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# Neighboring mapping points theorem

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We introduce and study a new family of theorems extending the class of Borsuk–Ulam and topological Radon type theorems. The defining idea for this new family is to replace requirements of the form "the image of a subset that is large in some sense is a singleton" with requirements of the milder form "the image of a subset that is large in some sense is a subset that is small in some sense". This approach covers the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ and extends to wider classes of spaces.

An example of a statement from this new family is the following theorem. Let $f$ be a continuous map of the boundary $\partial\Delta^n$ of the $n$–dimensional simplex $\Delta^n$ to a contractible metric space $M$. Then $\partial\Delta^n$ contains a subset $E$ such that $E$ (is "large" in the sense that it) intersects all facets of $\Delta^n$ and the image $f(E)$ (is "small" in the sense that it) is either a singleton or a subset of the boundary $\partial B$ of a metric ball $B \subset M$ whose interior does not meet $f(\partial\Delta^n)$.

We generalize this theorem to noncontractible normal spaces via covers and deduce a series of its corollaries. Several of these corollaries are similar to the topological Radon theorem.

55M20, 55M25, 55P05

## 1 Introduction

We introduce and study a new family of theorems extending the class of Borsuk–Ulam- and topological Radon-type theorems (though none of our theorems is a generalization of the Borsuk–Ulam or topological Radon theorem itself). By the Borsuk–Ulam- and topological Radon-type theorems we mean those stating that a continuous map takes a "wide" set of some specific kind to a point. Let us list several of the most influential examples:

- *The Borsuk–Ulam theorem* itself says that every continuous map of a Euclidean $n$–sphere $\mathbb{S}^n$ into Euclidean $n$–space $\mathbb{R}^n$ identifies two antipodes.

- *The Hopf theorem* states that, if $X$ is a compact Riemannian $n$–manifold and $f\colon X \to \mathbb{R}^n$ is a continuous map, then, for any $\delta > 0$, there exists a geodesic $\gamma\colon [0,\delta] \to X$ of length $\delta$ such that $f(\gamma(0)) = f(\gamma(\delta))$.
- *The topological Radon theorem* says that if $P$ is a convex $n$–polytope, then any continuous map $\partial P \to \mathbb{R}^{n-1}$ identifies two points from disjoint faces.
- *The topological Tverberg theorem* says that, if $d \geq 1$ is an integer, $r$ is a prime power and $P$ is a convex $(r-1)(d+1)$–polytope, then any continuous map $\partial P \to \mathbb{R}^d$ identifies $r$ points from $r$ pairwise disjoint faces.

See eg Steinlein [35; 36], Matoušek [22], Karasev [17], Akopyan, Karasev and Volovikov [3], Frick [11], Bárány, Blagojević and Ziegler [5], Skopenkov [33] and Bárány and Soberón [6] for more examples, including various extensions and generalizations for $\mathbb{Z}_p$–spaces, maps between manifolds, matroids and colored versions. Another related family is Knaster's conjecture-type theorems (see Matschke [23]).

All of these examples involve rigid dimensional restrictions. It is a natural question whether the maps not satisfying these restrictions have any properties of the Borsuk–Ulam kind. In particular, we are interested in whether the Borsuk–Ulam theorem has any reasonable extensions to the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ (a related idea appears in Adams, Bush and Frick [2]).

Extensions of this kind are found in a new class we study. This class emerges by replacing conditions of the form "*the image of a subset that is large in some sense is a singleton*" with conditions of the milder form "*the image of a subset that is wide in some sense is a subset that is restricted in some sense*". This approach covers the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ and extends to wider classes of spaces.

Here is an example for the simplest nondegenerate case $\mathbb{S}^1 \to \mathbb{R}^2$:

**Proposition 1** (Malyutin [20]) *Let $a$, $b$ and $c$ be three closed arcs covering the circle $\mathbb{S}^1$ such that no two of them cover $\mathbb{S}^1$, and let $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a continuous map. Then either $f(a) \cap f(b) \cap f(c) \neq \varnothing$ or each of $f(a)$, $f(b)$ and $f(c)$ touches a closed Euclidean disk $D^2 \subset \mathbb{R}^2$ whose interior does not meet $f(\mathbb{S}^1)$.*

Proposition 1 works for plane curves and knot diagrams and has a corollary with applications in knot theory (see [20]). We formulate this corollary here. Let $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a regular smooth plane curve in general position (that is, its only singularities are transversal double points). By an *edge* of $\gamma$ we mean the closure of a component of the set $\gamma(\mathbb{S}^1) \setminus V$, where $V$ is the set of double points of $\gamma$. We say that two edges $I$ and $J$

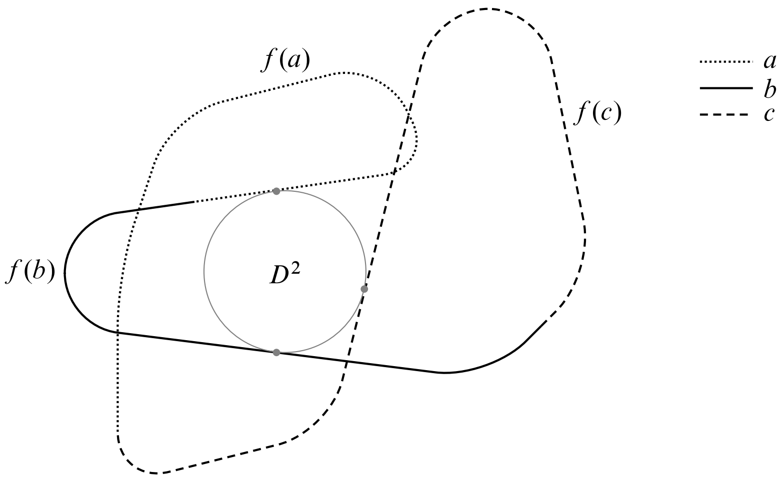


Figure 1: For Proposition 1. The circle $\partial D^2$ touches $f(a)$, $f(b)$ and $f(c)$

of $\gamma$ are *neighboring edges* or *neighbors* if there exists a component $Q$ of $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$ such that the boundary $\partial Q$ contains both $I$ and $J$. We say that two edges $I$ and $J$ of $\gamma$ are *consecutive* if the union $I \cup J$ coincides with the image $\gamma(\alpha)$ of a (connected) arc $\alpha$ in $\mathbb{S}^1$. We denote by $\rho$ the maximal metric on the set $E(\gamma)$ of edges of $\gamma$ in the class of metrics satisfying the condition "$\rho(I, J) = 1$ whenever $I$ and $J$ are consecutive edges of $\gamma$".

**Proposition 2** [20] *If the curve $\gamma$ has $k$ double points, then $\gamma$ has a pair of neighboring edges $I$ and $J$ with $\rho(I, J) \geq \frac{2}{3}k$.*

Proposition 1 readily implies Proposition 2 if we choose the arcs $a$, $b$ and $c$ appropriately. Proposition 2 appears in [20] as an auxiliary lemma (Lemma 5.1) needed to obtain a series of statements related to knot theory. In [20], this lemma is deduced from the topological Helly theorem (see Bogatyĭ [7] and Montejano [24]). The statement of Proposition 2 was one of the starting points for our study.

Here, we generalize Proposition 1 to noncontractible normal spaces via covers. The generalizations and their corollaries will be formulated in the next sections, after definitions. Our method is based on obstruction theory and uses a variation of the concept of non-nullhomotopic covers introduced by Musin [27; 28].

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# 2 Definitions and results

Throughout this paper we mainly consider normal topological spaces,[1] all simplicial complexes and covers will be finite, all manifolds will be both compact and PL, $\mathbb{S}^n$ will denote the $n$–dimensional sphere, $\Delta^n$ will denote the $n$–dimensional simplex and $\mathrm{sk}_k(\Delta^n)$ will denote the $k$–skeleton of $\Delta^n$. We shall denote the set of homotopy classes of continuous maps $V \to W$ by $[V, W]$. The nerve of a (finite) collection $\mathcal{S}$ of sets will be denoted by $\mathcal{N}(\mathcal{S})$. When this does not cause confusion we use the same notation for an abstract simplicial complex and its underlying space (carrier).

The further exposition in this section is structured as follows: first we give a chain of successively stronger generalizations of Proposition 1 (Theorem 4 is the weakest, Theorem 36 is the strongest); then we present a family of corollaries (all but one of which follow from Theorem 4).

## 2.1 Spherical $f$–neighbors

All of the following generalizations and corollaries replace the condition "the image is a singleton" appearing in the Borsuk–Ulam-type theorems with the following milder condition of "spherical neighboring":

**Definition 3** (spherical $f$–neighbors) Let $X$ be a set, let $Y$ be a metric space and let $f : X \to Y$ be a map. We say that a subset $N \subset X$ is a *set of spherical $f$–neighbors* if $N$ contains at least two points and the image $f(N)$ is either a point or a subset of the boundary $\partial B$ of a metric ball[2] $B \subset Y$ whose interior does not meet $f(X)$. If a two-point set $\{p, q\}$ is a set of spherical $f$–neighbors, we say that $p$ and $q$ are *spherical $f$–neighbors*. (See Figure 2.)

The first extension generalizes Proposition 1 to the case of spheres of arbitrary dimension and replaces Euclidean spaces with arbitrary contractible metric spaces. (Recall that *facets* of a polytope of dimension $n$ are its faces of dimension $n-1$.)

