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On the wheeled PROP of stable cohomology of Aut(F_n) with bivariant coefficients

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We show that the stable cohomology of automorphism groups of free groups with coefficients obtained by applying Hom$(-,-)$ to tensor powers of the abelianization, is equipped with the structure of a wheeled PROP $\mathcal{H}$. We define another wheeled PROP $\mathcal{E}$ by Ext–groups in the category of functors from the category of finitely generated free groups to $\Bbbk$–modules. The main result of this paper is the construction of a morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ such that $\varphi(\mathcal{E})$ is the wheeled PROP generated by the cohomology class h_1 constructed by the first author.

20F28; 18M85, 20J06

1 Introduction

This paper concerns the cohomology of automorphism groups of free groups $\mathrm{Aut}(\mathbb{Z}^{*n})$ for $n \in \mathbb{N}$, with coefficients given by the $\Bbbk$–modules

$$B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}) := \mathrm{Hom}_{\mathcal{V}}((\Bbbk^n)^{\otimes l}, (\Bbbk^n)^{\otimes q}),$$

where $l, q \in \mathbb{N}$, $\Bbbk$ is a commutative ring and $\mathcal{V}$ is the category of $\Bbbk$–modules, and where the structure of $\mathrm{Aut}(\mathbb{Z}^{*n})$–module on $B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ is given by the diagonal action.

In [8] — see also [9] — for $n \geq 2$, the first author introduced a nonzero cohomology class

$$h_1 \in H^1(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2}))$$

and constructed, from h_1, cohomology classes

$$h_p \in H^p\big(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes p+1})\big)$$

for $p > 1$ and $\bar{h}_p \in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}), (\Bbbk^n)^{\otimes p})$ for $p \geq 1$, even in the unstable range. The construction of these classes is inspired by previous works of Morita [19] and Morita with the first author [10; 11] concerning cohomology classes of the mapping class group with trivial coefficients $\mathbb{Q}$; see Remark 7.3.

By a classical construction (see Section 4), there are group morphisms

$$H^*(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B_{l,q}(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{\alpha_n} H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})).$$

The stable cohomology of the automorphism groups of free groups with coefficients given by $B_{l,q}$ is defined by

$$H^*_{\mathrm{st}}(B_{l,q}) := \lim_{n \in \mathbb{N}} H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}))$$

where the limit is taken over the group morphisms α_n.

By a result of Randal-Williams and Wahl [21], this cohomology stabilizes so that the stable cohomology $H^i_{\mathrm{st}}(B_{l,q})$ is isomorphic to $H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}))$ for n big enough. It follows from the stability that stable cohomology is equipped with a cup product map

$$\cup\colon H^*_{\mathrm{st}}(B_{l_1,q_1}) \otimes H^*_{\mathrm{st}}(B_{l_2,q_2}) \to H^*_{\mathrm{st}}(B_{l_1,q_1} \otimes B_{l_2,q_2})$$

for $l_1, q_1, l_2, q_2 \in \mathbb{N}$.

In Definition 6.1 we define the PROP $\mathcal{H}$, where the morphisms are the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}(q,l) = H^*_{\mathrm{st}}(B_{l,q})$$

where the action of symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by place permutation on the tensor product $(-)^{\otimes q}$ (resp. $(-)^{\otimes l}$) and where the horizontal composition is given by the cup product map for stable cohomology and the vertical composition is induced by the composition in $\mathcal{V}$.

We show that this PROP is equipped with further structure:

Proposition 1 (Proposition 6.2) *The PROP $\mathcal{H}$ is a wheeled PROP, ie it is equipped with contraction maps*

$$\xi^i_j \colon \mathcal{H}(q,l) \to \mathcal{H}(q-1, l-1)$$

for $1 \leq i \leq q$ and $1 \leq j \leq l$ compatible with the structure of PROP.

Wheeled PROPs were introduced by Markl, Merkulov and Shadrin in [15] to treat PROPs equipped with trace maps. The typical example of a wheeled PROP is the PROP of endomorphism of a free finitely generated module where the contractions are given by partial trace maps; see Example 2.2. The wheeled PROP structure on the PROP $\mathcal{H}$ should be viewed as a cohomological version of the wheeled endomorphism PROP.

In the stable range, for $p > 1$, the cohomology classes h_p are obtained from h_1 using the horizontal and vertical composition in the PROP $\mathcal{H}$, and for $p \geq 1$, the classes $\bar{h}_p$ are obtained from h_p using the contraction maps. We deduce that the classes h_p and $\bar{h}_p$ are in the subwheeled PROP $\mathcal{K}$ of $\mathcal{H}$ generated by the class h_1.

Understanding the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients given by $B_{l,q}$ is equivalent to giving a description of the wheeled PROP $\mathcal{H}$ in terms of generators and relations. This is open; in particular, it is unknown whether the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{H}$ is strict.

By the results of Djament and the second author [1; 3], we know that the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with nonconstant coefficients is closely related to Ext–groups in the category $\mathcal{F}(\mathbf{gr}; \mathbb{k})$ of functors from the category $\mathbf{gr}$ of finitely generated free groups to the category $\mathcal{V}$ of $\mathbb{k}$–modules. More precisely, the main result of [3], obtained using functor homology methods, implies that $\mathcal{H}(0,l) = 0$ for $l > 0$ and [1, Théorème 4] gives, for $\mathbb{k} = \mathbb{Q}$, a natural isomorphism

$$\mathcal{H}(q,0) \simeq \bigoplus_{j \in \mathbb{N}} \mathrm{Ext}^{*-j}_{\mathcal{F}(\mathbf{gr};\mathbb{k})}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}), \tag{1-1}$$

where $\mathfrak{a}^{\otimes q}$ is the q^{th} tensor power of the abelianization functor and $\Lambda^j \mathfrak{a}$ is the j^{th} exterior power of the abelianization functor; see Section 3. The Ext–groups on the right-hand side of the isomorphism (1-1) are computed in [22] (the result is recalled in Proposition 10.1) giving the explicit computation of $\mathcal{H}(q,0)$.

Note that Randal-Williams obtained in [20] the computation of $\mathcal{H}(0,l)$ and $\mathcal{H}(q,0)$ using geometric techniques independent of the approach in [1; 3].

Inspired by a conjecture given in [1] we define in Section 10, for $\mathbb{k} = \mathbb{Q}$, a PROP $\mathcal{E}$ where the morphisms are the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{E}(q,l) = \bigoplus_{j \in \mathbb{N}} \mathrm{Ext}^{*-j}_{\mathcal{F}(\mathbf{gr};\mathbb{k})}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}).$$

We give an explicit description of the PROP $\mathcal{E}$ (see Theorem 5) and deduce from it that the PROP $\mathcal{E}$ inherits a structure of *wheeled* PROP.

The main result of this paper is the following:

Theorem 2 (Theorem 11.1) *There is a morphism of wheeled PROPs*,

$$\varphi: \mathcal{E} \to \mathcal{H},$$

such that $\varphi(\mathcal{E}) \simeq \mathcal{K}$.

Djament's conjecture can be rephrased in the following way:

Conjecture 3 *The morphism* φ *is an isomorphism of wheeled PROPs.*

This conjecture would imply that the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{H}$ is an equivalence of categories, ie the class h_1 would generate the wheeled PROP $\mathcal{H}$.

The sub-PROP $\mathcal{E}_0$ of $\mathcal{E}$ given by

$$\mathcal{E}_0(q,l) = \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$$

has been studied by the second author; in particular, by [22, Proposition 3.5] the PROP $\mathcal{E}_0$ is generated by its underlying operad $\mathcal{P}_0$. In Proposition 9.13 we give an explicit description of this operad by generators and relations. A more conceptual description of the operad $\mathcal{P}_0$ is the following:

Proposition 4 (Proposition 9.14) *The operad* $\mathcal{P}_0$ *is the operadic suspension of the operad* $\mathcal{C}om$ *of* ***nonunital*** *commutative algebras.*

The previous results on $\mathcal{E}_0$ and $\mathcal{P}_0$ are also true for $\Bbbk = \mathbb{Z}$.

The forgetful functor from wheeled PROP to operads has a left adjoint. We denote by $\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$ the wheeled PROP associated to the operad $\mathcal{P}_0$ by this functor. We obtain the following result.

Theorem 5 (Theorem 10.11) *There is an isomorphism of PROPs*

$$\chi: \mathcal{C}_{\mathcal{P}_0^{\circlearrowright}} \xrightarrow{\simeq} \mathcal{E}.$$

In particular, $\mathcal{E}$ inherits a structure of *wheeled* PROP via this isomorphism. The existence of a wheeled structure on the PROP $\mathcal{E}$ is quite surprising and is very specific to the situation studied in this paper; see Remark 10.12. We deduce from Theorem 5 a description of the wheeled PROP $\mathcal{E}$ by generators and relations.

Let $\mathcal{E}_w$ (resp. $\mathcal{H}'$) be the sub-PROP of $\mathcal{E}$ (resp. $\mathcal{H}$) keeping only the morphisms to 0 and the endomorphisms in degree 0 in $\mathcal{E}$ (resp. $\mathcal{H}$). Djament's result can be rephrased in the following way:

Proposition 6 [1, Théorème 4] *By restriction*, φ *induces an isomorphism of PROPs*, $\varphi': \mathcal{E}_w \xrightarrow{\simeq} \mathcal{H}'$.

Notation We denote by $\mathbb{N} = \{0, 1, \dots\}$ and by $\boldsymbol{q}$ the set $\{1, \dots, q\}$.

Throughout the paper, $\Bbbk$ is a commutative ring which will be, most of the time, a field of characteristic zero or $\mathbb{Z}$. We denote by $\mathcal{V}$ the category of $\Bbbk$–modules and $\mathcal{V}^f$ its full subcategory of free finitely generated modules.

A *homological* $\mathbb{Z}$–graded $\Bbbk$–module is denoted by $V_\bullet = \bigoplus_n V_n$, and a morphism *of homological degree* d, $f: V_\bullet \to W_\bullet$, is a family of linear maps $f_n: V_n \to W_{n+d}$ for all $n \in \mathbb{Z}$. To $V_\bullet$ a homological $\mathbb{Z}$–graded $\Bbbk$–module, we associate a *cohomological* $\mathbb{Z}$–graded $\Bbbk$–module $V^\bullet$ by $V^n := V_{-n}$. A morphism of homological degree d corresponds to a morphism of *cohomological degree* $-d$.

Graded $\Bbbk$–modules and morphisms of degree 0 form a category denoted by $\mathrm{gr}\,\mathcal{V}$. For $\otimes$, the tensor product of $\mathbb{Z}$–graded $\Bbbk$–modules, the category $(\mathrm{gr}\,\mathcal{V}, \otimes, \Bbbk)$ is equipped with the symmetry given by the maps $\tau: V \otimes W \to W \otimes V$ defined by $\tau(v \otimes w) := (-1)^{pq} w \otimes v$ where $v \in V_p$ and $w \in W_q$.

For V a $\Bbbk$–module we denote again by V the graded $\Bbbk$–module concentrated in degree 0, where it is equal to V.

Let $\Bbbk s$ be the graded $\Bbbk$–module concentrated in degree one and such that $(\Bbbk s)_1$ is spanned by s; the *suspension* of a graded $\Bbbk$–module V is $sV := \Bbbk s \otimes V$, so that $(sV)_i = V_{i-1}$.

The duality functor, denoted by $-^*: (\mathcal{V})^{\mathrm{op}} \to \mathcal{V}$ is defined by $\mathrm{Hom}_{\mathcal{V}}(-, \Bbbk)$.

Nonspecified tensor products are taken over $\Bbbk$.

For objects C and C' of a category $\mathcal{C}$, the set of morphisms from C to C' is denoted by $\mathrm{Hom}_{\mathcal{C}}(C, C')$ or $\mathcal{C}(C, C')$.

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Part I Recollections

2 Recollections on PROPs and operads

PROPs and operads arose in the work of Mac Lane [13]. Since then, they have turned out to be very important algebraic structures, especially in algebraic topology.

In this section we recall some basic facts that we will use in the paper on PROPs and operads, as well as their wheeled versions introduced more recently by Markl, Merkulov and Shadrin [15].

2.1 Classical PROPs and operads

For PROPs, we refer the reader to [14] for further details. The notion of PROP is closely related to the notion of operad. For operads, we refer the reader to [12].

A PROP is a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ with objects the natural numbers whose symmetric monoidal structure $\otimes$ is given by the sum of integers on objects.

In this paper we will consider PROPs enriched over $\mathrm{gr}\,\mathcal{V}$, called graded linear PROPs. Such a PROP $\mathcal{C}$ is a collection $\{\mathcal{C}(m,n)\}_{m,n\in\mathbb{N}}$ of graded $(\mathfrak{S}_m, \mathfrak{S}_n)$–bimodules (ie graded left $\mathfrak{S}_m \otimes (\mathfrak{S}_n)^{\mathrm{op}}$–modules) together with two types of compositions: the horizontal composition

$$\mathcal{C}(m_1,n_1)\otimes\mathcal{C}(m_2,n_2)\to\mathcal{C}(m_1+m_2,n_1+n_2)$$

induced by the monoidal product, and the vertical composition

$$\mathcal{C}(n,l)\otimes\mathcal{C}(m,n)\to\mathcal{C}(m,l)$$

given by the categorical composition.

