AGT

Algebraic & Geometric Topology

Volume 23 (2023)

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Giuseppe De Nittis

Kiyonori Gomi

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Algebraic & Geometric Topology **23**:7 (2023) 2925–2974
DOI: 10.2140/agt.2023.23.2925
Published: 26 September 2023

Differential geometric invariants for time-reversal symmetric Bloch bundles II: The low-dimensional "quaternionic" case

GIUSEPPE DE NITTIS
KIYONORI GOMI

This paper is devoted to the construction of differential geometric invariants for the classification of "quaternionic" vector bundles. Provided that the base space is a smooth manifold of dimension two or three endowed with an involution that leaves fixed only a finite number of points, it is possible to prove that the *Wess–Zumino term* and the *Chern–Simons invariant* yield topological invariants able to distinguish between inequivalent realizations of "quaternionic" structures. This is a nontrivial generalization of results previously known only in the case of tori with time-reversal involution.

57R22; 53A55, 53C80, 55N25

1 Introduction

The present paper continues the study of the classification of *"quaternionic" vector bundles* started in [8; 10; 11]. The main novelty with respect to the previous papers consists of the use of differential geometric invariants to classify inequivalent isomorphism classes of "quaternionic" structures. In this sense, as expressed by the title, this paper represents a continuation of [9] where differential geometric techniques have been used to classify "real" vector bundles.

At a topological level, "quaternionic" vector bundles, or *Q–bundles* for short, are complex vector bundles defined over spaces with involution and endowed with a further structure at the level of the total space. An involution τ on a topological space X is a homeomorphism of period 2, ie $\tau^2 = \mathrm{Id}_X$. The pair (X, τ) will be called an involutive space. The *fixed point* set of the involutive space (X, τ) is by definition

$$X^\tau := \{x \in X \mid \tau(x) = x\}.$$

A Q–bundle over (X,τ) is a pair $(\mathscr{E},\Theta)$, where $\mathscr{E}\to X$ denotes the underlying complex vector bundle and $\Theta\colon \mathscr{E}\to\mathscr{E}$ is an *antilinear* map which covers the action of τ on the base space and such that Θ^2 acts fiberwise as multiplication by -1. A more precise description is given in Definition 2.2. Q–bundles were introduced for the first time by J L Dupont in [12] (under the name of symplectic vector bundles). They form a category of topological objects which is significantly different from the category of complex vector bundles. For this reason the problem of the classification of Q–bundles over a given involutive space requires the use of tools which are structurally different from those typically used in the classification of complex vector bundles. The aim of the present work is to define differential geometric invariants able to distinguish the elements of $\mathrm{Vec}_Q^m(X,\tau)$, where the latter symbol denotes the set of isomorphism classes of rank m Q–bundles over (X,τ).

The interest for the classification of Q–bundles has increased in the last years because of the connection with the study of *topological insulators*. Although this work does not focus on the theory of topological insulators — the interested reader is referred to the recent reviews by Ando and Fu [2] and Hasan and Kane [25] — it is worth mentioning that the first example of topological effects in condensed matter related to a "quaternionic" structure dates back to the seminal works by L Fu, C L Kane and E J Mele [18; 31]. The existence of distinguished topological phases for the so-called Kane–Mele model is the result of the simultaneous presence of two symmetries. The first symmetry is given by the invariance of the system under spatial translations. This fact allows the use of the *Bloch–Floquet theory* — see Kuchment [36] — for the analysis of the spectral properties of the system. As a result, a well-established procedure provides the construction of a vector bundle, usually known as *Bloch bundle*, from each gapped energy band of the system. Even though the details of the construction of the Bloch bundle will be omitted in this work — the interested reader is referred to Panati [42] or the authors [7, Section 2] — it is important to remark that the Bloch bundle is a complex vector bundle over the torus $\mathbb{T}^d\simeq\mathbb{R}^d/(2\pi\mathbb{Z})^d$. The integer d represents the dimensionality of the system and the physically relevant dimensions are $d=2,3$. The second crucial ingredient for the topology of the Kane–Mele model is the fermionic (or odd) *time-reversal symmetry* (TRS). In terms of the Bloch bundle the TRS translates into the involution $\tau_{\mathrm{TR}}\colon\mathbb{T}^d\to\mathbb{T}^d$ of the base space given by

$$\tau_{\mathrm{TR}}(k_1,\dots,k_d):=(-k_1,\dots,-k_d)$$

and into an antilinear map Θ of the total space such that $\Theta^2=-1$ fiberwise. Therefore, one concludes that the different topological phases of the Kane–Mele model are labeled by the inequivalent realization of Q–bundles over the torus $\mathbb{T}^d$ with involution τ_{TR}, namely by the distinct elements of $\mathrm{Vec}^m_Q(\mathbb{T}^d, \tau_{\mathrm{TR}})$.

The classification of the topological phases of the Kane–Mele model given in [18; 31] is summarized by

$$\mathrm{Vec}^m_Q(\mathbb{T}^d, \tau_{\mathrm{TR}}) = \begin{cases} \mathbb{Z}_2 & \text{if } d = 2, \\ \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3 & \text{if } d = 3, \end{cases} \tag{1-1}$$

where $\mathbb{Z}_2 := \{\pm 1\}$ is the cyclic group of order 2 presented in multiplicative notation. The topological classification (1-1) has been rigorously derived with the use of different techniques in various papers — see eg [8], Fiorenza, Monaco and Panati [14], and Graf and Porta [24] — and generalized to any (low-dimensional) involutive space (X, τ) by Lawson, Lima-Filho, Michelsohn and dos Santos [37; 45] and in [10; 11], independently. However, the topological classification based on the construction of homotopy invariants (such as characteristic classes) has the disadvantage of being difficult to compute. For this reason one is naturally inclined to look for different types of invariants.

A special role in the classification of complex vector bundles is played by the *Chern classes*. The latter, in view of the Chern–Weil homomorphism, can be represented via differential forms and integrated over suitable cocycles. The resulting *Chern numbers* are enough to provide a complete classification of complex vector bundles in several situations of interest in condensed matter. This is, for instance, the case of the *quantum Hall effect* and the related *TKNN numbers*; see Thouless, Kohmoto, Nightingale and den Nijs [46]. Using this observation as Ariadne's thread, one expects to find differential and integral invariants able to classify Q–bundles at least under some reasonable hypotheses. Indeed, "gauge-theoretic invariants" have already been used to reproduce the classification (1-1). The first pioneering works in this direction are Essin, Moore and Vanderbilt [13], Fu and Kane [17], and Qi, Hughes, Wang and Zhang [44; 47], where the Chern–Simons field theory has been used to relate the topological phases of the Kane–Mele model in $2+1$ and $3+1$ space-time dimensions with integral quantities like the (time-reversal) *polarization*. Afterwards, these results have been revisited and put in a solid mathematical background in various works like Carpentier, Delplace, Fruchart, Gawędzki, Monaco and Tauber [6; 5; 21; 22; 41], Freed and Moore [16], and Kaufmann, Li and Wehefritz-Kaufmann [32], just to mention some of them. If one ignores the differences due to the use of distinct mathematical techniques, it is possible to recognize a common outcome from all the papers listed above: the topological phases of the two-dimensional Kane–Mele model are governed by the *Wess–Zumino term* [15; 20; 21] while in the three-dimensional case the relevant object is the *Chern–Simons invariant* [15; 21; 28]. The present work is inspired by the latter consideration and it aims to provide a general and rigorous description of the relation between the classification of Q–bundle and the Wess–Zumino term, or the Chern–Simons invariant. The main achievements are presented below.

The two-dimensional case will be described first. In this case the relevant family of base spaces is restricted by the following:

Definition 1.1 (oriented two-dimensional FKMM–manifold) An *oriented two-dimensional FKMM–manifold* is an involutive space (Σ, τ) subject to the following conditions:

- (a′) Σ is an oriented two-dimensional compact Hausdorff manifold without boundary.
- (b′) The involution τ preserves the manifold structure and the orientation.
- (c′) The fixed point set $\Sigma^\tau \neq \varnothing$ consists of a finite collection of points.

Let us point out that *manifold structure* in (b′) shall be eventually assumed to be a smooth one as is stated at the beginning of Section 3. An example of oriented two-dimensional FKMM–manifold is provided by the torus $\mathbb{T}^2$ with the involution τ_{TR}. The set of oriented two-dimensional FKMM–manifolds forms a subclass of the *FKMM–spaces* defined in Definition 2.8 below. Q–bundles over these spaces are completely classified by a characteristic class called *FKMM–invariant*; see Theorem 2.9.

The crucial result for the classification of Q–bundles over two-dimensional FKMM–manifolds is expressed by the chain of isomorphisms

$$\mathrm{Vec}_Q^{2m}(\Sigma,\tau) \overset{\imath_1}{\simeq} [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \overset{\imath_2}{\simeq} \mathbb{Z}_2. \tag{1-2}$$

The first isomorphism $\imath_1$ is essentially proved in Theorem 2.13 for $m=1$ and justified in Remark 2.16 for every $m \in \mathbb{N}$. Elements of $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ are $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps $\xi\colon \Sigma \to \mathbb{SU}(2)$ constrained by the *equivariance* condition $\xi(\tau(x)) = \xi(x)^{-1}$ for all $x \in \Sigma$. The set $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ consists of $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps $\phi\colon X \to \mathbb{U}(1)$ such that $\phi(\tau(x)) = \overline{\phi(x)} = \phi(x)^{-1}$. The action of $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ over $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ is specified in the statement of Theorem 2.13. The second isomorphism $\imath_2$ is described in Section 2.7 and is given by the composition of two identifications: The first isomorphism,

$$[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \overset{\Phi_\kappa}{\simeq} \mathrm{Map}(\Sigma^\tau, \{\pm 1\})/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

proved in Proposition 2.18, shows that the "new" description of Q–bundles in terms of maps $\xi\colon \Sigma \to \mathbb{SU}(2)$ agrees with the "old" description in terms of the FKMM–invariant given in Proposition 2.10. The second identification,

$$\mathrm{Map}(\Sigma^\tau, \{\pm 1\})/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \overset{\Pi}{\simeq} \mathbb{Z}_2,$$

is described in Theorem 2.11 and it is induced by the *product sign* map (also known as the *Fu–Kane–Mele index*).

The isomorphism $\imath_1$ in (1-2) expresses the fact that an element of $\mathrm{Vec}^{2m}_Q(\Sigma, \tau)$ can be completely identified with an equivariant map $\xi\colon \Sigma \to \mathbb{SU}(2)$ that, in many situations, can be built explicitly; see Remark 2.19. Therefore, the relevant question is whether there is a way to access directly the isomorphism $\imath_2$ from the knowledge of the classifying map ξ without passing through the FKMM–invariant and the product sign map. The answer is positive. First of all it is important to point out that, without loss of generality, the map ξ can be chosen smooth. This allows us to define the *Wess–Zumino term*

$$\mathcal{WZ}_\Sigma(\xi) := -\frac{1}{24\pi^2}\int_{X_\Sigma} \mathrm{Tr}(\tilde{\xi}^{-1}\cdot \mathrm{d}\tilde{\xi})^3 \mod \mathbb{Z}, \tag{1-3}$$

where X_Σ is any compact three-dimensional oriented manifold whose boundary coincides with Σ and $\tilde{\xi}\colon X_\Sigma \to \mathbb{SU}(2)$ is any smooth extension of ξ; see Definition 3.16 for more details. The first main result of this paper is:

Theorem 1.2 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Let $(\mathscr{E}, \Theta)$ be a Q–bundle of rank $2m$ over (Σ, τ) and $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ any map which represents $(\mathscr{E}, \Theta)$ in the sense of the isomorphism $\imath_1$ in* (1-2)*. Then the map*

$$\mathrm{Vec}^{2m}_Q(\Sigma, \tau) \ni [(\mathscr{E}, \Theta)] \to \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} \in \mathbb{Z}_2$$

provides a realization of the isomorphism $\mathrm{Vec}^{2m}_Q(\Sigma, \tau) \simeq \mathbb{Z}_2$ in (1-2).

The proof of Theorem 1.2 is postponed to Section 3.6. Theorem 1.2 clearly applies to the classification of Q–bundles over the involutive torus $(\mathbb{T}^2, \tau_{\mathrm{TR}})$, reproducing in this way results already existing in the literature. In this regard the result [21, (2.9)], previously announced in [22, II.25, page 19], deserves a special mention. The latter is in agreement with Theorem 1.2 above in view of the equality $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(w)} = \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)}$ (justified by the Polyakov–Wiegmann formula, see Lemma 3.17) where the map w employed in [22] is related to the map ξ of Theorem 1.2 by the relation $w = \xi Q$, with Q the constant matrix in (2-2). However, it is worth pointing out that the validity of

Theorem 1.2 goes far beyond the standard case $(\mathbb{T}^2, \tau_{\mathrm{TR}})$. For instance, Theorem 1.2 extends the classification of Q–bundles over Riemann surfaces of genus g endowed with an orientation-preserving involution with a finite set of fixed points [8, Appendix A] and this application seems to be new in the literature.

In order to describe the three-dimensional case it is worth mentioning that any Q–bundle $(\mathscr{E}, \Theta)$ over the involutive space (X, τ) can be equivalently described by a *principal Q–bundle* $(\mathscr{P}, \widehat{\Theta})$ over the same base space (see Section 3.1) and that for principal Q–bundles there exists a notion of *equivariant Q–connection* (see Section 3.2). Given a Q–connection $\omega \in \Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m))$ one can define the associated *Chern–Simons 3–form*

$$\mathcal{CS}(\omega) := \frac{1}{8\pi^2}\,\mathrm{Tr}\Big(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\Big)$$

and the *intrinsic Chern–Simons invariant*

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) := \int_X s^*\mathcal{CS}(\omega) \mod \mathbb{Z} \tag{1-4}$$

as specified in Definitions 3.9 and 3.14. Remarkably, under the hypotheses stipulated in Proposition 3.12, the quantity in the right-hand side of (1-4) turns out to be independent of the choice of the invariant connection ω or of the global section $s\colon X \to \mathscr{P}$, and therefore defines an invariant for the underlying principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$, or equivalently for the associated Q–bundle $(\mathscr{E}, \Theta)$.

Let us recall that when (X, τ) is a three-dimensional FKMM–manifold in the sense of Definition 2.8, Proposition 2.10 applies and we have an isomorphism

$$\mathrm{Vec}^{2m}_Q(X, \tau) \stackrel{\kappa}{\simeq} \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \quad \text{for all } m \in \mathbb{N}.$$

In the formula above, $\mathrm{Map}(X^\tau, \{\pm 1\}) \simeq \mathbb{Z}_2{}^{|X^\tau|}$ denotes the set of maps from X^τ to $\{\pm 1\}$ (recall that X^τ is a set of finitely many points). The group action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $\mathrm{Map}(X^\tau, \{\pm 1\})$ is given by multiplication and restriction. The map κ which implements the isomorphism is the FKMM–invariant; see Section 2.3. Given a Q–bundle $(\mathscr{E}, \Theta)$ over (X, τ), its FKMM–invariant $\kappa(\mathscr{E}, \Theta)$ can be represented in terms of a map $\phi \in \mathrm{Map}(X^\tau, \{\pm 1\})$ and one can use the product sign map to define the so-called *strong* Fu–Kane–Mele index

$$\kappa_{\mathrm{s}}(\mathscr{E}, \Theta) := \Pi[\phi] = \prod_{x_j \in X^\tau} \phi(x_j) \in \mathbb{Z}_2. \tag{1-5}$$

It turns out that the definition above is well-posed in the sense that $\kappa_{\mathrm{s}}(\mathscr{E}, \Theta)$ only depends on the equivalence class of ϕ in $\mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$; hence it defines a topological invariant for $(\mathscr{E}, \Theta)$. This fact is a consequence of the second main result of this paper:

Theorem 1.3 *Let (X, τ) be a three-dimensional FKMM–manifold in the sense of Definition 2.8 such that $X^\tau \neq \varnothing$. Assume in addition that*:

(e) *X is oriented and τ* **reverses** *the orientation.*

Let $(\mathscr{E}, \Theta)$ be a Q–bundle over (X, τ) with FKMM–invariant

$$\kappa(\mathscr{E}, \Theta) \in \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$$

according to Proposition 2.10. For a given representative $\phi \in \mathrm{Map}(X^\tau, \{\pm 1\})$ of $\kappa(\mathscr{E}, \Theta)$, let $\Pi[\phi]$ be as in (1-5). Then, independent of the choice of ϕ,

$$\mathrm{e}^{\mathrm{i}2\pi\, \mathfrak{cs}(\mathscr{P}, \widehat{\Theta})} = \Pi[\phi], \tag{1-6}$$

where $(\mathscr{P}, \widehat{\Theta})$ is the principal Q–bundle associated to $(\mathscr{E}, \Theta)$ and $\mathfrak{cs}(\mathscr{P}, \widehat{\Theta})$ is the intrinsic Chern–Simons invariant of Definition 3.14.

The proof of Theorem 1.3 is postponed to Section 3.7. Along with Corollary 3.32, it expresses the fact that the strong index

$$\kappa_{\mathrm{s}}(\mathscr{E}, \Theta) = \mathrm{e}^{\mathrm{i}2\pi\, \mathfrak{cs}(\mathscr{P}, \widehat{\Theta})} \tag{1-7}$$

is a topological invariant which allows us to, at least partially, classify Q–bundles. In the case of the involutive torus $(\mathbb{T}^3, \tau_{\mathrm{TR}})$ described by (1-1) the invariant $\kappa_{\mathrm{s}}(\mathscr{E}, \Theta)$ takes values in the first (strong) summand of $\mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3$. For a more recent review of the topological interpretation of the (strong) Fu–Kane–Mele index we refer to [4].

Theorems 1.2 and 1.3 show that the differential geometric gauge invariants (1-3) and (1-4) can be used as tools for the classification of Q–bundles in dimension two and three, provided that the base space meets some restrictive conditions. The results contained in Theorems 1.2 and 1.3 are valid for base spaces which are much more general than the involutive tori $(\mathbb{T}^d, \tau_{\mathrm{TR}})$ usually considered in literature. However, these results are still not completely satisfactory in view of the restrictions on the nature of the base space that we need to assume. There are two questions which are still open, and that it would be interesting to answer: *Is it possible to extend Theorems 1.2 and 1.3 to involutive base spaces (X, τ) such that X^τ is a submanifold of dimension bigger than zero*? In the case of Theorem 1.2, *is it possible to construct the classifying map ξ directly from the projection which represents the Q–bundle in K–theory without relying on the use of a predetermined global frame*?

