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JENS HARLANDER
STEPHAN ROSEBROCK

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Wirtinger presentations of deficiency 1 appear in the context of knots, long virtual knots, and ribbon 2–knots. They are encoded by labeled oriented trees and, for that reason, are also called LOT presentations. These presentations are a well known and important testing ground for the validity (or failure) of Whitehead's asphericity conjecture. We define LOTs of Coxeter type and show that for every given n there exists a prime LOT of Coxeter type with group of rank n. We also show that label separated Coxeter LOTs are aspherical.

20F05, 20F06, 20F65; 57K20, 57K45

Dedicated to the memory of Stephen Pride

1 Introduction

Wirtinger presentations of deficiency 1 appear in the context of knots, long virtual knots, and ribbon 2–knots; see Harlander and Rosebrock [9]. They are encoded by labeled oriented trees and, for that reason, are also called LOT presentations. Adding a generator to the set of relators in a Wirtinger presentation P gives a balanced presentation of the trivial group. Thus the associated 2–complex $K(P)$ is a subcomplex of an aspherical (in fact contractible) 2–complex. Wirtinger presentations are a well-known and important testing ground for the validity (or failure) of Whitehead's asphericity conjecture, which states that a subcomplex of an aspherical 2–complex is aspherical. For more on the Whitehead conjecture see Bogley [3], Berrick and Hillman [1] and Rosebrock [18].

If P is a Wirtinger presentation and the group $G(P)$ defined by P is a 1–relator group, then $G(P)$ admits a 2–generator 1–relator presentation P' and the corresponding 2–complex $K(P')$ is aspherical. Since $K(P')$ and $K(P)$ have the same Euler characteristic and the same fundamental group, it follows (using Schanuel's lemma and Kaplansky's theorem, which states that finitely generated free $\mathbb{Z}G$–modules are Hopfian) that $K(P)$

is also aspherical. Thus, when investigating the asphericity of $K(P)$ for a given Wirtinger presentation P, the first thing to ask is if $G(P)$ is a 1–relator group.

Composite knot groups require more than two generators; see Norwood [17].[1] However, many knots have 2–generator 1–relator knot groups. Prime knots whose groups need more than two generators were known to Crowell and Fox in 1963. As one example, Crowell and Fox consider a certain prime 9–crossing knot, show that its Wirtinger presentation simplifies to

$$P = \langle x, y, z \mid y^{-1}xyx^{-1}y = x^{-1}zx^{-1}zxz^{-1}x,\ x^{-1}zxz^{-1}x = y^{-1}zyz^{-1}y \rangle,$$

and show that the length of the chain of elementary ideals for this knot group is 2. It follows that the rank (the minimal number of generators) of $G(P)$ is greater than 2 and therefore equal to 3. This can also be seen without the use of elementary ideals. We have an epimorphism

$$G(P) \to \Delta(3,3,3) = \langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (xz)^3, (yz)^3 \rangle$$

sending $x \to x$, $y \to y$ and $z \to z$. Since the rank of the Euclidean triangle group $\Delta(3,3,3)$ is 3 (see Klimenko and Sakuma [13]), we have $\operatorname{rank}(G(P)) = 3$.

This example motivates the first part of this article. It is easier to construct high-rank ribbon 2–knot groups than classical knot groups, because we do not have to verify that a given Wirtinger presentation can be read off a knot projection (a 4–regular planar graph). Below we define labeled oriented trees of Coxeter type and show that, given a Coxeter group W which abelianizes to $W_{\mathrm{ab}} = \mathbb{Z}_2$, there exists a Coxeter-type LOT group that maps onto W. Using this we give examples of prime LOT groups of arbitrarily high rank.

In the second part of the paper we investigate the question of asphericity of LOTs of Coxeter type. We show that label-separated LOTs of Coxeter type are aspherical. It turns out that the study of asphericity can be translated into questions concerning free subgroups of 1–relator LOT groups of dihedral type.

2 Groups defined by graphs

A *labeled oriented graph* (LOG) is an oriented finite graph Γ on vertices $\boldsymbol{x}$ and edges $\boldsymbol{e}$, where each oriented edge is labeled by a word in $\boldsymbol{x}^{\pm 1}$. Associated with a LOG Γ is

[1] The central Lemma 3 in Norwood's paper has a gap which can be filled, as was pointed out by Menasco and Reid [15, page 223, Remark 4] and also Bleiler [2].

the presentation

$$P(\Gamma) = \langle \boldsymbol{x} \mid \boldsymbol{r} = \{r_e \mid e \in \boldsymbol{e}\}\rangle,$$

where $r_e = xw(wy)^{-1}$ when $e = (x \xrightarrow{w} y)$ is the edge of Γ starting at x, ending at y, and labeled with the word w on letters in $\boldsymbol{x}^{\pm 1}$. We remark that what we call a labeled oriented graph is elsewhere called a *weakly labeled oriented graph* or *word-labeled oriented graph*. See Howie [12] and Harlander and Rosebrock [10].

Denote by $K(\Gamma)$ and $G(\Gamma)$ the standard 2–complex and the group defined by $P(\Gamma)$, respectively. The case where Γ is a tree, now called a *labeled oriented tree* (LOT), is special. It is known that the groups $G(\Gamma)$ where Γ is a LOT are precisely the ribbon 2–knot groups (see Yajima [22], Howie [12] and also Hillman [11, Section 1.7]), since, in that case, $G(\Gamma)$ is a group of weight 1 (normally generated by a single element, in fact by each generator) that has a deficiency 1 presentation $P(\Gamma)$. The 2–complexes $K(\Gamma)$ with Γ a LOT are of central importance to Whitehead's asphericity conjecture, since adding a generator to the set of relators in $P(\Gamma)$ gives a balanced presentation of the trivial group. So $K(\Gamma)$ is a subcomplex of a 2–dimensional contractible complex. A question that has been open for a long time asks if $K(\Gamma)$ is aspherical, ie $\pi_2(K(\Gamma)) = 0$. See Bogley [3], Berrick and Hillman [1] and Rosebrock [18].

A subtree $\Gamma_0 \subseteq \Gamma$ of a LOT is a *sub-LOT* if the label w of an edge in Γ_0 is a word in the vertices of Γ_0. A sub-LOT $\Gamma_0 \subseteq \Gamma$ is called *proper* if it has more than one vertex and is not all of Γ. A LOT is called *prime* if it does not contain proper sub-LOTs.