An *operation of biarity* (m,n) in a PROP $\mathcal{C}$ is an element in $\mathcal{C}(m,n)$.

In the rest of this paper all the operads and PROPs will be *graded linear*. To simplify the terminology we will call them simply operads and PROPs.

An important example of a PROP is the endomorphism PROP of a graded $\Bbbk$–module:

Example 2.1 To an object V of $\mathrm{gr}\,\mathcal{V}$, we associate the PROP, denoted by $\mathcal{E}nd_V$, defined by

$$\mathcal{E}nd_V(m,n) = \mathrm{Hom}_{\mathrm{gr}\,\mathcal{V}}(V^{\otimes m}, V^{\otimes n}),$$

with the action of the symmetric groups given by the action on the tensor product by place permutations. The horizontal composition is given by the tensor product of linear maps and the vertical composition by the composition in $\mathcal{V}$.

Every PROP $\mathcal{C}$ has an underlying operad $\mathcal{P}_{\mathcal{C}}$ given by $\mathcal{P}_{\mathcal{C}}(n) = \mathrm{Hom}_{\mathcal{C}}(n,1)$.

Conversely, every operad $\mathcal{P}_0$ generates a PROP $\mathcal{C}_{\mathcal{P}_0}$ where

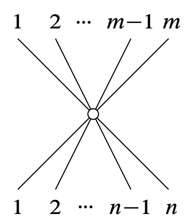
$$\mathcal{C}_{\mathcal{P}_0}(q,l) = \bigoplus_{f:\boldsymbol{q}\to\boldsymbol{l}} \bigotimes_{i=1}^{l} \mathcal{P}_0(f^{-1}(i)).$$

For two PROPs, $\mathcal{C}$ and $\mathcal{C}'$, a morphism of PROPs is a strict monoidal functor $F:\mathcal{C}\to\mathcal{C}'$ which is the identity on the objects and graded linear (ie the maps between the Hom–sets are morphisms of degree 0).

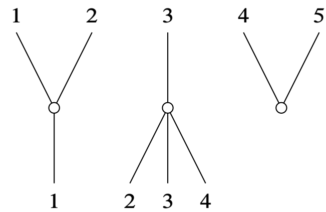
For a PROP $\mathcal{C}$, a morphism in $\mathcal{C}(m,n)$ can be represented by a *directed* (m,n)*–graph*, ie a finite, not necessary connected, graph such that each edge is equipped with a direction, there are no directed cycles and the set of legs is divided into the set of inputs labeled by $\{1,\dots,m\}$ and outputs labeled by $\{1,\dots,n\}$.

In our pictures the graphs are oriented from top to bottom.

Using the horizontal composition in $\mathcal{C}$, each morphism in $\mathcal{C}$ is the disjoint union of (m,n)*–corollas* which are (connected) graphs of the form



For example, the following depicts a morphism in $\mathcal{C}(5,4)$:



2.2 Wheeled PROPs and wheeled operads

In this section we recall the wheeled versions of PROPs and operads, introduced by Markl, Merkulov and Shadrin in [15], in order to encode algebras equipped with traces; see Example 2.2. Note that wheeled PROPs are particular cases of traced monoidal categories introduced by Joyal, Street and Verity in [7].

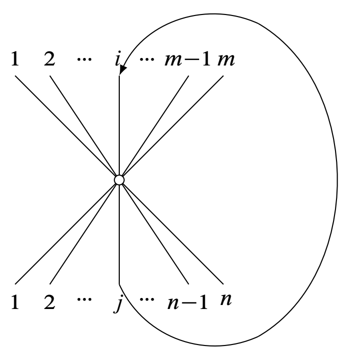
A wheeled PROP is a PROP equipped with *contractions*

$$\xi_j^i : \mathcal{C}(m,n) \to \mathcal{C}(m-1,n-1)$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. These contractions satisfy compatibility axioms.

For a wheeled PROP $\mathcal{C}$, a morphism in $\mathcal{C}(m,n)$ can be represented by a directed (m,n)–graph having possibly wheels and loops.

The contraction ξ_j^i can be viewed as connecting the i^{th} input and the j^{th} output. For example, for an (m,n)–corolla we have the following picture:



In a wheeled PROP, vertical composition is determined by the horizontal composition and the contractions by the formula

$$\mathcal{C}(n,l) \otimes \mathcal{C}(m,n) \to \mathcal{C}(n+m, l+n) \xrightarrow{(\xi_{l+1}^1)^n} \mathcal{C}(m,l); \tag{2-1}$$

see [15, (17)].

A fundamental example of a wheeled PROP is the wheeled endomorphism PROP associated with a free finitely generated $\mathbb{k}$–module where the contractions are given by the trace map:

Example 2.2 By classical linear algebra, for objects E and F in $\mathcal{V}$, we have canonical homomorphisms $E^* \otimes F \to \mathrm{Hom}_{\mathcal{V}}(E, F)$ and $E^* \otimes F^* \to (E \otimes F)^*$ which are isomorphisms if E is a free finitely generated $\mathbb{k}$–module.

For V an object of $\mathcal{V}^f$, the previous observations give rive to the isomorphisms

$$\theta_{m,n} \colon \mathrm{Hom}_{\mathcal{V}^f}(V^{\otimes m}, V^{\otimes n}) \xrightarrow{\simeq} (V^*)^{\otimes m} \otimes V^{\otimes n}.$$

By Example 2.1, $\mathcal{E}nd_V(m,n) = \mathrm{Hom}_{\mathcal{V}}(V^{\otimes m}, V^{\otimes n})$ defines a PROP. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the contractions $\xi^i_j \colon \mathcal{E}nd_V(m,n) \to \mathcal{E}nd_V(m-1,n-1)$ correspond through the previous isomorphisms to the maps

$$\varphi^i_j \colon (V^*)^{\otimes m} \otimes V^{\otimes n} \to (V^*)^{\otimes m-1} \otimes V^{\otimes n-1}$$

given by

$$\begin{aligned}&\varphi^i_j(f_1 \otimes \cdots \otimes f_m \otimes v_1 \otimes \cdots \otimes v_n)\\ &\quad = f_i(v_j)(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_m \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots \otimes v_n)\end{aligned}$$

where $f_i \in V^*$ and $v_i \in V$.

Note that $\varphi^1_1 \colon V^* \otimes V \to \mathbb{k}$ is the evaluation and $\varphi^1_1 \circ \theta_{1,1} \colon \mathrm{Hom}_{\mathcal{V}^f}(V, V) \to \mathbb{k}$ is the trace map Tr which associates to an endomorphism of V its trace.

A morphism of wheeled PROPs is a morphism of PROPs that is compatible with the contractions.

The forgetful functor from the category of wheeled PROPs to the category of PROPs has a left adjoint denoted by $(-)^{\circlearrowright}$. For $\mathcal{C}$ a PROP, $\mathcal{C}^{\circlearrowright}$ is called the *wheeled completion* of $\mathcal{C}$; see [15, Definition 2.1.9].

Recall from [15, Definition 5.1.1], that a *wheeled operad* $\mathcal{P} = \{\mathcal{P}(n,m)\}_{m,n}$, where $n \in \mathbb{N}$ and $m \in \{0, 1\}$, consists of

(1) an ordinary operad $\mathcal{P}_0 := \{\mathcal{P}(n,1)\}_{n\geq 0}$;

(2) a right $\mathcal{P}_0$–module $\mathcal{P}_w := \{\mathcal{P}(n,0)\}_{n\geq 0}$;

(3) for $1 \leq i \leq n$, contractions $\xi^i \colon \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$, satisfying compatibility conditions with the structures given in (1) and (2).

The operad $\mathcal{P}_0$ is called the *operadic part* of $\mathcal{P}$ and $\mathcal{P}_w$ its *wheeled* part.

Recall that the operad $\mathcal{P}_0$ is itself a right $\mathcal{P}_0$–module.

Every wheeled PROP $\mathcal{C}$ has an underlying wheeled operad $\mathcal{P}^{\mathcal{C}}$ where the operadic part is $\mathcal{P}_0^{\mathcal{C}} = \{\mathrm{Hom}_{\mathcal{C}}(n,1)\}$, the wheeled part is $\mathcal{P}_w^{\mathcal{C}} = \{\mathrm{Hom}_{\mathcal{C}}(n,0)\}$ and the contractions $\xi^i : \mathcal{P}_0^{\mathcal{C}}(n) \to \mathcal{P}_w^{\mathcal{C}}(n-1)$ are the contractions $\xi_1^i : \mathrm{Hom}_{\mathcal{C}}(n,1) \to \mathrm{Hom}_{\mathcal{C}}(n,0)$ of the wheeled PROP.

Conversely, every wheeled operad $\mathcal{P}$ generates a wheeled PROP. Let $\mathcal{C}_{\mathcal{P}}$ be the free wheeled PROP generated by the wheeled operad $\mathcal{P}$. The following explicit description of the wheeled PROP $\mathcal{C}_{\mathcal{P}}$ follows from the description of the free wheeled PROP generated by a collection of $(\mathfrak{S}_m, \mathfrak{S}_n)$–bimodules given in [17, Section 2.1.6] — see also [16, Section 2.3].

Proposition 2.3 *The wheeled PROP $\mathcal{C}_{\mathcal{P}}$ associated to a wheeled operad $\mathcal{P}$ is given by the $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules*

$$\mathcal{C}_{\mathcal{P}}(q,l) = \bigoplus_{J \subset \boldsymbol{q}} \Bigg(\bigoplus_{f : J \twoheadrightarrow \boldsymbol{l}} \bigotimes_{i=1}^{l} \mathcal{P}_0(f^{-1}(i)) \Bigg) \otimes \Bigg(\bigoplus_{k \in \mathbb{N}} \Bigg(\bigoplus_{g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}} \bigotimes_{i=1}^{k} \mathcal{P}_w(g^{-1}(i)) \Bigg)_{\mathfrak{S}_k} \Bigg)$$

where $\mathfrak{S}_k$ acts on $\bigoplus_{g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}} \bigotimes_{i=1}^{k} \mathcal{P}_w(g^{-1}(i))$ by postcomposition on $g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}$. The symmetric group $\mathfrak{S}_l$ acts by postcomposition on $f : J \to \boldsymbol{l}$ and $\mathfrak{S}_q$ by precomposition on $f : J \to \boldsymbol{l}$ and on $g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}$.

Horizontal composition is induced by disjoint union of maps and partitions.

The contractions $\xi_j^i : \mathcal{C}_{\mathcal{P}}(q,l) \to \mathcal{C}_{\mathcal{P}}(q-1,l-1)$ for $1 \le i \le q$ and $1 \le j \le l$ are induced by

(i) *the contractions $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$,*

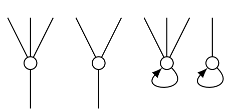
(ii) *the composition in the operad $\mathcal{P}_0$,*

(iii) *the right $\mathcal{P}_0$–module structure on $\mathcal{P}_w$.*

To illustrate the contractions defined in the previous proposition consider in $\mathcal{C}_{\mathcal{P}}(9,2)$ the summand corresponding to $J = \{1,2,3,4,5\}$, $f : J \to \mathbf{2}$ given by $f(1) = f(2) = f(3) = 1$ and $f(4) = f(5) = 2$, $g : \{6,7,8,9\} \to \mathbf{2}$ given by $g(6) = g(7) = g(8) = 1$ and $g(9) = 2$ and consider the element

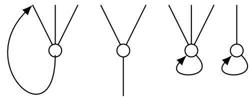
$$X \in \mathcal{P}_0(\{1,2,3\}) \otimes \mathcal{P}_0(\{4,5\}) \otimes \big(\mathcal{P}_w(\{6,7,8\}) \otimes \mathcal{P}_w(\{9\})\big)_{\mathfrak{S}_2}$$

given by the graph

$X =$ 

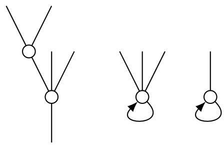
Case (i) is illustrated by

$$\xi_1^1(X) =$$



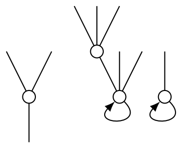
Case (ii) is illustrated by

$$\xi_2^1(X) =$$



Case (iii) is illustrated by

$$\xi_1^6(X) =$$



Remark 2.4 Note that

$$\mathcal{C}_{\mathcal{P}}(n,n) = \bigoplus_{f \in \mathfrak{S}_n} (\mathcal{P}_0(1))^{\otimes n}.$$

Considering the identity operation $\mathrm{Id} \in \mathcal{P}_0(1)$, we obtain a monomorphism of $\mathfrak{S}_n$–bimodules

$$\Bbbk[\mathfrak{S}_n] \to \mathcal{C}_{\mathcal{P}}(n,n).$$

Remark 2.5 Recall that from (2-1) vertical composition in a wheeled PROP is induced by horizontal composition and contractions.

The forgetful functor from the category of wheeled operads to the category of operads has a left adjoint denoted by $(-)^{\circlearrowright}$. For $\mathcal{P}_0$ an operad, $(\mathcal{P}_0)^{\circlearrowright}$ is called the *wheeled completion* of $\mathcal{P}_0$.

