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The Devinatz–Hopkins theorem via algebraic geometry

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We show how a continuous action of the Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n, satisfying the conclusion $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$ of the Devinatz–Hopkins theorem, may be obtained by monodromy on the stack of oriented deformations of formal groups in the context of formal spectral algebraic geometry.

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A classical and computationally invaluable result in chromatic homotopy theory, the Morava change-of-rings theorem — see for instance Devinatz [5] — identifies the second page of the $K(n)$–local Adams spectral sequence for the Lubin–Tate spectrum E_n as continuous group cohomology,

$$E_2^{s,t} \simeq \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

A conceptual spectrum-level explanation for this isomorphism is given by the Devinatz–Hopkins theorem [6]. It asserts the existence of a (suitably interpreted) continuous action of the Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n, such that its continuous homotopy fixed points are

$$E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S. \tag{1}$$

The proof of the equivalence (1) has by now become largely standard, using nilpotence technology applied to the $K(n)$–local Amitsur complex of E_n, and ultimately stemming from the key observation of Hopkins and others that the Adams spectral sequence of E_n possesses a horizontal vanishing line. The somewhat less straightforward part is instead identifying said Amitsur complex with the simplicial bar resolution of a suitably interpreted continuous action of $\mathbb{G}_n$ on E_n. That was accomplished in a somewhat ad hoc manner in [6], and in various contexts of continuous group actions of spectra such as Behrens and Davis [2] and Quick [21]; though these approaches ostensibly amount to enriching the construction from [6]. A formalization using the condensed set technology of [23] to tackle continuity has also been announced by Clausen and Scholze.

In contrast, we propose to side-step the issue of continuous actions altogether. Instead, we exhibit the action in an appropriate context of formal spectral algebraic geometry. Our results may be summarized as follows.

Theorem *The Morava stabilizer group $\mathbb{G}_n$ admits a canonical action on the formal spectral stack $\mathrm{Spf}(E_n)$. For continuous homotopy fixed points of this action defined as $E_n^{h\mathbb{G}_n} := \mathcal{O}(\mathrm{Spf}(E_n)/\mathbb{G}_n)$, there is a canonical equivalence $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$. Furthermore, the three resulting spectral sequences coincide*:

(1) *The descent spectral sequence for the structure sheaf on* $\mathrm{Spf}(E_n)/\mathbb{G}_n$,

$$E_2^{s,t} = \mathrm{H}^s(\mathrm{Spf}(E_n)/\mathbb{G}_n; \pi_t(\mathcal{O})) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

(2) *The homotopy fixed point spectral sequence for the $\mathbb{G}_n$–action on E_n*,

$$E_2^{s,t} = \mathrm{H}^s(\mathrm{B}\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

(3) *The $K(n)$–local Adams spectral sequence for E_n*,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(L_{K(n)}(E_n \otimes E_n))}(\pi_*(E_n), \pi_*(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

Our approach is based on a theorem of Lurie [17, Theorem 5.1.5], identifying $\mathrm{Spf}(E_n)$ with the moduli stack of oriented deformation of a height n formal group. We show that the Morava stabilizer group action arises as an instance of monodromy actions on de Rham spaces. To establish the above version of the Devinatz–Hopkins theorem in our setting, we employ similar arguments to the analogous considerations in classical formal algebraic geometry from Goerss [8, Chapter 7].

The computational underpinning of the proof (somewhat obscured in our account) is the fundamental observation from [6] that the $K(n)$–local Adams spectral sequence for E_n possesses a horizontal vanishing line. Ours is in that sense analogous to all of the currently known approaches to the Devinatz–Hopkins theorem, including, to the best of the author's understanding, the forthcoming work of Clausen and Scholze. The latter construct the continuous (or in their setting, more precisely, condensed) Morava stabilizer group action similarly to us, in that they employ results[1] from [17].

[1] Though unlike our account, where the algebrogeometric aspect of the results in [17] are center-stage, the approach of Clausen and Scholze only relies on the more flexible functoriality of Lubin–Tate theory (in particular, that its base can be taken to be an arbitrary perfect $\boldsymbol{F}_p$–algebra as base, as opposed to only a perfect field) afforded by Lurie's construction, as compared to the traditional one by Goerss, Hopkins and Miller.

In particular, we wish to make it clear that the majority of our proof of the Devinatz–Hopkins theorem follows the same reasoning and insights as the original account in [6].

In general, many of the results in this paper follow without much difficulty from the existing literature. We nonetheless believe that a streamlined conceptual proof of the Devinatz–Hopkins theorem, which this paper provides, is worthwhile. Other than in the presentation, our primary contribution is a novel way to obtain the Morava stabilizer group action by way of formal spectral algebraic geometry, building on Lurie's work in [15; 17]. Related applications of those results to topics in chromatic homotopy theory, primarily concerning Gross–Hopkins duality, are considered by Devalapurkar [4].

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1 Background on formal spectral algebraic geometry

We begin by summarizing some notions and results from [15; 17] which are key for the purpose of this note.

1.1 Adic $\mathbb{E}_\infty$–rings and formal SAG

From the functor of points perspective, formal spectral algebraic geometry, in the form relevant to us and in [17] (but slightly differently from [15, Definition 8.1.1.5], where a connectivity assumption is imposed throughout), concerns functors $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$.

Here $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denotes the ∞–category of *complete adic* $\mathbb{E}_\infty$*–rings* in the sense of [17, Definition 0.0.11]. That is, an object of $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ consists of an $\mathbb{E}_\infty$–ring A, together with a topology on $\pi_0(A)$ which admits a finitely generated ideal of definition $I \subseteq \pi_0(A)$, such that the topology on $\pi_0(A)$ is equivalent to the I–adic topology, and finally such that the $\mathbb{E}_\infty$–ring A is I–complete, in the sense of [14, Definition 7.2.3.22].

Since the notion of completeness for $\mathbb{E}_\infty$–rings and modules over them features prominently in this note, let us recall (an equivalent rephrasing of) the definition:

Definition 1.1 [15, Proposition 7.3.2.1, Corollary 7.3.3.3] Let A be an adic $\mathbb{E}_\infty$–ring with an ideal of definition $I \subseteq \pi_0(A)$. Then an A–module M is *I–complete* if for every element $a \in I$, the canonical map

$$M \to \varprojlim_n M/a^n$$

is an equivalence of A–modules, where $M/a^n = \mathrm{cofib}(M \xrightarrow{a^n} M)$. Let $\mathrm{Mod}_A^{\mathrm{cplt}} \subseteq \mathrm{Mod}_A$ denote the full subcategory spanned by I–complete A–modules. The adic $\mathbb{E}_\infty$–ring A is *complete* if it is I–complete as a module over itself.

Given a complete adic $\mathbb{E}_\infty$–ring A in the sense discussed above, we define its *formal spectrum* to be the corepresentable functor $\mathrm{Spf}(A) \colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ given by

$$B \mapsto \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(A, B) := \mathrm{Map}_{\mathrm{CAlg}}(A, B) \times_{\mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\pi_0(A), \pi_0(B))} \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{CAlg}^\heartsuit}(A, B).$$

Of course, this (Yoneda) embedding $(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$ is fully faithful, and its codomain is a convenient place to do formal spectral algebraic geometry.

1.2 Formal groups over $\mathbb{E}_\infty$–rings

As an instance of that motto, the theory of formal groups over $\mathbb{E}_\infty$–rings is developed in [17, Chapter 1]. We give a slightly informal account, and refer to [loc. cit.] for a precise and detailed account.

Definition 1.2 A *formal group over an $\mathbb{E}_\infty$–ring A* is an abelian group object in the ∞–category of 1–dimensional fiber-smooth formal spectral A–schemes.

Remark 1.3 There are a number of caveats concerning the above definition:

(1) The notion of an *abelian group object* must be understood in the sense of Section 1.2 of [16]. That is to say, we must equip its Yoneda presheaf with a factorization through the functor $\Omega^\infty \colon \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \to \mathcal{S}$, or equivalently, the forgetful functor $\mathcal{T}\mathrm{op}\mathcal{A}\mathrm{b} \to \mathcal{S}$. This is a *strictified* version of the more familiar notion of a grouplike $\mathbb{E}_\infty$–algebra objects, since the Yoneda presheaf is in the latter case asked to factor through $\Omega^\infty \colon \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$, or equivalently, the forgetful functor $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}) \to \mathcal{S}$.

(2) The requirement of *fiber-smoothness* on a formal A–scheme X is taken in the sense of [15, Definition 11.2.3.1], and roughly amounts to asking for X to be étale-

locally isomorphic to the formal affine space $\widehat{A}^n_A = \mathrm{Spf}(A[\![t_1,\dots,t_n]\!])$ (since we are working in the 1–dimensional case in Definition 1.2, it suffices to take $n = 1$). In particular, this implies that X is a flat over A. This differs from the notion of differential smoothness in the sense of [15, Definition 11.2.2.2], which imposes conditions on the cotangent complex $L_{X/A}$, but is incompatible with flatness unless A is a $\boldsymbol{Q}$–algebra. Since we want ordinary formal groups over commutative rings to be special cases of Definition 1.2, and they are indeed flat, we therefore have no choice but to use fiber-smoothness instead of differential smoothness.

