathcal{C}_{+,\le i}^{R_i,J_i}) & \longrightarrow & \varinjlim_i H^{*+k+1}(\mathcal{C}_{+,\le i}^{R_i,J_i}) \\
\| & & \| \\
\varinjlim_i H^*(\mathcal{C}_{+,\le i}^{R_i,J_i'}) & \longrightarrow & \varinjlim_i H^{*+k+1}(\mathcal{C}_{+,\le i}^{R_i,J_i'}) \\
\downarrow & & \downarrow \\
H^{*+1}(W) & \xrightarrow{\wedge(-e(E))} & H^{*+k+2}(W)
\end{array}$$

By [30, Proposition 2.17], the vertical arrows in the bottom square do not depend on the choice of J_i'. $\square$

3.5 Neck-stretching and independence of the positive Gysin sequence

Let (Y,α) be a k–ADC contact manifold with two topologically simple fillings W_1 and W_2. Note that $\widehat{W}_1$ and $\widehat{W}_2$ both contain the symplectization $(Y \times (0,\infty)_r, d(r\alpha))$. Since Y is k–ADC, there exist nested contact type surfaces $Y_i \subset Y \times (0,1)$ such that Y_i lies outside of Y_{i+1} and contractible Reeb orbits of contact form $r\alpha|_{Y_i}$ have the property that the degree is greater than k if the period is smaller than D_i.

We now define neck-stretching near Y_i. Assume domains of the form $Y_i \times [1-\epsilon_i, 1+\epsilon_i]_{r_i}$ are disjoint for some small ϵ_i, where r_i is the coordinate determined by the Liouville vector field near Y_i such that $r_i|_{Y_i} = 1$. Assume $J|_{Y_i\times[1-\epsilon_i,1+\epsilon_i]_{r_i}} = J_0$ where J_0 is independent of S^1 and r_i, and $J_0(r_i\partial_{r_i}) = R_i$ and $J_0\xi_i = \xi_i$ where $\xi_i = \ker r\alpha|_{Y_i}$ and R_i is the associated Reeb vector field. Then we pick a family of diffeomorphisms $\phi_R\colon [(1-\epsilon_i)e^{1-1/R}, (1+\epsilon_i)e^{1/R-1}] \to [1-\epsilon_i, 1+\epsilon_i]$ for $R \in (0,1]$ such that $\phi_1 = \mathrm{id}$ and ϕ_R near the boundary is linear with slope 1. Then the stretched almost complex

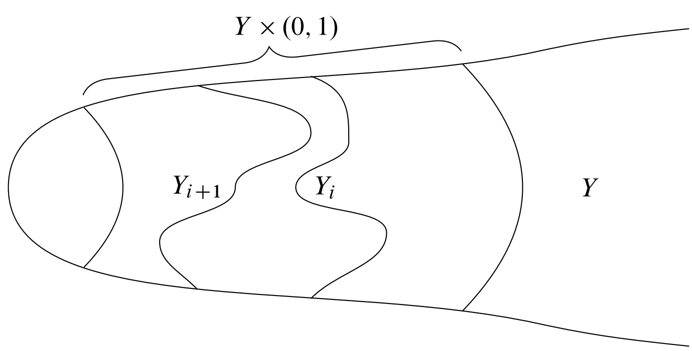

Figure 2: $Y_i \subset \widehat{W}_*$.

structure $NS_{i,R}(J)$ is defined to be J outside $Y_i \times [1-\epsilon_i, 1+\epsilon_i]$ and is $(\phi_R \times \mathrm{id})_* J_0$ on $Y_i \times [1-\epsilon_i, 1+\epsilon_i]$. Then $NS_{i,1}(J) = J$ and $NS_{i,0}(J)$ gives almost complex structures on the completions of the cobordism X_i between Y and Y_i, the filling of Y_i, and the symplectization $Y_i \times \mathbb{R}_+$.

Since we need to stretch along different contact surfaces, we assume the $NS_{i,R}(J)$ have the property that $NS_{i,R}(J)$ will modify the almost complex structure near Y_{i+1} to a cylindrical almost complex structure for R from 1 to $\frac{1}{2}$, and for $R \le \frac{1}{2}$ we only keep stretching along Y_i. We use $\mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le i}(H_0)$ to denote the set of admissible regular J, ie almost complex structures satisfying Definition 3.2 on the completion of W outside Y_i and asymptotic (in a prescribed way as in the stretching process) to J_0 on the negative cylindrical end such that the following moduli space up to dimension k is cut out transversely:

$$\left\{ u\colon \mathbb{R} \times S^1 \backslash Z \to \widehat{X}_i \;\middle|\; \begin{array}{c} \partial_s u + J(\partial_t u - X_{H_0}) = 0, \\ \lim_{s\to\infty} u = x, \quad \lim_{s\to\infty} u = y, \\ \mathcal{A}_{H_0}(x), \mathcal{A}_{H_0}(y) > -D_i, \\ Z = \{z_1, \dots, z_I\}, \\ \lim_{z\to z_j} u = \gamma_j \times \{-\infty\}, \forall 1 \le j \le I, \end{array} \right\} \Big/ \mathbb{R}.$$

