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# New bounds on maximal linkless graphs

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We construct a family of maximal linklessly embeddable graphs on $n$ vertices and $3n-5$ edges for all $n \geq 10$, and another family on $n$ vertices and $m < \frac{25}{12}n - \frac{1}{4}$ edges for all $n \geq 13$. The latter significantly improves the lowest edge-to-vertex ratio for any previously known infinite family. We construct a family of graphs showing that the class of maximal linklessly embeddable graphs differs from the class of graphs that are maximal without a $K_6$ minor studied by L Jørgensen. We give necessary and sufficient conditions for when the clique sum of two maximal linklessly embeddable graphs over $K_2$, $K_3$ or $K_4$ is a maximal linklessly embeddable graph, and use these results to prove our constructions yield maximal linklessly embeddable graphs.

57M15; 05C10

## 1 Introduction

All graphs in this paper are finite and simple. A graph is *intrinsically linked* (IL) if every embedding of it in $\mathbb{R}^3$ (or, equivalently, $S^3$) contains a nontrivial 2–component link. A graph is *linklessly embeddable* if it is not intrinsically linked (nIL). A nIL graph $G$ is *maxnil* if it is not a proper subgraph of a nIL graph of the same order. The combined work of Conway and Gordon [2], Sachs [11] and Robertson, Seymour and Thomas [9] fully characterized IL graphs: a graph is IL if and only if it contains a graph in the Petersen family as a minor. The Petersen family consists of seven graphs obtained from $K_6$ by $\nabla Y$–moves and $Y\nabla$–moves, as described in Figure 1. The $\nabla Y$–move and the $Y\nabla$–move preserve the IL property.

The property of being maxnil is, in a way, analogous to the property of being maximal planar. While it is well known that every maximal planar graph with $n$ vertices has $3n-6$ edges, an analogous statement for maxnil graphs does not exist. For example,

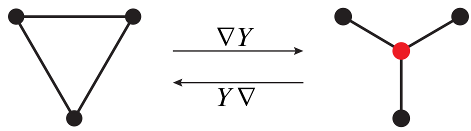


Figure 1: $\nabla Y$– and $Y\nabla$–moves.

start with a maximal planar graph $G$ and add one vertex $v$ together with all the edges from $v$ to the vertices of $G$. Such a graph is maxnil by [11], and if it has $n$ vertices, then it has $4n-10$ edges. In fact, $4n-10$ is an upper bound on the number of edges of a maxnil graph on $n$ vertices. This follows from work of Mader [7], who proved that having more than $4n-10$ edges implies the existence of a $K_6$ minor, which implies the graph is IL.

On the other hand, Jørgensen [5] and Dehkordi and Farr [3] constructed maxnil graphs with $n$ vertices and $3n-3$ edges. Jørgensen's maxnil graphs are obtained from the Jørgensen graph in Figure 2, left, by subdividing the highlighted edge incident to the vertex $y$ and then adding edges that connect every new vertex to $u$ and $v$. We denote the graph obtained this way through $i$ subdivisions by $J_i$ for $i \geq 1$. See Figure 2, right.

Recently, Aires [1] found a family of graphs with fewer than $3n-3$ edges. For each value $n \geq 13$ with $n \equiv 3 \pmod{10}$, he constructed a maxnil graph with $\frac{14}{5}n - \frac{27}{5}$ edges. He also proved that, if $G$ is a maxnil graph with $n \geq 5$ vertices and $m$ edges, then $m \geq 2n$. This bound is sharp: the maxnil graph $Q(13,3)$ described by Maharry [8] has 26 edges and 13 vertices.

In Section 2, we present two constructions of maxnil graphs. The first one is a family of maxnil graphs with $n \geq 10$ vertices and $3n-5$ edges. This construction builds upon a maxnil graph on 10 vertices and 25 edges and uses edge subdivisions. The second

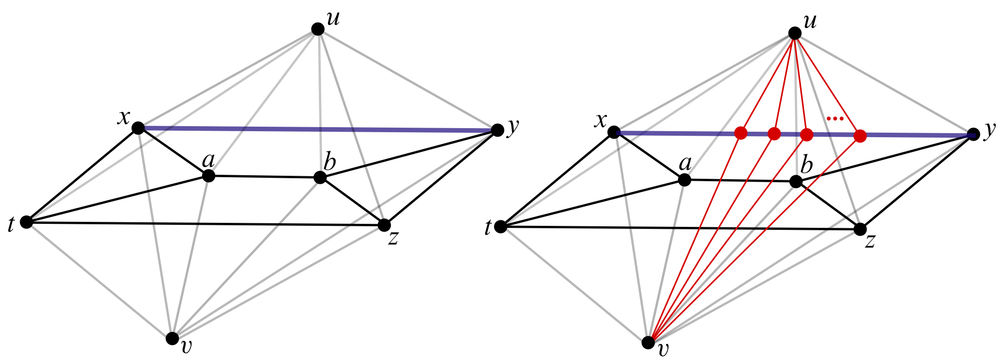


Figure 2: Left: the Jørgensen graph. Right: the graph $J_i$ in Jørgensen's $3n-3$ family.

construction significantly improves on Aires' result on the number of edges. Using clique sums of copies of $Q(13,3)$, we construct examples with a smaller "edge-to-vertex ratio", as in the following theorem:

**Theorem** *For each $n \geq 13$, there exists a maxnil graph $G$ with $n$ vertices and $m < \frac{25}{12}n - \frac{1}{4}$ edges.*

In Section 3, we study the properties of maxnil graphs under clique sums. Some of these results are used in the constructions of Section 2. We give sufficient and necessary conditions for when the clique sum of two maxnil graphs over $K_2$, $K_3$ or $K_4$ is maxnil. Jørgensen [5] studied clique sums of graphs that are maximal without a $K_6$ minor. We give examples showing that the class of maxnil graphs and the class of graphs that are maximal without a $K_6$ minor are distinct.