Let Υ be a simplicial graph on vertices $\boldsymbol{x}$, and suppose edges e are labeled with integers $m_e \geq 2$. Define

$$P(\Upsilon) = \langle \boldsymbol{x} \mid x^2 \text{ for } x \in \boldsymbol{x},\ (xy)^{m_e} \text{ if } e = \{x, y\} \text{ is an edge}\rangle.$$

The group $W = W(\Upsilon)$ defined by this presentation is called a *Coxeter group*. We refer to Υ as the *defining graph* for the Coxeter group. We remark that the graph Υ shows up in Davis [6, Example 7.1.6] (the Coxeter system associated to a labeled simplicial graph). It should not be confused with the Dynkin diagram, another labeled graph that appears in connection with Coxeter groups. Let $K = K(\Upsilon)$ be the 2–complex associated with $P(\Upsilon)$. Consider the universal covering $\widetilde{K}(\Upsilon)$. The 1–skeleton of $\widetilde{K}(\Upsilon)$ is the Cayley graph for $(W, \boldsymbol{x})$. All edges in $\widetilde{K}(\Upsilon)$ are double edges: for every $g \in W$ and $x \in \boldsymbol{x}$, we have an edge (g, x) connecting g to gx, and an edge (gx, x) connecting gx to $gxx = g$. Note that a double edge pair bounds two 2–cells in $\widetilde{K}(\Upsilon)$, coming from the relator x^2. We remove one and collapse the other one to an edge. This turns each double edge into a single unoriented edge. Every relator $(xy)^{m_e}$ gives rise to

$2m_e$ 2–cells with the same boundary. We remove all but one from this set. We denote the 2–complex obtained in this fashion by $\Sigma^{(2)}(\Upsilon)$. It is the 2–skeleton of the Davis complex $\Sigma(\Upsilon)$. See [6, Proposition 7.3.4]. We remark that the Davis complex is closely related to the Coxeter complex, but the complexes are not the same. For the definition of Coxeter complex, see [6, Example 5.2.7]. Under certain conditions, for example when the defining graph Υ is a tree, the Davis complex is 2–dimensional: $\Sigma(\Upsilon) = \Sigma^{(2)}(\Upsilon)$. See [6, Example 7.4.2].

Proposition 2.1 *Let Υ be a defining tree with associated Coxeter group $W(\Upsilon)$. Then*:

1. *For every edge $e = \{x, y\}$ of Υ we have a 2–cell κ_e in $\Sigma(\Upsilon)$ attached along a $2m_e$–gon whose edge labels read $(xy)^{m_e}$.*
2. *$\Sigma(\Upsilon)$ is the union of the 2–cells $w\kappa_e$ for $e \in \{\text{edges of } \Upsilon\}$ and $w \in W(\Upsilon)$. Furthermore, if $w_1\kappa_{e_1} \cap w_2\kappa_{e_2} \neq \varnothing$, then $e_1 \cap e_2 \neq \varnothing$; if $x = e_1 \cap e_2$, then the edge $w_1\kappa_{e_1} \cap w_2\kappa_{e_2}$ carries the label x.*
3. *$\Sigma(\Upsilon)$ is a tree of 2–cells: if we connect the barycenters of the 2–cells with the barycenters of their boundary edges, we obtain a tree. In particular, if M is a finite connected union of Coxeter 2–cells $w_i\kappa_{e_i}$ in $\Sigma(\Upsilon)$, then there exists a 2–cell $w\kappa_e$ in M that intersects with the rest of M in a single edge.*

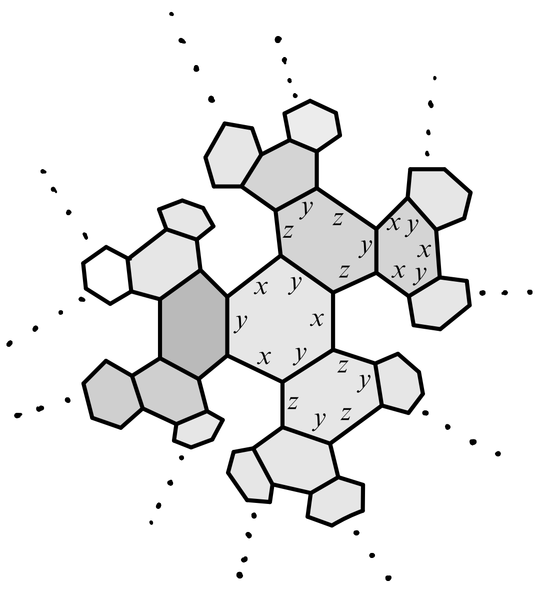

Figure 1: The Davis complex $\Sigma(\Upsilon)$ for $\Upsilon = x \overset{3}{—} y \overset{3}{—} z$. It is a tree of Coxeter cells.

Proof The statements (1) and (2) are clear from the construction of Υ. For an edge $e=\{x,y\}$, let $P(e)=\langle x,y \mid x^2,y^2,(xy)^{m_e}\rangle$. Let D_{m_e} be the dihedral group defined by $P(e)$. Since Υ is a tree, $W(\Upsilon)$ is an amalgamated product of the D_{m_e}. The associated Bass–Serre tree can be seen inside the Davis complex $\Sigma(\Upsilon)$. The vertices of that tree are the barycenters of the 2–cells and 1–cells, and the edges connect barycenters of 2–cells to the barycenters of the 1–cells in the boundary of that 2–cell. We can think of $\Sigma(\Upsilon)$ as a tree of Coxeter 2–cells. An example is shown in Figure 1.

Suppose $M=\bigcup_{i=0}^k D_i$ is a union of 2–cells. Let d_i be the barycenter of D_i. Let d_p be a vertex in the Bass–Serre tree furthest away from d_0 with $p\in\{0,\dots,k\}$. Consider a geodesic from d_0 to d_p and let d_q be the barycenter that is encountered just before getting to d_p when traveling along the geodesic. Then $\bigl(\bigcup_{i\neq p} D_i\bigr)\cap D_p=D_q\cap D_p$, which is a single edge. $\square$

Lemma 2.2 *Let Γ be a LOT $e=(x\xrightarrow{w}y)$ an edge such that the word w contains letters and $z\neq x,y$ with even (positive or negative) exponent only. Then the relator $r_e=xw(wy)^{-1}$ reduces (up to cyclic permutation) to $\bar r_e=(yx)^{m_e}$, with $m_e\geq 1$ and odd, in $\langle \boldsymbol{x}\mid x^2$ for $x\in\boldsymbol{x}\rangle$.*

Proof The word w reduces to an alternating word $\overline{w}$ in the letters x and y. If $\overline{w}$ is the empty word, then $\bar r_e=xy$. There are four remaining cases to consider:

(1) $\overline{w}$ starts with x and has even length.

(2) $\overline{w}$ starts with x and has odd length.

(3) $\overline{w}$ starts with y and has even length.

(4) $\overline{w}$ starts with y and has odd length.

In case (1) we have $\overline{w}=xyxy$, say. So $x(xyxy)y(xyxy)=xxyxyyxyxy=xy$. In case (2) we have $\overline{w}=xyxyx$, say. So $x(xyxyx)y(xyxyx)=xxyxyxyxyxyx=(yx)^5$. In case (3) we have $\overline{w}=yxyx$, say. So $x(yxyx)y(yxyx)=xy$. In case (4) we have $\overline{w}=yxyxy$, say. So $x(yxyxy)y(yxyxy)=(xy)^5$. $\square$

Definition 2.3 Let Γ be a LOT with vertex set $\boldsymbol{x}$. We say Γ is of *Coxeter type* if:

(1) For every edge $e=(x\xrightarrow{w}y)$, the word w contains letters $z\neq x,y$ with even (positive or negative) exponent only.