(3) Definition 1.2 is really correct *as stated* when the $\mathbb{E}_\infty$–ring A is connective. For a nonconnective $\mathbb{E}_\infty$–ring A, we should instead define formal groups over A to be formal groups in the above sense over the connective cover $\tau_{\ge 0}(A)$ — see [17, Variant 1.6.2]. However, certain constructions associated to a formal group $\widehat{\boldsymbol{G}}$, for instance the $\mathbb{E}_\infty$–algebra of functions $\mathcal{O}_{\widehat{\boldsymbol{G}}}$ of [17, Notation 1.5.12] and Remark 1.6, depend on whether we are considering it as existing over A or over $\tau_{\ge 0}(A)$. See the thorough treatment in [17, Section 1.2] for precise details.

Example 1.4 The following are the only classes of formal groups that we will be concerned with in this note:

- Over a commutative ring A, viewed as a discrete $\mathbb{E}_\infty$–ring, Definition 1.2 reproduces the usual meaning of (as always, 1–dimensional smooth) formal groups over A.
- Let A be a *complex periodic* $\mathbb{E}_\infty$–ring, ie complex orientable and $\pi_2(A)$ is a locally free $\pi_0(A)$ module of rank 1. Then the *Quillen formal group of A* is
$$\widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_A := \mathrm{Spf}(C^*(\boldsymbol{CP}^\infty; A)),$$
which indeed gives rise to a formal group over A by [17, Section 4.1.3].

Formal groups over A form an ∞–category $\mathcal{M}_{\mathrm{FG}}(A)$, and this construction is functorial in A by base change:

Definition 1.5 Let $f\colon A \to B$ be a map of $\mathbb{E}_\infty$–rings, and $\widehat{\boldsymbol{G}}$ a formal group over A. The pullback of formal spectral schemes along $\mathrm{Spec}(f)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives rise to a formal group over B, which we denote by $f^*\widehat{\boldsymbol{G}}$.

There is also another slightly different form of functoriality afforded to formal groups. Sending
$$\widehat{\boldsymbol{G}} \mapsto \widehat{\boldsymbol{G}}{}^0 := \mathrm{Spf}(\pi_0(\mathcal{O}_{\widehat{\boldsymbol{G}}}))$$

gives rise to a functor $\mathcal{M}_{\mathrm{FG}}(A) \to \mathcal{M}_{\mathrm{FG}}(\pi_0(A))$. Informally, this sends a spectral formal group to its underlying ordinary formal group.

Remark 1.6 When the $\mathbb{E}_\infty$–ring A is connective, the preceding construction is a special case of Definition 1.5. Indeed, in that case there exists a map of $\mathbb{E}_\infty$–rings $t\colon A \to \pi_0(A)$, and $\widehat{\boldsymbol{G}}^0 \simeq t^*\widehat{\boldsymbol{G}}$. For a nonconnective $\mathbb{E}_\infty$–ring A on the other hand, the connection between A and $\pi_0(A)$ is only through the span $A \leftarrow \tau_{\geq 0}(A) \to \pi_0(A)$, and so $\widehat{\boldsymbol{G}} \mapsto \widehat{\boldsymbol{G}}^0$ is not merely an instance of base change. This is closely related to the subtleties alluded to in item (3) of Remark 1.3.

1.3 Orientations and deformations of formal groups

The class of formal groups singled out by the following definition is of special importance in relation to chromatic homotopy theory. Here an $\mathbb{E}_\infty$–ring A is called *complex periodic* [15, Definition 4.1.8] if it is both complex orientable and weakly 2–periodic.

Definition 1.7 [17, Proposition 4.3.23] A formal group $\widehat{\boldsymbol{G}}$ over an $\mathbb{E}_\infty$–ring A is *oriented* if and only if A is complex periodic and $\widehat{\boldsymbol{G}} \simeq \widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_A$ is its Quillen formal group. We denote by $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A) \subseteq \mathcal{M}_{\mathrm{FG}}(A)$ the subspace of oriented formal groups over A.

Remark 1.8 Though the above form is the most practical for our purposes, we would be remiss not to summarize an equivalent but better-motivated approach to defining oriented formal groups [17, Definition 4.3.9]. To any formal group $\widehat{\boldsymbol{G}}$ over an $\mathbb{E}_\infty$–ring A we may by [17, Sections 5.2.1–5.2.3] associate an A–module $\omega_{\widehat{\boldsymbol{G}}}$, its *dualizing line*, and the analogue of the module of invariant differentials on a classical formal group. An *orientation of* $\widehat{\boldsymbol{G}}$ then amounts to an A–linear equivalence $\omega_{\widehat{\boldsymbol{G}}} \simeq \Sigma^{-2}(A)$. This is in spirit a 2–shifted analogue of the various notions of orientation in classical geometric contexts, where it usually means some kind of trivialization of a bundle of volume forms.

The space of *deformations of* $\widehat{\boldsymbol{G}}_0$ *over* A is defined as

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(A) := \varinjlim_{I} \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\kappa, \pi_0(A)/I) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/I)} \mathcal{M}_{\mathrm{FG}}(A),$$

with the colimit ranging over all the ideals of definition $I \subseteq \pi_0(A)$. Informally, this consists of a ring homomorphism $f\colon \kappa \to \pi_0(A)/I$, a formal group $\widehat{\boldsymbol{G}}$ over the $\mathbb{E}_\infty$–ring A, and an isomorphism $f^*\widehat{\boldsymbol{G}}_0 \simeq q^*\widehat{\boldsymbol{G}}^0$ of formal groups over $\pi_0(A)/I$, where

$q\colon \pi_0(A) \to \pi_0(A)/I$ is the quotient projection. *Oriented deformations* are defined analogously as

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(A) := \varinjlim_I \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}}(\kappa, \pi_0(A)/I) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/I)} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A).$$

Both of these construction respect pullback along maps of adic $\mathbb{E}_\infty$–rings, and as such give rise to functors $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}, \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$.

The following theorem of Lurie, a cousin of the Goerss–Hopkins–Miller theorem, may be taken as the definition of Lubin–Tate spectra, and is the bedrock of this note.

Theorem 1.9 [17, Theorem 5.1.5, Remark 6.0.7] *Let $\widehat{\boldsymbol{G}}_0$ be a formal group of finite height over a perfect field κ of characteristic $p > 0$. Let $E(\kappa, \widehat{\boldsymbol{G}})$ be the Lubin–Tate spectrum of $\widehat{\boldsymbol{G}}_0$, viewed as an adic $\mathbb{E}_\infty$–ring with respect to the n^{th} Landweber ideal $\mathfrak{I}_n \subseteq \pi_0(E(\kappa, \widehat{\boldsymbol{G}}_0))$. There is a natural equivalence*

$$\mathrm{Spf}(E(\kappa, \widehat{\boldsymbol{G}}_0)) \simeq \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$$

in the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$.

Remark 1.10 Lurie formulates his result (which also works over more general perfect base rings than a field) in terms of deformations of p–divisible groups instead of formal groups. This has the advantage of being more general, applying for instance also to étale p–divisible groups, and is crucial in the follow-up paper [18] on Hopkins–Kuhn–Ravenel character theory and transchromatic ambidexterity. Alas, for our purposes, since all the p–divisible groups in sight would be connected, the analogue of Tate's theorem in [17, Section 2.3], allows us to restrict to formal groups instead. Ultimately however, this is nothing more than an aesthetic preference, and this note could well have been written with the functor $\mathcal{M}_{\mathrm{BT}}$ everywhere in place of $\mathcal{M}_{\mathrm{FG}}$.

2 Morava stabilizer group action and fixed points

2.1 Complete Noetherian local $\mathbb{E}_\infty$–rings

For the remainder of this note, κ will be a perfect field of characteristic $p > 0$. We find it convenient to restrict to a smaller subcategory of $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, consisting roughly of complete Noetherian local $\mathbb{E}_\infty$–rings with residue field κ.

Definition 2.1 Let $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denote the subcategory spanned by complete adic $\mathbb{E}_\infty$–rings A for which the commutative ring $\pi_0(A)$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$, topologized with respect to the $\mathfrak{m}$–adic topology, and such that there exists an abstract (ie nonspecified) isomorphism $\pi_0(A)/\mathfrak{m} \simeq \kappa$.

Remark 2.2 The notation $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ is potentially misleading. Indeed, unlike what it may seem to indicate, said ∞–category *is not* equivalent to a subcategory of the overcategory $\mathrm{CAlg}_{/\kappa}$. That would hold if we restricted to connective object, but we cannot do so, since our primary interest rests with the nonconnective complex periodic $\mathbb{E}_\infty$–rings.

Remark 2.3 Similarly to the preceding remark, the connective objects in $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ are not the Noetherian $\mathbb{E}_\infty$–rings in the sense of [14, Definition 7.2.4.30]. We could not have used that notion of Noetherianness in the above definition, since it again only applies to connective $\mathbb{E}_\infty$–rings. It would be possible to imitate such a definition, by also imposing finiteness assumptions on the homotopy groups $\pi_i(A)$ in Definition 2.1 for $i \neq 0$. But since we can make do without, we choose to only impose the (unavoidable) π_0–level assumption. That is to say, the notion of a complete Noetherian local $\mathbb{E}_\infty$–ring from Definition 2.1 is only guaranteed to be adequate for the purposes of this paper. For most other purposes in spectral algebraic geometry where a Noetherian assumption might be desirable, stronger finiteness assumptions would probably need to be imposed.

