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Maximal knotless graphs

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A graph is maximal knotless if it is edge maximal for the property of knotless embedding in $\mathbb{R}^3$. We show that such a graph has at least $\frac{7}{4}|V|$ edges, and construct an infinite family of maximal knotless graphs with $|E| < \frac{5}{2}|V|$. With the exception of $|E| = 22$, we show that for any $|E| \geq 20$ there exists a maximal knotless graph of size $|E|$. We classify the maximal knotless graphs through nine vertices and 20 edges. We determine which of these maxnik graphs are the clique sum of smaller graphs and construct an infinite family of maxnik graphs that are not clique sums.

05C10; 57K10, 57M15

1 Introduction

A graph G is *maximal planar* if it is edge maximal for the property of being a planar graph. That is, G is either a planar complete graph, or else adding any missing edge to G results in a nonplanar graph. Maximal planar graphs are triangulations and are characterized by the number of edges: a planar graph with $|V| \geq 3$ is maximal planar if and only if $|E| = 3|V| - 6$.

Naturally, planarity is not the only property of graphs that can be studied with respect to edge maximality. A graph is *intrinsically linked* if every embedding of the graph in $\mathbb{R}^3$ contains a nonsplit link. Some early results on *maximal linkless* (or *maxnil*) graphs — those that are edge maximal for the property of not being intrinsically linked — include a family of maximal linkless graphs with $3|V| - 3$ edges (see Jørgensen [10]), and the fact that the graph $Q(13, 3)$ is a splitter for intrinsic linking, a property that implies it is maximal linkless; see Maharry [13]. Recently there have been several new results including families of maxnil graphs with $3|V| - 3$ edges (rediscovering Jørgensen's examples, see Dehkordi and Farr [4]), with $\frac{14}{5}|V|$ edges (see Aires [1]) and with $\frac{25}{12}|V|$

edges (see Naimi, A Pavelescu, and E Pavelescu [16]). Lower bounds for the number of edges required for a maxnil graph have been established [1], and methods for creating new maxnil graphs via clique sum have been developed [16].

We extend this work with what appears to be the first study of *maximal knotless* graphs. A graph is *intrinsically knotted* (*IK*) if every embedding in $\mathbb{R}^3$ includes a nontrivially knotted cycle, and a graph is *not IK* or *nIK* if it has a knotless embedding, that is, an embedding in which every cycle is a trivial knot. We will call a graph that is edge maximal for the nIK property maximal knotless or *maxnik*.

In Section 2, we establish a connection between maximal 2–apex graphs and maxnik graphs, specifically that a 2–apex graph is maxnik if and only if it is maximally 2–apex. This connection is instrumental in allowing the identification of all maxnik graphs with nine or fewer vertices, and with 20 or fewer edges. We remark that there is an analogous connection between maximal apex graphs and maxnil graphs that may be of independent interest.

We consider clique sums of maxnik graphs in Section 3, and are able to establish similar, if weaker, results to those of [16]. Most importantly, we show that the edge sum of two maxnik graphs G_1 and G_2 on an edge e is maxnik if e is nontriangular (ie not part of a 3–cycle) in at least one G_i. Similarly, we provide conditions that ensure that the clique sum over K_3 of two maxnik graphs is again maxnik. These results are used in Section 4 to construct new maxnik graphs from those found in Section 2.

We then turn to studying general properties of maxnik graphs in Section 4. We establish a lower bound for the number of edges in a maxnik graph of $\frac{7}{4}|V|$, and construct an infinite family of maxnik graphs with fewer than $\frac{5}{2}|V|$ edges. A maximal planar graph has $|E| = 3|V| - 6$, and maximal k–apex graphs also have a fixed number of edges depending on $|V|$. In contrast, the number of edges in maxnil and maxnik graphs can vary. We show that, except for $|E| = 22$, given any $|E| \geq 20$, there exists a maxnik graph of size $|E|$.

We will call a maxnik graph *composite* if it is the clique sum of two smaller graphs. Otherwise we say it is *prime*. These terms are analogous to knots, where a knot is composite if is the connected sum of two nontrivial knots, and prime otherwise. The infinite families of maxnik graphs constructed in Section 4 are all composite, as they are clique sums of smaller maxnik graphs. In Section 5, we classify the maxnik graphs found in Section 2 and construct an infinite family of prime maxnik graphs.

2 Classification through order nine and size 20

For a graph G, let $\delta(G)$ denote the minimal degree, the smallest degree among the vertices of G. Similarly, $\Delta(G)$ is the maximal degree.

Theorem 2.1 *A maxnik graph is* 2–*connected. If* $|V| \geq 3$, *then* $\delta(G) \geq 2$. *If* $|V| \geq 7$, *then* $20 \leq |E| \leq 5|V| - 15$.

Proof Suppose G is maxnik. If G is not connected, then, in a knotless embedding, add an edge e to connect two components. This is a knotless embedding of $G + e$, contradicting G being maximal knotless.