[1] A topological space $X$ is *normal* if any two disjoint closed sets of $X$ are contained in disjoint open sets of $X$; see [32, page 446] for equivalent definitions via the Urysohn and shrinking lemmas.

[2] By a *metric ball* in a metric space $(Y, d)$ with metric $d$ we mean a subset of the form $\{y \in Y \mid d(y, x) \leq R\}$ with $x \in Y$ and $R \geq 0$.

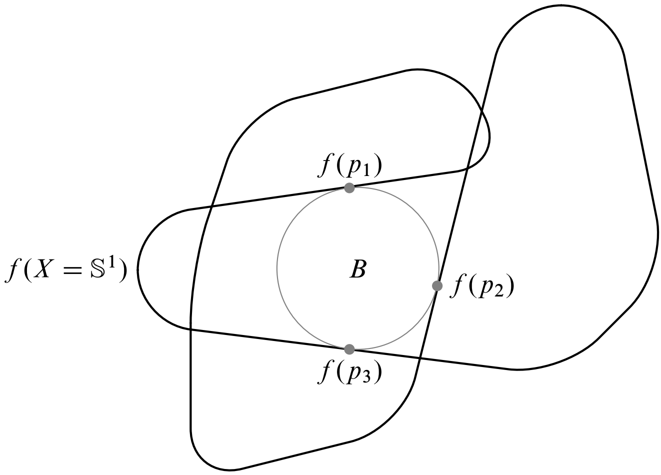


Figure 2: The image of a set $\{p_1, p_2, p_3\}$ of spherical $f$–neighbors

**Theorem 4** *Let $f$ be a continuous map of the boundary $\partial\Delta^n$ of the $n$–dimensional simplex $\Delta^n$ to a contractible metric space $M$. Then a set of spherical $f$–neighbors intersects all facets of $\Delta^n$.*

**Proof** Let $d$ denote the metric on $M$. If $z$ is a point and $N$ is a subset in $M$, we write

$$d(z, N) := \inf_{p \in N} d(z, p).$$

Let $\Delta_1, \dots, \Delta_{n+1}$ be the facets of $\Delta^n$. For each $i \in \{1, \dots, n+1\}$, we set

$$E_i := \{z \in M \mid d(z, f(\Delta_i)) = d(z, f(\partial\Delta^n))\}.$$

Observe that $E_i$ contains $f(\Delta_i)$ and is closed, so $\{E_1, \dots, E_{n+1}\}$ is a closed cover of $M$. Since $M$ is contractible, $f$ extends to a continuous map $F\colon \Delta^n \to M$. Then $\{F^{-1}(E_1), \dots, F^{-1}(E_{n+1})\}$ is a closed cover of $\Delta^n$ extending the closed cover $\{\Delta_1, \dots, \Delta_{n+1}\}$ of $\partial\Delta^n$. By the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma, the elements of $\{F^{-1}(E_1), \dots, F^{-1}(E_{n+1})\}$ have a common point $p$. Then $x = F(p)$ lies in $E_1 \cap \dots \cap E_{n+1}$. Then either $x \in f(\partial\Delta^n)$, so that $x$ belongs to all of $f(\Delta_i)$ by the definition of $E_i$; or $r = d(x, f(\partial\Delta^n)) > 0$ and the ball $B_r(x)$ of radius $r$ centered at $x$ touches all of $f(\Delta_i)$ while its interior does not meet $f(\partial\Delta^n)$. □

We generalize Theorem 4 by replacing the set of facets with a more general class of covers as in the KKM lemma.

## 2.2 KKM covers and spherical $f$–neighbors

**Definition 5** (KKM covers) Let $\Delta^{n+1}$ be an $(n+1)$–dimensional simplex with vertices labeled $v_1, \dots, v_{n+2}$. A closed cover $\{C_1, \dots, C_{n+2}\}$ of the $n$–sphere $\mathbb{S}^n$ is called a *KKM cover* if there exists a homeomorphism $h\colon \mathbb{S}^n \to \partial\Delta^{n+1}$ such that, for each $J \subset \{1, \dots, n+2\}$, the convex hull of the vertices $v_j$ with $j \in J$ is covered by the union $\bigcup_{j \in J} h(C_j)$.

The argument in the proof of Theorem 4 also proves the following theorem:

**Theorem 6** *Let $\mathcal{C}$ be a KKM cover of the $n$–sphere $\mathbb{S}^n$, and let $f\colon \mathbb{S}^n \to M$ be a continuous map to a contractible metric space $M$. Then a set of spherical $f$–neighbors intersects all elements of $\mathcal{C}$.*

The key role in Theorem 6 is played by the properties of the cover, and not by the fact that the underlying space is a sphere. To move on to the next generalization, we define non-nullhomotopic covers (we generalize the concept of non-nullhomotopic covers given in [27; 28]).

## 2.3 Non-nullhomotopic covers and spherical $f$–neighbors

Let $X$ be a normal topological space and let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of $X$. Let $\mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$. Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a partition of unity subordinate to $\mathcal{U}$. Let $v_1, \dots, v_n$ be the vertices of the $(n-1)$–dimensional unit simplex $\Delta^{n-1}$, where

$$\Delta^{n-1} := \{x \in \mathbb{R}^n \mid x_i \geq 0,\ x_1 + \cdots + x_n = 1\}.$$

For each $i$, we identify the vertex of $\mathcal{N}(\mathcal{U})$ corresponding to $U_i$ with $v_i$, so that $\mathcal{N}(\mathcal{U})$ becomes a subcomplex of $\Delta^{n-1}$. We set

$$h_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i.$$

Then $h_{\mathcal{U},\Phi}$ is a continuous map from $X$ to $\mathcal{N}(\mathcal{U}) \subset \Delta^{n-1}$. Since the linear homotopy $\Theta(t) = (1-t)\Phi + t\Psi$ of two partitions of unity $\Phi$ and $\Psi$ subordinate to $\mathcal{U}$ induces a homotopy between the corresponding maps, it follows that the homotopy class $[h_{\mathcal{U},\Phi}]$ in $[X, \mathcal{N}(\mathcal{U})]$, where by $[V, W]$ we denote the set of homotopy classes of continuous maps $V \to W$, does not depend on $\Phi$ (see [27, Lemma 1.6]). We denote this class in $[X, \mathcal{N}(\mathcal{U})]$ by $[\mathcal{U}]$.

The homotopy classes of covers are also well defined for closed sets. Indeed, in a normal space, any finite closed cover has an open extension with the same nerve (see eg [25, Theorem 1.3; 16, pages 31–33]). Furthermore, if $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of a normal space $X$ and $\mathcal{S} = \{S_1, \dots, S_n\}$ and $\mathcal{U} = \{U_1, \dots, U_n\}$ are two open covers such that $S_i \cap U_i$ contains $C_i$ for all $i$ that have the same nerve $\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathcal{U}) = \mathcal{N}(\mathcal{C})$, then each partition of unity subordinate to the open cover

$$\mathcal{T} := \{S_1 \cap U_1, \dots, S_n \cap U_n\}$$

is also subordinate to both $\mathcal{S}$ and $\mathcal{U}$. This implies that $[\mathcal{S}] = [\mathcal{T}] = [\mathcal{U}]$ in $[X, \mathcal{N}(\mathcal{C})]$ due to the independence of the choice of partition of unity mentioned above. Then we set

$$[\mathcal{C}] := [\mathcal{S}] = [\mathcal{T}] = [\mathcal{U}].$$

**Definition 7** (non-nullhomotopic covers) We say that an open or closed cover $\mathcal{C}$ of a normal topological space $X$ is *non-nullhomotopic* if the corresponding homotopy class $[\mathcal{C}]$ in $[X, \mathcal{N}(\mathcal{C})]$ contains no constant map.

**Remark 8** Any non-nullhomotopic map $X \to K$ to a finite simplicial complex yields non-nullhomotopic covers on $X$; to obtain an example, take the inverse images of all elements in one of the collections

- open stars of vertices of $K$,
- stars of vertices of $K$ in its first barycentric subdivision,
- maximal simplexes of $K$.

**Definition 9** (homotopy ranks of maps) Let $X$ be a topological space, let $K$ be a finite simplicial complex and let $h\colon X \to K$ be a continuous map. Let $\Delta_K$ be the simplex spanned by the vertices of $K$, so $K$ is a subcomplex of $\Delta_K$. We define the *homotopy rank* $\operatorname{rk}(h)$ of $h$ to be the least nonnegative integer $k$ such that $h$ is nullhomotopic in $K \cup \operatorname{sk}_k(\Delta_K)$, where $\operatorname{sk}_k$ stands for the $k$–skeleton.[3] (Since $\Delta_K$ is contractible, the homotopy rank is well defined and does not exceed the dimension of $\Delta_K$.)

**Remark 10** In terms of Definition 9, $h$ is nullhomotopic if and only if $\operatorname{rk}(h) = 0$.

**Definition 11** (ranks of covers) We define the (*homotopy*) *rank* $\operatorname{rk}(\mathcal{C})$ of a (closed or open) finite cover $\mathcal{C}$ of a normal space $X$ to be the homotopy rank of maps $X \to \mathcal{N}(\mathcal{C})$ in the class $[\mathcal{C}]$ determined by $\mathcal{C}$.