From here on, we will consider the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$ as the setting for formal spectral algebraic geometry. In particular, we will implicitly restrict the domain of the functor $\mathrm{Spf}(A)$ to the subcategory $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ for any adic $\mathbb{E}_\infty$–ring A.

Remark 2.4 The restriction functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$, induced from the subcategory inclusion $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, preserves both limits and colimits. The Yoneda embedding $(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$ also preserves limits, and the coproduct in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ is given by the completed smash product of [14, Corollary 7.3.5.2]. It follows that we have for any pair of complete adic $\mathbb{E}_\infty$–rings A and B a canonical equivalence

$$\mathrm{Spf}(A) \times \mathrm{Spf}(B) \simeq \mathrm{Spf}(A \mathbin{\widehat{\otimes}} B)$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$. That is to say, restriction to complete Noetherian local $\mathbb{E}_\infty$–rings does not change the products of affine formal spectral schemes.

The functors of the *ring of functions* $\mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ and the ∞–category of *quasicoherent sheaves* $\mathrm{QCoh}\colon \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ are defined by right Kan extension from the subcategory of affines $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}}$ (ie representable functors) on which they are defined as

$$\mathcal{O}(\mathrm{Spf}(A)) := A, \quad \mathrm{QCoh}(\mathrm{Spf}(A)) := \mathrm{Mod}^{\mathrm{cpl}}_A.$$

For a detailed treatment of such an approach to quasicoherent sheaves (in a slightly different but closely related setting), see [15, Section 6.2.2].

Remark 2.5 Because we are not equipping $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ with a Grothendieck topology, questions of descent are beyond our reach. Fortunately, as explained for QCoh in [15, Proposition 6.2.3.1] (in only a slightly different setting), both $\mathcal{O}$ and QCoh are agnostic regarding sheafification, making their definition unambiguous.

Remark 2.6 In defining the functors $\mathcal{O}$ and QCoh by Kan extension, we are being slightly imprecise regarding set-theoretical considerations. The issue is that the category $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ is not small. This may be circumvented by the usual trick of universe enlargement, at the cost of eg the ∞–category $\mathrm{QCoh}(X)$ being not necessarily small. For a precise treatment along those lines in a closely related setting, see [15, Section 6.2]. On the other hand, all functors which we will ultimately be interested in will all be given in explicit ways as small colimits of representables. In principle, we could in each individual such case redefine the functors $\mathcal{O}$ and QCoh by indexing them on appropriate small indexing categories, and verify *post factum* that the choice didn't matter. With this understanding, we will ignore questions of smallness, and set-theoretical technicalities alike, from now on.

Noting that we may have equivalently replaced the ∞–category $\mathcal{C}\mathrm{at}_\infty$ with $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ in the definition of quasicoherent sheaves (with the caveat of Remark 2.6 in mind), we see that any map of functors $f\colon X \to Y$ induces adjoint functors

$$f^*\colon \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*,$$

the familiar pullback and pushforward functoriality. In particular, we call pushforward along the terminal map $p\colon X \to *$ *global sections* and denote $\Gamma(X;\mathcal{F}) := p_*(\mathcal{F})$ for any $\mathcal{F} \in \mathrm{QCoh}(X)$. For the structure sheaf $\mathcal{F} = \mathcal{O}_X$, global sections $\Gamma(X;\mathcal{O}_X) \simeq \mathcal{O}(X)$ recover the ring of functions.

Remark 2.7 The pushforward functor $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is not necessarily very well behaved without some additional assumptions on the morphism $f\colon X \to Y$ (such as being quasicompact and separated); eg the Beck–Chevalley push–pull formula for base change, and the projection formula may both fail in general. In particular, this "functor-of-points" pushforward it might in that case not coincide with a "ringed space" pushforward, if such exists. See also [15, Warning 6.3.4.2].

Remark 2.8 It follows from the definition of global sections that there is a chain of homotopy equivalences

$$\Gamma(X;\mathscr{F}) \simeq \mathrm{Map}_{\mathrm{Sp}}(S, p_*(\mathscr{F})) \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(p^*(S), \mathscr{F}) \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \mathscr{F}).$$

As consequence, we cannot expect the global sections functor $\mathscr{F} \mapsto \Gamma(X;\mathscr{F})$ to preserve filtered colimits unless the structure sheaf $\mathscr{O}_X$ is a compact object of the ∞–category $\mathrm{QCoh}(X)$. Since $\Gamma(X;-) = p_*$ is a pushforward along the terminal map, this ties into the more general ill-behavedness of the pushforward discussed in Remark 2.7. Fortunately, such issues will not arise in the (rather simplistic) applications we discuss in this paper.

2.2 A short digression on monodromy

In the proof of Proposition 2.12 in the next subsection, we will need a certain result, which becomes particularly simple and natural when viewed in a slightly more general context than strictly necessary for our purposes.

Recall that monodromy is classically understood to be the action of the fundamental group $\pi_1(X, x)$ of a base space X on the fiber $\mathcal{L}_x$ of a local systems $\mathcal{L}$ on X, acting through parallel transport around loops. The following is a simple incarnation of that idea in the setting of an ∞–topos, but with the notion of a "point" being understood in the generalized sense of algebraic geometry.

Lemma 2.9 *Let $x\colon P \to X$ be a morphism in an ∞–topos $\mathcal{X}$.*

(i) *The "based loop space" $\Omega_x(X) := P \times_X P$ admits a canonical group structure in the overtopos $\mathcal{X}_{/P}$, exhibiting it as an object $\Omega_x(X) \in \mathrm{Grp}(\mathcal{X}_{/P})$. There is a canonical equivalence of simplicial objects*

$$\mathrm{B}^\bullet_{\Omega_x(X)}(P, P) \simeq \check{\mathrm{C}}^\bullet(P \xrightarrow{x} X)$$

between its bar construction in $\mathcal{X}_{/P}$ and the Čech nerve of x in $\mathcal{X}$.

(ii) *For any object $Y \in \mathcal{X}_{/X}$, we define its "fiber over x" through the pullback square*

$$\begin{tikzcd} x^*(Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ P \arrow[r, "x"] & X \end{tikzcd} \tag{2}$$

in $\mathcal{X}$. This "fiber" $x^*(Y) \in \mathcal{X}_{/P}$ admits a canonical $\Omega_x(X)$–action, whose bar constriction in $\mathcal{X}_{/P}$ is equivalent to the Čech nerve in $\mathcal{X}$,

$$\mathrm{B}^\bullet_{\Omega_x(X)}(P, x^*(Y)) \simeq \check{\mathrm{C}}^\bullet(x^*(Y) \to Y).$$

Proof Recall from [12, Section 6.1.2; 14, Proposition 2.4.2.5] that group objects and group actions (or their common generalization, groupoid objects) in an ∞–topos are completely and equivalently encoded by their bar constructions. Thus it is necessary and sufficient to verify that the Čech complexes in question are of the appropriate forms for a group object and group action respectively.

For (i), we rewrite the Čech complex of the morphism x as

$$\check{\mathrm{C}}^\bullet(x) \simeq \underbrace{P \times_X \cdots \times_X P}_{\bullet+1} \simeq \underbrace{(P \times_X P) \times_P \cdots \times_P (P \times_X P)}_{\bullet} \simeq \underbrace{\Omega_x(X) \times_P \cdots \times_P \Omega_x(X)}_{\bullet}.$$

It follows clearly that it satisfies the Segal condition and exhibits $\Omega_x(X) \in \mathrm{Grp}(\mathcal{X}_{/P})$.

For (ii), observe that we may compare the two Čech nerves in sight via (degreewise) pullback of simplicial objects. Combining that with point (i), we get equivalences of simplicial objects

$$\begin{aligned}
\check{\mathrm{C}}^\bullet(x^*(Y) \to Y) &\simeq \check{\mathrm{C}}^\bullet(P \to X) \times_X Y \\
&\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, P) \times_X Y \\
&\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, P \times_X Y) \\
&\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, x^*(Y)),
\end{aligned}$$

exhibiting the desired $\Omega_x(X)$–action on the fiber $x^*(Y)$. □

Remark 2.10 In the setting of Lemma 2.9, passage to geometric realizations from (i) gives an equivalence $\mathrm{B}\Omega_x(X) \simeq X^\wedge_x$ between the classifying space for $\Omega_x(X)$ (in the overtopos $\mathcal{X}_{/P}$) and the so-called *nilpotent completion* of X at x, defined as $X^\wedge_x = |\check{\mathrm{C}}^\bullet(x)|$. This is a not necessarily affine variant of the notion of nilpotent completion of ring spectra, first introduced by Bousfield [3, Theorem 6.5]. When $x\colon P \to X$ is an effective epimorphism, we have $X^\wedge_x \simeq X$. Then Lemma 2.9(ii) shows that $Y \simeq x^*(Y)/\Omega_x(X)$, generalizing the classical fact that a local system on a connected base space is completely determined by its monodromy representation.