Suppose G has connectivity one with cut vertex v. Label the two components of $G \setminus v$ as A and B. Let a be a neighbor of v in A and b be a neighbor of v in B. These must exist as G is connected. We will argue that $G + ab$ is also nIK, a contradiction.

Let A' denote the subgraph induced by A and the vertex v and similarly define B'. Since G is maxnik, the subgraphs A' and B' are both nIK. Embed A' and B' so that they are knotless and disjoint with a plane separating them. Isotope the edges va and vb to lie on the plane so that the two copies of the vertex v are identified. In this way, we obtain an embedding of G with A' on one side of the separating plane and B' on the other side of the plane so that v, a, and b are the only vertices on the separating plane and va and vb the only edges. Next add edge ab so that the triangle abv bounds a disk.

Any cycle contained in A (or in B) is an unknot. Any cycle c that uses vertices from both A and B must use at least two vertices in the triangle abv. Since abv bounds a disk, this means the cycle c is a connected sum of a cycle in A and a cycle in B. Since those are unknots, c must be as well. This shows $G + ab$ is nIK, contradicting G being maxnik. So a maxnik graph cannot have connectivity one and must be 2–connected.

Suppose G is a maxnik graph with $|V| \geq 3$. Since G is connected, $\delta(G) > 0$. If $v \in V(G)$ has degree one, let u be the neighbor of v. Since G is connected and $|V| \geq 3$, u must have another neighbor $w \neq v$. In a knotless embedding of G, we can introduce the edge vw that closely follows the path vuw. This gives a knotless embedding of $G + vw$, contradicting the maximality of G.

Suppose G is maxnik with $|V| \geq 7$. The lower bound on size is a consequence of the observation [9; 14] that an IK graph has at least 21 edges. The upper bound follows, as a graph with $|E| \geq 5|V| - 14$ has a K_7 minor and is therefore IK [3; 12]. □

In Theorem 4.3 below, we construct an infinite family of maxnik graphs, each with $\delta(G) = 2$.

We say that a graph is *apex* if it is planar or it becomes planar on deletion of a single vertex (the apex). Similarly, a graph is 2–*apex* if it is apex or becomes apex on deletion of a single vertex and a *maximal* 2–*apex* graph is one that is edge maximal for the 2–apex property.

Theorem 2.2 *A* 2*–apex graph is maxnik if and only if it is maximal* 2*–apex.*

Proof Let G be 2–apex. If G is not maximal 2–apex, then there is an edge e such that $G + e$ is 2–apex, hence nIK [2; 17]. This shows that G is not maxnik. Conversely, if G is maximal 2–apex there are two cases, depending on $|V|$. If $n = |V| < 7$, then K_n is 2–apex, so $G = K_n$. But, K_n is also nIK and therefore maxnik. If $|V| \geq 7$, then $|E| = 5|V| - 15$. Since G is 2–apex, it is nIK. Adding any edge e, we have $G + e$ with $5|V| - 14$ edges. It follows that G has a K_7 minor and is IK [3; 14]. This shows that G is maxnik. □

A similar result, with essentially the same proof, holds for maxnil.

Theorem 2.3 *An apex graph is maxnil if and only if it is maximal apex.*

Theorem 2.4 *For $|V| = n \leq 6$, K_n is the only maxnik graph. The only maxnik graphs for $n = 7$ and 8 are the three* 2*–apex graphs derived from triangulations on five and six vertices.*

Proof In [14, Proposition 1.4] it's shown that every nIK graph of order eight or less is 2–apex. So, the maxnik graphs are the maximal 2–apex graphs. For $n \leq 6$, all graphs are 2–apex, so K_n is the only maximal knotless graph. For $n = 7$, the maximal 2–apex graph is K_7^-, formed by adding two vertices to the unique graph with a planar triangulation on five vertices, K_5^-. The two maximal planar graphs on eight vertices are formed by adding two vertices to the two triangulations on six vertices, the octahedron and a graph whose complement is a 3–path. We will call these graphs $K_8 - 3$ disjoint edges and $K_8 - P_3$. □

Let E_9 (called N_9 in [8]) be the nIK nine vertex graph in the Heawood family. Figure 2 in Section 4 below shows a knotless [14] embedding of E_9.

Theorem 2.5 *The graph E_9 is maxnik.*

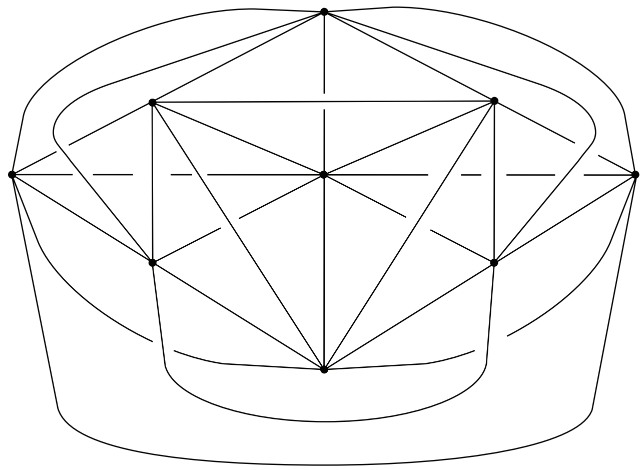

Figure 1: A knotless embedding of $G_{9,29}$.