## 2 Two families of maxnil graphs

We note that the Jørgensen graph is 2–apex, ie removing the vertices $u$ and $v$ leaves a planar graph $P$. Furthermore, the embedding of $P$ in $\mathbb{R}^2$ shown in Figure 2, left, has no separating cycles, ie for every cycle $C$ in $P$, one of the components of $\mathbb{R}^2 \setminus C$ contains no vertices of $P$. These properties are generalized in the next lemma, which we use to prove the graphs in the $3n-5$ family are nIL.

**Lemma 1** *Let $G$ be a graph with two nonadjacent vertices $u, v$ such that there exists an embedding $\Sigma$ of $G - \{u, v\}$ in $\mathbb{R}^2$, where, for every cycle $C$ in $\Sigma$, $\mathbb{R}^2 \setminus C$ has a component $X$ such that $X \cup C$ separates $u$ and $v$ (ie every path in $G$ from $u$ to $v$ contains a vertex in $X \cup C$). Then embedding $u$ as $(0,0,1)$ and $v$ as $(0,0,-1)$ and connecting each of them to its neighbors in $\Sigma$ with straight edges yields a linkless embedding of $G$ in $\mathbb{R}^3$.*

**Proof** Let $\Gamma$ denote the embedding of $G$ as described in the lemma, and let $K \cup K'$ be a 2–component link in $\Gamma$. We consider two cases.

**Case 1** (neither $K$ nor $K'$ contains both $u$ and $v$) Then we have three subcases: zero, one or both of $K$ and $K'$ are in $\Sigma$. In each of these three subcases it is easy to see that $K \cup K'$ is a trivial link. We prove this for one of the three subcases here; the other two are similar and easier. Suppose $K$ contains $u$ but not $v$, and $K' \subset \Sigma$. Then $K$ consists of two edges incident to $u$ and a path $P \subset \Sigma$. Connecting $u$ with straight line segments to every point in $P$ gives us a $\Gamma$–panel for $K$. On the other hand, $K'$ bounds a disk $D$ in $\mathbb{R}^2$. We isotop $D$, while keeping its boundary fixed, by pushing its interior slightly below $\mathbb{R}^2$, to make it disjoint from $K$ (since $K$ contains no points below $\mathbb{R}^2$). It follows that $K \cup K'$ is a trivial link.

**Case 2** (one of the link's components, say $K$, contains both $u$ and $v$) Then $K' \subset \Sigma$. So $\mathbb{R}^2 \setminus K'$ has two components such that one of them, $X$, separates $u$ and $v$. Therefore all vertices of $K$ except $u$ and $v$ lie in $X$. Now, $K$ has exactly two vertices, call them $a$ and $b$, that are adjacent to $u$, and two vertices, $c$ and $d$, adjacent to $v$. Note that $\{a, b\}$ is not necessarily disjoint, or even distinct, from $\{c, d\}$. Furthermore, $K \cap X$ consists of two components, $P_1$ and $P_2$, each of which is a path of length zero or greater. We can assume $a, c \in P_1$ and $b, d \in P_2$. We consider three subcases.

**Case 2.1** ($a = c$ and $b = d$) Join $a$ to $b$ by an arc $\beta \subset X$ (not necessarily in $\Sigma$), and then connect each of $u$ and $v$ by straight line segments to every point in $\beta$. See Figure 3, left. This gives us a disk bounded by $K$ and disjoint from $K'$. Similarly to Case 1 above, $K'$ also bounds a disk disjoint from $K$. Hence $K \cup K'$ is a trivial link.

**Case 2.2** ($a = c$ and $b \neq d$) Join $a$ to each of $b$ and $d$ by disjoint arcs $\beta$ and $\delta$ respectively, both in $X$, such that $\beta \cup \delta \cup P_2$ is a simple closed curve. See Figure 4, right. Connect each of $u$ and $v$ by straight line segments to every point in $\beta$ and $\delta$ respectively. This gives us two disks whose union with the disk bounded by $\beta \cup \delta \cup P_2$ in $X$ is a disk bounded by $K$ and disjoint from $K'$. As before, $K'$ bounds a disk disjoint from $K$. Hence, $K \cup K'$ is a trivial link.

**Case 2.3** ($a \neq c$ and $b \neq d$) This case is similar to Case 2.2, except that we join $a$ to $b$ and $c$ to $d$ by disjoint arcs $\beta$ and $\delta$ in $X$ such that $\beta \cup \delta \cup P_1 \cup P_2$ is a simple closed curve. □

## 2.1 The $3n-5$ family

We construct a family of graphs with $n$ vertices and $3n - 5$ edges for $n \geq 10$. This family is obtained from the graph $G$ pictured in Figure 4, left, through a sequence of subdivisions and edge additions. The graph $G$ is obtained from the Jørgensen graph by splitting (the opposite of contracting edges) the vertices $a$ and $b$ into the edges $ad$ and $bc$. See Figures 2, left, and 4, left. With the notation in Figure 4, left, construct the graph $G_1$ by subdividing the edge $xy$ with a new vertex $z_1$, then adding edges $z_1u$ and $z_1v$. Construct graphs $G_i$ for $i \geq 2$ as follows: subdivide the edge $z_{i-1}y$ of $G_{i-1}$ with a new vertex $z_i$, then add edges $z_iu$ and $z_iv$ to $G_{i-1}$. Notice that $G_i$ has one more vertex and three more edges than $G_{i-1}$. The graph $G_i$ has $10 + i$ vertices and $25 + 3i = 3(10 + i) - 5$ edges. We note that the graphs $G_i$ can also be obtained by successive splittings of the vertex $y$ into the edge $yz_i$.

