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The mod 2 cohomology of the infinite families of Coxeter groups of type B and D as almost-Hopf rings

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We describe a Hopf ring structure on the direct sum of the cohomology groups $\bigoplus_{n\geq 0} H^*(W_{B_n};\mathbb{F}_2)$ of the Coxeter groups of type W_{B_n}, and an almost-Hopf ring structure on the direct sum of the cohomology groups $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$ of the Coxeter groups of type W_{Dn}, with coefficients in the field with two elements $\mathbb{F}_2$. We give presentations with generators and relations, determine additive bases and compute the Steenrod algebra action. The generators are described both in terms of a geometric construction by De Concini and Salvetti and their restriction to elementary abelian 2–subgroups.

20F55, 20J06; 20J05

1 Introduction

The Coxeter groups of type W_{B_n} and W_{D_n} are two infinite families of finite reflection groups. Coxeter groups are traditionally described via Coxeter diagrams, ie graphs in which each edge e has a weight $m_e \geq 3$. Given such an object, the associated Coxeter group has a generator s_v for every vertex v, with relations of the form $s_v^2 = 1$, $(s_v s_w)^{m_e} = 1$ for every edge $e = (v, w)$, and $(s_v s_w)^2 = 1$ if v and w are not connected by an edge. For an exhaustive introduction to the geometry and topology of these groups we refer to Davis's book [3]. The reflection groups of type W_{B_n} and W_{D_n} are the finite Coxeter groups associated with the Coxeter diagrams in Figure 1.

The goal of this paper is to provide an effective description of the mod 2 cohomology of these groups. Other authors have previously computed these cohomology groups. Most notably, Swenson, in his thesis [18], adapted techniques used by Hu'ng [12] and Feshbach [5], stemming from the analysis of the restriction maps to elementary abelian 2–subgroups, to compute generators and relations for the mod 2 cohomology algebra of a finite reflection group. However, his presentation is involved and intrinsically recursive. Borrowing ideas from Giusti, Salvatore and Sinha [7; 9], we exploit additional

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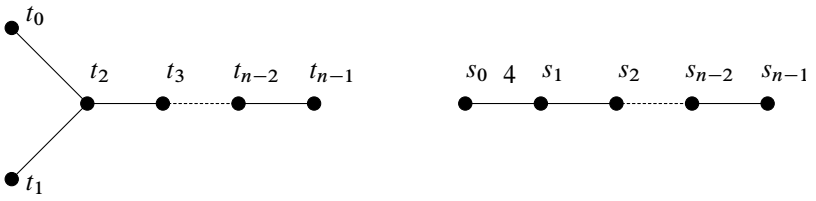

Figure 1: Diagrams of type D_n (left) and B_n (right).

structures to provide a simpler description of the cup product. Our approach also has the advantage of being more easily readable from the well-known chain-level geometric and combinatorial description of a resolution for Coxeter groups by De Concini and Salvetti [4].

The sequences of Coxeter groups of type B and D have standard embeddings

$$W_{B_n} \times W_{B_m} \to W_{B_{n+m}}, \quad W_{D_n} \times W_{D_m} \to W_{D_{n+m}}$$

that are, in a certain sense, compatible. The homomorphisms induced by these maps on mod 2 cohomology define a coproduct Δ. The cohomology transfer maps associated with them determine a product $\odot$. There is also a canonical embedding of W_{D_n} into W_{B_n} as an index-2 subgroup, which induces an involution $\iota\colon H^*(W_{D_n};\mathbb{F}_2) \to H^*(W_{D_n};\mathbb{F}_2)$.

In the B case, the resulting structure is modeled on that of the symmetric groups, the Coxeter groups of type A, as described by Giusti, Salvatore and Sinha [7] (mod 2) and by the author [10] (modulo odd primes). Together with the usual cup product $\cdot$, these maps form a ring in the category of $\mathbb{F}_2$–coalgebras, ie a Hopf ring over $\mathbb{F}_2$. More explicitly, given a ring R, a (graded) Hopf ring over R is a graded R–module with a coproduct Δ and two products, $\odot$ and $\cdot$, such that

- $(A, \Delta, \odot)$ is a Hopf algebra, with an antipode S;
- $(A, \Delta, \cdot)$ is a bialgebras over R;
- if $x, y, z \in A$ and $\Delta(x) = \sum_i x_i' \otimes x_i''$, then the distributivity formula

$$x \cdot (y \odot z) = \sum_i (-1)^{\deg(y)\deg(x_i'')} (x_i' \cdot y) \odot (x_i'' \cdot z)$$

holds.

In the D case, Δ, $\odot$ and $\cdot$ satisfy the last two axioms in the definition of a Hopf ring, and Δ and $\cdot$ form a bialgebra. However, as we will explain later, Δ and $\odot$ do not form a bialgebra. We call this weaker structure an *almost-Hopf ring* over $\mathbb{F}_2$. Due to this fact, the study of the cohomology of W_{D_n}, with the cup product, the transfer product,

and the coproduct, is more complicated. The reader will find similarities between the cohomology of W_{D_n} and that of the alternating groups, as described by Giusti and Sinha [9]. Such structures stem from the seminal work of Strickland and Turner [17], in which the authors discovered a Hopf ring structure on the cohomology of symmetric groups, even with generalized cohomology theories.

The main results of this paper are Theorems 5.9 and 5.15, stated in Section 5.2, consisting of a presentation in terms of generators and relations of the mod 2 cohomology of the Coxeter groups of type B_n as a Hopf ring and of type D_n as an almost-Hopf ring respectively. We provide here self-contained statements for clarity and reference.

Theorem 5.9 (main theorem for type B) *The Hopf ring $\bigoplus_{n\geq 0} H^*(W_{B_n};\mathbb{F}_2)$ over $\mathbb{F}_2$ admits a presentation with two families of generators, $\gamma_{k,n}\in H^{n(2^k-1)}(W_{B_{n2^k}};\mathbb{F}_2)$ for $k\geq 0$ and $n>0$, and $\delta_n\in H^n(W_{B_n};\mathbb{F}_2)$ for $n>0$, and the following relations*:

- $\Delta(\gamma_{k,m})=\sum_{i+j=m}\gamma_{k,i}\otimes\gamma_{k,j}$;
- $\gamma_{k,n}\odot\gamma_{k,m}=\binom{n+m}{n}\gamma_{k,n+m}$;
- $\Delta(\delta_n)=\sum_{k+l=n}\delta_k\otimes\delta_l$;
- $\delta_n\odot\delta_m=\binom{n+m}{n}\delta_{n+m}$;
- *the cup product, $\cdot$, of classes in different components is* 0;
- *$\gamma_{0,n}$ is the $\cdot$–unit of $H^*(W_{B_n};\mathbb{F}_2)$.*

The generators are explicitly characterized, both combinatorially at the cochain level (see Definition 5.1) and geometrically, as suitable Thom classes (see Proposition 5.3). The classes $\gamma_{k,n}$ and the relations among them arise from the presentation of the mod 2 cohomology of the symmetric groups as a Hopf ring. The only new generators are δ_n and their behavior is governed by the third and fourth relations above.

The almost-Hopf ring constructed from the cohomology rings of the Coxeter groups of type D is more complicated. The relations are intricate, and the behavior of generators is more easily understood with the aid of a "polarized" basis $\mathcal{B}^+\sqcup\mathcal{B}^-\sqcup\mathcal{B}^0$ (see Proposition 5.22). For instance, the bialgebra axiom for $\odot$ and Δ is replaced with a different compatibility identity involving the projection p^+ onto the addend $(\mathrm{Span}(\mathcal{B}^+)\otimes A_D^{\otimes 3})\oplus(\mathrm{Span}(\mathcal{B}^0)\otimes\mathrm{Span}(\mathcal{B}^+)\otimes A_D^{\otimes 2})$:

$$\Delta(x\odot y)=(\odot\otimes\odot)\tau(p^+)(\Delta(x)\otimes\Delta(y))\quad\text{for all } x,y,$$

where τ is the transposition of the second and third factors. Nevertheless, this surrogate axiom can be expressed directly in terms of the generators, without explicit reference to the additive basis (see Proposition 5.14).

The main presentation theorem in this regard is the following.

Theorem 5.15 (main theorem for type D) *The almost-Hopf ring structure over $\mathbb{F}_2$ of $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$ extends uniquely to a graded almost-Hopf ring structure with components on the $\mathbb{F}_2$–vector space $\mathbb{F}_2 1^+ \oplus \mathbb{F}_2 1^- \oplus \bigoplus_{n\geq 1} H^*(W_{D_n};\mathbb{F}_2)$ such that*

- $1^+ \odot _ = \mathrm{id}$, $1^- \odot _ = \iota$, $1^+ \cdot 1^+ = 1^+$, $1^- \cdot 1^- = 1^-$, *and* $1^+ \cdot 1^- = 0$;
- $\Delta(1^\pm) = 1^+ \otimes 1^\pm + 1^- \otimes 1^\mp$;
- $\Delta(x) = 1^+ \otimes x + 1^- \otimes \iota(x) + \bar{\Delta}(x) + \iota(x) \otimes 1^- + x \otimes 1^+$ *for all x in $\bigoplus_{n\geq 1} H^*(W_{D_n};\mathbb{F}_2)$, where $\bar{\Delta}$ is the reduced coproduct in $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$.*

This almost-Hopf ring admits a presentation with two families of generators,

$$\begin{aligned} &\gamma^+_{k,n} \in H^{n(2^k-1)}(W_{D_{n2^k}};\mathbb{F}_2) && \text{for } k,n>0,\\ &\delta^0_{n:m} \in H^n(W_{D_{n+m}};\mathbb{F}_2) && \text{for } n\neq 1 \text{ and } m\geq 0, \end{aligned}$$

together with 1^-. The compatibility identity above and the following list of equalities provide a complete set of relations, where 1^+ is the $\odot$–unit:

- $1^- \odot 1^- = 1^+$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^- = 0$, *and* $\Delta(1^-) = 1^+ \otimes 1^- + 1^- \otimes 1^+$;
- $\Delta(\gamma^+_{k,m}) = \sum_{l=0}^{m} \big(\gamma^+_{k,l} \otimes \gamma^+_{k,m-l} + (1^- \odot \gamma^+_{k,l}) \otimes (1^- \odot \gamma^+_{k,m-l})\big)$;
- $\Delta(\delta^0_{n:m}) = \sum_{i=0}^{n} \sum_{j=0}^{m} \delta^0_{i:j} \otimes \delta^0_{n-i:m-j}$;
- $\gamma^+_{k,a} \odot \gamma^+_{k,b} = \binom{a+b}{a} \gamma^+_{k,a+b}$ *and* $\delta^0_{n:m} \odot 1^- = \delta^0_{n:m}$,
- $b \odot b' = 0$ *if b and b' are cup products of generators of the form $\delta^0_{n:m}$;*
- $\gamma^+_{k,n} \cdot (1^- \odot \gamma^+_{h,m}) = 0$ *for all $n,m,k \geq 1$ and $h \geq 2$;*
- $\gamma^+_{1,m} \cdot (1^- \odot \gamma^+_{1,m}) = (\gamma^+_{1,m-1})^2 \odot \delta^0_{2:0}$ *for all $m \geq 1$;*
- *the cup product, $\cdot$, of generators belonging to different components is* 0;
- $\delta^0_{0:m}$ *is the $\cdot$–product unit of the m^{th} component*;
- $\delta^0_{n:m} \cdot \gamma^+_{k,\frac{n+m}{2^k}} = \delta^0_{n:0} \cdot \gamma^+_{k,\frac{n}{2^k}} \odot \gamma^+_{k,\frac{m}{2^k}}$ *for all $k>0$ and $m,n \geq 0$ with $n \neq 1$.*

In this case, too, the generators are explicitly described (see Definitions 5.4 and 5.5).

The relations are spread out in a few lemmas to prove the identities concerning coproduct, transfer product, and cup product separately. Building on these core theorems, we also describe convenient additive bases for the cohomology of these groups, with a graphical description via skyline diagrams similar to that obtained for the symmetric group in [7],

and compute the Steenrod algebra action. Our formulation of the cohomology of W_{B_n} and W_{D_n} yields without additional effort many features of these cohomology algebras. For instance, Hepworth's homological stability results [11] in these particular cases follow directly.

We obtain our presentation via three technical tools. First, we exploit De Concini and Salvetti's geometric combinatorial model to realize such (almost) Hopf rings structures at the cochain level. Specializing their construction to the families of groups of our interest, we observe that a resolution for W_{B_n} is obtained from the symmetrized version of the planar level trees used by Giusti and Sinha [9] for the symmetric groups. The cohomology of W_{D_n} is governed by an oriented version of these objects. We describe cochain representatives of the structural maps in detail. Our treatment follows the paper cited above closely. However, we note that while the transfer product is realized very similarly to the Σ_n case, coproducts are more complicated and require the combinatorial operation of "pruning" symmetric planar level trees. This cochain-level description allows us to quickly retrieve some of our relations and give a more geometric flavor to our generating classes. For instance, they can be interpreted as Thom classes in a suitable sense.

Second, we use the existence of well-behaved maps between W_{B_n}, W_{D_n} and Σ_n. These homomorphisms preserve parts of our structures. Therefore, we exploit them to build our presentations on the known result for the cohomology of the symmetric groups. We provide a cochain-level description of these morphisms, and we determine both their action on generators and their relations to the coproduct and transfer product.

Third, we reconcile with Swenson's approach, and we investigate restrictions to elementary abelian 2–subgroups. The mod 2 cohomology of finite reflection groups is known to be detected by this family of subgroups. We effectively compute the action of these restriction maps on our additive bases. The multiplicative structure on the cohomology of (the invariant subalgebras of) such subgroups is known. Thus, these calculations allow us to deduce cup product relations that would be otherwise difficult to obtain.

We organize the paper as follows. After describing the structures on the cohomology of W_{B_n} and W_{D_n} in Section 2, we devote the following two sections to developing our geometric tools. In Section 3, we review De Concini and Salvetti's construction, and we specialize it to W_{B_n} and W_{D_n}. In Section 4, we investigate the combinatorics of pruning operations, and we retrieve cochain-level representatives of our structural

and connecting homomorphisms. Section 5 is devoted to our main theorems. We define generators and we discuss relations between them. In this context, we also deduce from our presentation additive bases, and we discuss the relations between the cohomology of Coxeter groups of type A, B and C. We postpone the proofs of the presentation theorem and some cup product relations. In Section 6, we turn our attention to the restriction to elementary abelian 2–subgroups. We review relevant results from Swenson's thesis, compute restriction maps, and use them to complete the proof of our cup-product relations. Section 7 is devoted to completing the proof of our main theorems. In Section 8, we calculate the Steenrod algebra action.

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2 (Almost) Hopf ring structures for the cohomology of W_{B_n} and W_{D_n}

We begin this paper by describing in detail how the desired algebraic structures on the cohomology of Coxeter groups of type B and D are obtained. Throughout this paper, we use several combinatorial descriptions of the groups W_{B_n} and W_{D_n}. We refer to [2, Chapter 8] for a thorough treatment, and we recall below what we need for our purposes.

With reference to Figure 1, we recall that there is an inclusion $j_n\colon W_{D_n} \hookrightarrow W_{B_n}$ defined by $t_0 \mapsto s_0 s_1 s_0$ and $t_i \mapsto s_i$ if $i > 0$. W_{B_n} can be seen as the group of signed permutation on n numbers, that is, the group of bijective functions f from the set $\{-n, \dots, -1, 1, \dots, n\}$ into itself that satisfy $f(-i) = -f(i)$ for every $1 \le i \le n$. Hence W_{B_n} is naturally a subgroup of Σ_{2n}, the symmetric group on $2n$ objects. The image of j_n is $W_{B_n} \cap \mathrm{Alt}(2n)$, the intersection of W_{B_n} with the alternating group $\mathrm{Alt}(2n)$, the subgroup of even permutations in Σ_{2n}. Note that Σ_n can be identified with the parabolic subgroup of W_{B_n} generated by $s_1, \dots, s_{n-1}$, corresponding to the signed permutations on $\{-n, \dots, n\}$ that preserve signs. There is also a standard projection $W_{B_n} \to \Sigma_n$, of which the previous inclusion is a section, whose kernel is

the normal subgroup generated by s_0. We observe that this provides an isomorphism between W_{B_n} and the wreath product $\mathbb{F}_2 \wr \Sigma_n$, a semidirect product of $\mathbb{F}_2^n$ and Σ_n. Therefore, the inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ extend naturally to monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$. These inclusions are associative and commutative up to conjugation.

Let $A_B = \bigoplus_{n\geq 0} H^*(W_{B_n}; \mathbb{F}_2)$. We define a coproduct Δ and two products, $\cdot$ and $\odot$, on A_B in the following way:

- Δ is induced by the obvious monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$;
- $\odot$ is induced by the cohomology transfer maps associated with these inclusions;
- $\cdot$ is the usual cup product.

Due to the associativity and the commutativity of the natural inclusions, these morphisms define an almost-Hopf ring structure. This is a general fact, as noticed in [9]. In this case, however, A_B is a full Hopf ring.

Proposition 2.1 *A_B, with these structural morphisms, is a Hopf ring.*

Proof The almost-Hopf ring axioms hold by [9, Theorem 2.3]. It remains only to prove that $(A_B, \Delta, \odot)$ forms a bialgebra. This claim follows from the fact — compare with [7, Section 3] — that this diagram is a pullback of finite coverings for all $n, m \in \mathbb{N}$,

$$\begin{array}{ccc}
\bigsqcup_{\substack{p+q=n\\ r+s=m}} \dfrac{E(W_{B_{n+m}})}{W_{B_p}\times W_{B_q}\times W_{B_r}\times W_{B_s}} & \xrightarrow{\bigsqcup \pi_{p+1,r+s}} & \dfrac{E(W_{B_{n+m}})}{W_{B_n}\times W_{B_m}} \\
\Big\downarrow{\scriptstyle \bigsqcup \pi_{p+r,q+s}} & & \Big\downarrow{\scriptstyle \pi_{n,m}} \\
\bigsqcup_{k+l=n+m} \dfrac{E(W_{B_{n+m}})}{W_{B_k}\times W_{B_l}} & \xrightarrow[\bigsqcup \pi_{k,l}]{} & \dfrac{E(W_{B_{n+m}})}{W_{B_{n+m}}}
\end{array}$$

where π indicates the projections. □

We remark that, since A_B with Δ and $\odot$ is a conilpotent bialgebra, the existence of the antipode comes for free. This antipodal morphism does not play a role in our treatment; thus, we will not discuss it further.

Similarly, we can construct an additional almost-Hopf ring structure on the cohomology of the Coxeter groups of type D_n. Indeed, on the direct sum $A_D = \bigoplus_{n\geq 0} H^*(W_{D_n}; \mathbb{F}_2)$, we can define a coproduct Δ and two products $\odot$ and $\cdot$ as done for A_B. However, these do not make A_D a full Hopf ring because, as we will see later, $(A_D, \Delta, \odot)$ fails to be a bialgebra.

With essentially the same proof used for A_B, we can prove the following easy proposition, which follows again from [9, Theorem 2.3].

Proposition 2.2 *A_D, with the coproduct and the two products defined before, is an almost-Hopf ring over $\mathbb{F}_2$.*

As we remarked in the introduction, there is a similar result for the mod 2 cohomology of the symmetric groups, obtained by Giusti, Salvatore and Sinha in [7]. We recall their statement here because we will build our computations upon it.

Theorem 2.3 [7, Theorems 1.2 and 3.2] *$A_\Sigma = \bigoplus_{n\geq 0} H^*(\Sigma_n;\mathbb{F}_2)$, together with a coproduct $\Delta\colon A_\Sigma \to A_\Sigma \otimes A_\Sigma$ induced by the obvious inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, a product $\odot\colon A_\Sigma \otimes A_\Sigma \to A_\Sigma$ given by the transfer maps associated with these inclusions, and a second product $\cdot\colon A_\Sigma \otimes A_\Sigma \to A_\Sigma$ defined as the usual cup product, is a Hopf ring over $\mathbb{F}_2$.*

A_Σ is generated, as a Hopf ring, by classes $\gamma_{k,n} \in H^{n(2^k-1)}(\Sigma_{n2^k};\mathbb{F}_2)$ for $k \geq 0$ and $n \geq 1$. The coproduct of these classes is given by the formula

$$\Delta(\gamma_{k,n}) = \sum_{l=0}^{n} \gamma_{k,l} \otimes \gamma_{k,n-l},$$

the cup product of generators belonging to different components is 0, and

$$\gamma_{k,n} \odot \gamma_{k,m} = \binom{n+m}{n} \gamma_{k,n+m}.$$

There are no more relations between these classes.