Acknowledgements De Nittis is supported by the grant FONDECYT regular 2019, 1190204. Gomi is supported by the JSPS KAKENHI grant 15K04871. The authors wish to thank Krzysztof Gawędzki for very useful discussions. De Nittis wants to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) of Vienna where the results described in this paper were presented for the first time during the thematic program *Topological phases of quantum matter* held in 2014.

2 "Quaternionic" vector bundles from a topological perspective

In this section base spaces will be considered only from a topological point of view. Henceforth, we will assume that:

Assumption 2.1 ($\mathbb{Z}_2$–CW–complex) X is a topological space which admits the structure of a $\mathbb{Z}_2$–CW–complex. The *dimension* d of X is, by definition, the maximal dimension of its cells, and X is called *low-dimensional* if $0 \leqslant d \leqslant 3$.

For the sake of completeness, let us recall that an involutive space (X, τ) has the structure of a $\mathbb{Z}_2$–CW–complex if it admits a skeleton decomposition given by gluing cells of different dimension in ascending order, and the involution permutes the cells. For a precise definition of the notion of $\mathbb{Z}_2$–CW–complex the reader can refer to [7, Section 4.5] or [1; 38]. Assumption 2.1 allows the space X to have several disconnected components. However, in the case of multiple components, we will tacitly assume that vector bundles built over X possess fibers of constant rank on the whole base space. Let us recall that a space with a CW–complex structure is automatically Hausdorff and paracompact, and it is compact exactly when it is constructed out of a finite number of cells [26]. Almost all the examples considered in this paper will concern spaces with a finite CW–complex structure.

2.1 Basic facts about "quaternionic" vector bundles

In this section we recall some basic facts about the topological category of "quaternionic" vector bundles. Furthermore, the necessary notation for the description of the various results will be fixed. We refer to [8; 10; 11; 12] for a more systematic presentation of the subject.

Definition 2.2 ("quaternionic" vector bundles) A "quaternionic" vector bundle, or Q–bundle, over (X,τ) is a complex vector bundle $\pi:\mathscr{E}\to X$ endowed with a homeomorphism $\Theta:\mathscr{E}\to\mathscr{E}$ such that

(Q_1) the projection π is *equivariant* in the sense that $\pi\circ\Theta=\tau\circ\pi$;

(Q_2) Θ is *antilinear* on each fiber, ie $\Theta(\lambda p)=\bar{\lambda}\Theta(p)$ for all $\lambda\in\mathbb{C}$ and $p\in\mathscr{E}$, where $\bar{\lambda}$ is the complex conjugate of λ;

(Q_3) Θ^2 acts fiberwise as multiplication by -1, namely $\Theta^2|_{\mathscr{E}_x}=-\mathbb{1}_{\mathscr{E}_x}$.

Let us recall that it is always possible to endow $\mathscr{E}$ with an (essentially unique) *equivariant* Hermitian metric $\mathfrak{m}$ with respect to which Θ is an *antiunitary* map between conjugate fibers [8, Proposition 2.5]. The equivariance is expressed by

$$\mathfrak{m}(\Theta(p_1),\Theta(p_2))=\mathfrak{m}(p_2,p_1)\quad\text{for all }\ (p_1,p_2)\in\mathscr{E}\times_\pi\mathscr{E},$$

where $\mathscr{E}\times_\pi\mathscr{E}:=\{(p_1,p_2)\in\mathscr{E}\times\mathscr{E}\mid\pi(p_1)=\pi(p_2)\}$.

A vector bundle *morphism* between two vector bundles $\pi:\mathscr{E}\to X$ and $\pi':\mathscr{E}'\to X$ over the same base space is a continuous map $f:\mathscr{E}\to\mathscr{E}'$ which is fiber preserving in the sense that $\pi=\pi'\circ f$ and that restricts to a *linear* map on each fiber $f|_x:\mathscr{E}_x\to\mathscr{E}'_x$. Complex vector bundles over X together with vector bundle morphisms define a category. The symbol $\mathrm{Vec}^m_{\mathbb{C}}(X)$ is used to denote the set of equivalence classes of isomorphic vector bundles of rank m. From these data, it is possible to define a category of Q–bundles and Q–morphisms. A Q–morphism between two Q–bundles $(\mathscr{E},\Theta)$ and $(\mathscr{E}',\Theta')$ over the same involutive space (X,τ) is a vector bundle morphism f commuting with the involutions, ie $f\circ\Theta=\Theta'\circ f$. The set of equivalence classes of isomorphic Q–bundles of rank m over (X,τ) will be denoted by $\mathrm{Vec}^m_Q(X,\tau)$.

Remark 2.3 ("real" vector bundles) By changing condition (Q_3) in Definition 2.2 to

(R) Θ^2 acts fiberwise as the multiplication by 1, namely $\Theta^2|_{\mathscr{E}_x}=\mathbb{1}_{\mathscr{E}_x}$,

one ends in the category of *"real" vector bundles*, or *R–bundles*. The set of isomorphism classes of rank m R–bundles over the involutive space (X,τ) is denoted by $\mathrm{Vec}^m_R(X,\tau)$. For more details we refer to [3; 7].

In the case of a trivial involutive space (X,Id_X), one has bijections

$$\mathrm{Vec}^{2m}_Q(X,\mathrm{Id}_X)\simeq\mathrm{Vec}^m_{\mathbb{H}}(X),\quad \mathrm{Vec}^m_R(X,\mathrm{Id}_X)\simeq\mathrm{Vec}^m_{\mathbb{R}}(X),\quad m\in\mathbb{N}, \tag{2-1}$$

where $\mathrm{Vec}^m_{\mathbb{F}}(X)$ is the set of equivalence classes of vector bundles over X with typical fiber $\mathbb{F}^m$ and $\mathbb{H}$ denotes the skew field of quaternions. The first isomorphism in (2-1) is proved in [12] — see also [8, Proposition 2.2] — while the proof of the second is provided in [3] — see also [7, Proposition 4.5]. These two results justify the names "quaternionic" and "real" for the related categories.

Let $x \in X^\tau$ and $\mathscr{E}_x \simeq \mathbb{C}^m$ be the related fiber. In this case the restriction $\Theta|_{\mathscr{E}_x} \equiv J$ defines an *antilinear* map $J: \mathscr{E}_x \to \mathscr{E}_x$ such that $J^2 = -\mathbb{1}_{\mathscr{E}_x}$. Said differently, the fibers $\mathscr{E}_x$ over fixed points $x \in X^\tau$ are endowed with a *quaternionic* structure; see [8, Remark 2.1]. This fact has an important consequence [8, Proposition 2.1]:

Proposition 2.4 *If $X^\tau \neq \varnothing$, then every Q–bundle over (X, τ) has even rank.*

The set $\mathrm{Vec}^{2m}_Q(X, \tau)$ is nonempty since it contains at least the *trivial* element in the "quaternionic" category. The rank $2m$ *product Q–bundle* over the involutive space (X, τ) is the complex vector bundle

$$X \times \mathbb{C}^{2m} \to X$$

endowed with the *product Q*–structure

$$\Theta_0(x, \mathrm{v}) = (\tau(x), Q\bar{\mathrm{v}}), \quad (x, \mathrm{v}) \in X \times \mathbb{C}^{2m},$$

where the matrix Q is given by

$$Q := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}_m = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{2-2}$$

A "quaternionic" vector bundle is called *Q–trivial* if it is isomorphic to the product Q–bundle.

A *section* of a complex vector bundle $\pi: \mathscr{E} \to X$ is a continuous map $s: X \to \mathscr{E}$ such that $\pi \circ s = \mathrm{Id}_X$. The set $\Gamma(\mathscr{E})$ of sections of $\mathscr{E}$ has the structure of a left $C(X)$–module with multiplication given by the pointwise product $(fs)(x) := f(x)s(x)$ for any $f \in C(X)$ and $s \in \Gamma(\mathscr{E})$ and for all $x \in X$. If $(\mathscr{E}, \Theta)$ is a Q–bundle over (X, τ) then $\Gamma(\mathscr{E})$ is endowed with a natural antilinear antiinvolution $\tau_\Theta: \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ given by

$$\tau_\Theta(s) := \Theta \circ s \circ \tau.$$

The compatibility with the $C(X)$–module structure is given by

$$\tau_\Theta(fs) = \tau_*(f)\tau_\Theta(s), \quad f \in C(X),\ s \in \Gamma(\mathscr{E}),$$

where the antilinear involution $\tau_*: C(X) \to C(X)$ is defined by $\tau_*(f)(x) := \overline{f(\tau(x))}$. The triviality of a "quaternionic" vector bundle can be characterized in terms of global Q–frames of sections [8, Definition 2.1 and Theorem 2.1].

2.2 Stable range in low dimension

The stable rank condition for vector bundles expresses the pretty general fact that the nontrivial topology can be concentrated in a subvector bundle of *minimal* rank. This minimal value depends on the dimensionality of the base space and on the category of vector bundles under consideration. For complex (as well as real or quaternionic) vector bundles the stable rank condition is a well-known result; see eg [29, Chapter 9, Theorem 1.2]. The proof of the latter is based on an "obstruction-type argument" which provides the construction of a certain *maximal* number of global sections [29, Chapter 2, Theorem 7.1].

The latter argument can be generalized to vector bundles over spaces with involution by means of the notion of $\mathbb{Z}_2$–CW–complex [1; 38] — see also [7, Section 4.5]. A $\mathbb{Z}_2$–CW–complex is a CW–complex with a $\mathbb{Z}_2$–action that permutes the cells. The action of $\mathbb{Z}_2$ on each cell is either trivial or free. Since this construction is modeled after the usual definition of CW–complex, just by replacing "points" with "$\mathbb{Z}_2$–points", (almost) all topological and homological properties valid for CW–complexes have their natural counterpart in the equivariant setting. The use of this technique is essential for the determination of the stable rank condition in the case of R–bundles [7, Theorem 4.25] and Q–bundles [10, Theorems 4.2 and 4.5].

In this section we recall the results about the stable range for R–bundles and (even rank) Q–bundles over low-dimensional base spaces. Indeed, these are the only cases of interest in the present work.

Theorem 2.5 (stable condition in low dimension) *Let (X,τ) be an involutive space such that X has a finite $\mathbb{Z}_2$–CW–complex decomposition of dimension d. Assume that X^τ is discrete. Then*:

- ***Stable condition for R–bundles*** *For all $m \in \mathbb{N}$,*

$$\begin{aligned} &\mathrm{Vec}^m_R(X,\tau) = 0 && \textit{if } d = 0,1,\\ &\mathrm{Vec}^m_R(X,\tau) \simeq \mathrm{Vec}^1_R(X,\tau) && \textit{if } 2 \leqslant d \leqslant 3. \end{aligned}$$

- ***Stable condition for Q–bundles*** *For all $m \in \mathbb{N}$,*

$$\begin{aligned} \mathrm{Vec}^{2m}_{Q}(X,\tau) &= 0 && \textit{if } d = 0,1, \\ \mathrm{Vec}^{2m}_{Q}(X,\tau) &\simeq \mathrm{Vec}^{2}_{Q}(X,\tau) && \textit{if } 2 \leqslant d \leqslant 5. \end{aligned}$$

In particular, under the hypotheses of validity of Theorem 2.5, the dimensions $d = 0, 1$ are trivial since in these cases only the trivial R– and Q–bundles (up to isomorphism) exist. In the cases $d = 2, 3$, which are the really interesting cases for this work, it is enough to study the sets $\mathrm{Vec}^1_R(X,\tau)$ and $\mathrm{Vec}^2_Q(X,\tau)$.

2.3 The FKMM–invariant

Q–bundles can be classified, at least partially, by means of a characteristic class called *FKMM–invariant*. This topological object was first introduced in [19] and then studied and generalized in [8; 10; 11]. In this section we review the main properties of the FKMM–invariant.

Let (X,τ) be an involutive space and $X^\tau \subseteq X$ its fixed point subset. In order to introduce the FKMM–invariant one needs the *equivariant Borel cohomology* group of (X,τ) with coefficients in the local system $\mathbb{Z}(1)$; ie

$$H^\bullet_{\mathbb{Z}_2}(X,\mathbb{Z}(1)) := H^\bullet(X_{\sim\tau},\mathbb{Z}(1)). \tag{2-3}$$

More precisely, each equivariant cohomology group $H^j_{\mathbb{Z}_2}(X,\mathbb{Z}(1))$ is given by the singular cohomology group $H^j(X_{\sim\tau},\mathbb{Z}(1))$ of the homotopy quotient

$$X_{\sim\tau} := X \times \mathbb{S}^\infty/(\tau\times\theta_\infty),$$

where θ_∞ is the antipodal map on the infinite sphere $\mathbb{S}^\infty$. The local system $\mathbb{Z}(1)$ over (X,τ) can be identified with the product space $\mathbb{Z}(1) \simeq X\times\mathbb{Z}$ made equivariant by the $\mathbb{Z}_2$–action $(x,l)\mapsto(\tau(x),-l)$. The fixed point subset X^τ is closed in X and τ–invariant. The inclusion $\imath\colon X^\tau \hookrightarrow X$ extends to an inclusion $\imath\colon X^\tau_{\sim\tau}\hookrightarrow X_{\sim\tau}$ of the respective homotopy quotients. The *relative* equivariant cohomology can be defined as usual by the identification

$$H^\bullet_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1)) := H^\bullet(X_{\sim\tau}|X^\tau_{\sim\tau},\mathbb{Z}(1)).$$

For a more detailed description of equivariant Borel cohomology we refer to Section 3.1 of [8].

The FKMM–invariant is a map

$$\kappa\colon \mathrm{Vec}^{2m}_Q(X,\tau)\to H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1)) \tag{2-4}$$

which associates the isomorphism class $[(\mathscr{E},\Theta)]$ of the Q–bundle $(\mathscr{E},\Theta)$ to a cohomology class $\kappa(\mathscr{E},\Theta)$ in the relative equivariant cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. The construction of the map κ was first described in [8, Section 3.3] and then generalized in [10, Section 2.5]. In this section we will skip the details of the construction of the FKMM–invariant and we will focus only on the relevant properties of the map (2-4):

(a) Isomorphic Q–bundles define the same FKMM–invariant.

(b) The FKMM–invariant is *natural* with respect to equivariant maps.

(c) If $(\mathscr{E},\Theta)$ is Q–trivial, then $\kappa(\mathscr{E},\Theta)=0$.

(d) The FKMM–invariant is additive with respect to the Whitney sum and the abelian structure of $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. More precisely,

$$\kappa(\mathscr{E}_1\oplus\mathscr{E}_2,\Theta_1\oplus\Theta_2)=\kappa(\mathscr{E}_1,\Theta_1)\cdot\kappa(\mathscr{E}_2,\Theta_2)$$

for each pair of Q–bundles $(\mathscr{E}_1,\Theta_1)$ and $(\mathscr{E}_2,\Theta_2)$ over the same involutive space (X,τ).

For the justification of these properties we refer to [10, Section 2.6].

2.4 Topological classification over low-dimensional FKMM–spaces

The FKMM–invariant is an extremely efficient tool for the classification of Q–bundles in low dimensions. The first observation is that, in great generality, the FKMM–invariant is injective in low dimensions, ie when the base space has dimension $0\leqslant d\leqslant 3$. More precisely, as a consequence of [10, Theorems 4.7 and 4.9] one has that:

Theorem 2.6 (injectivity in low dimensions) *Let (X,τ) be an involutive space of dimension $d=0,1,2,3$ which satisfies Assumption 2.1. Then the map* (2-4) *is injective.*

This result suggests that in low dimensions the invariant κ can be used to label inequivalent classes of Q–bundles by means of elements of the cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. The next natural question is about the surjectivity of the map κ. In this case it is possible to provide a general positive answer only if $0\leqslant d\leqslant 2$. As proved in [11, Corollary 4.2 and Proposition 4.9] one has that:

Theorem 2.7 (surjectivity in dimension two) *Let (X,τ) be an involutive space of dimension $d=2$ which satisfies Assumption 2.1. Then*

$$\mathrm{Vec}^{2m}_Q(X,\tau)\simeq H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))\quad \textit{for all } m\in\mathbb{N},$$

namely the map (2-4) *is bijective.*

Theorem 2.7 can be juxtaposed with the stable condition described in Theorem 2.5,

$$\mathrm{Vec}_Q^{2m}(X,\tau) = 0 \quad \text{for all } m \in \mathbb{N} \text{ if } d = 0, 1,$$

to obtain a complete classification of Q–bundles in dimension $d = 0, 1, 2$.

In the case $d = 3$, the surjectivity of the FKMM–invariant can be recovered by requiring some extra properties for the base space (X,τ). In the next part of this work we will mainly focus on spaces of the following type:

Definition 2.8 (FKMM–manifold) An involutive space (X,τ) is called an FKMM–manifold if

(a) X is a compact Hausdorff manifold without boundary;

(b) the involution τ preserves the manifold structure;

(c) the fixed point set X^τ consists at most of a finite collection of points;

(d) $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1)) = 0$.

Let us observe that an involutive space (X,τ) which fulfills conditions (a) and (b) in Definition 2.8 is a *closed manifold* which automatically admits the structure of a $\mathbb{Z}_2$–CW–complex; see eg [39, Theorem 3.6]. Then an FKMM–manifold meets all the requirements stated in Assumption 2.1. The conditions (c) and (d) are the crucial ingredients for the definition of a *topological* FKMM–space according to the original definition [8, Definition 1.1]. The requirement of a manifold structure has a twofold justification: first of all it allows the use of a technical tool (the slice theorem) in the proof of the crucial result [11, Proposition 4.13]; second, the main aim of this work is the study of the classification of Q–bundles over involutive manifolds (see Section 3). The manifold structure and the map τ are tacitly assumed to be of some given regularity (eg C^r or smooth). The next result provides the topological classification of Q–bundles over low-dimensional FKMM–manifolds.

Theorem 2.9 (classification of FKMM–manifolds) *Let (X,τ) be an FKMM–manifold of dimension $0 \leqslant d \leqslant 3$. Then, for all $m \in \mathbb{N}$,*

$$\begin{aligned} \mathrm{Vec}_Q^{2m}(X,\tau) &= 0 && \text{if } d = 0, 1,\\ \mathrm{Vec}_Q^{2m}(X,\tau) &\simeq H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1)) && \text{if } d = 2, 3, \end{aligned}$$

and the isomorphism (in the nontrivial cases) is given by the FKMM–invariant κ.