Remark 2.6 The wheeled PROP $\mathcal{C}_{\mathcal{P}}$ generated by a wheeled operad $\mathcal{P}$ has two distinguished sub-PROPs:

(1) the sub-PROP $\mathcal{C}_{\mathcal{P}_0}$ generated by the operad $\mathcal{P}_0$, corresponding to forgetting the wheeled part of $\mathcal{P}$;

(2) the sub-PROP denoted by $\mathcal{C}_w$ such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{C}_w(n,0) &= \mathcal{C}_{\mathcal{P}}(n,0), \\ \mathcal{C}_w(n,n) &= \Bbbk[\mathfrak{S}_n] && \text{for } n \geq 1, \\ \mathcal{C}_w(n,m) &= 0 && \text{for } m \notin \{0,n\}, \end{aligned}$$

corresponding to forgetting the operadic part in the PROP $\mathcal{C}_{\mathcal{P}}$.

3 Recollections on the functor category $\mathcal{F}(\mathbf{gr};\Bbbk)$

The purpose of this section is to review results on covariant and contravariant functors from the category $\mathbf{gr}$ of finitely generated free groups to $\Bbbk$–modules. We refer the reader to [2; 3; 5] for more details.

We denote by $\mathbb{Z}^{*n}$ the free group on n generators. The category $\mathbf{gr}$ has a skeleton with objects $\mathbb{Z}^{*n}$ for $n \in \mathbb{N}$. Consequently, $\mathbf{gr}$ is essentially small and we denote by $\mathcal{F}(\mathbf{gr};\Bbbk)$ (resp. $\mathcal{F}(\mathbf{gr}^{\mathrm{op}};\Bbbk)$) the category of covariant (resp. contravariant) functors from $\mathbf{gr}$ to $\mathcal{V}$.

A fundamental example of functor in $\mathcal{F}(\mathbf{gr};\Bbbk)$ is the abelianization functor $\mathfrak{a}\colon \mathbf{gr} \to \mathcal{V}$ that sends a free group G to $(G/[G,G]) \otimes_{\mathbb{Z}} \Bbbk$.

Composing $\mathfrak{a}$ with the duality functor $-^*\colon \mathcal{V} \to (\mathcal{V})^{\mathrm{op}}$, we obtain a functor from $\mathbf{gr}$ to $\mathcal{V}^{\mathrm{op}}$. The category of functors from $\mathbf{gr}$ to $\mathcal{V}^{\mathrm{op}}$ is equivalent to $(\mathcal{F}(\mathbf{gr}^{\mathrm{op}};\Bbbk))^{\mathrm{op}}$. We will denote by $\mathfrak{a}^\vee\colon \mathbf{gr}^{\mathrm{op}} \to \mathcal{V}$ the functor corresponding to $-^* \circ \mathfrak{a}$ by this equivalence.

By the Yoneda lemma, the category $\mathcal{F}(\mathbf{gr};\Bbbk)$ has enough projective objects and a set of projective generators is given by the functors, for $n \in \mathbb{N}$,

$$P_n := \Bbbk[\mathbf{gr}(\mathbb{Z}^{*n}, -)]$$

where $\Bbbk[-]$ is the linearization functor from the category of sets to $\mathcal{V}$.

Each functor $F\colon \mathbf{gr} \to \mathcal{V}$ can be decomposed naturally as a direct sum $F = F(0) \oplus \bar{F}$ where $F(0)$ is the constant functor equal to $F(0)$ and $\bar{F}$ is a *reduced* functor, ie $\bar{F}(0) = 0$. For simplicity we denote $\bar{P} := \bar{P}_1$.

The notion of cross-effects and polynomial functors introduced by Eilenberg and Mac Lane for categories of modules can be extended to functors on $\mathbf{gr}$ and on $\mathbf{gr}^{\mathrm{op}}$. The d^{th} cross-effect defines an exact functor $\mathrm{cr}_d\colon \mathcal{F}(\mathbf{gr};\Bbbk) \to \mathcal{F}(\mathbf{gr}^{\times n};\Bbbk)$, where $\mathcal{F}(\mathbf{gr}^{\times n};\Bbbk)$ is the category of functors from $\mathbf{gr}^{\times n}$ to $\mathcal{V}$. A functor $F\colon \mathbf{gr} \to \mathcal{V}$ is polynomial of degree d if $\mathrm{cr}_{d+1}(F) = 0$ and $\mathrm{cr}_d(F) \neq 0$. Similarly, we can define polynomial functors on $\mathbf{gr}^{\mathrm{op}}$.

The functors $\mathfrak{a}$ and $\mathfrak{a}^\vee$ are reduced polynomial functors of degree one.

The reduced functor $\bar{P}$ and the cross-effects are related by the following result:

Proposition 3.1 *For $d \in \mathbb{N}$ and $F \in \mathcal{F}(\mathbf{gr};\Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes d}, F) \simeq \mathrm{cr}_d(F)(\mathbb{Z}, \dots, \mathbb{Z}).$$

We deduce the following corollary:

Corollary 3.2 *For $d \in \mathbb{N}$ and $F \in \mathcal{F}(\mathbf{gr};\Bbbk)$ a polynomial functor of degree $< d$,*

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes d}, F) = 0.$$

Since the abelianization functor $\mathfrak{a}$ takes its values in $\mathcal{V}^f$, for F a functor from $\mathcal{V}^f$ to $\mathcal{V}$, we can postcompose $\mathfrak{a}$ with F to obtain a functor of $\mathcal{F}(\mathbf{gr};\Bbbk)$. An important example of functor from $\mathcal{V}^f$ to $\mathcal{V}$ is the d^{th} tensor product functor $T^d : \mathcal{V}^f \to \mathcal{V}$, for $d \in \mathbb{N}$, defined on objects by $V \mapsto V^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts by place permutations on T^d. The functor $\mathfrak{a}^{\otimes d} := T^d \circ \mathfrak{a}$ is a polynomial covariant functor of degree d and $(\mathfrak{a}^\vee)^{\otimes d} := T^d \circ \mathfrak{a}^\vee$ is a polynomial contravariant functor of degree d. The notion of exponential functors is a powerful tool for computation; see [4]. A graded exponential functor is a monoidal functor from $(\mathcal{V}^f, \oplus, 0)$ to $(\mathrm{gr}\,\mathcal{V}, \otimes, \Bbbk)$.

If $\Bbbk$ is a field of characteristic 0, the d^{th} exterior power functor is defined, on a vector space V, by $\Lambda^d(V) = (T^d(V) \otimes \mathrm{sgn}_n)_{\mathfrak{S}_d}$ where sgn_n is the signature module and $\mathfrak{S}_d$ acts diagonally. The functor Λ^d is a direct summand of T^d. The functor $\Lambda^d \mathfrak{a} := \Lambda^d \circ \mathfrak{a}$ is a polynomial covariant functor of degree d. The exterior powers define a graded exponential functor $\Lambda^\bullet$. In particular, there are natural commutative products $\Lambda^i \otimes \Lambda^j \to \Lambda^{i+j}$ and cocommutative coproducts $\Lambda^{i+j} \to \Lambda^i \otimes \Lambda^j$, for $i, j \in \mathbb{N}$. Composing with the abelianization functor, we obtain a natural transformation of functors in $\mathcal{F}(\mathbf{gr}, \Bbbk)$,

$$\Lambda^{i+j}\mathfrak{a} \to \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}, \tag{3-1}$$

which will be used later.

For $G, F \in \mathcal{F}(\mathbf{gr};\Bbbk)$, the *exterior tensor product* of G and F is the functor

$$G \boxtimes F : \mathbf{gr} \times \mathbf{gr} \to \Bbbk\text{–Mod}$$

given on objects by

$$(G \boxtimes F)(\mathbb{Z}^{*n}, \mathbb{Z}^{*m}) = G(\mathbb{Z}^{*n}) \otimes F(\mathbb{Z}^{*m}).$$

Similarly, for $G \in \mathcal{F}(\mathbf{gr}^{\mathrm{op}};\Bbbk)$ and $F \in \mathcal{F}(\mathbf{gr};\Bbbk)$ we define $G \boxtimes F : \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr} \to \Bbbk\text{–Mod}$.

We denote by $\amalg_d : \mathbf{gr}^{\times d} \to \mathbf{gr}$ the functor obtained by iteration of the free product (which is the coproduct in $\mathbf{gr}$) and $\delta_d : \mathbf{gr} \to \mathbf{gr}^{\times d}$ the diagonal functor. The functor δ_d is right adjoint to the functor $\amalg_d$. It follows that the functor $\delta_d^* : \mathcal{F}(\mathbf{gr}^{\times d};\Bbbk) \to \mathcal{F}(\mathbf{gr};\Bbbk)$ given by precomposition is left adjoint of the functor $\pi_d^* : \mathcal{F}(\mathbf{gr};\Bbbk) \to \mathcal{F}(\mathbf{gr}^{\times d};\Bbbk)$ given by precomposition.

Tensor powers $T^\bullet$ do not define an exponential functor but we have a similar property using induction of symmetric groups; see [22, (3)]. In particular,

$$\pi_2^*(\mathfrak{a}^{\otimes q}) \simeq \bigoplus_{J \subset \boldsymbol{q}} \mathfrak{a}^{\otimes |J|} \boxtimes \mathfrak{a}^{\otimes |\boldsymbol{q} \setminus J|}. \tag{3-2}$$

Part II Stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients given by a bifunctor

4 Definition of stable homology of $\mathrm{Aut}(\mathbb{Z}^{*n})$

Let $I_n\colon \mathrm{Aut}(\mathbb{Z}^{*n}) \to \mathrm{Aut}(\mathbb{Z}^{*n+1})$ be the group monomorphism induced by $- * \mathbb{Z}$. By restriction along I_n we obtain a functor $U^{I_n}\colon \mathrm{Aut}(\mathbb{Z}^{*n+1})$–Mod $\to \mathrm{Aut}(\mathbb{Z}^{*n})$–Mod where $\mathrm{Aut}(\mathbb{Z}^{*n})$–Mod is the category of modules over $\mathrm{Aut}(\mathbb{Z}^{*n})$.

For $B\colon \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr} \to \mathcal{V}$ a functor and $n \in \mathbb{N}$, $B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ is an $\mathrm{Aut}(\mathbb{Z}^{*n})^{\mathrm{op}} \times \mathrm{Aut}(\mathbb{Z}^{*n})$–module. Let $p_n\colon \mathbb{Z}^{*n+1} \to \mathbb{Z}^{*n}$ be the group epimorphism given by the projection on the first n variables and $i_n\colon \mathbb{Z}^{*n} \to \mathbb{Z}^{*n+1}$ be the group monomorphisms given by the inclusion of the first n variables.

The previous data give rise to $\mathrm{Aut}(\mathbb{Z}^{*n})$–homomorphisms

$$U^{I_n}(B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{B(i_n, p_n)} B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$$

where the structure of $\mathrm{Aut}(\mathbb{Z}^{*n})$–module on $B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ and $U^{I_n}(B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1}))$ is given by the diagonal action.

This gives group morphisms

$$H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})\big) \xrightarrow{\alpha_n} H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}); B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big).$$

The stable cohomology of the automorphism groups of free groups with coefficients given by B is then defined by

$$H^*_{\mathrm{st}}(B) := \lim_{n \in \mathbb{N}} H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}); B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big),$$

where the limit is taken over the group morphisms α_n.

In this paper we consider the family of coefficients $B_{l,q} = (\mathfrak{a}^\vee)^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q}$, where $l, q \in \mathbb{N}$ and $\boxtimes$ is the exterior tensor product. By the usual canonical homomorphism

$E^* \otimes F \to \mathrm{Hom}(E, F)$ for two $\Bbbk$–modules E and F, which is an isomorphism if E is free finitely generated, we obtain isomorphisms

$$B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*m}) = ((\Bbbk^n)^*)^{\otimes l} \otimes (\Bbbk^m)^{\otimes q} \simeq \mathrm{Hom}_{\mathcal{V}}((\Bbbk^n)^{\otimes l}, (\Bbbk^m)^{\otimes q}).$$

Based on this isomorphism, $B_{l,q}$ will sometimes be denoted by $\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$.

5 Stability

Let $\mathcal{G}$ be the category having as objects the finitely generated free groups and where a morphism from A to B is a pair (u, H) where $u\colon A \hookrightarrow B$ is an injective homomorphism and H is a subgroup of B such that $B = H * u(A)$. Recall from [3, Définition 4.2] that there is a functor $\iota\colon \mathcal{G} \to \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr}$ sending an object A to (A, A) and a map $(u, H)\colon A \to B$ to $(B = H * u(A) \to u(A) \xrightarrow{u^{-1}} A, u\colon A \to B)$.