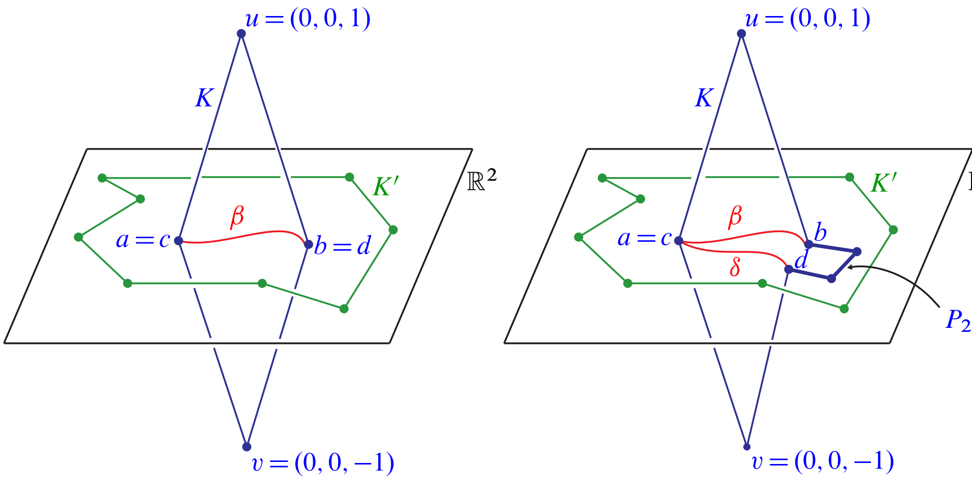


Figure 3: Left: configuration for Case 2.1. Right: configuration for Case 2.2.

**Proposition 2** *The graphs $G$ and $G_i$ in Figure 4 are linklessly embeddable.*

**Proof** It is straightforward to check that these graphs satisfy the hypotheses of Lemma 1 and hence are nIL. □

**Proposition 3** *The graph $G$ in Figure 4, left, is maxnil.*

**Proof** Since $G$ is linklessly embeddable, it remains to show that adding any edge to $G$ gives an IL graph.

Note that both of the minors $G/(ab \cup cd)$ and $G/(ad \cup bc)$ are isomorphic to the Jørgensen graph. If an edge $e$ other than $bd$ is added to $G - \{u, v\}$, then $e$ is an edge

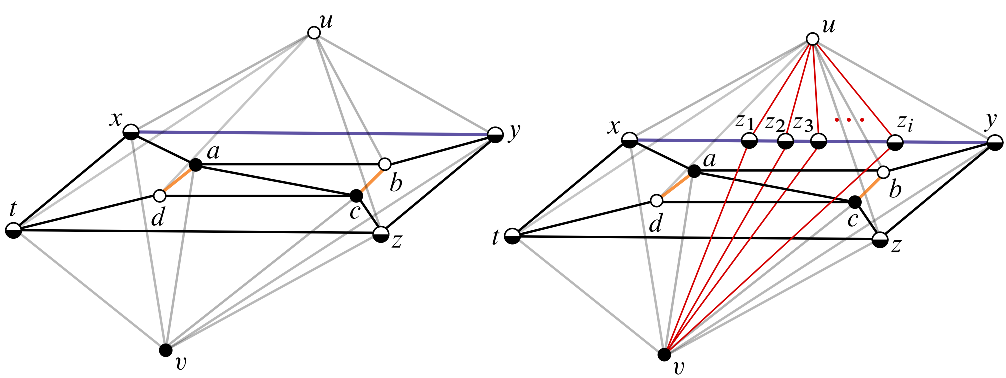


Figure 4: Left: the graph $G$ is maxnil with 10 vertices and 25 edges. Right: the graph $G_i$ is obtained through $i$ edge subdivisions and edge additions.

in $(G+e)/(ab \cup cd)$ or $(G+e)/(ad \cup bc)$. Thus $G+e$ contains a minor that itself contains the Jørgensen graph plus an edge.

Since the Jørgensen graph is maxnil, $G+e$ is IL. The same holds if $e = uv$ is added to $G$. If the edge $bd$ is added, then contracting the edges $dt$, $cz$, $ux$ and $vy$ creates a $K_6$ minor of $G+bd$.