Remark 2.11 Let us take for $\mathcal{X}$ the presheaf ∞–topos $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$, the usual setting for "functor of points" nonconnective spectral algebraic geometry (once again ignoring questions of descent). An $\mathbb{E}_\infty$–ring A gives rise to the terminal map of nonconnective affines $x_A\colon \mathrm{Spec}(A) \to \mathrm{Spec}(S)$, which we may view as an A–point of $\mathrm{Spec}(S)$. It follows from Lemma 2.9(i) that the loop space $\Omega_{x_A}(\mathrm{Spec}(S))$ admits a group structure over $\mathrm{Spec}(A)$. That amounts to an appropriately interpreted (see [24] for a thorough discussion of appropriate coalgebras in this setting) Hopf algebroid structure on $\mathcal{O}(\Omega_{x_A}(\mathrm{Spec}(S)) \simeq A \otimes A$ over A. Upon passage to homotopy groups, this recovers the usual "generalized dual Steenrod algebra" Hopf algebroid

$$(\pi_*(A), \pi_*(A \otimes A)) = (A_*, A_*A).$$

Similarly, given any $\mathbb{E}_\infty$–ring A, the $\Omega_{x_A}(\mathrm{Spec}(S))$–action on the fiber $x_A^*(\mathrm{Spec}(X))$ described in Lemma 2.9(ii), gives rise on homotopy groups to the usual "generalized Steenrod comodule" structure on $\pi_*(A \otimes X) = A_*(X)$. This hints at the relationship between the monodromy construction of Lemma 2.9 and generalized Adams spectral sequences, which we partly elucidate in Section 3.2, and in Remark 3.4 in a bit more detail in the case of the Adams–Novikov spectral sequence.

2.3 Morava stabilizer group action on oriented deformations

Fix a formal group $\widehat{\boldsymbol{G}}_0$ of finite height over $\kappa = \overline{\boldsymbol{F}}_p$ and let $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ be its (big, ie extended) Morava stabilizer group, viewed as an algebraic group, and hence a functor $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \to \mathcal{S}$, as explained in [8, Remark 5.29] and reviewed in Remark 2.17.

Proposition 2.12 *There exists a canonical action of the Morava stabilizer group* $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ *on the oriented deformations* $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$, *whose two-sided bar construction is equivalent as a simplicial object in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$,

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to *) \simeq \mathrm{B}^\bullet_{\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)}(*, \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}),$$

to the Čech nerve of (*the terminal map of*) $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$.

Before embarking on the proof, let us outline its logical structure. We successively reduce the statement to simpler ones, until we end up with an explicit verification. The first reduction, from the oriented statement of Proposition 2.12 to a nonoriented version, Lemma 2.13, is completely formal. The proof of Lemma 2.13 is where we use the monodromy ideas from the previous subsection. Using them, or more precisely Lemma 2.9, we are reduced to identifying the naturally occurring automorphism group with the Morava stabilizer group. That is something of a classical observation, eg [13, Lectrure 19] or [8, Theorem 7.18], and is the content of Lemma 2.16. Its proof, after reducing from the ∞–categorical to a classical 1–categorical setting, is an explicit point-set-level comparison.

Proof of Proposition 2.12 By the definition of oriented deformations,

$$\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}. \tag{3}$$

The factor $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}$ in this fibered product may be replaced with $\{\widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_{A}\}$ when A is complex oriented, and with $\varnothing$ when A is not. It follows from this observation that

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to *) \simeq \check{\mathrm{C}}^\bullet(\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}) \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}$$

as the base change of simplicial objects. Consequently, pulling back the equivalence of simplicial objects from the next Lemma 2.13 along the inclusion $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}} \to \mathcal{M}_{\mathrm{FG}}$ gives rise to a $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$–action on $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$ with the desired bar construction. □

Lemma 2.13 *There exists a canonical action of the Morava stabilizer group* $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ *on the unoriented deformations* $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$, *whose two-sided bar construction is equivalent as a simplicial object in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$,

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}) \simeq \mathrm{B}^\bullet_{\mathbb{G}(\kappa,\widehat{\boldsymbol{G}}_0)}(*, \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}),$$

to the Čech nerve of the map $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}$.

Proof Unlike oriented deformations, unoriented deformations of formal groups are as a functor determined (as Kan extension) by its restriction to connective $\mathbb{E}_\infty$–rings by [17, Proof of Theorem 3.4.1]. Therefore, let us implicitly restrict all functors to the full subcategory $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \subseteq \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ spanned by connective $\mathbb{E}_\infty$–rings for the rest of this proof.

There, we have by [17, Proof of Proposition 3.4.3] a natural identification

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \simeq (\mathrm{Spec}(\kappa)/\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$$

with the relative de Rham space of the morphism $\mathrm{Spec}(\kappa) \to \mathcal{M}_{\mathrm{FG}}$ classifying $\widehat{\boldsymbol{G}}_0$. Recall from [15, Definition 18.2.1.1] that the *relative de Rham space* of a map of functors $X \to Y$ is defined as the pullback

$$(X/Y)_{\mathrm{dR}} \simeq X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y, \tag{4}$$

where the *absolute de Rham space* of a functor X is given by[2] $X_{\mathrm{dR}}(A) = X(\pi_0(A)/\mathfrak{m})$.

[2]Restricting to the subcategory $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ helps substantially here, as no colimiting over nilpotent ideals of definition is necessary.

Observe that we have at this point found ourselves in the setting of Lemma 2.9, with the pullback square

$$\begin{tikzcd} \mathrm{Def}_{\widehat{G}_0} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{FG}} \arrow[d] \\ \mathrm{Spec}(\kappa)_{\mathrm{dR}} \arrow[r] & (\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}} \end{tikzcd}$$

playing the role of (2). More precisely, we have

- an ambient ∞–topos $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})$,
- a "point" $\mathrm{Spec}(\kappa)_{\mathrm{dR}} \to (\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$,
- an object $\mathcal{M}_{\mathrm{FG}}$ over the "base space" $(\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$,
- and its "fiber" $\mathrm{Spec}(\kappa)_{\mathrm{dR}} \times_{(\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}} \mathcal{M}_{\mathrm{FG}} \simeq \mathrm{Def}_{\widehat{G}_0}$.

Lemma 2.9(i) thus exhibits the "based loop space", which is the de Rham space $\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}$ of

$$\mathrm{Spec}(\kappa) \times_{\mathcal{M}_{\mathrm{FG}}} \mathrm{Spec}(\kappa) \simeq \Omega_{\widehat{G}_0}(\mathcal{M}_{\mathrm{FG}}) \simeq \underline{\mathrm{Aut}}(\widehat{G}_0),$$

the automorphism group of the formal group $\widehat{G}_0$, as a group object in the overcategory $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$. Thus Lemma 2.9(ii) equips the "fiber" $\mathrm{Def}_{\widehat{G}_0}$ with the "monodromy" $\underline{\mathrm{Aut}}(\widehat{G}_0)$–action over $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$, whose bar construction is

$$\mathrm{B}^{\bullet}_{\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}}(\mathrm{Spec}(\kappa)_{\mathrm{dR}}, \mathrm{Def}_{\widehat{G}_0}) \simeq \check{\mathrm{C}}^{\bullet}(\mathrm{Def}_{\widehat{G}_0} \to \mathcal{M}_{\mathrm{FG}}).$$

In light of Lemma 2.16, this $\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}$–action on the deformation (pre)stack $\mathrm{Def}_{\widehat{G}_0}$ in the overcategory $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$ is equivalent to a $\mathbb{G}(\kappa, \widehat{G}_0)$–action on it in $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})$, exhibited on the level of bar constructions (see Remark 2.14) by the equivalence

$$\mathrm{B}^{\bullet}_{\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}}(\mathrm{Spec}(\kappa)_{\mathrm{dR}}, \mathrm{Def}_{\widehat{G}_0}) \simeq \mathrm{B}^{\bullet}_{\mathbb{G}(\kappa, \widehat{G}_0)}(*, \mathrm{Def}_{\widehat{G}_0}). \tag{5}$$ □

Remark 2.14 We must clarify that the two bar constructions appearing on each side of the equivalence (5) are formed in different ∞–categories. That is to say, the products comprising the simplices on the left-hand side are all taken over $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$, while on the right-hand side, the products are absolute, ie taken over the terminal object $*$.

Remark 2.15 The de Rham space $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$ that we encountered above in the proof of Lemma 2.13 is equivalent to the affine formal scheme $\mathrm{Spf}(W^+(\kappa))$, where $W^+(\kappa)$ the $\mathbb{E}_\infty$–ring of *spherical Witt vectors over* κ, as defined in [17, Example 5.2.7]. Indeed, in [17, Proof of Theorem 5.2.5] the spherical Witt vectors are defined to corepresent as an affine formal scheme the relative de Rham space $(\mathrm{Spec}(\kappa)/\mathrm{Spec}(S))_{\mathrm{dR}}$. But since clearly $\mathrm{Spec}(S) \simeq \mathrm{Spec}(S)_{\mathrm{dR}}$, it follows that $(\mathrm{Spec}(\kappa)/\mathrm{Spec}(S))_{\mathrm{dR}} \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$, as claimed. More concretely, the universal property of the spherical Witt vectors may be written as

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(W^+(\kappa), A) \simeq \varinjlim_I \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\kappa, \pi_0(A)/I)$$

for any adic $\mathbb{E}_\infty$–ring A, and with the colimit ranging over all of the finitely generated ideals of definition in $\pi_0(A)$. Another characterization of it is that $W^+(\kappa)$ is a flat p–complete $\mathbb{E}_\infty$–ring and $\pi_0(W^+(\kappa)) = W(\kappa)$ recovers the usual ring of (p–typical) Witt vectors.