The cases $d = 0, 1$ are a consequence of the stable condition described in Theorem 2.5. The case $d = 2$ follows from Theorem 2.7. Finally the new case $d = 3$ is proved in [11, Proposition 4.13].

Let us observe that Theorem 2.9 also holds trivially in the case of a free involution, that is, when $X^\tau = \varnothing$. In this case, as a consequence of condition (d) in Definition 2.8 one has that $H^2_{\mathbb{Z}_2}(X|\varnothing, \mathbb{Z}(1)) \simeq H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$. Therefore, as a consequence of Theorem 2.9, one concludes that an FKMM–manifold with free involution only supports the trivial Q–bundle. In order to focus on the nontrivial situations we will assume henceforth that $d = 2, 3$ and $X^\tau \neq \varnothing$.

When (X, τ) is an FKMM–manifold, the cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1))$ has an explicit representation in terms of equivalence classes of maps. As proved in [8, Lemma 3.1] one has the isomorphism

$$H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \simeq \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}, \tag{2-5}$$

where $\mathrm{Map}(X^\tau, \{\pm 1\}) \simeq {\mathbb{Z}_2}^{|X^\tau|}$ is the set of maps from X^τ to $\{\pm 1\}$ (recall that X^τ is a set of finitely many points) and $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ denotes the set of classes of $\mathbb{Z}_2$–homotopy equivalent equivariant maps between the involutive space (X, τ) and the group $\mathbb{U}(1)$ endowed with the involution given by complex conjugation. The group action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $\mathrm{Map}(X^\tau, \{\pm 1\})$ is given by restriction and multiplication. More precisely, let $[u] \in [X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ and $s \in \mathrm{Map}(X^\tau, \{\pm 1\})$. Then the action of $[u]$ on s is given by $[u](s) := u|_{X^\tau} \cdot s$. By combining Theorem 2.9 with the isomorphism (2-5) one gets the following result:

Proposition 2.10 *Let (X, τ) be an FKMM–manifold of dimension $d = 2, 3$ and assume that $X^\tau \neq \varnothing$. Then, the FKMM–invariant κ induces the isomorphism*

$$\mathrm{Vec}^{2m}_Q(X, \tau) \simeq \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \quad \textit{for all } m \in \mathbb{N}.$$

In summary, the content of Theorem 2.9 and Proposition 2.10 is the following: Every Q–bundle $(\mathscr{E}, \Theta)$ over an FKMM–space (X, τ) of dimension $d = 2, 3$ such that $X^\tau \neq \varnothing$ is classified by its FKMM–invariant $\kappa(\mathscr{E}, \Theta)$. The latter can be represented as a map

$$s_{(\mathscr{E},\Theta)} \colon X^\tau \to \{\pm 1\}$$

modulo the (right) multiplication by the restriction over X^τ of an equivariant function $u \colon X \to \mathbb{U}(1)$. The map $s_{(\mathscr{E},\Theta)}$ is called the *canonical section* associated to $(\mathscr{E}, \Theta)$ and its construction is described in [8, Section 3.2] or [10, Section 2.2].

2.5 The Fu–Kane–Mele index

Let us focus on the nontrivial case of an FKMM–manifold (X,τ) of dimension $d=2,3$ such that $X^\tau\neq\varnothing$. At the end of last section we observed that every Q–bundle $(\mathscr{E},\Theta)$ over (X,τ) is classified by the canonical section $s_{(\mathscr{E},\Theta)}\in\mathrm{Map}(X^\tau,\{\pm1\})$ modulo the action (multiplication and restriction) of an equivariant map $u\colon X\to\mathbb{U}(1)$. Clearly $(\mathscr{E},\Theta)$ is equivalently classified by any other map $\phi\in\mathrm{Map}(X^\tau,\{\pm1\})$ in the same equivalence class of $s_{(\mathscr{E},\Theta)}$, namely by any representative of

$$[s_{(\mathscr{E},\Theta)}]\in\mathrm{Map}(X^\tau,\{\pm1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Consider now the *product sign* map

$$\Pi\colon\mathrm{Map}(X^\tau,\{\pm1\})\to\{\pm1\}\tag{2-6}$$

defined by

$$\Pi(\phi):=\prod_{x_j\in X^\tau}\phi(x_j),\qquad \phi\in\mathrm{Map}(X^\tau,\{\pm1\}).\tag{2-7}$$

The value $\Pi(\phi)$ is called the *Fu–Kane–Mele index* of ϕ. There is no reason to suspect a priori that the Fu–Kane–Mele index is well defined on the equivalence classes in $\mathrm{Map}(X^\tau,\{\pm1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$. In fact, if ϕ_1 and ϕ_2 were two representatives of the same class $[\phi]\in\mathrm{Map}(X^\tau,\{\pm1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ related by an equivariant function $u\colon X\to\mathbb{U}(1)$ which takes an odd number of times the value -1 on X^τ, then one would have that $\Pi[\phi_1]=-\Pi[\phi_2]$. For this reason the following result, proved in [8, Proposition 4.5 and Theorem 4.2] is quite surprising, at first glance, from a topological point of view.

Theorem 2.11 (Fu–Kane–Mele formula, $d=2$) *Let (X,τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then (X,τ) is an FKMM–manifold according to Definition 2.8. Moreover,*

$$H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))\simeq\mathbb{Z}_2,\tag{2-8}$$

where $\mathbb{Z}_2$ is identified with the multiplicative group $\{\pm1\}$. Moreover, any Q–bundle $(\mathscr{E},\Theta)$ over (X,τ) is classified by the FKMM–invariant $\kappa(\mathscr{E},\Theta)\in\{\pm1\}$ which can be computed by $\kappa(\mathscr{E},\Theta)=\Pi(\phi)$, where Π is the product sign map (2-6) *and ϕ is any representative of the class $[s_{(\mathscr{E},\Theta)}]\in\mathrm{Map}(X^\tau,\{\pm1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ of the canonical section.*

Proof (sketch) Clearly conditions (a′), (b′) and (c′) of Definition 1.1 imply conditions (a), (b) and (c) of Definition 2.8. Moreover, Proposition 4.4 of [8] assures that (a′), (b′)

and (c′) imply condition (d) of Definition 2.8, ie $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$ along with isomorphism (2-8). The rest of the claim is proved in [8, Proposition 4.5 and Theorem 4.2]. □

As a byproduct of Theorem 2.11 one has that the Fu–Kane–Mele index is unambiguously defined on the whole equivalence class $[s_{(\mathscr{E},\Theta)}]$, and the Q–bundle $(\mathscr{E},\Theta)$ is classified, up to isomorphism, by the sign $\Pi(\phi)\in\{\pm1\}$ where $\phi\in\mathrm{Map}(X^\tau,\{\pm1\})$ is any map which differs from $s_{(\mathscr{E},\Theta)}$ by the multiplication with the restriction of an equivariant map $u\colon X\to\mathbb{U}(1)$.

Although with some differences, the next result pairs Theorem 2.11 in dimension $d=3$. It can be considered one of the main achievements of this work.

Theorem 2.12 (Fu–Kane–Mele formula, $d=3$) *Let (X,τ) be an FKMM–manifold of dimension $d=3$ with $X^\tau\neq\varnothing$. Assume in addition that:*

(e) *X is oriented and τ* ***reverses*** *the orientation.*

Let $(\mathscr{E},\Theta)$ be a Q–bundle over (X,τ) with FKMM–invariant $\kappa(\mathscr{E},\Theta)$ represented by the class $[s_{(\mathscr{E},\Theta)}]\in\mathrm{Map}(X^\tau,\{\pm1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ according to Proposition 2.10. Then the sign

$$\kappa_{\mathrm{s}}(\mathscr{E},\Theta):=\Pi[\phi] \tag{2-9}$$

is independent of the choice of the representative $\phi\in[s_{(\mathscr{E},\Theta)}]$ and provides a topological invariant for $(\mathscr{E},\Theta)$.

Theorem 2.12 is a direct consequence of Theorem 1.3, which will be proved in Section 3.7. It is worth noting that even though Theorems 2.11 and 2.12 seem to be of topological nature, they need the manifold structure of X. In particular, Theorem 1.3 relies on differential geometric techniques.

In general the quantity $\kappa_{\mathrm{s}}(\mathscr{E},\Theta)$ in Theorem 2.12 does not completely specify the FKMM–invariant of $(\mathscr{E},\Theta)$, but only a part of it. We refer to $\kappa_{\mathrm{s}}(\mathscr{E},\Theta)$ as the *strong component* of the FKMM–invariant.

2.6 Alternative presentation of "quaternionic" vector bundles in low dimensions

This section is focused on an alternative description of rank 2 Q–bundles over low-dimensional involutive spaces (X,τ) such that $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$. It is worth mentioning that under these conditions the complex vector bundle underlying each Q–bundle is necessarily trivial [8, Proposition 4.1].

Let $\mathrm{Map}(X, \mathbb{SU}(2))$ be the space of (smooth) maps from X into $\mathbb{SU}(2)$. Given $\xi \in \mathrm{Map}(X, \mathbb{SU}(2))$, let $\tau^*\xi$ be the map defined by $\tau^*\xi(x) := \xi(\tau(x))$ for all $x \in X$. The space of equivariant maps from X into $\mathbb{SU}(2)$ is defined by

$$\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} := \{\xi \in \mathrm{Map}(X, \mathbb{SU}(2)) \mid \tau^*\xi = \xi^{-1}\}. \tag{2-10}$$

The set of $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps will be denoted by

$$[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}.$$

Let us consider also the groups

$$\begin{aligned} \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2} &:= \{\psi \in \mathrm{Map}(X, \mathbb{U}(2)) \mid \det(\tau^*\psi) = \det(\bar{\psi})\}, \\ \mathrm{Map}(X, \mathbb{U}(1))_{\mathbb{Z}_2} &:= \{\phi \in \mathrm{Map}(X, \mathbb{U}(1)) \mid \tau^*\phi = \bar{\phi}\}, \end{aligned} \tag{2-11}$$

where $\bar{\psi}$ and $\bar{\phi}$ are the complex conjugates of ψ and ϕ, respectively, and the group structures are given by pointwise multiplication. The related sets of equivalence classes under $\mathbb{Z}_2$–homotopy are denoted by $[X, \mathbb{U}(2)]'_{\mathbb{Z}_2}$ and $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$, respectively.

By construction one has an inclusion $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} \subset \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$. Moreover, the group $\mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$ acts on $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2}$ as follows: given $\psi \in \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$ let G_ψ be the automorphism of $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2}$ given by

$$G_\psi(\xi) := -(\tau^*\psi^{-1})\xi Q \bar{\psi} Q, \quad \xi \in \mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} \tag{2-12}$$

where Q is the (size 2×2) matrix (2-2). In fact, given that

$$\det(\tau^*\psi^{-1}) = \det(\tau^*\psi)^{-1} = \det(\bar{\psi})^{-1},$$

it follows that $\det(G_\psi(\xi)) = \det(\xi) = 1$. Moreover, the equality $\tau^* G_\psi(\xi) = G_\psi(\xi)^{-1}$ follows from a direct calculation along with the equality $Q\xi = \bar{\xi} Q$ valid for maps with values in $\mathbb{SU}(2)$.

The main aim of this section is to prove the following result:

Theorem 2.13 *Let (X, τ) be an involutive space of dimension $0 \leqslant d \leqslant 2$ satisfying Assumption 2.1. Assume in addition that $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ in the case $d = 2$. Then there is a natural bijection*

$$\mathrm{Vec}^2_Q(X, \tau) \simeq [X, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [X, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

where the action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ is defined as follows: given $[\phi]$ in $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$, let $L_{[\phi]}$ be the automorphism of $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ defined by

$$L_{[\phi]}([\xi]) := \left[\begin{pmatrix} \tau^*\bar{\phi} & 0 \\ 0 & 1 \end{pmatrix} \xi \begin{pmatrix} 1 & 0 \\ 0 & \bar{\phi} \end{pmatrix} \right].$$

We start with a couple of preliminary results which are valid in dimension $0 \leqslant d \leqslant 3$.

Lemma 2.14 *Let (X, τ) be a low-dimensional involutive space satisfying Assumption 2.1. Assume in addition that $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ in the case $d = 2, 3$. Then there is a natural bijection*

$$\mathrm{Vec}^2_Q(X, \tau) \simeq \mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} / \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2} \tag{2-13}$$

where the action of $\mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$ on $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2}$ is given by the automorphisms (2-12).

Proof Let $\pi\colon \mathscr{E} \to X$ be a rank 2 Q–bundle. The low-dimensionality of the base space implies that the underlying complex vector bundle $\mathscr{E}$ is isomorphic to the product bundle $X \times \mathbb{C}^2$ [8, Proposition 4.1]. The induced Q–structure Θ on $X \times \mathbb{C}^2$ is then expressed through a function $\xi\colon X \to \mathbb{U}(2)$ of the form $\Theta\colon (x, v) \mapsto (\tau(x), \xi(x) Q \bar{v})$ and the "quaternionic" condition is guaranteed by the constraint $\tau^* \xi = -Q \bar{\xi}^{-1} Q$. Let us introduce the subset

$$\mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2} := \{\xi \in \mathrm{Map}(X, \mathbb{U}(2)) \mid \tau^* \xi = -Q \bar{\xi}^{-1} Q\} \subset \mathrm{Map}(X, \mathbb{U}(2)).$$

Two Q–structures Θ and Θ' on $X \times \mathbb{C}^2$, induced respectively by the maps ξ and ξ' in $\mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2}$, are isomorphic if there exists a map $\psi \in \mathrm{Map}(X, \mathbb{U}(2))$ such that $\tau^* \psi \xi' Q = \xi Q \bar{\psi}$. Consider the action of $\mathrm{Map}(X, \mathbb{U}(2))$ on $\mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2}$ defined as follows: for any $\psi \in \mathrm{Map}(X, \mathbb{U}(2))$ let G_ψ be the automorphism of $\mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2}$ given by the formula (2-12). From the argument above it follows that

$$\mathrm{Vec}^2_Q(X, \tau) \simeq \mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2} / \mathrm{Map}(X, \mathbb{U}(2))$$

where the equivalence relation is induced by the action of the automorphisms G_ψ. Since $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ by hypothesis, any "real" line bundle over X is automatically trivial [30]. This applies in particular to the determinant line bundle of the Q–bundle $(\mathscr{E}, \Theta)$. The triviality of the "real" structure $(x, u) \mapsto (\tau(x), \det(\xi)(x)\bar{u})$ on $X \times \mathbb{C}$ implies the existence of a map $\phi\colon X \to \mathbb{U}(1)$ such that $\det(\xi) = \tau^* \phi \phi$. Consider the map $\psi_0 \in \mathrm{Map}(X, \mathbb{U}(2))$ given by

$$\psi_0(x) := \begin{pmatrix} \phi(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

A direct computation shows that

$$\det(G_{\psi_0}(\xi)) = \det(\tau^* \psi_0)^{-1} \det(\xi) \det(\psi_0)^{-1} = 1. \tag{2-14}$$

As a result, it is possible to choose $\xi \in \mathrm{Map}(X, \mathbb{U}(2))_{\mathbb{Z}_2} \cap \mathrm{Map}(X, \mathbb{SU}(2))$ as the representative for the element of $\mathrm{Vec}^2_Q(X, \tau)$. Since it holds that $-Q \bar{\xi} Q = \xi$ for maps with values in $\mathbb{SU}(2)$, one has that the intersection $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}\cap\mathrm{Map}(X,\mathbb{SU}(2))$ coincides with the set $\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$ as described by (2-10). Finally, it is straightforward to see that $\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ described by (2-11) is the maximal subgroup of $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}$ preserving such representatives. □

Lemma 2.15 *Under the hypotheses of Lemma 2.14 there is a natural bijection*

$$\mathrm{Vec}^2_Q(X,\tau)\simeq[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}. \tag{2-15}$$

Proof Consider the natural surjection onto the equivalence classes

$$\varpi\colon\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}\hookrightarrow[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}.$$

The action of $\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ on $\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$ given by (2-12) induces an action of the group $[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}$ on $[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}$. Under these actions, ϖ is equivariant, and one gets

$$\mathrm{Vec}^2_Q(X,\tau)\simeq\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}/\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}\xrightarrow{\varpi}[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}.$$

The latter is a bijection. Indeed, given $\xi\in\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$, let $\mathscr{E}_\xi=X\times\mathbb{C}^2$ be the Q–bundle of rank 2 with Q–structure given by $(x,v)\mapsto(\tau(x),\xi(x)Q\bar{v})$. In view of the homotopy property of Q–bundles [8, Theorem 2.3], if ξ and ξ' are $\mathbb{Z}_2$–homotopy equivalent, then $\mathscr{E}_\xi$ and $\mathscr{E}_{\xi'}$ are isomorphic. □

We are now in position to complete the proof of Theorem 2.13. For this purpose the restriction to dimensions $d\leqslant 2$ will be crucial.

Proof of Theorem 2.13 We will begin with the case $m=1$. Consider the exact sequence

$$1\to[X,\mathbb{SU}(2)]\xrightarrow{\imath}[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}\xrightarrow{\det}[X,\mathbb{U}(1)]_{\mathbb{Z}_2}\to 1$$

where $\imath$ is induced by the natural inclusion $\mathrm{Map}(X,\mathbb{SU}(2))\hookrightarrow\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ and det stands for the determinant. The latter sequence is right-split in view of the map $s\colon[X,\mathbb{U}(1)]_{\mathbb{Z}_2}\to[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}$ induced (with a slight abuse of notation) by

$$\mathrm{Map}(X,\mathbb{U}(1))_{\mathbb{Z}_2}\ni\phi\overset{s}{\longmapsto}\begin{pmatrix}\phi&0\\0&1\end{pmatrix}\in\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}. \tag{2-16}$$

Indeed, it is straightforward to check $\det\circ s=\mathrm{Id}$. Consequently, one has a group isomorphism

$$[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}\simeq[X,\mathbb{SU}(2)]\rtimes[X,\mathbb{U}(1)]_{\mathbb{Z}_2},$$

where $\rtimes$ denotes the semidirect product. Since $\pi_k(\mathbb{SU}(2))=0$ if $k=0,1,2$, it follows that $[X,\mathbb{SU}(2)]=0$ whenever X has dimension $0\leqslant d\leqslant 2$. In these three cases, the isomorphism above reduces to $[X, \mathbb{U}(2)]'_{\mathbb{Z}_2} \simeq [X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ and the combination of the action G described by (2-12) with the homomorphism s in (2-16) produces the action L of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ as described in the claim. In view of the stable rank condition described in Theorem 2.5, the bijection generalizes to

$$\mathrm{Vec}^{2m}_{Q}(X, \tau) \simeq [X, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}, \quad m \in \mathbb{N}$$

and this concludes the proof for the general case. □

Remark 2.16 (higher rank case) A representative map $\xi\colon X \to \mathbb{SU}(2)$ for a given Q–bundle $(\mathscr{E}, \Theta)$ of rank $2m$ can be constructed in this way: The Q–structure of $(\mathscr{E}, \Theta)$ is coded in an equivariant map $\xi'\colon X \to \mathbb{SU}(2m)$ which, for instance, can be constructed from a global frame according to the prescription described in Remark 2.19. The stable rank condition implies that ξ' can be always reduced to the form

$$\xi' \simeq \begin{pmatrix} \xi & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^{2(m-1)}} \end{pmatrix}$$

up to conjugation with an equivariant map with values in $\mathbb{U}(2m)$. The reduced map $\xi\colon X \to \mathbb{SU}(2)$ obtained in this way provides a representative of the Q–bundle $(\mathscr{E}, \Theta)$ as an element of the group $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$.