Lastly, suppose an edge $e$ from $u$ or $v$ to $G - \{u, v\}$ is added; by symmetry, we can assume that $e = ua$ or $e = vb$. If $e = ua$, then contracting the edges $cd$, $dt$, $by$ and $uz$ creates a $K_6$ minor of $G+ua$. If $e = vb$, then contracting the edges $ax$, $cz$, $du$ and $dt$ creates a $K_6$ minor of $G+vb$. □

**Proposition 4** *All graphs $G_i$ for $i \geq 1$ are maxnil.*

**Proof** Since $G_i$ is linklessly embeddable, it remains to show that adding any edge to $G_i$ gives an IL graph. Adding any edge $e$ different from $xy$ and disjoint from $\{z_1, z_2, \dots, z_i\}$ to $G_i$ gives a graph $G_i + e$ that contains $G+e$ as a minor (obtained by contracting the path $xz_1z_2\dots z_i$). Since $G$ is maxnil, $G+e$ is IL and so is $G_i+e$. Adding an edge $e$ that is either $xy$ or has at least one endpoint in $\{z_1, z_2, \dots, z_i\}$ to $G_i$ gives a graph $G_i+e$ that contains $J_i+e$ as a minor (obtained by contracting the edges $ad$ and $bc$). Since $J_i$ is maxnil, $J_i+e$ is IL and so is $G_i+e$. □

## 2.2 The $Q(13,3)$ family

A graph $G$ is called *triangular* if each edge of $G$ belongs to at least one triangle. In a nontriangular graph, an edge that is not part of a triangle is a *nontriangular edge*. In Section 3, we study the properties of maxnil graphs under the operation of clique sum (defined in Section 3). For the construction presented in the next theorem we use the result of Lemma 10 about clique sums of maxnil graphs over $K_2$.

**Theorem 5** *For each $n \geq 13$, there exists a maxnil graph $G$ with $n$ vertices and $m < \frac{25}{12}n - \frac{1}{4}$ edges.*

**Proof** The construction is based on the maxnil graph $Q_{13,3}$ described by Maharry [8]. See Figure 5, left. This graph has 13 vertices and 26 edges, and it is triangle free.

For each $n$ with $13 \leq n \leq 39$, we construct a set of maxnil graphs with $n$ vertices and $2n$ edges by adding $n-13$ new vertices, and then choosing $n-13$ edges in $Q_{13,3}$ and

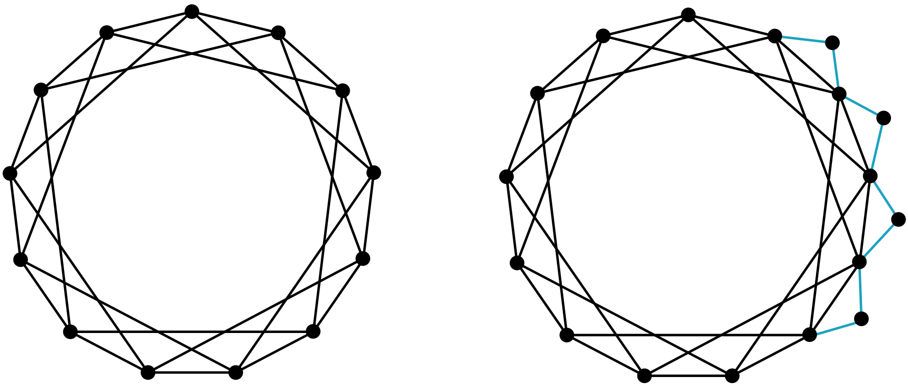

Figure 5: Left: $Q_{13,3}$ is a maxnil graph with 13 vertices and 26 edges. Right: a maxnil graph with 17 vertices and 34 edges obtained from $Q_{13,3}$ by adding four vertices of degree 2 and eight edges.

connecting the two endpoints of each of them to one of the new vertices. Equivalently, we are taking the clique sum of $Q_{13,3}$ with $n-13$ disjoint triangles over $n-13$ copies of $K_2$. See Figure 5, right. By Lemma 10, the resulting graph is maxnil.

The graph on 39 vertices obtained this way is triangular, so the construction cannot proceed further. To build graphs with a larger number of vertices, we use multiple copies of $Q_{13,3}$ joined along an edge (clique sum over $K_2$). Consider $k \geq 1$ copies of $Q_{13,3}$ and choose one edge in each copy. Then join the $k$ graphs together by identifying the $k$ chosen edges into one edge. This graph, which we denote by $H_k$, is maxnil (by repeated application of Lemma 10) and has $11k+2$ vertices and $25k+1$ edges. All edges of $H_k$ are nontriangular and adding vertices of degree 2 (as above) along any subset of the edges of $H_k$ gives a maxnil graph.

For $n \geq 13$, let $k = \lceil \frac{1}{36}(n-3) \rceil$ and add $n-(11k+2)$ vertices of degree 2 along any $n-(11k+2)$ edges of $H_k$. With every added vertex of degree 2, the number of edges is increased by 2. This gives a maxnil graph with $n$ vertices and $m = (25k+1) + 2[n-(11k+2)] = 2n+3k-3$ edges. Moreover,

$$m = 2n + 3\lceil \tfrac{1}{36}(n-3) \rceil - 3 < 2n + 3\big(\tfrac{1}{36}(n-3)+1\big) - 3 = \tfrac{25}{12}n - \tfrac{1}{4}. \qquad \square$$

**Remark 6** The above shows there exist maxnil graphs of arbitrarily large order $n$ with an edge-to-vertex ratio of less than $\frac{25}{12} - 1/(4n)$. Whether this edge-to-vertex ratio can be lowered further is an open question.

## 3 Clique sums of maxnil graphs

In this section we study the properties of maxnil graphs under taking clique sums. A set $S \subset V(G)$ is a *vertex cut set* of a connected graph $G$ if $G - S$ is disconnected. We say a vertex cut set $S \subset V(G)$ is *minimal* if no proper subset of $S$ is a vertex cut set of $G$. A graph $G$ is the *clique sum* of $G_1$ and $G_2$ over $K_t$ if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and the subgraphs induced by $V(G_1) \cap V(G_2)$ in both $G_1$ and $G_2$ are complete of order $t$. Since the vertices of the clique over which a clique sum is taken form a vertex cut set in the resulting graph, the vertex connectivity of a clique sum over $K_t$ is at most $t$. For a set of vertices $\{v_1, v_2, \dots, v_k\} \subset V(G)$, $\langle v_1, v_2, \dots, v_k \rangle_G$ denotes the subgraph of $G$ induced by this set of vertices. By abuse of notation, the subgraph induced in $G$ by the union of the vertices of subgraphs $H_1, H_2, \dots, H_k$ is denoted by $\langle H_1, H_2, \dots, H_k \rangle_G$.