**Remark 12** A cover is non-nullhomotopic if and only if it is of nonzero rank.

[3]We say that $K \cup \operatorname{sk}_k(\Delta_K)$ is the *$k$–exoskeleton* of $K$.

**Remark 13** Since $\mathrm{sk}_m(\Delta^m) = \Delta^m$ is contractible, it follows that the rank of an $n$–element cover does not exceed $n-1$.

**Definition 14** (principal covers) An $n$–element cover ($n \geq 2$) of rank $n-1$ is said to be *principal*.

**Remark 15** Since any proper nonempty subcomplex of $\partial\Delta^m$ is contractible in $\partial\Delta^m$, it follows that a cover is principal if and only if it is non-nullhomotopic and its nerve is the boundary of a simplex.

**Remark 16** Remark 15 implies that no principal cover has a proper subcollection of elements with empty intersection; in particular, no principal cover has disjoint elements.

**Remark 17** Any non-nullhomotopic map $X \to \mathbb{S}^k$ to the $k$–sphere yields a principal cover of $X$ of rank $k+1$ (see Remark 8 and [27, Theorem 1.5]). Thus, a space can have principal covers of distinct ranks.

**Remark 18** (conditions for cover non-nullhomotopicity) If the composition of continuous maps is non-nullhomotopic, then each of them is non-nullhomotopic.

- On the one hand, this implies that any refinement of a cover of rank $k$ has rank at least $k$. In particular, any refinement of a non-nullhomotopic cover is non-nullhomotopic.
- On the other hand, this implies that, if $f\colon X \to Y$ is a continuous map of normal spaces and $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of $Y$ such that the dimension of the nerve $\mathcal{N}(\mathcal{C})$ is less than the rank $\mathrm{rk}(\widetilde{\mathcal{C}})$ of the induced cover $\widetilde{\mathcal{C}} = \{f^{-1}(C_1), \dots, f^{-1}(C_n)\}$, then $\mathrm{rk}(\mathcal{C}) \geq \mathrm{rk}(\widetilde{\mathcal{C}})$. In particular, if the induced cover is principal and $\bigcap_{i=1}^n C_i = \varnothing$, then $\mathcal{C}$ is principal. (See Lemma 52.)

We now have all the definitions needed to replace spheres in Theorem 6 with general "noncontractible" spaces.

**Theorem 19** *Let $X$ be a compact normal space, let $M$ be a contractible metric space and let $f\colon X \to M$ be a continuous map. Then, for any non-nullhomotopic cover $\mathcal{C}$ of $X$, a set of spherical $f$–neighbors intersects at least $\mathrm{rk}(\mathcal{C})+1$ elements of $\mathcal{C}$. In particular, for any principal cover, a set of spherical $f$–neighbors intersects all elements of the cover.*

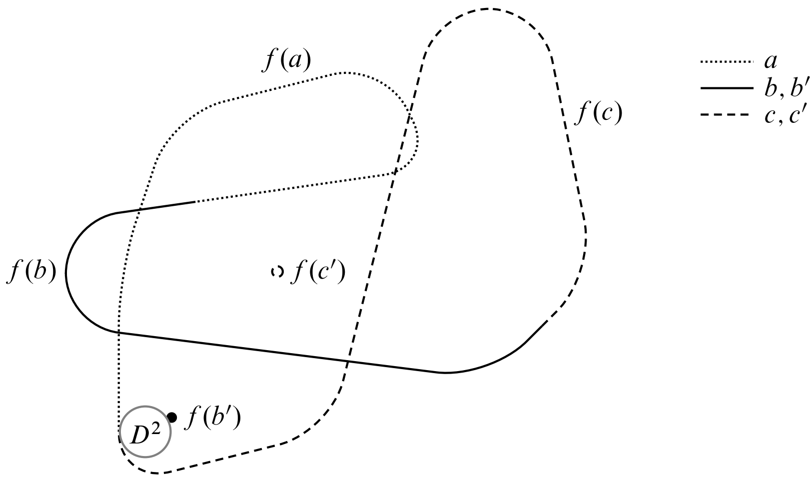


Figure 3: An example with disconnected $X = \mathbb{S}^1 \cup \{b', c'\}$.

Theorem 19 implies Theorem 6 because each KKM cover either is principal or all of its elements have a common point; furthermore, the maps in the homotopy class $[\mathcal{C}]$ corresponding to each principal KKM cover $\mathcal{C}$ are of degree one, so $[\mathcal{C}]$ contains a homeomorphism (see [28, Corollaries 2.1–2.3]).

**Remark 20** $X$ in Theorem 19 is not assumed to be connected. Figure 3 shows an example with $X = \mathbb{S}^1 \cup \{b', c'\}$ (compare Figure 1).

**Remark 21** Combining the idea that $X$ in Theorem 19 is not necessarily connected with switching attention to the image of the cover leads to generalizations of Helly's theorem and the KKM lemma. See also Lemma 52 below. We do not develop this line here.

## 2.4 EP triples and ranks, and spherical $f$–neighbors

We are going to upgrade Theorem 19 to the more general Theorem 36, which covers the case of maps to not necessarily contractible spaces. In order to state and prove Theorem 36, we introduce concepts of Eilenberg–Pontryagin and Knaster–Kuratowski–Mazurkiewicz ranks.

**Definition 22** (Eilenberg–Pontryagin triples and ranks) Let $Z$ be a topological space with a subspace $A$, let $K$ be a finite simplicial complex and let $[h]$ be a homotopy class in $[A, K]$. We say that $(Z, A, [h])$ is an *Eilenberg–Pontryagin triple* if no map in $[h]$ extends to a continuous map $Z \to K$.

We define the *Eilenberg–Pontryagin rank* (*EP rank*) $\mathrm{rk}(Z, A, [h])$ of the triple $(Z, A, [h])$ to be the least nonnegative integer $k$ such that there exists a continuous map $H\colon Z \to K \cup \mathrm{sk}_k(\Delta_K)$ whose restriction $H|_A$ is homotopic in $K \cup \mathrm{sk}_k(\Delta_K)$ to the maps of $[h]$, where $\Delta_K$ is the simplex spanned by the vertices of $K$ and containing $K$ as a subcomplex. (Since $\Delta_K$ is contractible, the EP rank is well defined and does not exceed the dimension of $\Delta_K$.)

**Remark 23** In terms of Definition 22, a triple $(Z, A, [h])$ is Eilenberg–Pontryagin if and only if it is of nonzero EP rank (because $K \cup \mathrm{sk}_0(\Delta_K) = K$).

**Remark 24** Since any constant map extends to any ambient space, it follows that in terms of Definitions 9 and 22, for any $Z$, $A$, $K$ and $h$,

$$\mathrm{rk}(Z, A, [h]) \le \mathrm{rk}(h).$$

Furthermore, if $A$ is contractible in $Z$, then

$$\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h).$$

In particular, if $\mathcal{C}$ is a finite closed cover of $A$ and $A$ is contractible in $Z$, then

$$\mathrm{rk}(Z, A, [\mathcal{C}]) = \mathrm{rk}(\mathcal{C}).$$

**Example 25**

- If $Z = \Delta^n$, $A = K = \partial\Delta^n$ and $h = \mathrm{id}$, then $\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h) = n$.
- If $Z = K = \Delta^n$, $A = \partial\Delta^n$, and $h = \mathrm{id}$, then $\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h) = 0$.

**Example 26** We have $\mathrm{rk}(Z, A, [h]) = 0$ whenever $A$ is a retract of $Z$.

**Example 27** Let $W$ be an orientable, compact PL $m$–manifold with connected nonempty boundary $\partial W$, and let $h\colon \partial W \to \partial\Delta^n$ be a continuous map. Then $\mathrm{rk}(h) \in \{0, n\}$, $\mathrm{rk}(W, \partial W, [h]) \in \{0, n\}$ and $\mathrm{rk}(W, \partial W, [h]) \le \mathrm{rk}(h)$.

- If $m = n$, then $\mathrm{rk}(W, \partial W, [h]) = \mathrm{rk}(h)$ (this follows from the Hopf degree theorem; see the proof of Corollary 39 below).
- If $W = \Delta^m$, then $\mathrm{rk}(W, \partial W, [h]) = \mathrm{rk}(h)$ (because $\Delta^m$ is contractible; see Remark 24).
- Results of [29] imply however that, for any $m$ and $n$ with nontrivial $\pi_{m-1}(\mathbb{S}^{n-1})$ and for any non-nullhomotopic $h\colon \mathbb{S}^{m-1} \to \partial\Delta^n$, there exists an $m$–manifold $W$ with $\partial W = \mathbb{S}^{m-1}$ such that $h$ extends to a continuous map $W \to \partial\Delta^n$, so $\mathrm{rk}(W, \partial W, [h]) = 0$ and $\mathrm{rk}(h) = n$.

**Definition 28** (Knaster–Kuratowski–Mazurkiewicz rank) Let $Z$ be a topological space and let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a collection of subsets in $Z$. We say that the pair $(Z, \mathcal{S})$ is a *Knaster–Kuratowski–Mazurkiewicz* (*KKM*) *system* if no closed cover $\{E_1, \dots, E_n\}$ of $Z$ with $S_i \subset E_i$ for all $i$ has the same nerve as $\mathcal{S}$.