Here γ_j is a Reeb orbit on Y_i and we write $\lim_{z\to z_j} u = \gamma_j \times \{-\infty\}$ if u is asymptotic to γ_j near the negative puncture $z_j \in \mathbb{R} \times S^1$. Then $\mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le i}(H_0)$ is an open dense subset of all admissible almost complex structures on $\widehat{X}_i$. To compare moduli spaces for two Liouville fillings W_1 and W_2 we can assume that H_R outside Y_i is the same for W_1 and W_2 whenever $R \le \frac{1}{i}$. The following is simply a variant of [30, Proposition 3.12]:

Proposition 3.26 *With the setup above there exist admissible J^1_* and J^2_* on $\widehat{W}_*$ for $* = 1, 2$, and positive real numbers $\epsilon_1, \epsilon_2, \dots \le 1$ and $\delta_1, \delta_2, \dots \le 1$ with $\delta_i \le \frac{1}{i}$ such that the following hold:*

(i) *For $R < \epsilon_i$ and any $R' \in [0, 1]$,*

$$NS_{i,R}(J^i_*) \in \mathcal{J}^{R\delta_i, \le k}_{+,\le i}(W_*) \quad \textit{and} \quad NS_{i+1,R'}(NS_{i,R}(J^i_*)) \in \mathcal{J}^{R'R\delta_i, \le k}_{+,\le i}(W_*).$$

Moreover, all moduli spaces $\mathcal{M}_{x,y}$ of dimension up to k are the same for both W_1 and W_2, and contained outside Y_i for $x, y \in \mathcal{P}^*(H)$ with action at least $-D_i$.

(ii) *$J^{i+1}_* = NS_{i,\epsilon_i/2}(J^i_*)$ on W^i_* and $\delta_{i+1} = \frac{1}{2}\epsilon_i \delta_i$.*

Proof We prove the proposition by induction. Firstly, we set $\delta_1 = 1$. We then choose a J^1 such that $NS_{1,0}(J^1) \in \mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le 1}(H_0)$. We will apply neck-stretching to J^1 at Y_1.

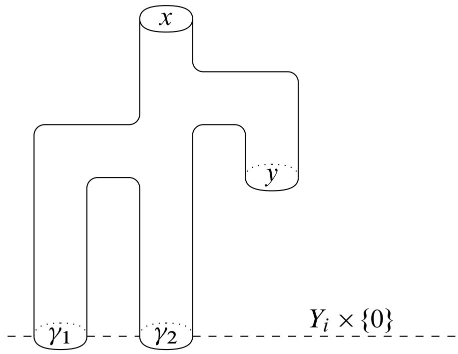

Figure 3: Moduli spaces for the definition of $\mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le i}(H_0)$.

Note that we need to arrange the Hamiltonian converging to a constant near Y_i. We consider moduli space $\mathcal{M}_{x,y,H_R}$ with expected dimension at most k for $NS_{1,R}(J^1)$. Assume $\mathcal{M}_{x,y,H_R}$ is not contained outside Y_1 in the stretching process. Then a limit curve u outside Y_1 has one component by [9, Lemma 2.4].[6] Moreover, by the argument in [9, Lemma 2.4], u can only be asymptotic to Reeb orbits $\{\gamma_i\}_{i\in I}$ that are contractible in W_* on Y_1 with period smaller than D_1. Since W_* is topological simple, $\{\gamma_i\}_{i\in I}$ are contractible in Y_1. In particular, they all have well-defined $\mathbb{Z}$–valued Conley–Zehnder indices with SFT degree greater than k. The expected dimension of the moduli spaces of such u is $\operatorname{ind}(u)-1=|y|-|x|-\sum_{i\in I}(\mu_{CZ}(\gamma_i)+n-3)-1<|y|-|x|-1-k<0$. Since $NS_{1,0}(J^1)\in \mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le 1}(H_0)$, we have that such a u is cut transversely. In particular, there is no such u as the expected dimension is negative. Then for $R \ll 1$, we have that $\mathcal{M}_{x,y,H_R}$ using $NS_{1,R}(J^1)$ is contained outside Y_1 whenever $\dim \mathcal{M}_{x,y,H_R}\le k$. Then $NS_{1,0}(J^1)\in \mathcal{J}^{\le k}_{\mathrm{reg},\mathrm{SFT},\le 1}(H_0)$ also implies that $NS_{1,R}(J^1)\in \mathcal{J}^{R,\le k}_{+,\le 1}(W_*)$ by the openness of transversality.