Lemma 2.16 *There is a canonical equivalence* $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} \simeq \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spf}(W^+(\kappa))$ *of group objects in* $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spf}(W^+(\kappa))}$.

Proof By unwinding the definitions, we find for any connective $A \in \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ that

$$\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \simeq \mathrm{Spec}(\kappa)(\pi_0(A))/\mathfrak{m}) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/\mathfrak{m})} \mathrm{Spec}(\kappa)(\pi_0(A)/\mathfrak{m})$$

consists of a pair of maps $f_1, f_2 : \kappa \to \pi_0(A)/\mathfrak{m}$ and an isomorphism $\varphi : f_1^* \widehat{\boldsymbol{G}}_0 \to f_2^* \widehat{\boldsymbol{G}}$ of formal groups over $\pi_0(A)/\mathfrak{m}$. In particular, it is a discrete space — indeed, this follows from the fact that the ordinary moduli stack of formal groups,

$$\mathcal{M}_{\mathrm{FG}}|_{\mathrm{CAlg}^\heartsuit} : \mathrm{CAlg}^\heartsuit \to \mathcal{S},$$

is actually a 1–stack, ie a groupoid-valued functor $\mathrm{CAlg}^\heartsuit \to \tau_{\le 1}(\mathcal{S}) \hookrightarrow \mathcal{S}$. Since the functor $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$ amounts, as observed above, to passing to internal automorphisms of this stack, and the essential image of the based loops functor $\Omega : \tau_{\le 1}(\mathcal{S}_*) \to \mathcal{S}$ belongs to the full subcategory of discrete spaces $\mathcal{S}\mathrm{et} \simeq \tau_{\le 0}(\mathcal{S}) \hookrightarrow \mathcal{S}$, it is a set-valued functor itself. In conclusion, the functor $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} : (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to \mathcal{S}$ factors through $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}$ in the target, and through $\pi_0 : (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit$ in the source. The same holds for $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ by definition of the de Rham space. Hence to prove the lemma, it suffices to exhibit an isomorphism between the two functors $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$ and $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ as group objects in the ordinary category $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit, \mathcal{S}\mathrm{et})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$.

Let us therefore construct a natural transformation $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} \to \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ as functors $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$ over $\mathrm{Spf}(W^+(\kappa)) \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$. Fix a complete Noetherian local commutative ring A with residue field κ. Recall from the above discussion that elements of $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A)$ consist of triples (f_1, f_2, φ) as above, and the base map $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \to \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A)$ is given by $(f_1, f_2, \varphi) \mapsto f_1$. Thus fixing the element $f_1 \in \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A)$ (since we wish to work over $\mathrm{Spf}(W^+(\kappa)) \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$), we obtain an element

$$(g, \psi) \in \mathrm{Gal}(\kappa/\boldsymbol{F}_p) \ltimes \mathrm{Aut}_{\mathrm{FGrp}(\kappa)}(\widehat{\boldsymbol{G}}_0) = \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$$

as follows. Thanks to the hypothesis that $A/\mathfrak{m} \simeq \kappa$, the field map f_1 may be abstractly identified with a field endomorphism of $\kappa = \overline{\boldsymbol{F}}_p$. Any such endomorphism must fix the prime subfield $\boldsymbol{F}_p$, and since the inclusion $\boldsymbol{F}_p \subseteq \overline{\boldsymbol{F}}_p$ is algebraic, this implies that it is actually an automorphism. It follows that $f_1 : \kappa \to A/\mathfrak{m}$ is a field isomorphism, so we can set $g := f_1^{-1} \circ f_2$. We obtain the formal group isomorphism over κ as

$$\psi : \widehat{\boldsymbol{G}}_0 \xrightarrow{\varphi} (f_1^{-1})^* f_2^* \widehat{\boldsymbol{G}}_0 \simeq g^* \widehat{\boldsymbol{G}}_0.$$

Sending $(f_1, f_2, \varphi) \mapsto ((g, \varphi), f_1)$ gives the desired map of sets

$$\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \to \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A).$$

It is clear from the description that this procedure is bijective, compatible with the group structure, functorial in A, and compatible with the maps to $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$; hence it gives rise to an equivalence of functors as claimed. □

Remark 2.17 The matter of viewing $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ as a profinite group scheme here comes from the classical observation that topology coincides with the usual Zariski topology on automorphisms. Indeed, as we noted in the proof, all the functors involved in Lemma 2.16 factor through the functor $\pi_0 : \mathrm{CAlg} \to \mathrm{CAlg}^\heartsuit$, and are as such a matter of classical algebraic geometry. In that context, see [8, Theorem 7.18], or [13, Lecture 19].

On the other hand, let us explain where the profinite structure on $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ *is coming from* from the algebrogeometric perspective. Let us view the fixed formal group as a functor $\widehat{\boldsymbol{G}}_0 : (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$ from Artinian local rings with residue field κ; ie infinitesimal extensions of the point $\mathrm{Spec}(\kappa)$. Consider the subcategory $\mathrm{Nil}^{\le n}_{/\kappa} \subseteq (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit$ of local Artinian rings with $\mathfrak{m}^{n+1} = 0$. Restriction and Kan extension back along this inclusion produces a functor $\widehat{\boldsymbol{G}}_0^{\le n} : (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$, which we may view as the n^{th} infinitesimal neighborhood; Goerss calls this the *n–bud* of the formal group $\widehat{\boldsymbol{G}}_0$, see in particular [8, Remark 3.24]. Since every ideal in an Artinian local ring is nilpotent, the tower

$$\mathrm{Nil}^{\le 0}_{/\kappa} \subseteq \mathrm{Nil}^{\le 1}_{/\kappa} \subseteq \mathrm{Nil}^{\le 2}_{/\kappa} \subseteq \mathrm{Nil}^{\le 3}_{/\kappa} \subseteq \cdots \subseteq (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit$$

is exhaustive and the canonical map $\varinjlim \widehat{\boldsymbol{G}}_0^{\leq n} \to \widehat{\boldsymbol{G}}_0$ is an equivalence. Furthermore, any morphism of formal groups $\widehat{\boldsymbol{G}} \to \widehat{\boldsymbol{G}}'$ induces a family of maps $\widehat{\boldsymbol{G}}^{\leq n} \to \widehat{\boldsymbol{G}}'^{\leq n}$ for all $n \geq 0$, which induces an isomorphism

$$\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) = \mathrm{Aut}(\widehat{\boldsymbol{G}}_0) \simeq \varprojlim \mathrm{Aut}(\widehat{\boldsymbol{G}}_0^{\leq n}).$$

Each factor in this filtered limit is finite, recovering the usual profinite structure on the Morava stabilizer group. In particular, this implies that the product

$$\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}\big(C^*_{\mathrm{cont}}(\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0); E(\kappa, \widehat{\boldsymbol{G}}_0))\big)$$

is the formal spectrum of an incarnation of continuous $E(\kappa, \widehat{\boldsymbol{G}}_0)$–valued cochains on the profinite group $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$.

Remark 2.18 Let us indicate an alternative approach to proving Proposition 2.12. Instead of using the identification (3), we can rather observe that we have for any $A \in \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ a natural equivalence

$$\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}(A) \simeq \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(\pi_0(A)) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A))} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A).$$

In light of that, it suffices to establish an appropriate $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$–action on unoriented deformations, when all functors in sight are postcomposed with the functor $A \mapsto \pi_0(A)$. That involves only classical (ie nonspectral) algebraic geometry, and as such avoids coherence issues. Therefore the desired bar construction claim follows inductively from finding an appropriately equivariant equivalence

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \times_{\mathcal{M}_{\mathrm{FG}}} \mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \simeq \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}, \tag{6}$$

with both sides restricted to the subcategory $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\heartsuit} \subseteq \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ of discrete objects. Since any complete Noetherian local ring may be written as a filtered limit of Artinian ones, and we are working in the "continuous" category, it further suffices to prove the result upon the further restriction to local Artinian rings; see [8, Remark 7.3]. For that, we can reference [8, Theorem 7.18].

There is one final small hitch: Goerss's analogue of (6) takes the fiber product over a moduli functor $\widehat{\mathcal{H}}(n) = (\mathcal{M}^{=n}_{\mathrm{FG}}/\mathcal{M}^{\leq n}_{\mathrm{FG}})_{\mathrm{dR}}$ instead of over $\mathcal{M}_{\mathrm{FG}}$. But since the forgetful functor $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}$ naturally factors through the substack inclusion $\widehat{\mathcal{H}}(n) \hookrightarrow \mathcal{M}_{\mathrm{FG}}$, this turns out not to effect the result. See [10, Section 3.5] for further discussion.