(2) For every edge $e=(x\xrightarrow{w}y)$, the relator $r_e=xw(wy)^{-1}$ reduces (up to cyclic permutation) to $\bar r_e=(yx)^{m_e}$, with $m_e\geq 2$, in $\langle\boldsymbol{x}\mid x^2$ for $x\in\boldsymbol{x}\rangle$.

Remark 2.4 Lemma 2.2 shows that, if Γ is a LOT of Coxeter type, then, for each edge e, $m_e \geq 3$ and is odd.

Let Γ be a LOT of Coxeter type. Define a tree Υ in the following way: erase orientations in Γ and, if $e = (x \xrightarrow{w} y)$ is an edge and the LOT relator r_e reduces to $\bar{r}_e = (yx)^{m_e}$ (up to cyclic permutation) in $\langle \boldsymbol{x} \mid x^2 \text{ for } x \in \boldsymbol{x} \rangle$, then label the (unoriented) edge e by m_e. Note that Υ is a defining tree for a Coxeter group. We have a map $P(\Gamma) \to P(\Upsilon)$ sending x to x which induces a group epimorphism $G(\Gamma) \to W(\Upsilon)$. This process can be reversed.

Lemma 2.5 *Let Υ be a defining tree for a Coxeter group where all m_e are odd. Then there exists a LOT of Coxeter type Γ such that the process described above produces Υ from Γ. In particular, $G(\Gamma)$ maps onto $W(\Upsilon)$.*

Proof Suppose $e = \{x, y\}$ is an edge in Υ. Orient it from x to y. Let $w = (yx)^{(m_e-1)/2}$. Let $e = (x \xrightarrow{w} y)$ be the corresponding edge in Γ. □

Note that the LOT Γ of Coxeter type constructed in the lemma is not prime. In fact, every edge is a sub-LOT. Note also that $G(\Gamma)$ is an Artin group. One can show that all Artin groups are LOG groups, but we will not pursue this here. Given a defining tree Υ, there are many LOTs of Coxeter type that give rise to Υ.

Lemma 2.6 *Let Υ be a defining tree where all m_e are odd. Suppose Γ is a LOT of Coxeter type such that the process just described produces Υ. Then there also exists a prime LOT that produces Υ.*

Proof Suppose Γ_0 is a proper sub-LOT of Γ. Let $e = (x \xrightarrow{w} y)$ be an edge in Γ_0 and z be a vertex not in Γ_0. Replace the label w by $z^2 w$ to obtain a new LOT Γ'. Then Γ_0 is not a sub-LOT of Γ', but Γ' still produces Υ. We can apply this procedure until we arrive at a LOT without proper sub-LOTs. □

3 LOT groups of high rank

Given two LOTs Γ_1 and Γ_2 and two valency-one vertices $x_i \in \Gamma_i$ for $i = 1, 2$, one can form a composite LOT $\Gamma = \Gamma_1 \cup_{x_1 = x_2} \Gamma_2$ by identifying the two vertices. The LOT group $G(\Gamma)$ is an amalgam $G(\Gamma_1) *_{\mathbb{Z}} G(\Gamma_2)$, and, avoiding trivial cases, the rank of $G(\Gamma)$ is greater than the rank of each $G(\Gamma_i)$ for $i = 1, 2$. This follows from a theorem of Karras and Solitar. See also [17]. However, Γ is not prime, and it is more difficult to provide lower bounds for the rank of prime LOT groups. This issue is already present in the classical knot world, as was discussed in the introduction. In this section we present a method for constructing prime LOTs with groups of arbitrarily high rank.

Theorem 3.1 (Carette and Weidmann [5]) *Let Υ be a defining graph with n vertices and assume that $m_e \geq 6 \cdot 2^n$ for each e. Then the rank of $W(\Upsilon)$ is n.*

Theorem 3.2 *Let $W = W(\Upsilon)$ be a Coxeter group such that $W_{\mathrm{ab}} = \mathbb{Z}_2$. There exists a prime labeled oriented tree Γ of Coxeter type such that $G = G(\Gamma)$ maps onto W.*

Proof Since $W_{\mathrm{ab}} = \mathbb{Z}_2$, the defining graph Υ is connected. In fact, the subgraph Υ_{odd} consisting of edges with odd label is connected, because an edge with an even label does not contribute a relation in W_{ab}, so $W(\Upsilon_{\mathrm{odd}})_{\mathrm{ab}} = W(\Upsilon)_{\mathrm{ab}}$. Thus Υ contains a maximal tree Υ_0 in which all labels m_e are odd. Then Υ and Υ_0 have the same set of vertices and we have an epimorphism $W(\Upsilon_0) \to W(\Upsilon)$. From Lemmas 2.5 and 2.6, we know that there is a prime LOT Γ of Coxeter type such that $G(\Gamma)$ maps onto $W(\Upsilon_0)$. $\square$

Corollary 3.3 *For any given n there exists a prime labeled oriented tree Γ of Coxeter type with n vertices such that $G(\Gamma)$ has rank n. In particular, if $n \geq 3$, then $G(\Gamma)$ is not a 1–relator group.*

Proof This follows from Theorem 3.2 together with the Carette–Weidmann theorem, Theorem 3.1. $\square$

Example 3.4 Let Γ be the prime LOT $x \xrightarrow{yz^2x} y \xrightarrow{zx^2y} z$. Note that $G(\Gamma)$ maps onto the amalgamated product $D_3 *_{\mathbb{Z}_2} D_3$, which cannot be generated by two elements. Thus the rank of $G(\Gamma)$ is 3 and it follows that this LOT group is not a 1–relator group.

Remark 3.5 If Γ is a LOT of Coxeter type and Υ is the associated defining tree, then $W(\Upsilon)$ is an amalgamated product of dihedral groups. A direct way to obtain upper bounds for the rank of $W(\Upsilon)$ without the full force of Theorem 3.1 is via Weidmann [21].

Remark 3.6 A *reorientation* of a LOT is obtained when changing signs on the exponents of letters that occur in the edge words, which has no effect on the quotient $W(\Upsilon)$. Thus, if $\mathrm{rk}(G(\Gamma)) = \mathrm{rk}(W(\Upsilon))$, then this equation holds also for all reorientations of Γ.

4 Largeness

A group is *large* if it has a subgroup of finite index that has a free quotient of rank ≥ 2. Large groups of deficiency 1 are studied in Button [4]. A list of properties can also be found there. If G is large, then:

(1) G contains free subgroups of rank ≥ 2.

(2) G is SQ-universal (every countable group is the subgroup of some quotient).

(3) G has finite-index subgroups with arbitrarily large first Betti number.

(4) G has uniformly exponential word growth.

(5) G has subgroup growth of strict type n^n (which is the largest possible growth for finitely generated groups).