Proof That E_9 is nIK is established in [14]. Up to symmetry, there are two types of edges that may be added. One type yields the graph $E_9 + e$, shown to be IK (in fact minor minimal IK or MMIK) in [7]. The other possible addition yields a graph that has as a subgraph F_9 in the Heawood family. Kohara and Suzuki [11] established that F_9 is MMIK. □

Theorem 2.6 *There are seven maxnik graphs of order nine.*

Proof The seven graphs are the five maximal 2–apex graphs with 30 edges, E_9, and the graph $G_{9,29}$, shown in Figure 1. Note that $G_{9,29}$ is the complement of $K_1 \sqcup K_2 \sqcup C_6$. Theorems 2.2 and 2.5 show that six of these seven graphs are maxnik. To see that $G_{9,29}$ is as well, note that the embedding shown in Figure 1, due to Ramin Naimi (personal communication, 2011), is knotless. Up to symmetry, there are two ways to add an edge to the graph. In either case, the new graph has a K_7 minor and is IK.

It remains to argue that no other graphs of order nine are maxnik. We know that order nine graphs with size 21 or less are either IK, the graph E_9, or else 2–apex; see [14, Propositions 1.6 and 1.7]. Using Theorem 2.2, this completes the argument for graphs with $|E| \leq 21$. Suppose G is maxnik of order nine with $|E| \geq 22$. By Theorem 2.1, we can assume $|E| \leq 30$. If G is 2–apex, by Theorem 2.2, it is one of the five maximal 2–apex graphs. So, we can assume G is not 2–apex. The minor minimal not 2–apex (MMN2A) graphs through order nine are classified in [15]. With a few exceptions these graphs are also MMIK. If G has an IK minor (including an MMIK minor) it is IK and not maxnik. So, we can assume G has as a minor a graph that is MMN2A, but not MMIK. There are three such graphs. One is E_9, the other two, G_{26} and G_{27}, have 26 and 27 edges. In Theorem 2.5, we showed that E_9 is maxnik. The other two are subgraphs of $G_{9,29}$. To complete the proof, we observe that any order nine graph that contains G_{26} is either a subgraph of $G_{9,29}$ or else IK and similarly for G_{27}. In fact, for those that are IK, we can verify this by finding an MMIK minor, either in the K_7 or $K_{3,3,1,1}$ family, or else the graph $G_{9,28}$ described in [7]. □

Theorem 2.7 *The only maxnik graph of size* 20 *is* K_7^-. *There are seven maxnik graphs with at most* 20 *edges.*

Proof Work above establishes this through order nine. The seven maxnik graphs with at most 20 edges are the seven on seven or fewer vertices. Suppose G of order ten or more and size 20 is maxnik. By [14, Theorem 2.1], G is 2–apex and therefore maximal 2–apex. But this means $|E| = 5|V| - 15 \geq 35$, a contradiction. □

Remark 2.8 A computer search suggests that E_9 is the only maxnik graph of size 21. The search makes use of the 92 known MMIK graphs of size 22; see [5].

3 Clique sums of maxnik graphs

Clique sums of maxnil graphs were studied in [16], and we will show similar, if weaker, versions in the case of maxnik graphs. These results are used in Section 4. A *clique* in a graph is a complete subgraph. When graphs G and H both contain the same clique K_n, we can form a new graph $G \cup_{K_n} H$, called the *clique sum*, from the disjoint union by identifying the vertices in the two copies of K_n.

Lemma 3.1 *For* $t \leq 2$, *the clique sum over* K_t *of nIK graphs is nIK.*

Proof Let G_1 and G_2 be nIK graphs, and let $\Gamma(G)$ denote the set of all cycles in G. Let G be the clique sum of G_i over a clique of size t. Let f_i be an embedding of G_i that contains no nontrivial knot.

Suppose $t = 1$. We may extend the f_i to an embedding of G by embedding $f_1(G_1)$ in 3–space with $z > 0$, and $f_2(G_2)$ with $z < 0$. $G = G_1 \cup_v G_2$, so by isotoping vertex v from each G_i to the plane $z = 0$ and identifying them there, we have an embedding $f(G)$. A closed cycle in G must be contained in a single G_i, and hence given $c \in \Gamma(G)$, then $c \in \Gamma(G_i)$ for some i. As the embeddings $f_i(G_i)$ contain no nontrivial knot, c must be the unknot, and hence G is nIK.