2.4 The Devinatz–Hopkins theorem

As before, fix $\kappa = \overline{\boldsymbol{F}}_p$ and let $\widehat{\boldsymbol{G}}_0$ be of height n, (which specifies it up to isomorphism). We denote by E_n and $\mathbb{G}_n$ the associated Lubin–Tate spectrum and Morava stabilizer group, respectively. Proposition 2.12 equips $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}(E_n)$ with an action[3] of $\mathbb{G}_n$ on $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}(E_n)$ in the ∞–topos $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})$. Let $\mathrm{Spf}(E_n)/\mathbb{G}_n$ denote the quotient of this action in this ∞–topos. We view its ring of functions

$$E_n^{h\mathbb{G}_n} := \mathcal{O}(\mathrm{Spf}(E_n)/\mathbb{G}_n)$$

as the *continuous homotopy fixed points* of the corresponding action of $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n. See Remark 3.5 for some further justification of this terminology.

Theorem 2.19 (Devinatz–Hopkins) *With continuous homotopy fixed points defined as above, the initial map $L_{K(n)}S \to E_n$ in $L_{K(n)}\,\mathrm{Sp}$ induces an equivalence $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$.*

Proof By definition of the ring of functions, we have $\mathcal{O}(\mathrm{Spf}(E_n)) \simeq E_n$. Similarly, for products we have $\mathcal{O}(\mathrm{Spf}(E_n)^{\times\bullet}) \simeq E_n^{\widehat{\otimes}\bullet}$, where $\widehat{\otimes}$ denotes the completed smash product of [14, Corollary 7.3.5.2], ie the coproduct in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ of complete adic $\mathbb{E}_\infty$–rings — see Remark 2.4. Therefore Proposition 2.12 implies that

$$E_n^{h\mathbb{G}_n} \simeq \mathcal{O}(\mathrm{Spf}(E_n)^{\times(\bullet+1)}) \simeq \mathrm{Tot}(E_n^{\widehat{\otimes}(\bullet+1)}).$$

It follows from [17, Corollary 4.5.4] that completion in the ∞–category of E_n–modules coincides with $K(n)$–localization, and so $E_n^{\widehat{\otimes}\bullet} \simeq L_{K(n)}(E_n^{\otimes\bullet})$. Thus it suffices to show that $L_{K(n)}S \to E_n$ induces an equivalence

$$\mathrm{Tot}(L_{K(n)}(E_n^{\otimes(\bullet+1)})) \simeq L_{K(n)}S. \tag{7}$$

That is a standard result, stemming from the nilpotence of $L_{K(n)}S$ in the ∞–category $L_{K(n)}\,\mathrm{Mod}_{E_n}$, and ultimately, the horizontal vanishing line in the $K(n)$–local Adams spectral sequence for E_n; see for instance [6, Proposition AI.3]. But for completeness, we sketch an argument anyway, following the account in [19].

The smashing product theorem of Hopkins and Ravenel [22, Theorem 7.5.6] asserts that the Bousfield localization functor $L_n := L_{E_n}$ is smashing, which, by Proposition 8.2.4

[3] Of course this is just the action of $\mathbb{G}_n$ on $\mathrm{Spf}(E_n)$ induced by the identification of the Morava stabilizer group as $\mathbb{G}_n \simeq \mathrm{Aut}(E_n) \simeq \mathrm{Aut}(\mathrm{Spf}(E_n))$, as observed in [17, Remark 5.0.8]. But from our way of obtaining it, its bar construction is more transparent.

of [22], is equivalent to $L_n S$ being E_n–nilpotent. That is further equivalent, by standard nilpotence technology, eg [13, Lectures 30 and 31], to the cosimplicial object $(E_n^{\otimes(\bullet+1)})$, whose totalization is $L_n S$, being pro-constant. Applying the functor $L_{K(n)}$ to this cosimplicial object then gives the desired equivalence. □

Remark 2.20 An explicit analysis of how the horizontal vanishing line in the $K(n)$–local Adams spectral sequence for E_n gives rise to the equivalence (7) is given in [6, Section 4 and Appendix I]. The argument that we gave, following [19], while phrased slightly differently, is merely a repackaging of the same fundamental idea — indeed, the proof of the Hopkins–Ravenel smashing product theorem is based on the existence of a uniform vanishing line; see [20, Section 3.4] for a sketch and relationship to the "standard nilpotence technology" referred to in the proof above.

Remark 2.21 The equivalence of Theorem 2.19 is a purely function-level statement. Indeed, the quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$ is not equivalent to the affine formal scheme $\mathrm{Spf}(L_{K(n)}S)$. The value of $\mathrm{Spf}(E_n)/\mathbb{G}_n$ on any noncomplex-periodic $K(n)$–local $\mathbb{E}_\infty$–ring is the empty set, while the value of $\mathrm{Spf}(L_{K(n)}S)$ is contractible for all $K(n)$–local $\mathbb{E}_\infty$–rings.

Despite the preceding remark, we may view quasicoherent sheaves on the quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$, which are by definition a derived version of Morava modules, as a natural incarnation in spectral formal algebraic geometry of the $K(n)$–local stable category.

Corollary 2.22 *There is a canonical equivalence of symmetric monoidal ∞–categories*

$$\mathrm{QCoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq L_{K(n)}\,\mathrm{Sp}.$$

Proof It follows from the proof of Theorem 2.19 that

$$\mathrm{QCoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq \mathrm{Tot}(\mathrm{Mod}^{\mathrm{cpl}}_{E_n^{\hat{\otimes}(\bullet+1)}}) \simeq \mathrm{Tot}(L_{K(n)}\,\mathrm{Mod}_{L_{K(n)}(E_n^{\otimes(\bullet+1)})}),$$

which is equivalent to the $K(n)$–local stable ∞–category in [19, Proposition 10.10]. □

2.5 Analogue over a general base

At the cost of replacing the Morava stabilizer group with the more involved algebrogeometric group $\mathcal{G} := \underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$, the contents of this section still hold after dropping the assumption that $\kappa = \overline{\boldsymbol{F}}_p$.

Proposition 2.23 *Let $\widehat{G}_0$ be a formal group of finite height over a perfect field κ of positive characteristic. Then there exists a canonical $\mathcal{G}$–action on $\mathrm{Def}^{\mathrm{or}}_{\widehat{G}_0}$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})_{/\mathrm{Spec}(W^+(\kappa))}$, whose two-sided bar construction in said ∞–category is equivalent as a simplicial object in the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$,*

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{G}_0} \to *) \simeq \mathrm{B}^\bullet_{\mathcal{G}}\big(\mathrm{Spf}(W^+(\kappa)), \mathrm{Def}^{\mathrm{or}}_{\widehat{G}_0}\big),$$

to the Čech nerve of (the terminal map of) $\mathrm{Def}^{\mathrm{or}}_{\widehat{G}_0}$.

Proof The only step of the proof of Proposition 2.12 that employs the assumption $\kappa = \overline{F}_p$ is in the proof of Lemma 2.16. The rest of the argument, including the proof of Lemma 2.13, goes through for any perfect field κ of positive characteristic, giving the stated result. □

Proposition 2.23 equips $\mathrm{Spf}(E(\kappa, \widehat{G}))$ with a $\mathcal{G}$–action, though this time we need to be working in the relative setting over $\mathrm{Spf}(W^+(\kappa))$. This may be viewed as an incarnation of a $\mathcal{G}$–action on the Lubin–Tate spectrum $E(\kappa, \widehat{G}_0)$ in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{/W^+(\kappa)}$. Just as before, we obtain a workable definition of continuous homotopy fixed points by setting

$$E(\kappa, \widehat{G}_0)^{h\mathcal{G}} := \mathcal{O}\big(\mathrm{Spf}(E(\kappa, \widehat{G}_0))/\mathcal{G}\big),$$

and the analogue of the Devinatz–Hopkins theorem holds as follows.

Proposition 2.24 *Let $\widehat{G}_0$ be a formal group of height $n < \infty$ over a perfect field κ of positive characteristic. With notation as above, the initial map $L_{K(n)}S \to E(\kappa, \widehat{G}_0)$ in $L_{K(n)}\,\mathrm{Sp}$ induces an equivalence of spectra $E(\kappa, \widehat{G}_0)^{h\mathcal{G}} \simeq L_{K(n)}S$.*

Proof The proof of Theorem 2.19 works just as well in this setting, provided we use [11, Proposition 5.2.6] for the nilpotence claim. □

Remark 2.25 As explained in [11, Notation 2.1.10], every Lubin–Tate spectrum $E(\kappa, \widehat{G}_0)$ gives rise to a Morava K–theory $K(\kappa, \widehat{G}_0)$. It might seem like we should have used the localization functor $L_{K(\kappa, \widehat{G}_0)}$ in Proposition 2.24, but alas this does not matter, since even though the spectra $K(\kappa, \widehat{G}_0)$ do depend on the base field and formal group used to define them, the induced localization functor does not. By [11, Remark 2.1.14], the Bousfield localization functor $L_{K(\kappa, \widehat{G}_0)} \simeq L_{K(n)}$ only depends on the characteristic of the field κ and the height n of the formal group $\widehat{G}_0$.