Next we will apply neck-stretching both at Y_1 and Y_2. By the same argument as above, for every $R'\in[0,1]$, we can find $\epsilon_{R'}>0$ and $\delta_{R'}>0$ such that for $\epsilon<\epsilon_{R'}$ and $|\delta-R'|<\delta_{R'}$,

(i) $NS_{2,\delta}(NS_{1,\epsilon}(J^1))\in \mathcal{J}^{\delta\epsilon,\le k}_{+,\le 1}(W_*)$, and

(ii) $\mathcal{M}_{x,y,H_{\epsilon\delta}}$ is contained outside Y_1 if the expected dimension is at most k.

Then compactness of $[0,1]_{R'}$ implies that there exists $\epsilon_1>0$ such that, for $R<\epsilon_i$ and any $R'\in[0,1]$, we have $NS_{1,R}(J^i_*)\in \mathcal{J}^{R,\le k}_{+,\le 1}(W_*)$ and $NS_{2,R'}(NS_{1,R}(J^1_*))\in \mathcal{J}^{R'R,\le k}_{+,\le 1}(W_*)$.

[6]Note that our symplectic action has the opposite sign compared to [9, Proposition 9.17].

Moreover, the moduli space $\mathcal{M}_{x,y,H_{R'R}}$ for $NS_{2,R'}(NS_{1,R}(J^1_*))$ is contained outside Y_1. We can certainly arrange ϵ_1 small enough so that $\delta_2=\frac{1}{2}\epsilon_1\delta_1=\frac{1}{2}\epsilon_1\le\frac{1}{2}$. Since moduli spaces in Figure 3 for $NS_{2,0}(NS_{1,R}(J^1_*))$ must be contained outside Y_1 for x and y with action at least $-D_1$ when $R\ll 0$ by the same neck-stretching argument along Y_1, we may assume $NS_{2,0}(NS_{1,\epsilon_1/2}(J^1))\in\mathcal{J}^{\le k}_{\mathrm{reg,SFT},\le 1}(H_0)$. Therefore we can perturb $NS_{1,\epsilon_1/2}(J^1)\in\mathcal{J}^{\epsilon_1/2,\le k}_{+,\le 1}(W_*)$ outside W^1_* near orbits in W^2_* to obtain J^2_* such that $NS_{2,0}(J^2)\in\mathcal{J}^{\le k}_{\mathrm{reg,SFT},\le 2}(H_0)$. This will not influence the previous regularity property for periodic orbits with action down to $-D_1$ by the integrated maximum principle. Then we can apply neck-stretching to J^2_* at Y_2 to obtain ϵ_2 with the desired properties and keep the induction going.

Since we require that H_R outside Y_i is the same for W_1 and W_2 whenever $R\le\frac{1}{i}$ and $\delta_i\le\frac{1}{i}$, it is clear that $\mathcal{M}_{x,y,H_{R'R\delta_i}}$ using $NS_{i+1,R'}(NS_{i,R}(J^i_*))$ can be identified for $R<\epsilon_i$ whenever the dimension is at most k and the action of x and y is greater than $-D_i$. This is because it is contained outside Y_i where all the geometric data are the same. □

Proposition 3.27 *Let Y be a k–ADC contact manifold with two topologically simple Liouville fillings W_1 and W_2. Then for $*=1,2$, there exists a sequence of almost complex structures $\widetilde{J}^1_*,\widetilde{J}^2_*,\dots$ and positive numbers $1>R_1>R_2>\dots>0$ such that for any oriented k–sphere bundles E_* over W_* with $E_1|_Y=E_2|_Y$, we have an isomorphism $\Phi\colon \varinjlim_i H(\mathcal{C}^{R_i,\widetilde{J}^i_1}_+)\simeq SH^*_+(W_1)\to SH^*_+(W_2)\simeq\varinjlim_i H(\mathcal{C}^{R_i,\widetilde{J}^i_2}_+)$ such that the following Euler part of the Gysin exact sequence commutes*:

$$\begin{array}{ccc}
\varinjlim_i H^*(\mathcal{C}^{R_i,\widetilde{J}^i_1}_{+,\le i}) & \longrightarrow & \varinjlim_i H^{*+k+1}(\mathcal{C}^{R_i,\widetilde{J}^i_1}_{+,\le i}) \\
\Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\Phi} \\
\varinjlim_i H^*(\mathcal{C}^{R_i,\widetilde{J}^i_2}_{+,\le i}) & \longrightarrow & \varinjlim_i H^{*+k+1}(\mathcal{C}^{R_i,\widetilde{J}^i_2}_{+,\le i})
\end{array}$$