Holst, Lovász and Schrijver [4, Theorem 2.10] studied the behavior of the Colin de Verdière $\mu$–invariant for graphs under clique sums. Since a graph $G$ is nIL if and only if $\mu(G) \le 4$ [6; 10], their theorem implies the following:

**Theorem 7** (Holst, Lovász and Schrijver [4]) *If $G$ is the clique sum over $S$ of two nIL graphs, then $G$ is IL if and only if one can contract two or three components of $G - S$ so that the contracted nodes together with $S$ form a $K_7$ minus a triangle.*

Theorem 7 implies that, for $t \le 3$, the clique sum over $K_t$ of nIL graphs is nIL. While Theorem 7 shows when a clique sum is nIL, it does not establish when a clique sum of maxnil graphs is maxnil.

**Lemma 8** *Any maxnil graph is 2–connected.*

**Proof** Let $G$ be a maxnil graph. If $G$ is disconnected, let $A$ and $B$ denote two of its connected components. Let $a \in V(A)$ and $b \in V(B)$. Then $G + ab$ is a nIL graph, as it can be obtained by performing two consecutive clique sums over $K_1$ of nIL summands, namely

$$G + ab = A \cup_{\{a\}} ab \cup_{\{b\}} (G - A).$$

But this contradicts the maximality of $G$.

If the vertex connectivity of $G$ is one, assume $x \in V(G)$ is a cut vertex; that is, $G - \{x\} = A \sqcup B$, with $A$ and $B$ nonempty, and no edges between vertices of $A$ and vertices of $B$. Let $a \in V(A)$ and $b \in V(B)$ be neighbors of $x$ in $G$. Then $G + ab$ is nIL, as it can be obtained by performing two consecutive clique sums over $K_2$ of nIL summands. If $\Delta$ denotes the triangle $axb$,

$$G + ab = \langle A, x\rangle_G \cup_{ax} \Delta \cup_{xb} \langle B, x\rangle_G.$$

But this contradicts the maximality of $G$. □

**Lemma 9** *Let $G$ be a maxnil graph with a vertex cut set $S = \{x, y\}$, and let $G_1, G_2, \dots, G_r$ denote the connected components of $G - S$. Then $xy \in E(G)$ and $\langle G_i, S\rangle_G$ is maxnil for all $1 \le i \le r$.*

**Proof** By Lemma 8, $x$ and $y$ are distinct and each of them has at least one neighbor in each $G_i$. Suppose $xy \notin E(G)$. Let $G' = G + xy$ and $G'_i = \langle G_i, S\rangle_{G'}$. Then, for every $i$, $G'_i$ is a minor of $G$ since, if we pick a $j \ne i$ and in $\langle G_i, G_j, S\rangle_G$ contract $G_j$ to $x$, we get a graph isomorphic to $G'_i$. So $G'_i$ is nIL. Then, by Theorem 7, $G' = G'_1 \cup_{xy} \cdots \cup_{xy} G'_r$ is nIL, contradicting the assumption that $G$ is maxnil. So $xy \in E(G)$.

For each $i$, we repeatedly add new edges to $\langle G_i, S\rangle_G$, if necessary, to get a maxnil graph $H_i$. Then $H := H_1 \cup_{xy} \cdots \cup_{xy} H_r$ is nIL and contains $G$ as a subgraph, so $H = G$ and every $\langle G_i, S\rangle_G$ is maxnil. □

**Lemma 10** *Let $G_1$ and $G_2$ be maxnil graphs. Pick an edge in each $G_i$ and label it $e$. Then $G = G_1 \cup_e G_2$ is maxnil if and only if $e$ is nontriangular in at least one $G_i$.*

**Proof** The graph $G$ is nIL by Theorem 7. Suppose $e$ is nontriangular in at least one $G_i$, say $G_2$. Denote the endpoints of $e$ in $G$ by $x$ and $y$. To prove $G$ is maxnil, it is enough to show that $G + b_1b_2$ is IL for all $b_i \in V(G_i) \setminus \{x, y\}$. By Lemma 8, $G_1$ is 2–connected, so each of $x$ and $y$ has at least one neighbor in $G_1$. So, if we contract $G_1$ to $b_1$ and then contract $b_1b_2$ to $b_2$, we obtain a graph $G'_2$ that contains $G_2$ as a proper subgraph, since $b_2x$ and $b_2y$ are both in $G'_2$, while $e$ is nontriangular in $G_2$. So $G'_2$ is IL since $G_2$ is maxnil. But $G'_2$ is a minor of $G$, which is nIL, so we have a contradiction.