In particular, we obtain by the same proof as Corollary 2.22 a "derived Morava module" presentation of the $K(n)$–local stable category for every height n formal group $\widehat{G}_0$ over a perfect field κ of positive characteristic.

Corollary 2.26 *Keeping all the notation from Proposition 2.24, there is a canonical equivalence of symmetric monoidal ∞–categories,*

$$\mathrm{QCoh}\big(\mathrm{Spf}(E(\kappa, \hat{G}_0)/\mathscr{G})\big) \simeq L_{K(n)}\,\mathrm{Sp}.$$

3 Spectral sequences

The goal of this section is to prove a version of the Morava change-of-rings theorem, identifying the $K(n)$–local Adams spectral sequence for E_n with the continuous fixed-point spectral sequence for the $\mathbb{G}_n$–action on E_n. The Devinatz–Hopkins Theorem 2.19 already guarantees that they converge to (filtrations on) the homotopy groups on the same spectrum, but the actual comparison of the spectral sequences (and interpretation of the second one) will take a little more work.

3.1 The descent spectral sequence

Unlike the fundamentally nonconnective $\mathrm{Spf}(E_n)$ and its intimidating-looking quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$, the classifying (pre)stack $\mathrm{B}\mathbb{G}_n = */\mathbb{G}_n$ is quite well behaved. In particular, it (or better, its sheafification; but since the difference between them does not matter for quasicoherent sheaves or functions, we will freely switch between them) is representable by a formal spectral stack which, while not quite Deligne–Mumford, is nonetheless quite manageable.

For instance, $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ admits an accessible t–structure by the (formal geometry analogue of) [15, Proposition 6.2.5.8]. Similarly, the descent spectral sequence, a piece of technology familiar from the theory of topological modular forms, applies to $\mathrm{B}\mathbb{G}_n$. The following proof is essentially a repetition of the one in [7, Chapter 5, Section 3], but since the setting is slightly different, we have opted to spell it out.

Lemma 3.1 *Let $X\colon (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to \mathcal{S}$ be a formal spectral fpqc stack*[4] *that admits a flat cover $U \to X$, such that all the (nontrivial) fiber products $U \times_X \cdots \times_X U$ are affine formal spectral schemes. For any quasicoherent sheaf $\mathscr{F}$ on X, there exists a canonical Adams-graded spectral sequence*

$$E_2^{s,t} = \mathrm{H}^s(X; \pi_t(\mathscr{F})) \Rightarrow \pi_{t-s}(\Gamma(X; \mathscr{F})),$$

called the ***descent spectral sequence****.*

[4] Here we are following [15], in that the absence of the adjective "nonconnective" automatically implies connectivity.

Proof Let $U \to X$ be a flat cover by an affine formal spectral scheme as postulated in the statement of the lemma. It gives rise to a Čech nerve $\check{\mathrm{C}}^\bullet(U \to X)$ and hence a cosimplicial spectrum $\Gamma(\check{\mathrm{C}}^\bullet(U \to X); \mathscr{F}|_{\check{C}^\bullet(U\to X)})$ with totalization $\Gamma(X; \mathscr{F})$. We claim that the Bousfield–Kan spectral sequence of this cosimplicial spectrum, see for instance [14, Remark 1.2.4.4], is the desired spectral sequence.

It converges (albeit only conditionally) to the homotopy groups of the totalization of the cosimplicial spectrum by (an opposite variant of) [14, Proposition 1.2.2.14]. Hence it remains to show that the E_2 page is of the desired form. If $C^*\colon \mathrm{Fun}(\Delta, \mathcal{A}\mathrm{b}) \to \mathrm{Ch}(\mathcal{A}\mathrm{b})^{\geq 0}$ denotes the cochain complex associated to a cosimplicial abelian group[5], then the second page of the Bousfield–Kan spectral sequence of a cosimplicial spectrum $M^\bullet$ may be expressed as cochain complex cohomology,

$$E_2^{s,t} = \mathrm{H}^s(C^*(\pi_t(M^\bullet))).$$

To apply this to the cosimplicial object in question, we must therefore determine the homotopy groups

$$\pi_t\Gamma(\check{\mathrm{C}}^\bullet(U \to X); \mathscr{F}|_{\check{C}^\bullet(U\to X)}) \simeq \pi_t\Gamma(U \times_X \cdots \times_X U; f^*(\mathscr{F})),$$

where $f\colon U \times_X \cdots \times_X U \to X$ is the canonical map. Since $U \to X$ is flat by assumption, the same holds for f, and so $f^* \circ \pi_t \simeq \pi_t \circ f^*$ — see [14, Proposition 7.2.2.13] for the affine case, from which it follows for an arbitrary flat morphism by the yoga of [15, Section 6.2.5]. Secondly, the fiber product $U \times_X \cdots \times_X U$ is affine by hypothesis, from which it follows that its global sections functor is t–exact. Putting all that together, we find that

$$\pi_t\Gamma(U \times_X \cdots \times_X U; f^*(\mathscr{F})) \simeq \Gamma\big(U \times_X \cdots \times_X U; f^*(\pi_t(\mathscr{F}))\big),$$

and so the E_2 page of the spectral sequence in question is just the standard Čech cohomology procedure for computing the s^{th} sheaf cohomology group of the quasicoherent sheaf $\pi_t(\mathscr{F})$ on X. □

Remark 3.2 Though the approach using a cover that we sketched above will be the most convenient for us in what follows, the descent spectral sequence does not depend on that choice from the second page onwards. It may alternatively even be obtained in an invariant way: the assumptions on the stack X ensure that $\mathrm{QCoh}(X)$ admits a well-behaved t–structure. Then the spectral sequence associated to the filtered object

[5] This is the functor that participates in one direction of the Dold–Kan correspondence; see [14, Definition 1.2.3.8] for the opposite version.

$\mathrm{N}(\boldsymbol{Z}) \ni n \mapsto \Gamma(\tau_{\geq -n}(X);\mathscr{F}) \in \mathrm{Sp}$ by [14, Definition 1.2.2.9] again gives rise to the descent spectral sequence after an appropriate reindexing; see [9, Construction 1.5.7] for details.

3.2 The Adams spectral sequence

We wish to apply the descent spectral sequence on the quotient stack $\mathrm{B}\mathbb{G}_n$, for which we need a quasicoherent sheaf on it. Consider the map $q\colon \mathrm{Spf}(E_n)/\mathbb{G}_n \to \mathrm{B}\mathbb{G}_n$, induced on quotients by the terminal projection $p\colon \mathrm{Spf}(E_n) \to *$. Using the pushforward functionality of quasicoherent sheaves, we define the desired sheaf as

$$\mathscr{E}_n := q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \in \mathrm{QCoh}(\mathrm{B}\mathbb{G}_n).$$

As we will need it in the subsequent proposition, let us identify the fiber of this quasicoherent sheaf at the point $i : * \to */\mathbb{G}_n \simeq \mathrm{B}\mathbb{G}_n$. By invoking base change along the pullback square

$$\begin{tikzcd} \mathrm{Spf}(E_n) \arrow[r, "p"] \arrow[d] & * \arrow[d, "i"] \\ \mathrm{Spf}(E_n)/\mathbb{G}_n \arrow[r, "q"] & \mathrm{B}\mathbb{G}_n \end{tikzcd}$$

we find this fiber to be

$$i^*(\mathscr{E}_n) \simeq i^* q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \simeq p_*(\mathcal{O}_{\mathrm{Spf}(E_n)}) \simeq E_n.$$

Proposition 3.3 *The descent spectral sequence for the quasicoherent sheaf $\mathscr{E}_n$ on $\mathrm{B}\mathbb{G}_n$ is isomorphic to*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(L_{K(n)}(E_n \otimes E_n))}(\pi_*(E_n), \pi_*(E_n)) \Rightarrow \pi_*(L_{K(n)}S),$$

the $K(n)$–local Adams spectral sequence for E_n.

Proof Observe that both spectral sequences in question may be obtained as Bousfield–Kan spectral sequences of certain cosimplicial spectra. Thus it suffices to exhibit an equivalence between those.

For the descent spectral sequence, we choose the flat cover $i : * \to \mathrm{B}\mathbb{G}_n$; indeed, this is a cover by the usual yoga of classifying stacks, and it is flat thanks to the Morava stabilizer group $\mathbb{G}_n$ being pro-étale and as such flat. Then the Čech nerve of i is given by $\check{\mathrm{C}}^\bullet(* \to \mathrm{B}\mathbb{G}_n) \simeq \mathbb{G}_n^{\times\bullet}$, and coincides with the bar construction of $\mathbb{G}_n$. Let $p_\bullet\colon \mathbb{G}_n^{\times\bullet} \to *$ denote the terminal map. Then it follows from the computation preceding the statement of the proposition that $\mathscr{F}|_{\check{\mathrm{C}}^\bullet(*\to \mathrm{B}\mathbb{G}_n)} \simeq p_\bullet^*(E_n)$, and so the cosimplicial spectrum that gives rise to the relevant descent spectral sequence is $\Gamma(\mathbb{G}^{\times\bullet}; p_\bullet^*(E_n))$, with the cosimplicial structure inherited from the bar construction of $\mathbb{G}_n$.