The unit of the algebra $H^*(\Sigma_n;\mathbb{F}_2)$ under the cup product is $\gamma_{0,n} \in H^0(\Sigma_n;\mathbb{F}_2)$. For this reason, we will often denote it with the symbol 1_n throughout the paper.

3 Review of a geometric construction of De Concini and Salvetti and Fox–Neuwirth type cell structures

3.1 De Concini and Salvetti resolution

In this section, we recall a geometric construction introduced by De Concini and Salvetti in [4], which we will require to describe the generators of the Hopf ring under consideration.

Given a finite reflection group $G \leq \mathrm{Gl}_n(\mathbb{R})$, there is a natural hyperplane arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$ associated with G, whose hyperplanes are the fixed points sets of reflections in G. The choice of a fundamental chamber C_0 of $\mathcal{A}_G$ gives rise to a Coxeter presentation (G, S) for G, whose set of generators S is composed by reflections with respect to hyperplanes that are supports of a face of C_0. Every finite Coxeter group arises this way.

For any $F \subseteq \mathbb{R}^n$, we can define

$$\mathcal{A}_F = \{H \in \mathcal{A}_G : F \subseteq H\}.$$

$\mathcal{A}_F$ gives rise to a stratification $\Phi(\mathcal{A}_F)$ of $\mathbb{R}^n$, in which the strata are the connected components of sets of the form $L \setminus \bigcup_{H \in \mathcal{A}_F, H \not\supseteq L} H$, where L is the intersection of some of the hyperplanes of $\mathcal{A}_F$. Let $\mathbb{R}^\infty$ be the direct limit of $\mathbb{R}^m$ under the inclusions $\mathbb{R}^m \hookrightarrow \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^{m+1}$. For all $m \in \mathbb{N} \cup \{\infty\}$, there is a stratification Φ_m (different from the product stratification) of the topological space,

$$Y_G^{(m)} = \mathbb{R}^n \otimes \mathbb{R}^m \setminus \bigcup_{H \in \mathcal{A}_G} (H \otimes \mathbb{R}^m) = (\mathbb{R}^n)^m \setminus \bigcup_{H \in \mathcal{A}_G} H^m.$$

The strata in Φ_m are defined as sets of the form $F_1 \times \cdots \times F_k \times \cdots$, with $F_k \in \Phi(\mathcal{A}_{F_{k-1}})$ for $k \geq 1$. Here we put, by convention, $F_0 = \{0\}$. In what follows, if there is no ambiguity, we will use the simpler notations $Y^{(m)}$ and Y to indicate $Y_G^{(m)}$ and $Y_G^{(\infty)}$ respectively.

De Concini and Salvetti construct a regular G–equivariant CW–complex $X \subseteq Y$ that is "dual" to the stratification Φ_∞, in the sense that for every stratum $F \in \Phi_\infty$ of codimension d, there exist a unique d–dimensional cell in X that intersects F, and they intersect transversally in a single point. For $m < \infty$, the intersection $X^{(m)}$ of X with $Y^{(m)}$ is a subcomplex of X whose cells are dual to strata in Φ_m. This construction is done equivariantly, in the sense that for every stratum $F \in \Phi_\infty$ and every $g \in G$, if $\varphi\colon D^d \to X$ is the cell dual to F in X, then $(g._) \circ \varphi\colon D^d \to X$ is the cell dual to $g.F$. The authors then show that X is a G–equivariant strong deformation retract of Y. Since Y is contractible and G–free, the quotient X/G is a cellular model for the classifying space $B(G)$ and the cellular chain complex $C_*^G = C_*^{\mathrm{CW}}(X)$ is a $\mathbb{Z}[G]$–free resolution of $\mathbb{Z}$.

The strata of Φ_∞ have a more compact combinatorial description in terms of the Coxeter presentation. For every $s \in S$ generating reflection for G, we let H_s be the hyperplane fixed by s. H_s divides the space $\mathbb{R}^n$ into two semispaces, H_s^+ and H_s^-. We let H_s^+ be the semispace that contains the chosen fundamental chamber C_0. To a

flag $\underline{\Gamma} = (S \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k = \varnothing)$ of subsets of S we can associate a stratum F of Φ_∞ such that $x = (x_1, \dots, x_n) \in (\mathbb{R}^\infty)^n$ belongs to F if and only if

$$\begin{aligned} &((x_1)_r, \dots, (x_n)_r) \in H_s \quad \text{if } s \in \Gamma_r, \\ &((x_1)_r, \dots, (x_n)_r) \in H_s^+ \quad \text{if } s \in \Gamma_{r-1} \setminus \Gamma_r \end{aligned}$$

is satisfied for every $s \in S$ and every $r \geq 1$. Thus, to a couple $(\underline{\Gamma}, g)$, where $\underline{\Gamma}$ is a flag as before and $g \in G$, we can associate the stratum $g.F$ obtained from the above F by applying g. This construction yields an algebraic-combinatorial description of the cellular chain complex of X. The main theorem of De Concini and Salvetti's paper is the following.

Theorem 3.1 [4, Section 3] *Let (G, S) be a finite Coxeter group, and consider the set*

$$\{(\underline{\Gamma}, \gamma) \mid \gamma \in G, \underline{\Gamma} = (\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supseteq \cdots), \Gamma_1 \subseteq S, \Gamma_k = \varnothing \text{ for some } k\}.$$

The function described above is a bijection between this set and the set of strata in Φ_∞ (and thus, by duality, with the set of cells in X). The codimension of the stratum (and the dimension of the corresponding dual cell) associated with $(\underline{\Gamma}, \gamma)$ is equal to $\sum_{r=1}^{\infty} |\Gamma_r|$, and the action of an element $g \in G$ on strata and cells corresponds to the function $(\underline{\Gamma}, \gamma) \mapsto (\underline{\Gamma}, g\gamma)$.

Let $c(\underline{\Gamma}, \gamma)$ be the cell dual to the stratum corresponding to $(\underline{\Gamma}, \gamma)$. The boundary homomorphism in $C_*^{\mathrm{CW}}(X)$ is given by the formula

$$\partial c(\underline{\Gamma}, \gamma) = \sum_{i \geq 1} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subseteq \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\underline{\Gamma}, i, \tau, \beta)} c(\underline{\Gamma}', \gamma\beta),$$

where α is an integer number easily computed in terms of $\underline{\Gamma}$, i, τ, β, $\Gamma_k' = \Gamma_k$ for $k < i$, $\Gamma_i' = \Gamma_i \setminus \{\tau\}$ and $\Gamma_k' = \beta^{-1}\Gamma_k\beta$ if $k > i$, and $W_{T'}^T$, for $T' \subseteq T \subseteq S$ is the set of minimal length coset representatives for the parabolic subgroup $W_{T'}$ in W_T.

We remark that in the case of Coxeter groups of type B or D, minimal coset representatives are explicitly known. For a complete description, we refer, for instance, to [14].

3.2 Alexander duality and Fox–Neuwirth complexes

We recall an alternative description of C_*^G. This description has been exposed in [8], where it is investigated in much detail in the A_n case. As observed in that paper, for

every $1 \leq m \leq \infty$, the strata of Φ_m are the interiors of cells in a G–equivariant cell structure on the Alexandroff compactification $(Y^{(m)})^+ = Y^{(m)} \cup \{\infty\}$.

Denote its augmented (G–equivariant) cellular chain complex with the symbol $\widetilde{\mathrm{FN}}_G^m$. Its cells are the closures $e(F)$ of strata $F \in \Phi_m$ (together with the basepoint $\{*\}$) and, from the construction of $X^{(m)}$ as a CW–complex dual to Φ_m — details in [4] — $e(F)$ is contained in the boundary of $e(F')$ if and only if the cell of X dual to F contains the cell dual to F' in its boundary. This fact implies that the complex $\widetilde{\mathrm{FN}}_G^m$ is, up to a shift of degrees, the dual of $C_*^{\mathrm{CW}}(X^{(m)})$, at least modulo 2 (in general, there are differences in some signs due to orientations). Explicitly, the closure in $\widetilde{\mathrm{FN}}_G^m$ of a stratum of dimension d correspond to the dual of a chain in $C_*^{\mathrm{CW}}(X^{(m)})$ of dimension $nm-d$. In the remaining sections of this paper, we will always implicitly assume this shift, and we will grade $\widetilde{\mathrm{FN}}_G^m$ to match the corresponding dimension of the dual cell.

In particular, $\widetilde{\mathrm{FN}}_G^m$ calculates the cohomology of $Y^{(m)}$ and is therefore acyclic up to dimension $nm-2$. Alternatively, we can see this, as explained in [8], by observing that the Atiyah duality theorem implies that the Spanier dual of $Y^{(m)}$ is $(Y^{(m)})^+$.

Passing to the limit for $m \to \infty$, we obtain an acyclic $\mathbb{F}_2$–complex $\widetilde{\mathrm{FN}}_G \otimes \mathbb{F}_2$, dual to $C_*^{\mathrm{CW}}(X) \otimes \mathbb{F}_2$, for which a basis $\{e(S)\}_{S \in \Phi_\infty}$ is parametrized by strata in Φ_∞. The degree of $e(S)$ as a cochain of X is equal to the codimension of F. This is an equivariant cochain model for $E(G)$. In particular, the quotient $\mathrm{FN}_G \otimes \mathbb{F}_2 = \widetilde{\mathrm{FN}}_G / G \otimes \mathbb{F}_2$ calculates $\widetilde{H}^*(G;\mathbb{F}_2)$. In the following, when we need to stress the Coxeter group G involved, we will use the heavier notation $\Phi_{\infty,G}$ instead of Φ_∞.

This description of the cochain complex FN_G calculating the cohomology of G fits particularly well with a chain-level interpretation of duality via intersection theory that we will occasionally use in proofs and that we briefly recall here. Given a manifold X and an immersion $i : W \to X$ of a codimension d manifold in X, we say that a smooth singular chain in X is transverse to i if, for every simplex $\sigma : \Delta^k \to X$ of the chain, σ is transverse on every face of Δ^k and subface, in the sense of manifolds with corners. It can be proved that the subcomplex consisting of chains that are transverse to i is chain equivalent to the full one. To every d–dimensional singular simplex $\sigma : \Delta^d \to X$ transverse to i we can associate the element $\tau_W(\sigma) \in \mathbb{F}_2$ given by the mod 2 cardinality of $\sigma^{-1}(W)$. This procedure defines a cochain dual in the complex dual to the chain complex of singular chains transverse to i. If i is a proper embedding, τ_W is a cocycle and defines a cohomology class. The most important constructions in cohomology can be understood geometrically using this model. In particular, if $f : Y \to X$ is

transverse to i, then $f^{\#}(\tau_W) = \tau_{f^{-1}(W)}$. The reader will find a complete reference of this approach to cohomology in [6].

In our particular context, each stratum $S \in \Phi_\infty$ defines such a cochain τ_S. We understand the coboundary of τ_S as $\tau_{\partial(S)}$, so we can identify $\widetilde{\mathrm{FN}}^*_G$, at least modulo 2, with the cochain complex spanned by τ_S for strata $S \in \Phi_\infty$. Suppose $W \subseteq Y_G^{(\infty)}$ is a proper submanifold of codimension d obtained as a union of strata. In that case, its associated cochain τ_W is the sum of τ_S for strata $S \subseteq W$ of minimal codimension, and $\delta(\tau_W) = 0$. If, in addition, the action of G preserves W, then, passing to the quotient, its image $\overline{W} \subseteq Y_G^{(\infty)}/G$ defines a Thom class represented in FN^*_G by the sum of strata contained in $\overline{W}$. This construction is made precise in [7, Definition 4.6].

3.3 The special case of Coxeter groups of type B

We conclude this section by further investigating the cases of our interest $G = W_{B_n}$ and, in the following subsection, $G = W_{D_n}$. The strata of Φ_m for the symmetric group Σ_n can be described in terms of leveled trees, as shown in [8] using ideas dating back to Vassiliev [19]. A straightforward adaptation of these ideas shows that, in the case of the Coxeter groups of type B_n, we can describe them in terms of symmetric leveled trees. This interpretation encodes geometrically and combinatorially the structure of W_{B_n} as a wreath product of Σ_n with a cyclic group of order 2. Below we provide the precise definitions.

First, we observe that, since W_{B_n} is generated by a set $S = \{s_0, \dots, s_{n-1}\}$ of n fundamental reflections as described in Figure 1, the Fox–Neuwirth complex $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ has a $\mathbb{Z}[W_{B_n}]$–basis $\{e(\underline{a})\}$ indexed by n–tuples of nonnegative integer numbers $(a_0, \dots, a_{n-1})$.

The reflection hyperplane arrangement associated with W_{B_n} can be described as $\mathcal{A}_{B_n} = \{H^{\pm}_{i,j}\}_{1 \le i < j \le n} \cup \{H^0_i\}$, where

$$H^{\pm}_{i,j} = \{\underline{x} \in \mathbb{R}^n \mid x_i = \pm x_j\}, \quad H^0_i = \{\underline{x} \in \mathbb{R}^n \mid x_i = 0\}.$$

Moreover, s_0 can be identified with the reflection with respect to H^0_1 and, for every $i > 0$, s_i with the reflection with respect to $H^+_{i,i+1}$. Thus the basis element corresponding to $\underline{a} = (a_0, \dots, a_{n-1})$ is described as the stratum

$$e(\underline{a}) = \big\{(\underline{x}_1, \dots, \underline{x}_n) \in (\mathbb{R}^\infty)^n \mid \forall 1 \le i \le n-1, \forall 1 \le j \le a_i : (x_i)_j = (x_{i+1})_j, \\ (x_i)_{a_i+1} < (x_{i+1})_{a_i+1}, \forall 1 \le k \le a_0 : (x_1)_k = 0, (x_1)_{a_0+1} > 0\big\}.$$

Passing to the quotient by the action of W_{B_n}, we see that $\mathrm{FN}^*_{W_{B_n}}$ has a $\mathbb{Z}$–basis constituted by $[a_0 : \cdots : a_{n-1}] = [e(a_0, \ldots, a_{n-1})]$.

The differential on $\mathrm{FN}^*_{W_{B_n}}$ is complicated, but it is combinatorially accessible via a description of its basis in terms of trees.

Definition 3.2 A *signed depth-ordering* is a sequence of labeled inequalities of the form $\Gamma = (0 <_{a_0} i_1 <_{a_1} \cdots <_{a_{n-1}} i_n)$, where $i_k \in \{-n, \ldots, -1, 1, \ldots, n\}$ for all $1 \leq k \leq n$, and these indices have pairwise different absolute values. By convention, we let $i_0 = 0$.

A *planar level tree* is a planarly embedded tree T satisfying the following conditions:

- it has a root vertex embedded in $(0, 0)$ and all the other vertices having their second coordinate (the "height") equal to a positive integer;
- two edges connected by an edge have heights whose difference is 1;
- the height along the unique minimal path from the root to every leaf is always increasing.

A *planar level tree with labels in I* is a couple (T, λ) defined as follows: T is a planar level tree, and λ is a bijective labeling of the leaves of T with elements of I.

A *symmetric planar level tree* is a planar level tree invariant under the reflection r along the y–axis and having an odd number of leaves.

An *antisymmetric planar level tree with labels in $\{-n, \ldots, n\}$* is a labeled planar level tree (T, λ) with labels in $\{-n, \ldots, n\}$ such that T is symmetric, and two leaves that correspond to each other under the application of r have labels opposite to each other.

The antisymmetric planar level tree associated with a depth ordering Γ is the antisymmetric planar level tree T_Γ, unique up to isotopy, defined by the following properties:

- the labels of the leaves, from left to right, are $-i_n, \ldots, -i_1, 0, i_1, \ldots, i_n$;
- the leaves labeled i_{k-1}, i_k, for $1 \leq k \leq n$, are separated by a vertex of height a_k but not by vertices of height less than a_k.

Let $k \geq 0$. The *k–symmetrization* $S_k(T)$ (resp. $\widetilde{S}_k(T)$) of a planar level tree T (with labels in $\{1, \ldots, n\}$) is a symmetric planar level tree S (resp. antisymmetric planar level tree with labels in $\{-n, \ldots, n\}$) obtained by the following procedure. Glue T from the right to a vertical linear planar level tree lying into the y–axis up to height k.

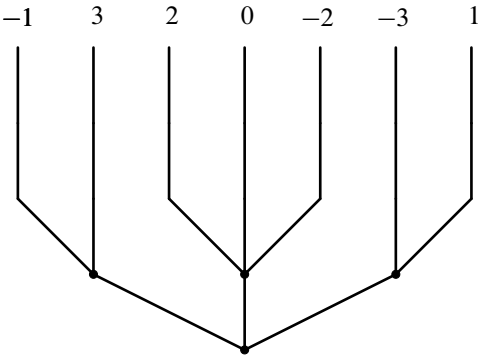

Figure 2: An example of antisymmetric planar level tree with labels in $[-3,3]$.

Then, add the mirror image of such tree under r to obtain a symmetric planar level tree (choosing the unique antisymmetric labeling that extends the labeling of T in the labeled case).

There is a free action of W_{B_n} on the set antisymmetric planar level trees with labels in $\{-n,\dots,n\}$ given by interpreting elements of W_{B_n} as signed permutations and permuting labels accordingly. We always assume that the edges of a level tree are oriented so that there is a unique oriented path from the root vertex to each leaf.

Similarly to the symmetric group case, we have the following immediate proposition.

Proposition 3.3 *The function $\Gamma \mapsto T_\Gamma$ is a bijection between the set of signed depth-orderings with n labels and the set of isotopy classes of antisymmetric planar level trees with labels in $\{-n,\dots,n\}$. Furthermore, to $\Gamma = (0 <_{a_0} i_1 <_{a_1} \cdots <_{a_{n-1}} i_n)$ is associated a stratum $\sigma e(\underline{a}) \in \Phi_{\infty,W_{B_n}}$, where $\sigma(k) = i_k$, $\underline{a} = (a_0,\dots,a_{n-1})$, and this provides a W_{B_n}–equivariant additive basis of $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ labeled by signed depth-orderings or, equivalently, by isotopy classes of antisymmetric planar level trees with labels in $\{-n,\dots,n\}$. W_{B_n} acts on this basis by permuting labels. Consequently, an additive basis for $\mathrm{FN}^*_{W_{B_n}}$ is given by symmetric planar level trees with $2n+1$ leaves.*

An example of an antisymmetric planar level tree (T,λ), with labels in $[-3,3]$, is given in Figure 2. The associated signed depth-ordering is $\Gamma = (0 <_1 -2 <_0 -3 <_1 1)$ and the corresponding stratum is $\sigma e([1,0,1])$, where $\sigma(1) = -2$, $\sigma(2) = -3$ and $\sigma(3) = 1$.

We observe that we can use Proposition 3.3 to reinterpret operations on (symmetric) level trees in terms of n–tuples or (signed) depth-orderings. For instance, the k–

symmetrization of trees provides a linear map $S_k\colon \mathrm{FN}^*_{\Sigma_n} \to \mathrm{FN}^*_{W_{B_n}}$ that we can interpret as $[a_1 : \cdots : a_{n-1}] \mapsto [k : a_1 : \cdots : a_{n-1}]$.

We can now describe the differential in terms of this basis.

Definition 3.4 [8] Let (T, λ) be a planar level tree. Let v be an internal vertex. Let $E(v)$ be the set of edges whose source vertex is v. The planar embedding of T induces an order on $E(v)$, defined from left to right. A vertex permutation of (T, λ) at v is another planar level tree that is isomorphic to (T, λ) as a labeled tree but with a different planar embedding that differs from the original one only by the ordering on $E(v)$.

Given a planar level tree (T, λ) and an internal vertex v, let (e, f), with $e < f$, be a couple of adjacent edges in $E(v)$. Let u_e and u_f be the targets of e and f, respectively. Let σ be a shuffle of the two sets $E(u_e)$ and $E(u_f)$. Let $d_{e,f,\sigma}(T, \lambda)$ be the planar level tree obtained by gluing together e and f, with common target $\bar{u}$, and then applying the vertex permutation that permutes the edges in $E(\bar{u})$ by σ.

Recall that, in the A_n case, the differential in $\widetilde{\mathrm{FN}}^*$ of the basis element corresponding to a planar level tree with labels (T, λ) is given by the sum over (v, σ) as above of $d_{v,\sigma}(T, \lambda)$. Similarly, we have the following proposition, which essentially states that a symmetrization of the previous construction gives the differential in the B_n case.