Suppose $t = 2$. We may extend the f_i to an embedding of G by embedding $f_1(G_1)$ in 3–space with $z > 0$, and $f_2(G_2)$ with $z < 0$. $G = G_1 \cup_e G_2$, so by shrinking the edge e in each G_i and then isotoping them to the plane $z = 0$ and identifying them there, we have an embedding $f(G)$. A closed cycle $c \in \Gamma(G)$ must either be an element of $\Gamma(G_i)$, or $c = c_1 \# c_2$, with $c_i \in \Gamma(G_i)$. As the embeddings $f_i(G_i)$ contain no nontrivial knot, in the first case c is the unknot, and in the second it is the connected sum of unknots and hence unknotted. Thus, G is nIK. □

For $H_1, H_2, \ldots, H_k$ subgraphs of graph G, let $\langle H_1, H_2, \ldots, H_k \rangle_G$ denote the subgraph induced by the vertices of the subgraphs.

Lemma 3.2 *Let G be a maxnik graph with a vertex cut set $S = \{x, y\}$, and let $G_1, G_2, \ldots, G_r$ denote the connected components of $G \setminus S$. Then $xy \in E(G)$ and $\langle G_i, S \rangle_G$ is maxnik for all $1 \le i \le r$.*

Proof As G is 2–connected by Theorem 2.1, each of x and y has at least one neighbor in each G_i. Suppose $xy \notin G$. Form $G' = G + xy$ and let $G'_i = \langle G_i, S \rangle_{G'}$. For each i, edge xy is in G'_i. But G'_i is a minor of G, as there exists G_j with $i \neq j$ since S is separating, and there exists a path from x to y in G_j as G_j is connected. Thus in $\langle G_i, G_j, S \rangle_G$, we may contract G_j to x to obtain a graph isomorphic to G'_i. Thus, G'_i is nIK. So, by Lemma 3.1, $G' = G'_1 \cup_{xy} G'_2 \cup_{xy} \cdots \cup_{xy} G'_r$ is nIK. This contradicts the fact that G is maxnik, and hence $xy \in E(G)$.

Suppose that one or more of the G_i are not maxnik. Then add edges as needed to each G_i to form graphs H_i that are maxnik. Then the graph $H = H_1 \cup_{xy} H_2 \cup_{xy} \cdots \cup_{xy} H_r$ is nIK by Lemma 3.1 and contains G as a subgraph. As G is maxnik, $G = H$ and hence $G_i = H_i$ for all i, so every G_i is maxnik as well. □

We say that an edge in a graph is *triangular* if it is part of a triangle or 3–cycle. Similarly, the edge is *nontriangular* if it is part of no 3–cycle in the graph.

Lemma 3.3 *Let G_1 and G_2 be maxnik graphs. Pick an edge in each G_i and label it e. Then $G = G_1 \cup_e G_2$ is maxnik if e is nontriangular in at least one G_i.*

Proof Suppose that e is nontriangular in G_1 and has endpoints x and y. Add an edge ab to the graph G. The graph G is nIK by Lemma 3.1. If both $a, b \in G_i$ for some i, then $G + ab$ is IK, as the G_i are each maxnik. Thus, we may assume that $a \in G_1$ and $b \in G_2$. The edge e is nontriangular in G_1, so vertex a is not adjacent to both endpoints of e. We may assume that a is not adjacent to x. As G_2 is connected, we construct a minor G' of $G + ab$ by contracting the whole of G_2 to vertex x. Note that as $b \in G_2$, we have the edge ax in G', and in fact $G' = G_1 + ax$. As G_1 is maxnik, G' is IK and so is $G + ab$. Thus, G is maxnik. □

Lemma 3.4 *For $i = 1, 2$, let G_i be maxnik, containing a 3–cycle C_i, and admitting a knotless embedding such that C_i bounds a disk whose interior is disjoint from the graph. Then the clique sum G over K_3 formed by identifying C_1 and C_2 is nIK. Moreover, G is maxnik if C_i is not part of a K_4 in at least one G_i.*

Proof Let f_i be the knotless embedding of G_i. Embed the $f_i(G_i)$ so that they are separated by a plane. We may then extend this to an embedding $f(G)$ by isotoping the C_i to the separating plane and identifying them there.

Let $\Gamma(G)$ denote the set of all cycles in G. As the cycles C_i bound a disk in $f(G)$, if a closed cycle $c \in \Gamma(G)$ is not contained in one of the $f_i(G_i)$, then $c = c_1 \# c_2$, with $c_i \in \Gamma(G_i)$. As the embeddings $f_i(G_i)$ contain no nontrivial knot, in the first case c is the unknot, and in the second, it is the connected sum of unknots and hence unknotted. Thus, G is nIK.

Suppose C_1 is not contained in a 4–clique in G_1. We will show $G + ab$ is IK, and hence G is maxnik. As the G_i are maxnik, we may assume that $a \in G_1$ and $b \in G_2$, as otherwise $G + ab$ is IK. As C_1 is not contained in a 4–clique in G_1, there exists a vertex x in C_1 that is not adjacent to a. As G_2 is connected, there is a path from b to x. Contract G_2 to x. This graph contains $G_1 + ax$ as a minor, and hence is IK, as G_1 is maxnik and does not contain edge ax. Thus, G is maxnik. □

4 Bounds on maximal knotless graphs

We now consider maximal knotless graphs in general and establish bounds on the possible number of edges, and the maximal and minimal degrees. We first show a lemma that will be useful for establishing a lower bound. A similar result holds for maximal linkless graphs as well.