(6) The word problem for G is solvable strongly generically in linear time.

Theorem 4.1 *Let Γ be a LOT of Coxeter type on at least three vertices. Then $G(\Gamma)$ is large.*

Proof The conditions imply that $W(\Upsilon)$ is an infinite group that is the fundamental group of a finite tree of groups where the vertex groups are either $\mathbb{Z}_2$ or dihedral groups D_m with $m \geq 3$ ($\mathbb{Z}_2$ vertex groups will appear when Υ has vertices of valency ≥ 3). Thus $W(\Upsilon)$ contains a free subgroup F of rank ≥ 2 of finite index (see Serre [20, Proposition 11, page 120)]. Let H be the preimage of F in $G(\Gamma)$. Then H is a subgroup of $G(\Gamma)$ of finite index that maps onto F. It follows that $G(\Gamma)$ is large. □

A characterization of virtual free Coxeter groups is given in Davis [6, Section 8.8]. When Υ is a tree, the characterization implies that $W(\Upsilon)$ is virtually free. This provides another proof for Theorem 4.1.

Example 4.2 As in Example 3.4 let Γ be the prime LOT $x \xrightarrow{yz^2x} y \xrightarrow{zx^2y} z$. We have $W(\Upsilon) = D_3 *_{\mathbb{Z}_2} D_3$. Let $\Delta(3,3,2)$ be the spherical triangle group (it is the symmetric group S_4) defined by $\langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (yz)^3, (xz)^2\rangle$. We have an epimorphism $W(\Upsilon) \to \Delta(3,3,2)$ and we claim that the kernel V is free of rank ≥ 2. Indeed, since both D_3's of $W(\Upsilon)$ are also subgroups of $\Delta(3,3,2)$, it follows that V intersects both D_3's trivially and thus V acts freely on the Bass–Serre tree T for $W(\Upsilon) = D_3 *_{\mathbb{Z}_2} D_3$, and hence is free. Note that the valency of every vertex in T is equal to 3 (since the index of $\mathbb{Z}_2$ in the D_3's is 3), and so V cannot be cyclic. Here is why: Note that $V = \pi_1(X)$, where $X = T/V$ is a finite graph in which every vertex has valency 3. Let $v(X)$ and $e(X)$ denote the number of vertices and edges, respectively. We have $v(X) = \frac{2}{3}e(X)$ and we obtain $\chi(X) = v(X) - e(X) = \frac{2}{3}e(X) - e(X) < 0$. Thus $\dim H_0(X) - \dim H_1(X) = 1 - \dim H_1(X) = \chi(X) < 0$. So $\dim H_1(X) > 1$ and hence $\dim V_{\mathrm{ab}} > 1$. One can also check directly that $(xz)^2$ and $x(xz)^2x^{-1} = (zx)^2$ generate a free subgroup of V of rank 2.

5 The question of asphericity

Let Γ be a labeled oriented tree of Coxeter type and let Υ be the associated defining tree for the Coxeter group $W(\Upsilon)$. Let $\overline{K}(\Gamma)$ be the normal covering space with fundamental group the kernel of the epimorphism $G(\Gamma) \to W(\Upsilon)$. We will analyze the structure of $\overline{K}(\Gamma)$. We have maps
$$\overline{K}(\Gamma) \to \widetilde{K}(\Upsilon) \to \Sigma(\Upsilon),$$
and note that $\overline{K}(\Gamma)$ and $\widetilde{K}(\Upsilon)$ have the same 1–skeleton. Let $e = (x \xrightarrow{w} y)$ be an edge in Γ. Let $P_e = \langle \boldsymbol{x}_e \mid r_e \rangle$, where $\boldsymbol{x}_e \subseteq \boldsymbol{x}$ is the subset of the vertices of Γ that occur in r_e. Let $\boldsymbol{z} = \boldsymbol{x}_e - \{x, y\}$. Then $P_e = \langle x, y, \boldsymbol{z} \mid xw = wy \rangle$. The complex $K(P_e)$ is a subcomplex of $K(\Gamma)$. Consider the preimage of $K(P_e)$ under the covering projection $\overline{K}(\Gamma) \to K(\Gamma)$. It is a union of finite subcomplexes $w\overline{K}_e$ for $w \in W(\Upsilon)$, which we will now describe in detail. The 1–skeleton of $\overline{K}_e$ is a $2m_e$–gon with double edges labeled in an alternating way by x and y. At each of the $2m_e$ vertices we have a double edge for every $z \in \boldsymbol{z}$. The situation is depicted in Figure 2. We have $2m_e$ 2–cells, attached along the loop with label r_e, starting at every vertex. The dihedral group D_{m_e}, the stabilizer of the cell κ_e in $\Sigma(\Upsilon)$, acts freely on $\overline{K}_e$. It is convenient to replace $\overline{K}_e$ by a complex with a single D_{m_e} orbit of vertices. Let $\overline{L}_e$ be the 2–complex obtained

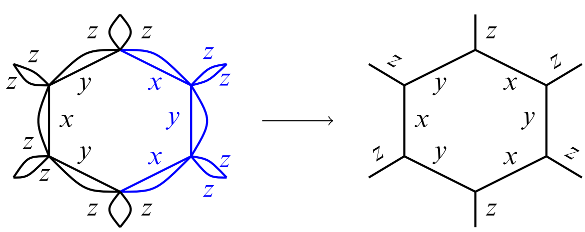

Figure 2: The complex $\overline{K}_e$ (on the left) in the case $e = (x \xrightarrow{w} y) \in \Gamma$ with corresponding edge $e = (x \xrightarrow{3} y) \in \Upsilon$, so the Coxeter relator is $(xy)^3$. On the right is the corresponding Coxeter cell κ_e together with z–edges. The blue part is a y–side in $\overline{K}_e$.

from $\overline{K}_e$ in the following way: at every vertex collapse one of the z–edges from the z–double edge for some $z \in \boldsymbol{z}$. The complex $\overline{L}_e$ is homotopy equivalent to $\overline{K}_e$. The 1–skeleton of $\overline{L}_e$ is a $2m_e$–gon with double edges labeled in an alternating way by x and y. At each of the $2m_e$ vertices we have a loop for every $z \in \boldsymbol{z}$. Let $\hat{r}_e$ be the word obtained from r_e by replacing every z^p for $z \in \boldsymbol{z}$ by $z^{p/2}$. Let $\widehat{P}_e = \langle x, y, \boldsymbol{z} \mid \hat{r}_e \rangle$. Note that the dihedral group D_{m_e} acts freely on $\overline{L}_e$ and we have a covering map $\overline{L}_e \to \overline{L}_e / D_{m_e} = K(\widehat{P}_e)$.