2.7 The FKMM–invariant for oriented two-dimensional FKMM–manifolds

Throughout this section we will assume that the pair (Σ, τ) is an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. The use of the letter Σ instead of X is motivated to easier connect the results discussed here with the theory developed in Section 3.4 and 3.6

When (Σ, τ) is an oriented two-dimensional FKMM–manifold, two presentations for $\mathrm{Vec}^2_Q(\Sigma, \tau)$ are available. The first description,

$$\mathrm{Vec}^2_Q(\Sigma, \tau) \simeq \mathrm{Map}(\Sigma^\tau, \{\pm 1\})/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

was proved in Proposition 2.10 and uses the FKMM–invariant. The second presentation,

$$\mathrm{Vec}^2_Q(\Sigma, \tau) \simeq [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

comes from Theorem 2.13. Therefore, there must exist an isomorphism of groups

$$[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq \mathrm{Map}(\Sigma^\tau, \{\pm 1\})/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$$

which associates the map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ with the FKMM–invariant of the Q–bundle $\mathscr{E}_\xi$ classified by ξ. Such a map can be constructed by means of the *Pfaffian* Pf; see Proposition 2.18.

The evaluation of a map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ on a fixed point $x \in \Sigma^\tau$ is an element of $\mathbb{SU}(2)$ which satisfies $\xi(x) = \xi(x)^{-1}$. This implies that $\xi(x) = \pm \mathbb{1}_{\mathbb{C}^2}$ if $x \in \Sigma^\tau$. Moreover, every matrix $\xi(x) \in \mathbb{SU}(2)$ satisfies the identity $Q\overline{\xi(x)} = \xi(x)Q$. Then, on a fixed point $x \in \Sigma^\tau$, the matrix $\xi(x)Q = \pm Q$ turns out to be skew-symmetric and the Pfaffian $\mathrm{Pf}(\xi(x)Q)$ is well defined. In particular one has that

$$-\mathrm{Pf}(\xi(x)\cdot Q) = \begin{cases} +1 & \text{if } \xi(x) = +\mathbb{1}_{\mathbb{C}^2}, \\ -1 & \text{if } \xi(x) = -\mathbb{1}_{\mathbb{C}^2}. \end{cases}$$

This suggests studying the mapping

$$\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2} \ni \xi \xrightarrow{\Phi_\kappa} -\mathrm{Pf}(\xi Q|_{\Sigma^\tau}) \in \mathrm{Map}(\Sigma^\tau, \{\pm 1\}). \tag{2-17}$$

Lemma 2.17 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then there is a bijection*

$$\Phi_\kappa \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\})$$

defined by $[\xi] \mapsto -\mathrm{Pf}(\xi Q)|_{\Sigma^\tau}$.

Proof We will start by proving the injectivity of Φ_κ. Suppose $\xi, \xi' \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ are such that $\Phi_\kappa(\xi) = \Phi_\kappa(\xi')$. We want to show the existence of a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi} \colon \Sigma \times [0,1] \to \mathbb{SU}(2)$ such that $\tilde{\xi}|_{\Sigma \times \{0\}} = \xi$ and $\tilde{\xi}|_{\Sigma \times \{1\}} = \xi'$. This can be done by a standard argument in homotopy theory. Let Σ_j be the j–skeleton of Σ with respect to a $\mathbb{Z}_2$–CW decomposition. The 0–skeleton Σ_0 consists of the 0–cells of the form e^0 (a fixed cell) or $\mathbb{Z}_2 \times e^0$ (a free cell), where $e^0 = *$ is a standard 0–cell. Accordingly, we can express Σ_0 as the disjoint union $\Sigma_0 = \Sigma_0^{\mathrm{fix}} \sqcup \Sigma_0^{\mathrm{free}}$. By assumption, we have $\Sigma_0^{\mathrm{fix}} = X^\tau$. Notice that the map Φ_κ factors through

$$[\Sigma, \mathbb{SU}(2))]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \mathbb{SU}(2)^\vartheta) \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\}),$$

where the involution ϑ on $\mathbb{SU}(2)$ is $\vartheta(\xi) = \xi^{-1}$, so that the fixed point set $\mathbb{SU}(2)^\vartheta = \{\pm \mathbb{1}_{\mathbb{C}^2}\}$ consists of two points. The first map is induced from the restriction $\xi \mapsto \xi|_{\Sigma^\tau}$. The second map is the bijection induced from the obvious identification $\mathbb{SU}(2)^\vartheta \simeq \{\pm 1\}$. It follows that $\xi|_{\Sigma_0^{\mathrm{fix}}} = \xi'|_{\Sigma_0^{\mathrm{fix}}}$. On the other hand, for each free 0–cell $\mathbb{Z}_2 \times e^0$ we can find a homotopy connecting $\xi|_{\{1\} \times e^0}$ and $\xi'|_{\{1\} \times e^0}$ because $\mathbb{SU}(2)$ is path connected. This homotopy extends to a $\mathbb{Z}_2$–equivariant homotopy connecting $\xi|_{\mathbb{Z}_2 \times e^0}$ and $\xi'|_{\mathbb{Z}_2 \times e^0}$ since the action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times e^0$ is free. In this way, we get a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi}_0 \colon \Sigma_0 \times [0,1] \to \mathbb{SU}(2)$ such that $\tilde{\xi}_0|_{\Sigma_0 \times \{0\}} = \xi|_{\Sigma_0}$ and $\tilde{\xi}_0|_{\Sigma_0 \times \{1\}} = \xi'|_{\Sigma_0}$.

By assumption again, the 1–skeleton Σ_1 is given by attaching only free 1–cells of the form $\mathbb{Z}_2 \times e^1$ to Σ_0. We already have a homotopy $\tilde{\xi}_0|_{\{1\}\times\partial e^1\times[0,1]}$. This homotopy, together with $\xi|_{\{1\}\times e^1}$ and $\xi'|_{\{1\}\times e^1}$, gives a map from

$$\partial(\{1\}\times e^1\times[0,1]) = (\{1\}\times\partial e^1\times[0,1]) \cup (\{1\}\times e^1\times\partial[0,1])$$

which can be extended to a map from $\{1\}\times e^1\times[0,1]$ in view of $\pi_1(\mathbb{SU}(2)) = 0$. Extending this map equivariantly, and gathering together the maps constructed in this way for each free 1–cell, one gets a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi}_1\colon \Sigma_1\times[0,1]\to\mathbb{SU}(2)$ which extends $\tilde{\xi}_0$ and connects $\xi|_{\Sigma_1}$ with $\xi'|_{\Sigma_1}$. Finally, the 2–skeleton $\Sigma_2 = \Sigma$ is given by attaching only free 2–cells of the form $\mathbb{Z}_2\times e^2$ to Σ_1. We already have a homotopy $\tilde{\xi}_1|_{\{1\}\times\partial e^2\times[0,1]}$. This homotopy, together with $\xi|_{\{1\}\times e^2}$ and $\xi'|_{\{1\}\times e^2}$, provides a map from $\partial(\{1\}\times e^2\times[0,1])$. This extends to a map from $\{1\}\times e^2\times[0,1]$, since $\pi_2(\mathbb{SU}(2)) = 0$. Extending this equivariantly and gathering together the resulting maps for each free 2–cell, one gets a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi}\colon\Sigma\times[0,1]\to\mathbb{SU}(2)$ connecting ξ with ξ'.

Now the surjectivity. The idea is to construct an element $\xi_\epsilon\in\mathrm{Map}(\Sigma,\mathbb{SU}(2))_{\mathbb{Z}_2}$ for each $\epsilon\in\mathrm{Map}(\Sigma^\tau,\mathbb{Z}_2)$ such that $\Phi_\kappa(\xi_\epsilon)=\epsilon$. A preliminary fact is necessary. Let $D\subset\mathbb{C}$ be the closed unit disk endowed with the involution $z\mapsto -z$. Then, the map $\xi_D\in\mathrm{Map}(D,\mathbb{SU}(2))_{\mathbb{Z}_2}$ given by

$$\xi_D(z) := \frac{1}{2(|z|^2-|z|)+1}\begin{pmatrix} 2|z|-1 & -2\bar{z}(|z|-1) \\ 2z(|z|-1) & 2|z|-1\end{pmatrix}$$

satisfies $\xi_D(0) = -\mathbb{1}_{\mathbb{C}^2}$ and $\xi_D(z) = +\mathbb{1}_{\mathbb{C}^2}$ if $z\in\partial D$. Let $\Sigma^\tau=\{x_1,\dots,x_n\}$ be a given labeling for the fixed points. The *slice theorem* [27, Chapter I, Section 3] assures that for each x_i there exists a closed disk $D_i\subset\Sigma$ such that $\tau(D_i)=D_i$, $x_i\in D_i$ and $D_i\cap D_j=\varnothing$ when $i\neq j$. Let $x_{i_1},\dots,x_{i_k}\in\Sigma^\tau$ be the set of points such that $\epsilon(x_{i_j})=-1$. Using an equivariant diffeomorphism $D\cong D_{i_j}$ one can induce the equivariant map $\xi_{D_{i_j}}$ on D_{i_j} from ξ_D. Extending these maps by $\mathbb{1}_{\mathbb{C}^2}$ outside of $D_{i_1}\cup\cdots\cup D_{i_k}$ one gets an equivariant map $\xi_\epsilon\in\mathrm{Map}(\Sigma,\mathbb{SU}(2))_{\mathbb{Z}_2}$ such that $\xi_\epsilon(x)=\epsilon(x)\mathbb{1}_{\mathbb{C}^2}$ for every $x\in\Sigma^\tau$. This ensures that $\Phi_\kappa(\xi_\epsilon)=\epsilon$. □

Proposition 2.18 *Let (Σ,τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then the bijection of Lemma 2.17 induces the bijection*

$$\Phi_\kappa\colon[\Sigma,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2}\to\mathrm{Map}(\Sigma^\tau,\{\pm1\})/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Proof Lemma 2.17 asserts the bijectivity of the homomorphism

$$\Phi_\kappa\colon[\Sigma,\mathbb{SU}(2)]_{\mathbb{Z}_2}\to\mathrm{Map}(\Sigma^\tau,\{\pm1\}).$$

The same group $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ acts on both sides and Φ_κ is equivariant. An inspection of the group actions shows that Φ_κ descends to a bijective homomorphism between the quotients. □

In view of Theorem 2.13, one can think of a map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ as a rank 2 Q–bundle on Σ. Then it makes sense to talk about the "FKMM–invariant of the map ξ". Proposition 2.18 shows that such an invariant is indeed built through the isomorphism Φ_κ. More precisely, by combining Proposition 2.18 with Theorem 2.11 one obtains

$$\kappa(\xi) := \Pi \circ \Phi_\kappa(\xi) \in \mathbb{Z}_2, \tag{2-18}$$

where $\kappa(\xi)$ represents the FKMM–invariant of the Q–bundle defined by the map ξ.

Remark 2.19 (construction of the classifying map from a frame) Let $(\mathscr{E}, \Theta)$ be a Q–bundle of rank 2 over an oriented two-dimensional FKMM–manifold. If the map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ classifies $(\mathscr{E}, \Theta)$ according to Theorem 2.11 then formula (2-18) provides the computation of the FKMM–invariant of $(\mathscr{E}, \Theta)$. Therefore, the relevant problem is how to extract ξ from the knowledge of $(\mathscr{E}, \Theta)$. This problem has a simple solution when a global trivializing frame of sections $t_1, t_2\colon \Sigma \to \mathscr{E}$ of the underlying (trivial) complex vector bundle is known. This situation has been described in detail [8, Section 4.2]. By a Gram–Schmidt orthonormalization if necessary, one can assume without loss of generality that the frame t_1, t_2 is orthonormal, ie $\mathfrak{m}(t_i, t_j) = \delta_{i,j}$ where $\mathfrak{m}$ is the (unique) Θ–equivariant Hermitian metric on $\mathscr{E}$. Then the classifying map $\xi = \{\xi_{ij}\}$ is given by the formula

$$\xi_{ij}(x) := \mathfrak{m}(\tau^* t_i(x), \Theta t_j(x)),$$

where $\tau^* t_i(x) := t_i(\tau(x))$ and $\Theta t_j(x) := \Theta(t_j(x))$ are short notations.

3 Differential geometric classification of "quaternionic" vector bundles

In this section we provide differential geometric realizations of the FKMM–invariant. However, this require more structure on the involutive space (X, τ). More properly, we need to pass from the *topological category* to the *smooth category*. In this section the quite general Assumption 2.1 will be replaced by the more restrictive:

Assumption 3.1 (smooth category) X is a compact, path-connected, Hausdorff *smooth* d–dimensional manifold without boundary and with a *smooth* involution τ.

In particular, a space X which fulfills Assumption 3.1 is a *closed* manifold and the pair (X,τ) automatically admits the structure of a $\mathbb{Z}_2$–CW–complex; see eg [39, Theorem 3.6]. Observe that the notion of FKMM–manifold given in Definition 2.8 is compatible with Assumption 3.1. It is worth pointing out that the *smooth* condition can be relaxed to a less demanding regularity condition; for instance it is sufficient to assume that the manifold structure is C^r–regular for some $r \in \mathbb{N}$. Anyway, this is only a technical detail and for a simpler presentation it is enough to focus only on the smooth case.

Let us point out that in Section 2.1 we introduced the notion of Q–bundle in the topological category meaning that all the maps involved in the various definitions are continuous functions between topological spaces. However, when the involutive space (X,τ) has an additional smooth manifold structure one can equivalently define Q–bundles in the smooth category by requiring that all spaces involved in the definitions carry a smooth manifold structure and maps are smooth functions. However, for what concerns the problem of the classification, the two categories are equivalent [9, Theorem 2.1], namely

$$^{\mathrm{top}}\,\mathrm{Vec}^m_Q(X,\tau) \simeq {}^{\mathrm{smooth}}\mathrm{Vec}^m_Q(X,\tau).$$

Clearly, the same holds true also in the "real" category. For more details on this point we refer to [9, Section 2].

3.1 Principal "quaternionic" bundles and related FKMM–invariant

The next definition was introduced in [9, Section 2.1].

Definition 3.2 (principal R– and Q–bundle) Let (X,τ) be an involutive space which satisfies Assumption 3.1 and $\pi : \mathscr{P} \to X$ a (smooth) principal $\mathbb{U}(m)$–bundle. We say that $\mathscr{P}$ has a *"real" structure* if there is a homeomorphism $\widehat{\Theta} : \mathscr{P} \to \mathscr{P}$ such that:

(Eq.) The bundle projection π is *equivariant* in the sense that $\pi \circ \widehat{\Theta} = \tau \circ \pi$.

(Inv.) $\widehat{\Theta}$ is an *involution*, ie $\widehat{\Theta}^2(p) = p$ for all $p \in \mathscr{P}$.

($\widehat{R}$) The right $\mathbb{U}(m)$–action on the fibers and the homeomorphism $\widehat{\Theta}$ fulfill the condition

$$\widehat{\Theta}(R_u(p)) = R_{\bar{u}}(\widehat{\Theta}(p)), \quad \text{for all } p \in \mathscr{P} \text{ and } u \in \mathbb{U}(m),$$

where $R_u(p) = p \cdot u$ denotes the right $\mathbb{U}(m)$–action and $\bar{u}$ is the complex conjugate of u.

We say that $\mathscr{P}$ has a *"quaternionic" structure* if the structure group $\mathbb{U}(2m)$ has even rank and condition $(\widehat{R})$ is replaced by:

($\widehat{Q}$) The right $\mathbb{U}(2m)$–action on the fibers and the homeomorphism $\widehat{\Theta}$ fulfill the condition
$$\widehat{\Theta}(R_u(p)) = R_{\sigma(u)}(\widehat{\Theta}(p)), \quad \text{for all } p \in \mathscr{P} \text{ and } u \in \mathbb{U}(2m),$$
where $\sigma\colon \mathbb{U}(2m) \to \mathbb{U}(2m)$ is the involution given by
$$\sigma(u) := Q \cdot \bar{u} \cdot Q^{-1} = -Q \cdot \bar{u} \cdot Q$$
and Q is the matrix (2-2).

We will often refer to principal "real" and "quaternionic" bundles with the abbreviations principal R–bundles and principal Q–bundles, respectively.

Remark 3.3 Let us notice that both the "real" and the "quaternionic" case require that $\widehat{\Theta}$ has to be an involution. This means that both principal R– and Q–bundles are examples of $\mathbb{Z}_2$–*equivariant* principal bundles (indeed, properties (Eq.) and (Inv.) define these objects). This is indeed a difference with respect to the vector bundle case; cf Definition 2.2.

Morphisms (and isomorphisms) between principal R– and Q–bundles are defined in a natural way: if $(\mathscr{P}, \widehat{\Theta})$ and $(\mathscr{P}', \widehat{\Theta}')$ are two such principal bundles over the same involutive space (X, τ) then an R– or Q–morphism is a principal bundle morphism $f\colon \mathscr{P} \to \mathscr{P}'$ such that $f \circ \widehat{\Theta}' = \widehat{\Theta} \circ f$. We will use the symbols $\mathrm{Prin}_R^{\mathbb{U}(m)}(X, \tau)$ and $\mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau)$ for the sets of equivalence classes of principal "real" and "quaternionic" bundles over (X, τ), respectively. A principal R–bundle over (X, τ) is called *trivial* if it is isomorphic to the product bundle $X \times \mathbb{U}(m)$ with *trivial* R–structure $\widehat{\Theta}_0\colon (x, u) \mapsto (\tau(x), \bar{u})$. In much the same way, a *trivial* principal Q–bundle is isomorphic to the product bundle $X \times \mathbb{U}(2m)$ endowed with the *trivial* Q–structure $\widehat{\Theta}_0\colon (x, u) \mapsto (\tau(x), \sigma(u))$.