The category $\mathcal{G}$ is homogeneous in the sense of [21, Definition 1.3] and the functor $B_{l,q} = (\mathfrak{a}^{\vee})^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q}$ is the exterior product between a polynomial contravariant functor of degree l and a polynomial covariant functor of degree q, so the composition $B \circ \iota$ is a coefficient system of degree $l+q$. Hence, by [21, Theorem 5.4], for $i \in \mathbb{N}$ the group morphism

$$H^i(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B_{l,q}(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{\alpha_n} H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \tag{5-1}$$

is an isomorphism for $n \geq N_{l,q,i}$ where $N_{l,q,i} = 2i + l + q + 3$. We deduce that, for n big enough, we have an isomorphism

$$H^i_{\mathrm{st}}((\mathfrak{a}^{\vee})^{\otimes l} \otimes \mathfrak{a}^{\otimes q}) \simeq H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})). \tag{5-2}$$

For l_1, q_1, l_2, q_2 in $\mathbb{N}$, the cup product gives morphisms

$$\begin{array}{c} H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_1,q_1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \otimes H^j(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_2,q_2}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \\ \Big\downarrow \cup \\ H^{i+j}(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_1,q_1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}) \otimes B_{l_2,q_2}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \end{array}$$

For $n > \mathrm{Max}(N_{l_1,q_1,i}, N_{l_2,q_2,j})$, the stability isomorphisms (5-2) give the following cup product map on the stable cohomology:

$$\cup\colon H^i_{\mathrm{st}}(B_{l_1,q_1}) \otimes H^j_{\mathrm{st}}(B_{l_2,q_2}) \to H^{i+j}_{\mathrm{st}}(B_{l_1,q_1} \otimes B_{l_2,q_2}). \tag{5-3}$$

6 The wheeled PROP $\mathcal{H}$ of stable cohomology

The aim of this section is to prove that the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients twisted by $B_{l,q} = (\mathfrak{a}^\vee)^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q} = \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ defines a wheeled PROP. This should be viewed as a cohomological version of the wheeled endomorphism PROP considered in Examples 2.1 and 2.2 using the stability isomorphism (5-2).

Definition 6.1 The PROP $\mathcal{H}$ is defined by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}(q,l) = H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}))$$

where the action of the symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by place permutations of the copies of $\mathfrak{a}$ (resp. $(\mathfrak{a}^\vee)$).

The horizontal composition $\otimes\colon \mathcal{H}(q_1,l_1) \otimes \mathcal{H}(q_2,l_2) \to \mathcal{H}(q_1+q_2, l_1+l_2)$ is given by

$$\begin{gathered}
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1}, \mathfrak{a}^{\otimes q_1})) \otimes H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_2}, \mathfrak{a}^{\otimes q_2})) \\
\big\downarrow \cup \\
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1}, \mathfrak{a}^{\otimes q_1}) \otimes \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_2}, \mathfrak{a}^{\otimes q_2})) \\
\big\downarrow \lambda \\
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1+l_2}, \mathfrak{a}^{\otimes q_1+q_2}))
\end{gathered}$$

where $\cup$ is the cup product map given in (5-3) and λ is the map induced by the tensor product of linear maps.

The vertical composition $\circ\colon \mathcal{H}(l,m) \otimes \mathcal{H}(q,l) \to \mathcal{H}(q,m)$ is given by

$$\begin{gathered}
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes l})) \otimes H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})) \\
\big\downarrow \cup \\
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes l}) \otimes \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})) \\
\big\downarrow \gamma \\
H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes q}))
\end{gathered}$$

where γ is the map induced by the composition in $\mathcal{V}$.

Proposition 6.2 *The PROP $\mathcal{H}$ is a wheeled PROP for the contractions, for $1 \le i \le q$ and $1 \le j \le l$,*

$$\xi^i_j\colon \mathcal{H}(q,l) \to \mathcal{H}(q-1,l-1)$$

induced, for $n \geq N_{l,q,*}$, *by the maps*

$$H^*(\mathrm{Aut}(\mathbb{Z}^{*n}), \varphi^i_j)\colon H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}), B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big) \\ \to H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}), B_{l-1,q-1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big),$$

where φ^i_j *are as defined in Example 2.2*

Proof The maps φ^i_j are $\mathrm{Aut}(\mathbb{Z}^{*n})$–equivariant. For $n \geq N_{l,q,*}$, since

$$N_{l,q,*} > N_{l-1,q-1,*},$$

$H^*(\mathrm{Aut}(\mathbb{Z}^{*n}), \varphi^i_j)$ induces a map $\xi^i_j\colon \mathcal{H}(q,l) \to \mathcal{H}(q-1,l-1)$.

We verify that the contraction maps satisfy commutativity conditions and that they are compatible with the horizontal composition.

The biequivariance condition for the contraction maps corresponds to the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}(q,l) \arrow[r, "\xi^i_j"] \arrow[d, "(\sigma_1,\sigma_2)"'] & \mathcal{H}(q-1,l-1) \arrow[d, "(\sigma_1^{(\sigma_1^{-1}(i))},\sigma_2^{(j)})"] \\
\mathcal{H}(q,l) \arrow[r, "\xi^{\sigma_1^{-1}(i)}_{\sigma_2(j)}"'] & \mathcal{H}(q-1,l-1)
\end{tikzcd}$$

where $\sigma_1 \in \mathfrak{S}_q$ and $\sigma_2 \in \mathfrak{S}_l$, $\sigma_2^{(j)} \in \mathfrak{S}_{l-1}$ is the permutation induced by σ_2 forgetting j and $\sigma_2(j)$ and reindexing, and $\sigma_1^{(\sigma_1^{-1}(i))} \in \mathfrak{S}_{q-1}$ is the permutation induced by σ_1 forgetting $\sigma_1^{-1}(i)$ and i and reindexing. □

Remark 6.3 For $l > 0$, $(\mathfrak{a}^\vee)^{\otimes l}$ is a reduced contravariant functor which is polynomial of degree l. It follows from the main result of [3] that $H^*_{\mathrm{st}}((\mathfrak{a}^\vee)^{\otimes l}) = 0$, so $\mathcal{H}(0,l) = 0$ for $l > 0$.

In order to relate our results to Djament's result obtained in [1] we introduce the following:

Definition 6.4 Let $\mathcal{H}'$ be the sub-PROP of $\mathcal{H}$ such that, for $n \in \mathbb{N}$

$$\mathcal{H}'(n,0) = \mathcal{H}(n,0), \quad \mathcal{H}'(n,n) = \Bbbk[\mathfrak{S}_n], \quad \mathcal{H}'(n,m) = 0 \text{ if } m \notin \{0,n\}.$$

Remark 6.5 Note that

$$\mathcal{H}(n,n)^0 = H^0_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n})) \simeq (\mathrm{Hom}_{\mathcal{V}}((\Bbbk^m)^{\otimes n}, (\Bbbk^m)^{\otimes n}))^{\mathrm{Aut}(\mathbb{Z}^{*m})}$$

where the last isomorphism is given by (5-2), for m big enough. Hence

$$\mathcal{H}(n,n)^0 \simeq \Bbbk[\mathfrak{S}_n]$$

and the endomorphisms in the sub-PROP $\mathcal{H}'$ correspond to the endomorphisms in $\mathcal{H}$ in degree 0.

Remark 6.6 The wheeled PROP structure on $\mathcal{H}$ comes from the wheeled endomorphism PROP and does not depend on the family of groups considered. Consequently, there are other families of groups for which we have a wheeled PROP structure on the stable cohomology with coefficients given by $B_{l,q} = \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$. For example, for the braid groups we have a wheeled PROP $\mathcal{H}^{B_\infty}$ and the group morphism $B_n \to \mathrm{Aut}(\mathbb{Z}^{*n})$ induces a morphism of wheeled PROP $\mathcal{H} \to \mathcal{H}^{B_\infty}$. Similarly, for $\Sigma_{g,1}$ a connected and oriented surface of genus g with 1 boundary component and $\mathcal{M}_{g,1}$ its mapping class group, we have a wheeled PROP $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ and the group morphism $\mathcal{M}_{g,1} \to \mathrm{Aut}(\mathbb{Z}^{*2g})$ gives a morphism of wheeled PROP $\mathcal{H} \to \mathcal{H}^{\mathrm{MCG}_{\infty,1}}$. The wheeled PROPs $\mathcal{H}^{B_\infty}$ and $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ have further structure. This will be developed elsewhere. Similarly, for a symmetric monoidal category $\mathcal{C}$ and a dualizable object in $\mathcal{C}$, the cohomology of the automorphism groups in $\mathcal{C}$ with appropriate coefficients has a wheeled PROP structure.

Remark 6.7 We have also wheeled PROP structures in the unstable ranges. More precisely, for $n \in \mathbb{N}$, we can define a wheeled PROP $\mathcal{H}^n$ given by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}^n(q,l) = H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})),$$

the wheeled PROP structure being defined in a similar way as in Definition 6.1 and Proposition 6.2. The stabilization morphism (5-1) gives a morphism of wheeled PROPs: $\mathcal{H}^{n+1} \to \mathcal{H}^n$. The PROP $\mathcal{H}$ considered in Definition 6.1 is the limit of these PROP morphisms. The PROPs $\mathcal{H}^n$ are, in general, more complicated than $\mathcal{H}$ since they can contain nonstable cohomological classes.

7 Cohomological classes

In [8] — see also [9] — the first author introduced cohomology classes that give nonzero morphisms in the PROP $\mathcal{H}$. In this section we show that these classes are obtained from the class h_1, recalled below, using the wheeled PROP structure on $\mathcal{H}$.

The q^{th} Johnson map induced by a Magnus expansion θ is a map

$$\tau_q^\theta\colon \operatorname{Aut}(\mathbb{Z}^{*n}) \to \operatorname{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes q+1}).$$

By [8, Lemma 2.1], τ_1^θ is a 1–cocycle and the cohomology class

$$h_1 = [\tau_1^\theta] \in H^1(\operatorname{Aut}(\mathbb{Z}^{*n}), \operatorname{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2}))$$

does not depend on the choice of Magnus expansion θ. For n big enough, h_1 gives a nonzero element in $\operatorname{Hom}_{\mathcal{H}}(2,1)$ in cohomological degree 1. Using [8, (4.4)] and the anticommutativity of the cup product we obtain that $\mathfrak{S}_2$ acts on h_1 by the signature.

By [8, (4.11)], we have the relation in $\operatorname{Hom}_{\mathcal{H}}(3,1)$

$$h_1 \circ (h_1 \otimes 1) + h_1 \circ (1 \otimes h_1) = 0 \tag{7-1}$$

where $\otimes$ is the horizontal composition in the PROP $\mathcal{H}$ and $\circ$ is the vertical composition in the PROP $\mathcal{H}$.

Let $\mathcal{K}$ be the subwheeled PROP of $\mathcal{H}$ generated by the class h_1.

Proposition 7.1 *For $p \in \mathbb{N}$, the classes $h_{p+1} \in \mathcal{K}(p+2,1)$, defined inductively by*

$$h_{p+1} = h_1 \circ (h_p \otimes 1),$$

and $\bar{h}_p \in \mathcal{K}(p,0)$, defined by

$$\bar{h}_p = \xi_1^1(h_p),$$

are the cohomological classes introduced in [8], in the stable range.

Proof For $p \geq 2$, using the cup product we obtain classes

$$(h_1)^{\cup p} \in H^p\big(\operatorname{Aut}(\mathbb{Z}^{*n}), \operatorname{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2})^{\otimes p}\big) \simeq H^p\big(\operatorname{Aut}(\mathbb{Z}^{*n}), \operatorname{Hom}_{\mathcal{V}}((\Bbbk^n)^{\otimes p}, (\Bbbk^n)^{\otimes 2p})\big)$$

where the isomorphism is induced by the canonical homomorphism of $\Bbbk$–modules given by tensor product of linear maps which is an isomorphism for free finitely generated modules.

In the stable range, we obtain $(h_1)^{\cup p} \in \mathcal{H}(2p,p)$ and the previous construction corresponds to the horizontal composition in the PROP $\mathcal{H}$ introduced in Definition 6.1.

Consider the maps

$$\varsigma_p\colon ((\Bbbk^n)^* \otimes (\Bbbk^n)^{\otimes 2})^{\otimes p} \simeq \operatorname{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2})^{\otimes p} \to \operatorname{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes p+1}) \simeq (\Bbbk^n)^* \otimes (\Bbbk^n)^{\otimes p+1}$$

given by

$$\varsigma_p(u_1\otimes u_2\otimes\cdots\otimes u_p):=(u_1\otimes 1_{(\Bbbk^n)^{\otimes p-1}})\circ(u_2\otimes 1_{(\Bbbk^n)^{\otimes p-2}})\circ\cdots\circ(u_{p-1}\otimes 1_{\Bbbk^n})\circ u_p$$

where $u_i\in \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n,(\Bbbk^n)^{\otimes 2})$ for $1\le i\le p$; see [8, (4.8)].