Lemma 5.1 *The* 2*–complex* $\overline{K}_e$ *is aspherical.*

Proof The complex $K(\widehat{P}_e)$ is aspherical because $\widehat{P}_e$ is a 1–relator presentation for which the relator is not a proper power. Thus $\overline{L}_e$ is aspherical, being a covering space of $K(\widehat{P}_e)$. Since $\overline{K}_e$ is homotopy equivalent to $\overline{L}_e$, it follows that $\overline{K}_e$ is aspherical. □

An x*–side* of $\overline{K}_e$ consists of a double edge with label x together with all the double edges connected to the two vertices of the x–double edge. A y–side is defined in the same way. See Figure 2, where the blue part on the left shows a y–side. Note that $\overline{K}_e$ has m_e x–sides and m_e y–sides. We refer to these as the *sides* of $\overline{K}_e$. We say $\overline{K}_e$ is *side injective* if the inclusion induced map $\pi_1(S) \to \pi_1(\overline{K}_e)$ is injective for every side S. An x–side in $\overline{L}_e$ is the image of an x–side under $\overline{K}_e \to \overline{L}_e$, etc.

Lemma 5.2 *The* 2*–complex* $\overline{K}_e$ *is* x*–side injective if and only if*

$$\langle x^2, y^2, z, xy^2x^{-1}, xzx^{-1} : z \in \boldsymbol{z} \rangle$$

is a free subgroup of $G(\widehat{P}_e)$ *on the given basis.*

Proof Recall that $m_e \geq 3$. An x–side S in $\overline{L}_e$ is an x–double edge, a y–double edge at each of the two vertices, and a loop for every $z \in \boldsymbol{z}$ at each of the two vertices. The image of $\pi_1(S)$ in $G(\widehat{P}_e)$ under the covering projection is the group in the statement of the lemma. □

Lemma 5.3 *If* T *is a subgraph of the* 1*–skeleton of* $\overline{K}_e$ *that does not involve every letter from* $\boldsymbol{x}_e = \{x, y, \boldsymbol{z}\}$*, then* $\pi_1(T) \to \pi_1(\overline{K}_e)$ *is injective.*

Proof We can argue with $\overline{L}_e$ instead of $\overline{K}_e$. A reduced loop γ in T gives a reduced word u in the generators of $\widehat{P}_e$ that does not involve all letters from $\boldsymbol{x}_e = \{x, y, \boldsymbol{z}\}$. The presentation $\widehat{P}_e$ has only one relator $\hat{r}_e$ that does involve all letters from the generating set $\boldsymbol{x}_e = \{x, y, \boldsymbol{z}\}$. The Freiheitssatz for 1–relator groups implies that u does not represent the trivial element in $G(\widehat{P})$. Thus γ is not trivial in $\pi_1(\overline{L}_e)$. □

We continue our analysis. The complex $\overline{K}(\Gamma)$ is a union of the complexes $w\overline{K}_e$ for $w \in W(\Upsilon)$ and $e \in$ edges of Γ. The maps

$$\overline{K}(\Gamma) \to \widetilde{K}(\Upsilon) \to \Sigma(\Upsilon)$$

give a one-to-one correspondence between the $w\overline{K}_e$ and Coxeter cells $w\kappa_e$. Since Υ is a tree, the Davis complex $\Sigma(\Upsilon)$ is a tree of Coxeter cells $w\kappa_e$ and so $\overline{K}(\Gamma)$ is a tree of complexes $w\overline{K}_e$. In complete analogy to Proposition 2.1, we have:

Proposition 5.4 *Consider* $\overline{K}(\Gamma) = \bigcup w\overline{K}_e \to \Sigma(\Upsilon) = \bigcup w\kappa_e$.

1. $\overline{K}(\Gamma)$ *is the union of the* 2*–complexes* $w\overline{K}_e$ *for* $e \in \{$*edges of* $\Gamma\}$ *and* $w \in W(\Upsilon)$. *Furthermore, if* $w_1\overline{K}_{e_1} \cap w_2\overline{K}_{e_2} \neq \varnothing$, *then* $e_1 \cap e_2 \neq \varnothing$; *if* $x = e_1 \cap e_2$, *then* $w_1\overline{K}_{e_1} \cap w_2\overline{K}_{e_2} = T$, *where* T *is the subgraph of an* x*–side* S *that carries the letters* $\boldsymbol{x}_{e_1} \cap \boldsymbol{x}_{e_2}$.
2. $\overline{K}(\Gamma)$ *is a tree of* 2*–complexes. In particular, if* $\overline{M}$ *is a finite connected union of* 2*–complexes* $w_i\overline{K}_{e_i}$ *in* $\overline{K}(\Gamma)$, *then there exists a* 2*–complex* $w\overline{K}_e$ *in* $\overline{M}$ *that intersects with the rest of* $\overline{M}$ *in a subgraph of a single side.*

Theorem 5.5 *Let* Γ *be a LOT of Coxeter type. Then* $K(\Gamma)$ *is aspherical if the* $\overline{K}_e$ *are side injective for every edge* e *in* Γ.

Proof We will show that $\overline{K}(\Gamma)$ is aspherical. It suffices to show that every finite connected union $\overline{M} = \bigcup_{i=1}^n w_i\overline{K}_{e_i}$ is aspherical. We first claim that the sides of the $w_i\overline{K}_{e_i}$ π_1–inject into the union $\overline{M}$. We do induction on n. If $n = 1$, the result follows from the hypothesis. Assume $n > 1$. Then, by Proposition 5.4(2), there exists a 2–complex $w\overline{K}_e$ in $\overline{M}$ that intersects with the rest of $\overline{M}$ in a subgraph T of a single side S (of course, T could be S). Now, by the induction hypothesis, the inclusion $S \subseteq \overline{M} - w\overline{K}_e = \overline{M}_0$ is π_1–injective, and the inclusion $S \subseteq w\overline{K}_e$ is π_1–injective by hypothesis. It follows that $\pi_1(\overline{M})$ is an amalgamated product $\pi_1(\overline{M}) = \pi_1(\overline{M}_0) *_{\pi_1(T)} \pi_1(w\overline{K}_e)$. Thus the inclusion $S \subseteq \overline{M}$ is π_1–injective. All other sides that occur in $\overline{M}$ are contained in either $\overline{M}_0$ or $w\overline{K}_e$. π_1–injectivity follows from the amalgamated product decomposition. Asphericity of $\overline{M}$ now follows from induction on n and the amalgamated product decomposition $\pi_1(\overline{M}) = \pi_1(\overline{M}_0) *_{\pi_1(T)} \pi_1(w\overline{K}_e)$. □

Remark The above proof shows more than asphericity. Since each $\pi_1(\overline{K}_e)$ is a finite-index subgroup of a 1–relator group, we see that $\pi_1(\overline{K})$ is a tree of groups, the vertex groups being finite-index subgroups of 1–relator groups, and the edge groups (over which we amalgamate) being finitely generated and free.

Definition 5.6 A labeled oriented tree Γ is called *label separated* if, for every pair of edges e_1 and e_2 that have a vertex in common, the intersection $\boldsymbol{x}_{e_1} \cap \boldsymbol{x}_{e_2}$ is a proper subset of both $\boldsymbol{x}_{e_1}$ and $\boldsymbol{x}_{e_2}$.