Proof Using the almost complex structures from Proposition 3.26, we define $\widetilde{J}^i_*$ to be $NS_{i,\epsilon_i/2}(J^i_*)$ for $*=1,2$. By Proposition 3.26, $\widetilde{J}^i_*\in\mathcal{J}^{\epsilon_i\delta_i/2,\le k}_{+,\le i}(W_*)=\mathcal{J}^{\delta_{i+1},\le k}_{+,\le i}(W_*)$. Therefore by Proposition 3.23, the direct limit of the following commutative sequence computes the Euler part of the Gysin exact sequence:

$$\begin{array}{ccccc}
H^*(\mathcal{C}^{\delta_2,\widetilde{J}^1_*}_{+,\le 1}) & \longrightarrow & H^*(\mathcal{C}^{\delta_3,\widetilde{J}^2_*}_{+,\le 2}) & \longrightarrow & \cdots \\
\Big\downarrow & & \Big\downarrow & & \\
H^{*+k+1}(\mathcal{C}^{\delta_2,\widetilde{J}^1_*}_{+,\le 1}) & \longrightarrow & H^{*+k+1}(\mathcal{C}^{\delta_3,\widetilde{J}^2_*}_{+,\le 2}) & \longrightarrow & \cdots
\end{array}$$

We first show that the continuation map $H^*(\mathcal{C}_{+,\le i}^{\delta_{i+1},\tilde{J}^i_*}) \to H^*(\mathcal{C}_{+,\le i+1}^{\delta_{i+2},\tilde{J}^{i+1}_*})$ is naturally identified for $*=1,2$. Note that the continuation map is decomposed into continuation maps

$$\Xi\colon H^*(\mathcal{C}_{+,\le i}^{\delta_{i+1},\tilde{J}^i_*}) \to H^*(\mathcal{C}_{+,\le i}^{\delta_{i+2},NS_{i+1,\epsilon_{i+1}/2}(\tilde{J}^i_*)})$$

and

$$\Psi\colon H^*(\mathcal{C}_{+,\le i}^{\delta_{i+2},NS_{i+1,\epsilon_{i+1}/2}(\tilde{J}^i_*)}) \to H^*(\mathcal{C}_{+,\le i+1}^{\delta_{i+2},\tilde{J}^{i+1}_*}).$$

Then Ξ is the identity by Proposition 3.22 using the regular homotopy $NS_{i+1,s}(\tilde{J}^i_*)$ for $s\in\left[\frac{1}{2}\epsilon_{i+1},1\right]$. Since J^{i+1}_* is the same as $\tilde{J}^i_*$ inside W^i, $NS_{i+1,\epsilon_{i+1}/2}(\tilde{J}^i_*)$ is $\tilde{J}^{i+1}_*$ inside W^i. Then the integrated maximum principle implies that Ψ is the composition

$$H^*(\mathcal{C}_{+,\le i}^{\delta_{i+2},NS_{i+1,\epsilon_{i+1}/2}(\tilde{J}^i_*)}) \xrightarrow{=} H^*(\mathcal{C}_{+,\le i}^{\delta_{i+2},\tilde{J}^{i+1}_*}) \xrightarrow{\subseteq} H^*(\mathcal{C}_{+,\le i+1}^{\delta_{i+2},\tilde{J}^{i+1}_*}),$$

which is the same for $*=1,2$. Therefore all the horizontal arrows in the diagram can be identified for both W_1 and W_2. We still need to identify the vertical arrow, ie the Euler part of Gysin sequence. For $\mathcal{C}_{+,\le i}^{\delta_{i+1},\tilde{J}^i_*}$, we pick the parallel transport outside Y_i such that they are identified for $*=1,2$, which is possible since $E|_{\partial W_1}=E|_{\partial W_2}$. Since the Euler part only requires $\mathcal{M}_{x,y}$ with $\dim\mathcal{M}_{x,y}\le k$ and parallel transport over them, Proposition 3.26(i) implies that whole diagram can be identified for $*=1,2$. Then Proposition 3.23 completes the proof □

Remark 3.28 Using that the almost complex structure satisfies the condition here and is close to the condition in [30, Theorem A], the isomorphism in Proposition 3.27 also yields the identification of the map $\delta_\partial\colon SH^\bullet_+(W_*)\to H^{\bullet+1}(Y)$ for $*=1,2$.

4 Proof of the main theorem and applications

Our method of proving Theorem 1.1 is to represent even degree cohomology classes as Euler classes of sphere bundles. The following result explains which class can be realized as the Euler class of a sphere bundle.