Proposition 3.5 *With the correspondence provided by Proposition 3.3, the differential of the cochain complex* $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2$ *is given in terms of antisymmetric level trees with labels* $d(T, \lambda) = \sum_{(e,f,\sigma)} \sum_{(e',f',\tau)} d_{e,f,\sigma} d_{e',f',\tau}(T, \lambda)$, *where the sum is over sextuples* $(e, f, \sigma, e', f', \tau)$ *such that* $d_{e,f,\sigma} d_{e',f',\tau}(T, \lambda)$ *is again an antisymmetric planar level tree. Equivalently,* $d(T, \lambda)$ *is obtained by performing an operation* $d_{e,f,\sigma}$ *starting from a couple of adjacent vertices* (e, f) *lying into the positive half-plane* $\{(x, y) \mid x \geq 0\}$, *and then perform the mirror operation on the mirror pair of adjacent edges* (e', f') *in the negative half-plane. If we call such symmetric operation* $d^S_{e,f,\sigma}$, *we have that*

$$d(T, \lambda) = \sum_{(e,f)} \sum_{\sigma} d^S_{e,f,\sigma}(T, \lambda),$$

where the sum is over couples (e, f) *of adjacent edges in the positive half-plane and shuffles* σ *of the two sets of vertices incident to the target of* e *and* f, *respectively.*

We can equivalently express this construction using planar level trees T with $n + 1$ leaves labeled by $(-n, \ldots, -1, 0, 1, \ldots, n)$, with labels having pairwise different absolute values, such that the leftmost leaf has label 0. We recover the corresponding

antisymmetric level tree as follows. We choose a representative of the isotopy class of (T, λ) in which the entire oriented path from the root vertex to the label 0 lies on the y–axis. Then we merge T with its mirror image along y with opposite labels. In this case, the differential is given by contracting a couple of adjacent edges and shuffling. When the result is a tree whose leftmost leaf is not labeled by 0, we replace the part of the tree belonging to the negative half-plane $\{(x, y) \mid x \leq 0\}$ with its mirror image in the positive half-plane, with opposite labels, and shuffle the corresponding edges in all possible ways.

3.4 The special case of Coxeter groups of type D

We now turn to the description of the complex $\mathrm{FN}^*_{W_{D_n}}$. Once again, since this Coxeter group has n fundamental reflections $t_0, \dots, t_{n-1}$, a $\mathbb{Z}[W_{D_n}]$–basis for $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ is indexed by n–tuples of nonnegative integers $\underline{a} = (a_0, \dots, a_{n-1})$.

The inclusion $j_n \colon W_{D_n} \to W_{B_n}$ identifies the reflection arrangement associated to W_{D_n} with the subarrangement of $\mathcal{A}_{W_{B_n}}$ composed by the hyperplanes $H^{\pm}_{i,j}$, for $1 \leq i < j \leq n$, and $t_i = s_i$ for $1 \leq i \leq n$, while t_0 is the reflection along $H^-_{1,2}$. Thus the basis element of $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ corresponding to $\underline{a}$ is described as the stratum

$$e(\underline{a}) = \big\{(\underline{x}_1, \dots, \underline{x}_n) \in (\mathbb{R}^\infty)^n \mid \forall 1 \leq i \leq n-1, 1 \leq j \leq a_i : (x_i)_j = (x_{i+1})_j,$$
$$(x_i)_{a_i+1} < (x_{i+1})_{a_i+1}, \forall 1 \leq k \leq a_0 : (x_2)_k = -(x_1)_k, (x_2)_{a_0+1} > -(x_1)_{a_0+1}\big\}.$$

Passing to the quotient by the action of W_{D_n}, we see that $\mathrm{FN}^*_{W_{D_n}}$ has a $\mathbb{Z}$–basis constituted by $[a_0 : \cdots : a_{n-1}] = [e(a_0, \dots, a_{n-1})]$.

The complex $\widetilde{\mathrm{FN}}^*_{W_{B_n}} / j_n(W_{D_n})$ also calculates the cohomology of W_{D_n}. Therefore, there is a cochain equivalence $\varphi \colon \mathrm{FN}^*_{W_{D_n}} \to \widetilde{\mathrm{FN}}^*_{W_{B_n}} / j_n(W_{D_n})$ between the two resolutions. In the subsequent section, we compute an explicit formula for φ that we will use to perform cochain-level computation in the following sections. For instance, we will prove the relations for coproduct of transfer products of Hopf ring generators by mapping them to $\widetilde{\mathrm{FN}}^*_{W_{B_n}} / j_n(W_{D_n})$, where their expressions are closer to the B_n case. As a notational convention, we denote this cochain complex by $\mathrm{FN}'^*_{W_{D_n}}$.

First, we observe that $[W_{B_n} : j_n(W_{D_n})] = 2$; thus $j_n(W_{D_n})$ is a normal subgroup of W_{B_n}. The two cosets of $j_n(W_{D_n})$ in W_{B_n} are represented by the identity and s_0, the only fundamental reflection of W_{B_n} that is not contained in $j_n(W_{D_n})$. Thus, given a $\mathbb{Z}[W_{B_n}]$–basis $\mathcal{B}$ for $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$, the classes of x and $s_0.x$, where $x \in \mathcal{B}$, provide a $\mathbb{Z}$–basis for $\mathrm{FN}'^*_{W_{D_n}}$. Let $\mathcal{B}$ be the basis defined above in terms of n–tuples or equivalently

of symmetric planar level trees, parametrized by n–tuples of nonnegative integers $\underline{a} = (a_0, \dots, a_{n-1})$. We denote by $[a_0 : \cdots : a_{n-1}]^+$ and $[a_0 : \cdots : a_{n-1}]^-$ the cochains in $\mathrm{FN}'^*_{W_{D_n}}$ arising from the basis element corresponding to $\underline{a}$ and $s_0\underline{a}$.

The complex $\mathrm{FN}'^*_{W_{D_n}}$ also has a description in terms of trees.

Definition 3.6 Let T be a symmetric planar level tree with $2n+1$ leaves. An *orientation* of T is the choice of an element of $L/\!\sim$, where L is the set of antisymmetric labelings of T with labels in $\{-n, \dots, n\}$, and $\sim$ is the equivalence relation defined by

$$\lambda \sim \lambda' \quad \Longleftrightarrow \quad \lambda' = \lambda\sigma \quad \text{for some } \sigma \in \mathrm{Alt}(2n+1).$$

An *oriented symmetric planar level tree* is a couple $(T, \mathcal{O})$, where T is a symmetric planar level tree and $\mathcal{O}$ is an orientation of T.

Note that if two antisymmetric labelings of a symmetric planar level tree T differ by a permutation $\sigma \in \Sigma_{\{-n,\dots,n\}}$, then σ must fix 0 and act as a signed permutation on $\{-n, \dots, -1, 1, \dots, n\}$. Hence, an orientation of T is the choice of an antisymmetric labeling up to the action of $j_n(W_{D_n})$. Since the index $[W_{B_n} : j_n(W_{D_n})]$ is 2, there are two possible orientations for a symmetric planar level tree T, determined by the parity of the number of negative labels of leaves in the positive half-plane. In particular, we can identify an orientation $\mathcal{O}$ with a sign $+$ or $-$, corresponding to labelings with an even or odd number of positively labeled leaves in the positive half-plane, respectively.

Moreover, from the fact that $\mathrm{Alt}(2n+1)$ is normal in Σ_{2n+1}, it follows that if T is a symmetric planar level tree, λ is a labeling of T and $\sigma(T)$ is a vertex permutation of T at a vertex v, then the orientation of the permuted labeled tree $\sigma(T, \lambda)$ only depends on the orientation determined by λ. Therefore, the rule for the differential in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ induces a formula for the differential in $\mathrm{FN}'^*_{W_{D_n}}$ in terms of trees. Hence, we have the following description.

Proposition 3.7 *$\mathrm{FN}'^*_{W_{D_n}}$ can be described as the cochain complex having additive basis indexed by oriented symmetric planar level trees with $2n+1$ leaves, with differential induced by the symmetric tree differential in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ by keeping track of orientations.*

The reader is encouraged to compare this description with the notion of "charged" configuration used in [9].

4 Geometry and combinatorics: chain-level formulas

We devote this section to developing some formulas that will allow us to perform calculations at the (co)chain level. These computations will be needed at points, especially when retrieving relations. We first compute some connecting maps between the Fox–Neuwirth complexes of Coxeter groups of type A, B and D. Then, we provide cochain representatives of the structural maps of our almost-Hopf ring structures.

4.1 The connecting homomorphisms

As $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ and $\widetilde{\mathrm{FN}}'^*_{W_{D_n}}$ are both free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}[W_{D_n}]$–module, they need to be W_{D_n}–equivariantly cochain equivalent. We begin by providing a formula for an explicit equivalence φ relating the two models $\mathrm{FN}^*_{W_{D_n}}$ and $\mathrm{FN}'^*_{W_{D_n}}$.

Lemma 4.1 *There is a cochain homotopy equivalence $\varphi^*\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}'^*_{W_{D_n}}$ defined by the formula*

$$\varphi^*[a_0:\cdots:a_{n-1}] = \begin{cases} [a_0:a_1:a_2:\cdots:a_{n-1}]^+ & \text{if } a_0 < a_1, \\ [a_0:a_1:a_2:\cdots:a_{n-1}]^+ + [a_1:a_0:a_2:\cdots:a_{n-1}]^- & \text{if } a_0 = a_1, \\ [a_1:a_0:a_2:\cdots:a_{n-1}]^- & \text{if } a_0 > a_1, \end{cases}$$

induced by the inclusion $Y^{(\infty)}_{W_{B_n}} \subseteq Y^{(\infty)}_{W_{D_n}}$ and yielding the identity in cohomology.

Proof We observe that the inclusion

$$Y^{(\infty)}_{W_{B_n}} \subseteq Y^{(\infty)}_{W_{D_n}}$$

is a W_{D_n}–equivariant homotopy equivalence. Moreover, the inverse image in $Y^{(\infty)}_{W_{B_n}}$ of each stratum of $\Phi_{\infty,W_{D_n}}$ is a union of strata in $\Phi_{\infty,W_{B_n}}$. Thus, passing to quotients, this yields a map

$$\varphi\colon \frac{Y^{(\infty)}_{W_{B_n}}}{W_{D_n}} \to \frac{Y^{(\infty)}_{W_{D_n}}}{W_{D_n}}$$

that induces a well-defined map between the cochain complexes $\varphi^*\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}'^*_{W_{D_n}}$.

We now check that φ^* satisfies the given formulas. It is sufficient to consider the finite approximations

$$\varphi^{(d)}\colon \frac{Y^{(d)}_{W_{B_n}}}{W_{D_n}} \to \frac{Y^{(d)}_{W_{D_n}}}{W_{D_n}}.$$

For any given stratum $S = e(a_0, \dots, a_{n-1})$ for W_{D_n}, since $\varphi^{(d)}$, being a 0–codimensional immersion, is transverse to S, we have that $(\varphi^{(d)})^*(\tau_S) = \tau_{(\varphi^{(d)})^{-1}(S)}$. We now distinguish three cases:

- if $a_0 < a_1$, then $(\varphi^{(d)})^{-1}(S) = e(a_0, a_1, a_2, \dots, a_{n-1})$;
- if $a_0 > a_1$, then $(\varphi^{(d)})^{-1}(S) = s_0.e(a_1, a_0, a_2, \dots, a_{n-1})$;
- if $a_0 = a_1$, then $(\varphi^{(d)})^{-1}(S)$ is the union of $e(a_0, \dots, a_{n-1})$, $s_0.e(a_0, \dots, a_{n-1})$ and strata of bigger codimension.

This implies that φ^* has the desired form. □

We also consider the following group homomorphisms:

- the standard inclusion $j : \Sigma_n \to W_{B_n}$ already considered in the previous section;
- the involution $c_{s_0} : W_{D_n} \to W_{D_n}$ given by conjugation by s_0, the unique generating reflection of W_{B_n} that does not belong to W_{D_n}, that fixes t_i for $2 \le i < n$ and switches t_0 and t_1;
- the two inclusions $i_+, i_- : \Sigma_n \to W_{D_n}$ given, in terms of the Coxeter generators $t_0, \dots, t_n$ of Figure 1, by $i_\pm(i, i+1) = t_i$ if $i \ge 2$, $i_+(1,2) = t_1$ and $i_-(1,2) = t_0$.

We denote by $\iota : H^*(W_{D_n}; \mathbb{F}_2) \to H^*(W_{D_n}; \mathbb{F}_2)$ the morphism induced by c_{s_0} on cohomology.

We note that the two following properties hold by construction:

- $\pi j = \mathrm{id}_{\Sigma_n}$;
- $\pi \circ i_+ = \pi \circ i_- = \mathrm{id}_{\Sigma_n}$, where $\pi : W_{D_n} \to \Sigma_n$ is the composition of the inclusion $j : W_{D_n} \to W_{B_n}$ with the projection $W_{B_n} \to \Sigma_n$;
- $c_{s_0} \circ i_+ = i_-$.

We compute cochain representatives of ι in the following lemmas.

Lemma 4.2 *ι is induced by the cochain-level map $\iota^\# : \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{W_{D_n}}$ defined by*

$$\iota^\#[a_0 : a_1 : a_2 : \cdots : a_{n-1}] = [a_1 : a_0 : a_2 : \cdots : a_{n-1}].$$

Proof Since the image under c_{s_0} of a fundamental reflection for $W = W_{D_n}$ is again a fundamental reflection, for every $\Gamma' \subseteq \Gamma \subseteq \{t_0, \dots, t_{n-1}\}$, the set of minimal-length coset representatives satisfies $c_{s_0}(W_\Gamma^{\Gamma'}) = W_{c_{s_0}(\Gamma)}^{c_{s_0}(\Gamma')}$. Thus,

$$e(\Gamma_1 \supseteq \cdots \supseteq \Gamma_k \supseteq \cdots) \mapsto e(c_{s_0}(\Gamma_1) \supseteq \cdots \supseteq c_{s_0}(\Gamma_k) \supseteq \cdots)$$

defines a c_{s_0}–equivariant chain map $C_*^{W_{D_n}} \to C_*^{W_{D_n}}$. This yields, dually, the desired cochain map $\mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{W_{D_n}}$. □

We can also describe ι in terms of $\mathrm{FN}'^*_{W_{D_n}}$. The proof of the following lemma is straightforward.

Lemma 4.3 *ι is induced by the cochain-level map $\iota'^{\#}\colon \mathrm{FN}'^*_{W_{D_n}} \to \mathrm{FN}'^*_{W_{D_n}}$ defined by*

$$\iota'^{\#}[a_0 : \cdots : a_{n-1}]^+ = [a_0 : \cdots : a_{n-1}]^-, \quad \iota'^{\#}[a_0 : \cdots : a_{n-1}]^- = [a_0 : \cdots : a_{n-1}]^+.$$

In terms of oriented symmetric planar level trees, the map $\iota'^{\#}$ acts on $(T, \mathcal{O})$ by replacing $\mathcal{O}$ with the opposite orientation.

The following identity is also proved by direct inspection.

Lemma 4.4 *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{FN}^*_{W_{D_n}} & \xrightarrow{\varphi^*} & \mathrm{FN}'^*_{W_{D_n}} \\ \Big\downarrow{\scriptstyle \iota^{\#}} & & \Big\downarrow{\scriptstyle \iota'^{\#}} \\ \mathrm{FN}^*_{W_{D_n}} & \xrightarrow[\varphi^*]{} & \mathrm{FN}'^*_{W_{D_n}} \end{array}$$

The formulas for the other connecting maps follow from a general remark.

Lemma 4.5 *Let G be a Coxeter group, with Coxeter generators $S = \{s_0, \dots, s_n\}$ and $H \leq G$ be a parabolic subgroup, generated by a subset $T = \{s_{i_0}, \dots, s_{i_m}\} \subseteq S$. The inclusion $H \hookrightarrow G$ is represented at the chain level by the chain map $C_*^H \to C_*^G$ given by $c(\underline{\Gamma}, \gamma) \mapsto c(\underline{\Gamma}, \gamma)$, for flags $\underline{\Gamma} = (\Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_k \supseteq \varnothing)$ with $\Gamma_0 \subseteq T \subseteq S$ and elements $\gamma \in H$.*

Dually, it is represented at the cochain level by the cochain map $\mathrm{FN}^*_G \to \mathrm{FN}^*_H$ given by

$$[e(a_0, \dots, a_n)] \in C_*^H \mapsto \begin{cases} [e(a_{i_0}, \dots, a_{i_m})] & \text{if } a_j = 0 \text{ for all } j \notin \{i_0, \dots, i_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Since the inclusion of parabolic subgroups preserves minimal coset representatives, the De Concini–Salvetti boundary formula of Theorem 3.1 implies that the given linear morphism $C_*^H \to C_*^G$ is an H–equivariant chain map. Dualizing this yields the cochain formula between Fox–Neuwirth complexes. □

As particular cases of this lemma, we retrieve cochain formulas for our connecting homomorphisms:

Corollary 4.6 *The following statements are true.*

(1) *The linear morphism* $j^{\#}\colon \mathrm{FN}^*_{W_{B_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_1,\dots,a_{n-1}] & \text{if } a_0 = 0,\\ 0 & \text{if } a_0 > 0,\end{cases}$$

represents j *at the cochain level.*

(2) *The linear morphism* $i_+^{\#}\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_1,\dots,a_{n-1}] & \text{if } a_0 = 0,\\ 0 & \text{if } a_0 > 0,\end{cases}$$

represents i_+ *at the cochain level.*

(3) *The linear morphism* $i_-^{\#}\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_0,a_2,\dots,a_{n-1}] & \text{if } a_1 = 0,\\ 0 & \text{if } a_1 > 0,\end{cases}$$

represents i_+ *at the cochain level.*

4.2 Structural morphisms: A_B

We want to describe the almost-Hopf ring structures presented in Section 2 in our geometric context. We begin with the coproduct map in A_B. In contrast with the symmetric group case, the cochain-level map inducing the coproduct is relatively complicated. Its underlying combinatorics is built upon elementary steps that we, mindful of the botanic analogy, suggestively call "prunings".

Definition 4.7 Let T be a planar level tree. An *elementary k–pruning* of T is a planar level tree T' obtained by the following procedure. Choose an internal vertex v of T of height k, and consider on $E(v)$ the order induced by the planar embedding. Let $1 \le l < |E(v)|$, consider the l biggest elements $e_1,\dots,e_l$ of $E(v)$ with respect to this order, and let v'_i be the target of e_i. T'' is the subtree of T spanned by v and all vertices that can be reached from one of the v'_i through an oriented path. T' is the complementary subtree of T'' in T. We call the planarly embedded subtree T'' the *scrap* of the elementary k–pruning. An elementary k–pruning is said to be *minimal* if $l = 1$. A *k–pruning* is a couple (T', T'') constructed as follows:

- T' is obtained from a sequence of elementary k–prunings

$$T \rightsquigarrow T'_1 \rightsquigarrow T'_2 \rightsquigarrow \cdots \rightsquigarrow T'_j = T'$$

  performed on pairwise different vertices $v_1, \ldots, v_j$ of T, with scraps $T''_1, \ldots, T''_j$;
- T'' is a planar level tree obtained by joining these scrap subtrees along a vertex w of height k and performing a vertex permutation at w that shuffles the edges of the scraps.

Let T be a symmetric planar level tree. An *elementary symmetric k–pruning* of T is a tree T' obtained as follows. Apply to T an elementary (nonsymmetric) k–pruning whose scrap T'' does not contain the central leaf belonging to the $\boldsymbol{y}$–axis. Then, remove the image of the subtree of T'' under the reflection r along the vertical axis. T'' is called the *scrap* of the elementary symmetric pruning. An elementary symmetric k–pruning is said to be *minimal* if it is obtained from a minimal elementary k–pruning. A *symmetric k–pruning* is a couple (T', T''), where

- T' is obtained from a sequence of elementary k–prunings

$$T \rightsquigarrow T'_1 \rightsquigarrow T'_2 \rightsquigarrow \cdots \rightsquigarrow T'_j = T'$$

  performed on pairwise different vertices of T, with scraps $T''_1, \ldots, T''_j$;
- T'' is a nonsymmetric planar level tree obtained by joining the scrap subtrees to a vertex w of height k and performing a vertex permutation at w that shuffles the edges of the scraps.