We define the *KKM rank* $\mathrm{rk}(Z, \mathcal{S})$ of the pair $(Z, \mathcal{S})$ to be the least integer $k$ such that there exists a closed cover $\mathcal{E} = \{E_1, \dots, E_n\}$ of $Z$ with $S_i \subset E_i$ for all $i$ such that the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{S})$ is $k$.

**Remark 29** In terms of Definition 28, a pair $(Z, \mathcal{S})$ is a KKM system if and only if it is of nonzero KKM rank.

**Example 30** If $Z = \Delta^m$ and $\mathcal{S} = \{S_1, \dots, S_{m+1}\}$ is a KKM cover of $\partial\Delta^m$, then $\mathrm{rk}(Z, \mathcal{S}) = m$ by the KKM lemma.

**Example 31** We have $\mathrm{rk}(Z, \mathcal{S}) = 0$ whenever $\mathcal{S}$ is a closed cover of a retract of $Z$.

**Example 32** We have $\mathrm{rk}(Z, \{S_1, \dots, S_n\}) = 0$ whenever $\bigcap_{i=1}^n S_i \neq \varnothing$.

**Lemma 33** *Let a normal space $Z$ contain a normal space $A$ as a subspace, let $\mathcal{C}$ be a closed cover of $A$, and let $[\mathcal{C}]$ be the corresponding homotopy class in $[A, \mathcal{N}(\mathcal{C})]$, where $\mathcal{N}(\mathcal{C})$ is the nerve. Then the EP rank of the triple $(Z, A, [\mathcal{C}])$ does not exceed the KKM rank of the system $(Z, \mathcal{C})$:*

$$\mathrm{rk}(Z, A, [\mathcal{C}]) \leq \mathrm{rk}(Z, \mathcal{C}).$$

*Furthermore, if $A$ is closed in $Z$, then*

$$\mathrm{rk}(Z, A, [\mathcal{C}]) = \mathrm{rk}(Z, \mathcal{C}).$$

Lemma 33 is proved in the next section.

**Example 34** (showing that the closedness requirement of $A$ in the second part of Lemma 33 is essential) If $X$ is a compact normal space, $\mathcal{C}' = \{C'_1, \dots, C'_n\}$ is a closed cover of $X$ with $\bigcap_{i=1}^n C_i = \varnothing$ and each $C'_i$ nonempty, $Z$ is the cone over $X$, $z_0$ is the top of $Z$, $A = Z \setminus \{z_0\}$, $C''_i$ is the subcone in $Z$ over $C'_i$, $C_i = C''_i \setminus \{z_0\}$, and $\mathcal{C} = \{C_1, \dots, C_n\}$, then the KKM rank $\mathrm{rk}(Z, \mathcal{C})$ is $n-1$ and the EP rank $\mathrm{rk}(Z, A, [\mathcal{C}])$ is one more than the dimension of the nerve $\mathcal{N}(\mathcal{C}')$. (See Figure 4 with $X = \mathbb{S}^1$.) For example, if $n > 2$ and the elements of $\mathcal{C}'$ are pairwise disjoint, then

$$\mathrm{rk}(Z, A, [\mathcal{C}]) = 1 < n - 1 = \mathrm{rk}(Z, \mathcal{C}).$$

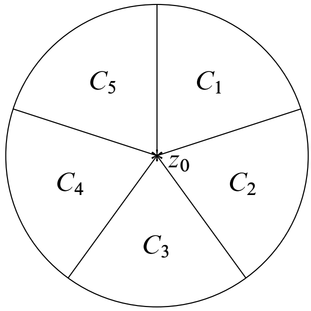


Figure 4: A disk with $\operatorname{rk}(Z, A, [\mathcal{C}]) = 2 < 4 = \operatorname{rk}(Z, \mathcal{C})$.

**Remark 35** Lemma 33 implies (see Remark 24) that, given a compact normal space $A$ with a finite closed cover $\mathcal{C}$, for any ambient normal space $Z \supset A$, $\operatorname{rk}(Z, A, [\mathcal{C}]) = \operatorname{rk}(Z, \mathcal{C}) \le \operatorname{rk}(\mathcal{C})$, while $\operatorname{rk}(Z, A, [\mathcal{C}]) = \operatorname{rk}(Z, \mathcal{C}) = \operatorname{rk}(\mathcal{C})$ if $A$ is contractible in $Z$. (This generalizes [27, Theorem 2.2].)

**Theorem 36** *Let $A$ be a compact normal space, let $\mathcal{C}$ be a closed cover of $A$, and let $[\mathcal{C}]$ be the corresponding homotopy class in $[A, \mathcal{N}(\mathcal{C})]$, where $\mathcal{N}(\mathcal{C})$ is the nerve. Let $Z$ be a normal space containing $A$ as a subspace. If the triple $(Z, A, [\mathcal{C}])$ is Eilenberg–Pontryagin, with EP rank $\operatorname{rk}(Z, A, [\mathcal{C}]) > 0$, then, for any metric space $M$ and any continuous map $f : A \to M$ that extends to a continuous map $Z \to M$, a set of spherical $f$–neighbors intersects at least $\operatorname{rk}(Z, A, [\mathcal{C}]) + 1$ elements of $\mathcal{C}$.*

Theorem 36 is proved in the next section.

**Proof of Theorem 19** We deduce Theorem 19 from Theorem 36. Let $X$, $M$, $f$ and $\mathcal{C}$ be as in Theorem 19. Set $\operatorname{Cone}(X) := (X \times [0, 1])/(X \times \{0\})$ and identify $X$ with $X \times \{1\} \subset \operatorname{Cone}(X)$. (The cone is normal because $X$ is compact and normal; see eg [30].) Definitions of ranks imply (see Remark 24) that

$$(***) \qquad \operatorname{rk}(\operatorname{Cone}(X), X, [\mathcal{C}]) = \operatorname{rk}(\mathcal{C}).$$

In particular, the triple $(\operatorname{Cone}(X), X, [\mathcal{C}])$ is Eilenberg–Pontryagin since $\mathcal{C}$ is non-nullhomotopic. Since $M$ is contractible, it follows that $f$ extends to a continuous map $F : \operatorname{Cone}(X) \to M$. We apply Theorem 36 to the Eilenberg–Pontryagin triple $(\operatorname{Cone}(X), X, [\mathcal{C}])$, with $Z = \operatorname{Cone}(X)$ and $A = X$ in the notation of Theorem 36, and see that a set of spherical $f$–neighbors intersects at least $\operatorname{rk}(\operatorname{Cone}(X), X, [\mathcal{C}]) + 1$ elements of $\mathcal{C}$. Then Theorem 19 follows by $(***)$. $\square$

**Remark 37** Theorem 36 has further refinements regarding the number of distinct sets of spherical $f$–neighbors intersecting the prescribed number of cover elements, but we do not develop this line here.

## 2.5 Corollaries

Next, we list several corollaries of Theorems 4, 6, 19 and 36. In fact, all of the following corollaries, except for Corollary 39, follow from Theorem 4.

**Definition 38** A continuous map $f: A \to Y$ of an orientable, connected, closed PL manifold $A$ to a space $Y$ is said to be *null-cobordant* if there exists an orientable, compact PL manifold $W$ with $\partial W = A$ and a continuous map $F: W \to Y$ such that $F|_A = f$.

**Corollary 39** (see [27, Theorem 2.6]) *Let $A$ be an orientable*, *connected*, *closed PL $n$–manifold and let $\mathcal{C}$ be a non-nullhomotopic cover of $A$ such that the nerve of $\mathcal{C}$ is homeomorphic to the $n$–sphere. Then*, *for any metric space $M$ and any null-cobordant map $f: A \to M$, a set of spherical $f$–neighbors intersects at least $n+2$ elements of $\mathcal{C}$. In particular*, *if $\mathcal{C}$ is principal and contains precisely $n+2$ elements*, *then a set of spherical $f$–neighbors intersects all elements of $\mathcal{C}$.*

**Proof** If $f: A \to M$ is null-cobordant, then there is an orientable, compact PL $(n+1)$–manifold $Z$ with $\partial Z = A$ and a continuous map $F: Z \to M$ with $F|_A = f$. A homological argument shows that, for each continuous map $H: Z \to \mathcal{N}(\mathcal{C}) \cong \mathbb{S}^n$, the restriction $H|_A$ is of zero degree. Then the Hopf degree theorem implies that $H|_A$ is nullhomotopic. This means that the triple $(Z, A, [\mathcal{C}])$ is Eilenberg–Pontryagin and the statement follows by Theorem 36. □

**Remark 40** (the dimensional restriction in Corollary 39 is essential) It is shown in [29] that any continuous map $\mathbb{S}^m \to \mathbb{S}^n$ is null-cobordant if $m > n$. Let $m$ and $n$ be such that $m > n$ and $\pi_m(\mathbb{S}^n)$ is nontrivial, and let $h: \mathbb{S}^m \to \partial\Delta^{n+1}$ be a non-nullhomotopic continuous map. Then there exists an orientable, compact PL $(m+1)$–manifold $W$ with $\partial W = \mathbb{S}^m$ and a continuous map $H: W \to \partial\Delta^{n+1}$ such that $H|_{\partial W} = h$. Let $\mathcal{C}$ be the closed cover of $\partial W$ composed of the inverse images of the facets of $\Delta^{n+1}$. Then $[\mathcal{C}] = [h]$ and $\mathcal{C}$ is principal. We embed $W$ into a Euclidean ball $B^N$ of large dimension and "tiny" diameter, then embed $W$ into the product $\partial\Delta^{n+1} \times B^N$ such that the projection of this embedding to $\partial\Delta^{n+1}$ yields $H$, and take the induced metric on $W$.