Lemma 4.1 *Suppose G is maxnik and contains a vertex v of degree three. Then all neighbors of v are adjacent to each other.*

Proof Label the neighbors of v as x_1, x_2, and x_3. Let $E_v = \{x_1x_2, x_1x_3, x_2x_3\}$ and $E = E(G)$. Delete the edges in $E \cap E_v$ to form $G_Y = G \setminus (E \cap E_v)$. Then add back all the edges of E_v to form $G' = G_Y + E_v$. We will show $G = G'$.

As G is maxnik, G_Y has an embedding f with no nontrivial knot. We may extend f to an embedding of G' by embedding each edge $x_i x_j$ so that the 3–cycle $x_i v x_j$ bounds a disk.

Let $\Gamma(G)$ denote the set of all cycles in the graph G. Suppose c is a cycle in $\Gamma(G')$. If c does not contain one or more edges $x_i x_j$, then $c \in \Gamma(G_Y)$, and hence is a trivial cycle in $f(G')$. Suppose that c does contain one or more edges $x_i x_j$. There are three possibilities: c is a 3–cycle $x_i v x_j$ and bounds a disk, c includes a path of the form $x_i x_j v x_k$ with $\{i, j, k\} = \{1, 2, 3\}$, or c does not include the vertex v. In the first case c is trivial as it bounds a disk. If c does not contain v, then, since the cycles $x_i v x_j$ bound disks, c is isotopic to $c' \in \Gamma(G_Y)$ and hence trivial. Similarly, if c includes a path $x_i x_j v x_k$, using the disk $x_i v x_j$ we can isotope the path to $x_i v x_k$ to make c isotopic to $c' \in \Gamma(G_Y)$ and hence trivial.

Thus, G' has an embedding with no nontrivial knot. As G is maxnik, G cannot be a proper subgraph of G', and hence $G = G'$. □

Theorem 4.2 *If G is maxnik with $|V| \geq 5$, then $|E| \geq \frac{7}{4}|V|$.*

Proof By Theorem 2.4, K_5 is the only maxnik graph with order five and it satisfies the conclusion of the theorem.

Suppose H has the least number of vertices among counterexamples to the theorem. We will consider a vertex v of minimal degree in H. If $\deg(v) \geq 4$, then H has $|E| \geq 2|V|$ and hence is not a counterexample, so $\deg(v) \leq 3$. By Theorem 2.1, $\deg(v) \geq 2$, so we need only consider v of degree two or three.

Suppose $\deg(v) = 2$. We will argue that $H' = H \setminus v$ is also maxnik with $|E'| < \frac{7}{4}|V'|$, contradicting our assumption that H was a minimal counterexample. Let $N(v) = \{w, x\}$ and note that $wx \in E(H)$. Otherwise, in an unknotted embedding of H, we could add the edge wx so that the 3–cycle vwx bounds a disk. This will not introduce a knot into the embedding and contradicts the maximality of H.

As a subgraph of H, H' is nIK. Suppose it is not maxnik because there is an edge ab such that $H' + ab$ remains nIK. In a knotless embedding of $H' + ab$, we can add the vertex v and its two edges so the 3–cycle vwx bounds a disk. This will not introduce a knot into the embedding and shows that $H + ab$ is also nIK, contradicting the maximality of H. Thus, no such graph H with a vertex of degree two can exist.

So we may assume that $\deg(v) = 3$. Here we cannot apply the techniques of [1], as $Y\nabla$ moves do not preserve intrinsic knotting [6]. However, Lemma 4.1 allows us to show the average degree of H is actually at least 3.5, and hence H is not a counterexample.

Divide the vertices of H into three sets: $A = \{\text{vertices of degree } 3\}$, $B = \{\text{vertices of degree} > 3 \text{ that are neighbors of vertices in } A\}$, and $C = \{\text{all other vertices of } H\}$. Form the graph $H' = H \setminus C$. All vertices in C have degree four or greater, so it suffices to show that the vertices in each connected component of H' have average degree 3.5 or higher.

A vertex a_{i1} of degree three has three neighbors, label them b_{i1}, b_{i2}, and a_{i2}, where a_{i2} is a neighbor of minimal degree. By Lemma 4.1, the neighbors of a_{i1} are mutually adjacent. If $\deg(a_{i2}) = 3$, we continue. If not, delete all edges incident on a_{i2} except those between a_{i2} and $\{a_{i1}, b_{i1}, b_{i2}\}$. This creates a subgraph of H' with strictly fewer edges; we will abuse notation and continue to call it H'. Vertex a_{i2} now has degree three in H', and we move it to set A.