We note that elementary k–prunings at different vertices commute, both in the symmetric and nonsymmetric cases. Hence, a k–pruning or symmetric k–pruning is uniquely determined by the set of elementary k–prunings that compose it, independently of the order in which they are performed.

There is also an alternative way to define (symmetric) k–prunings in terms of minimal k–prunings instead of elementary ones. A (symmetric) k–pruning is obtained by performing a sequence of minimal elementary (symmetric) k–prunings, not necessarily at pairwise different vertices, and then joining the scraps at a vertex of height k without shuffling the edges.

We now consider three linear morphisms that we will need to define the cochain-level coproduct map:

- the k–pruning map
$$P_k : \mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{a+b=n} \mathrm{FN}^*_{W_{B_a}} \otimes \mathrm{FN}^*_{W_{B_b}} \otimes \mathbb{F}_2$$
that maps a symmetric planar level tree T to the sum $\sum T' \otimes S_k(T'')$ over all the possible symmetric k–prunings (T', T'') of T;
- the minimal k–pruning map
$$P_k^{\min} : \mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{a+b=n} \mathrm{FN}^*_{W_{B_a}} \otimes \mathrm{FN}^*_{W_{B_b}} \otimes \mathbb{F}_2$$
that maps a symmetric planar level tree T to the sum $\sum T' \otimes S_k(T'')$ over all the possible minimal elementary symmetric k–prunings (T', T'') of T;
- the concatenation map $C : \mathrm{FN}^*_{W_{B_n}} \otimes \mathrm{FN}^*_{W_{B_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}^*_{W_{B_{n+m}}}$ such that $C([a_0 : \cdots : a_{n-1}] \otimes [b_0 : \cdots : b_{m-1}]) = [a_0 : \cdots : a_{n-1} : b_0 : \cdots : b_{m-1}]$.

The map P_k is exemplified in Figure 3. We understand C at the level of symmetric planar level trees as the function given by the following procedure. Take a couple of such objects (T, S). Cut S along its central vertical axis. Finally, glue the right piece of S onto the right side of T and the left part onto its left side to obtain a new symmetric planar level tree. We remark that these linear morphisms are degree-preserving, but they are not chain maps.

In the A_n case, we can define a similar k–pruning map P'_k by summing all nonsymmetric k–prunings. For $k = 0$, P'_0 is a chain map, and it is shown in [8] to induce the coproduct in cohomology. This statement is not true in the B_n case because the differential of an antisymmetric planar level tree with labels behaves badly near the central "trunk" labeled 0. Nevertheless, at each level k, away from this central stem, this is essentially true. For this intuitive reason, we must define our cochain-level coproduct map differently: prune a symmetric planar level tree at every level and tensor it with a symmetric planar level tree whose principal k–blocks, as defined below in Definition 4.10, are the scraps of the performed prunings. To prove this statement, we need some preliminary calculations.

Suppose that a symmetric planar level tree T corresponds to $[a_0 : \cdots : a_{n-1}] \in \mathrm{FN}_{W_{B_n}}$. In that case, consider the set of couples of adjacent edges (e, f), with $e < f$, in T having the same source vertex and belonging to the positive half-plane $\{(x, y) \mid x \geq 0\}$. This set in bijective correspondence with $\{0, \ldots, n-1\}$, and the height of the common vertex of the couple (e, f) corresponding to i is a_i. This bijection is explicitly given by counting the leaves in the positive half-plane that lie on the left of e. For $0 \leq i \leq n-1$,

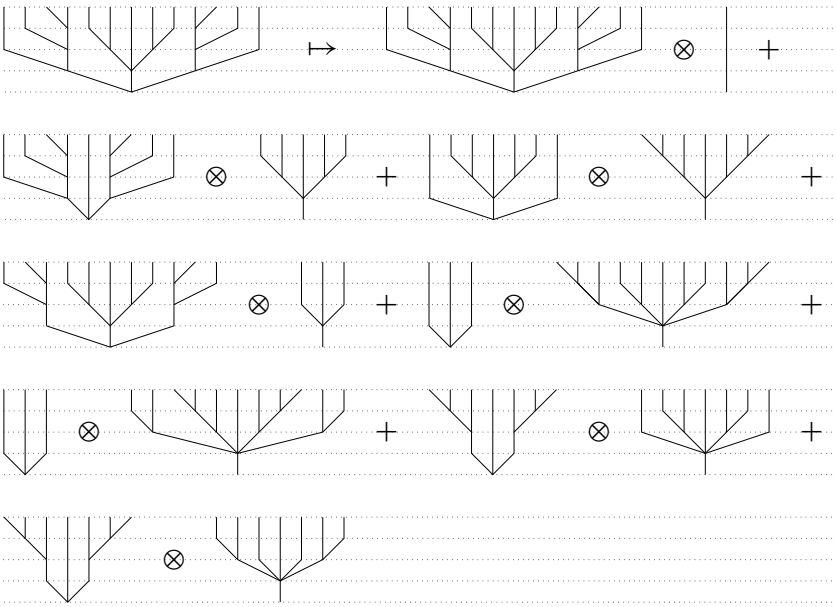

Figure 3: The map P_1, defined as the sum of all possible symmetric 1–pruning, on a given symmetric planar level tree.

we denote by $d_i(T)$ or equivalently by $d_{e,f}$ the sum of the addends $d^S_{e,f,\sigma}$ of the differential d, as expressed in Proposition 3.5, in which a vertex shuffle constructed from the couple (e, f) corresponding to i appear. Thus, $d(T) = \sum_{i=0}^{n-1} d_i(T)$.

Lemma 4.8 *Let T be a symmetric planar level tree corresponding to $[a_0 : \cdots : a_{n-1}]$. Let m_k be the smallest index such that $a_{m_k} = k$. Let I be the trivial symmetric planar level tree. Then the following statements are true*:

(1) *the pruning maps and the differential satisfy the equality*

$$P_k d + dP_k + (\mathrm{id} \otimes d_0)(P_k - \mathrm{id} \otimes I) = (\mathrm{id} \otimes d_{m_{k-1}})(\mathrm{id} \otimes C)(P_k \otimes \mathrm{id})P^{\min}_{k-1};$$

(2) $P_k(T) = T \otimes I$ *if* $a_i < k$ *for all* $0 \le i < n$;

(3) *for all* $\boldsymbol{a} = [a_0 : \cdots : a_{n-1}]$ *and* $\boldsymbol{b} = [b_0 : \cdots : b_{m-1}]$ *with* $b_0 < \min\{a_0, \dots, a_{n-1}\}$,

$$d_i C(\boldsymbol{a} \otimes \boldsymbol{b}) = \begin{cases} C(d_i \otimes \mathrm{id})(\boldsymbol{a} \otimes \boldsymbol{b}) & \text{if } 0 \le i < n, \\ C(\mathrm{id} \otimes d_{i-n})(\boldsymbol{a} \otimes \boldsymbol{b}) & \text{if } n < i < n + m, \end{cases}$$

and the latter also holds for $i = n$ if $b_0 < \min\{a_0, \dots, a_{n-1}\} - 1$;

(4) $(\mathrm{id} \otimes C)(P^{\min}_k \otimes \mathrm{id})P_k(T) = P_k(T) - T \otimes I$;

(5) $C(C \otimes \mathrm{id}) = C(\mathrm{id} \otimes C)$.

Proof The statements from (2) to (5) are easy. Regarding (2), if $a_i < k$ for all i, T has no vertex of height k with more than one outgoing edge. Thus the only possible symmetric k–pruning is the trivial one. Regarding (3), the bijection

$$\varphi\colon \{0,\dots,n-1\} \sqcup \{0,\dots,m-1\} \to \{0,\dots,n+m-1\}$$

that shifts elements of $\{0,\dots,m-1\}$ by n yields a bijection between pairs (e, f) of adjacent edges of the symmetric planar level tree T corresponding to $C(\boldsymbol{a}\otimes\boldsymbol{b})$ and those of the symmetric planar level trees T' and T'' corresponding to $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. If $b_0 < \min\{a_0,\dots,a_{n-1}\}$, then for all $i \in \{0,\dots,n+m-1\}$, with the only possible exception of $i = n$, this bijection preserves $E(v_{e_i})$ and $E(v_{f_i})$, where v_{e_i} and v_{f_i} are the target vertices of the corresponding pair of edges (e_i, f_i). The edges in $E(v_{e_i})$ and $E(v_{f_i})$ of the corresponding pair come either both from T' or both from T''. Hence $d_i C(\boldsymbol{a}\otimes\boldsymbol{b}) = d_{\varphi^{-1}(i)}\boldsymbol{a}\otimes\boldsymbol{b}$. If $b_0 < \min\{a_0,\dots,a_{n-1}\}-1$ the same is also true for the edges e_n and f_n, so the equality is satisfied also in this case. Statement (4) is immediate from the definition of k–prunings and the combinatorics of shuffles, and (5) is obvious.

On the contrary, (1) is more complicated and requires a more detailed proof. As a notational convention, let $d_l^h = \sum_{i:a_i=l} d_i$, the sum of the contributions to the differential coming from vertices at height k. We compare $d_l^h P_k(T)$ with $P_k d_l^h(T)$. We consider different cases depending on the difference between k and l.

- If $l > k$, d_l^h is computed by gluing together a pair (e, f) of adjacent edges of height bigger than k (and its mirror pair) and performing a shuffle at the new target vertex. These operations only change a connected subtree whose vertices all have height bigger than k, and, by construction, k–prunings commute with such operations. Hence $d_l^h P_k = P_k d_l^h$.

- If $l = k$, then we can write $d_k^h P_k(T) = \sum_{(T',T'')} \sum_{(e,f)} d_{e,f}(T' \otimes S_k(T''))$, where the sum is over symmetric k–prunings (T', T'') of T and pairs of adjacent edges (e, f) in the positive half-plane with a common source vertex of height k in T' or $S_k(T'')$. We also note that $S_k(T'')$ has a unique vertex w of height k. There is an obvious bijection

$$\bigsqcup_{\substack{v\in V(T)\\ h(v)=k}} E(v) \leftrightarrow \bigsqcup_{\substack{u\in V(T')\\ h(v)=k}} E(u) \sqcup (E(w)\setminus\{e_0\})$$

that maps an edge to its image in T' (if it is not pruned away) or in $S_k(T'')$ (if it is), and that arises from the fact that, for elementary prunings, $T = T' \cup T'' \cup r(T'')$. The

edge e_0 is the unique edge belonging to the central vertical stem whose source vertex is w. Moreover, this bijection preserves the properties of belonging to the positive and negative half-plane. Therefore, we can write the summation above expressing $d_k^h P_k(T)$ as the sum of three pieces:

- The first piece is the sum of the terms corresponding to (e, f) such that (e, f) come from adjacent edges in T. These terms correspond to symmetric k–prunings of $d^S_{e,f,\sigma}(T)$, for shuffles σ at the common vertex of e and f. Hence, their sum yields $P_k d_k^h(T)$.
- The second piece is the sum of the terms corresponding to (e, f) in $S_k(T'')$ such that $e \neq e_0$ and (e, f) do not come from adjacent vertices of T. Under this condition, the symmetric vertex permutation $\sigma(S_k(T''))$ of $S_k(T'')$ at w that switched the positions of e and f still produces a shuffle of the scraps of the elementary prunings involved in (T', T''). Every tree in $d_{(e,f)}(S_k(T''))$ cancel out with a tree in $d_{(f,e)}(\sigma(S_k(T'')))$. Hence, this second piece is 0.
- The third piece is given by the terms corresponding to (e, f) with $e = e_0$. These terms yield $(\mathrm{id} \otimes d_0) P_k(T)$.

Finally, we deduce that $d_k^h P_k(T) = (\mathrm{id} \otimes d_0) P_k(T) + P_k d_k^h(T)$.

• If $l = k-1$, $P_k d^h_{k-1}(T) = \sum_{(e,f)} \sum_{(T',T'')} T' \otimes S_k(T'')$, where the sum is taken over couples (e, f) of adjacent edges in T whose common source v has height $k-1$, and symmetric k–prunings (T', T'') of trees in $d_{(e,f)}(T)$. Let v_e and v_f be the targets of e and f, respectively. By construction, $d_{(e,f)}(T)$ glues v_e and v_f to a single vertex $\bar{v}$, such that $E(\bar{v}) = E(v_e) \sqcup E(v_f)$, suitably shuffled. Let A be the set of edges removed by the corresponding elementary symmetric prunings at $\bar{v}$ and at $r(\bar{v})$, the mirror vertex of $\bar{v}$ (which might coincide). We retrieve symmetric k–prunings for which $E(v_e) \not\subseteq A$ and $E(v_f) \not\subseteq A$ from symmetric k–prunings (T', T'') of T by applying $d_{(e,f)}$ to T'. Now assume that v is not on the central stem of the tree. If $e \neq \min(E(v))$, it is the successor of an edge $g \in E(v)$, and the terms of $P_k d_{e,f}(T)$ for which $E(v_e) \subseteq A$ cancel out with the terms of $P_k d_{g,e}(T)$ for which $E(v_e) \subseteq A$. Similarly, all the terms for which $E(v_f) \subseteq A$ and $f \neq \max(E(v))$ cancel out. The only remaining terms are those in which we remove an entire subtree corresponding to $\min(E(v))$ — which is the mirror image of $\max(E(r(v))$. If v belongs to the central axis, we must slightly modify the argument to take into account only edges in the positive half-plane and shows that the surviving terms are those in which an entire

subtree stemming from $\max(E(v))$ is removed. The sum of all these elements is exactly equal to the correcting term $(\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})P^{\min}_{k-1}(T)$. We deduce that

$$d^h_{k-1}P_k(T) = P_k d^h_{k-1}(T) + (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})P^{\min}_{k-1}(T).$$

- If $l < k-1$, since k–prunings only depend on the part of the tree above height k and d^h_l does not change it, the same argument used for $l > k$ shows that $d^h_l P_k = P_k d^h_l$.

Combining the equalities obtained in these cases yields (1). □

We are now ready to construct a cochain representative of the cohomological coproduct map $H^*(W_{B_n}) \to \bigoplus_{i=0}^n H^*(W_{B_i})\otimes H^*(W_{B_{n-i}})$.

Proposition 4.9 *Let $\Delta_k : \mathrm{FN}^*_{W_{B_n}}\otimes\mathbb{F}_2 \to \bigoplus_{i=0}^n \mathrm{FN}^*_{W_{B_i}}\otimes\mathrm{FN}^*_{W_{B_{n-i}}}\otimes\mathbb{F}_2$ be the linear maps defined recursively by the formulas*

- $\Delta_0 = P_0$,
- *for $k>0$, $\Delta_k = (\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})\Delta_{k-1}$.*

Then

(1) *the limit $\Delta = \varinjlim \Delta_k$ exists,*

(2) *Δ is a cochain map,*

(3) *Δ represents the cohomology coproduct map at the cochain level.*

Proof (1) Let $\boldsymbol{a}\in\mathrm{FN}^*_{W_{B_n}}$ and let $m = \max\{a_0,\dots,a_{n-1}\}$. Statement (2) of Lemma 4.8 guarantees that $\Delta_k(\boldsymbol{a}) = \Delta_m(\boldsymbol{a})$ for all $k > m$. Thus, the sequence $\{\Delta_k\}_{k=0}^\infty$ stabilizes and consequently has a limit.

(2) We first observe that Lemma 4.8(4) and (5) imply that

$$(\mathrm{id}\otimes C)(P^{\min}_k\otimes\mathrm{id})\Delta_k = \Delta_k - \Delta_{k-1}$$

for all $k\geq 0$, with the convention that $\Delta_{-1}(T) = T\otimes I$. Combining this remark with Lemma 4.8(3) and (5), we obtain that, for all $k\geq 1$,

$$\begin{aligned}(\mathrm{id}\otimes C)&(\mathrm{id}\otimes d_{m_{k-1}}\otimes\mathrm{id})(\mathrm{id}\otimes C\otimes\mathrm{id})(P_k\otimes\mathrm{id}\otimes\mathrm{id})(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\ &= (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(\mathrm{id}\otimes C\otimes\mathrm{id})(P_k\otimes\mathrm{id}\otimes\mathrm{id})(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\ &= (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes C)(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\ &= (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}).\end{aligned}$$

We use this to prove by induction on k that $\Delta_k d = d\Delta_k + (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1})$. For $k = 0$ this identity is the content of the first statement of Lemma 4.8. For $k > 0$, we deduce from the identity above and the previous lemma that

$$\begin{aligned}
\Delta_k d &= (\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})\Delta_{k-1}d \\
&= (\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})d\Delta_{k-1} + (\mathrm{id}\otimes C)(P_k\otimes d_0)(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})d\Delta_{k-1} + (\mathrm{id}\otimes C)d(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&\qquad + (d - d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C)d(P_k\otimes\mathrm{id})\Delta_{k-1} + (\mathrm{id}\otimes C)(\mathrm{id}\otimes d_0\otimes\mathrm{id})[(P_k - \mathrm{id}\otimes I)\otimes\mathrm{id}]\Delta_{k-1} \\
&\quad + (\mathrm{id}\otimes C)d(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) + d(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C)d(P_k\otimes\mathrm{id})\Delta_{k-2} + (\mathrm{id}\otimes d_0)(\mathrm{id}\otimes C)[(P_k - \mathrm{id}\otimes I)\otimes\mathrm{id}]\Delta_{k-1} \\
&\qquad + d\Delta_k - d(1\otimes C)(P_k\otimes\mathrm{id})\Delta_{k-2} \\
&= d\Delta_k - (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1}) + (\mathrm{id}\otimes C)d(P_k\otimes\mathrm{id})\Delta_{k-2} \\
&\qquad - d(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})\Delta_{k-2} \\
&= d\Delta_k - (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1}).
\end{aligned}$$

To justify the last equality, we observe that $(P_k\otimes\mathrm{id})\Delta_{k-2}$ is a sum of terms of the form $\boldsymbol{c}\otimes\boldsymbol{a}\otimes\boldsymbol{b}$ with $b_0 < \min\{a_i\} - 1$, and we apply the stronger clause of Lemma 4.8(3).

Now the identity $d\Delta = \Delta d$ follows by passing to the limit, and using that the sequence $\{\Delta_k\}_{k=0}^{\infty}$ stabilizes.

(3) Consider the dg-module U over $\mathbb{F}_2$ with basis given by symmetric planar level trees with antisymmetric labels in any finite subset $I \subseteq \mathbb{N}$, not necessarily $\{-n,\dots,n\}$, with the symmetric tree differential. Note that $\bigoplus_{n\geq 0}\widetilde{\mathrm{FN}}^*_{W_{B_n}}\otimes\mathbb{F}_2$ embeds in U in the obvious way. We observe that the linear maps P_k, $P_k^{\min}$ and C lift to linear maps $\widetilde{P}_k, \widetilde{P}_k^{\min}\colon U\to U\otimes U$ and $\widetilde{C}\colon U\otimes U\to U$. $\widetilde{P}_k$ and $\widetilde{P}_k^{\min}$ are still defined via prunings, but we additionally keep track of the labels of the subtrees involved. We compute $\widetilde{C}$ on $T'\otimes T''$ by splitting T'' symmetrically along the vertical axis, keeping labels, and symmetrically attach the two parts to T' to obtain a new basis element of U. Lemma 4.8 still holds for this labeled version of the morphisms by the same proof. Consequently, there is a labeled version $\widetilde{\Delta}\colon U\to U\otimes U$ of Δ, constructed recursively via finite approximations $\widetilde{\Delta}_k$, that still commutes with the differential. Note that we can also embed

$$\widetilde{\mathrm{FN}}^*_{W_{Bn}}\otimes\widetilde{\mathrm{FN}}^*_{W_{Bm}}\otimes\mathbb{F}_2$$

into $U \otimes U$ by keeping the labels of trees in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ and relabeling trees in $\widetilde{\mathrm{FN}}^*_{W_{B_m}}$ via the bijection $\{0,\dots,m-1\} \to \{n,\dots,n+m-1\}$ that raises numbers by n. There is also a projection $U \times U \to \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ that maps every basis element of $U \otimes U$ that does not belong to $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ to 0. By induction, we easily see that restricting each $\widetilde{\Delta}_k$ for all k (and, consequently, $\widetilde{\Delta}$) to $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ and composing with this projection we obtain linear maps $\bigoplus_{n\geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{n\geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \bigoplus_{n\geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2$ that are equivariant with respect to the monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$ and satisfy the same formal relation with respect to the differential. By identifying $\mathrm{FN}^*_{W_{B_n}}$ with the invariant subspace $(\widetilde{\mathrm{FN}}^*_{W_{B_n}})^{W_{B_n}}$, the limit map $\widetilde{\Delta}$ restricts to Δ, which is thus a cochain-level realization of the coproduct map. □

We now turn our attention to the transfer map. We need a preliminary definition.