Theorem 5.7 *Let Γ be a label separated LOT of Coxeter type. Then $K(\Gamma)$ is aspherical.*

Proof The proof is very much the same as the proof of Theorem 5.5. Let $\overline{M} = \bigcup_{i=1}^{n} w_i \overline{K}_{e_i}$ as before. Again it suffices to show that $\overline{M}$ is aspherical. If $n = 1$ then the proof is clear. It is instructive to look at the case $n = 2$. The intersection $w_1 \overline{K}_{e_1} \cap w_2 \overline{K}_{e_2} = T$ is the subgraph of a side that carries the letters $\boldsymbol{x}_{e_1} \cap \boldsymbol{x}_{e_2}$, which is a proper subset of both $\boldsymbol{x}_{e_1}$ and $\boldsymbol{x}_{e_2}$. Thus π_1–injectivity for the inclusions $T \subseteq w_i \overline{K}_{e_i}$ for $i = 1, 2$ follows from Lemma 5.3. We have $\pi_1(\overline{M}) = \pi_1(w_1 \overline{K}_{e_1}) *_{\pi_1(T)} \pi_1(w_2 \overline{K}_{e_2})$ and $\overline{M}$ is aspherical. For $n \geq 2$ we argue by induction and obtain (as in the proof of Theorem 5.5) a decomposition $\pi_1(\overline{M}) = \pi_1(\overline{M}_0) *_{\pi_1(T)} \pi_1(w \overline{K}_e)$, which proves asphericity of $\overline{M}$. □

6 Side injectivity

Let $P = \langle a, b, \boldsymbol{c} \mid r \rangle$, be a 1–relator group, where $\boldsymbol{c}$ is a finite set of letters (which could be empty). We assume that r is cyclically reduced and contains all generators. Assume further that $r = (ab)^m$ for some $m \geq 0$ modulo the relations $a^2 = b^2 = c = 1$ for $c \in \boldsymbol{c}$ and cyclic permutation. The number m is called *the dihedral type* of P.

Let $Q = \langle a, b, \boldsymbol{c} \mid (ab)^m, a^2, b^2 \text{ for } c \in \boldsymbol{c} \rangle$. We have an epimorphism $\phi\colon G(P) \to G(Q) = D_m$. Let $\overline{K}(P)$ be the covering of $K(P)$ associated with the kernel. Note that $\overline{K}(P)^{(1)} = \widetilde{K}(Q)^{(1)}$, which is the Cayley graph for D_m on the generating set $\{a, b, \boldsymbol{c}\}$. So $\overline{K}(P)^{(1)}$ is a $2m$–gon, consisting of double edges labeled in an alternating way with a and b, and at every vertex we have a c–loop for every $c \in \boldsymbol{c}$. An a–side of $\overline{K}(P)$ is a connected subgraph of the 1–skeleton that consists of a double edge with label a, together with all the b–double edges and c–loops connected to the two vertices of the a–double edge. A b–side is defined in an analogous way. We say P is *side injective* if the inclusion of any side $S \to \overline{K}(P)$ is π_1–injective.

Lemma 6.1 *Assume that P is of dihedral type $m \geq 3$. Suppose that, for every cyclically reduced word w in $\{a, b, \boldsymbol{c}\}^{\pm 1}$ which represents the trivial element in $G(P)$, some cyclic permutation of w contains a reduced subword u of the form*

$$a^{\pm 1} d_1 b^{\beta} d_2 a^{\alpha} d_3 b^{\pm 1} \quad \text{or} \quad b^{\pm 1} d_1 a^{\alpha} d_2 b^{\beta} d_3 a^{\pm 1},$$

where α and β are odd integers and the d_i are words in the generators containing a and b with even exponents (the d_i could be trivial). *Then P is side injective.*

Proof We begin with some notation. If w is a word in $\{a,b,\boldsymbol{c}\}^{\pm 1}$, then we denote by $\overline{w}$ the element of D_m that it represents. If $w = x_1 \dots x_n$ with $x_i \in \{a,b,\boldsymbol{c}\}^{\pm 1}$, then the lift $\gamma(w,V)$ of w into $\overline{K}(P)^{(1)}$, starting at a vertex V, is a path with vertices $V, \overline{x_1}V, \overline{x_1x_2}V, \dots, \overline{x_1 \dots x_n}V$. Now let w be a reduced word as in the statement. We assume without loss of generality that $w = w_1(a^{\pm 1}d_1b^{\beta}d_2a^{\alpha}d_3b^{\pm 1})w_2$. Consider $\gamma(w,V)$. Let $V' = \overline{w_1}V$. Among the vertices of $\gamma(w,V)$ we find V', $\bar{a}V'$, $\overline{ab}V'$, $\overline{aba}V'$, and $\overline{abab}V'$. These are five distinct vertices. A side of $\overline{K}(P)$ contains exactly four vertices. It follows that $\gamma(w,V)$ is not contained in a side. We conclude that P is side injective. □

Example 6.2 $P = \langle a,b \mid (ab)^m \rangle$ for $m \geq 3$ is side injective. This is because 1–relator presentations with torsion are Dehn presentations (in particular, $G(P)$ is hyperbolic). See Newman [16]. A word w that is trivial in the group contains a subword of length more than $\frac{1}{2}$ that of a cyclic permutation of the relator or its inverse; hence, it contains a cyclic permutation of $abab$, or its inverse. The result follows from Lemma 6.1.

Example 6.3 More generally, if $P = \langle a,b,\boldsymbol{c} \mid r(a,b,\boldsymbol{c}) \rangle$ ($\boldsymbol{c}$ could be empty) is a Dehn presentation of dihedral type $m \geq 3$ such that more than half of a cyclic permutation of the relator or its inverse contains a subword u as in Lemma 6.1, then P is side injective. Recall that P is a Dehn presentation for instance if it satisfies the small cancellation condition $C'\big(\frac{1}{6}\big)$ or $C'\big(\frac{1}{4}\big) - T(4)$ (see Lyndon and Schupp [14, Chapter V, Theorem 4.4]). For example, if

$$r(a,b,\boldsymbol{c}) = a^{\alpha_1}d_1b^{\beta_1}d_2a^{\alpha_2}d_3b^{\beta_2}d_4a^{\alpha_3}d_5b^{\beta_3}d_6a^{\alpha_4}d_7b^{\beta_4}d_8,$$

where the α_i and β_i are odd integers satisfying $|\alpha_i| = |\alpha_j|$ and $|\beta_i| = |\beta_j|$ for all $i,j \leq 4$ and the d_i are words of the same length containing a and b with even exponents, and P satisfies the small cancellation condition $C'\big(\frac{1}{6}\big)$ or $C'\big(\frac{1}{4}\big) - T(4)$, then P is side injective. Concrete examples are

$$\langle a,b,c \mid (acbc^{-1}ac^{-1}bc)^2 \rangle \quad \text{and} \quad \langle a,b,c \mid acbc^{-1}acbcac^{-1}bc^{-1}ac^{-1}bc \rangle,$$

which are $C'\big(\frac{1}{4}\big) - T(4)$, and

$$\langle a,b,c \mid acbca^{-1}cbc^{-1}a^{-1}c^{-1}bcac^{-1}bc^{-1} \rangle,$$

which is $C'\big(\frac{1}{6}\big)$. These presentations were checked with the help of GAP (see [7]) and the package SMALLCANCELLATION by Ivan Sadofschi Costa (see [19]).