To prove the converse, suppose $e$ is triangular in $G_1$ and $G_2$. Let $t_i \in V(G_i)$ be adjacent to both endpoints of $e$. Let $K$ be a complete graph on four vertices, with vertices labeled $x$, $y$, $t_1$ and $t_2$. Denote the triangles induced by $x$, $y$ and $t_i$ in $K$ and in $G_i$ by $\Delta_i$. Then, by Theorem 7, $G' := G_1 \cup_{\Delta_1} K \cup_{\Delta_2} G_2$ is nIL. But $G'$ is isomorphic to $G + t_1t_2$, so $G$ is not maxnil. □

**Lemma 11** *Let $G$ be a maxnil graph with vertex connectivity 3 and a vertex cut set $S = \{x, y, z\}$. Let $G_1, G_2, \dots, G_r$ denote the connected components of $G - S$. Then $\langle S\rangle_G \simeq K_3$ and $\langle G_i, S\rangle_G$ is maxnil for all $1 \le i \le r$.*

**Proof** Suppose $\langle S\rangle_G \not\simeq K_3$. Let $G'$ be the graph obtained from $G$ by adding one or more edges to $\langle S\rangle_G$ so that $S$ induces a triangle $T$ in $G'$. For $1 \le i \le r$, let $G_i' = \langle G_i, T\rangle_{G'}$. We see that $G_i'$ is nIL as follows. Pick any $j \ne i$ and, in the graph $\langle G_i, G_j, S\rangle_G$, contract $G_j$ to an arbitrary vertex $v$ in $G_j$. Then $v$ is connected to each of $x$, $y$ and $z$ since $G$ is 3–connected and hence each of $x$, $y$ and $z$ has at least one neighbor in $G_j$. The graph $M_i$ obtained this way is a minor of $G$, and hence is nIL. Performing a $\nabla Y$–move on $T \subset G_i'$ we obtain a subgraph of $M_i$. Since $M_i$ is nIL, so is $G_i'$. By Theorem 7, $G' = G_1' \cup_T \cdots \cup_T G_r'$ is nIL, which contradicts the maximality of $G$. So $T = \langle S\rangle_G \simeq K_3$.

To show $\langle G_i, S\rangle_G$ is maxnil, repeatedly add new edges to it, if necessary, to get a maxnil graph $H_i$. Then $H := H_1 \cup_T \cdots \cup_T H_r$ is nIL by Theorem 7 and contains $G$ as a subgraph, so $H = G$ and every $\langle G_i, S\rangle_G$ is maxnil. □

Let $G$ be a graph and let $T = \langle x, y, z, t\rangle_G$ be an induced $K_4$ subgraph (*tetrahedral graph*). We say $T$ is *strongly separating* if $G - T$ has at least two connected components $C_1$ and $C_2$ such that every vertex of $T$ has a neighbor in each $C_i$.

**Lemma 12** *Let $G_1$ and $G_2$ be maxnil graphs and let $G = G_1 \cup_\Delta G_2$ be the clique sum of $G_1$ and $G_2$ over a $K_3$ subgraph $\Delta = \langle x, y, z\rangle_G$. Assume $\Delta$ is a minimal vertex cut set in $G$. Then $G$ is maxnil if and only if, for some $i \in \{1, 2\}$, every induced $K_4$ subgraph of the form $\langle x, y, z, t\rangle_{G_i}$ is strongly separating.*

**Proof** By Theorem 7, $G := G_1 \cup_\Delta G_2$ is nIL. Then $G$ is maxnil if and only if, for every $t_1 \in V(G_1) \setminus V(\Delta)$ and $t_2 \in V(G_2) \setminus V(\Delta)$, the graph $G' := G + t_1t_2$ is IL.

First, suppose for some $i$ at least one of $x$, $y$ and $z$ is not connected to $t_i$, say $xt_2 \notin E(G_2)$. Contracting $G_1 - \{y, z\}$ to $x$ produces $G_2 + t_2x$ as a minor of $G'$. Since $G_2$ is maxnil, this minor is IL, and hence $G'$ is IL, as desired. So we can assume $\langle x, y, z, t_i\rangle_{G_i}$ is a tetrahedral graph for both $i = 1, 2$.

Assume every tetrahedral graph in $G_2$ that contains $\Delta$ is strongly separating. So $G_2 - \langle x, y, z, t_2\rangle_{G_2}$ has at least two connected components each of which, when contracted to a single vertex, is adjacent to all four vertices $x$, $y$, $z$ and $t_2$. In Figure 6,

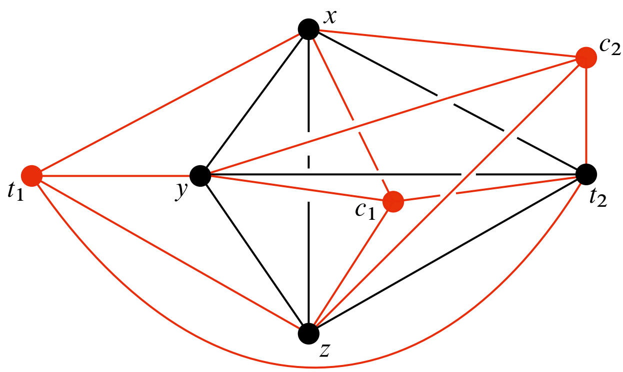


Figure 6: A $K_7$ minus a triangle minor of the graph $G$.

these vertices are denoted by $c_1$ and $c_2$. Now, if the component of $G_1 - \Delta$ that contains $t_1$ is contracted to $t_1$, this vertex too will be adjacent to $x$, $y$, $z$ and $t_2$. So we get a minor of $G'$ isomorphic to $K_7$ minus a triangle, which is IL since it contains a Petersen family graph (the one obtained by one $\nabla Y$–move on $K_6$) as a minor. It follows that $G'$ is IL, and therefore $G$ is maxnil.