Definition 4.10 (partially from [8]) Let $\boldsymbol{a} = [a_0 : \cdots : a_{n-1}] \in \mathrm{FN}^*_{W_{B_n}}$ be as defined above. In what follows, we assume, by convention, that $a_{-1} = a_n = 0$. We say that the chain $[a_i : \cdots : a_j]$ is a *k–block* of $\boldsymbol{a}$ if $a_l > k$ for all $i \leq l \leq j$ and $\max\{a_{i-1}, a_{j+1}\} \leq k$. We say that a k–block $[a_i : \cdots : a_j]$ of $\boldsymbol{a}$ is *principal* if, in addition, $\min_{0\leq r<i} a_r = k$. We denote by $\mathrm{PBl}_k(\boldsymbol{a})$ the tuple of the principal k–blocks of $\boldsymbol{a}$, ordered from left to right.

For example, the basis element $\boldsymbol{a} = [3:2:3:1:2:1:3:2:0:3]$ has four 1–blocks: $B_{1,1} = [3:2:3]$, $B_{1,2} = [2]$, $B_{1,3} = [3:2]$, and $B_{1,4} = [3]$. $\mathrm{PBl}_1(\boldsymbol{a}) = (B_{1,2}, B_{1,3})$.

Note that a basis element $\boldsymbol{a}$ is uniquely determined by $\{\mathrm{PBl}_k(\boldsymbol{a})\}_{k=0}^{\infty}$, the collection of its principal blocks. To retrieve $\boldsymbol{a}$ from these data, we can use the following procedure. First, for all $k \geq 0$, add an entry equal to k before each principal k–block and concatenate all such tuples to obtain $\boldsymbol{a}_k$. Then, we obtain $\boldsymbol{a}$ as the concatenation of $\dots, \boldsymbol{a}_k, \boldsymbol{a}_{k-1}, \dots, \boldsymbol{a}_0$. This sequence is necessarily finite because for $k > \max_{i=0}^{n-1} a_i$, $\mathrm{PBl}(\boldsymbol{a})$ is the empty 0–tuple. With this method, we can construct a basis element $\boldsymbol{a}$ from an eventually empty collection of tuples $\{B_k\}$, where the entries of B_k are tuples of natural numbers strictly bigger than k.

We also observe that k–blocks can be retrieved from the corresponding symmetric level tree T. They are given by the connected components of $T \cap \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\, y > k\}$. Interpreted this way, a k–block is principal if and only if it does not intersect the central vertical axis but is contained in the $(k-1)$–block intersecting it.

Proposition 4.11 *Given $\boldsymbol{a} \in \mathrm{FN}^*_{W_{B_n}}$, $\boldsymbol{b} \in \mathrm{FN}^*_{W_{B_m}}$ and $k \geq 0$, let $n_{a,k}$ and $n_{b,k}$ be the lengths of $\mathrm{PBl}_k(\boldsymbol{a})$ and $\mathrm{PBl}_k(\boldsymbol{b})$, respectively. Given a sequence $\underline{\sigma} = \{\sigma_k\}_{k=0}^{\infty}$ of permutations $\sigma_k \in \Sigma_{n_{a,k}+n_{b,k}}$, define $\underline{\sigma}(\boldsymbol{a},\boldsymbol{b})$ as the unique basis elements of $\mathrm{FN}^*_{W_{B_{n+m}}}$ such that, for all $k \geq 0$, $\mathrm{PBl}_k(\underline{\sigma}(\boldsymbol{a},\boldsymbol{b})) = \sigma_k(\mathrm{PBl}_k(\boldsymbol{a}), \mathrm{PBl}_k(\boldsymbol{b}))$, where $(\mathrm{PBl}_k(\boldsymbol{a}), \mathrm{PBl}_k(\boldsymbol{b}))$ is the concatenated $(n_{a,k}+n_{b,k})$–tuple and σ_k acts on $(n_{a,k}+n_{b,k})$–tuples by permuting the entries. Let $\odot\colon \mathrm{FN}^*_{W_{B_n}} \otimes \mathrm{FN}^*_{W_{B_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}^*_{W_{B_{n+m}}} \otimes \mathbb{F}_2$ be the homomorphism that maps $\boldsymbol{a} \otimes \boldsymbol{b}$ to the sum $\sum_{\underline{\sigma}} \underline{\sigma}(\boldsymbol{a},\boldsymbol{b})$ over sequences of permutations $\underline{\sigma} = \{\sigma_k\}_{k=0}^{\infty}$ such that σ_k is a $(n_{a,k}, n_{b,k})$–shuffle for all $k \geq 0$. Informally, $\boldsymbol{a} \odot \boldsymbol{b}$ is the sum of basis elements whose principal k–blocks are obtained by shuffling the principal k–blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$. This defines a morphism of complexes that induces the transfer product in cohomology.*

Proof The reflection arrangement of $W_{B_n} \times W_{B_m}$, with its product reflection action on $\mathbb{R}^n \times \mathbb{R}^m$, is $\mathcal{A}_{B_n} \times \mathcal{A}_{B_m} = \{H \times \mathbb{R}^m\}_{H \in \mathcal{A}_{B_n}} \cup \{\mathbb{R}^n \times H'\}_{H' \in \mathcal{A}_{B_m}}$. Being it a subarrangement of $\mathcal{A}_{B_{n+m}}$, we have a natural inclusion

$$Y^{(\infty)}_{W_{B_{n+m}}} \to Y^{(\infty)}_{\mathcal{A}_{B_n} \times \mathcal{A}_{B_m}} \cong Y^{(\infty)}_{W_{B_n}} \times Y^{(\infty)}_{W_{B_m}}.$$

We can explicitly obtain such inclusion by splitting a configuration of $n+m$ points into the two subconfigurations consisting of its first n points and its last m points, respectively, and relabeling the indices of the second one. This map is a $(W_{B_n} \times W_{B_m})$–equivariant homotopy equivalence.

Therefore, passing to quotients, this yields a map

$$\pi\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}} \times \frac{Y^{(\infty)}_{W_{B_m}}}{W_{B_m}}$$

that models the standard homotopy equivalence $B(W_{B_n} \times W_{B_m}) \simeq B(W_{B_n}) \times B(W_{B_m})$. Moreover, the obvious quotient map

$$\pi'\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}$$

is a covering model for $B(W_{B_n} \times W_{B_m}) \to B(W_{B_{n+m}})$.

Let $x = [a_0 : \cdots : a_{n-1}] \otimes [b_0 : \cdots : b_{m-1}]$ be a basis element for the Fox–Neuwirth complex $\mathrm{FN}^*_{B_n} \otimes \mathrm{FN}^*_{B_m} \otimes \mathbb{F}_2$. Let σ be a smooth singular simplex transverse to our strata. By construction, the evaluation of $[a_0 : \cdots : a_{n-1}] \odot [b_0 : \cdots : b_{m-1}]$ on σ is the sum of the evaluations of x on $\pi(\tilde{\sigma})$, as $\tilde{\sigma}$ varies among all liftings of σ. A direct

calculation shows that some $\pi(\tilde{\sigma})$ intersects the stratum corresponding to x if and only if σ intersects some stratum $e(\boldsymbol{c})$, where the k–principal blocks of $\boldsymbol{c}$ are obtained by shuffling the k–principal blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$. □

We conclude the treatment of the structural maps on the cohomology of W_{B_n} with some potentially helpful remarks. Since we will not use these facts in this paper, we will not provide complete statements nor proofs of these last claims. Nevertheless, it should be straightforward, although notationally heavy, to fill in the details.

Remark 4.12 (1) The transfer and coproduct maps commute already at the cochain level. To see this, you can observe that, by construction, $\Delta(\boldsymbol{a})$ is a sum of tensors $\boldsymbol{a}' \otimes \boldsymbol{a}''$, where $\mathrm{PBl}_k(\boldsymbol{a}'')$ is given by the leftovers of symmetric k–prunings of $\boldsymbol{a}$, suitably shuffled, and that the pruning map P_k itself commute with $\odot$.

(2) The same constructions of the coproduct map in terms of prunings and the transfer map in terms of principal block shuffles can be generalized to the cohomology with integral coefficients. In these cases, additional signs that we can compute from those appearing in Theorem 3.1 are required.

4.3 Structural morphisms: A_D

The coproduct and the transfer product for W_{D_n} are described geometrically, similarly to what we did for W_{B_n}. However, some complications arise. For example, we cannot repeat the proof of Proposition 4.11 as it is for $\mathrm{FN}^*_{W_{D_n}}$, because, in this case, a product of strata $S \times S' \subseteq Y^{(\infty)}_{W_{D_n}} \times Y^{(\infty)}_{W_{D_m}}$ is not necessarily the closure of a union of strata in $Y^{(\infty)}_{W_{D_{n+m}}}$. However, these ideas adapt well to the cochain complex $\mathrm{FN}'^*_{W_{D_n}}$, which we will use in the following as a cochain model. We can retrieve the identities we need in $\mathrm{FN}^*_{W_{D_n}}$ by using the equivalence φ of Lemma 4.1.

We can now state the formulas parallel to Propositions 4.9 and 4.11 for W_{D_n}. First, we consider the following oriented versions of the pruning and concatenation maps. Given a symmetric k–pruning (T', T'') of a symmetric planar level tree T, let $\mathcal{O}$ and $\mathcal{O}'$ be orientations of T and T' respectively. Fix an antisymmetric labeling λ' of T' inducing $\mathcal{O}'$, and an antisymmetric labeling λ of T inducing $\mathcal{O}$ such that its restriction to T', seen as a subtree of T, is λ'. By keeping track of the labels of scraps, λ induces an antisymmetric labeling λ'' on $S_k(T'')$ and, consequently, an orientation $\mathcal{O}''$. Unless the k–pruning is trivial, it is always possible to find such labelings λ and λ', and the resulting orientation $\mathcal{O}''$ only depends on $\mathcal{O}$ and $\mathcal{O}'$.

Definition 4.13 Let $k \in \mathbb{N}$ and let $(T, \mathcal{O})$ be an oriented symmetric planar level tree. An *oriented k–pruning* of $(T, \mathcal{O})$ is a quadruple $(T', \mathcal{O}', T'', \mathcal{O}'')$ where

- (T', T'') is a k–pruning of T,
- $\mathcal{O}'$ is an orientation of T',
- $\mathcal{O}''$ is the orientation of $S_k(T'')$ determined from $\mathcal{O}$ and $\mathcal{O}'$ via the procedure above.

An oriented k–pruning of $(T', \mathcal{O}', T'', \mathcal{O}'')$ of T is called *positive* (resp. *negative*) if $\mathcal{O}'$ is the positive (resp. negative) orientation of T'.

Given a nontrivial k–pruning (T', T'') of T, there are precisely two ways to extend it to an oriented k–pruning $(T', \mathcal{O}, T'', \mathcal{O}'')$, one positive and one negative.

We now mimic the construction we produced for W_{B_n} to describe the coproduct. We thus consider the following maps:

- the positive and negative k–pruning maps

$$P_k^+, P_k^- : \bigoplus_{n\geq 0} \mathrm{FN}'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{n\geq 0} \mathrm{FN}'^*_{W_{D_n}} \otimes \bigoplus_{m\geq 0} \mathrm{FN}'^*_{W_{D_m}} \otimes \mathbb{F}_2$$

given by the formula

$$P_k^{\pm}(T) = \sum_{(T',\mathcal{O}',T'',\mathcal{O}'')} (T', \mathcal{O}') \otimes (S_k(T''), \mathcal{O}''),$$

where the sum runs over all positive and negative oriented k–prunings of T, respectively;

- $\widehat{C} : \mathrm{FN}'^*_{W_{D_n}} \otimes \mathrm{FN}'^*_{W_{D_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}'^*_{W_{D_{n+m}}} \otimes \mathbb{F}_2$, the oriented concatenation map, given by the formulas

$$\widehat{C}((\boldsymbol{a}, +) \otimes (\boldsymbol{b}, +)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), +), \quad \widehat{C}((\boldsymbol{a}, +) \otimes (\boldsymbol{b}, -)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), -),$$
$$\widehat{C}((\boldsymbol{a}, -) \otimes (\boldsymbol{b}, +)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), -), \quad \widehat{C}((\boldsymbol{a}, -) \otimes (\boldsymbol{b}, -)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), +).$$

We can also define $\Delta_k^+, \Delta_k^- : \mathrm{FN}'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^n \mathrm{FN}'^*_{W_{D_i}} \otimes \mathrm{FN}'^*_{W_{D_{n-i}}} \otimes \mathbb{F}_2$ by the recursive formulas

- $\Delta_0^{\pm} = P_0^{\pm}$,
- $\Delta_k^{\pm} = (\mathrm{id} \otimes \widehat{C})(P_k^{\pm} \otimes \mathrm{id})\Delta_{k-1}^{\pm}$ if $k \geq 1$.

Let Δ be the direct limit $\varinjlim_k (\Delta_k^+ + \Delta_k^-)$.

Proposition 4.14 *The oriented pruning coproduct Δ is a cochain map and induces the coproduct $\Delta : A_D \to A_D \otimes A_D$ in cohomology.*

Proof It is enough to observe that, looking at the proof of Proposition 4.9, we can obtain the map $\Delta\colon \mathrm{FN}'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^{n} \mathrm{FN}'^*_{W_{D_i}} \otimes \mathrm{FN}'^*_{W_{D_{n-i}}} \otimes \mathbb{F}_2$ from

$$\widetilde{\Delta}\colon \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^{n} \widetilde{\mathrm{FN}}^*_{W_{B_i}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_{n-i}}} \otimes \mathbb{F}_2$$

by restricting to W_{D_n}–invariants. □

Proposition 4.15 *Let $\boldsymbol{a}^{\pm}$ and $\boldsymbol{b}^{\pm}$ be generic basis elements of $\mathrm{FN}'^*_{W_{D_n}}$ and $\mathrm{FN}'^*_{W_{D_m}}$ respectively, where $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) is defined by an n–tuple $\underline{a}$ (resp. an m–tuple $\underline{b}$) of nonnegative integers. Let $\odot\colon \mathrm{FN}'^*_{W_{D_n}} \otimes \mathrm{FN}'^*_{D_m} \otimes \mathbb{F}_2 \to \mathrm{FN}'^*_{W_{D_{n+m}}} \otimes \mathbb{F}_2$ be the homomorphism that maps $\boldsymbol{a}^{\pm} \otimes \boldsymbol{b}^{\pm}$ to the sum of all elements $\boldsymbol{c}^{\pm}$, such that the principal k–blocks of $\boldsymbol{c}$ are obtained by shuffling the principal k–blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$ for all $k \geq 0$, and the sign of $\boldsymbol{c}$ is deduced from the signs of $\boldsymbol{a}$ and $\boldsymbol{b}$ by applying the multiplication sign rule $(+,+) \mapsto +$, $(+,-) \mapsto -$, $(-,+) \mapsto -$, and $(-,-) \mapsto +$. This map is a morphism of complexes and induces the transfer product in cohomology.*

Proof The proof is essentially the same as that of Proposition 4.11. □

5 The almost-Hopf ring presentations

This section contains the statements of the Hopf ring presentation for A_B and the almost-Hopf ring presentation for A_D. We thus state our main theorems, whose proof will be postponed until Section 7 because we still need to develop some necessary algebraic machinery. In the first subsection, we construct our generators, providing cochain representatives and a geometric interpretation. In the second one, we explain our relations and state Theorems 5.9 and 5.15. We then apply these results to extract combinatorially accessible additive bases for A_B and A_D in Section 5.2. Finally, the last subsection is devoted to the link between all these almost-Hopf ring structures.

5.1 Generators

We define certain cohomology classes that we will later prove to generate our (almost) Hopf rings. We begin with A_B.

Definition 5.1 In $\mathrm{FN}^*_{W_{B_n}}$, the following cochains are defined for $k \geq 0$, $m > 0$, and $n > 0$:

- $\gamma_{k,m} = [0 : \underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \cdots : 0 : \underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}}]$, (m times)
- $\delta_n = [\underbrace{1:1:\cdots:1}_{n \text{ times}}]$.

A direct calculation shows that both $\gamma_{k,m}$ and δ_n have trivial differential, and thus define cohomology classes $\gamma_{k,n} \in H^{m(2^k-1)}(W_{B_{m2^k}}, \mathbb{F}_2)$ and $\delta_n \in H^n(W_{B_n}; \mathbb{F}_2)$, that we still denote, with a slight abuse of notation, by the same symbols. While the proof of this fact is entirely straightforward, we provide a proof for the sake of completeness.

Lemma 5.2 *$\gamma_{k,m}$ and δ_n are cocycles in $\mathrm{FN}^*_{W_{B_{m2^k}}} \otimes \mathbb{F}_2$ and $\mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2$, respectively.*

Proof $\gamma_{k,m}$ is represented by the symmetric planar level tree in Figure 4. We prove that $d_i(\gamma_{k,m}) = 0$ for all $0 \le i < m2^k$ by considering different cases:

- If $i \neq l2^k$ for $0 \le l < m$, the addend d_i of the differential identifies two edges adjacent in a vertex v_j for $1 \le j \le m$, and performs a vertex shuffle at the new vertex. Exactly two possible vertex shuffles yield the same tree. Hence $d_i(\gamma_{k,m}) = 0$.

- If $i = l2^k$ for some $1 \le l < m$, then $d_i(\gamma_{k,m})$ is obtained by gluing together v_l and v_{l+1} and shuffling the outgoing edges of these two vertices. Since all these shuffles yield the same tree, and there is an even number of them — precisely $\binom{2^{k+1}}{2^k}$ — we have again that $d_i(\gamma_{k,m}) = 0$.

- If $i = 0$, v_1 and its mirror vertex are glued to the central axis of the tree, and the corresponding outgoing edges are permuted with a symmetric shuffle. Again, there is an even number of them (precisely 2^{2^k}), and thus $d_0(\gamma_{k,m}) = 0$.

δ_n is represented by a symmetric planar level tree with $2n+1$ leaves and a single internal vertex of height 1. The same proof used in the second case of $\gamma_{k,m}$ shows that $d_i(\delta_n) = 0$ for all $0 \le i < n$. □

Another possible point of confusion is that the symbol $\gamma_{k,m}$ is used in [8] to indicate a class in $H^{m(2^k-1)}(\Sigma_{m2^k}; \mathbb{F}_2)$. The class we define is the image of this cohomology class of the symmetric group in $H^{m(2^k-1)}(W_{B_{m2^k}}; \mathbb{F}_2)$ via the map induced by the projection $\pi : W_{B_{m2^k}} \to \Sigma_{m2^k}$, as we will prove later (Proposition 5.26).

We can interpret all the cohomology classes that we defined above geometrically.

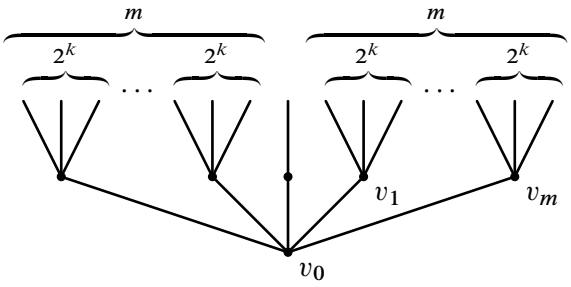

Figure 4: The planar symmetric level tree representing $\gamma_{k,m}$.

Proposition 5.3 *The following statements are true*:

(1) *Consider the proper submanifold* $\Gamma_{k,m}$ *of* $Y^{(\infty)}_{W_{B_{m2^k}}}/W_{B_{m2^k}}$ *consisting of* 2^m *points that can be partitioned into* m *sets of* 2^k *points, where all the points in the same subset share the first coordinate. Then* $\gamma_{k,m}$ *is the Thom class of* $\Gamma_{k,m}$ *in* $Y^{(\infty)}_{W_{B_{m2^k}}}$, *in the sense of* [7, Definition 4.6].

(2) *Consider the vector bundle* $\eta\colon E(W_{B_n})\times_{W_{B_n}}\mathbb{R}^n \to B(W_{B_n})$, *where* W_{B_n} *acts on* $\mathbb{R}^n$ *via its irreducible reflection representation. Then* δ_n *is the* n*–dimensional Stiefel–Whitney class of* η (*the nonoriented version of the Euler class*).