For the Adams spectral sequence, let us apply the functor $\mathcal{O}$ to the equivalence of simplicial objects of Proposition 2.12. We obtain an equivalence of cosimplicial spectra

$$L_{K(n)}(E_n^{\otimes(\bullet+1)}) \simeq \mathcal{O}(\mathbb{G}_n^\bullet \times \mathrm{Spf}(E_n)).$$

The left-hand side (for recognizing which, we have made use of a calculation from the proof of Theorem 2.19), gives rise to the $K(n)$–local Adams spectral sequence for E_n. To tackle the left-hand side, consider the Cartesian diagram

$$\begin{tikzcd} \mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Spf}(E_n) \arrow[d, "p"] \\ \mathbb{G}_n^{\times\bullet} \arrow[r, "p_\bullet"] & * \end{tikzcd}$$

Using base change along it, we have a series of equivalences

$$\begin{aligned} \mathcal{O}(\mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n)) &\simeq \Gamma(\mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n); \mathcal{O}_{\mathbb{G}_n^{\times\bullet}\times \mathrm{Spf}(E_n)}) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; (\mathrm{pr}_1)_*\mathrm{pr}_2^*(\mathcal{O}_{\mathrm{Spf}(E_n)})) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; p_\bullet^* p_*(\mathcal{O}_{\mathrm{Spf}(E_n)})) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; p_\bullet^*(E_n)), \end{aligned}$$

and because the cosimplicial structure comes at each step from the bar construction on $\mathbb{G}_n$, this is an equivalence of cosimplicial spectra. Since we already saw that the thus-obtained cosimplicial spectrum gives rise to the descent spectral sequence for $\mathrm{B}\mathbb{G}_n$, we are done. □

Remark 3.4 By working in a nonformal setting, we may argue similarly to the above in order to obtain the Adams–Novikov spectral sequence as a special case of a descent spectral sequence — this is also explained in [15, Remark 9.3.1.9]. Indeed, consider the $\mathbb{E}_\infty$–ring MP, the periodic complex bordism spectrum. As we saw in Remark 2.11, it gives rise to a "based loop space" $\Omega_{x_{\mathrm{MP}}}(\mathrm{Spec}(S))$ in nonconnective spectral stacks over Spec(MP). Let $\mathcal{X}$ denotes the classifying (pre)stack of this nonconnective spectral group scheme. Its underlying ordinary stack is given by

$$\mathcal{X}^\heartsuit \simeq \mathrm{Spec}(\pi_0(\mathrm{MP}))/\mathrm{Spec}(\pi_0(\mathrm{MP}\otimes \mathrm{MP})) \simeq \mathcal{M}_{\mathrm{FG}}^\heartsuit,$$

which is identified with the ordinary stack of formal groups by a celebrated theorem of Quillen. On the other hand, its (derived) $\mathbb{E}_\infty$–ring of functions is given by

$$\mathcal{O}(\mathcal{X}) \simeq \mathrm{Tot}(\mathrm{MP}^{\otimes(\bullet+1)}) = S^{\wedge}_{\mathrm{MP}} \simeq S,$$

which is by definition the MP–*nilpotent completion* of the sphere spectrum of [3], as already discussed in Remark 2.10. This nilpotent completion is well-known to agree with the sphere spectrum itself. Since the smash product $\mathrm{MP}\otimes\mathrm{MP}$ is a flat MP–module (see [17, Theorem 5.3.13]), a variant of Lemma 3.1 applies to the cover $\mathrm{Spec}(\mathrm{MP}) \to \mathcal{X}$. The resulting descent spectral sequence converges to $\pi_*(S)$, while by an argument analogous to our proof of Proposition 3.3, its second page is

$$E_2^{s,t} = \mathrm{H}^s(\mathcal{M}^{\heartsuit}_{\mathrm{FG}}; \pi_t(\mathcal{O}_{\mathcal{X}})) \simeq \mathrm{Ext}^{s,t}_{\pi_*(\mathrm{MP}\otimes\mathrm{MP})}(\pi_*(\mathrm{MP}), \pi_*(\mathrm{MP})),$$

viewable either as sheaf cohomology on the underlying ordinary stack, or as the Adams–Novikov spectral sequence. See [9] (where the content of this remark is expanded upon in [9, Section 2.6]) for a further development of these ideas.

3.3 Homotopy fixed-point spectral sequence

Let us say a few words about the interpretation of Proposition 3.3. We may view $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ as a version of continuous discrete representations of the Morava stabilizer group over the sphere spectrum. From that perspective, the underlying spectrum of a quasicoherent sheaf $\mathscr{F}$ on $\mathrm{B}\mathbb{G}_n$ is given by the fiber $i^*(\mathscr{F}) = M$ (keeping the notation $i : * \to \mathrm{B}\mathbb{G}_n$ from the previous subsection), and the sheaf structure on $\mathscr{F}$ witnesses the $\mathbb{G}_n$–action on M. The (continuous) fixed points of this action are incarnated as global sections $M^{h\mathbb{G}_n} := \Gamma(\mathrm{B}\mathbb{G}_n; \mathscr{F})$, and continuous group cohomology is given in terms of sheaf cohomology as

$$\mathrm{H}^i_{\mathrm{cont}}(\mathbb{G}_n; M) := \mathrm{H}^i(\mathrm{B}\mathbb{G}_n; \mathscr{F}) \simeq \pi_{-i}(M^{h\mathbb{G}_n}).$$

Under these identifications, the descent spectral sequence for $\mathrm{B}\mathbb{G}_n$ corresponds to the fixed-point spectral sequence for $\mathbb{G}_n$,

$$E_2^{s,t} = \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(E_n^{h\mathbb{G}_n}).$$

Remark 3.5 In line with the preceding discussion, the sheaf $\mathscr{E}_n$ on $\mathrm{B}\mathbb{G}_n$ encodes a continuous $\mathbb{G}_n$–action on the Lubin–Tate spectrum E_n. Its continuous homotopy fixed-points, in the above sense, are given by

$$E_n^{h\mathbb{G}_n} \simeq \Gamma(\mathrm{B}\mathbb{G}_n; \mathscr{E}_n) \simeq p_*q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \simeq \Gamma(\mathrm{Spf}(E_n)/\mathbb{G}_n; \mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}).$$

That agrees with (and perhaps justifies) our definition in Section 2.4, and its use in the Devinatz–Hopkins Theorem 2.19 in particular.

Remark 3.6 There exist a number of precise incarnations of the ∞–category of continuous $\mathbb{G}_n$–spectra in the literature, eg of [2] or [21]. In each, the construction from [6] is enhanced (relying heavily on Devinatz and Hopkins's detailed study of finite subgroup actions) to produce a version of E_n in the respective category. Instead, we claim that $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ should be viewed as an incarnation of continuous $\mathbb{G}_n$–spectra, sufficient for our purposes, but not intended to supplant the more sophisticated theories mentioned above (a careful comparison with which we decline to carry out).

Remark 3.7 In spite of the preceding remark, let us observe that our model at least gives rise to spectra with a $\mathbb{G}_n$–action in the sense of [1, Definition 2.2], referred to there as "a simple sense of continuity". Indeed, in light of Remark 2.17, a $\mathbb{G}_n$–action on M in our sense gives rise to an augmented cosimplicial diagram $M \to C^*_{\mathrm{cont}}(\mathbb{G}_n^{\times(\bullet+1)}; M)$. In fact, our approach to continuous $\mathbb{G}_n$–actions is, via the bar resolution $\mathrm{B}\mathbb{G}_n \simeq |\mathbb{G}_n^{\times\bullet}|$, essentially equivalent to the one of [1]. Their restriction to the $K(n)$–local setting is mirrored in our setup by working in the setting of formal algebraic geometry, ie inside the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$ instead of say $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$.

Remark 3.8 With M as in the previous remark, we find by unwinding the proof of Proposition 3.3 that the descent spectral sequence for the corresponding sheaf on $\mathrm{B}\mathbb{G}_n$ is obtained as the Bousfield–Kan spectral sequence of the cosimplicial object $C^*_{\mathrm{cont}}(\mathbb{G}_n^{\times\bullet}; M)$. That is also one traditional approach to defining the homotopy fixed-point spectral sequence (for a compact Lie group, say), somewhat justifying our identification of the two.

With all the notation in place, the following is a formal consequence of Proposition 3.3.

Corollary 3.9 (Morava's change-of-rings isomorphism) *The second page of the $K(n)$–local Adams spectral sequence of the Lubin–Tate spectrum E_n may be expressed as continuous group cohomology $E_2^{s,t} = \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n))$.*

Remark 3.10 One difference between our approach and [6] is that they make use of a form of Morava's change-of-rings isomorphism from [5] to set up their theory. For us, on the other hand, that result did not feed into the construction of $E_n^{h\mathbb{G}_n}$ nor its identification with $L_{K(n)}S$, and we could instead derive it from our considerations.

Of course, that is largely a cosmetic difference; Morava's theorem, even if classically phrased differently, ultimately boils down to algebrogeometric considerations regarding the moduli of formal groups of the sort that we based our approach on.

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