Now, let $f\colon \partial W \to W$ be the identity map. Then $f$ is null-cobordant but no set of spherical $f$–neighbors intersects all elements of $\mathcal{C}$ if the diameter of $B^N$ is sufficiently small.

**Corollary 41** *Let $M$ be a contractible metric space, let $\mathbb{S}^n$ be the Euclidean unit $n$–sphere in Euclidean $(n+1)$–space $\mathbb{R}^{n+1}$ and let $f\colon \mathbb{S}^n \to M$ be a continuous map. Then there exists a pair $\{p, q\}$ of spherical $f$–neighbors such that the Euclidean distance $\|p-q\|$ is at least $\sqrt{(n+2)/n}$.*

Corollary 41 is proved in the next section.

**Remark 42** In [21] we show that, if $M = \mathbb{R}^m$ with $m > n$, then the constant $\sqrt{(n+2)/n}$ in Corollary 41 (the Euclidean distance between the centers of adjacent $(n-1)$–simplices of the regular triangulation of $\mathbb{S}^n$) can be replaced with $\sqrt{2(n+2)/(n+1)}$ (the Euclidean distance between vertices of the regular triangulation of $\mathbb{S}^n$), which is the best possible. Our proof for the Euclidean case $M = \mathbb{R}^m$ is based on the Delaunay triangulations and we do not know whether it extends to all contractible $M$.

**Corollary 43** *Let $M$ be a contractible metric space, let $P$ be a convex $n$–polytope and let $f\colon \partial P \to M$ be a continuous map. Then a set of spherical $f$–neighbors intersects at least $n+1$ facets of $P$.*

**Proof via Theorem 4** Corollary 43 is an "equivalent generalization" of Theorem 4 because the $(n-2)$–skeleton of any convex $n$–polytope contains the $(n-2)$–skeleton of the $n$–simplex as a topological subspace (see [14]). □

**Proof via Theorem 19** Clearly, the cover $\mathcal{C}$ of $\partial P$ composed of the facets of $P$ is non-nullhomotopic of rank $n$ because $\mathcal{C}$ is a *good* cover (that is, any intersection of elements in $\mathcal{C}$ is contractible), so the nerve of $\mathcal{C}$ has homotopy type of $\partial P \cong \mathbb{S}^{n-1}$ by the nerve theorem, while the maps in the class $[\mathcal{C}]$ are homotopy equivalences. Then a set of spherical $f$–neighbors intersects at least $n+1$ facets of $P$ by Theorem 19. □

Since any collection of $n+1$ facets of the $n$–cube contains a pair of antipodal facets, Corollary 43 implies the following:

**Corollary 44** *Let $M$ be a contractible metric space, let $\partial[0,1]^m$ be the boundary of the $m$–dimensional cube $[0,1]^m$ and let $f\colon \partial[0,1]^m \to M$ be a continuous map. Then there is a pair of spherical $f$–neighbors intersecting antipodal facets of $[0,1]^m$.*

There exists an example of continuous map $\mathbb{S}^2 \to \mathbb{R}^3$ showing that the statement of Corollary 44 about spherical $f$–neighbors lying on antipodal facets holds for neither regular octahedra nor regular dodecahedra nor regular icosahedra. A weaker version of Corollary 44 where "antipodal" is replaced with "disjoint" holds for many polytopes.

## 2.6 Radon-type theorems

**Definition 45** (weak Radon polytopes) We say that an $n$–polytope $P$ is *weakly Radon* if, for any continuous map $f : \partial P \to M$ into any contractible metric space $M$, there is a pair of spherical $f$–neighbors intersecting two disjoint faces of $P$.

We recall some standard definitions. A *flag* polytope is a convex polytope such that every collection of pairwise intersecting facets has a nonempty intersection. A (*combinatorial*) *fullerene* is a simple 3–polytope with all facets pentagons and hexagons.

A "visual" simply checked sufficient condition for weakly Radon polytopes is provided by the so-called *belts*. A *$k$–belt* (or a *prismatic $k$–circuit*) in a 3–polytope is a cyclic sequence $(F_1, \dots, F_k)$ of $k \geq 3$ facets in which pairs of consecutive facets (including $\{F_k, F_1\}$) are adjacent, other pairs of facets do not intersect, and no three facets have a common vertex.

**Corollary 46** (1) *If the $(n-2)$–skeleton of a convex $n$–polytope $P$ contains the $(n-2)$–skeleton of the $n$–cube as a topological subspace, then $P$ is weakly Radon.*

(2) *Each convex 3–polytope having a $k$–belt with $k \geq 4$ is weakly Radon.*

(3) *Each flag 3–polytope is weakly Radon.*

(4) *Each fullerene is weakly Radon.*

(5) *The regular dodecahedron and the regular icosahedron are weakly Radon.*

**Proof** Assertion (1) follows from Corollary 44 in an obvious way. Assertions (2) and (5) directly follow from assertion (1). Assertion (3) follows from Corollary 51(2) below. Assertion (4) is a particular case of assertion (3). □

**Definition 47** (weak Radon rank) If $P$ is an $n$–polytope, $Y$ is a metric space and $f : \partial P \to M$ is a map, we say that two facets $F_1$ and $F_2$ of $P$ are *spherical $f$–neighbors* (or that the pair $\{F_1, F_2\}$ is a *pair of spherical $f$–neighbors*) if there is a pair $\{p, q\}$ of spherical $f$–neighbors with $p \in F_1$ and $q \in F_2$. We say that $f : \partial P \to M$ *has weak Radon rank $m$* if there are exactly $m$ distinct pairs of facets of $P$ such that each of these pairs is a pair of disjoint spherical $f$–neighbors. By the *weak Radon rank* of a polytope $P$ we mean the least of the weak Radon ranks of continuous maps $f\colon \partial P \to M$ into contractible metric spaces.

Corollary 44 allows us to obtain rough lower bounds on the weak Radon rank.

**Definition 48** (cubic hemisphere) Let $H$ be a subset of the boundary $\partial P$ of a convex $n$–polytope $P$. We say that $H$ is a *cubic hemisphere* if there exists a homeomorphism $h\colon [0,1]^n \to P$ such that the restriction of $h$ to the $(n-2)$–skeleton of $[0,1]^n$ is a topological embedding to the $(n-2)$–skeleton of $P$ and $H$ is the image of the union of $n$ facets of $[0,1]^n$ that have a common vertex.

**Definition 49** (lighthouse independence number) We say that a set $Z$ of vertices of an $n$–polytope is *lighthouse independent* if no two vertices in $Z$ share a facet (equivalently, the corresponding facets of the dual polytope are pairwise disjoint). The *lighthouse independence* number $\operatorname{lin}(P)$ of an $n$–polytope $P$ is the cardinality of a largest lighthouse independent set of $P$.

**Remark 50** The lighthouse independence number of an $n$–polytope equals the cardinality of a largest set of pairwise disjoint facets of the dual polytope.

**Corollary 51** (1) *Let $P$ be a convex $n$–polytope. If $\partial P$ contains $k$ cubic hemispheres with pairwise disjoint interiors, then the weak Radon rank of $P$ is at least $\frac{1}{2}k$.*

(2) *Let $P$ be a flag 3–polytope (eg a fullerene). Then the weak Radon rank of $P$ is at least half the lighthouse independence number of $P$.*

(3) *If $P$ is a flag simple 3–polytope with $\psi$ facets and $g$ is the largest number of edges in a facet of $P$, then the weak Radon rank of $P$ is at least*

$$\frac{1}{2}\left\lfloor \frac{2\psi-7}{3g-8} \right\rfloor,$$

*where $\lfloor\cdot\rfloor$ stands for the floor function.*

(4) *If $P$ is a fullerene with $\psi$ facets, then the weak Radon rank of $P$ is at least*

$$\tfrac{1}{2}\left\lfloor \tfrac{1}{5}(\psi-3) \right\rfloor.$$

(5) *The weak Radon rank of the regular dodecahedron is at least 2.*

(6) *The weak Radon rank of the regular icosahedron is at least 2.*

(7) *The weak Radon rank of the cube is 1.*

Corollary 51 is proved in the next section.