Proof The description of $\gamma_{k,n}$ is a direct consequence of the conclusions of the geometric arguments of the previous section.

Regarding the second point, consider the vector bundle $\eta\colon E(\eta)\to B(\eta)$ above, with zero section $\sigma_0\colon B(\eta)\to E(\eta)$, and let $T(\eta)\in H^n(E(\eta), E(\eta)\setminus\sigma_0(B(\eta)))$ be its Thom class. Define

$$\widetilde{X}_n = \{(\underline{x}_1,\dots,\underline{x}_n)\in Y^{(\infty)}_{B_n} \mid (x_1)_1 = \cdots = (x_n)_1 = 0\}.$$

We observe that $\widetilde{X}_n$ is a proper submanifold of $Y^{(\infty)}_{W_{B_n}}$ and that the Thom class of the image X_n of $\widetilde{X}_n$ in $Y^{(\infty)}/W_{B_n}$ is δ_n. We observe that the normal bundle of X_n in $Y^{(\infty)}/W_{B_n}$ is isomorphic to $\eta|_{X_n}$. Since restriction of vector bundles to subspaces preserve Thom classes, we deduce that, if we take $Y^{(\infty)}/W_{B_n}$ as a model for $B(W_{B_n})$, then $j^*(l^*)^{-1}k^*(T(\eta)) = \delta_n$, where

- $k\colon (E(\eta|_{X_n}), E(\eta|_{X_n})\setminus\sigma_0(X_n)) \to (E(\eta), E(\eta)\setminus\sigma_0(B(\eta)))$,
- $l\colon (E(\eta|_{X_n}), E(\eta|_{X_n})\setminus\sigma_0(X_n)) \to (B(W_{B_n}), B(W_{B_n})\setminus X_n)$ is a tubular neighborhood of X_n in $B(\eta)$, and
- $j\colon (B(W_{B_n}), \varnothing) \to (B(W_{B_n}), B(W_{B_n})\setminus X_n)$.

Note that the induced map l^* in cohomology is invertible by excision.

Let $\Phi\colon H^*(B(\eta);\mathbb{F}_2) \to H^*(E(\eta), E(\eta)\setminus\sigma_0(B(\eta));\mathbb{F}_2)$ be the Thom isomorphism. We recall that $\Phi(\alpha) = p^*(\alpha)\cup T(\eta)$, where $p\colon E(\eta)\to B(\eta)$ is the projection. We know, for instance from Milnor and Stasheff's book [13, page 91], that the Thom class $T(\eta)$ and $w_n(\eta)$ are linked by the formula

$$w_n(\eta) = \Phi^{-1}(\mathrm{Sq}^n(T(\eta))) = \Phi^{-1}(T(\eta)^2).$$

Therefore, in order to prove that $w_n(\eta)=\delta_n$, it is sufficient to show that

$$i^*(T(\eta)) = p^* j^*(l^*)^{-1}k^*(T(\eta)),$$

where i is the obvious inclusion map between pairs of spaces,

$$i : (E(\eta),\varnothing) \to \big(E(\eta), E(\eta)\setminus\sigma_0(B(\eta))\big).$$

To prove this claim, we first observe that we can use a slightly different model for $B(\eta)$. We recall that there is a tubular neighborhood $\widetilde{N}$ of $\widetilde{X}_n$ in $Y^{(\infty)}_{W_{B_n}}$ determined by an embedding of the total space of the normal bundle. Explicitly, we can define the embedding by the formula

$$(\underline{x}_1,\dots,\underline{x}_n)\times(\lambda_1,\dots,\lambda_n)\in\widetilde{X}_n\times\mathbb{R}^n \mapsto (\underline{x}_1+\lambda_1\boldsymbol{e}_1,\dots,\underline{x}_n+\lambda_n\boldsymbol{e}_1),$$

where $\boldsymbol{e}_1$ is the first element of the canonical basis of $\mathbb{R}^\infty$. Hence

$$\widetilde{N} = \big\{(\underline{x}_1,\dots,\underline{x}_n) \mid (\underline{x}_i-(x_i)_1\boldsymbol{e}_1) \neq \pm(\underline{x}_j-(x_j)_1\boldsymbol{e}_1) \text{ for all } 1\le i<j\le n,\\ (\underline{x}_i-(x_i)_1\boldsymbol{e}_1)\neq 0 \text{ for all } 1\le i\le n\big\}.$$

Note that the action of W_{B_n} preserves $\widetilde{N}$, and $\widetilde{N}$ is provided with a stratification induced from that on $Y^{(\infty)}_{W_{B_n}}$ by restriction. Further, every stratum of N is obtained from a stratum of $Y^{(\infty)}_{W_{B_n}}$ by removing an infinite-codimensional affine subspace. Thus, $\widetilde{N}\to Y^{(\infty)}_{W_{B_n}}$ is a homotopy equivalence. $\widetilde{N}$ is still contractible, and therefore we can use its quotient $N=\widetilde{N}/W_{B_n}$ as an alternative model for $B(W_{B_n})$. In this model, the inclusion l is an isomorphism. Thus we do not need to worry about excision maps, and this simplifies the argument. The claim now follows by observing that i and kjp are homotopic. An explicit homotopy is

$$((\underline{x}_1,\dots,\underline{x}_n),\underline{\lambda},t)\in\widetilde{N}\times_{W_{B_n}}\mathbb{R}^n\times[0,1]\\ \mapsto\big((\underline{x}_1-(1-t)(x_1)_1\boldsymbol{e}_1,\dots,\underline{x}_n-(1-t)(x_n)_1\boldsymbol{e}_1),((1-t)\underline{\lambda}+t((x_1)_1,\dots,(x_n)_1))\big). \quad\square$$

We now turn our attention to W_{D_n}. First, we give the following definition.

Definition 5.4 Let $n \geq 1$ and $m \geq 0$. We define $\delta^0_{n;m} \in H^n(W_{D_{n+m}}; \mathbb{F}_2)$ as the restriction of $\delta_n \odot 1_m \in H^*(W_{B_{n+m}}; \mathbb{F}_2)$ to the cohomology of $W_{D_{n+m}}$. We also let $\delta^0_{0:m}$ be the unique nonzero class in $H^0(W_{D_m}; \mathbb{F}_2)$ for all $m \geq 0$.

We will require some other generators that do not arise as restrictions of cohomology classes of W_{B_n}.

Definition 5.5 Given $k, m \geq 1$, we define two cochains in $\mathrm{FN}^{m2^k}_{W_{D_n}}$:

- $\gamma^+_{k,m} = [0 : \underbrace{\underbrace{1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \underbrace{1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \cdots : \underbrace{1:\cdots:1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$;
- $\gamma^-_{k,m} = [1 : 0 : \underbrace{\underbrace{1:\cdots:1}_{2^k-2 \text{ times}} : 0 : \underbrace{1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \cdots : \underbrace{1:\cdots:1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$.

Lemma 5.6 *$\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ are cocycles.*

Proof The cochain equivalence φ^* of Lemma 4.1 maps $\gamma^\pm_{k,m}$ to

$$[0 : \underbrace{\underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}} : 0 : \cdots : 0 : \underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}}}_{m \text{ times}}]^\pm.$$

The same proof used for Lemma 5.2, with the additional requirement of keeping track with orientations, shows that these cochains in $\mathrm{FN}'^{m2^k}_{W_{D_n}}$ are cocycles. As φ^* is injective, $\gamma^\pm_{k,m}$ must also be a cocycle.

An alternative proof can be obtained by directly using the De Concini and Salvetti description of the boundary in $C^{W_{D_n}}_*$ and dualizing. □

A consequence of the previous lemma is that $\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ represent cohomology classes, that, once again, we denote by the same symbols with a slight abuse of notation.

To adapt our notation to Giusti and Sinha's for the alternating groups, we will refer to $\gamma^+_{k,m}$ (resp. $\gamma^-_{k,m}$) for some k and m as positively (resp. negatively) charged generators, and to $\delta^0_{n:m}$ for some n and m as neutral generators.

5.2 Relations

This subsection is devoted to deriving algebraic relations between the generators defined above. We will mainly obtain the relations as a consequence of the results in Section 4.

We first focus on A_B. We can retrieve in A_B the same relations among the classes $\gamma_{k,m}$ that appear in Giusti, Salvatore and Sinha's Theorem 2.3.

Proposition 5.7 *The following formulas hold in A_B*:

$$\Delta(\gamma_{k,m}) = \sum_{i+j=m} \gamma_{k,i} \otimes \gamma_{k,j}, \quad \gamma_{k,n} \odot \gamma_{k,m} = \binom{n+m}{n} \gamma_{k,n+m}.$$

Proof We use the chain-level formulas computed in Propositions 4.9 and 4.11.

To compute the coproduct, we represent $\gamma_{k,m}$ by the symmetric planar level tree depicted in Figure 4. Note that $P_l(\gamma_{k,m})$ is trivial for $l \geq 2$ and that the 0–pruning map gives $P_0(\gamma_{k,m}) = \sum_{i+j=m} \gamma_{k,i} \otimes \gamma_{k,j}$. Therefore it is enough to prove that $P_1(\gamma_{k,n}) = \gamma_{k,n} \otimes I$, where I is the trivial symmetric level tree, for all $k \geq 0$ and $n > 0$. Consider a 1–pruning (T', T'') of $\gamma_{k,m}$. Every vertex v_i, for $1 \leq i \leq m$, as depicted in Figure 4 corresponds to a vertex u_i of height 1 in T'. Let $2^k - n_i$ be the number of outgoing edges of u_i for some integer $0 \leq n_i < 2^k$. We can obtain the pruning (T', T'') from $\gamma_{k,m}$ by applying a sequence of elementary 1–prunings at each vertex v_i and their mirror vertices $r(v_i)$ that prunes away a_i outgoing edges from v_i and $n_i - a_i$ outgoing edges from $r(v_i)$, for some $0 \leq a_i \leq n_i$. Therefore, summing over all the possible shuffles of leftovers, whose number is

$$\frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m a_i! \prod_{i=1}^m (n_i - a_i)!},$$

we deduce that $(T', S_1(T''))$ appears in $P_1(\gamma_{k,m})$ with coefficient

$$\sum_{0 \leq a_1 \leq n_1, \ldots, 0 \leq a_m \leq n_m} \frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m a_i! \prod_{i=1}^m (n_i - a_i)!} = \frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m n_i!} \prod_{i=1}^m 2^{n_i}.$$

This number is even unless $n_i = 0$ for all $1 \leq i \leq m$, yielding the trivial 1–pruning.

The transfer product formula follows directly from the application of the cochain-level map of Proposition 4.11, by observing that $\gamma_{k,m}$ has m principal 0–blocks all equal to $[1, \ldots, 1]$, where the entry 1 is repeated $2^k - 1$ times, and that it has no principal l–block for $l \geq 1$. Thus $\gamma_{k,n} \odot \gamma_{k,m}$ is given by a single basis element in $\mathrm{FN}^*_{W_{B2^k}(n+m)}$ (representing $\gamma_{k,n+m}$) counted as many times as the number of (n, m)–shuffles, that is the binomial coefficient appearing in the equation. □

We can obtain coproduct formulas for δ_n via the same geometric description. The following is again a consequence of the formulas in Lemmas 4.9 and 4.11.

Proposition 5.8 $\Delta(\delta_n) = \sum_{k+l=n} \delta_k \otimes \delta_l, \quad \delta_n \odot \delta_m = \binom{n+m}{n}\delta_{n+m}.$

Proof Since all the entries of δ_n are equal to 1, the cochain-level coproduct map on δ_n reduces to the 1–pruning map P_1 and provides the desired formula. We compute the transfer product as in Proposition 5.7, by observing that δ_n has no principal l–blocks for $l \neq 1$, and has n principal 1–blocks all empty. □

These relations will suffice to describe A_B completely. We restate here our main result, which we will prove in Section 7.

Theorem 5.9 (main theorem for type B) *The Hopf ring A_B is generated by classes $\gamma_{k,n}$ (with $k \geq 0$ and $n > 0$) and δ_n (with $n > 0$) with the relations described in Propositions 5.7 and 5.8, together with the following additional relations*:

- *the product $\cdot$ of generators from different components is* 0;
- *$\gamma_{0,n}$ is the $\cdot$–product unit of the n^{th} component.*

We now turn our attention to A_D. A trick borrowed from [9, page 9] can be used to simplify the presentation of this almost-Hopf ring. We recall that there is an involution $\iota\colon A_D \to A_D$. We can define A'_D to be the bigraded $\mathbb{F}_2$–vector space defined by $(A'_D)_{n,d} = H^d(W_{D_n}; \mathbb{F}_2)$ if $(n,d) \neq (0,0)$ and $(A'_D)_{0,0} = \mathbb{F}_2\{1^+, 1^-\}$. We can embed A_D as a vector space in A'_D by identifying the nonzero class in $H^0(W_{D_0}; \mathbb{F}_2)$ with $1^+ + 1^-$.

Lemma 5.10 *The following statements are true in A_D*:

(1) $\iota(x \odot y) = \iota(x) \odot y = x \odot \iota(y)$,

(2) $(\iota \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \iota)\Delta(x) = \Delta\iota(x)$,

(3) $\iota(x \cdot y) = \iota(x) \cdot \iota(y)$.

Proof (1) $\odot$ is commutative, and the following diagram induces a pullback of finite coverings at the level of classifying spaces:

$$\begin{array}{ccc} W_{D_n} \times W_{D_m} & \longrightarrow & W_{D_{n+m}} \\ {\scriptstyle c_{s_0}\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle c_{s_0}} \\ W_{D_n} \times W_{D_m} & \longrightarrow & W_{D_{n+m}} \end{array}$$

(2) This follows from the cocommutativity of Δ and the commutativity of the diagram above.

(3) It is the cohomological consequence of the diagonal map being equivariant with respect to the conjugation c_{s_0}. □

Proposition 5.11 *Write the coproduct of every element $x \in A_D$ in A_D as*

$$x \otimes 1_0 + \bar{\Delta}(x) + 1_0 \otimes x,$$

so $\bar{\Delta}$ is the reduced coproduct. By letting $1^- \cdot 1^+ = 0$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^+ = 1^+$, $1^- \odot 1^- = 1^+$ and $\Delta(1^\pm) = 1^+ \otimes 1^\pm + 1^- \otimes 1^\mp$, the almost-Hopf ring structure on A_D extends to an almost-Hopf ring structure on A'_D such that $1^- \odot x = \iota(x)$, $1^- \cdot x = 0$ and $\Delta(x) = 1^+ \otimes x + 1^- \otimes \iota(x) + \bar{\Delta}(x) + x \otimes 1^+ + \iota(x) \otimes 1^-$ for every $x \in A'_D$ of positive degree.

Proof Using the formulas in the statement of this proposition, we can extend $\odot$ and $\cdot$ uniquely to two commutative products on A'_D and Δ to a unique cocommutative coproduct on A'_D. The coassociativity of Δ follows from Lemma 5.10(3). The associativity of $\cdot$ on A'_D is obvious. The bialgebra structure of A'_D with $\cdot$ and Δ follows from the bialgebra structure on A_D and (2) of the previous lemma. Moreover, the fact that the transfer product with 1^- is associative follows from (1). Hopf ring distributivity with classes involving a transfer product with 1^- follows again from (3) of the result referenced above. □

Instead of determining a presentation for A_D, we calculate a presentation for A'_D because we can write it more concisely. For example, $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$ in A'_D; thus the formulas for $\gamma^-_{k,m}$ arise as a direct consequence of the formulas for $\gamma^+_{k,m}$ and the almost-Hopf ring structure of A'_D. The two approaches are equivalent.

Proposition 5.12 *Let $k, m \geq 1$ and $n \geq 0$. The following coproduct formulas hold in A'_D, where $\gamma^-_{k,m-l} = 1^- \odot \gamma^+_{k,m-l}$:*

$$\Delta(\gamma^+_{k,m}) = \sum_{l=0}^{m} \gamma^+_{k,l} \otimes \gamma^+_{k,m-l} + \gamma^-_{k,l} \otimes \gamma^-_{k,m-l},$$

$$\Delta(\delta^0_{n:m}) = \sum_{i=0}^{n} \sum_{j=0}^{m} \delta^0_{i:j} \otimes \delta^0_{n-i:m-j}.$$

Moreover, the transfer product in A'_D satisfies the following formulas for every choice of indexes:

$$\gamma^+_{k,a} \odot \gamma^+_{k,b} = \binom{a+b}{a} \gamma^+_{k,a+b},$$

$$b \odot b' = 0 \text{ if } b \text{ and } b' \text{ are cup products of neutral generators (ie } \delta^0_{n:m}),$$

$$\delta^0_{n:m} \odot 1^- = \delta^0_{n:m}.$$

Proof Note that $\gamma^-_{k,m} = 1_- \odot \gamma^+_{k,m} = \iota(\gamma^+_{k,m})$ as a direct consequence of Lemma 4.2 and the definition of the cochain representatives of these classes. The coproduct formulas for $\gamma^+_{k,m}$ follow from Lemma 4.1 and Proposition 4.14. More precisely, we observe that mapping $\gamma^+_{k,m}$ into $\mathrm{FN}'^*_{W_{Dm2^k}} \otimes \mathbb{F}_2$ via φ^* yields a cohomology class represented by the same symmetric planar level tree of Figure 4, with positive orientation. The same proof of Proposition 5.7 holds in this case by keeping track of orientations.

The coproduct formula for $\delta^0_{n:m}$ is a consequence of Proposition 5.8, the Hopf ring properties of A_B, and the fact that the restriction map $\rho\colon A_B \to A_D$ preserves coproducts.

Regarding transfer product, we prove the first identity using Proposition 4.15 precisely in the same way as the second part of Proposition 5.7.

Let $\rho\colon A_B \to A_D$ be the restriction map. For every $x \in H^*(W_{B_n}; \mathbb{F}_2)$ and $y \in H^*(W_{B_m}; \mathbb{F}_2)$, we can prove that $\rho(x) \odot \rho(y) = 0$ in $H^*(W_{D_{n+m}}; \mathbb{F}_2)$ with the same argument used in [9, Proposition 3.14]. Essentially, it is sufficient to observe that both the restriction $H^*(W_{B_n} \times W_{B_m}; \mathbb{F}_2) \to H^*(W_{D_n} \times W_{D_m}; \mathbb{F}_2)$ and the transfer $H^*(W_{D_n} \times W_{D_m}; \mathbb{F}_2) \to H^*(W_{D_{n+m}}; \mathbb{F}_2)$ factor through the cohomology of the subgroup $G = W_{D_{n+m}} \cap (W_{B_n} \times W_{B_m})$, and that the composition

$$H^*(G; \mathbb{F}_2) \xrightarrow{\mathrm{res}} H^*(W_{D_n} \times W_{D_m}; \mathbb{F}_2) \xrightarrow{\mathrm{tr}} H^*(G; \mathbb{F}_2)$$

is 0 for mod 2 coefficients because $W_{D_n} \times W_{D_m}$ has even index in G. In particular, nontrivial transfer products of blocks obtained by cup-multiplying neutral generators must be 0. The last relation also follows from the invariance of $\delta^0_{n:m}$ with respect to the involution ι. □

After these coproduct and transfer product formulas, we will also need some cup product relations. Since the Fox–Neuwirth type cell complex does not behave well with cup products, we found that it is simpler to obtain these formulas via restriction to elementary abelian subgroups. This approach is fruitful because of a detection theorem for these subgroups. We postpone the proof of the following proposition to Section 6, where we will explain this in detail.