The cohomological classes $h_p\in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\mathrm{Hom}_{\mathcal{V}}(\Bbbk^n,(\Bbbk^n)^{\otimes p+1}))$ are defined in [8, Theorem 4.1] by

$$h_p=H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\varsigma_p)(h_1^{\cup p}).$$

Note that

$$\varsigma_p(u_1\otimes u_2\otimes\cdots\otimes u_p)=(\varsigma_{p-1}(u_1\otimes u_2\otimes\cdots\otimes u_{p-1})\otimes 1_{\Bbbk^n})\circ u_p$$

It follows that, for n big enough, the classes h_p can be defined recursively by

$$h_{p+1}=h_1\circ(h_p\otimes 1)\in\mathcal{H}(p+2,1).$$

Consider the map

$$\varphi_1^1\colon(\Bbbk^n)^*\otimes(\Bbbk^n)^{\otimes p+1}\to(\Bbbk^n)^{\otimes p}$$

introduced in Example 2.2. The reduced class $\bar{h}_p\in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),(\Bbbk^n)^{\otimes p})$ is defined, in [8, (4.7)], from the class h_p by

$$\bar{h}_p=H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\varphi_1^1)(h_p).$$

In the stable range, this corresponds to considering the contraction

$$\xi_1^1\colon\mathcal{H}(p+1,1)\to\mathcal{H}(p,0)$$

introduced in Proposition 6.2, so we have $\bar{h}_p=\xi_1^1(h_p)$. □

Remark 7.2 By the biequivariance condition for the contraction map,

$$\xi_1^2(h_1)=-\bar{h}_1.$$

Remark 7.3 The wheeled PROP $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ evoked in Remark 6.6 is related to the graph description of the (twisted) Mumford–Morita–Miller classes by Morita and the first author [11; 19]. Morita [18] extended the first Johnson homomorphism of the Torelli group $\mathcal{I}_{g,1}$ to a twisted cohomology class $\tilde{k}\in H^1\big(\mathcal{M}_{g,1};\frac{1}{2}\Lambda^3\mathfrak{a}(\pi_1(\Sigma_{g,1}))\big)$. The class h_1 restricts to $\tilde{k}$ on the mapping class group $\mathcal{M}_{g,1}$. Morita [19] constructed cohomology classes of the mapping class group with trivial coefficients $\mathbb{Q}$ by contracting a power of the class $\tilde{k}$ in terms of trivalent graphs. More precisely, any trivalent graph Γ with $2n$ vertices defines an $\mathrm{Sp}(\mathfrak{a}(\pi_1(\Sigma_{g,1})))$–invariant linear map $\alpha_\Gamma\colon\Lambda^{2n}(\Lambda^3\mathfrak{a}(\pi_1(\Sigma_{g,1}))\otimes\mathbb{Q})\to\mathbb{Q}$ by using the intersection pairing on $\mathfrak{a}(\pi_1(\Sigma_{g,1}))$. Then we obtain a cohomology class $\alpha_{\Gamma *}(\tilde{k}^{2n})\in H^{2n}(\mathcal{M}_{g,1};\mathbb{Q})$. Morita and the first author proved all these classes are polynomials in the Mumford–Morita–Miller classes [10], and generalized his construction to all finite graphs and twisted Mumford–Morita–Miller classes [11]. Here any graph with n univalent vertices defines a cohomology class of $\mathcal{M}_{g,1}$ with coefficients in $\Lambda^n \mathfrak{a}(\pi_1(\Sigma_{g,1})) \otimes \mathbb{Q}$, which is proved to be a polynomial of twisted Mumford–Morita–Miller classes.

Part III Functor cohomology in $\mathcal{F}(\mathbf{gr}; \Bbbk)$

The aim of this part is to introduce the wheeled PROP $\mathcal{E}$ given by Ext–groups in the functor category $\mathcal{F}(\mathbf{gr})$.

8 Projective resolution of the abelianization functor

The abelianization functor $\mathfrak{a}$ has an explicit projective resolution in $\mathcal{F}(\mathbf{gr}; \Bbbk)$. This resolution occurs in [6, Proposition 5.1] and plays a crucial rôle in [3] and [22]. In this section we recall the construction of this projective resolution.

Recall (see Section 3) that for $n \in \mathbb{N}$, the functors $P_n := \Bbbk[\mathbf{gr}(\mathbb{Z}^{*n}, -)]$ form a set of projective generators of the category $\mathcal{F}(\mathbf{gr}; \Bbbk)$. Consider the simplicial object in $\mathcal{F}(\mathbf{gr}; \Bbbk)$

$$\cdots \overset{\cdots}{\rightrightarrows} P_{n+1} \overset{\cdots}{\rightrightarrows} P_n \overset{\cdots}{\rightrightarrows} \cdots \rightrightarrows P_2 \rightrightarrows P_1 \rightrightarrows P_0 \tag{8-1}$$

where $\delta_i \colon P_{n+1} \to P_n$ for $0 \le i \le n+1$ are defined by

$$\begin{aligned} \delta_0[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_2, \dots, g_n, g_{n+1}], \\ \delta_i[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_1, \dots, g_i g_{i+1}, \dots, g_n, g_{n+1}] \quad \text{for } 1 \le i \le n, \\ \delta_{n+1}[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_1, g_2, \dots, g_n], \end{aligned}$$

and $\varepsilon_i \colon P_n \to P_{n+1}$ for $1 \le i \le n+1$ are defined by

$$\varepsilon_i[g_1, \dots, g_n] = [g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n].$$

We denote by $C_\bullet$ the unnormalized chain complex associated to this simplicial object and $D_\bullet$ the complex defined by $D_i = C_{i+1}$ for $i \ge 0$ and $D_i = 0$ for $i < 0$.

Since the homology of a free group is naturally isomorphic to its abelianization in degree 1 and is zero in degree > 1, $D_\bullet$ is a resolution of $\mathfrak{a}$ and we obtain that the exact sequence in $\mathcal{F}(\mathbf{gr})$

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_2 \to P_1$$

is a projective resolution of the abelianization functor $\mathfrak{a} \colon \mathbf{gr} \to \mathrm{Ab}$.

Considering the normalized version we obtain a variant of the previous resolution of the form

$$\cdots \to \overline{P}^{\otimes n+1} \xrightarrow{d_n} \overline{P}^{\otimes n} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \overline{P}^{\otimes 2} \xrightarrow{d_1} \overline{P} \tag{8-2}$$

where $\overline{P}$ is the reduced part of P_1 (see Section 3) and the map $\pi : \overline{P} \to \mathfrak{a}$ corresponds to $1 \in \mathbb{Z}$ via the isomorphism $\mathrm{Hom}(\overline{P}, \mathfrak{a}) \simeq \mathbb{Z}$ obtained using Proposition 3.1.

9 The PROP $\mathcal{E}^0$

The aim of this section is to describe the structure of the following graded PROP which can be viewed as an Ext–version of the endomorphism PROP:

Definition 9.1 The graded linear PROP $\mathcal{E}^0$ is defined by the $(\mathfrak{S}_q, \mathfrak{S}_l)$–graded bimodules

$$\mathcal{E}^0(q,l) = \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$$

where the action of the symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by the permutations of the copies of $\mathfrak{a}$ in the first (resp. second) variable.

The horizontal composition $\mathcal{E}^0(q_1,l_1) \otimes \mathcal{E}^0(q_2,l_2) \to \mathcal{E}^0(q_1+q_2, l_1+l_2)$ is given by the exterior product and the vertical composition $\mathcal{E}^0(q,l) \otimes \mathcal{E}^0(l,m) \to \mathcal{E}^0(q,m)$ is given by the Yoneda product.

Remark 9.2 We warn the reader that $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ should not be confused with $\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ introduced at the end of Section 4.

In [22], the second author obtained the following results:

Theorem 9.3 [22, Theorem 2.3] *For $l, q \in \mathbb{N}$, we have an isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}) \simeq \begin{cases} \Bbbk[\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})] & \text{if } * = q-l, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})$ is the set of surjections from $\boldsymbol{q}$ to $\boldsymbol{l}$.

Theorem 9.4 [22, Proposition 2.5] *The symmetric groups $\mathfrak{S}_q$ and $\mathfrak{S}_l$ act on*

$$\mathrm{Ext}^{q-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}) \simeq \Bbbk[\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})]$$

in the following way: for $\sigma \in \mathfrak{S}_q$, $\tau_{a,b} \in \mathfrak{S}_l$, *the transposition of* a *and* b *where* $a, b \in \{1, \dots, l\}$, *and* $f \in \mathrm{Surj}(\boldsymbol{q}, \boldsymbol{l})$

$$[f].\sigma = \prod_{1 \le i \le l} \varepsilon(\overline{\sigma|_{(f\circ\sigma)^{-1}(i)}})[f \circ \sigma],$$

where $\sigma|_{(f\circ\sigma)^{-1}(i)} \colon (f\circ\sigma)^{-1}(i) \to \sigma((f\circ\sigma)^{-1}(i))$, *and*

$$\tau_{a,b}.[f] = (-1)^{(|f^{-1}(a)|-1)(|f^{-1}(b)|-1)}[\tau_{a,b} \circ f].$$

Proposition 9.5 [22, Proposition 3.1] *The external product*

$$e \colon \mathrm{Ext}^{m-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes m}) \otimes \mathrm{Ext}^{n-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes p}, \mathfrak{a}^{\otimes n}) \to \mathrm{Ext}^{m+n-l-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l+p}, \mathfrak{a}^{\otimes m+n})$$

is induced by the disjoint union of sets via the isomorphism of Theorem 9.3.

For c_{m-l} (resp. c_{n-p}) a cocycle representing a generator of $\mathrm{Ext}^{m-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes m})$ (resp. $\mathrm{Ext}^{n-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes p}, \mathfrak{a}^{\otimes n})$), we will denote $e([c_{m-l}], [c_{n-p}])$ by $[c_{m-l}] \otimes [c_{n-p}]$.

Note that in the description of the Yoneda product in terms of surjection given in [22, Proposition 3.1] the signs are not correct. One of the aim of Sections 9.1 and 9.2 is to give a corrigendum of this statement.

9.1 Explicit classes in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$

The aim of this section is to construct explicit cocycles representing the generators in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ and to study their behavior via the Yoneda product

$$\mathcal{Y} \colon \mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \otimes \mathrm{Ext}^{1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1}) \to \mathrm{Ext}^{n}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n+1})$$

We begin by introducing explicit classes in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$.

Proposition 9.6 *For* $n \in \mathbb{N} \setminus \{0\}$, *the morphism* $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ *is a cocycle representing a generator of* $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$.

Proof Using the normalized bar resolution (8-2), $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ is the homology of the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n-1}, \mathfrak{a}^{\otimes n}) \xrightarrow{d} \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}) \xrightarrow{d} \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) \to \cdots.$$

By Corollary 3.2, $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) = 0$ since $\mathfrak{a}^{\otimes n}$ is a polynomial functor of degree n. It follows that $d(\pi^{\otimes n}) = 0$.

Moreover, the morphism $\pi^{\otimes n}$ represents $[\mathrm{Id}_n]$ via the isomorphism

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}) \simeq \mathrm{cr}_n \mathfrak{a}^{\otimes n}(\mathbb{Z}, \dots, \mathbb{Z}) \simeq \Bbbk[\mathfrak{S}_n]$$

using the external product $\mathrm{Hom}(\bar{P}, \mathfrak{a}) \otimes \cdots \otimes \mathrm{Hom}(\bar{P}, \mathfrak{a}) \to \mathrm{Hom}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n})$. We deduce from the previous complex an exact sequence of $\mathfrak{S}_n$–modules

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n-1}, \mathfrak{a}^{\otimes n}) \to \Bbbk[\mathfrak{S}_n] \to \mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \to 0.$$

It follows that $[\mathrm{Id}_n]$ gives a generator of the $\mathfrak{S}_n$–module $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$. □

In the next proposition we introduce particular classes in $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1})$. Let $Q_n^\bullet \to \mathfrak{a}^{\otimes n}$ be a projective resolution of $\mathfrak{a}^{\otimes n}$ and consider the resolution

$$\bar{P}^{\otimes \bullet+1} \otimes \mathfrak{a}^{\otimes n-1} \to \mathfrak{a}^{\otimes n}$$

obtained by tensoring the complex (8-2) with $\mathfrak{a}^{\otimes n-1}$. By standard homological algebra — see [23, Comparison Theorem 2.2.6] — there is a chain map

$$\alpha^\bullet \colon Q_n^\bullet \to \bar{P}^{\otimes \bullet+1} \otimes \mathfrak{a}^{\otimes n-1}$$

lifting $\mathrm{Id}_{\mathfrak{a}^{\otimes n}} \colon \mathfrak{a}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ which is unique up to chain homotopy equivalence.

Lemma 9.7 *The map*

$$(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}) \circ \alpha^1 \colon Q_n^1 \to \mathfrak{a}^{\otimes n+1}$$

represents the class of $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1})$ *corresponding to the exterior product* $[\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}]$.

Proof This is a direct consequence of the definition of the exterior product of classes. □

Lemma 9.8 *The functor* $\mathrm{Im}(d_{n-1})$ *has projective resolution*

$$\bar{P}^{\otimes \bullet+n} \colon \cdots \to \bar{P}^{\otimes n+1} \xrightarrow{d_n} \bar{P}^{\otimes n} \tag{9-1}$$

given by truncating (8-2). Moreover, the map $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ *factorizes through* $\mathrm{Im}(d_{n-1})$, *giving rise to a morphism* $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$.