# 3 Proofs

**Proof of Lemma 33** (1) We show that $\operatorname{rk}(Z, A, [\mathcal{C}]) \le \operatorname{rk}(Z, \mathcal{C})$.

Let $\mathcal{C} = \{C_1, \dots, C_n\}$, let $\Delta^{n-1}$ denote the simplex spanned by the vertices of the nerve $\mathcal{N}(\mathcal{C})$ of $\mathcal{C}$, so $\mathcal{N}(\mathcal{C})$ is a subcomplex of $\Delta^{n-1}$, and let $r := \operatorname{rk}(Z, \mathcal{C})$. By the definition of the KKM rank, there exists a closed cover $\mathcal{E} = \{E_1, \dots, E_n\}$ of $Z$ with $C_i \subset E_i$ for all $i$ such that the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{C})$ is $r$. Therefore, the union $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1})$ contains $\mathcal{N}(\mathcal{E})$. Set

$$\mathcal{E}_A := \{E_1 \cap A, \dots, E_n \cap A\}.$$

Since $C_i \subset E_i$ for all $i$, it follows that the nerve $\mathcal{N}(\mathcal{E}_A)$ contains $\mathcal{N}(\mathcal{C})$. We have

$$\mathcal{N}(\mathcal{C}) \subset \mathcal{N}(\mathcal{E}_A) \subset \mathcal{N}(\mathcal{E}) \subset \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1}).$$

Let $[\mathcal{E}]$ be the homotopy class in $[Z, \mathcal{N}(\mathcal{E})]$ determined by $\mathcal{E}$ and let $F\colon Z \to \mathcal{N}(\mathcal{E})$ be a map in $[\mathcal{E}]$. Let $[\mathcal{E}_A]$ be the homotopy class in $[A, \mathcal{N}(\mathcal{E}_A)]$ determined by $\mathcal{E}_A$ and let $f'\colon A \to \mathcal{N}(\mathcal{E}_A)$ be a map in $[\mathcal{E}_A]$. Let $f\colon A \to \mathcal{N}(\mathcal{C})$ be a map in $[\mathcal{C}] \in [A, \mathcal{N}(\mathcal{C})]$.

Since $E_i \cap A$ contains $C_i$ for each $i$, the argument preceding Definition 7 shows that $f$ and $f'$ are homotopic in $\mathcal{N}(\mathcal{E}_A)$. By construction, $F|_A$ and $f'$ are homotopic in $\mathcal{N}(\mathcal{E})$. Thus, $F|_A$ and $f$ are homotopic in $\mathcal{N}(\mathcal{E})$ and hence in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1})$ as well. By the definition of the EP rank this means that $\operatorname{rk}(Z, A, [\mathcal{C}]) \le r = \operatorname{rk}(Z, \mathcal{C})$.

(2) We show that $\operatorname{rk}(Z, \mathcal{C}) \le \operatorname{rk}(Z, A, [\mathcal{C}])$ whenever $A$ is closed in $Z$.

We start by constructing a specific map $A \to \mathcal{N}(\mathcal{C})$ from the class $[\mathcal{C}]$. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ and let $\mathcal{N}(\mathcal{C})$ be a subcomplex in $\Delta^{n-1}$ (as in the first part of the proof). Since $A$ is normal, there exists an open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of $A$ such that $U_i$ contains $C_i$ for each $i$ and the nerve of $\mathcal{U}$ coincides with that of $\mathcal{C}$ (see eg [25, Theorem 1.3; 16, pages 31–33]). The Urysohn lemma for normal spaces implies that, for each $i$, there exists a continuous function $f_i\colon A \to [0,1]$ with $f_i(C_i) = 1$ and $f_i(A \setminus U_i) = 0$. Then $\Phi = \{\varphi_1, \dots, \varphi_n\}$, where $\varphi_i := f_i / \sum_j f_j$, is a partition of unity subordinate to $\mathcal{U}$ such that $\varphi_i^{-1}[1/n, 1]$ contains $C_i$ for all $i$. Let

$$h_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i$$

be the corresponding map $A \to \mathcal{N}(\mathcal{C})$ representing the class $[\mathcal{C}] = [\mathcal{U}]$ (see the construction preceding Definition 7).

Now, let $p := \operatorname{rk}(Z, A, [\mathcal{C}])$. Then, by the definition of the EP rank, there exists a continuous map $F\colon Z \to \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$ such that the restriction $F|_A$ is homotopic to $h_{\mathcal{U},\Phi}$ in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$. The generalizations of Borsuk's homotopy extension theorem obtained in [26; 34] imply that, since $A$ is closed in $Z$, there exists a continuous map $G\colon Z \to \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$ with $G|_A = h_{\mathcal{U},\Phi}$. Then the collection of subsets

$$\mathcal{G} := \{G_1^{-1}[1/n, 1], \dots, G_n^{-1}[1/n, 1]\},$$

where $G_1, \dots, G_n$ are the coordinate functions of $G$, is a closed cover of $Z$ such that $G_i^{-1}[1/n, 1]$ contains $C_i$ for all $i$ and the nerve $\mathcal{N}(\mathcal{G})$ is contained in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$, so the dimension of $\mathcal{N}(\mathcal{G}) \setminus \mathcal{N}(\mathcal{C})$ is at most $p$. By the definition of the KKM rank, this means that $\operatorname{rk}(Z, [\mathcal{C}]) \le p = \operatorname{rk}(Z, A, [\mathcal{C}])$. $\square$

Now we state and prove Lemmas 52 and 54, and then deduce Theorem 36 from Lemmas 33, 52 and 54.

**Lemma 52** *Let $(Z, \mathcal{C} = \{C_1, \dots, C_n\})$ be a KKM system of rank $r > 0$, let $f : Z \to Z'$ be a continuous map to a topological space $Z'$, and let $\mathcal{C}' = \{C_1', \dots, C_n'\}$ be a family of subsets in $Z'$ such that $f(C_i) \subset C_i'$ for all $i$. Then either $\{1, \dots, n\}$ contains a subset $J$ of cardinality $r+1$ such that $\bigcap_{j \in J} C_j = \varnothing$ and $\bigcap_{j \in J} C_j' \neq \varnothing$ or $(Z', \mathcal{C}')$ is a KKM system of rank at least $r$.*

**Remark 53** In Lemma 52, two key special cases are $C_i' = f(C_i)$ and $f = \mathrm{id}$.

**Proof** If neither $\operatorname{rk}(Z', \mathcal{C}') \ge r$ nor $\{1, \dots, n\}$ contains $J$ with $|J| = r+1$ such that $\bigcap_{j \in J} C_j = \varnothing$ and $\bigcap_{j \in J} C_j' \neq \varnothing$, then

(i) there exists a closed cover $\mathcal{E}' = \{E_1', \dots, E_n'\}$ of $Z'$ with $C_i' \subset E_i'$ for all $i$ such that the dimension of $\mathcal{N}(\mathcal{E}') \setminus \mathcal{N}(\mathcal{C}')$ is less than $r$ (by definition), and

(ii) the dimension of $\mathcal{N}(\mathcal{C}') \setminus \mathcal{N}(\mathcal{C})$ is less than $r$.

Consequently, the dimension of $\mathcal{N}(\mathcal{E}') \setminus \mathcal{N}(\mathcal{C})$ is less than $r$. The collection $\mathcal{E} = \{E_1, \dots, E_n\}$ with $E_i := f^{-1}(E_i')$ is a closed cover of $Z$ such that $C_i \subset E_i$ for all $i$. The nerve $\mathcal{N}(\mathcal{E}')$ contains $\mathcal{N}(\mathcal{E})$. Therefore, the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{C})$ is less than $r$. This contradicts the assumption that $r = \operatorname{rk}(Z, \mathcal{C})$. $\square$

**Lemma 54** *Let $(Z, \mathcal{C} = \{C_1, \dots, C_n\})$ be a KKM system of rank $r > 0$ with metrizable $Z$ and all $C_i$ compact, and let $d$ be a metric on $Z$. Then there exists a closed metric ball*

$$B_R(x) := \{z \in Z \mid d(z, x) \le R\}, \quad x \in Z, \ R > 0,$$

*whose interior intersects no element of $\mathcal{C}$ and whose boundary sphere touches at least $r+1$ elements of $\mathcal{C}$.*

**Proof** If $z$ is a point and $N$ is a subset in $Z$, we write

$$d(z,N) := \inf_{p\in N} d(z,p).$$

Let $A$ denote the union $\bigcup_{i=1}^{n} C_i$. For each $i \in \{1,\dots,n\}$, we set

$$E_i := \{z \in Z \mid d(z,C_i) = d(z,A)\}.$$

Observe that $E_i$ contains $C_i$ and is closed, so $\{E_1,\dots,E_n\}$ is a closed cover of $Z$. Since $(Z,\mathcal{C})$ is a KKM system of rank $r>0$, it follows by the definition of KKM rank that the set $\{1,\dots,n\}$ contains a subset $J$ of cardinality $r+1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} E_j \neq \varnothing$. Let $x$ be a point in $\bigcap_{j\in J} E_j \neq \varnothing$. Then the ball $B_{d(x,A)}(x)$ of radius $d(x,A)$ centered at $x$ meets the requirements of the lemma (since each of $C_i$ is compact). □

**Proof of Theorem 36** Since $(Z,A,[\mathcal{C}])$ is an Eilenberg–Pontryagin triple, it follows by Lemma 33 that $(Z,\mathcal{C})$ is a KKM system of rank $\operatorname{rk}(Z,\mathcal{C}) = \operatorname{rk}(Z,A,[\mathcal{C}])$.

Let $\mathcal{C} = \{C_1,\dots,C_n\}$. Set $\mathcal{F} := \{F(C_1),\dots,F(C_n)\}$. Then Lemma 52 implies that we have two possibilities:

1. The dimension of $\mathcal{N}(\mathcal{F})\backslash\mathcal{N}(\mathcal{C})$ is at least $\operatorname{rk}(Z,\mathcal{C})$, so $\{1,\dots,n\}$ contains a subset $J$ of cardinality $\operatorname{rk}(Z,\mathcal{C})+1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} F(C_j) \neq \varnothing$.
2. The pair $(M,\mathcal{F})$ is a KKM system of rank at least $\operatorname{rk}(Z,\mathcal{C})$.