Proposition 5.13 *Let $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$ as an element of A'_D. Then the following formulas hold in A_D:*

(1) $\gamma^+_{k,n} \cdot \gamma^-_{h,m} = 0$ *for all* $n, m, k \geq 1$ *and* $h \geq 2$;

(2) $\gamma^+_{1,m}\gamma^-_{1,m} = (\gamma^+_{1,m-1})^2 \odot \delta^0_{2:0}$ *for all* $m \geq 1$;

(3) *the* $\cdot$ *product of generators belonging to different components is* 0 *and* $\delta^0_{0:m}$ *is the* $\cdot$*–product unit of the* m^{th} *component*

(4) $\delta^0_{1:m} = 0$ *for all* $m \geq 0$;

(5) $\delta^0_{n:m} \cdot \gamma^+_{k,\frac{n+m}{2^k}} = \delta^0_{n:0} \cdot \gamma^+_{k,\frac{n}{2^k}} \odot \gamma^+_{k,\frac{m}{2^k}}$ *for all* $k > 0$ *and* $m, n \geq 0$, *where we understand that* $\gamma^+_{k,r} = 0$ *if* r *is not an integer.*

The last relation we require involves the behavior of the coproduct with the transfer product. We need a preliminary remark. Let $b \in A'_D$ be an element obtained as a cup-product of positively and neutrally charged generators (ie $\gamma^+_{k,m}$ or $\delta^0_{n:m}$), with at least one positively charged generator. Note that, by Propositions 5.12 and 5.13, $\Delta(b)$ can be written as a sum $\sum_i b'_i \otimes b''_i$ where b'_i and b''_i are elements obtained as iterated transfer products of elements of the same form, or the images of such elements via the involution $\iota = 1^- \odot _$. We let $\Delta'(b)$ be that sum restricted only to addends $b'_i \otimes b''_i$ in which the involution is not performed to obtain b'_i or b'_i is fixed by ι and the involution is not performed to obtained b''_i. As Δ is $(\iota \otimes \iota)$–invariant, this intuitively amounts to keeping half of the addends of the coproduct in A'_D.

Proposition 5.14 (cf [9, Theorem 3.21]) *Let $\tau : \alpha \otimes \beta \in A'_D \otimes A'_D \mapsto \beta \otimes \alpha \in A'_D \otimes A'_D$ be the map that exchanges the two factors. For all $b \in A'_D$ the cup-product of positively and neutrally charged generators, with at least a positively charged generator appearing, and for all $x \in A'_D$, we have that*

$$\Delta(b \odot x) = (\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta'(b) \otimes \Delta(x)),$$

where Δ' is the expression described above.

The proof of the analog of this proposition is done in [9] by a careful examination of certain spectral sequences. It can be done this way also for A_D. Still, we decide to argue here using detection by elementary abelian subgroups that for finite Coxeter groups comes for free and leads to a shorter proof. Therefore, we postpone the proof of this proposition until the next section.

We restate our presentation theorem for A'_D, whose proof we postpone.

Theorem 5.15 (main theorem for type D) *A'_D is generated, as an almost-Hopf ring, by the classes $\delta^0_{n;m}$ for $n \geq 0$ and $m \geq 0$, $\gamma^+_{k,m}$ for $k, m \geq 1$, and 1^- defined above, under the relations described in Propositions 5.12, 5.13 and 5.14 and the relations $1^- \odot 1^- = 1^+$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^- = 0$ and $\Delta(1^-) = 1^+ \otimes 1^- + 1^- \otimes 1^+$ coming from Proposition 5.11.*

5.3 Additive bases

We describe here additive bases for A_B and A_D. In this subsection, we assume that the statements of Theorems 5.9 and 5.15 are true. They do not rely logically on the existence of such bases in A_B and A_D. Thus this choice does not invalidate their proof.

We begin with A_B.

Definition 5.16 (cf [7]) A *gathered block* in A_B is an element of the form

$$b = \delta_m^{t_0} \prod_{k=1}^{n} \gamma_{k,\frac{m}{2^k}}^{t_k},$$

where m is a positive integer, 2^n divides m, and n is the maximal index such that $\gamma_{n,\frac{m}{2^n}}$ appears in b with a nonzero exponent. The *profile* of b is the $(n+1)$–tuple $(t_0, \dots, t_n)$. We also allow $n = 0$: in this case, $b = \delta_m^{t_0}$ for some $t_0 \geq 0$.

A *Hopf monomial* is a transfer product of gathered blocks $x = b_1 \odot \cdots \odot b_r$. We denote by $\mathcal{M}_B$ the set of Hopf monomials whose constituent gathered blocks have pairwise different profiles.

Note that, given a possible profile $(t_0, \dots, t_n)$, for all $l \geq 1$, there is a unique gathered block b in the $(l2^n)^{\text{th}}$ component having that profile. As a notational convention, we denote it $b_{l,\underline{t}}$.

We can describe elements of $\mathcal{M}_B$ graphically. We represent $\gamma_{k,n}$ as a rectangle of width $n2^k$ and height $1 - 2^{-k}$ and δ_n as a rectangle of width n and height 1. The width of a box is the number of the component to which the class belongs. Its area is its cohomological dimension. We understand the cup product of two generators as stacking the corresponding boxes on top of the other. In contrast, their transfer product corresponds graphically to placing them next to each other horizontally. The profile of a gathered block is described by the height of the rectangles of the corresponding column. Thus, we can represent every gathered block as a column made of boxes with the same width. Hence, an element of $\mathcal{M}_B$ is a diagram consisting of columns placed next to each other, such that there are not two columns that consist of rectangles of the

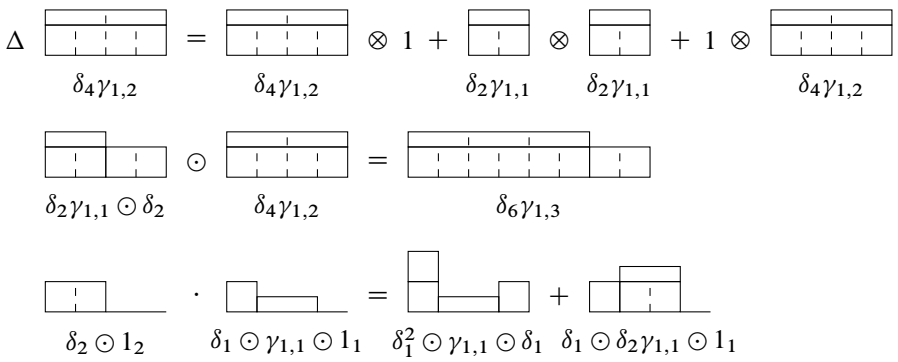

Figure 5: Computations via skyline diagrams.

same height. Following the notation of Giusti, Salvatore and Sinha [7], we call these objects *B–skyline diagrams* or, more concisely, *skyline diagrams* where it is clear that we are considering the Hopf ring A_B.

As in [7], the coproduct and the two products in $\mathcal{M}_B$ have a graphical description, derived from our relations:

- We divide rectangles corresponding to δ_n or $\gamma_{k,n}$ in n equal parts via vertical dashed lines. The coproduct is then given by dividing along all vertical lines (dashed or not) of full height and then partitioning the new columns into two to make two new skyline diagrams.

- The transfer product of two skyline diagrams is given by placing them next to each other and merging every two columns with constituent boxes of the same heights, with a coefficient of 0 if the widths of these columns share a 1 in their binary expansion.

- To compute the cup product of two diagrams, we consider all possible ways to split each into columns, along vertical lines (dashed or not) of full height. We then match columns of each in all possible ways up to automorphism and stack the resulting matched columns to build a new diagram.

We depict some examples of calculations with skyline diagrams in Figure 5.

Proposition 5.17 (cf [7, Proposition 6.4]) *$\mathcal{M}_B$ is an additive basis for A_B and Δ, and $\odot$ and $\cdot$ of basis elements are computed graphically via the algorithmic procedures described above.*

Proof We first prove the correctness of the graphical interpretation of the structural morphisms. The coproduct of a gathered block $b_{m,\underline{t}}$ with profile $(t_0, \dots, t_n)$ is of the form

$$\Delta(b_{m,\underline{t}}) = \sum_{i+j=m} b_{i,\underline{t}} \otimes b_{j,\underline{t}}.$$

We prove this formula by induction on the number of cup-product generators constituting $b_{i,\underline{t}}$: for single generators δ_m or $\gamma_{k,\frac{m}{2^k}}$ the formula appears in the set of relations for A_B, and the induction step is a consequence of the bialgebra structure formed by $\cdot$ and Δ. Thus, the graphical procedure for the calculation of the coproduct is correct on single-column skyline diagrams. As a general skyline diagram represents the transfer product of its columns, the general algorithm is justified because Δ and $\odot$ form a bialgebra.

Regarding $\odot$, the transfer product of two Hopf monomials corresponds to the horizontal juxtaposition of the corresponding skyline diagrams. Thus, we only need to justify the merging of columns. In formulas, this reads as follows. Fix a profile $\underline{t} = (t_0, \dots, t_n)$, with $t_k \geq 0$ for $0 \leq k < n$ and $t_n > 0$. Then

$$b_{i,\underline{t}} \odot b_{j,\underline{t}} = \binom{i+j}{i} b_{i+j,\underline{t}}.$$

Again, we prove this by induction on $r = t_0 + \cdots + t_n$. For $r = 1$, gathered blocks with profile $\underline{t}$ are single generators, and the formula above is exactly our transfer product relation among them. For $r > 1$, the induction step is proved by combining the coproduct formula for $\gamma_{l,(i+j)2^{n-l}}$, Hopf ring distributivity, and the fact that cup products of elements in different components is 0 to deduce that

$$\gamma_{l,(i+j)2^{n-l}}(b_{i,\underline{t}} \odot b_{j,\underline{t}}) = (b_{i,\underline{t}} \cdot \gamma_{l,i2^{n-l}}) \odot (b_{j,\underline{t}} \cdot \gamma_{l,j2^{n-l}}),$$

or the analogous formula with δ_{i+j} in place of $\gamma_{n,(i+j)2^{n-l}}$ if $n = 0$.

The $\cdot$–product algorithm above graphically encodes Hopf ring distributivity.

Finally, we prove that $\mathcal{M}_B$ is an additive basis for A_B. We consider the bigraded vector space V over $\mathbb{F}_2$ with skyline diagrams or, equivalently, $\mathcal{M}_B$ as a basis. Define linear maps $\odot, \cdot : V \otimes V \to V$ and $\Delta\colon V \to V \otimes V$ by computing their values on basis elements via the algorithm above. Note that these maps define a Hopf ring structure on V. There is a map $V \to A_B$ that realizes every Hopf monomial as the corresponding element of A_B. Since the procedures to compute the structural morphisms on $\mathcal{M}_B$ are deduced from the Hopf ring structure of A_B and the relations of Theorem 5.9, this map is a morphism of Hopf rings. We also note that V is generated as a Hopf ring by

single rectangles, corresponding to $\gamma_{k,n}$ and δ_n, and that the relations of Theorem 5.9 are satisfied in V. Since A_B is presented by such generators and relations, it follows that the map $V \to A_B$ is an isomorphism. □

We now construct an additive basis for A_D, assuming Theorem 5.15. The first step is to identify the subalgebra of A_D under the cup product generated by neutral generators. Let $\widetilde{\mathcal{B}}^0$ be the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$, ordered with $a_1 > \cdots > a_r$ and $k_1 \geq 2$. These correspond to skyline diagrams in which only boxes of height 1 appear and in which the highest column has width strictly bigger than 1.

Lemma 5.18 *Every element of $\widetilde{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2)$ lies in the cup product subalgebra generated by $\delta_n, \delta_{n-1} \odot 1_1, \ldots, \delta_1 \odot 1_{n-1}$. Moreover, the images in A_D of elements of $\widetilde{\mathcal{B}}^0$ are a vector space basis for the subalmost-Hopf ring generated by elements of the form $\delta^0_{n:m}$ for $n, m \geq 0$.*

Proof Let $\overline{\mathcal{B}}^0$ be the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$ ordered with $a_1 > \cdots > a_r$, without the condition $k_1 \geq 2$. We can define an injective function $\varepsilon_n \colon \overline{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2) \to \mathbb{N}^n$ given by

$$\varepsilon_n(\delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}) = (\underbrace{a_1, \ldots, a_1}_{k_1 \text{ times}}, a_2, \ldots, a_{r-1}, \underbrace{a_r, \ldots, a_r}_{k_r \text{ times}})$$

By identifying $\overline{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2)$ with a subset of $\mathbb{N}^n$ this way, the lexicographic ordering on $\mathbb{N}^n$ induces a total order on $\overline{\mathcal{B}}^0$. We observe that $\prod_{i=1}^n (\delta_i \odot 1_{n-i})^{a_i}$ is a linear combination of elements of $\overline{\mathcal{B}}^0$. In this linear combination, the maximal nonzero Hopf monomial corresponds to $\left(\sum_{i=1}^n a_i, \sum_{i=2}^n a_i, \ldots, a_{n-1} + a_n, a_n\right)$. Moreover, this belongs to $\widetilde{\mathcal{B}}^0$ if and only if $a_1 = 0$, ie if and only if $\delta_1 \odot 1_{n-1}$ does not appear as a factor. Since these are all different, $\delta_n, \delta_{n-1} \odot 1_1, \ldots, \delta_1 \odot 1_{n-1}$ generate, under the cup product, a polynomial subalgebra with basis $\overline{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2)$. By Proposition 5.13, the kernel of the restriction map to $H^*(W_{D_n}; \mathbb{F}_2)$ on this subalgebra is the ideal generated by $\delta_1 \odot 1_{n-1}$. Consequently, the images of elements of $\widetilde{\mathcal{B}}^0$ in A'_D are a basis for the cup product subalgebra generated by the elements $\delta^0_{n:m}$. Since the transfer products of these elements are trivial and this subalgebra is closed under coproduct by Proposition 5.12, this is a subalmost-Hopf ring. □

Definition 5.19 We call a *neutral gathered block* in A_D an element $b \in A'_D$ obtained as the image in A'_D of an element of the set $\widetilde{\mathcal{B}}^0$ considered in the previous lemma. A *positively charged gathered block*, or simply *positive gathered block*, is an element of

the form $b = (\delta^0_{2^n m:0})^{t_0} \prod_{i=1}^n (\gamma^+_{k,m2^{n-k}})^{t_k}$, for some $n, m \geq 1$, $t_k \geq 0$ for $0 \leq k < n$ and $t_n > 0$. The *profile* of b is $(t_0, \ldots, t_n)$. A *negatively charged gathered block*, or simply *negative gathered block*, is an element of the form $b = (\delta^0_{2^n m:0})^{t_0} \prod_{i=1}^n (\gamma^+_{k,m2^{n-k}})^{t_k}$, for some $n, m \geq 1$, $t_k \geq 0$ for $0 \leq k < n$ and $t_n > 0$. The *profile* of b is $(t_0, \ldots, t_n)$. A *Hopf monomial* is a transfer product of gathered blocks.

Note that, given a possible profile $\underline{t} = (t_0, \ldots, t_n)$, for all $l \geq 1$, there is a unique positively (resp. negatively) charged gathered block in the $(l2^n)^{\text{th}}$ component having that profile. As a notational convention, we denote it by $b^+_{l,\underline{t}}$ (resp. $b^-_{l,\underline{t}}$). Moreover, we stress that we require that a positively charged generator and a negatively charged one do not appear in the same gathered block. This is not a restriction since, due to Proposition 5.13, a cup product of two such generators is 0, or we can write it as a transfer product of gathered blocks. Therefore Hopf monomials generate A'_D as an $\mathbb{F}_2$–vector space.

We also define a filtration of A'_D that we will use to extract an additive basis from this set of (linear) generators.

Definition 5.20 Define the *weight* of a neutral gathered block b as $\mathrm{w}(b) = 0$. Define the weight of a positively or negatively charged gathered block $b^\pm_{l,\underline{t}}$, with profile $\underline{t} = (t_0, \ldots, t_n)$, $n \geq 1$, as $\mathrm{w}(b^\pm_{l,\underline{t}}) = l2^{n-1}t_1$. Define the weight of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ as the sum $\mathrm{w}(x) = \mathrm{w}(b_1) + \cdots + \mathrm{w}(b_r)$ of the weights of its constituent gathered blocks. Define the *weight filtration* as the increasing filtration $F(A'_D) = \{F_n(A'_D)\}_{n=0}^\infty$ of A'_D such that $F_n(A'_D)$ is the linear span of Hopf monomials in A'_D of weight at most n.

We first compute formulas for the coproduct and transfer product of gathered blocks in A'_D. These are essentially the charged versions of the corresponding identities in A_B, except for gathered blocks involving the generators $\gamma^\pm_{1,n}$, for which this is true only in the graded space $\mathrm{gr}_F(A'_D)$ associated with the weight filtration. Complete formulas in A'_D are complicated and can be retrieved recursively on the filtration F.

Lemma 5.21 *Let $n \geq 1$. Let $\underline{t} = (t_0, \ldots, t_n)$ with $t_k \geq 0$ for all $0 \leq k < n$ and $t_n > 0$. In any almost-Hopf ring satisfying the relations of Theorem 5.15 the following statements are true for all $i, j > 0$:*

(1) $1^- \odot b^+_{i,\underline{t}} = b^-_{i,\underline{t}}$ *and* $1^- \odot b^-_{i,\underline{t}} = b^+_{i,\underline{t}}$;

(2) *if $n > 1$, the coproduct satisfies* $\Delta(b^+_{m,\underline{t}}) = \sum_{i+j=n} (b^+_{i,\underline{t}} \otimes b^+_{j,\underline{t}} + b^-_{i,\underline{t}} \otimes b^-_{j,\underline{t}})$ *and* $\Delta(b^-_{m,\underline{t}}) = \sum_{i+j=n} (b^+_{i,\underline{t}} \otimes b^-_{j,\underline{t}} + b^-_{i,\underline{t}} \otimes b^+_{j,\underline{t}})$;

(3) *if* $n > 1$, $\odot$ *satisfies* $b^+_{i,\underline{t}} \odot b^+_{j,\underline{t}} = b^-_{i,\underline{t}} \odot b^-_{j,\underline{t}} = \binom{i+j}{i} b^+_{i+j,\underline{t}}$ *and* $b^+_{i,\underline{t}} \odot b^-_{j,\underline{t}} = b^-_{i,\underline{t}} \odot b^+_{j,\underline{t}} = \binom{i+j}{i} b^-_{i+j,\underline{t}}$;

(4) *for all neutral gathered block* b^0, $b^+_{i,\underline{t}} \odot b^0 = b^-_{i,\underline{t}} \odot b^0$;

(5) *for all profiles* $\underline{u}$, *possibly different from* $\underline{t}$, $b^+_{i,\underline{t}} \odot b^+_{j,\underline{u}} = b^-_{i,\underline{t}} \odot b^-_{j,\underline{u}}$, $b^+_{i,\underline{t}} \odot b^-_{j,\underline{u}} = b^-_{i,\underline{t}} \odot b^+_{j,\underline{u}}$;

(6) *if* $n > 1$, *for all profiles* $\underline{u}$, $b^+_{i,\underline{t}} \cdot b^-_{j,\underline{u}} = b^-_{i,\underline{t}} \cdot b^+_{j,\underline{u}} = 0$;

(7) *for all neutral gathered block* b', $b^-_{i,\underline{t}} \cdot b' = 1^- \odot (b^+_{i,\underline{t}} \cdot b')$.

Moreover, (2), (3) *and* (6) *are true in* $\mathrm{gr}_F(A'_D)$ *even if* $n = 1$.

Proof (1) Recall that, by definition, $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$. Combining the link between transfer product and coproduct provided by Proposition 5.14 with the coproduct formula for 1^- and $\gamma^+_{k,m}$, we deduce that

$$\Delta(\gamma^-_{k,m}) = \sum_{i+j=m} (\gamma^+_{k,i} \otimes \gamma^-_{k,j} + \gamma^-_{k,i} \otimes \gamma^+_{k,j}),$$

with the convention that $\gamma^\pm_{k,0} = 1^\pm$. Then, we can prove that $1^- \odot b^+_{i,\underline{t}} = b^-_{i,\underline{t}}$ by induction on the number of cup-product factors of the involved gathered block. If $b^+_{i,\underline{t}}$ is a single generator $\gamma^+_{k,m}$, the statement holds by definition. The induction step

$$(b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}}) = (1^- \odot b^+_{i,\underline{t}}) \cdot \gamma^-_{k,\frac{i2^n}{2^k}} = 1^- \odot (b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}})$$

is deduced from Hopf ring distributivity and the coproduct formula derived above for $\gamma^-_{k,m}$, using that $1^- \cdot 1^- = 1^-$, $1^- \cdot 1^+ = 0$ and that the cup product of elements in different components is zero. The statement for negatively charged gathered blocks is obtained from its analog for positively charged ones by taking the transfer product of both members of the identity with 1^- and using the relation $1^- \odot 1^- = 1^+$.