If a_{i2} had degree greater than three in H, then, since it has the minimal degree among the neighbors of a_{i1}, $\deg(b_{ij}) \geq 4$ and $b_{i1}, b_{i2} \in B$. If $\deg(a_{i2}) = 3$ in H, vertices a_{ij} are adjacent only to each other and the b_{ij}. If either of the b_{ij} have degree three in H, then H can be disconnected by deleting the other b_{ij}. This is a contradiction as H is maxnik and must be 2–connected by Theorem 2.1. Thus, the b_{ij} are in B.

Consider the connected component of v in H', call it H'_1. We will calculate the total degree of the vertices in H'_1 and divide by the number of vertices. Suppose there are n vertices from set A and m vertices from set B in H'_1 for a total of $n + m$ vertices. Each vertex from set A has degree three, so the contribution to total degree from set A is $3n$. Each vertex in A is adjacent to exactly two of the b_{ij}, so the total degree contribution for set B is at least $2n$ from edges to set A. Further, H'_1 is connected. As a_{ij} is only adjacent to $b_{i'j'}$ if $i = i'$, there must be at least $m - 1$ edges between the b_{ij}, which adds $2(m - 1)$ to the total degree. This gives an average degree of $(5n + 2m - 2)/(n + m)$ in H'_1. However, H is 2–connected by Theorem 2.1, so there must be at least two edges from H'_1 to its complement in H. So within H, these

$\lvert V\rvert$	1	2	3	4	5	6	7	8	9
$\min(\lvert E\rvert/\lvert V\rvert)$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{20}{7}$	$\frac{25}{8}$	$\frac{21}{9}$

Table 1: The least ratios of size to order for maxnik graphs through order nine.

vertices must have average degree greater than or equal to $(5n+2m)/(n+m)$. Note that $2 \le m \le n$, and $(5n+2m)/(n+m)$ attains its minimum at $m = n$. The minimum is $\frac{7}{2}$, and hence H must have $|E| \ge \frac{7}{4}|V|$. □

Theorem 4.3 *There exist maxnik graphs with $|E| < \frac{5}{2}|V|$ edges for arbitrarily large $|V|$.*

Proof Let e be an edge of E_9 connecting a degree four vertex to one of degree five. Edge e is nontriangular and there are five other edges symmetric to it. Using Lemma 3.3, take k copies of E_9 glued along edge e. The resulting graph has $7k+2$ vertices and $20k+1$ edges. Gluing on five K_3 graphs in each E_9 on the other nontriangular edges gives an additional $5k$ vertices and $10k$ edges. So, for each $k \ge 1$, we have a graph G with $n = 12k+2$ vertices and $m = 30k+1$ edges. Then $m = 30(n-2)\frac{1}{12}+1 = \frac{5}{2}n-5$. □

These two theorems suggest the following question. In Table 1 we give the least ratios through order nine.

Question 4.4 What is the minimal number of edges for a maxnik graph of n vertices?

For maximal planar graphs, $|E| = 3|V|-6$. Similarly, maximal k–apex graphs have a fixed number of edges depending on $|V|$. In contrast, as with maximal linkless graphs, the number of edges in a maxnik graph can vary. In fact, with the exception of $|E| = 22$, for any $|E| \ge 20$, there exists a maxnik graph of that size.

Theorem 4.5 *Let $n \ge 20$ and $n \ne 22$. Then there exists a maxnik graph with $|E| = n$.*

Proof The graph K_7^- is maxnik of size 20 by Theorem 2.4. The graph E_9 has a knotless embedding where the 3–cycle abc bounds a disk [14], shown in Figure 2. As no vertex in E_9 is adjacent to all three of these vertices, we may use Lemma 3.4 to construct a maxnik graph of size 24 by taking a clique sum over K_3 of E_9 and K_4. So, we may assume $n \ge 21$ and $n \notin \{22, 24\}$.

The graph E_9 has size 21 and six nontriangular edges. Let G_i denote the maxnik graph obtained from i copies of E_9 by gluing along nontriangular edges.

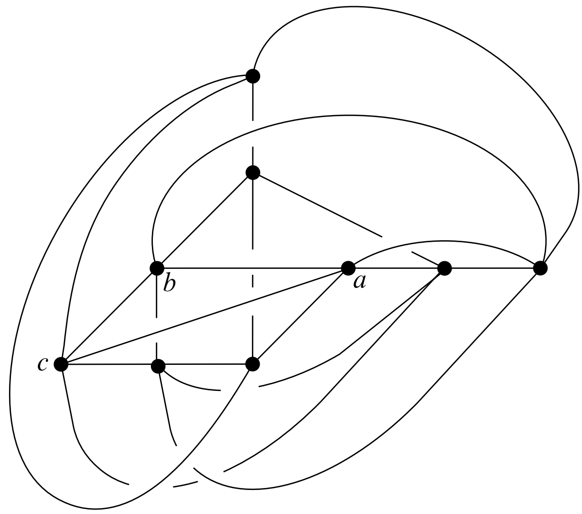

Figure 2: A knotless embedding of E_9.