Theorem 4.1 [14, Theorem 4.1] *Given $k,m\in\mathbb{N}$, let $K(\mathbb{Z},2k)^m$ be the m–skeleton of the Eilenberg–Mac Lane space $K(\mathbb{Z},2k)$, with inclusion $i\colon K(\mathbb{Z},2k)^m\hookrightarrow K(\mathbb{Z},2k)$. Then there is an integer $N(k,m)>0$ and an oriented $2k$–dimensional vector bundle $\xi_{k,m}$ over $K(\mathbb{Z},2k)^m$ with $e(\xi_{k,m})=N(k,m)\cdot i^*u$, where u is the generator of $H^{2k}(K(\mathbb{Z},2k);\mathbb{Z})$.*

As a corollary of Theorem 4.1, let W be a manifold of dimension $2n$ and $\alpha \in H^{2k}(W;\mathbb{Z})$. Then α is uniquely represented by the homotopy class of a map $f_\alpha : M \to K(\mathbb{Z},2k)^{2n+1}$, so the Euler class of $f_\alpha^* \xi_{k,2n+1}$ is $N(k,2n+1)\cdot\alpha$. We first obtain the following proposition, which may have some independent interest:

Proposition 4.2 *Let Y be a k–ADC manifold with a topologically simple Liouville filling W_1 such that $SH^*(W_1)=0$. Then for any other topologically simple Liouville filling W_2, we have that $H^{2m}(W_2)\to H^{2m}(Y)$ is injective for $2m\le k+1$.*

Proof Note that the abelian group $H^{2m}(W_2;\mathbb{Z})$ has a noncanonical decomposition into free and torsion parts $H^{2m}(W_2;\mathbb{Z}) = H^{2m}_{\mathrm{free}}(W_2;\mathbb{Z}) \oplus H^{2m}_{\mathrm{tor}}(W_2;\mathbb{Z})$. To prove the injectivity of $H^{2m}(W_2)\to H^{2m}(Y)$ for real cohomology, it suffices to show that for any decomposition $H^{2m}_{\mathrm{free}}(W_2;\mathbb{Z})\to H^{2m}(Y;\mathbb{Z})$ is an injection for $2m\le k+1$. Assume otherwise, so there is an element $\alpha\in H^{2m}_{\mathrm{free}}(W_2;\mathbb{Z})\subset H^{2m}(W_2;\mathbb{Z})$ such that $\alpha|_Y=0$ in $H^{2m}(Y;\mathbb{Z})$. By Theorem 4.1, there exist $N\in\mathbb{N}$, a $2m$–dimensional vector bundle $\xi_{m,2n+1}$ over the $2n+1$–skeleton $K(\mathbb{Z},2m)^{2n+1}$ of the Eilenberg–Mac Lane space $K(\mathbb{Z},2m)$, and $f_\alpha : W_2\to K(\mathbb{Z},2m)^{2n+1}$ such that the Euler class of $E_2 := f_\alpha^*\xi_{m,2n+1}$ is $N\alpha$. Since $\alpha|_Y=0$ in $H^{2m}(Y;\mathbb{Z})$, we have that $f_\alpha|_Y : Y\to K(\mathbb{Z},2m)^{2n+1}$ is contractible. Hence $E_2|_Y$ is a trivial bundle. Let $E_1\to W_1$ be the trivial sphere bundle. Therefore by Proposition 3.27, there exist almost complex structures $J^1_*, J^2_*,\dots$ and $1>R_1>R_2>\cdots>0$ such the Euler part for the positive symplectic cohomology for $E_1\to W_1$ and $E_2\to W_2$ can be identified. By [30, Corollary B], we have $SH^*(W_2)=0$. Then Proposition 3.25 implies the commutative diagram

$$\begin{tikzcd}
H^{*+1}(W_1) \arrow[r, "0"] & H^{*+k+2}(W_1) \\
\varinjlim H^*(\mathcal{C}^{R_i,J_1^i}_{+,\le i}) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & H^{*+k+1}(\mathcal{C}^{R_i,J_1^i}_{+,\le i}) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\varinjlim H^*(\mathcal{C}^{R_i,J_2^i}_{+,\le i}) \arrow[r] \arrow[d, "\simeq"] & H^{*+k+1}(\mathcal{C}^{R_i,J_2^i}_{+,\le i}) \arrow[d, "\simeq"] \\
H^{*+1}(W_2) \arrow[r, "\wedge(-N\alpha)"] & H^{*+k+2}(W_2)
\end{tikzcd}$$

We arrive at a contradiction, since $N\alpha\ne 0$. □

Proposition 4.2 says that if we have extra room in the positivity of the SFT degree and also the vanishing of symplectic cohomology, then $H^*(W) \to H^*(Y)$ is necessarily injective for low even degrees. For example, if Y^{2n-1} is a flexibly fillable contact manifold, then Y is $(n-3)$–ADC [18]. In this case, we have that $H^*(W) \to H^*(Y)$ is always injective for even degree with $* \le n-2$. Note that such a property also follows from [30, Corollary B]: $H^*(W) \to H^*(Y)$ is independent of fillings and for $* < n-2$ we have that $H^*(W) \to H^*(Y)$ is an isomorphism for Weinstein fillings. However, Proposition 4.2 holds for very different reasons. Note that we do not assume $H^{2m}(W_1) \to H^{2m}(Y)$ is injective in Proposition 4.2.