Proof By the projective resolution (9-1) we obtain the complex

$$\cdots \leftarrow \mathrm{Hom}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) \xleftarrow{d_n^*} \mathrm{Hom}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}),$$

computing $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathrm{Im}(d_{n-1}), \mathfrak{a}^{\otimes n})$. The map $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ satisfies $\pi^{\otimes n} \circ d_n = 0$, so it represents a cocycle in $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\mathrm{Im}(d_{n-1}), \mathfrak{a}^{\otimes n})$. We deduce that $\pi^{\otimes n}$ factorizes through $\mathrm{Im}(d_{n-1})$, giving rise to a morphism $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$ □

Lemma 9.9 *We have a morphism of exact chain complexes*

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes k+1} \otimes \overline{P}^{\otimes n-1} \arrow[r, "d_{n+k-1}"] \arrow[d, "\mathrm{Id} \otimes \pi^{\otimes n-1}"] & \cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes 2} \otimes \overline{P}^{\otimes n-1} \arrow[r, "d_n"] \arrow[d, "\mathrm{Id} \otimes \pi^{\otimes n-1}"] & \overline{P} \otimes \overline{P}^{\otimes n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\mathrm{Id} \otimes \pi^{\otimes n-1}"] & \mathrm{Im}(d_{n-1}) \arrow[r] \arrow[d, "\bar{\pi}^{\otimes n}"] & 0 \\
\cdots \arrow[r] & \overline{P}^{\otimes k+1} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r, "d_k \otimes \mathrm{Id}"] & \cdots \arrow[r] & \overline{P}^{\otimes 2} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r, "d_1 \otimes \mathrm{Id}"] & \overline{P} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r, "\pi \otimes \mathrm{Id}"] & \mathfrak{a}^{\otimes n} \arrow[r] & 0
\end{tikzcd}$$

Proof The square on the right commutes since

$$\bar{\pi}^{\otimes n} \circ d_{n-1} = \pi^{\otimes n} = (\pi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \pi^{\otimes n-1}).$$

For $k \in \mathbb{N}$, a direct computation using the differential in the reduced bar resolution gives the commutativity of the diagram

$$\begin{tikzcd}
\overline{P}^{\otimes k+2} \otimes \overline{P}^{\otimes n-1} \arrow[r, "d_{n+k}"] \arrow[d, "\mathrm{Id} \otimes \pi^{\otimes n-1}"] & \overline{P}^{\otimes k+1} \otimes \overline{P}^{\otimes n-1} \arrow[d, "\mathrm{Id} \otimes \pi^{\otimes n-1}"] \\
\overline{P}^{\otimes k+2} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r, "d_{k+1} \otimes \mathrm{Id}"'] & \overline{P}^{\otimes k+1} \otimes \mathfrak{a}^{\otimes n-1}
\end{tikzcd}$$

□

Remark 9.10 The reader's attention is drawn to the fact that the following diagram is only commutative *up to a sign*:

$$\begin{tikzcd}
\overline{P}^{\otimes n-1} \otimes \overline{P}^{\otimes k+2} \arrow[r, "d_{n+k}"] \arrow[d, "\pi^{\otimes n-1} \otimes \mathrm{Id}"] & \overline{P}^{\otimes n-1} \otimes \overline{P}^{\otimes k+1} \arrow[d, "\pi^{\otimes n-1} \otimes \mathrm{Id}"] \\
\mathfrak{a}^{\otimes n-1} \otimes \overline{P}^{\otimes k+2} \arrow[r, "\mathrm{Id} \otimes d_k"'] & \mathfrak{a}^{\otimes n-1} \otimes \overline{P}^{\otimes k+1}
\end{tikzcd}$$

Proposition 9.11 *For* $n \geq 2$

$$\mathcal{Y}([\pi^{\otimes n}], [\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}]) = [\pi^{\otimes n+1}].$$

Proof By [23, Comparison Theorem 2.2.6] there is a chain map $\beta^\bullet$ lifting

$$\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}.$$

We obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes n+1} \arrow[r, "d_n"] \arrow[d, "\beta^1"] & \overline{P}^{\otimes n} \arrow[r, "d_{n-1}"] \arrow[d, "\beta^0"] & \mathrm{Im}(d_{n-1}) \arrow[r] \arrow[d, "\bar{\pi}^{\otimes n}"] & 0 \\
\cdots \arrow[r] \arrow[d] & Q_n^1 \arrow[r] \arrow[d, "\alpha^1"] & Q_n^0 \arrow[r] \arrow[d, "\alpha^0"] & \mathfrak{a}^{\otimes n} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\cdots \arrow[r] & \overline{P}^{\otimes 2} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r] & \overline{P}^{\otimes 1} \otimes \mathfrak{a}^{\otimes n-1} \arrow[r] & \mathfrak{a}^{\otimes n} \arrow[r] & 0
\end{tikzcd}$$

By the construction of the Yoneda product, $(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1 \circ \beta^1$ is a cocycle representing $\mathcal{Y}([\pi^{\otimes n}], [(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1])$.

Since the chain map lifting $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$ is unique up to chain homotopy equivalence, using Lemma 9.9 we have

$$[(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1 \circ \beta^1] = [(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ (\mathrm{Id} \otimes \pi^{\otimes n-1})] = [\pi^{\otimes n+1}]. \qquad \square$$

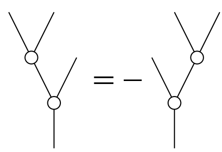
9.2 The operad $\mathcal{P}_0$

In [22, Proposition 3.5] the second author proved that the graded PROP $\mathcal{E}^0$ is freely generated by its underlying operad $\mathcal{P}_0$. Using Theorems 9.3 and 9.4, $\mathcal{P}_0$ is the graded operad such that $\mathcal{P}_0(0) = 0$ and for $k > 0$, $\mathcal{P}_0(k)$ is the sign representation of $\mathfrak{S}_k$ placed in cohomological degree $k-1$ and 0 in other degrees. The aim of this section is to give an explicit description of this operad, in particular to describe the composition which is induced by the Yoneda product.

Definition 9.12 The operad $\mathcal{Q}$ is the quadratic graded operad generated by one antisymmetric binary operation μ in degree 1 subject to the quadratic relation

$$\mu \circ_1 \mu = -\mu \circ_2 \mu.$$

Pictorially, we have



Proposition 9.13 *The operad $\mathcal{P}_0$ is isomorphic to $\mathcal{Q}$.*

Proof We show that there is a morphism of operads $f \colon \mathcal{Q} \to \mathcal{P}_0$ given on the generator μ by $f_2(\mu) = [\pi^{\otimes 2}]$ where $[\pi^{\otimes 2}]$ is a generator, in degree 1, of $\mathcal{E}^0(2,1) = \mathcal{P}_0(2)$ defined in Proposition 9.6 which is antisymmetric by Theorem 9.4.

Before proving that $[\pi^{\otimes 2}]$ satisfies the quadratic relation satisfied by μ, note that the partial composition operations in $\mathcal{P}_0$ are given by the restriction of the categorical composition induced by the Yoneda product

$$\mathcal{Y} \colon \mathcal{P}_0(n) \otimes_{\mathfrak{S}_n} \mathcal{E}^0(n+1, n) \to \mathcal{P}_0(n+1).$$

More explicitly the partial composition $\circ_i$ is obtained by restricting to the inclusion of $\Bbbk$–modules

$$\xi_i^n : \mathcal{P}_0(2) \simeq \mathcal{P}_0(1)^{\otimes i-1} \otimes \mathcal{P}_0(2) \otimes \mathcal{P}_0(1)^{\otimes n-i} \hookrightarrow \mathcal{E}^0(n+1,n)$$

given by the external product.

For $c_2 \in \mathfrak{S}_2$ and $c_3 \in \mathfrak{S}_3$ the cyclic permutations given by $i \mapsto i+1$, by Theorem 9.4 we have in $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes 2}, \mathfrak{a}^{\otimes 3})$,

$$c_2.([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]).c_3 = [\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}}],$$

so

$$\begin{aligned}
\mathcal{Y}([\pi^{\otimes 2}], [\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}}]) &= \mathcal{Y}([\pi^{\otimes 2}], c_2.([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]).c_3) \\
&= \mathcal{Y}([\pi^{\otimes 2}].c_2, ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}])).c_3 \\
&= \mathcal{Y}([\pi^{\otimes 2}](-1), ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}])) \\
&= -\mathcal{Y}([\pi^{\otimes 2}], ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]))
\end{aligned}$$

using that $\mathcal{P}_0(k)$ is the sign representation in degree $k-1$. We deduce that $[\pi^{\otimes 2}]$ satisfies the quadratic relation.

The fact that the operad $\mathcal{P}_0$ is binary follows from Proposition 9.11 and the fact that $\mathcal{P}_0(n+1)$ is $\Bbbk$ concentrated in degree n.

We deduce that the morphism of operads $f : \mathcal{Q} \to \mathcal{P}_0$ is an isomorphism. $\square$

In Proposition 9.14 we give a more conceptual description of the operad $\mathcal{P}_0$ using the notion of *operadic suspension*, recalled in the next paragraphs following [12, Section 7.2.2].

Let $\mathcal{S}$ be the underlying operad of the endomorphism PROP (see Example 2.1) associated with the graded vector space $s\Bbbk$ (ie the graded vector space concentrated in homological degree one and such that $(s\Bbbk)_1 = \Bbbk$). Explicitly, as a representation of $\mathfrak{S}_n$, $\mathcal{S}(n) = \mathrm{Hom}_{\mathrm{gr}\mathcal{V}}((s\Bbbk)^{\otimes n}, s\Bbbk)$ is the signature representation concentrated in homological degree $-n+1$.

For $\mathcal{P}$ and $\mathcal{Q}$ two operads, the Hadamard tensor product $\mathcal{P} \otimes_H \mathcal{Q}$ is an operad such that

$$(\mathcal{P} \otimes_H \mathcal{Q})(n) = \mathcal{P}(n) \otimes_H \mathcal{Q}(n),$$

where the action of $\mathfrak{S}_n$ is the diagonal action. The unit of the Hadamard product is the operad $u\mathcal{C}om$ of *unital* commutative algebras. In particular we have $u\mathcal{C}om(0) = \Bbbk$.

For $\mathcal{P}$ an operad, the *operadic suspension* of $\mathcal{P}$ is the operad $\mathcal{S} \otimes_H \mathcal{P}$.

The operad $\mathcal{P}_0$ being defined using Ext–groups, it is naturally graded with cohomological degree. So $\mathcal{P}_0(n)$ is a graded $\mathbb{k}$–module concentrated in *cohomological* degree $n-1$ and so in *homological* degree $1-n$.

Let $\mathcal{C}om$ be the operad of *nonunital* commutative algebras (thus $\mathcal{C}om(0)=0$). In the next proposition we consider $\mathcal{P}_0$ with its homological degree.

Proposition 9.14 *The operad $\mathcal{P}_0$ is the operadic suspension of the operad $\mathcal{C}om$. In other words, we have an isomorphism of operads*

$$\mathcal{P}_0 \simeq \mathcal{S} \otimes_H \mathcal{C}om.$$

Proof Recall that $\mathcal{P}_0(0)=0$ and for $n>0$, $\mathcal{P}_0(n)$ is the sign representation of $\mathfrak{S}_n$ placed in homological degree $1-n$ and 0 in other degrees. The underlying $\mathfrak{S}_n$–modules are isomorphic since $\mathcal{C}om(0)=0$ and for $n \geq 1$, $\mathcal{P}_0(n) \simeq \mathcal{S}(n)$.

By Proposition 9.13, $\mathcal{P}_0$ is a quadratic operad generated by μ such that $\mu \circ_1 \mu = -\mu \circ_2 \mu$. Define $\mathcal{P}_0(2) \to \mathcal{S}(2)$ sending μ to the generator ν of $\mathcal{S}(2)$. By a similar argument as in the proof of Proposition 9.13 for the endomorphism PROP associated with the graded vector space $s\mathbb{k}$, one proves that ν satisfies the relation $\nu \circ_1 \nu = -\nu \circ_2 \nu$. This implies the isomorphism in the statement. □

Remark 9.15 We denote by $s_i^n \in \mathrm{Surj}(n, n-1)$ the unique surjection preserving the natural order and such that $s_i^n(i) = s_i^n(i+1) = i$. The Yoneda product gives, via the isomorphism given in Theorem 9.3, a map

$$Y : \mathbb{k}[\mathrm{Surj}(m,l)] \otimes \mathbb{k}[\mathrm{Surj}(n,m)] \to \mathbb{k}[\mathrm{Surj}(n,l)].$$

For $m=2$, $l=1$ and $n=3$, the quadratic relation in the operad $\mathcal{P}_0$ corresponds to $Y([s_1^2] \otimes [s_2^3]) = -Y([s_1^2] \otimes [s_1^3])$ showing that the signs given in [22, Proposition 3.1] are not correct.

The following maps will be used in Proposition 10.7 in order to define the contraction maps in the wheeled operad $\mathcal{P}$.

Definition 9.16 For $1 \leq i \leq n$, let $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_0(n-1)$ be the morphism of graded vector spaces of degree -1 given by

$$\xi^i([\pi^{\otimes n}]) = (-1)^i [\pi^{\otimes n-1}].$$

Proposition 9.17 *For* $1 \leq i \leq n$*, the contraction map* $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_0(n-1)$ *is equivariant; ie for* $\sigma \in \mathfrak{S}_n$*, the diagram*

$$\begin{tikzcd} \mathcal{P}_0(n) \arrow[r, "\xi^i"] \arrow[d, "\sigma"'] & \mathcal{P}_0(n-1) \arrow[d, "\sigma^{(i)}"] \\ \mathcal{P}_0(n) \arrow[r, "\xi^{\sigma(i)}"'] & \mathcal{P}_0(n-1) \end{tikzcd}$$

is commutative, where $\sigma^{(i)} \in \mathfrak{S}_{n-1}$ *is the permutation* $\sigma : \boldsymbol{n}\setminus\{i\} \to \boldsymbol{n}\setminus\{\sigma(i)\}$ *considered as reindexed.*

Proof Since $\mathfrak{S}_n$ acts on $\mathcal{P}_0(n)$ by the signature, we need to prove that

$$\varepsilon(\sigma)(-1)^{\sigma(i)} = \varepsilon(\sigma^{(i)})(-1)^i.$$

Let $\alpha \in \mathfrak{S}_n$ be the permutation given by

$$\alpha = (1,2)\circ(2,3)\circ\cdots\circ(\sigma(i)-1,\sigma(i))\circ\sigma\circ(i-1,i)\circ\cdots\circ(1,2)$$

where $(l, l+1)$ is the permutation exchanging l and $l+1$. We have

$$\varepsilon(\alpha) = (-1)^{\sigma(i)-1}\varepsilon(\sigma)(-1)^{i-1},$$

and for $\alpha^{(1)}$ the permutation $\alpha : \boldsymbol{n}\setminus\{1\} \to \boldsymbol{n}\setminus\{\alpha(1)=1\}$ considered as reindexed. We have $\alpha^{(1)} = \sigma^{(i)}$ and $\alpha = \mathrm{Id}_{\{1\}} \oplus \alpha^{(1)}$. Hence

$$\varepsilon(\sigma^{(i)}) = \varepsilon(\alpha^{(1)}) = \varepsilon(\alpha) = (-1)^{\sigma(i)-1}\varepsilon(\sigma)(-1)^{i-1}. \qquad \square$$

10 The PROP $\mathcal{E}$

In the rest of the paper $\Bbbk$ is a field of characteristic zero. The previous condition on $\Bbbk$ allows us to use the computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q})$ given in [22] (see Proposition 10.1 below) obtained by taking the coinvariants by the action of the symmetric groups, twisted by the signature, in the result of Theorem 9.3.