To prove the converse, for $i = 1, 2$ let $t_i$ be a vertex in $G_i$ such that $T_i := \langle x, y, z, t_i \rangle_{G_i}$ is a tetrahedral graph that is not strongly separating. Let $G' = G + t_1 t_2$. Then $G' = G_1 \cup_{T_1} \langle x, y, z, t_1, t_2 \rangle_{G'} \cup_{T_2} G_2$. Each of these clique sums is over a $K_4$, each summand is nIL, and each of $T_1$ and $T_2$ is nonstrongly separating; so, by Theorem 7, $G'$ is nIL, and hence $G$ is not maxnil. $\square$

Unlike the vertex connectivity 2 and 3 cases, it is not true that a minimal vertex cut set in a 4–connected maxnil graph must be a clique. The four neighbors of $b$ in the graph depicted in Figure 4, left, form a vertex cut set, but the graph induced by its vertices has exactly two edges. The four neighbors of any vertex in the graph $Q_{13,3}$ in Figure 5, left, form a discrete vertex cut set. However, if a maxnil graph $G$ has vertex connectivity 4, the following lemma provides some restrictions on the shape of the subgraph induced by the vertices of any minimal vertex cut set:

**Lemma 13** *Let $G$ be a maxnil graph and assume $\{x, y, z, t\}$ is a minimal vertex cut. Let $S = \langle x, y, z, t \rangle_G$. Then $S$ is either a clique or a subgraph of a 4–cycle.*

**Proof** Assume that $S$ is neither a clique nor a subgraph of a 4–cycle. This implies that, if every vertex of $S$ has degree less than 3, then $S$ contains $K_3$ as a subgraph; and if $S$ has a vertex of degree at least 3, then it contains $K_{1,3}$ as a subgraph. Below, we consider these two cases separately. In both cases, we use the fact that since $\{x, y, z, t\}$ is a minimal vertex cut set in $G$, each of $x$, $y$, $z$ and $t$ has at least one neighbor in each component of $G - S$.

**Case 1** ($S$ has a $K_3$ subgraph) We can assume that $x$, $y$ and $z$ induce a triangle in $G$. If $G - S$ has at least three connected components, contracting each of them to a single node would produce a minor of $G$ which has a subgraph isomorphic to $G_7$, the graph in the Petersen family obtained by one $\nabla Y$ move on $K_6$. This contradicts the fact that $G$ is nIL.

It follows that $G - S$ has at most two components, $G_1$ and $G_2$. For each $i = 1, 2$, contract $\langle G_i, t\rangle_G$ to $t$ to produce a minor of $G$, denoted by $G_i'$, which must be nIL. Then $\{x, y, z, t\}$ induces a 4–clique $K$ in both $G_1'$ and $G_2'$. By Theorem 7, the clique sum $G' = G_1' \cup_K G_2'$ is nIL since $G' - K$ has only two components and $K$ has only four vertices. But $G'$ strictly contains $G$ as a subgraph; this implies $G$ is not maxnil, a contradiction.

**Case 2** ($S$ has a $K_{1,3}$ subgraph) We can assume that $t$ is adjacent to $x$, $y$ and $z$ in $G$. If $G - S$ has at least three connected components, contracting each of them to a single node would produce a minor of $G$ containing a subgraph isomorphic to $K_{3,3,1}$; thus, $G$ is IL. So $G - S = G_1 \sqcup G_2$, with $G_1$ and $G_2$ connected. For $i = 1, 2$, contracting each of $G_i$ to a single node $t_i$, deleting the edge $t_i t$, deleting any existing edges of $\langle x, y, z\rangle_G$, and then performing a $Y\nabla$–move at $t_i$ produces a nIL graph, denoted by $G_i'$. Let $G' = G_1' \cup_{K_4} G_2'$ be the clique sum over the complete graph with vertices $x$, $y$, $z$ and $t$. By Theorem 7, $G'$ is nIL since $G' - S = G_1 \sqcup G_2$; but $G'$ strictly contains $G$ as a subgraph, a contradiction. $\square$

**Lemma 14** *Let $G = G_1 \cup_S G_2$ be the clique sum of maxnil graphs $G_1$ and $G_2$ over $S = \langle x, y, z, t\rangle_G \simeq K_4$. Assume $S$ is a minimal vertex cut set in $G$. Then $G$ is maxnil if and only if, in both $G_1$ and $G_2$, $S$ is not strongly separating.*

**Proof** If $S$ is strongly separating in $G_1$ or $G_2$, then $G - S$ has at least three connected components and contracting each of them to a single node produces a minor isomorphic to $K_7$ minus a triangle.

If, in both $G_1$ and $G_2$, $S$ is not strongly separating, then $G - S$ has only two connected components. Contracting each of the two components to a single node produces $K_6$

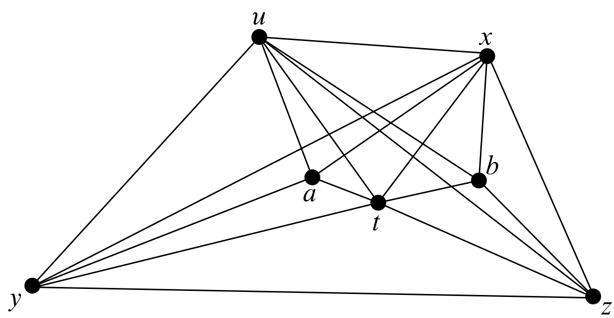


Figure 7: A maxnil graph that is a clique sum over $K_5$.