Proof of Theorem 1.1 By Proposition 3.27, we can pick $1 > R_1 > R_2 > \dots > 0$ and $J^1_*, J^2_*, \dots$ such that the Euler part of the positive symplectic cohomology for W_1 and W_2 can be identified as long as $E_1|_Y = E_2|_Y$. Since

$$SH^*(W_1) = SH^*(W_2) = 0,$$

we can define ϕ to be the composition

$$H^*(W_1) \xrightarrow{\simeq} \varinjlim_i H^{*-1}(\mathcal{C}_{+,\le i}^{R_i,J_1^i}) \xrightarrow{\simeq} \varinjlim_i H^{*-1}(\mathcal{C}_{+,\le i}^{R_i,J_2^i}) \xrightarrow{\simeq} H^*(W_2).$$

In other words, ϕ is the identification in [30, Corollary B] such that

$$\begin{tikzcd} H^*(W_1) \arrow[rr, "\phi"] \arrow[dr] & & H^*(W_2) \arrow[dl] \\ & H^*(Y) & \end{tikzcd} \tag{8}$$

is commutative. In particular, $\phi(1) = 1$[7] and ϕ is actually induced from an isomorphism $\phi_{\mathbb{Z}}$ for $\mathbb{Z}$–coefficient cohomology. We pick an element $\alpha_1 \ne 0 \in H^{2k}(W_1;\mathbb{Z})$ for $2k \le n-2$. By [30, Corollary B], let $\alpha_2 = \phi_{\mathbb{Z}}(\alpha_1) \in H^{2k}(W_2;\mathbb{Z})$. Then we have $\alpha_2|_Y = \alpha_1|_Y \in H^*(Y;\mathbb{Z})$ by the $\mathbb{Z}$–coefficient version of (8), [30, Corollary B]. By Theorem 4.1, there exist $N \in \mathbb{N}$ and a bundle $\xi_{k,2n+1}$ such that $E_* := f^*_{\alpha_*}\xi_{k,2n+1}$ is a vector bundle over W_* with Euler class $N\alpha_*$ for $* = 1, 2$, and where the map $f_{\alpha_*}: W_* \to K(\mathbb{Z}, 2k)^{2n+1}$ represents α_*. Since $\alpha_2|_Y = \alpha_1|_Y \in H^*(Y;\mathbb{Z})$, we have that $f_{\alpha_1}|_Y$ is homotopic to $f_{\alpha_2}|_Y$. As a consequence, we have $E_1|_Y = E_2|_Y$, $e(E_1) = N\alpha_1$ and $e(E_2) = N\alpha_2$. Then by the same argument as in Proposition 4.2, we have the

[7]Without [30, Corollary B], $\phi(1) = \pm 1$ can already be obtained by grading.

commutative diagram

$$
\begin{tikzcd}
H^{*+1}(W_1) \arrow[r, "\wedge(-N\alpha_1)"] \arrow[ddd, bend right=60, "\phi"'] & H^{*+k+2}(W_1) \arrow[ddd, bend left=60, "\phi"] \\
\varinjlim H^*(\mathcal{C}^{R_i,J_1^i}_{+,\le i}) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & H^{*+k+1}(\mathcal{C}^{R_i,J_1^i}_{+,\le i}) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\varinjlim H^*(\mathcal{C}^{R_i,J_2^i}_{+,\le i}) \arrow[r] \arrow[d, "\simeq"] & H^{*+k+1}(\mathcal{C}^{R_i,J_2^i}_{+,\le i}) \arrow[d, "\simeq"] \\
H^{*+1}(W_2) \arrow[r, "\wedge(-N\alpha_2)"] & H^{*+k+2}(W_2)
\end{tikzcd}
$$

So $\phi(N\alpha_1 \wedge \beta) = N\alpha_2 \wedge \phi(\beta)$. Since $\phi(1) = 1$, it follows that $\phi(N\alpha_1) = N\alpha_2$ and $\phi(N\alpha_1 \wedge \beta) = N\alpha_2 \wedge \phi(\beta) = \phi(N\alpha_1) \wedge \phi(\beta)$. □

Remark 4.3 Combining the argument in this paper with [30], one can prove that the following commutative diagram for a k sphere bundle E is independent of fillings and extensions of $E|_Y$ to W as long as Y is k–ADC:

$$
\begin{tikzcd}
SH^*_+(W) \arrow[r, "\delta_\partial"] \arrow[d, "e(E)"] & H^{*+1}(Y) \arrow[d, "e(E)"] \\
SH^{*+k+1}_+(W) \arrow[r, "\delta_\partial"] & H^{*+k+2}(Y)
\end{tikzcd}
$$

As a corollary, $\operatorname{im}\delta_\partial$ is closed under multiplication by the Euler class $e(E|_Y)$. Then the argument of Theorem 1.1 implies that $\operatorname{im}\delta_\partial$ is closed under multiplication by even elements of degree at most $k+1$ in $\operatorname{im}\delta_\partial$. Note that $\operatorname{im}\delta_\partial$ is an interesting invariant of ADC manifolds and can be used to define obstructions to Weinstein fillability.