The aim of this section is to describe the structure of the graded PROP $\mathcal{E}$ introduced in Definition 10.2. We will prove in Theorem 10.11 that the PROP $\mathcal{E}$ is a wheeled PROP. The PROP $\mathcal{E}$ extends the PROP $\mathcal{E}^0$ in the sense that $\mathcal{E}^0$ is a sub-PROP of $\mathcal{E}$ (see Remark 10.3).

Since the exterior powers intervene in the PROP $\mathcal{E}$ we need the following result:

Proposition 10.1 [22, Theorem 4.2] *For $\Bbbk$ a field of characteristic 0 and $n,m \in \mathbb{N}$, we have isomorphisms*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \simeq \begin{cases} \Bbbk^{S(q,j)} & \text{if } * = q-j, \\ 0 & \text{otherwise,} \end{cases}$$

where $S(q,j)$ denotes the number of ways to partition a set of q elements into j nonempty subsets

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^n \mathfrak{a}, \Lambda^m \mathfrak{a}) \simeq \begin{cases} \Bbbk^{\rho(m,n)} & \text{if } * = m-n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho(m,n)$ denotes the number of partitions of m into n parts.

Since $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \Lambda^j \mathfrak{a}) \simeq \Bbbk$, the external product gives a morphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes l}) \xrightarrow{E} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m} \otimes \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}). \tag{10-1}$$

Recall that for $V^\bullet$ a cohomologically graded module, for $i \in \mathbb{N}$ the i^{th} desuspension of $V^\bullet$ is the graded module $s^{-i} V^\bullet$ such that $s^{-i} V^n = V^{n-i}$.

Definition 10.2 The PROP $\mathcal{E}$ is defined by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{E}(q,l) = \bigoplus_{j \in \mathbb{N}} s^{-j} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}),$$

where the action of the symmetric group $\mathfrak{S}_l$ (resp. $\mathfrak{S}_q$) is given by place permutation of the copies of $\mathfrak{a}$ in the first (resp. second) variable.

The horizontal composition $\otimes : \mathrm{Hom}_{\mathcal{E}}(q_1, l_1) \otimes \mathrm{Hom}_{\mathcal{E}}(q_2, l_2) \to \mathrm{Hom}_{\mathcal{E}}(q_1+q_2, l_1+l_2)$ is given by

$$\begin{array}{c}
\bigoplus_{j \in \mathbb{N}} s^{-j} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q_1}) \otimes \bigoplus_{i \in \mathbb{N}} s^{-i} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_2} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_2}) \\
\downarrow \beta \\
\bigoplus_{i,j \in \mathbb{N}} s^{-j-i} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1} \otimes \Lambda^j \mathfrak{a} \otimes \mathfrak{a}^{\otimes l_2} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2}) \\
\downarrow T \\
\bigoplus_{i,j \in \mathbb{N}} s^{-j-i} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1+l_2} \otimes \Lambda^j \mathfrak{a} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2}) \\
\downarrow c \\
\bigoplus_{j+i \in \mathbb{N}} s^{-j-i} \,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1+l_2} \otimes \Lambda^{j+i} \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2})
\end{array}$$

where β is the exterior product, T is induced by the permutation of $\Lambda^i \mathfrak{a}$ and $\mathfrak{a}^{\otimes l_2}$ and c by the natural transformation $\Lambda^{i+j} \mathfrak{a} \to \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}$ (see (3-1)).

The vertical composition $\circ\colon \mathrm{Hom}_{\mathcal{E}}(q,l)\otimes \mathrm{Hom}_{\mathcal{E}}(l,m)\to \mathrm{Hom}_{\mathcal{E}}(q,m)$ is given by

$$\begin{array}{c}
\bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes\bigoplus_{i\in\mathbb{N}} s^{-i}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes\Lambda^i\mathfrak{a},\mathfrak{a}^{\otimes l})\\
\big\downarrow{\simeq}\\
\bigoplus_{i,j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes s^{-i}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes\Lambda^i\mathfrak{a},\mathfrak{a}^{\otimes l})\\
\big\downarrow\\
\bigoplus_{i,j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes s^{-i}\,\mathrm{Ext}^*(\mathfrak{a}^{\otimes m}\otimes\Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a})\\
\big\downarrow{\mathcal{Y}}\\
\bigoplus_{i,j\in\mathbb{N}} s^{-i-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes\Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\\
\big\downarrow{c}\\
\bigoplus_{i+j\in\mathbb{N}} s^{-i-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes\Lambda^{i+j}\mathfrak{a},\mathfrak{a}^{\otimes q})
\end{array}$$

where the second morphism is induced by the map (10-1), the third by the Yoneda product and the last one by the canonical natural transformation $\Lambda^{i+j}\mathfrak{a}\to\Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a}$ given in (3-1).

Remark 10.3 The PROP $\mathcal{E}^0$ (see Definition 9.1) is the sub-PROP of $\mathcal{E}$ having the same objects and such that $\mathcal{E}^0(q,l)$ is the direct summand of $\mathcal{E}(q,l)$ for $j=0$.

10.1 Calculation of $\mathcal{E}(q,l)$

The aim of this section is to prove the following result:

Proposition 10.4 *For $q,l\in\mathbb{N}$ we have an isomorphism of graded $(\mathfrak{S}_q,\mathfrak{S}_l)$–bimodules*

$$\mathcal{E}(q,l)=\bigoplus_{J\subset\boldsymbol{q}}\Bigg(\bigoplus_{f\colon J\twoheadrightarrow\boldsymbol{l}}\bigotimes_{i=1}^{l}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes|f^{-1}(i)|})\Bigg)\otimes\Bigg(\bigoplus_{j\in\mathbb{N}}\Bigg(\bigoplus_{g\colon\boldsymbol{q}\setminus J\twoheadrightarrow\boldsymbol{j}}\bigotimes_{i=1}^{j}s^{-1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes|g^{-1}(i)|})\Bigg)_{\mathfrak{S}_j}\Bigg).$$

The proof of this proposition relies on the following lemma:

Lemma 10.5 *For $q,l,j\in\mathbb{N}$, we have the following isomorphisms of graded $(\mathfrak{S}_q,\mathfrak{S}_l)$–bimodules*:

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\simeq\bigoplus_{J\subset\boldsymbol{q}}\big(\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes|J|})\otimes\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes|q\setminus J|})\big),$$

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}) \simeq \bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{l}} \Bigl(\bigotimes_{k=1}^{l} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes |f^{-1}(k)|})\Bigr),$$

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \simeq \Bigl(\bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}} \Bigl(\bigotimes_{k=1}^{j} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes |f^{-1}(k)|})\Bigr)\Bigr)_{\mathfrak{S}_j},$$

where $\mathfrak{S}_j$ acts on $\bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}} (\bigotimes_{k=1}^{j} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes |f^{-1}(k)|}))$ by postcomposition on $f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}$.

Proof For the first isomorphism,

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\otimes \Lambda^j\mathfrak{a}, \mathfrak{a}^{\otimes q}) &\simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes \Lambda^j\mathfrak{a}, \pi_2^*(\mathfrak{a}^{\otimes q}))\\ &\simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}\Bigl(\mathfrak{a}^{\otimes l}\boxtimes \Lambda^j\mathfrak{a}, \bigoplus_{J\subset \boldsymbol{q}} \mathfrak{a}^{\otimes |J|}\boxtimes \mathfrak{a}^{\otimes |\boldsymbol{q}\setminus J|}\Bigr)\\ &\simeq \bigoplus_{J\subset \boldsymbol{q}} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes \Lambda^j\mathfrak{a}, \mathfrak{a}^{\otimes |J|}\boxtimes \mathfrak{a}^{\otimes |\boldsymbol{q}\setminus J|}),\end{aligned}$$

where the first isomorphism is given by the adjunction between δ_2^* and π_2^* (see Section 3) and the second by (3-2).

Using the resolution given in Section III, we obtain that $\mathfrak{a}^{\otimes l}$ and $\Lambda^j\mathfrak{a}$ have resolutions by finitely generated projective functors. Moreover, the values of $\mathfrak{a}^{\otimes n}$ and $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes |J|})$ (by Theorem 9.3) are torsion free. It follows, by the Künneth formula, that the graded morphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes |J|}) \otimes \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a}, \mathfrak{a}^{\otimes |\boldsymbol{q}\setminus J|}) \xrightarrow{\simeq} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes \Lambda^j\mathfrak{a}, \mathfrak{a}^{\otimes |J|}\boxtimes \mathfrak{a}^{\otimes |\boldsymbol{q}\setminus J|})$$

is an isomorphism.

For the second and third isomorphisms, we refer the reader to the proof of [22, Theorems 2.3 and 4.2]. □

Remark 10.6 By Proposition 10.4,

$$\mathcal{E}(n,n) \simeq \bigoplus_{\sigma\in\mathfrak{S}_n} \Bigl(\bigotimes_{i=1}^{n} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a})\Bigr) \simeq \Bbbk[\mathfrak{S}_n]$$

where the last isomorphism is given by Theorem 9.3. Hence the graded bimodule $\mathcal{E}(n,n)$ is $\Bbbk[\mathfrak{S}_n]$ concentrated in degree 0.

10.2 The wheeled PROP $\mathcal{C}_{\mathcal{P}}$

The aim of this section is to give an isomorphism between the PROP $\mathcal{E}$ and the wheeled PROP associated to the following wheeled operad $\mathcal{P}$.

Proposition 10.7 *The following data define a wheeled operad denoted by* $\mathcal{P}$:

1. *the operadic part of* $\mathcal{P}$ *is the operad* $\mathcal{P}_0$ *considered in Section 9.2*;
2. *the wheeled part* $\mathcal{P}_w$ *is given by*
$$\mathcal{P}_w(n) = s^{-1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes n});$$
3. *for* $1 \le i \le n$, *the contractions* $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$ *are the degree* 0 *maps induced by Definition 9.16.*

Proof We have $\mathcal{P}_w \simeq s^{-1}\mathcal{P}_0$, so $\mathcal{P}_w$ is a right $\mathcal{P}_0$–module. The contraction maps $\xi_i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$ are equivariant by Proposition 9.17. $\square$

Let $\mathcal{Q}^{\circlearrowright}$ be the wheeled completion of the operad $\mathcal{Q}$ given in Definition 9.12. In the following, we give an explicit description of $\mathcal{Q}^{\circlearrowright}$.

Proposition 10.8 *The wheeled operad* $\mathcal{Q}^{\circlearrowright}$ *is given by the following data*:

1. ***The operadic part*** $(\mathcal{Q}^{\circlearrowright})_0$ $(\mathcal{Q}^{\circlearrowright})_0(n)$ *is the graded vector space concentrated in degree* $n-1$ *and generated by* μ_n *defined inductively by* $\mu_1 = \mathrm{Id}$, $\mu_2 = \mu$ *and for* $n \ge 2$,
$$\mu_{n+1} = \mu \circ_1 \mu_n.$$
2. ***The wheeled part*** $(\mathcal{Q}^{\circlearrowright})_w$ $(\mathcal{Q}^{\circlearrowright})_w(n)$ *is the graded vector space concentrated in degree* n *generated by*
$$\xi^1(\mu) \circ_1 \mu_n.$$

Proof We have $(\mathcal{Q}^{\circlearrowright})_0 = \mathcal{Q}$, so the description of the operadic part follows from Definition 9.12.

The proof of the description of $(\mathcal{Q}^{\circlearrowright})_w(n)$ is similar to that of $(\mathcal{C}om^{\circlearrowright})_w(n)$ given in [15, Example 5.2.5], replacing the commutativity property by the commutativity up to signs and taking into account the fact that we have graded modules. $\square$

Proposition 10.9 *The wheeled operad* $\mathcal{P}$ *is isomorphic to* $\mathcal{Q}^{\circlearrowright}$. *In particular*, $\mathcal{P}$ *is isomorphic to the wheeled completion of the quadratic operad* $\mathcal{P}_0$, *ie* $\mathcal{P} \simeq \mathcal{P}_0^{\circlearrowright}$.