A standard result says that there is an equivalence of categories between principal $\mathbb{U}(m)$–bundles and complex vector bundles. This equivalence is realized by the *associated bundle* construction along its inverse, called *orthonormal frame bundle* construction; see [9, Appendix B] for more details. A similar result extends to the "real" and the "quaternionic" categories [9, Proposition 2.4] leading to

$$\mathrm{Prin}_R^{\mathbb{U}(m)}(X, \tau) \simeq \mathrm{Vec}_R^m(X, \tau), \quad \mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau) \simeq \mathrm{Vec}_Q^{2m}(X, \tau). \tag{3-1}$$

We can take advantage of the above isomorphisms to carry the notion of FKMM–invariant from vector bundles to principal bundles.

Definition 3.4 (FKMM–invariant: principal bundle version) Let $(\mathscr{P}, \widehat{\Theta})$ be a rank $2m$ principal Q–bundle over the involutive space (X, τ). Let $[(\mathscr{E}, \Theta)] \in \mathrm{Vec}_Q^{2m}(X, \tau)$ be the unique class associated with $[(\mathscr{P}, \widehat{\Theta})] \in \mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau)$ by the isomorphism (3-1). One defines the *FKMM–invariant* of $(\mathscr{P}, \widehat{\Theta})$ as the FKMM–invariant of the associated Q–bundle $(\mathscr{E}, \Theta)$, namely

$$\kappa(\mathscr{P}, \widehat{\Theta}) := \kappa(\mathscr{E}, \Theta).$$

Remark 3.5 Let us briefly discuss the consistency of Definition 3.4 with the construction of the FKMM–invariant presented in [10]. In view of the isomorphisms (3-1) to each $\mathbb{U}(2m)$ principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$, one can associate a unique (up to isomorphism) $\mathbb{U}(1)$ principal R–bundle $(\det(\mathscr{P}), \det(\widehat{\Theta}))$ which is defined as the unique (up to isomorphism) $\mathbb{U}(1)$ principal R–bundle associated with the rank one R–bundle $(\det(\mathscr{E}), \det(\Theta))$. Moreover, there is a one-to-one correspondence between sections of a $\mathbb{U}(1)$ principal R–bundle and sections of a rank one R–bundle of unit norm. Then, the quantity $\kappa(\mathscr{P}, \widehat{\Theta})$ turns out to be determined by the equivalence class of the pair $(\det(\mathscr{P}), s_{\mathscr{P}})$ where $s_{(\mathscr{P},\widehat{\Theta})} \equiv s_{(\mathscr{E},\Theta)}$ is the canonical section associated to $(\mathscr{E}, \Theta)$. For more details about the relation between the FKMM–invariant and the canonical section we refer to [8, Section 3.2] or [10, Section 2.2].

3.2 "Quaternionic" connections and curvatures

Connections with "quaternionic" and "real" structures have been studied in Section 2.2 of [9]. We review here the basic definitions and the main properties of these objects. For a reminder about the theory of connections we refer to the classic monographs [33; 34]; see also [9, Appendix B].

We consider principal bundles in the smooth category $\pi: \mathscr{P} \to X$ endowed with a "real" or "quaternionic" structure $\widehat{\Theta}: \mathscr{P} \to \mathscr{P}$ over the involutive space (X, τ). The structure group is $\mathbb{U}(m)$ (m even in the "quaternionic" case) and $\mathfrak{u}(m)$ is the related Lie algebra. The symbol $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ will be used for the *connection* 1–forms associated to given horizontal distributions $p \mapsto H_p$ of $\mathscr{P}$. We observe that the Lie algebra $\mathfrak{u}(m)$ has two natural involutions: a *real* involution $\mathfrak{u}(m) \ni \xi \mapsto \bar{\xi} \in \mathfrak{u}(m)$ and a *quaternionic* involution $\mathfrak{u}(2m) \ni \xi \mapsto \sigma(\xi) := -Q \cdot \bar{\xi} \cdot Q \in \mathfrak{u}(2m)$. Here $\xi \in \mathfrak{u}(m)$ is any anti-Hermitian matrix of size m and the matrix Q was defined in (2-2). Finally, given a k–form $\phi \in \Omega^k(\mathscr{P}, \mathscr{A})$ with values in some structure $\mathscr{A}$ (module, ring, algebra, group, etc) and a smooth map $f: \mathscr{P} \to \mathscr{P}$, we denote by $f^*\phi := \phi \circ f_*$ the *pullback* of ϕ with respect to the map f (and $f_*: T\mathscr{P} \to T\mathscr{P}$ is the differential, or *pushforward*, of vector fields). Given a $\mathfrak{u}(m)$–valued k–form $\phi \in \Omega^k(\mathscr{P}, \mathfrak{u}(m))$, we define the *complex conjugate* form $\bar{\phi}$ pointwise, ie $\bar{\phi}_p(\mathrm{w}_p^1, \dots, \mathrm{w}_p^k) := \overline{\phi_p(\mathrm{w}_p^1, \dots, \mathrm{w}_p^k)}$ for every k–tuple $\{\mathrm{w}_p^1, \dots, \mathrm{w}_p^k\}$ of tangent vectors at $p \in \mathscr{P}$. It follows that $f^*\bar{\phi} = \overline{f^*\phi}$ for every smooth map $f : \mathscr{P} \to \mathscr{P}$. Similarly, if $\phi \in \Omega^k(\mathscr{P}, \mathfrak{u}(2m))$, we define $\sigma(\phi)$ pointwise by $\sigma(\phi)_p(\mathrm{w}_p^1, \dots, \mathrm{w}_p^k) := -Q \cdot \overline{\phi_p(\mathrm{w}_p^1, \dots, \mathrm{w}_p^k)} \cdot Q$. Hence, one has that $\sigma(f^*\phi) = f^*\sigma(\phi)$.

Definition 3.6 ("real" and "quaternionic" equivariant connections) Let (X, τ) be an involutive space that satisfies Assumption 3.1 and $\pi : \mathscr{P} \to X$ a smooth principal $\mathbb{U}(m)$–bundle over X endowed with a "real" or a "quaternionic" structure $\widehat{\Theta} : \mathscr{P} \to \mathscr{P}$ as in Definition 3.2. A connection 1–form $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ is said to be *equivariant* if $\bar{\omega} = \widehat{\Theta}^*\omega$ in the "real" case or $\sigma(\omega) = \widehat{\Theta}^*\omega$ in the "quaternionic" case. Equivariant connections in the "real" case are called "real" connections (or R–connections). Similarly, the "quaternionic" connections (or Q–connections) are the equivariant connections in the "quaternionic" category.

Let $\mathfrak{A}_R(\mathscr{P}) \subset \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ be the space of R–connections on the principal R–bundle $(\mathscr{P}, \widehat{\Theta})$. Similarly, $\mathfrak{A}_Q(\mathscr{P}) \subset \Omega^1(\mathscr{P}, \mathfrak{u}(2m))$ will denote the space of Q–connections on the principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$. Let us introduce the sets of equivariant 1–forms

$$
\begin{aligned}
\Omega^1_R(\mathscr{P}, \mathfrak{u}(m)) &:= \{\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m)) \mid \bar{\omega} = \widehat{\Theta}^*\omega\}, \\
\Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m)) &:= \{\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(2m)) \mid \sigma(\omega) = \widehat{\Theta}^*\omega\}.
\end{aligned}
\tag{3-2}
$$

A 1–form is called *horizontal* if it vanishes on vertical vectors. The set of $\mathfrak{u}(m)$–valued 1–forms on $\mathscr{P}$ which are horizontal and which transform according to the adjoint representation of the structure group is denoted by $\Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(m), \mathrm{Ad})$. Let us introduce the sets

$$
\begin{aligned}
\mathcal{V}^1_R(\mathscr{P}) &:= \Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(m), \mathrm{Ad}) \cap \Omega^1_R(\mathscr{P}, \mathfrak{u}(m)), \\
\mathcal{V}^1_Q(\mathscr{P}) &:= \Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(2m), \mathrm{Ad}) \cap \Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m)).
\end{aligned}
$$

Proposition 3.7 [9, Propositions 2.11 and 2.12] *The sets $\mathfrak{A}_R(\mathscr{P})$ and $\mathfrak{A}_Q(\mathscr{P})$ are nonempty and are closed under convex combinations with real coefficients. Moreover, they are affine spaces modeled on the vector spaces $\mathcal{V}^1_R(\mathscr{P})$ and $\mathcal{V}^1_Q(\mathscr{P})$, respectively.*

Let F_ω be the curvature associated to the equivariant connection ω by the *structural equation*

$$F_\omega := \mathrm{d}\omega + \tfrac{1}{2}[\omega \wedge \omega].$$

According to [9, Proposition 2.22] one has that F_ω obeys the equivariant constraints

$$\begin{aligned} \overline{F}_\omega &= \widehat{\Theta}^* F_\omega \quad \text{(“real” case)}, \\ \sigma(F_\omega) &= \widehat{\Theta}^* F_\omega \quad \text{(“quaternionic” case)}. \end{aligned} \tag{3-3}$$

Let $\{\mathcal{F}_\alpha \in \Omega^2(\mathcal{U}_\alpha, \mathfrak{g})\}$ be the collection of local 2–forms which provides the local description of the curvature F_ω — in the sense of [9, Theorem C.2]. When ω is equivariant, it holds true that

$$\begin{aligned} \overline{\mathcal{F}}_\alpha &= \tau^* \mathcal{F}_\alpha \quad \text{(“real” case)}, \\ \sigma(\mathcal{F}_\alpha) &= \tau^* \mathcal{F}_\alpha \quad \text{(“quaternionic” case)}. \end{aligned} \tag{3-4}$$

3.3 Chern–Simons form and “quaternionic” structure

In this section we discuss some aspects of Chern–Simons theory defined over (compact) manifolds without boundary in the presence of a Q–structure. For a comprehensive introduction to Chern–Simons theory we refer to [15; 28].

Let $\pi : \mathscr{P} \to X$ be a (smooth) principal $\mathbb{U}(m)$–bundle and $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ a connection 1–form. The *Chern–Simons* 3*–form* $\mathcal{CS}(\omega) \in \Omega^3(\mathscr{P})$ associated to ω is defined by

$$\mathcal{CS}(\omega) := \frac{1}{8\pi^2} \operatorname{Tr}\Big(\omega \wedge \mathrm{d}\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega\Big), \tag{3-5}$$

where Tr is the usual trace on $m \times m$ matrices. The 3–form $\mathcal{CS}(\omega)$ is sometimes called *Chern–Simons Lagrangian*. A direct computation shows that the result of applying the exterior differential to $\mathcal{CS}(\omega)$ can be expressed in terms of the curvature $F_\omega \in \Omega^2(\mathscr{P}, \mathfrak{u}(m))$ according to

$$\mathrm{d}\mathcal{CS}(\omega) := \frac{1}{4\pi^2} \operatorname{Tr}(F_\omega \wedge F_\omega) \in \Omega^4(\mathscr{P}). \tag{3-6}$$

The following result will be used several times in the continuation of this work.

Lemma 3.8 *Assume that $\pi : \mathscr{P} \to X$ admits a (smooth) section $s : X \to \mathscr{P}$ and let $g : X \to \mathbb{U}(m)$ be a (smooth) map. Define a new section $s_g : X \to \mathscr{P}$ using the right action of $\mathbb{U}(m)$, that is, $s_g(x) := s(x) \cdot g(x)$. Then the two pullbacks $s_g^*\mathcal{CS}(\omega), s^*\mathcal{CS}(\omega) \in \Omega^3(X)$ are related by the equation*

$$s_g^*\mathcal{CS}(\omega) = s^*\mathcal{CS}(\omega) + \frac{1}{8\pi^2} \mathrm{d}\operatorname{Tr}(s^*\omega \wedge \mathrm{d}g^{-1} g) + \Lambda(g), \tag{3-7}$$

where $\Lambda(g) \in \Omega^3(X)$ is given by

$$\Lambda(g) := -\frac{1}{24\pi^2} \operatorname{Tr}((g^{-1}\mathrm{d}g)^{\wedge 3}). \tag{3-8}$$

Proof The proof is essentially a computation which is based on the two relations $s_g^*\mathcal{CS}(\omega) = \mathcal{CS}(s_g^*\omega)$ and $s_g^*\omega = g^{-1}(s^*\omega)g + g^{-1}\mathrm{d}g$. Therefore, by exploiting the cyclicity of the trace, one can check that

$$\begin{aligned}&\mathcal{CS}(g^{-1}(s^*\omega)g + g^{-1}\mathrm{d}g)\\ &\qquad = \mathcal{CS}(s^*\omega) - \frac{1}{8\pi^2}\,\mathrm{d}\mathrm{Tr}(s^*\omega \wedge g^{-1}\mathrm{d}g) - \frac{1}{24\pi^2}\,\mathrm{Tr}((g^{-1}\mathrm{d}g)^{\wedge 3}).\end{aligned}$$

The identity $0 = \mathrm{d}(g^{-1}g) = \mathrm{d}g^{-1}g + g^{-1}\mathrm{d}g$ concludes the computation. □

Definition 3.9 (Chern–Simons invariant) Let X be a compact oriented 3–dimensional manifold without boundary and $\pi : \mathscr{P} \to X$ a principal $\mathbb{U}(m)$–bundle equipped with a connection ω. Assume that there is a global section $s : X \to \mathscr{P}$. Then the quantity

$$\mathfrak{cs}(\omega) := \int_X s^*\mathcal{CS}(\omega) \mod \mathbb{Z}$$

is called the *Chern–Simons invariant* $\mathfrak{cs}(\omega) \in \mathbb{R}/\mathbb{Z}$ associated to ω.

The following result shows that the Chern–Simons invariant is well defined.

Proposition 3.10 *The Chern–Simons invariant does not dependent on the choice of a particular global section $s : X \to \mathscr{P}$, and depends only on the equivalence class of ω up to gauge transformations.*

Proof Two global sections of s_1 and s_2 of $\mathscr{P}$ are related by a unique map $g : X \to \mathbb{U}(m)$ such that $s_2(x) = s_1(x) \cdot g(x)$. Lemma 3.8, Stokes' theorem and the fact that X has no boundary imply

$$\int_X (s_1^*\mathcal{CS}(\omega) - s_2^*\mathcal{CS}(\omega)) = \int_X \Lambda(g) =: N_g \in \mathbb{Z},$$

where the integer N_g defines the "degree" of the map g. With a similar argument one can show that $\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega')$ if ω and ω' are related by the transformation induced by an element of the gauge group. □

When a principal $\mathbb{U}(2m)$–bundle $\pi : \mathscr{P} \to X$ is endowed with a Q–structure $\widehat{\Theta}$, it is natural to use an equivariant Q–connection $\omega \in \mathfrak{A}_Q(\mathscr{P})$ to define the Chern–Simons 3–form $\mathcal{CS}(\omega)$. The Q–structure $\widehat{\Theta}$ induces a symmetry of $\mathcal{CS}(\omega)$.

Lemma 3.11 *Let $(\mathscr{P}, \widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X, τ) which satisfies Assumption 3.1. Let $\omega \in \mathfrak{A}_Q(\mathscr{P})$ be an equivariant connection and $\mathcal{CS}(\omega) \in \Omega^3(\mathscr{P})$ the associated Chern–Simons 3–form. Then*

$$\widehat{\Theta}^*\mathcal{CS}(\omega) = \mathcal{CS}(\omega).$$

Proof The equivariance of ω means that $\widehat{\Theta}^*\omega = Q\bar{\omega}Q^{-1} = -Q({}^t\omega)Q^{-1}$, where we used $\bar{\omega} = -{}^t\omega$ since the form ω takes value in the Lie algebra $\mathfrak{u}(2m)$. The cyclicity of the trace provides

$$\widehat{\Theta}^*\mathcal{CS}(\omega) = \mathcal{CS}(\widehat{\Theta}^*\omega) = \frac{1}{8\pi^2}\,\mathrm{Tr}\Big({}^t\omega \wedge \mathrm{d}^t\omega + \frac{2}{3}\,{}^t\omega \wedge {}^t\omega \wedge {}^t\omega\Big).$$

The identity ${}^t\omega_1 \wedge {}^t\omega_2 = (-1)^{q_1q_2\,t}(\omega_2 \wedge \omega_1)$ is valid for each pair $\omega_1 \in \Omega^{q_1}(\mathscr{P}, \mathfrak{u}(2m))$ and $\omega_2 \in \Omega^{q_2}(\mathscr{P}, \mathfrak{u}(2m))$ and the invariance of the trace under the operation of taking the *transpose* imply

$$\begin{aligned}\widehat{\Theta}^*\mathcal{CS}(\omega) &= \frac{1}{8\pi^2}\,\mathrm{Tr}\Big(\mathrm{d}\omega \wedge \omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\Big)\\ &= \mathcal{CS}(\omega) + \frac{1}{8\pi^2}\,\mathrm{Tr}(\mathrm{d}\omega \wedge \omega - \omega \wedge \mathrm{d}\omega)\\ &= \mathcal{CS}(\omega) + \frac{1}{8\pi^2}\,\mathrm{d}\mathrm{Tr}(\omega \wedge \omega).\end{aligned}$$

To conclude the proof it is enough to observe that $\mathrm{Tr}\,(\omega \wedge \omega) = 0$ due to the anticommutation relation of 1–forms. □

The invariance of $\mathcal{CS}(\omega)$ expressed in Lemma 3.11 has an important implication on the Chern–Simons invariant in low dimensions, provided that certain conditions are met.