Theorem 1.1 can be applied to examples listed in Example 2.6; the major class would be flexibly fillable contact manifolds. In the following, we list several cases where the whole real cohomology ring is unique. For simplicity, we only consider simply connected contact manifolds. Note that the following corollary includes Corollary 1.2:

Corollary 4.4 *Let Y be a simply connected flexibly fillable contact manifold satisfying one the following conditions*:

(i) *Y is $4n+1$–dimensional for $n \ge 1$.*

(ii) *Y is $4n+3$–dimensional for $n \geq 1$, and the flexible filling W has the property that for every $\alpha \wedge \beta \in H^{2n+2}(W)$ with $\deg(\alpha)$ and $\deg(\beta)$ odd, then α or β can be decomposed into a nontrivial product.*

Then $H^*(W)$ as a ring is unique for any Liouville filling W with $c_1(W) = 0$.

Proof By Theorem 1.1 and [30, Corollary B], the ring structure on $H^*(W)$ is unique if one of the factors is of even degree at most $\frac{1}{2}\dim W - 2$, or if the degree of the product is at most $\frac{1}{2}\dim W - 1$. When $\dim Y = 4n+1$, if the degree of the product is in the undetermined region, ie $\frac{1}{2}\dim W = 2n+1$, then one of the factors must be of even degree. If Y is simply connected then $H^1(W)$ is 0 by [30, Corollary B]. As a consequence, the other odd degree factor must have degree at least 3. Therefore the even degree factor has degree at most $\frac{1}{2}\dim W - 3$. In particular, all products fall in the above two cases. Therefore the ring structure is unique. In case (ii), the undetermined case is when the product has degree $2n+2$. If the product is from two classes of even degree, then we can apply Theorem 1.1. If the product is from two classes of odd degree, then by assumption one of them can be reduced to a nontrivial product. Since the ring structure in that degree is unique, the decomposition exists for any other filling. Therefore the product can be rewritten as a product of two even elements. Hence the ring structure is unique. □

Proof of Corollary 1.4 By Corollary 4.4, the real cohomology ring of the filling is unique. Moreover, it is straightforward to verify that the cohomology ring of products of $\mathbb{CP}^n$, $\mathbb{HP}^n$, S^{2n} and at most one copy of S^{2n+1} for $n \geq 1$ have unique minimal models. By [30, Theorem E], any exact filling of $\partial(\mathrm{Flex}(T^*M))$ with vanishing first Chern class is necessarily simply connected, in which case the real homotopy type is determined by the minimal model by [5, Section 19]. □

Theorem 1.1 can only be applied when there is one filling with vanishing symplectic cohomology. In some cases, symplectic cohomology vanishes with nontrivial local systems [2]. Here we only give one special example in such a case.

Proposition 4.5 *Assume W is a Liouville filling of $Y := \partial T^*\mathbb{CP}^n$ for $n \geq 3$ odd, which is $(2n-4)$–ADC. If $c_1(W) = 0$ and $H^2(W;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is not zero, then the real cohomology ring $H^*(W)$ is isomorphic to $H^*(T^*\mathbb{CP}^n)$.*

Proof Since $\mathbb{CP}^n$ is spin for n odd, by [30, Theorem D] there is a local system on both W and $T^*\mathbb{CP}^n$ such that they are the same on Y and the twisted symplectic cohomology vanishes for both W and $T\mathbb{CP}^n$. Then we can apply the same argument of Theorem 1.1 to the case with local systems to finish the proof. □

References

[1] **M Abouzaid**, **P Seidel**, *An open string analogue of Viterbo functoriality*, Geom. Topol. 14 (2010) 627–718 MR Zbl

[2] **P Albers**, **U Frauenfelder**, **A Oancea**, *Local systems on the free loop space and finiteness of the Hofer–Zehnder capacity*, Math. Ann. 367 (2017) 1403–1428 MR Zbl

[3] **J-F Barraud**, **O Cornea**, *Lagrangian intersections and the Serre spectral sequence*, Ann. of Math. 166 (2007) 657–722 MR Zbl

[4] **K Barth**, **H Geiges**, **K Zehmisch**, *The diffeomorphism type of symplectic fillings*, J. Symplectic Geom. 17 (2019) 929–971 MR Zbl