Proof Recall that the wheelification $(-)^{\circlearrowright}$ is the left adjoint of the forgetful functor F from wheeled operads to operads. Since $\mathcal{P}$ is a wheeled operad whose operadic part is $\mathcal{P}_0$, we have a morphism of operads $\mathcal{P}_0 \to F(\mathcal{P})$. The composition of the isomorphism of operads $f: \mathcal{Q} \to \mathcal{P}_0$ constructed in the proof of Proposition 9.13, with the morphism of operads defined above, gives a morphism of operads $\mathcal{Q} \to F(\mathcal{P}_0)$. By adjunction, this morphism induces a morphism of wheeled operads

$$f^{\circlearrowright}: \mathcal{Q}^{\circlearrowright} \to \mathcal{P}.$$

By Proposition 9.13, the restriction of $f^{\circlearrowright}$ to the operadic parts is an isomorphism of operads given explicitly on the generators of $(\mathcal{Q}^{\circlearrowright})_0(n)$ by

$$f^{\circlearrowright}(\mu_n) = [\pi^{\otimes n}].$$

For the wheeled part, by Theorem 9.3, $\mathcal{P}_w(n) = s^{-1}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ is the graded vector space concentrated in degree n generated by $[\pi^{\otimes n}]$. By Proposition 10.8 it follows that $\mathcal{P}_w(n)$ and $(\mathcal{Q}^{\circlearrowright})_w(n)$ are isomorphic as graded vector spaces.

Since $f^{\circlearrowright}$ is a morphism of wheeled operads, the compatibility with the contractions gives:

$$f^{\circlearrowright}(\xi^1(\mu)) = \xi^1(f^{\circlearrowright}(\mu)) = \xi^1([\pi^{\otimes 2}]) = -[\pi]$$

where the last equality is given by Definition 9.16. We deduce that

$$f^{\circlearrowright}: (\mathcal{Q}^{\circlearrowright})_w(1) \to \mathcal{P}_w(1)$$

is an isomorphism. By Proposition 10.8, the generator of $(\mathcal{Q}^{\circlearrowright})_w(n)$ is obtained by composition of $\xi^1(\mu)$ with μ_n. It follows from the compatibility of $f^{\circlearrowright}$ with the composition that, for all $n \geq 1$,

$$f^{\circlearrowright}: (\mathcal{Q}^{\circlearrowright})_w(n) \to \mathcal{P}_w(n)$$

is an isomorphism. □

Corollary 10.10 *The PROP $\mathcal{C}_{\mathcal{P}}$ is isomorphic to the wheeled PROP generated by one antisymmetric operation μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation*

$$\mu(\mu \otimes 1) = -\mu(1 \otimes \mu).$$

The following theorem relates the PROP $\mathcal{C}_{\mathcal{P}}$ to the PROP $\mathcal{E}$ of extension groups introduced in Section 10.

Theorem 10.11 *There is an isomorphism of PROPs*

$$\chi: \mathcal{C}_{\mathcal{P}} \xrightarrow{\simeq} \mathcal{E}.$$

In particular, $\mathcal{E}$ inherits a structure of wheeled PROP via this isomorphism.

Proof Forgetting the wheeled structure on $\mathcal{C}_{\mathcal{P}}$, the PROP $\mathcal{C}_{\mathcal{P}}$ is generated by one antisymmetric operation μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation

$$\mu(\mu\otimes 1)=-\mu(1\otimes\mu)$$

and one operation $\bar{\mu}=\xi^1(\mu)$ of biarity $(1,0)$ in degree 1.

The functor χ is defined on these generators by

$$\chi(\mu)=[\pi^{\otimes 2}]\quad\text{and}\quad\chi(\bar{\mu})=[\pi]\in s^{-1}\,\mathrm{Ext}^*(\Lambda^1\mathfrak{a},\mathfrak{a}),$$

and $[\pi^{\otimes 2}]$ satisfies the quadratic relation by Proposition 9.13. This defines an isomorphism of PROPs since

$$\mathcal{C}_{\mathcal{P}}(n,m)\simeq\mathcal{E}(n,m),$$

comparing the formulas given in Propositions 2.3 and 10.4. □

Remark 10.12 The existence of a wheeled structure on the PROP $\mathcal{E}$ is quite surprising since it is induced by a morphism (of degree 0)

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2})\to s^{-1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}).$$

By Theorem 9.3, $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2})\simeq\Bbbk$, thus the Yoneda product with a generator in $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2})$ gives a morphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a})\to s^{1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2}).$$

By Theorem 9.3, this morphism is an isomorphism and the wheeled structure on the PROP $\mathcal{E}$ is given by the inverse on this morphism.

It follows that the existence of a wheeled structure is very specific to the situation studied in this paper (ie Ext–groups in the category $\mathcal{F}(\mathbf{gr};\Bbbk)$ between the tensor powers of the functor $\mathfrak{a}$) and, in general, there is no such natural map.

Remark 10.13 Theorem 10.11 should be viewed as an extension, in the wheeled world, of the isomorphism of PROPs

$$C_{\mathcal{P}_0}\simeq\mathcal{E}^0$$

given in [22, Proposition 3.5]. More precisely there is a commutative diagram

$$\begin{tikzcd} C_{\mathcal{P}} \arrow[r, "\simeq"] & \mathcal{E} \\ C_{\mathcal{P}_0} \arrow[u] \arrow[r, "\simeq"] & \mathcal{E}^0 \arrow[u] \end{tikzcd}$$

where the vertical maps are the inclusion functors.

Part IV Comparaison of the PROPs $\mathcal{H}$ and $\mathcal{E}$

11 The morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$

In this section we define a morphism from the wheeled PROP $\mathcal{E}$ to the wheeled PROP $\mathcal{H}$ of stable cohomology considered in Section 6.

Theorem 11.1 *Let μ be the generator of the wheeled PROP $\mathcal{E}$. There is a morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ given on generators by*

$$\varphi(\mu) = h_1.$$

Proof By the isomorphism given in Theorem 10.11 the PROP $\mathcal{E}$ is generated by the antisymmetric element μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation

$$\mu(\mu \otimes 1) = -\mu(1 \otimes \mu)$$

and one operation $\bar{\mu} = \xi^1(\mu)$ of biarity $(1,0)$ in degree 1. The functor φ is defined on these generators by

$$\varphi_{2,1}(\mu) = h_1 \quad \text{and} \quad \varphi_{1,0}(\bar{\mu}) = \bar{h}_1.$$

By Section 7, h_1 is antisymmetric and satisfies the quadratic relation.

For $i \in \{1,2\}$, by Proposition 7.1 and Remark 7.2, we have $\xi_1^i(h_1) = (-1)^{i+1}\bar{h}_1$. It follows that the diagram

$$\begin{tikzcd}
\mathcal{E}(2,1) \arrow[r, "\varphi_{2,1}"] \arrow[d, "\xi_1^i"'] & \mathcal{H}(2,1) \arrow[d, "\xi_1^i"] \\
\mathcal{E}(1,0) \arrow[r, "\varphi_{1,0}"'] & \mathcal{H}(1,0)
\end{tikzcd}$$

is commutative, giving the compatibility of the wheeled PROP structures. □

Corollary 11.2 *The subwheeled PROP $\mathcal{K}$ of $\mathcal{H}$ is equivalent to the wheeled PROP associated to the wheeled completion of the operadic suspension of the operad $\mathcal{C}om$.*

Proof By Theorem 10.11 and Proposition 10.9, $\mathcal{E}$ is the wheeled PROP generated by the wheeled completion of the operad $\mathcal{P}_0$, which is the suspension of the operad $\mathcal{C}om$ by Proposition 9.14. By Section 7, $\varphi(\mathcal{E}) \simeq \mathcal{K}$. □

The morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ induces an explicit graded morphism on Hom–sets,

$$\varphi_{q,l}\colon \bigoplus_{j=0}^{q-l} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \to H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})). \tag{11-1}$$

We denote by $\mathcal{E}_w$ the sub-PROP of $\mathcal{E}$ corresponding to forgetting the operadic part in the wheeled PROP $\mathcal{E}$ (see Remark 2.6). By restriction, the morphism $\varphi\colon \mathcal{E} \to \mathcal{H}$ induces a morphism $\varphi'\colon \mathcal{E}_w \to \mathcal{H}'$, where $\mathcal{H}'$ is defined in Definition 6.4.

One of the main and difficult results of Djament [1, Théorème 4] gives a graded isomorphism

$$\bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \simeq H^*_{\mathrm{st}}(\mathfrak{a}^{\otimes q}).$$

By [1, Corollaire 3.7] this isomorphism is induced by the morphism

$$\varphi_{q,0}\colon \bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \to H^*_{\mathrm{st}}(\mathfrak{a}^{\otimes q}).$$

We deduce the following result:

Theorem 11.3 [1, Théorème 4] *The functor* $\varphi'\colon \mathcal{E}_w \to \mathcal{H}'$ *is an equivalence of categories.*

Note that [1, Proposition 3.5] corresponds to the compatibility of the isomorphisms $\varphi_{q,0}$ with the horizontal composition in the PROPs $\mathcal{E}_w$ and $\mathcal{H}'$. Theorem 11.3 gives also the compatibility of the isomorphisms $\varphi_{q,0}$ with the action of the symmetric groups $\mathfrak{S}_q$.

For stable cohomology with coefficients given by a bivariant functor, Djament gives a conjecture in [1, Théorème 7.4]. In particular, Djament conjectures that there exist graded isomorphisms

$$\bigoplus_{j=0}^{q-l} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \simeq H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})).$$

Natural candidates for maps giving these isomorphisms are the maps $\varphi_{q,l}$. By functoriality, these maps are compatible with horizontal and vertical compositions in the PROPs and with the contractions.

Djament's conjecture can be rephrased in the following way

Conjecture 11.4 *The morphism* φ *is an isomorphism of wheeled PROPs.*

Remark 11.5 Let $\Gamma_r = \Gamma_r(\mathbb{Z}^{*n})$ for $r \geq 1$ be the lower central series of the free group $\mathbb{Z}^{*n}$; $\Gamma_1 := \mathbb{Z}^{*n}$ and $\Gamma_{r+1} := [\Gamma_r, \Gamma_1]$ for $r \geq 1$. The Andreadakis filtration, $A(r)$ for $r \geq 0$, of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{*n})$ is defined to be the kernel of the natural homomorphism $\mathrm{Aut}(\mathbb{Z}^{*n}) \to \mathrm{Aut}(\Gamma_1/\Gamma_{r+1})$. In particular, $A(0) = \mathrm{Aut}(\mathbb{Z}^{*n})$, and $A(1)$, which is called the IA–automorphism group of the free group $\mathbb{Z}^{*n}$, is the kernel of the abelianization homomorphism $\alpha\colon A(0) \to \mathrm{Aut}(\mathbb{Z}^{\oplus n}) = \mathrm{GL}_n(\mathbb{Z})$ induced by the abelianization $\Gamma_1/\Gamma_2 \cong \mathbb{Z}^{\oplus n}$. For $r \geq 1$, we have a group homomorphism $\tau_r\colon A(r) \to \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(r+1)})$ called the r^{th} Johnson homomorphism, which induces a group embedding $A(r)/A(r+1) \hookrightarrow \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(r+1)})$. Any linear map $\Bbbk^n \to \bigoplus_{s=2}^{r+1}(\Bbbk^n)^{\otimes s}$ defines an algebra automorphism of the truncated tensor algebra $T_{\leq r+1}(\Bbbk^n) := \bigoplus_{s=0}^{r+1}(\Bbbk^n)^{\otimes s}$ by extending the linear map multiplicatively. This makes the direct sum $\bigoplus_{s=1}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})$ a subgroup of the algebra automorphism group of the algebra $T_{\leq r+1}(\Bbbk^n)$. The group $\mathrm{GL}_n(\Bbbk)$ acts on the subgroup $\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})$ in an obvious way, so that we can take the semidirect product $\bigl(\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})\bigr) \rtimes \mathrm{GL}_n(\Bbbk)$. As was shown in [8, Theorem 3.1, page 13], there exists a group homomorphism

$$(\tau_1^\theta, \dots, \tau_r^\theta, \alpha)\colon A(0) \to \left(\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})\right) \rtimes \mathrm{GL}_n(\Bbbk)$$

such that $\tau_s^\theta|_{A(s)}$ equals the s^{th} Johnson homomorphism

$$\tau_s\colon A(s) \to \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(s+1)})$$

for each $1 \leq s \leq r$.

The multiplicativity of the group homomorphism implies that τ_1^θ is a cocycle, and τ_2^θ defines the same quadratic relation for τ_1^θ as in (7-1). The cohomology class h_1 equals that of the cocycle τ_1^θ. Hence the class h_1 comes from $A(0)/A(2)$, and the quadratic relation holds on $A(0)/A(3)$. This means the map $\varphi_{q,l}$ lifts to the cohomology group of $A(0)/A(3)$,

$$\begin{tikzcd}
 & H^*\bigl(A(0)/A(3), ((\Bbbk^n)^*)^{\otimes q} \otimes (\Bbbk^n)^{\otimes l}\bigr) \arrow[d] \\
\mathcal{E}(q,l) \arrow[ur] \arrow[r, "\varphi_{q,l}"] & \mathcal{H}(q,l)
\end{tikzcd}$$

where the vertical arrow means the homomorphism induced by the quotient map $\mathrm{Aut}(\mathbb{Z}^{*n}) = A(0) \to A(0)/A(3)$ and n is big enough.

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