Proposition 3.12 *Let $(\mathscr{P}, \widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X, τ) which satisfies Assumption 3.1. Assume in addition that*

(a) *X has dimension $d = 3$ and τ reverses the orientation of X;*

(b) *there is a global section $s\colon X \to \mathscr{P}$ (not necessarily equivariant).*

Then

(i) *if $\omega \in \mathfrak{A}_Q(\mathscr{P})$ is an equivariant connection then the associated Chern–Simons invariant $\mathfrak{cs}(\omega)$ takes values in the set $\{0, \frac{1}{2}\}$;*

(ii) *$\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega')$ for each pair of equivariant connections $\omega, \omega' \in \mathfrak{A}_Q(\mathscr{P})$;*

(iii) *if $(\mathscr{P}, \widehat{\Theta})$ admits a global equivariant section then $\mathfrak{cs}(\omega) = 0$ for any $\omega \in \mathfrak{A}_Q(\mathscr{P})$.*

Proof (i) Let $s\colon X \to \mathscr{P}$ be a global section. Since $\tau_\Theta(s) := \widehat{\Theta} \circ s \circ \tau$ generally differs from s, there is a (unique) map $g\colon X \to \mathbb{U}(2m)$ such that $\tau_\Theta(s) = s \cdot g$. Then

$$\tau^*(s^*\mathcal{CS}(\omega)) = (s \circ \tau)^*\mathcal{CS}(\omega) = (s \cdot g)^*(\widehat{\Theta}^*\mathcal{CS}(\omega)) = (s \cdot g)^*\mathcal{CS}(\omega),$$

where in the last equality we used the result of Lemma 3.11. By exploiting the fact that τ reverses the orientation of X, one has

$$\int_X s^*\mathcal{CS}(\omega) = -\int_X \tau^*(s^*\mathcal{CS}(\omega)) = -\int_X (s\cdot g)^*\mathcal{CS}(\omega) = -\int_X s^*\mathcal{CS}(\omega) + N_g,$$

where $N_g := \int_X \Lambda(g) \in \mathbb{Z}$. This implies that $2\mathfrak{cs}(\omega) = 0$, ie $\mathfrak{cs}(\omega) \in \{0, \frac{1}{2}\}$.

(ii) Let ω' be a second equivariant connection and consider the map

$$[0,1] \ni t \mapsto \omega_t := (1-t)\omega + t\omega' \in \mathfrak{A}_Q(\mathscr{P}).$$

Clearly $\mathfrak{cs}(\omega_t)$ is a polynomial (hence continuous) function in t. On the other hand $\mathfrak{cs}(\omega_t) \in \{0, \frac{1}{2}\}$ since ω_t is equivariant. This implies that $\mathfrak{cs}(\omega_{t_1}) = \mathfrak{cs}(\omega_{t_2})$ for all $t_1, t_2 \in [0,1]$ and in particular $\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega')$.

(iii) If s is a global equivariant section, one has

$$\tau^*(s^*\mathcal{CS}(\omega)) = \tau^*(s^*\mathcal{CS}(\omega)) = \tau^*(s^*(\widehat{\Theta}^*\mathcal{CS}(\omega))) = \tau_\Theta(s)^*\mathcal{CS}(\omega) = s^*\mathcal{CS}(\omega).$$

Hence,

$$\int_X s^*\mathcal{CS}(\omega) = \int_X \tau^*(s^*\mathcal{CS}(\omega)) = -\int_X s^*\mathcal{CS}(\omega),$$

which implies $\int_X s^*\mathcal{CS}(\omega) = 0$. □

Remark 3.13 Due to the low-dimensional assumption (a) in Proposition 3.12, the assumption (b) about the existence of a global section is completely equivalent to the condition of vanishing of the first Chern class of the principal bundle. This condition is guaranteed by the stronger requirements: (1) $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$, or (2) $H^2(X, \mathbb{Z}) = 0$.

The following definition is justified by item (ii) of Proposition 3.12.

Definition 3.14 (intrinsic Chern–Simons invariant) Let $(\mathscr{P}, \widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X, τ) such that X has dimension $d = 3$, τ reverses the orientation of X and $\mathscr{P}$ admits a global section. Then the quantity

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) := \mathfrak{cs}(\omega) \quad \text{for some } \omega \in \mathfrak{A}_Q(\mathscr{P})$$

does not depend on the choice of $\omega \in \mathfrak{A}_Q(\mathscr{P})$ and defines an *intrinsic* (Chern–Simons) invariant for $(\mathscr{P}, \widehat{\Theta})$.

Remark 3.15 (a formula for the Chern–Simons invariant) Let (X,τ) be a three-dimensional involutive manifold satisfying the assumption $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$. As a consequence of Lemma 2.14 and the isomorphism (3-1), any $\mathbb{U}(2m)$ Q–bundle $(\mathscr{P},\widehat{\Theta})$ over (X,τ) can be represented by a smooth map $\xi\colon X\to\mathbb{U}(2m)$ such that $\tau^*\xi=-Q\bar{\xi}^{-1}Q$. The average construction applied to the trivial connection on the product bundle [9, Example 2.15] gives an equivariant connection ω, whose pullback under the trivial section s is $s^*\omega=\frac{1}{2}\sigma(\xi^{-1}d\xi)$. We then have $s^*\mathcal{CS}(\omega)=\frac{1}{2}\Lambda(\xi)$, and hence the formula

$$\mathfrak{cs}(\mathscr{P},\widehat{\Theta})=\frac{1}{2}\int_X\Lambda(\xi)\mod\mathbb{Z}.$$

This formula can be compared with [16, Proposition 11.21].

3.4 Wess–Zumino term in absence of boundaries

In the last section we described the Chern–Simons invariant in the case of three-dimensional base manifolds without boundary. In the case of manifolds with boundary the Chern–Simons invariant itself depends on the choice of a section while the difference of the values of the Chern–Simons invariants depends only on the topological information on the boundary. This information is detected by the so-called *Wess–Zumino term*. The latter is a topological quantity initially defined in the context of certain two-dimensional conformal field theories known as Wess–Zumino–Witten models. An excellent introduction to the theory of Wess–Zumino–Witten models is provided by the lecture notes [20]. The presentation of the properties of the Wess–Zumino term given here follows mainly [15].

Definition 3.16 (Wess–Zumino term) Let Σ be a compact oriented manifold of dimension $d=2$ without boundary. For any map $\xi\colon\Sigma\to\mathbb{SU}(2)$, the *Wess–Zumino term* $\mathcal{WZ}_\Sigma(\xi)\in\mathbb{R}/\mathbb{Z}$ is defined by

$$\mathcal{WZ}_\Sigma(\xi):=\int_{X_\Sigma}\Lambda(\tilde{\xi})\mod\mathbb{Z},$$

where

$$\Lambda(\tilde{\xi}):=-\frac{1}{24\pi^2}\operatorname{Tr}(\tilde{\xi}^{-1}\mathrm{d}\tilde{\xi})^3$$

according to the notation (3-8), X_Σ is any compact three-dimensional oriented manifold whose boundary coincides with Σ, ie $\partial X_\Sigma=\Sigma$, and $\tilde{\xi}\colon X_\Sigma\to\mathbb{SU}(2)$ is any extension of ξ.

Notice that the extended manifold X_Σ and the extended section $\tilde{\xi}$ in Definition 3.16 always exist. The existence of X_Σ follows from the vanishing of the second *bordism group*,[1] $\Omega_2 = 0$ [40, Section 7]. The existence of $\tilde{\xi}$ is due to $\pi_k(\mathbb{SU}(2)) = 0$ for $k = 0, 1, 2$ plus a standard application of the Oka's (type) principle to pass from continuous sections to smooth sections. Finally, the condition $\xi\colon \Sigma \to \mathbb{SU}(2)$ can be relaxed by asking that the section $\xi\colon \Sigma \to \mathbb{U}(2)$ possesses a determinant section $\det(\xi)\colon \Sigma \to \mathbb{U}(1)$ which is nullhomotopic.

The well-posedness of Definition 3.16 is justified in the following result.

Lemma 3.17 (Polyakov–Wiegmann formula) *The Wess–Zumino term is independent of the choice of the extensions X_Σ and $\tilde{\xi}$. Moreover, for every pair of sections $\xi_j : \Sigma \to \mathbb{SU}(2)$, $j = 1, 2$, the* ***Polyakov–Wiegmann formula***

$$\mathcal{WZ}_\Sigma(\xi_1\xi_2) = \mathcal{WZ}_\Sigma(\xi_1) + \mathcal{WZ}_\Sigma(\xi_2) + \frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(\xi_1^{-1}\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2\xi_2^{-1})$$

holds in $\mathbb{R}/\mathbb{Z}$.

Proof Given Σ and $\xi\colon \Sigma \to \mathbb{SU}(2)$ as in Definition 3.16, consider two extended manifolds X_Σ and X'_Σ such that $\partial X_\Sigma = \Sigma = \partial X'_\Sigma$, and two extended sections $\tilde{\xi}$ and $\tilde{\xi}'$ such that $\tilde{\xi}|_\Sigma = \xi = \tilde{\xi}'|_\Sigma$. By reversing the orientation of X'_Σ and then gluing it with X_Σ along Σ one obtains a compact oriented three-dimensional manifold $X := (-X'_\Sigma) \sqcup X_\Sigma$, where the minus sign indicates the reversal of the orientation. Similarly, $\tilde{\xi}$ and $\tilde{\xi}'$ can be glued together to define a section $\xi_X := (\tilde{\xi} \sqcup \tilde{\xi}')\colon X \to \mathbb{SU}(2)$. It is well known that

$$\int_X \Lambda\,(\xi_M) = -\frac{1}{24\pi^2}\int_X \mathrm{Tr}\left(\xi_X^{-1}\mathrm{d}\xi_X\right)^{\wedge 3} \in \mathbb{Z}.$$

On the other hand, one has that

$$\int_X \Lambda(\xi_X) = \int_{X_\Sigma} \Lambda(\tilde{\xi}) - \int_{X'_\Sigma} \Lambda(\tilde{\xi}') \in \mathbb{Z},$$

where the minus sign is justified by the inversion of the orientation. Thus, since the Wess–Zumino term $\mathcal{WZ}_\Sigma(\xi)$ is defined modulo an integer, it can be computed equivalently through the pair $X_\Sigma, \tilde{\xi}$ or the pair $X'_\Sigma, \tilde{\xi}'$. The Polyakov–Wiegmann formula for $\mathcal{WZ}_\Sigma(\xi_1\xi_2)$ follows from an explicit computation. By taking extensions of ξ_1 and ξ_2 one computes $\Lambda(\xi_1\xi_2) - \Lambda(\xi_1) - \Lambda(\xi_2)$ directly. Then, integration over

[1]The existence of X_Σ can be also justified by observing that closed oriented two-dimensional manifolds are classified by the genus, and a genus g surface is always the boundary of a three-dimensional manifold. For instance the sphere $\mathbb{S}^2$ is the boundary of the three-dimensional disk $\mathbb{D}^3$. Similarly the torus $\mathbb{T}^2$ is the boundary of the manifold $\mathbb{S}^1 \times \mathbb{D}^2$. The same occurs for higher genus surfaces.

X_Σ and an application of Stokes' theorem to obtain the integral on the boundary Σ provide the final result. □

From formula (3-7) and Stokes' theorem one immediately deduces the following result:

Lemma 3.18 *Let X be a compact oriented manifold of dimension $d = 3$ with nonempty boundary $\Sigma := \partial X$. Let $\pi : \mathscr{P} \to X$ be a principal $\mathbb{U}(2)$–bundle equipped with a connection ω and a global (smooth) section $s\colon X \to \mathscr{P}$. Let $g\colon X \to \mathbb{U}(2)$ be any (smooth) map such that $\det(g)\colon X \to \mathbb{U}(1)$ is nullhomotopic. Then*

$$\int_X s_g^*\mathcal{CS}(\omega) - \int_X s^*\mathcal{CS}(\omega) = -\frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(s^*\omega \wedge \mathrm{d}g^{-1}g) + \mathcal{WZ}_\Sigma\,(g|_\Sigma) \mod \mathbb{Z}.$$

3.5 Wess–Zumino term in presence of boundaries

In the rest of this work we will be interested in calculating the Wess–Zumino term through "cutting and pasting". To set up the machinery, we need to extend the definition of the Wess–Zumino term for two-dimensional manifolds with boundary. To do that, let us observe that associated to a compact oriented one-dimensional manifold S without boundary (a union of circles), there exists a Hermitian line bundle $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$. The specific structure of this line bundle will be not used in this work and for this reason the details of the construction of $\mathscr{L}_S$ will be only sketched. The interested reader can refer to [15, Appendix A] or to [35, Section 1.3] for a more rigorous presentation.

Given S, consider a two-dimensional manifold D_S (a disjoint union of disks) with boundary $\partial D_S = S$ along with the space $\mathrm{Map}(D_S, \mathbb{SU}(2))$. Given an element $\tilde{\gamma}$ in $\mathrm{Map}(D_S, \mathbb{SU}(2))$, its restriction $\gamma := \tilde{\gamma}|_S$ defines an element in $\mathrm{Map}(S, \mathbb{SU}(2))$. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathrm{Map}(D_S, \mathbb{SU}(2))$ be two maps which agree on the boundary S, namely such that $\gamma_1 = \gamma_2$. Such two maps can be glued together to produce a map $\xi_{(1,2)} := \tilde{\gamma}_1 \sqcup \tilde{\gamma}_2$ on the two-dimensional manifold without boundary $\Sigma_S := (-D_S) \sqcup D_S$ obtained by gluing two copies of D_S (with opposite orientation) along the common boundary. As a consequence the quantity $\mathcal{WZ}_{\Sigma_S}(\xi_{(1,2)})$ turns out to be well defined according to Definition 3.16. Consider now the space

$$\mathscr{L}_S := (\mathrm{Map}(D_S, \mathbb{SU}(2)) \times \mathbb{C})/\sim,$$

where the equivalence relation $\sim$ is defined as follows: let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathrm{Map}(D_S, \mathbb{SU}(2))$ and $z_1, z_2 \in \mathbb{C}$; then

$$(\tilde{\gamma}_1, z_1) \sim (\tilde{\gamma}_2, z_2) \iff \gamma_1 = \gamma_2,\ z_1 = z_2 \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_S}(\xi_{(1,2)})}.$$

The space $\mathscr{L}_S$ defined in this way turns out to be the total space of a complex line bundle over $\mathrm{Map}(S, \mathbb{SU}(2))$ with projection $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ given by

$$\mu\colon [\tilde{\gamma}, z] \mapsto \gamma := \tilde{\gamma}|_S,$$

where $\gamma := \tilde{\gamma}|_S$ is independent of the choice of the representative by construction.

Henceforth, only the following properties of the line bundle $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ will be relevant [15, Proposition A.1]:

(i) For $\gamma_1, \gamma_2 \in \mathrm{Map}(S, \mathbb{SU}(2))$ let $\gamma_1\gamma_2 \in \mathrm{Map}(S, \mathbb{SU}(2))$ defined by pointwise multiplication. Then, there is an isometry

$$\mu^{-1}(\gamma_1) \otimes \mu^{-1}(\gamma_2) \to \mu^{-1}(\gamma_1\gamma_2). \tag{3-9}$$

(ii) The *product of fibers* (3-9) defined by the isometry is associative.

(iii) If $\gamma_0 \in \mathrm{Map}(S, \mathbb{SU}(2))$ is the constant map, there is a trivialization $\mu^{-1}(\gamma_0) \simeq \mathbb{C}$ which respects (3-9).

All the ingredients are now available for extending Definition 3.16 to manifolds with boundary.

Definition 3.19 (Wess–Zumino term with boundary) Let Σ be a compact oriented manifold of dimension $d = 2$ with one-dimensional (compact and oriented) boundary $S := \partial\Sigma$. Let $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ be the associated line bundle. Every $\xi\colon \Sigma \to \mathbb{SU}(2)$ gives rise to a point $\xi|_S \in \mathrm{Map}(S, \mathbb{SU}(2))$ and an associated fiber $\mu^{-1}(\xi|_S) \subset \mathscr{L}_S$. Let D_S be a disjoint union of disks (contractible two-dimensional manifolds) with boundary $\partial D_S = S = \partial\Sigma$. Given any $\zeta_{D_S}\colon D_S \to \mathbb{SU}(2)$ such that $\zeta_{D_S}|_S = \xi|_S$, let $\xi \sqcup \zeta_{D_S}$ be the map defined on the closed manifold $\Sigma_D := \Sigma \sqcup (-D_S)$ by the gluing of the functions ζ_{D_S} and ξ along the common boundary S. The Wess–Zumino term $\mathcal{WZ}_\Sigma(\xi)$ is then defined by the equation

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} := [\zeta_{D_S}, \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_D}(\xi \sqcup \zeta_{D_S})}] \in \mu^{-1}(\xi|_S).$$

To introduce the next result it is worth mentioning that given a complex vector bundle $\mathscr{E} \to X$, its *conjugate* $\overline{\mathscr{E}} \to X$ is the complex vector bundle whose underlying total space agrees with $\mathscr{E}$ as a set, but with inverted complex structure with respect to the multiplication by scalars $z \in \mathbb{C}$. If $\mathscr{E}$ is endowed with a Hermitian metric, then so is $\overline{\mathscr{E}}$. This allows the identification of $\overline{\mathscr{E}}$ with the *dual* vector bundle $\mathscr{E}^*$.

Proposition 3.20 (orientation) (i) *Let S be a compact oriented one-dimensional manifold without boundary, and $-S$ the same manifold with reversed orientation. Then there exists a natural isometric isomorphism*

$$\mathscr{L}_{-S} \simeq \overline{\mathscr{L}}_S.$$

(ii) *Let Σ be a compact oriented two-dimensional manifold with boundary, and $-\Sigma$ the same manifold with reversed orientation. Then for any $\xi\colon \Sigma \to \mathbb{SU}(2)$,*

$$\mathcal{WZ}_{-\Sigma}(\xi) = -\mathcal{WZ}_{\Sigma}(\xi).$$

Property (i) of Proposition 3.20 is a direct consequence of the construction of the space $\mathscr{L}_S$. Property (ii) follows from Definition 3.19 under the isometry described in (i).