[5] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Math. 82, Springer (1982) MR Zbl

[6] **F Bourgeois**, **A Oancea**, *An exact sequence for contact- and symplectic homology*, Invent. Math. 175 (2009) 611–680 MR Zbl

[7] **F Bourgeois**, **A Oancea**, *Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces*, Duke Math. J. 146 (2009) 71–174 MR Zbl

[8] **K Cieliebak**, **Y Eliashberg**, *From Stein to Weinstein and back*, American Mathematical Society Colloquium Publications 59, Amer. Math. Soc., Providence, RI (2012) MR Zbl

[9] **K Cieliebak**, **A Oancea**, *Symplectic homology and the Eilenberg–Steenrod axioms*, Algebr. Geom. Topol. 18 (2018) 1953–2130 MR Zbl

[10] **R L Cohen**, **J D S Jones**, **G B Segal**, *Floer's infinite-dimensional Morse theory and homotopy theory*, from "The Floer memorial volume" (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 297–325 MR Zbl

[11] **B Filippenko**, **K Wehrheim**, *A polyfold proof of the Arnold conjecture*, Selecta Math. 28 (2022) art. id. 11 MR Zbl

[12] **J Fish**, **H Hofer**, *Applications of polyfold theory*, *II*: *The polyfolds of symplectic field theory*, in preparation

[13] **M Gromov**, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985) 307–347 MR Zbl

[14] **L Guijarro**, **T Schick**, **G Walschap**, *Bundles with spherical Euler class*, Pacific J. Math. 207 (2002) 377–391 MR Zbl

[15] **H Hofer**, **K Wysocki**, **E Zehnder**, *Integration theory on the zero sets of polyfold Fredholm sections*, Math. Ann. 346 (2010) 139–198 MR Zbl

[16] **H Hofer**, **K Wysocki**, **E Zehnder**, *Polyfold and Fredholm theory*, Ergebnesse der Math. (3) 72, Springer (2021) MR Zbl

[17] **M Hutchings**, **J Nelson**, *Axiomatic S^1 Morse–Bott theory*, Algebr. Geom. Topol. 20 (2020) 1641–1690 MR Zbl

[18] **O Lazarev**, *Contact manifolds with flexible fillings*, Geom. Funct. Anal. 30 (2020) 188–254 MR Zbl

[19] **P Massot**, **K Niederkrüger**, **C Wendl**, *Weak and strong fillability of higher dimensional contact manifolds*, Invent. Math. 192 (2013) 287–373 MR Zbl

[20] **D McDuff**, *The structure of rational and ruled symplectic* 4*–manifolds*, J. Amer. Math. Soc. 3 (1990) 679–712 MR Zbl

[21] **D McDuff**, *Symplectic manifolds with contact type boundaries*, Invent. Math. 103 (1991) 651–671 MR Zbl

[22] **A Oancea**, **C Viterbo**, *On the topology of fillings of contact manifolds and applications*, Comment. Math. Helv. 87 (2012) 41–69 MR Zbl

[23] **A F Ritter**, *Topological quantum field theory structure on symplectic cohomology*, J. Topol. 6 (2013) 391–489 MR Zbl

[24] **P Seidel**, *A biased view of symplectic cohomology*, from "Current developments in mathematics, 2006" (B Mazur, T Mrowka, W Schmid, R Stanley, S-T Yau, editors), International, Somerville, MA (2008) 211–253 MR Zbl

[25] **G Walschap**, *The Euler class as a cohomology generator*, Illinois J. Math. 46 (2002) 165–169 MR Zbl

[26] **K Wehrheim**, *Fredholm notions in scale calculus and Hamiltonian Floer theory*, preprint (2012) arXiv 1209.4040 To appear in J. Symplectic Geom.

[27] **K Wehrheim**, *Smooth structures on Morse trajectory spaces, featuring finite ends and associative gluing*, from "Proceedings of the Freedman Fest" (R Kirby, V Krushkal, Z Wang, editors), Geom. Topol. Monogr. 18, Geom. Topol. Publ., Coventry (2012) 369–450 MR Zbl

[28] **Z Zhou**, *Morse–Bott cohomology from homological perturbation theory*, preprint (2019) arXiv 1902.06587 To appear in Algebr. Geom. Topol.

[29] **Z Zhou**, *Vanishing of symplectic homology and obstruction to flexible fillability*, Int. Math. Res. Not. 2020 (2020) 9717–9729 MR Zbl

[30] **Z Zhou**, *Symplectic fillings of asymptotically dynamically convex manifolds, I*, J. Topol. 14 (2021) 112–182 MR Zbl

[31] **Z Zhou**, *Symplectic fillings of asymptotically dynamically convex manifolds, II: k–dilations*, Adv. Math. 406 (2022) art. id. 108522 MR Zbl

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