In case (1), for any point $x \in \bigcap_{j\in J} F(C_j)$, the set $F^{-1}(x)$ is a set of spherical $F|_A$–neighbors that intersects all elements of $\{C_j\}_{j\in J}$, which proves the theorem.

In case (2), the required statement follows by Lemma 54 applied to $(M,\mathcal{F})$. □

**Proof of Corollary 41** We use a spherical version of Theorem 4. Let $T$ be a regular triangulation of the unit sphere $\mathbb{S}^n$ and let $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2}$, be the $n$–simplices of $T$; all of $\widetilde{\Delta}_i$ are regular spherical simplices with Euclidean distances between vertices

$$d_{n,\mathrm{Eu}} = \sqrt{\frac{2(n+2)}{n+1}} \tag{1}$$

and angular edge length

$$d_{n,\mathrm{A}} = 2\arcsin\left(\tfrac{1}{2}d_{n,\mathrm{Eu}}\right) = 2\arcsin\sqrt{\frac{n+2}{2(n+1)}} = \arccos\left(\frac{-1}{n+1}\right). \tag{2}$$

We recall that the *circumradius* of a compact set $Q$ in a metric space is defined as the radius of a least metric ball containing $Q$. If $Q$ is a compact subset of $\mathbb{S}^n$ we denote by $\operatorname{circ}_{\mathrm{A}} Q$ and $\operatorname{diam}_{\mathrm{A}} Q$, respectively, the circumradius and diameter of $Q$ with respect to the angular metric, and $\operatorname{diam}_{\mathrm{Eu}} Q$ will stand for the Euclidean diameter of $Q$ in $\mathbb{R}^{n+1} \supset \mathbb{S}^n$. Under this notation, Dekster's extension [9] of the Jung theorem says that, for any compact subset $Q$ of $\mathbb{S}^n$,

$$2\arcsin\left(\sqrt{\frac{n+1}{2n}}\sin(\operatorname{circ}_{\mathrm{A}} Q)\right) \le \operatorname{diam}_{\mathrm{A}} Q.$$

This immediately implies that, in the case where $\operatorname{circ}_{\mathrm{A}} Q \le \frac{\pi}{2}$,

$$\sqrt{\frac{2(n+1)}{n}}\sin(\operatorname{circ}_{\mathrm{A}} Q) \le \operatorname{diam}_{\mathrm{Eu}} Q. \tag{3}$$

Another auxiliary fact we need is that

$$\operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i = \tfrac{1}{2}(\pi - d_{n,\mathrm{A}}). \tag{4}$$

Indeed, observe that $\widetilde{\Delta}_i$ is the intersection of a finite number of closed hemispheres and hence its boundary is composed of fragments of great hyperspheres, which are geodesic in $\mathbb{S}^n$. Therefore, if $a$ and $b$ are two points in $\widetilde{\Delta}_i$ such that neither $a$ nor $b$ is a vertex of $\widetilde{\Delta}_i$, then $\operatorname{diam}_{\mathrm{A}}\{a,b\} < \operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i$ because $\widetilde{\Delta}_i$ contains two geodesic arcs[4] $\alpha$ and $\beta$ such that $\alpha$ contains $a$ in its relative interior and $\beta$ contains $b$ in its relative interior. Since $\widetilde{\Delta}_i$ is contained in the interior of a hemisphere, so $a$ and $b$ are not antipodal, it follows that there exist $a' \in \alpha \subset \widetilde{\Delta}_i$ and $b' \in \beta \subset \widetilde{\Delta}_i$ with $\operatorname{diam}_{\mathrm{A}}\{a,b\} < \operatorname{diam}_{\mathrm{A}}\{a',b'\}$ (imagine the interposition of $\alpha$, $\beta$ and the metric ball $D \subset \mathbb{S}^n$ of diameter $\operatorname{diam}_{\mathrm{A}}\{a,b\}$ containing $a$ and $b$). Thus, if $a$ and $b$ are two points in $\widetilde{\Delta}_i$ such that $\operatorname{diam}_{\mathrm{A}}\{a,b\} = \operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i$, then one of $a$ and $b$ is a vertex of $\widetilde{\Delta}_i$ and we easily obtain (4) by considering the regular triangulation of $\mathbb{S}^n$ dual (antipodal) to $T$.

Now, we pass to the proof of Corollary 41. If we have a continuous map $f\colon \mathbb{S}^n \to M$, then Theorem 4 implies that a finite set $\mathscr{P}$ of spherical $f$–neighbors intersects all of $\widetilde{\Delta}_i$. We need to prove that

$$\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} \ge \sqrt{\frac{n+2}{n}}. \tag{5}$$

Let $B \subset \mathbb{S}^n$ be a metric ball with angular radius $\operatorname{circ}_{\mathrm{A}} \mathscr{P}$ containing $\mathscr{P}$, let $C \in \mathbb{S}^n$ be the center of $B$, let $A \in \mathbb{S}^n$ be the antipode of $C$, let $\widetilde{\Delta}_k$ be a simplex of $T$ containing $A$ and

[4]By geodesic arcs in $\mathbb{S}^n$ we mean arcs of great circles.

let $B_2 \subset \mathbb{S}^n$ be the metric ball centered at $A$ of angular radius $\operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_k = \frac{1}{2}(\pi - d_{n,\mathrm{A}})$ (see (4)). Then $B_2$ contains $\widetilde{\Delta}_k$. Since $\mathscr{P}$ intersects $\widetilde{\Delta}_k$ while $\mathscr{P} \subset B$ and $\widetilde{\Delta}_k \subset B_2$, it follows that $B$ intersects $B_2$. Therefore,

$$\operatorname{circ}_{\mathrm{A}} \mathscr{P} = \operatorname{circ}_{\mathrm{A}} B \geq \pi - \operatorname{circ}_{\mathrm{A}} B_2 = \tfrac{1}{2} d_{n,\mathrm{A}}. \tag{6}$$

The situation splits into two cases:

(i) $\operatorname{circ}_{\mathrm{A}} \mathscr{P} > \frac{\pi}{2}$ (ie no hemisphere contains $\mathscr{P}$).

(ii) $\operatorname{circ}_{\mathrm{A}} \mathscr{P} \leq \frac{\pi}{2}$.

In case (i), we observe that, since no hemisphere contains $\mathscr{P}$, it follows that no Euclidean ball in $\mathbb{R}^n$ of radius less than 1 contains $\mathscr{P}$. Then the Jung theorem[5] says that $\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} \geq d_{n,\mathrm{Eu}}$, which implies the required (5).

In case (ii), (3) is applicable and yields

$$\begin{aligned}
\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} &\geq \sqrt{\frac{2(n+1)}{n}} \cdot \sin(\operatorname{circ}_{\mathrm{A}} \mathscr{P}) && \text{(by (3))} \\
&\geq \sqrt{\frac{2(n+1)}{n}} \cdot \sin\bigl(\tfrac{1}{2} d_{n,\mathrm{A}}\bigr) && \text{(by (6) and (ii))} \\
&= \sqrt{\frac{2(n+1)}{n}} \cdot \tfrac{1}{2} d_{n,\mathrm{Eu}} && \text{(by (2))} \\
&= \sqrt{\frac{2(n+1)}{n}} \cdot \sqrt{\frac{n+2}{2(n+1)}} = \sqrt{\frac{n+2}{n}} && \text{(by (1)).}
\end{aligned}$$

□

**Remark 55** It would be interesting to find a way to upgrade the above proof of Corollary 41 by considering the family of all regular triangulations of the unit sphere $\mathbb{S}^n$.

**Proof of Corollary 51** (1) Corollary 44 implies that, if $M$ is a contractible metric space and $f : \partial P \to M$ is a continuous map, then each cubic hemisphere in $\partial P$ contains a facet that is a member of a pair of disjoint facets that are spherical $f$–neighbors. The statement follows.

(2) Proposition 56 below implies that, if the lighthouse independence number of a flag 3–polytope $P$ is $k$, then $\partial P$ contains $k$ cubic hemispheres with pairwise disjoint interiors. This implies the required assertion by assertion (1) of the corollary.

(3)–(4) These follow from assertion (2) and Proposition 57 below.

[5] For a discussion and materials concerning the Jung theorem and containment in hemispheres, see [39; 15; 8, pages 112, 113, 132–136, 38; 19; 4; 1, Proposition 2.4].

(5)–(6) These follow from (2) because a direct check shows that the lighthouse independence number of the regular dodecahedron is 4 and the lighthouse independence number of the regular icosahedron is 3.

(7) Corollary 44 shows that the weak Radon rank of the cube is at least 1 and an example where $\partial[0,1]^3$ is mapped to an oblate spheroid in $\mathbb{R}^3$ shows that it is at most 1. □

We say that a vertex $v$ of a polytope is *cubical* if the union of the facets containing $v$ is a cubic hemisphere.