Remark 3.21 (central extension of the loop group) Definition 3.19 will be mainly applied to two-dimensional manifolds Σ such that $\partial\Sigma \simeq \mathbb{S}^1$. In this case we will write $\mathscr{L}_{\mathbb{S}^1}$ instead of $\mathscr{L}_{\partial\Sigma}$. The set $\mathrm{Map}(\mathbb{S}^1, \mathbb{SU}(2))$ endowed with the pointwise multiplication is known as the *loop group* of $\mathbb{SU}(2)$ [43], and will be denoted here by $\mathrm{Loop}_{\mathbb{SU}(2)}$. The total space $S(\mathscr{L}_{\mathbb{S}^1})$ of the principal $\mathbb{U}(1)$–bundle (also known as *circle bundle*) associated to $\mathscr{L}_{\mathbb{S}^1}$ inherits a group structure from the product of fibers (3-9). This gives rise to a central extension of $\mathrm{Loop}_{\mathbb{SU}(2)}$,

$$1 \to \mathbb{U}(1) \to S(\mathscr{L}_{\mathbb{S}^1}) \to \mathrm{Loop}_{\mathbb{SU}(2)} \to 1.$$

Let $\xi_0\colon \Sigma \to \mathbb{SU}(2)$ be the constant map with value the identity matrix $\mathbb{1}_{\mathbb{C}^2} \in \mathbb{SU}(2)$. By definition of the product of fibers (3-9) one has that $[\xi_0, \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_D}(\xi_0 \sqcup \xi_0)}]$ acts as the unit of the group $S(\mathscr{L}_{\mathbb{S}^1})$. Therefore, by invoking Definition 3.19 one obtains that $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma}(\xi_0)} \in \mathscr{L}_{\mathbb{S}^1}$ provides the unit of the central extension $S(\mathscr{L}_{\mathbb{S}^1})$. For a more complete description of this central extension the reader is referred to [15; 35; 43].

The link between Definitions 3.16 and 3.19 is provided by the following result.

Proposition 3.22 (gluing property) *Let Σ be a compact oriented two-dimensional manifold without boundary. Assume that Σ can be cut along an embedded circle $\mathbb{S}^1$ to get two compact oriented two-dimensional manifolds Σ_1 and Σ_2 such that $\partial\Sigma_1 \simeq -\mathbb{S}^1$ and $\partial\Sigma_2 \simeq \mathbb{S}^1$ in such a way that $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Then for any $\xi\colon \Sigma \to \mathbb{SU}(2)$,*

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma}(\xi)} = \langle \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1})}; \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_1})} \rangle, \tag{3-10}$$

where $\langle\,\cdot\,;\,\cdot\,\rangle$ denotes the contraction between

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1})} \in \mathscr{L}_{\mathbb{S}^1} \quad \textit{and} \quad \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_2})} \in \mathscr{L}^*_{\mathbb{S}^1}.$$

Equation (3-10) can be reformulated in the suggestive formula

$$\mathcal{WZ}_{\Sigma}(\xi) = \mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1}) - \mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_2}) \mod \mathbb{Z}.$$

A proof of a generalized version of Proposition 3.22 can be found in [35, Section 1.3].

Although simplified, the version of the gluing property described in Proposition 3.22 is sufficient for the purposes of this work. Indeed, the gluing property will be mainly applied to the situation described below.

Remark 3.23 Let Σ_1 and Σ_2 be compact oriented two-dimensional manifolds without boundary. Assume that an embedded disk D can be cut out from both the manifolds in such a way that $\Sigma_1 = \Sigma'_1 \sqcup D$ and $\Sigma_2 = \Sigma'_2 \sqcup D$ where Σ'_1 and Σ'_2 are two-dimensional manifolds with boundaries $\partial\Sigma_1 \simeq \partial\Sigma_2 \simeq -\partial D \simeq -\mathbb{S}^1$. Let $\xi_1 \colon \Sigma_1 \to \mathbb{SU}(2)$ and $\xi_2 \colon \Sigma_2 \to \mathbb{SU}(2)$ be two maps such that $\xi_1|_D = \xi_2|_D$ and both ξ_1 and ξ_2 have constant value $\mathbb{1}_{\mathbb{C}^2}$ on a neighborhood of $\Sigma'_1 \subset \Sigma_1$ and $\Sigma'_2 \subset \Sigma_2$, respectively. Under this setting it holds that

$$\mathcal{WZ}_{\Sigma_1}(\xi_1) = \mathcal{WZ}_{\Sigma_2}(\xi_2) \mod \mathbb{Z}. \tag{3-11}$$

In fact both $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma'_1}(\xi_1|_{\Sigma'_1})} \in \mathscr{L}^*_{\mathbb{S}^1}$ and $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma'_2}(\xi_2|_{\Sigma'_2})} \in \mathscr{L}^*_{\mathbb{S}^1}$ describe the unit of the central extension $S(\mathscr{L}_{\mathbb{S}^1})$ as discussed in Remark 3.21. Therefore,

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma'_1}(\xi_1|_{\Sigma'_1})} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma'_2}(\xi_2|_{\Sigma'_2})}, \quad \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{D}(\xi_1|_{D})} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{D}(\xi_2|_{D})},$$

where the second equality follows from the assumption $\xi_1|_D = \xi_2|_D$. By applying the gluing property (3-10) one gets $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1}(\xi_1)} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2}(\xi_2)}$ which justifies (3-11).

3.6 Classification via Wess–Zumino term in dimension two

In this section the description of rank 2 Q–bundles over an oriented two-dimensional FKMM–manifold (Σ, τ) obtained in Sections 2.6 and 2.7 will be combined with the theory of the Wess–Zumino term described in Sections 3.4 and 3.5 in order to prove that the Wess–Zumino term completely classifies $\mathrm{Vec}^2_Q(\Sigma, \tau)$.

The following three preliminary results are needed.

Lemma 3.24 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Let* $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ *be the set of equivariant maps described by* (2-10) *and* $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ *be the set of equivalence classes under the $\mathbb{Z}_2$–homotopy equivalence. Then*:

(i) *The exponentiated Wess–Zumino term of* $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ *takes values in* $\mathbb{Z}_2$*, so one gets a map*

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma}\colon \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

(ii) *The map above is invariant under* $\mathbb{Z}_2$*–homotopy, and hence induces a map*

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma}\colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

Proof (i) For every $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ the quantity $\mathcal{WZ}_\Sigma(\xi) \in \mathbb{R}/\mathbb{Z}$ is defined according to Definition 3.16. Since ξ satisfies $\tau^*\xi = \xi^{-1}$, the diffeo-invariance (functoriality) of the Wess–Zumino term [15] implies

$$\mathcal{WZ}_\Sigma(\xi) = \mathcal{WZ}_\Sigma(\tau^*\xi) = \mathcal{WZ}_\Sigma(\xi^{-1}).$$

From the relation $\zeta^{-1}\mathrm{d}\zeta = -\zeta\mathrm{d}\zeta^{-1}$, valid for generic maps with values in $\mathbb{SU}(2)$, it follows that $\mathrm{Tr}(\zeta^{-1}\mathrm{d}\zeta)^n = (-1)^n\,\mathrm{Tr}(\zeta\mathrm{d}\zeta^{-1})^n$. The application of this identity to the Wess–Zumino term implies $\mathcal{WZ}_\Sigma(\xi^{-1}) = -\mathcal{WZ}_\Sigma(\xi)$. In conclusion, one obtains that $\mathcal{WZ}_\Sigma(\xi) = -\mathcal{WZ}_\Sigma(\xi)$ modulo $\mathbb{Z}$, ie $2\mathcal{WZ}_\Sigma(\xi) \in \{0, 1\}$. This proves that the exponential map in (i) takes values in $\mathbb{Z}_2$.

(ii) If $\hat{\xi}\colon \Sigma \times [0,1] \to \mathbb{SU}(2)$ is a $\mathbb{Z}_2$–homotopy, then the map

$$[0,1] \ni t \mapsto \mathcal{WZ}_\Sigma(\hat{\xi}|_{\Sigma\times\{t\}}) \in \mathbb{R}/\mathbb{Z}$$

is continuous. Hence, the value of the exponential $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma}(\hat{\xi}|_{\Sigma\times\{t\}})$ must be constant for all t in view of the discreteness of the target space. This concludes the proof. □

Lemma 3.25 *Let* (Σ, τ) *be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. For each* $\epsilon \in \mathrm{Map}(\Sigma^\tau, \{\pm1\})$ *there exists* $\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ *such that* $\Phi_\kappa(\xi_\epsilon) = \epsilon$ *and*

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma(\xi_\epsilon)} = \Pi(\epsilon),$$

where Π *is the product sign map defined by* (2-7).

Proof The proof of Lemma 2.17 contains the recipe to construct a map

$$\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$$

for each $\epsilon \in \mathrm{Map}(\Sigma^\tau, \{\pm1\})$ such that $\Phi_\kappa(\xi_\epsilon) = \epsilon$. Let $\Sigma^\tau = \{x_1, \dots, x_n\}$ be a labeling for the fixed point set. Let $\epsilon_i \in \mathrm{Map}(\Sigma^\tau, \{\pm1\})$ be defined by $\epsilon_i(x_j) = 1 - 2\delta_{ij}$. Let $\xi_i := \xi_{\epsilon_i}$ be the element in $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ such that $\Phi_\kappa(\xi_i) = \epsilon_i$. Note that ξ_i takes the value $\mathbb{1}_{\mathbb{C}^2}$ outside the disk D_i. It follows that $\xi_1, \dots, \xi_n$ commute pointwise, and the pointwise product of the ξ_i is in $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$. Then, by construction, each ξ_ϵ can be expressed as the pointwise product of a certain number of ξ_i. Let us assume that $\xi_\epsilon = \xi_{i_1} \cdots \xi_{i_k}$. Since the supports of the differential forms $\xi_i^{-1}\mathrm{d}\xi_i$ are pairwise disjoint, the Polyakov–Wiegmann formula (see Lemma 3.17) provides

$$\mathcal{WZ}_\Sigma(\xi_\epsilon) = \mathcal{WZ}_\Sigma(\xi_{i_1}) + \cdots + \mathcal{WZ}_\Sigma(\xi_{i_k}) \mod \mathbb{Z}.$$

The next task is to evaluate the generic term $\mathcal{WZ}_\Sigma(\xi_i)$. For that, the construction in Remark 3.23 will be applied. Given $x_i \in \Sigma^\tau$, consider a small disk $D_i \subset \Sigma$ such that $\tau(D_i) = D_i$ and $x_i \in D_i$ is the only fixed point. The restriction $\xi_i|_{D_i}$ has by construction the following property: $\xi_i|_{D_i}(x_i) = -\mathbb{1}_{\mathbb{C}^2}$ and $\xi_i|_{D_i}(x) = +\mathbb{1}_{\mathbb{C}^2}$ if $x \in \partial D_i$. By an equivariant diffeomorphism, D_i can be identified with the closed unit disk $D \subset \mathbb{C}$ endowed with the involution $z \mapsto -z$ and the map $\xi_i|_{D_i}$ can be identified with the map ξ_D described in the proof of Lemma 2.17. By gluing two copies D and D' of the same disk along the common boundary $\mathbb{S}^1$ one obtains that $D \sqcup D'$ is identifiable with the equivariant sphere $\mathbb{S}^2$ with involution $(k_0, k_1, k_2) \mapsto (k_0, -k_1, -k_2)$ which fixes only the two poles $(\pm 1, 0, 0)$. Consequently, given the constant map $\xi_0 \colon D' \to \mathbb{1}_{\mathbb{C}^2}$, one has that the map $\xi_D \sqcup \xi_0 \colon D \sqcup D' \to \mathbb{SU}(2)$ can be identified with the equivariant map $\chi \colon \mathbb{S}^2 \to \mathbb{SU}(2)$ such that $\chi(\pm 1, 0, 0) = \pm \mathbb{1}_{\mathbb{C}^2}$. Since the conditions described in Remark 3.23 are met, one has that

$$\mathcal{WZ}_\Sigma(\xi_i) = \mathcal{WZ}_{\mathbb{S}^2}(\chi) \mod \mathbb{Z}.$$

A possible realization for χ is

$$\chi(k_0, k_1, k_2) = \begin{pmatrix} k_0 & -k_1 + \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & k_0 \end{pmatrix}. \tag{3-12}$$

Recall that $[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq H^1_{\mathbb{Z}^2}(\mathbb{S}^2, \mathbb{Z}(1))$ [23, Proposition A.2] and

$$H^1_{\mathbb{Z}_2}(\mathbb{S}^2, \mathbb{Z}(1)) \simeq H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \oplus H^{-1}_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$$

by [7, Lemma 5.6]. Since $H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq [*, \mathbb{U}(1)]_{\mathbb{Z}_2}$, it follows that $[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2}$ is represented by constant maps. Then the bijection $[\mathbb{S}^2, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2$ obtained from Proposition 2.18 assures that, up to a $\mathbb{Z}_2$–homotopy if necessary, one can always choose the equivariant map χ as given in (3-12). The computation of $\mathcal{WZ}_{\mathbb{S}^2}(\chi)$ with χ given by (3-12) is as follows: Consider the map $\tilde{\chi} \colon \mathbb{S}^3 \to \mathbb{SU}(2)$ defined by

$$\tilde{\chi}(k_0, k_1, k_2, k_3) = \begin{pmatrix} k_0 + \mathrm{i}k_3 & -k_1 + \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & k_0 - \mathrm{i}k_3 \end{pmatrix}. \tag{3-13}$$

Let $\mathbb{S}^3_+ := \{k \in \mathbb{S}^3 \mid k_3 \geqslant 0\}$ be the upper hemisphere. Then $\partial \mathbb{S}^3_+ \simeq \mathbb{S}^2$ and $\tilde{\chi}|_{\partial \mathbb{S}^3_+} = \chi$. Since $\mathbb{S}^3_+$ is just a half-sphere, one gets by a direct computation that

$$\mathcal{WZ}_{\mathbb{S}^2}(\chi) = \frac{-1}{48\pi^2} \int_{\mathbb{S}^3_+} \mathrm{Tr}(\tilde{\chi}^{-1}\mathrm{d}\tilde{\chi})^3 = \frac{1}{2}.$$

As a consequence, $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi_i)} = \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\mathbb{S}^2}(\chi)} = -1$ and

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi_\epsilon)} = \prod_{x_{i_1},\dots,x_{i_k}} (-1) = \Pi(\epsilon). \qquad \square$$

Lemma 3.26 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. The Wess–Zumino term induces a well-defined map*

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma} \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

Proof The claim is proved if one can show that for any $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ and $\phi \in \mathrm{Map}(\Sigma, \mathbb{U}(1))_{\mathbb{Z}_2}$, it holds that $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} = \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi')}$ where

$$\xi' = \begin{pmatrix} \tau^*\phi & 0 \\ 0 & 1 \end{pmatrix} \cdot \xi \cdot \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix}.$$

Let $\epsilon := \Phi_\kappa(\xi)$ and $\epsilon' := \Phi_\kappa(\xi')$. Associated with the maps $\epsilon, \epsilon' \in \mathrm{Map}(\Sigma^\tau, \mathbb{Z}_2)$ one can construct the associated maps $\xi_\epsilon, \xi_{\epsilon'} \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ according to Lemma 3.25. Lemma 2.17 assures that ξ and ξ' are $\mathbb{Z}_2$–homotopy equivalent to ξ_ϵ and $\xi_{\epsilon'}$, respectively. Thus,

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} = \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi_\epsilon)} = \Pi(\epsilon) = \Pi(\Phi_\kappa(\xi))$$

and similarly $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi')} = \Pi(\Phi_\kappa(\xi'))$. Since Proposition 2.18 assures that $\Phi_\kappa(\xi) = \Phi_\kappa(\xi')$, it follows that $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} = \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi')}$. $\square$

We are now in position to prove the first main result of this work.

Proof of Theorem 1.2 The case $m = 1$ will be treated first. In view of the bijection proved in Theorem 2.13 and the resulting equality (2-18), it is enough to show that $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma} = \Pi \circ \Phi_\kappa$ maps from $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ into $\mathbb{Z}_2$. By Proposition 2.18 and Theorem 2.11, $\Pi \circ \Phi_\kappa$ is a bijection. Thus, it is enough to prove the equality $\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma} = \Pi \circ \Phi_\kappa$ on $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$. However, this is clear from Lemma 3.25. The generalization to the case of Q–bundles of rank $2m$ follows by using the arguments in Remark 2.16. $\square$

3.7 Classification via Chern–Simons invariant in dimension three

The main aim of this section is to provide the proof of Theorem 1.3. This proof is facilitated by a particular presentation of principal Q–bundles over (X, τ). Suppose that $X^\tau = \{x_1, \dots, x_n\}$ consists of n points. Thanks to the *slice theorem* [27, Chapter I, Section 3], for each $i = 1, \dots, n$ one can find a closed τ–invariant disk D_i centered at x_i such that $D_i \cap D_j = \varnothing$ for $i \neq j$ and each D_i is equivariantly homotopic to the standard unit disk in $\mathbb{R}^3$ with antipodal involution $\tau(x) = -x$. Define

$$X_D := \bigsqcup_{i=1,\dots,n} D_i, \quad X' := X \backslash \mathrm{Int}(X_D),$$

so that $X = X' \cup X_D$. Given any map $\varphi : X' \cap X_D \to \mathbb{U}(2)$ one can glue together the product bundles over X' and X_D to form a principal $\mathbb{U}(2)$–bundle over X,

$$\mathscr{P}_\varphi := (X' \times \mathbb{U}(2)) \sqcup_\varphi (X_D \times \mathbb{U}(2)). \tag{3-14}$$

Assume that $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$; namely, φ is equivariant with respect to the involution $\tau^*\varphi = -Q\bar{\varphi}Q$. Then the principal $\mathbb{U}(2)$–bundle $\mathscr{P}_\varphi$ gives rise to a principal Q–bundle.

Lemma 3.27 *Assume that the hypotheses of Theorem 1.3 are met. Any principal $\mathbb{U}(2)$ Q–bundle $(\mathscr{P}, \widehat{\Theta})$ over (X, τ) is isomorphic to a principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_\varphi$ of the type* (3-14) *for a map $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ which satisfies the following property: Let $\varphi_i := \varphi|_{\partial D_i}$ be the restriction of φ on the boundary $\partial D_i \simeq \mathbb{S}^2$ of the disk D_i for every $i = 1, \dots, n$. Then, either φ_i is equivariantly homotopic to the equivariant map $\varphi_* : \mathbb{S}^2 \to \mathbb{U}(2)$ defined by*

$$\varphi_*(x_1, x_2, x_3) := \mathrm{i} \begin{pmatrix} x_1 & -x_2 + \mathrm{i}x_3 \\ x_2 + \mathrm{i}x_3 & x_1 \end{pmatrix},$$

where $\mathbb{S}^2$ is a $\mathbb{Z}_2$–space with the antipodal involution, or φ_i is the constant map at $\mathbb{1}_{\mathbb{C}^2} \in \mathbb{U}(2)$.