Note that $|E(G_{i+1})| - |E(G_i)| = 20$, and that G_i contains at least six nontriangular edges for any i. We will now work by induction. Suppose that maxnik graphs exist for size $n < |E(G_i)|$ and for size $|E(G_i)|+1$ and size $|E(G_i)|+3$. Then it suffices to show that there exist maxnik graphs of size $|E(G_i)|+k$ for $4 \le k \le 19$ and $k \in \{0, 2, 21, 23\}$.

Clearly a maxnik graph of size $|E(G_i)|+0$ exists, as G_i is maxnik. We may form a new maxnik graph from G_i by gluing a copy of K_m (for $3 \le m \le 6$) along a nontriangular edge of G_i by Lemma 3.3. As G_i has at least six nontriangular edges, we can glue on up to six such graphs, each adding $\binom{m}{2} - 1$ edges. Thus, to prove the result we need only to be able to form the desired values of k using six or fewer addends from the set $\{2, 5, 9, 14\}$. This is clearly possible.

In the base case $i = 1$, we have a maxnik graph of size $|E(G_1)| = 21$, and we excluded graphs of size 22 and 24 ($|E(G_1)| + 1$ and $|E(G_1)| + 3$) above. Thus we may form maxnik graphs of size $|E(G_1)| + k$ for the k of interest as before. □

Remark 4.6 A computer search shows there are no size 22 maxnik graphs. Our strategy is based on the classification through size 22 of the obstructions to 2–apex in [15]. Let's call such graphs MMN2A (minor minimal not 2–apex). All but eight of the graphs in the classification are MMIK. Two exceptions are 4–regular of order 11, the other six are in the Heawood family.

A maximal 2–apex graph has $5n-15$ edges where n is the number of vertices. By Theorem 2.2 a maxnik graph G of size 22 is not 2–apex and therefore has an MMN2A minor. Since G is nIK, it must have one of the eight exceptions as a minor. Using a computer, we verified that no size 22 expansion of any of these eight graphs is maxnik.

Theorem 4.5 implies that there are maxnik graphs of nearly every size. Note that there are maxnik graphs of any order, as there exist maximal 2–apex graphs of any order and by Theorem 2.2 these graphs are maxnik.

We have considered the minimal number and the possible number of edges in a maxnik graph. We now consider other aspects of maxnik graphs' structure, in particular, the maximal and minimal degree. Since $\Delta(G) = |V|-1$ for maximal 2–apex graphs, there are maxnik graphs with arbitrarily large $\Delta(G)$.

Proposition 4.7 *The complete graph K_3 is the only maxnik graph with maximal degree two.*

Proof Suppose G is maxnik with $\Delta(G) = 2$. Then $|G| \geq 3$ and, by Theorem 2.1, $\delta(G) = 2$ and G is connected. So G is a cycle. Now, a cycle is planar, hence 2–apex, and by Theorem 2.2 G is maximal 2–apex. However, a cycle is not maximal 2–apex unless it is K_3. □

Note that Lemma 4.1 has the following two immediate corollaries:

Corollary 4.8 *If a graph G is maxnik and has $\Delta(G) = 3$, then G is 3–regular.*

Corollary 4.9 *If a graph G is maxnik and 3–regular, then $G = K_4$.*

These results motivate the following question:

Question 4.10 Do there exist regular maxnik graphs other than K_n with $n < 7$?

A maximal 2–apex graph will have $\Delta(G) = |V|-1$ and $\delta(G) \leq 7$, so if there is such a regular maxnik graph with $|V| \geq 7$ it is not 2–apex. However, through order nine, our two examples of maxnik non-2–apex graphs are both close to regular, having $\Delta(G) - \delta(G) \leq 2$. This suggests the answer to our question is likely yes.

For $\delta(G)$, Theorem 2.1 gives a lower bound of two that is realized by the infinite family of Theorem 4.3. On the other hand, by starting with a planar triangulation of minimum

$\|V\|$	1	2	3	4	5	6	7	8	9
$\delta(G)$	0	1	2	3	4	5	5	5 or 6	4 to 7
$\Delta(G)$	0	1	2	3	4	5	6	7	5 to 8

Table 2: Maximal and minimal degrees of maxnik graphs through order nine.

degree five, we can construct graphs with $\delta(G) = 7$ that are maximal 2–apex, and hence maxnik. At the same time, since a graph with $|E| \geq 5|V| - 14$ has a K_7 minor and is IK [3; 12], a maxnik graph must have $\delta(G) \leq 9$. It seems likely that there are examples that realize this upper bound on $\delta(G)$. Table 2 records the range of degrees for maxnik graphs through order nine.

5 Prime and composite maxnik graphs

We will call a graph *composite* if it is the clique sum of two graphs. Otherwise it is *prime*. These terms are analogous to knots, where a knot is composite if is the connected sum of two nontrivial knots, and prime otherwise. In this section, we classify the maxnik graphs described earlier in this paper as prime and composite. We remark that it may be of interest to study other instances of prime graphs, for example, prime maximal planar or prime maxnil.

The infinite families of maxnik graphs constructed in Section 4 are all composite, as they are clique sums of smaller maxnik graphs.