Example 6.4 The Artin presentation $P = \langle a, b \mid \mathrm{prod}(a, b, m) = \mathrm{prod}(b, a, m)\rangle$ is not side injective for $m = 3$, but is side injective for $m \geq 4$:

***m* = 3** We show that $P = \langle a, b \mid aba = bab\rangle$ is not side injective. We have $a^2(aba^2ba)a^{-2} = aba^2ba$ in $G(P)$ because $(aba)^2 = aba^2ba$ is central. So

$$w = a^2ba^2ba^{-2}b^{-1}a^{-2}b^{-1} = 1$$

in $G(P)$. Note that w lifts into a b–side of $\overline{K}(P)$.

***m* = 4** We show that $P = \langle a, b \mid abab = baba\rangle$ is side injective. Note that $x = abab$ is a central element. The quotient $G(P)/\langle x\rangle$ has a presentation $\langle a, b \mid (ab)^2\rangle$. Let $y = ba$; then the presentation rewrites to $\langle a, y \mid y^2\rangle$. In order to show that P is a–side injective, we have to show that a^2, b^2 and ab^2a^{-1} generate a free group of rank 3 in $G(P)$. We will do this by showing that $A = a^2$, $B = (ya^{-1})^2 = ya^{-1}ya^{-1}$ and $C_0 = a(ya^{-1})^2a^{-1} = aya^{-1}ya^{-1}a^{-1}$ generate a free group in the quotient presented by $Q = \langle a, y \mid y^2\rangle = \mathbb{Z} * \mathbb{Z}_2$. Let $C_1 = BC_0$. We have

$$\begin{aligned} C_1 &= ya^{-1}ya^{-1}aya^{-1}ya^{-1}a^{-1} = ya^{-1}yya^{-1}ya^{-1}a^{-1} = ya^{-1}a^{-1}ya^{-1}a^{-1} \\ &= ya^{-2}ya^{-2}. \end{aligned}$$

And, finally, let $C = C_1A = ya^{-2}y$. In summary we have

$$A = a^2, \quad B = ya^{-1}ya^{-1}, \quad C = ya^{-2}y.$$

The group $H = \langle A, B, C\rangle$ is a normal free subgroup of $G(Q)$ of rank 3 and index 4. Figure 3 shows a covering space $p\colon \overline{K}(Q) \to K(Q)$ such that $\pi_1(\overline{K}(Q))$ is free of rank 3 and $p_*\big(\pi_1(\overline{K}(Q))\big) = \langle A, B, C\rangle \leq \pi_1(K(Q))$. The argument for b–side injectivity is analogous.

***m* ≥ 6 and even** This case is easy. Let $x = \mathrm{prod}(a, b, m)$. The quotient $G(P)/\langle x\rangle$ is presented by $\langle a, b \mid (ab)^{m/2}\rangle$, which is a Dehn presentation, being a 1–relator presentation with torsion. Since $m \geq 6$, we have $\frac{1}{2}m \geq 3$. Side injectivity follows from Example 6.2.

***m* ≥ 5 and odd** Let $x = \mathrm{prod}(a, b, m)$ and $y = ba$. Note that $x = ay^{(m-1)/2}$. Using $a = xy^{(-m+1)/2}$ and $b = y^{(m+1)/2}x^{-1}$, the presentation P can be rewritten to $\langle x, y \mid x^2 = y^m\rangle$. Thus $G(P)/\langle x^2\rangle$ is presented by $\langle x, y \mid x^2, y^m\rangle$, which is the hyperbolic group $\mathbb{Z}_2 * \mathbb{Z}_m$. In the original generators, this is $\langle a, b \mid \mathrm{prod}(a, b, m)^2, (ba)^m\rangle$. If this were a Dehn presentation, we could proceed as in the previous case (at least for $m \geq 7$), but we do not know. Instead we argue as for $m = 4$. For simplicity we assume $m = 5$; the cases $m \geq 7$ go along the same lines. In order to show that P

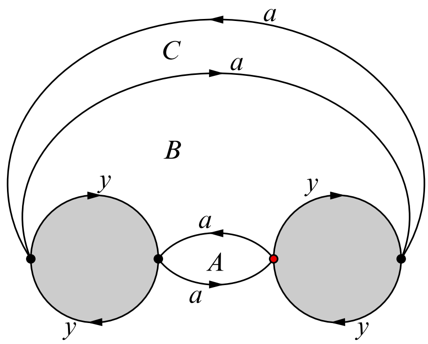

Figure 3: If $Q = \langle a, y \mid y^2 \rangle$ then the universal covering $\widetilde{K}(Q)$ is a tree with spheres attached. Here we see the intermediate covering $\overline{K}(Q)$ corresponding to the subgroup $H = \langle A, B, C \rangle$. The gray discs with boundary y^2 indicate 2–spheres.

is a–side injective we have to show that a^2, b^2 and ab^2a^{-1} generate a free group of rank 3 in $G(P)$. In terms of x and y, it suffices to show that $xy^{-2}xy^{-2}$, $y^3x^{-1}y^3x^{-1}$ and $(xy^{-2})y^3x^{-1}y^3x^{-1}(xy^{-2})^{-1}$ generate a free subgroup of rank 3 in the quotient presented by $Q = \langle x, y \mid x^2, y^5 \rangle$. Let

$$A = xy^3xy^3, \quad B = y^3xy^3x, \quad C_0 = xyxy^3xy^2x.$$

Let $C = C_0A = (xyxy^3xy^2x)(xy^3xy^3) = xyxy$.

Note that

$$C(y^{-1}Cy) = (xyxy)y^{-1}(xyxy)y = xy^2xy^2 = B^{-1}$$

and

$$C(y^{-1}Cy)(y^{-2}Cy^2) = xy^2xy^2y^{-2}xyxyy^2 = xy^3xy^3 = A.$$

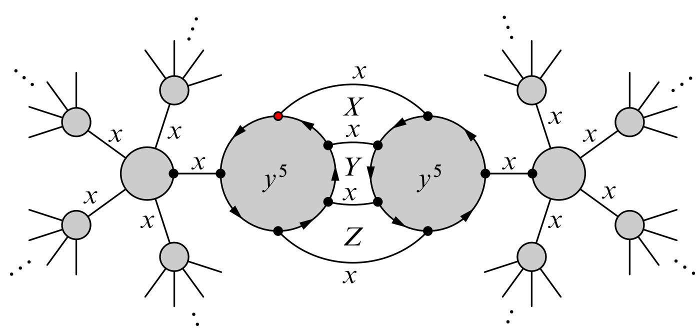

Figure 4: A rendering of the covering space $\overline{K}(Q)$. Each x–edge represents a double x–edge into which two discs with boundary x^2 are glued. Each gray disc represents five discs with boundary y^5.