minus an edge as a minor (not $K_7$ minus a triangle); hence, $G$ is nIL by Theorem 7. Adding an edge between a vertex in $G_1 - S$ and a vertex in $G_2 - S$ and contracting $G_1 - S$ and $G_2 - S$ to single nodes produces a $K_6$ minor. It follows that $G$ is maxnil in this case. □

The graph $G$ of Figure 7 is maxnil since $G - \{u\}$ is a maximal planar graph. If $S = \langle x, y, z, t, u\rangle$, $G_1 = \langle a, x, y, z, t, u\rangle$ and $G_2 = \langle b, x, y, z, t, u\rangle$, then $S \simeq K_5$, $G_1 \simeq G_2 \simeq K_6^-$ ($K_6$ minus one edge) and $G = G_1 \cup_S G_2$. This shows it is possible for the clique sum of two maxnil graphs over $S \simeq K_5$ to be nIL (and maxnil). However, no clique $S$ of order 5 can be a minimal vertex cut set in a nIL graph $G$, since then any connected component of $G - S$ would form a $K_6$–minor together with $S$, which would imply $G$ is IL. For $t \geq 6$, any clique sum over $K_t$ is IL since $K_6$ is IL.

Jørgensen studied clique sums of graphs that are maximal without a $K_6$ minor [5]. These are graphs that do not contain a $K_6$ minor and a $K_6$ minor is created by the addition of any edge. The class of maxnil graphs and the class of graphs that are maximal without a $K_6$ minor are not the same, as shown in the following proposition:

**Proposition 15** *The graph in Figure 8 is maxnil, and it is not maximal without a $K_6$ minor.*

**Proof** The graph $G$ in Figure 8 is obtained by adding vertices $v$ and $w$ to the plane triangulation $H$: the vertex $v$ connects to all nine vertices of $H$ and the vertex $w$ connects to the vertices $a$, $b$ and $c$ of $H$. The graph $H + v$ is maxnil since it is a cone over a maximal planar graph [11]. The graph $G$ is the clique sum over $K_3 = \langle a, b, c\rangle_G$ of maxnil graphs $H + v$ and $K_4 = \langle a, b, c, w\rangle_G$. The graph $\langle a, b, c, v\rangle_{H+v}$ is the only induced $K_4$ subgraph in $H + v$ containing $a, b$ and $c$ and it is strongly separating

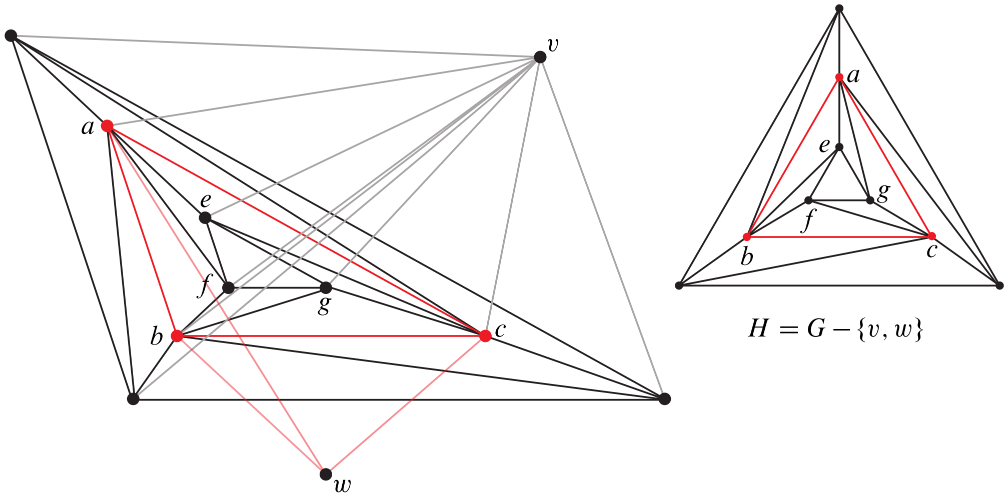


Figure 8: A maxnil graph $G$ (left) that is not maximal without a $K_6$ minor is obtained by adding two vertices to a plane triangulation with nine vertices (right).

in $H+v$. So, by Lemma 12, $G$ is maxnil; in particular, it has no $K_6$ minor. The graph $G+vw$ is a clique sum over $K_4 = \langle a,b,c,v\rangle_G$ of graphs $H+v$ and $K_5 = \langle a,b,c,v,w\rangle$, both of which are $K_6$ minor free. Hence, by [5], $G+vw$ is $K_6$ minor free, so $G$ is not maximal without a $K_6$ minor. The graph $G+vw$ has order 11 and size 34, so it is maximal without a $K_6$ minor by Mader's result [7], since $34 = 4\times 11 - 10$. $\square$

**Remark 16** Starting with the graph $G$ in Proposition 15, one can construct graphs $G_n$ with $n \geq 11$ vertices that are maxnil but not maximal without a $K_6$ minor. Take $G_{11} = G$ and construct $G_{11+k}$ from $G$ by triangulating the disk bounded by the triangle $efg$ with $k$ new vertices, and then adding edges between $v$ and these new vertices. The argument used in the proof of Proposition 15 shows that $G_n$ for $n \geq 11$ is maxnil but not maximal without a $K_6$ minor. Furthermore, $n = 11$ is the minimal order of a graph with this property, ie every maxnil graph with $n \leq 10$, vertices is maximal without a $K_6$ minor. We used Mathematica to generate all 136 maxnil graphs of orders between 6 and 10 and we confirmed that all of them are maximal without a $K_6$ minor.

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