Note that K_n is prime, so all maxnik graphs of order six or less are prime.

Proposition 5.1 *The following maxnik graphs are composite*: K_7^-, $K_8 - P_3$, *and four of the five maximal* 2*–apex graphs on nine vertices*, *specifically big-*Y, *long-*Y, *hat and house.*

Proof The graph K_7^- is formed from two copies of K_6 summed over a 5–clique.

The graph $K_8 - P_3$ is formed from K_7^- clique sum K_6 over a 5–clique, where the 5–clique contains exactly one endpoint of the missing edge.

Big-Y is formed from $K_8 - P_3$ clique sum K_6 over a 5–clique, where the 5–clique contains both of the terminal vertices of the 3–path.

Long-Y is formed from K_8- 3 disjoint edges clique sum K_6 over a 5–clique.

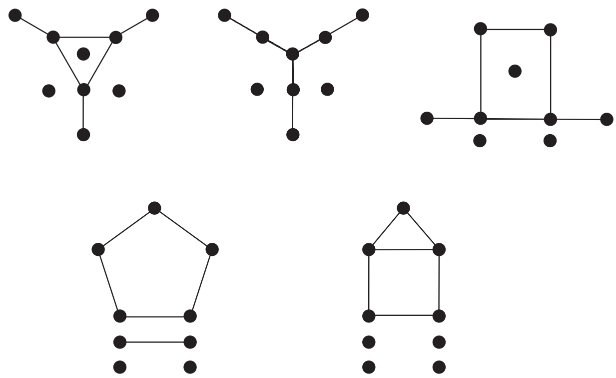

Figure 3: Complements of the maximal 2–apex graphs of order nine. Top row, left to right: big-Y, long-Y, and hat. Bottom row: pentagon–bar and house.

Hat is formed from $K_8 - P_3$ clique sum K_6 over a 5–clique, where the 5–clique contains one terminal vertex and one (nonadjacent) interior vertex of the 3–path.

House is formed from $K_8 - P_3$ clique sum K_6 over a 5–clique, where the 5–clique contains one interior vertex of the 3–path. □

Lemma 5.2 *If G^c is of the form $K_2 \amalg H$, then either G is prime, or G is the clique sum of two copies of K_n over an $n-1$ clique.*

Proof Call the two vertices of the K_2 in G^c v_1 and v_2. Suppose that G is a clique sum of G_1 and G_2 over a clique C. We cannot have both v_1 and v_2 in C, as edge v_1v_2 is in G^c. Without loss of generality, we may assume that v_1 is in $G_1 \setminus C$. So, in G^c, v_1 must be adjacent to every vertex of $G_2 \setminus C$. Thus $G_2 \setminus C$ is v_2. As the only neighbor of v_1 in G^c is v_2, v_1 is adjacent to every vertex in C. Similarly for v_2. Thus if G is composite, it is the clique sum of K_n and K_n over an $n-1$ clique. □

Corollary 5.3 *The following maxnik graphs are prime: pentagon–bar, $G_{9,29}$ and K_8- 3 disjoint edges.*

Proof Each of these graphs has a complement of the form $K_2 \amalg H$. As these graphs are not of the form K_n- a single edge, they are prime by Lemma 5.2. □

Note that if G is a clique sum over a t–clique, it is not $(t+1)$–connected.

Proposition 5.4 *The maxnik graph E_9 is prime.*

Proof The largest clique in E_9 is a 3–clique, but E_9 is 4–connected and hence must be prime. □

Lemma 5.5 *If $G = H * K_2$, and G is 2–apex, then G is prime maxnik if and only if H is prime maximal planar.*

Proof As G is 2–apex, it is maxnik if and only if it is maximal 2–apex, and G is maximal 2–apex if and only if H is maximal planar.

If H is composite, then H is the clique sum of H_1 and H_2 over a t–clique. So G is the clique sum of $H_1 * K_2$ and $H_2 * K_2$ over a $(t+2)$–clique, and hence G is composite.

As G is maxnik, it must be 2–connected. Hence if G is composite, it must be G_1 clique sum G_2 over a t–clique C, with $t \geq 2$. Label two of the vertices in C as v_1 and v_2. Then H is the clique sum of $G_1 \setminus \{v_1, v_2\}$ and $G_2 \setminus \{v_1, v_2\}$ over $C \setminus \{v_1, v_2\}$, and thus composite. □

Corollary 5.6 *There exist prime maxnik graphs of arbitrarily large size, and of any order ≥ 8.*

Proof The octahedron graph is maximal planar and 4–connected. The largest clique it contains is a 3–clique, so it is prime. New triangulations formed by repeated subdivision of a single edge are 4–connected and maximal planar, but have no 4–clique, hence are prime as well. Thus all of these graphs give prime maxnik examples when joined with K_2. □

We remark that the construction of this family of graphs is similar to the maxnil families with $3n-3$ edges due to Jørgensen [10] and $3n-5$ edges due to Naimi, Pavelescu, and Pavelescu [16].

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