So it suffices that to show that

$$X = C, \quad Y = y^{-1}Cy, \quad Z = y^{-2}Cy^2$$

generate a free subgroup of rank 3. Figure 4 shows a covering space $p\colon \overline{K}(Q) \to K(Q)$ such that $\pi_1(\overline{K}(Q))$ is free of rank 3 and $p_*\big(\pi_1(\overline{K}(Q))\big) = \langle X, Y, Z\rangle \le \pi_1(K(Q))$. The argument for b–side injectivity is analogous.

Example 6.5 Let $P = \langle a, b, c \mid a(babcaba) = (babcaba)b\rangle$. Then P is side injective by Theorem 6.6.

Theorem 6.6 *Suppose P has dihedral type $m \ge 3$ and*

- $P = \langle a, b, \boldsymbol{c} \mid a(u_1c^{\epsilon}u_3) = (u_1c^{\epsilon}u_3)b\rangle$, or
- $P = \langle a, b, \boldsymbol{c} \mid a(u_1c^{\epsilon}u_2c^{\epsilon}u_3) = (u_1c^{\epsilon}u_2c^{\epsilon}u_3)b\rangle$,

where

(1) $c \in \boldsymbol{c}$ *and* $\epsilon = \pm 1$,

(2) *the words* u_1 *and* u_3 *do not contain* c *while* u_2 *is arbitrary*, *and*

(3) *both* $u_1^{-1}a$ *and* u_3b^{-1} *contain a subword* u *as in Lemma 6.1.*

Then P is side injective.

Proof We assume we are in the second case and $\epsilon = 1$. The first case is shown in an analogous way. Envision the relator disc placed in the plane as a rectangle, where the a on the very left of the equation and the b on the very right of the equation are horizontal edges, and the word $u_1cu_2cu_3$ is a vertical edge sequence. Connect the midpoints of c–edges on the left and right by horizontal red edges. See Figure 5. Suppose that w is a cyclically reduced word that represents the trivial element in $G(P)$. Let D be a reduced Van Kampen diagram with boundary w. We may assume that D is a topological disc.

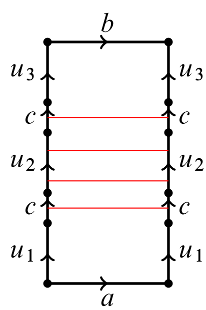

Figure 5: The relator disc drawn as a rectangle.

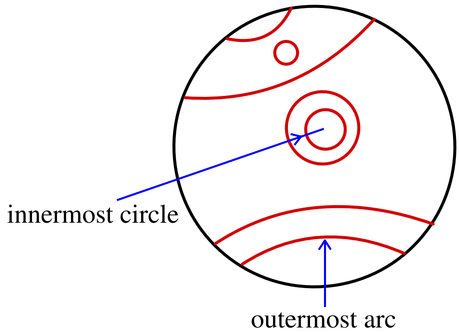

Figure 6: A disc with red arcs, indicating innermost circles and outermost arcs.

The red edges in our relator disc will form red circles and red arcs connecting points on the boundary of D. See Figure 6. Consider an innermost red circle. Going around the inside, we read off a word that freely reduces to $u_1^{-1}a^k u_1$ or $u_3 b^k u_3^{-1}$ for some $k \in \mathbb{Z}$. If $k = 0$, then D is not reduced. If $k \neq 0$, then $G(P)$ has torsion. Neither is the case; hence, there are no red circles in D. Consider an outermost red arc α. Let E be the component of $D - \alpha$ that does not contain anything red. Reading along the part of the boundary of D which belongs to E gives a reduced word (a subword of the reduced word w) equal to $u_1^{-1}a^k u_1$ or $u_3 b^k u_3^{-1}$. Because D is reduced, k cannot be zero. If k is positive then $u_1^{-1}a^k u_1$ contains $u_1^{-1}a$ and hence a word u as in Lemma 6.1. Also, $u_3 b^k u_3^{-1}$ contains bu_3^{-1}, and, since $u_3 b^{-1}$ contains a word u as in Lemma 6.1, so does $(u_3 b^{-1})^{-1} = bu_3^{-1}$. The case where k is negative goes the same way. It now follows from Lemma 6.1 that P is side injective. □

7 Last words about LOT applications

Theorem 7.1 *Let Γ be a LOT of Coxeter type. Suppose that, for every edge $e = (a \xrightarrow{w_e} b)$, the word w_e is of the form $u_1 c^\epsilon u_3$ or $u_1 c^\epsilon u_2 c^\epsilon u_3$ for some letter $c \neq a, b$, $\epsilon = \pm 1$, and words u_1, u_2 and u_3 as in Theorem 6.6. Then $K(\Gamma)$ is aspherical.*

Proof Each $\widehat{P}_e$ is side injective. This follows from Theorem 6.6. Thus each $\overline{K}_e$ is side injective. The result follows from Theorem 5.5. □

What if side injectivity fails?

Theorem 7.2 *Suppose Γ is a LOT of Coxeter type and e_1 and e_2 are two edges of Γ. Let $\overline{K}_{e_1} \cap \overline{K}_{e_2} = S$, which is a subgraph of a side of $\overline{K}_{e_1}$ and a side of $\overline{K}_{e_2}$. Let*

$N_1 = \ker(\pi_1(S) \to \pi_1(\overline{K}_{e_1}))$ and $N_2 = \ker(\pi_1(S) \to \pi_1(\overline{K}_{e_2}))$. Assume that

$$\frac{N_1 \cap N_2}{[N_1, N_2]} \neq 1.$$

Then Whitehead's asphericity conjecture is false.

Proof Suppose Whitehead's conjecture is true. Then $K(\Gamma)$ and hence $\overline{K}(\Gamma)$ is aspherical. Note that $\overline{K}_{e_1} \cup \overline{K}_{e_2}$ is a subcomplex of $\overline{K}(\Gamma)$. Let w be a reduced edge loop in S that represents a nontrivial element in the quotient $(N_1 \cap N_2)/[N_1, N_2]$. It is the boundary of a Van Kampen diagram D_1 for $\overline{K}_{e_1}$ and also the boundary of a Van Kampen diagram D_2 for $\overline{K}_{e_2}$. The two diagrams can be glued together to form a nontrivial element in $\pi_2(\overline{K}_{e_1} \cup \overline{K}_{e_2})$ (see Gutierrez and Ratcliffe [8]). This is a contradiction. □

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Department of Mathematics, Boise State University
Boise, ID, United States

Pädagogische Hochschule Karlsruhe
Karlsruhe, Germany

`jensharlander@boisestate.edu`, `rosebrock@ph-karlsruhe.de`

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