**Proposition 56** *All vertices of a flag* 3*–polytope are cubical.*

**Proof** Let $v$ be a vertex of a flag 3–polytope $P$. Observe that no facet of $P$ is a triangle (because any triangular facet together with the three adjacent ones form a collection of four pairwise intersecting facets with no common point). Therefore, each facet of $P$ containing $v$ has a vertex that is not adjacent to $v$. Let $v_1$, $v_2$ and $v_3$ be three such vertices lying on three distinct facets containing $v$. Let $D$ denote the union of the facets of $P$ that do not contain $v$. Then $D$ is a topological disk with the points $v_1$, $v_2$ and $v_3$ on its boundary. Since $P$ is flag, we see that

- no facet contained in $D$ intersects three of the facets not contained in $D$,
- no facet of $P$ splits $D$ (in the sense that $D \setminus F$ is connected for each facet $F$).

This implies that

- each of the vertices $v_1$, $v_2$ and $v_3$ is incident to an edge of $P$ whose second endpoint is contained in the interior of $D$ (in particular, the interior of $D$ contains at least one vertex of $P$); and
- the subgraph $G_D$ in the 1–skeleton $P_1$ of $P$ induced by the vertices of $P$ contained in the interior of $D$ is connected.

Thus, each of $v_1$, $v_2$ and $v_3$ is adjacent to a vertex of the connected subgraph $G_D$ in $P_1$. This easily implies that $P_1$ contains a $Y$–homeomorphic subgraph $Y'$ that is contained in $D$ and intersects the boundary $\partial D$ exactly in the set $\{v_1, v_2, v_3\}$.

Furthermore, since $v_1$, $v_2$ and $v_3$ belong to three distinct facets containing $v$, it follows that there exists a triple of edges in $P_1$ incident to $v$ whose endpoints split $\partial D$ into three arcs each of which contains exactly one of $v_1$, $v_2$ and $v_3$. Clearly, the union of these edges with $\partial D$ and $Y'$ is a graph homeomorphic to the cube 1–skeleton. This shows that $v$ is cubical. □

**Proposition 57** (1) *Let $P$ be a flag simple 3–polytope with $\psi$ facets and let $g$ be the largest number of edges in a facet of $P$. Then the lighthouse independence number of $P$ is at least*
$$\left\lfloor \frac{2\psi-7}{3g-8} \right\rfloor.$$

(2) *If $P$ is a fullerene with $\psi$ facets, then the lighthouse independence number of $P$ is at least*
$$\left\lfloor \tfrac{1}{5}(\psi-3) \right\rfloor.$$

**Proof** In the proof, if $v$ is a vertex of $P$, we denote by $L(v)$ the union of facets of $P$ that contain $v$.

We construct a lighthouse independent set by the following algorithm. First we choose a vertex $v_1$ of $P$ such that the number of vertices in $L(v_1)$ is the least possible and set $W_1 = L(v_1)$. The number of vertices in $L(v_1)$ is at most $3g-5$.

At each next step, being given $W_i \subset P$ such that a vertex of $P$ is not in $W_i$, we take a vertex $v_{i+1}$ of $P$ in $P \setminus W_i$ such that the number of vertices in $L(v_{i+1}) \setminus W_i$ is the least possible and set $W_{i+1} = W_i \cup L(v_{i+1})$. Observe that, if a vertex $v$ of $P$ is not in $W_i$ and adjacent to a vertex in $W_i$, then $L(v)$ shares at least three vertices with $W_i$. This implies that the number of vertices in $L(v_{i+1}) \setminus W_i$ is at most $3g-8$.

Therefore, if $P$ has $N$ vertices, this algorithm produces a lighthouse independent set $v_1, v_2, \dots$ with at least
$$1 + \left\lfloor \frac{N-(3g-5)}{3g-8} \right\rfloor = \left\lfloor \frac{N-3}{3g-8} \right\rfloor$$
elements. Since $P$ is simple, Euler's formula yields $N = 2\psi - 4$. This proves the required estimate.

The case of fullerenes follows if we observe that, when $v_1$ is a vertex of a pentagon, the number of vertices in $L(v_1)$ is at most 12. □

# 4 Concluding remarks

Now we discuss several concepts and open questions.

(1) **The Hopf theorem** The trefoil curve in Figure 5 shows that there exists a continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$ with no pair of spherical $f$–neighbors having distance less than $\sqrt{3}$ between them. This means that the direct analog of the aforementioned Hopf theorem for spherical $f$–neighbors does not hold for small distances. It would be

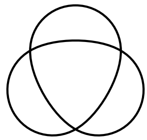

Figure 5

interesting to find more properties of the set of distances between spherical $f$–neighbors for a continuous map $f$ of given metric spaces. For example:

**Question** *Is it true that, for any continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^{n+k}$, the set*

$$\Gamma_f := \{\delta \in \mathbb{R} \mid \delta = d(p,q) \text{ for a pair } \{p,q\} \text{ of spherical } f\text{–neighbors}\}$$

*contains a nondegenerate interval? Is there a nonzero lower bound for the diameter of $\Gamma_f$?*

(2) **Topological Tverberg theorems** Projecting a Euclidean $n$–sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ into a hyperplane in $\mathbb{R}^{n+1}$ shows that there exists a continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ with no set of spherical $f$–neighbors of cardinality exceeding 2. Consequently, each convex $n$–polytope $P$ has a map $f\colon \partial P \to \mathbb{R}^{n-1}$ with no set of spherical $f$–neighbors intersecting three disjoint faces of $P$. This means that no direct analog of the topological Tverberg theorems with three or more disjoint faces holds for spherical $f$–neighbors. This correlates with the property (see Remark 16) that no principal cover has disjoint elements. Nevertheless, we have some analogs of the topological Radon theorem, which is the topological Tverberg theorem for two disjoint faces; see Corollaries 44, 46 and 51.

**Problem** *Find extensions of topological Tverberg theorems for spherical $f$–neighbors with additional restrictions.*

(See also Van Kampen–Flores- and Conway–Gordon–Sachs-type results [33].)

(3) **Weak Radon rank** It would be interesting to:

**Problem** *Describe the set of polyhedra that are not weakly Radon. Find the weak Radon rank for fullerenes.*

(4) **Minimaxes, I** Let $(X,\rho)$ and $(M,d)$ be metric spaces and let $f\colon X \to M$ be a continuous map. Let $P_f$ be the set of all pairs of spherical $f$–neighbors in $X$. We set

$$D_f := \sup_{\{x,y\}\in P_f} \rho(x,y), \qquad \mu(X,M) := \inf_{f\in C(X,M)} D_f,$$

where $C(X, M)$ stands for continuous maps. Suppose $X = \mathbb{S}^n$ and $M = \mathbb{R}^m$. If $m \le n$, then $\mu(\mathbb{S}^n, \mathbb{R}^m) = 2$ by the Borsuk–Ulam theorem. For $n < m$, it is shown in [21] that

$$\mu(\mathbb{S}^n, \mathbb{R}^m) = \sqrt{\frac{2(n+2)}{n+1}}.$$

**Problem** *Find $\mu(X, M)$ and its lower bounds in general and some special cases. In particular, find $D_f$ and $\mu$ for the case where $M = \mathbb{R}^n$ and $X$ is an $n$–dimensional Riemannian manifold.*

(5) **Minimaxes, II** Let us fix $[\mathcal{C}]$ in $[X, \mathbb{S}^{n-2}]$ (see Definition 7), for instance $[\mathcal{C}] \neq 0$ in $\pi_3(\mathbb{S}^2)$.

**Question** *What is the min–max distance between the points of a set intersecting each element of a cover of this class*?

(6) **Widths, distortion, filling radius, etc** Similarly to $\mu(X, M)$, we consider infima of $D_f$ over families of homotopic maps, over all continuous maps of a given space to certain classes of spaces (eg contractible spaces), etc. This generates a series of new metric "$\mu$–invariants" of maps and metric spaces. These $\mu$–invariants are similar to such invariants as distortion, filling radius and various widths (see [37; 12; 13; 10; 31; 18; 3]).

**Problem** *Find and describe relations between $\mu$–invariants and classical ones.*

(7) **Topological and visual $f$–neighbors** Let $f: X \to Y$ be a map of topological spaces. We say that two points $a$ and $b$ in $X$ are *topological $f$–neighbors* if $f(a)$ and $f(b)$ belong to the boundary of the same connected component of the complement $Y \setminus f(X)$. If $Y$ is a geodesic metric space, we say that $a$ and $b$ in $X$ are *visual $f$–neighbors* if $f(a)$ and $f(b)$ are connected by a geodesic, in $Y$, whose interior does not meet $f(X)$.

**Problem** *Translate the above constructions and questions to these new types of $f$–neighbors.*

(8) **Helly-type sufficient conditions for principal covers** Remark 18 implies some Helly-type sufficient conditions for principal covers. For example, if $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of a normal space $Y$ such that $\mathcal{N}(\mathcal{C}) = \partial\Delta^{n-1}$ and, for each $J \subset \{1, \dots, n\}$ with $|J| \le n-2$, any continuous map $\mathbb{S}^{n-2-|J|} \to \bigcap_{j \in J} C_j$ is null-homotopic, then there exists a map $f: \partial\Delta^{n-1} \to Y$ such that the image of each

facet is contained in an element of $\mathcal{C}$, so $\mathcal{C}$ is principal. (See the proofs of Theorems 5 and 6 in [7].)

**Question** *Which of the other versions of topological Helly theorem* (see eg [7; 24]) *give sufficient conditions for principal and non-nullhomotopic covers*?

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