Proof Since each connected component D_i of X_D is equivariantly contractible, the principal Q–bundle $\mathscr{P}|_{X_D}$ is trivial. By construction, the involution on X' is free; thus $\mathscr{P}|_{X'}$ is trivial as well. This fact follows from [10, Theorem 4.7(2)] along with the assumption $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ which implies the triviality of even rank Q–bundles over spaces with free involutions. The passage from vector bundles to principal bundles is then justified by the isomorphism (3-1). Let s_{X_D} and $s_{X'}$ be global sections (ie trivializations) of $\mathscr{P}|_{X_D}$ and $\mathscr{P}|_{X'}$, respectively. From these sections one gets the map $\varphi: X' \cap X_D \to \mathbb{U}(2)$ defined by the restriction on $X' \cap X_D$ of the (pointwise) product $s_{X_D}^{-1} s_{X'}$. The map φ is equivariant by construction and defines the principal Q–bundle $\mathscr{P}_\varphi$ as given in (3-14). The isomorphism $\mathscr{P} \simeq \mathscr{P}_\varphi$ is a manifestation of the fact that $\mathscr{P}$ and $\mathscr{P}_\varphi$ have the same system of transition functions. By the homotopy property of Q–bundles, the Q–isomorphism class of $\mathscr{P}_\varphi$ only depends on the $\mathbb{Z}_2$–homotopy class of φ. By [8, Corollary 4.1] one has $[\mathbb{S}^2, \mathbb{U}(2)]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2$, meaning that every equivariant map from the sphere $\mathbb{S}^2$ with the antipodal involution into the space $\mathbb{U}(2)$ with involution $g \mapsto -Q\bar{g}Q$ is $\mathbb{Z}_2$–homotopy equivalent to the constant map at $\mathbb{1}_{\mathbb{C}^2}$ or to the map φ_*. Since $X' \cap X_D$ is a disjoint union of antipodal spheres, the map φ restricted to each disconnected component can be equivariantly deformed to one of these two maps. □

Remark 3.28 Lemma 3.27 deserves two comments. First of all it is worth noticing that the map φ constructed in the proof of the lemma can be always deformed to a smooth map providing in this way a smooth principal Q–bundle $\mathscr{P}_\varphi$ which represents $\mathscr{P}$ in the smooth category. This is a manifestation of the equivalence between continuous and smooth category discussed in [9, Theorem 2.1]. The second observation refers to the content of Remark 2.16. In fact in view of the stable rank condition described in Theorem 2.5 one has that the representation (3-14) must be valid also for principal $\mathbb{U}(2m)$ Q–bundle. In the higher rank case the isomorphism reads

$$\mathscr{P} \simeq \mathscr{P}_\varphi := (X' \times \mathbb{U}(2m)) \sqcup_{\varphi'} (X_D \times \mathbb{U}(2m)), \tag{3-15}$$

where the equivariant map $\varphi'\colon X' \cap X_D \to \mathbb{U}(2m)$ factors as

$$\varphi' \simeq \begin{pmatrix} \varphi & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^{2(m-1)}} \end{pmatrix}$$

and the map $\varphi\colon X' \cap X_D \to \mathbb{U}(2)$ in the upper-left corner satisfies the properties of Lemma 3.27.

In view of the Lemma 3.27 one can assume that $\mathscr{P}$ has been of the form (3-14) since the beginning. With this presentation in hand, the next task is to compute the FKMM–invariant of $\mathscr{P}$. As a preliminary fact, let us recall that the FKMM–invariant of a principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$ is defined as the FKMM–invariant of the associated Q–bundle $(\mathscr{E}, \Theta)$; see Definition 3.4. The FKMM–invariant measures the difference of two trivializations of the sphere bundle of $\det(\mathscr{E})|_{X^\tau}$. This is the same as measuring the difference of two trivializations of $\det(\mathscr{E})|_{X^\tau}$.

Lemma 3.29 *Assume that the hypotheses of Theorem 1.3 are met. Let $(\mathscr{P}, \widehat{\Theta})$ be a principal $\mathbb{U}(2)$ Q–bundle and $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ the equivariant map which represents the principal Q–bundle according to Lemma 3.27. Then the FKMM–invariant of $(\mathscr{P}, \widehat{\Theta})$ is represented by the function $\phi := \det(\varphi)|_{X^\tau}$. More precisely, one has that*

$$\kappa(\mathscr{P}, \widehat{\Theta}) = [\phi] \in \mathrm{Map}(X^\tau, \{\pm 1\}) / [X, \mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Proof Starting from the representation (3-14), one has that

$$\det(\mathscr{P}) = (X' \times \mathbb{U}(1)) \sqcup_{\det(\varphi)} (X_D \times \mathbb{U}(1)).$$

From this expression one infers that the canonical invariant section $s_{(\mathscr{P},\widehat{\Theta})}$ of $\det(\mathscr{P})|_{X^\tau}$ is given by

$$s_{(\mathscr{P},\widehat{\Theta})} = (x,1) \in X^\tau \times \mathbb{U}(1) \subset \det(\mathscr{P}),$$

while a global invariant section s of $\det(\mathscr{P})$ is given by

$$s(x) = \begin{cases} (x, u_{X'}(x)) & \text{if } x \in X', \\ (x, u_D(x)) & \text{if } x \in X_D, \end{cases}$$

where $u_{X'}\colon X' \to \mathbb{U}(1)$ and $u_D\colon X_D \to \mathbb{U}(1)$ are two equivariant maps satisfying $u_{X'} = u_D \cdot \det(\varphi)$ on $X' \cap X_D$. Accordingly, it follows from Lemma 3.27 that $\det(\varphi_i)\colon X' \cap D_i \to \mathbb{U}(1)$ is a constant map at 1 or -1, where $\varphi_i := \varphi|_{D_i}$. Therefore, one can choose $u_{X'}$ to be the constant map at 1 and u_D to be the locally constant map such that $u_D|_{D_i} = \pm 1$ if $\det(\varphi_i) = \pm 1$. Then, it follows that the FKMM–invariant is represented by $u_D|_{X^\tau} = \det(\varphi)|_{X^\tau}$. □

The next goal is to compute the Chern–Simons invariant of $(\mathscr{P}, \widehat{\Theta})$. Let $s_{X'}$ and s_{X_D} be the invariant sections of $\mathscr{P}|_{X'} = X' \times \mathbb{U}(2)$ and $\mathscr{P}|_{X_D} = X_D \times \mathbb{U}(2)$ defined by

$$\begin{aligned} s_{X'}(x) &= (x, \mathbb{1}_{\mathbb{C}^2}) \quad \text{if } x \in X', \\ s_{X_D}(x) &= (x, \mathbb{1}_{\mathbb{C}^2}) \quad \text{if } x \in X_D, \end{aligned} \tag{3-16}$$

respectively. Then, any section s of $\mathscr{P}$ is described as

$$s(x) = \begin{cases} s_{X'}(x)\psi_{X'}(x)^{-1} = (x, \psi_{X'}(x)^{-1}) & \text{if } x \in X', \\ s_{X_D}(x)\psi_D(x)^{-1} = (x, \psi_D(x)^{-1}) & \text{if } x \in X_D, \end{cases} \tag{3-17}$$

for a pair of maps $\psi_{X'}\colon X' \to \mathbb{U}(2)$ and $\psi_D\colon X_D \to \mathbb{U}(2)$ such that $\psi_{X'} = \psi_D\varphi$ on $X' \cap X_D$. The maps $\psi_{X'}$ and ψ_D can be chosen smooth in such a way that the section s is smooth as well. Moreover, the choice of $\psi_{X'}$ and ψ_D can be further specified in view of the following result:

Lemma 3.30 *The smooth maps $\psi_{X'}$ and ψ_D in* (3-17) *can be chosen such that $\psi_D = \mathbb{1}_{\mathbb{C}^2}$ is the constant map.*

Proof By construction, $\psi_{X'} = \psi_D\varphi$ on $X' \cap X_D$. Thus, the proof of the claim reduces to the problem of extending $\varphi\colon \partial X' \to \mathbb{U}(2)$ to a smooth map $\tilde{\varphi}\colon X' \to \mathbb{U}(2)$ such that $\tilde{\varphi}|_{\partial X'} = \varphi$. Indeed, given such a $\tilde{\varphi}$, the proof can be completed by setting

$\psi_D = \mathbb{1}_{\mathbb{C}^2}$ and $\psi_{X'} = \tilde{\varphi}$. To prove the existence of $\tilde{\varphi}$, notice that the three-manifold X' admits a CW decomposition in which the dimension of each cell is at most 3. The homotopy groups $\pi_i(\mathbb{U}(2))$ are trivial for $i = 0, 2$. The map $\det\colon \mathbb{U}(2) \to \mathbb{U}(1)$ induces an isomorphism $\pi_1(\mathbb{U}(2)) \simeq \pi_1(\mathbb{U}(1)) \simeq \mathbb{Z}$. Since $\det(\varphi)$ is nullhomotopic by construction, one concludes that φ extends to a continuous map $\tilde{\varphi}'\colon X' \to \mathbb{U}(2)$. However, the isomorphism between continuous category and smooth category ensures the existence of a smooth map $\tilde{\varphi}\colon X' \to \mathbb{U}(2)$, approximating the continuous map $\tilde{\varphi}'$, that satisfies $\tilde{\varphi}|_{\partial X'} = \varphi$. □

Given an invariant connection ω on $(\mathcal{P}, \widehat{\Theta})$, one sets

$$\omega_{X'} := s_{X'}^* \omega, \quad \omega_{X_D} := s_{X_D}^* \omega.$$

The two local expressions are related by

$$\omega_{X'} = \varphi^{-1} \omega_{X_D} \varphi + \varphi^{-1} d\varphi. \tag{3-18}$$

The following result contains the key computation for the proof of Theorem 1.3.

Lemma 3.31 *Assume that the hypotheses of Theorem 1.3 are met. Let $(\mathcal{P}, \widehat{\Theta})$ be a principal $\mathbb{U}(2)$ Q–bundle and $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ the equivariant map which represents the principal Q–bundle according to Lemma 3.27. Then the Chern–Simons invariant of $(\mathcal{P}, \widehat{\Theta})$ is given by*

$$\mathfrak{cs}(\mathcal{P}, \widehat{\Theta}) = \mathcal{WZ}_{\partial X_D}(\varphi) + \frac{1}{8\pi^2} \int_{\partial X_D} \mathrm{Tr}(\omega_{X_D} \wedge d\varphi\varphi^{-1}) \mod \mathbb{Z},$$

where ω_{X_D} is defined by (3-18) from any invariant connection ω.

Proof Let us start with an observation. By construction, $\varphi = \bigsqcup_{i=1,\dots,n} \varphi_i$ and each $\det(\varphi_i)\colon \partial D_i \to \mathbb{U}(1)$ is constant at ± 1. Hence, $\det(\varphi)$ is nullhomotopic and $\mathcal{WZ}_{\partial D}(\varphi)$ makes sense. Now, the computation. Given the section described in (3-17), one has that

$$\begin{aligned}\int_X s^* \mathcal{CS}(\omega) &= \int_{X'} s^* \mathcal{CS}(\omega) + \int_{X_D} s^* \mathcal{CS}(\omega) \\ &= \int_{X'} (s_{X'} \psi_{X'}^{-1})^* \mathcal{CS}(\omega) + \int_{X_D} (s_{X_D} \psi_D^{-1})^* \mathcal{CS}(\omega).\end{aligned}$$

With the help of formula (3-7) one has that

$$\begin{aligned}&(s_{X'} \psi_{X'}^{-1})^* \mathcal{CS}(\omega) \\ &\qquad = s_{X'}^* \mathcal{CS}(\omega) + \frac{1}{8\pi^2} \mathrm{dTr}(s_{X'}^* \omega \wedge \psi_{X'} \mathrm{d}\psi_{X'}^{-1}) - \frac{1}{24\pi^2} \mathrm{Tr}((\psi_{X'} \mathrm{d}\psi_{X'}^{-1})^{\wedge 3}).\end{aligned}$$

Since

$$\int_{X'} s_{X'}^{*}\mathcal{CS}(\omega) = \int_{X'} \mathcal{CS}(\omega_{X'}) = 0,$$

in view of Proposition 3.12(iii) one gets

$$\int_{X'} (s_{X'}\psi_{X'}^{-1})^{*}\mathcal{CS}(\omega) = \frac{1}{8\pi^2}\int_{X'} \mathrm{dTr}(\omega_{X'}\wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) + \mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) \mod \mathbb{Z},$$

where Definition 3.16 has been used. With a similar computation one gets also

$$\int_{X_D} (s_{X_D}\psi_D^{-1})^{*}\mathcal{CS}(\omega) \\ = \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D}\wedge \mathrm{d}\psi_D\psi_D^{-1}) + \mathcal{WZ}_{\partial X_D}(\psi_D^{-1}|_{\partial X_D}) \mod \mathbb{Z}$$

and, after putting all the pieces together, one obtains

$$\int_{X} s^{*}\mathcal{CS}(\omega) = \frac{1}{8\pi^2}\int_{X'} \mathrm{dTr}(\omega_{X'}\wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) + \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D}\wedge \mathrm{d}\psi_D\psi_D^{-1}) \\ + \mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) + \mathcal{WZ}_{\partial X_D}(\psi_D^{-1}|_{\partial X_D}) \mod \mathbb{Z}.$$

Notice that the orientation on $\partial X' = X' \cap X_D$ induced from X is opposite to that on ∂X_D. Therefore, modulo $\mathbb{Z}$, one gets the equality

$$\mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) \\ = -\mathcal{WZ}_{\partial X_D}((\psi_D|_{\partial X_D}\varphi)^{-1}) \\ = -\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) - \mathcal{WZ}_{\partial X_D}(\psi_D|_{\partial X_D}^{-1}) - \frac{1}{8\pi^2}\int_{\partial X_D} \mathrm{Tr}(\varphi \mathrm{d}\varphi^{-1}\wedge \mathrm{d}\psi_D^{-1}\psi_D),$$

which is justified by the relation $\psi_{X'} = \psi_D\varphi$ on $\partial X' = \partial X_D$ and by the use of the Polyakov–Wiegmann formula proved in Lemma 3.17. The local relation between $\psi_{X'}$ and ψ_D also implies

$$\mathrm{Tr}(\omega_{X'}\wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) = \mathrm{Tr}(\omega_{X_D}\wedge \psi_D^{-1}\mathrm{d}\psi_D + \omega_{X_D}\wedge \mathrm{d}\varphi\varphi^{-1} + \mathrm{d}\varphi\varphi^{-1}\wedge \psi_D^{-1}\mathrm{d}\psi_D).$$

Summarizing, one finally gets

$$\int_{X} s^{*}\mathcal{CS}(\omega) = -\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) + \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D}\wedge \mathrm{d}\varphi\varphi^{-1}) \mod \mathbb{Z}.$$

The proof is completed by the general equality $\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) = -\mathcal{WZ}_{\partial X_D}(\varphi)$ and the use of Definition 3.14. □

We are now in position to provide the proof of the second main result of this work.

Proof of Theorem 1.3 Let us choose the maps $\psi_{X'}$ and ψ_D as in Lemma 3.30. Then $\omega_{X_D} := s_{X_D}^*\omega = (s\psi_D)^*\omega = 0$, since ψ_D is constant. Thus, from the formula in Lemma 3.31 and the definition of the map φ, one gets

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) = \mathcal{WZ}_{\partial X_D}(\varphi) = \sum_{i=1}^{n} \mathcal{WZ}_{\partial D_i}(\varphi_i) \mod \mathbb{Z}.$$

It holds that $\mathcal{WZ}_{\partial D_i}(\varphi_i) = 1$ when $\varphi_i = \mathbb{1}_{\mathbb{C}^2}$ (obvious!) and $\mathcal{WZ}_{\partial D_i}(\varphi_i) = \frac{1}{2}$ when φ_i is homotopic to the map φ_* in Lemma 3.27. The proof of the latter equality is contained in the proof of Lemma 3.25. In fact the map φ_* coincides with the map (3-12) and a possible extension $\tilde{\varphi}_*$ on the upper hemisphere of $\mathbb{S}^3$ can be realized by the prescription (3-13). In conclusion, one obtains that

$$\mathrm{e}^{\mathrm{i}2\pi\,\mathfrak{cs}(\mathscr{P},\widehat{\Theta})} = \Pi(\det(\varphi)|_{X^\tau}).$$

The proof is finally completed by the result in Lemma 3.29. $\square$

Theorem 1.3 has a surprising consequence.

Corollary 3.32 *Under the assumptions in Theorem 1.3, the homomorphism*

$$\Pi\colon \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \to \mathbb{Z}_2$$

induced by the product sign map (2-7) *is well defined.*

Proof One needs to shows that the homomorphism $\Pi\colon \mathrm{Map}(X^\tau, \{\pm 1\}) \to \mathbb{Z}_2$ given by the product sign map satisfies $\Pi(\phi\psi|_{X^\tau}) = \Pi(\phi)$ for any map $\phi\colon X^\tau \to \mathbb{Z}_2$ and any equivariant map $\psi\colon X \to \mathbb{U}(1)$. Consider the principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_\varphi$ generated according to (3-14) where the map φ is related to ϕ as follows: φ is the constant map at $\mathbb{1}_{\mathbb{C}^2}$ on the disk D_i if $\phi(x_i) = 1$ or φ agrees with φ_* on the boundary of D_i if $\phi(x_i) = -1$. By construction the map ϕ provides a representative of the FKMM–invariant of $\mathscr{P}_\varphi$; see Lemma 3.27. In a similar way the map $\phi' := \phi\psi$ represents the FKMM–invariant of an associated principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_{\varphi'}$. Since φ and φ' belong to the same class in $\mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ it follows that $\mathscr{P}_\varphi$ and $\mathscr{P}_{\varphi'}$ have the same FKMM–invariant. However, under the hypotheses of Theorem 1.3 the FKMM–invariant is an isomorphism (Proposition 2.10); hence $\mathscr{P}_\varphi$ and $\mathscr{P}_{\varphi'}$ are isomorphic. By the naturality of the Chern–Simons invariant, $\mathfrak{cs}(\mathscr{P}_\varphi, \widehat{\Theta}_\varphi) = \mathfrak{cs}(\mathscr{P}_{\varphi'}, \widehat{\Theta}_{\varphi'})$. The proof of the claim then follows in view of formula (1-6). $\square$

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Facultad de Matemáticas & Instituto de Física, Pontificia Universidad Católica de Chile
Santiago, Chile

Department of Mathematics, Tokyo Institute of Technology
Tokyo, Japan

gidenittis@mat.uc.cl, kgomi@math.titech.ac.jp

Received: 23 October 2018 Revised: 2 March 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 7 (pages 2925–3415) 2023