(2) We begin with the case of positively charged gathered blocks $b^+_{m,\underline{t}}$. We proceed, again, by induction on the number of $\cdot$ generators appearing in the expression of $b^+_{m,\underline{t}}$. If $b^+_{m,\underline{t}}$ is a single generator, then the statement holds by the coproduct identities of Proposition 5.12. The induction step follows from the fact that $\cdot$ and Δ form a bialgebra, and relations 1,2,3 of Proposition 5.13. For instance, for $k \geq 2$, we explicitly have

$$\Delta(b^+_{m,\underline{t}} \cdot \gamma^+_{k,\frac{m2^n}{2^k}}) = (\cdot \otimes \cdot)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \Bigg[\sum_{i+j=m} (b^+_{i,\underline{t}} \otimes b^+_{j,\underline{t}} + b^-_{i,\underline{t}} \otimes b^-_{j,\underline{t}}) \otimes \sum_{r+s=\frac{m2^n}{2^k}} (\gamma^+_{k,r} \otimes \gamma^+_{k,s} + \gamma^-_{k,r} \otimes \gamma^-_{k,s}) \Bigg]$$

$$\begin{aligned}
&= \sum_{i+j=m} \big(b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}} \\
&\qquad + b^+_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}}\big) \\
&= \sum_{i+j=m} \big(b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}}\big).
\end{aligned}$$

We only need to be careful when $k_1 > 1$ because $\gamma^+_{1,l}\gamma^-_{1,l}$ is not necessarily 0. Note that for $k \geq 2$ we have by Hopf ring distributivity

$$\gamma^\pm_{k,r}\gamma^+_{1,2^{k-1}r}\gamma^-_{1,2^{k-1}r} = \gamma^\pm_{k,r}((\gamma^+_{1,2^{k-1}r-1})^2 \odot \delta^0_{2:0}) = 0,$$

because the coproduct of $\gamma^\pm_{k,r}$ does not have an addend $x' \otimes x''$ with the component of x'' equal to 2. This observation guarantees that, if $n > 1$, the mixed-charge terms vanish. Even if $n = 1$, we obtain the additional terms by applying relation 2 of Proposition 5.13 to expressions of this form, and this procedure lowers weights. Thus, the desired formula holds in $\mathrm{gr}_F(A'_D)$ in this case.

The formulas for negatively charged gathered blocks are, once again, obtained by applying the transfer product with 1^-.

(3) The formula is easily deduced from the coproduct formulas (2) by induction on the number of $\cdot$–product generators appearing in $b^\pm_{i,\underline{t}}$. In the case $n = 1$, we use the obvious fact that $\odot$ preserves the weight filtration to deduce that the desired formula holds in the graded space.

(4) This is a combination of (1) and the relations $1^- \odot \delta^0_{n:m} = \delta^0_{n:m}$.

(5) This is a combination of (1) and the relations $1^\pm \odot 1^\pm = 1^+$ and $1^\pm \odot 1^\mp = 1^-$.

(6) If $n > 1$, it follows directly from relation (1) of Proposition 5.13. If $n = 1$, assume that $b^+_{i,\underline{t}} \in F_a$ and $b^-_{j,\underline{u}} \in F_b$. Relation (2) of Proposition 5.13 provides a way to write $b^+_{i,\underline{t}} \cdot b^-_{j,\underline{u}}$ as a product of the form $((\gamma^+_{1,l-1})^2 \odot \delta^0_{2:0}) \cdot b^+_{i,\underline{t}'} \cdot b^-_{j,\underline{u}'}$ for some $l \geq 1$, where $b^+_{i,\underline{t}'} \in F_{a-l}$ and $b_{i,\underline{u}'} \in F_{b-l}$. By relation (5) of the same proposition, these $\cdot$ products preserve the weight filtration. Therefore the statement is true in $\mathrm{gr}_F(A'_D)$.

(7) We argue as we did for (1), combining the formula given in (1) with the relation $1^- \odot \delta_{n:m} = \delta_{n:m}$, which implies that neutral gathered blocks are invariant by the action of $1^- \odot _$. □

Using this lemma, we can use Hopf monomials in the additive basis for A_B to construct basis elements of A_D by adding charges.

Proposition 5.22 *Let $\mathcal{M}_{\mathcal{B}}$ be the Hopf monomial basis for $A_{\mathcal{B}}$ of Proposition 5.17. Let $\widetilde{\mathcal{B}}^0 \subseteq \mathcal{M}_{\mathcal{B}}$ be as in Lemma 5.18. Let $\widetilde{\mathcal{B}}^c$ be the subset of $\mathcal{M}_{\mathcal{B}}$ consisting of nontrivial Hopf monomials in which every constituent gathered block has profile $\underline{t} = (t_0, \dots, t_n)$ with $n \geq 1$. Let $\mathcal{M}_D = \mathcal{B}^0 \sqcup \mathcal{B}^+ \sqcup \mathcal{B}^- \subseteq A'_D$, where*

- $\mathcal{B}^+ = \{x^+ = \bigodot_{i=1}^k b^+_{l_i,\underline{t}_i}\}_{x=\bigodot_{i=1}^k b_{l_i,\underline{t}_i} \in \widetilde{\mathcal{B}}^c} \cup \{1^+\}$,
- $\mathcal{B}^- = \{x^- = \bigodot_{i=1}^{k-1} b^+_{l_i,\underline{t}_i} \odot b^-_{l_k,\underline{t}_k}\}_{x=\bigodot_{i=1}^k b_{l_i,\underline{t}_i} \in \widetilde{\mathcal{B}}^c} \cup \{1^-\}$,
- $\mathcal{B}^0 = \{x^0 = \rho(y) \odot z^+\}_{x=y\odot z, y\in\widetilde{\mathcal{B}}^0\setminus\{1_0\}, z\in\widetilde{\mathcal{B}}^c\cup\{1_0\}}$.

Then $\mathcal{M}_D$ is an additive basis for A'_D.

Before providing a proof of this statement, we make a remark that clarifies the cumbersome identity of Proposition 5.14.

Remark 5.23 Proposition 5.22 provides a direct sum decomposition of A'_D as an $\mathbb{F}_2$–vector space with three addends, V^+, V^- and V^0, defined as the linear span of $\mathcal{B}^+$, $\mathcal{B}^-$ and $\mathcal{B}^0$, respectively. Note that the involution $\iota = 1^- \odot _$ switches V^+ and V^- and fixes all elements of V^0 by Lemma 5.21. We can consider the linear projection $\pi : V \to V^+$ defined as the identity on V^+ and as 0 on V^- and V^0. With this notation, we can rewrite Proposition 5.14 as

$$\Delta(b \odot x) = (\odot \otimes \odot)(\pi \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)(b \otimes x)$$

for all $x \in A'_D$ and b charged gathered block.

A further reduction can be performed. We can consider the free $\mathbb{F}_4$–module $\widetilde{V}$ with basis $\mathcal{M}_D$ and define a $\mathbb{Z}$–linear map $\widetilde{\Delta} : A'_D \otimes A'_D \to \widetilde{V}$ as follows. Given $x, y \in \mathcal{M}_D$, first compute the expansions of the coproducts $\Delta(x) = \sum_i x'_i \otimes x''_i$ and $\Delta(y) = \sum_j y'_j \otimes y''_j$ on the basis $\mathcal{M}_D \otimes \mathcal{M}_D$ of $A'_D \otimes A'_D$. Then, let

$$\widetilde{\Delta}(x \otimes y) = \sum_i \sum_j (x'_i \odot y'_j) \otimes (x''_i \odot y''_j),$$

where, this time, the sum is computed in $\widetilde{V}$. Recall that both $\odot$ and Δ are $(\iota\otimes\iota)$–invariant. Hence, each addend appears twice, except possibly the elements of the form $(x'_i \odot y'_j) \otimes (x''_i \odot y''_j)$, where x'_i, x''_i, y'_j and y''_j are all fixed by ι. But this implies that these classes belong to $\mathcal{B}^0$, and thus their transfer product is zero. Consequently, such addends do not appear in the summation. This implies that $\widetilde{\Delta}(x \otimes y)$ is killed by the multiplication by 2, and thus $\widetilde{\Delta}$ extends linearly to a map as desired. The image of $\widetilde{\Delta}$ is contained in the image of the embedding $\xi : A'_D \hookrightarrow \widetilde{V}$ that maps every $x \in \mathcal{M}_D$ to

$2x \in \widetilde{V}$. We can rephrase Proposition 5.14 by saying that

$$(\Delta\odot)\colon x \otimes y \in A'_D \otimes A'_D \mapsto \Delta(x \odot y) \in A'_D$$

is the unique linear map satisfying $\xi \circ (\Delta\odot) = \widetilde{\Delta}$. We immediately see that this statement is equivalent to the formulation above when x or y is a charged gathered block. If both x and y belong to $\mathcal{B}^0$, then $\widetilde{\Delta}(x \otimes y) = 0$ because the transfer product of two neutral gathered blocks is always zero. The general case follows by induction on the number of $\odot$–factors in the Hopf monomials involved.

Proof of Proposition 5.22 We can write every element in an almost-Hopf ring with generators $\gamma^+_{k,n}$, 1^- and $\delta^0_{n:m}$ satisfying the relations of Theorem 5.15 as a linear combination of addends in $\mathcal{M}_D$ due to Lemmas 5.18 and 5.21. Therefore $\mathcal{M}_D$ is a set of linear generators for A'_D.

The fact that Hopf monomials in $\mathcal{M}_D$ are linearly independent is a byproduct of the proof of Theorem 5.15. It is nevertheless possible to provide a fully independent proof that a basis for the almost-Hopf ring with the presentation of Theorem 5.15 has an additive basis given by $\mathcal{M}_D$, but we will not provide it, as it would be uselessly long. □

5.4 Comparison between A_Σ, A_B and A_D

In this subsection, we compute the action of the connecting homomorphisms on the elements of the additive bases determined in the previous subsection.

We first start with the link between A_Σ and A_B. We recall that there are a natural injection $j : \Sigma_n \to W_{B_n}$ and a natural projection $\pi : W_{B_n} \to \Sigma_n$, providing linear maps linking A_B and A_Σ. We begin by analyzing the relationship between A_Σ and A_B.

Proposition 5.24 *Let $j : \Sigma_n \to W_{B_n}$ and $\pi : W_{B_n} \to \Sigma_n$ be the natural homomorphisms. The induced maps $j^*\colon A_B \to A_\Sigma$ and $\pi^*\colon A_\Sigma \to A_B$ are Hopf-ring homomorphisms.*

Proof It is obvious from the fact that the diagrams

$$\begin{array}{ccc}
\dfrac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_n \times \Sigma_m} & \xrightarrow{j} & \dfrac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \\
\Big\downarrow p & & \Big\downarrow p \\
\dfrac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_{n+m}} & \xrightarrow{j} & \dfrac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}
\end{array}
\qquad
\begin{array}{ccc}
\dfrac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{(\mathbb{F}_2 \wr \Sigma_n) \times (\mathbb{F}_2 \wr \Sigma_m)} & \xrightarrow{\pi} & \dfrac{E(\Sigma_{n+m})}{\Sigma_n \times \Sigma_m} \\
\Big\downarrow p & & \Big\downarrow p \\
\dfrac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{\mathbb{F}_2 \wr \Sigma_{n+m}} & \xrightarrow{\pi} & \dfrac{E(\Sigma_{n+m})}{\Sigma_{n+m}}
\end{array}$$

are pullbacks of finite coverings, where p indicates covering maps. □

The following proposition is a direct consequence of Corollary 4.6 and Proposition 5.24.

Proposition 5.25 *With reference to the notation of Theorem 2.3, $j^*(\gamma_{k,n}) = \gamma_{k,n}$ and $j^*(\delta_n) = 0$. More generally, given a B–skyline diagram $x \in \mathcal{M}_B$, $j^*(x)$ is zero if x contains a rectangle of height 1. Otherwise, it is obtained by interpreting x as a skyline diagram in A_Σ.*

We can now use our algebraic description to compute the action of π^* on generators.

Proposition 5.26 *$\pi^*(\gamma_{k,n}) = \gamma_{k,n}$. For a skyline diagram $x \in A_\Sigma$, $\pi^*(x)$ is obtained by interpreting x as a B–skyline diagram without rectangles of height 1.*

Proof We proceed by induction on n. If $n = 1$, since $\pi \circ j = \mathrm{id}$, π^* is injective. Hence $\pi^*(\gamma_{k,1})$ is a nonzero class in $H^{2^k-1}(W_{B_{2^k}}; \mathbb{F}_2)$. Thanks to Proposition 5.24, $\pi^*(\gamma_{k,1})$ is primitive. From our description of A_B in terms of skyline diagrams, formalized with the statement of Proposition 5.17, we see that the only nontrivial primitive of A_B in the right component and cohomological degree is $\gamma_{k,1}$. For $n > 1$, Proposition 5.24 guarantees that π^* preserves coproducts. Hence we inductively have that $\pi^*(\gamma_{k,n}) + \gamma_{k,n}$ is primitive. However, there are no nonzero primitive in that bidegree, thus $\pi^*(\gamma_{k,n}) = \gamma_{k,n}$. □

We now turn to A_D. There is a restriction map $\rho\colon A_B \to A_D$ induced by the inclusions $W_{D_n} \hookrightarrow W_{B_n}$. Moreover, we recall that we have natural injections $i_+, i_-\colon \Sigma_n \to W_{D_n}$ determining maps $A_D \to A_\Sigma$ and an involution $\iota\colon A_D \to A_D$ induced on $H^*(W_{D_n}; \mathbb{F}_2)$ by the conjugation with $s_0 \in W_{B_n}$. We analyzed these maps in Section 4.1.

First, we explain the relation between $\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ and the natural maps between W_{D_n}, W_{B_n} and Σ_n.

Proposition 5.27 *For all $n, k \geq 1$ and $m \geq 0$,*

$$\begin{aligned} &i_+^*(\gamma^+_{k,n}) = \gamma_{k,n}, \quad i_+^*(\gamma^-_{k,n}) = 0, \\ &i_-^*(\gamma^-_{k,n}) = \gamma_{k,n}, \quad i_-^*(\gamma^+_{k,n}) = 0, \end{aligned} \qquad i_+^*(\delta^0_{n:m}) = i_-^*(\delta^0_{n:m}) = 0.$$

More generally, with reference to Proposition 5.22, i_+^* (resp. i_-^*) is zero on all neutral or negatively (resp. neutral or positively) charged Hopf monomials. We obtain the value of positively (resp. negatively) charged Hopf monomials under i_+^* (resp. i_-^*) by forgetting the charge to get a Hopf monomial in $\mathcal{M}_B$ and then applying j^* as described in Proposition 5.25.

Proof The formulas involving $\gamma_{k,n}^{\pm}$ are a direct consequence of Corollary 4.6 and the form of the chain representative of $\gamma_{k,n} \in \mathrm{FN}^*_{\Sigma_n} \otimes \mathbb{F}_2$ retrieved in [8, Definition 4.9]. To deduce that $i_+^*(\delta^0_{n:m}) = 0$, we recall that $\delta^0_{n:m} = \rho(\delta_n \odot 1_m)$ and that the composition $\Sigma_n \xhookrightarrow{i_+} W_{D_n} \hookrightarrow W_{B_n}$ is equal to j. By Proposition 5.25 $j^*(\delta_n \odot 1_m) = 0$; therefore $i_+^*(\delta^0_{n:m}) = 0$. The same is also true for $i_-^*(\delta^0_{n:m})$ because i_- is obtained by composing i_+ with the conjugation with an element of W_{B_n}, whose action is trivial on elements coming from A_B. □

Since we identify the involution ι with the transfer product with 1^-, the following proposition is essentially a restatement of the description of the previous subsection.

Proposition 5.28 *If x^0 is a neutral Hopf monomial in $\mathcal{M}_D$, then $\iota(x^0) = x^0$. If $x^{\pm}$ is a charged Hopf monomial in $\mathcal{M}_D$, we get $\iota(x^{\pm})$ by inverting the charge.*

To complete the description of the homomorphisms connecting our structures, we need to compute the restriction $\rho\colon A_B \to A_D$ and transfer $\mathrm{tr}\colon A_D \to A_B$ maps. To do this, we need to establish preliminary identities.

Lemma 5.29 *For all $x, x' \in A_D$ and for all $y \in A_B$, the following identities are satisfied*:

(1) $\rho(\mathrm{tr}(x) \odot y) = x \odot \rho(y)$,

(2) $\mathrm{tr}(x) \cdot \mathrm{tr}(x') = \mathrm{tr}(x \cdot x' + \iota(x) \cdot x')$,

(3) $\mathrm{tr}(x \cdot \rho(y)) = \mathrm{tr}(x) \cdot y$,

(4) $\mathrm{tr}(x \odot x') = \mathrm{tr}(x) \odot \mathrm{tr}(x')$.

Proof The first statement follows from the fact that this commutative diagram induces a pullback of covering spaces at the level of classifying spaces:

$$\begin{array}{ccc} W_{D_n} \times W_{D_m} & \longrightarrow & W_{D_n} \times W_{B_m} \\ \downarrow & & \downarrow \\ W_{D_{n+m}} & \longrightarrow & W_{B_{n+m}} \end{array}$$

Regarding the second statement, since the conjugation by s_0 is an endomorphism of the covering $B(W_{D_n} \times W_{D_n}) \to B(W_{D_{2n}} \cap (W_{B_n} \times W_{B_n}))$,

$$\mathrm{tr}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}_{W_{D_n} \times W_{D_n}} c_{s_0}^* = c_{s_0}^* \, \mathrm{tr}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}_{W_{D_n} \times W_{D_n}}.$$

Moreover, the classifying space functor applied to the following square produces a diagram homotopy equivalent to a pullback of covering, where d and d' are diagonal maps:

$$\begin{array}{ccc} W_{D_n} & \xrightarrow{\ d\ } & W_{D_n} \times W_{D_n} \\ {\scriptstyle j}\downarrow & & \downarrow \\ W_{B_n} & \xrightarrow[\ d'\ ]{} & W_{D_{2n}} \cap (W_{B_n} \times W_{B_n}) \end{array}$$

Hence $\operatorname{tr} d^* = d'^* \operatorname{tr}_{W_{D_n}\times W_{D_n}}^{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}$. These facts imply that, denoting by d the diagonal subgroups,

$$\begin{aligned}
&\operatorname{tr}(x)\cdot\operatorname{tr}(x') \\
&= \rho_{d(W_{B_n})}^{W_{B_n}\times W_{B_n}} \operatorname{tr}_{W_{D_n}\times W_{D_n}}^{W_{B_n}\times W_{B_n}}(x\otimes x') \\
&= \rho_{d(W_{B_n})}^{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})} \rho_{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}^{W_{B_n}\times W_{B_n}} \operatorname{tr}_{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}^{W_{B_n}\times W_{B_n}} \operatorname{tr}_{W_{D_n}\times W_{D_n}}^{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}(x\otimes x') \\
&= d'^*(\mathrm{id}+c_{s_0}^*) \operatorname{tr}_{W_{D_n}\times W_{D_n}}^{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}(x\otimes x') \\
&= d'^* \operatorname{tr}_{W_{D_n}\times W_{D_n}}^{W_{D_{2n}}\cap(W_{B_n}\times W_{B_n})}(\mathrm{id}+c_{s_0}^*)(x\otimes x') \\
&= \operatorname{tr} d^*(\mathrm{id}+c_{s_0}^*)(x\otimes x') \\
&= \operatorname{tr}(x\cdot x' + \iota(x)\cdot x').
\end{aligned}$$

Similarly, the last two statements follow from the diagrams below, where the vertical maps of the first one are the diagonal morphisms:

$$\begin{array}{ccc} W_{D_n} & \longrightarrow & W_{B_n} \\ \downarrow & & \downarrow \\ W_{D_n}\times W_{B_n} & \longrightarrow & W_{B_n}\times W_{B_n} \end{array} \qquad \begin{array}{ccc} W_{D_n}\times W_{D_m} & \longrightarrow & W_{D_{n+m}} \\ \downarrow & & \downarrow \\ W_{B_n}\times W_{B_m} & \longrightarrow & W_{B_{n+m}} \end{array} \qquad \square$$

Proposition 5.30 *The transfer map* $\operatorname{tr}\colon A_D \to A_B$ *is such that* $\operatorname{tr}(\gamma_{k,n}^{\pm}) = \gamma_{k,n}$ *and* $\operatorname{tr}(\delta_{n:m}^0) = 0$. *More generally, if* $b_{l,\underline{t}}^{\pm}$ *is a charged gathered block with profile* $\underline{t} = (t_0, t_1, \dots, t_n)$, *then* $\operatorname{tr}(b_{l,\underline{t}}^{\pm}) = b_{l,\underline{t}}$ *if* $n \geq 2$, *while if* $n = 1$ *the transfer of gathered blocks is computed inductively by the formula*

$$b_{l,\underline{t}} = \sum_{a=0}^{\lfloor t_1/2 \rfloor} \operatorname{tr}(b_{l,(t_0,t_1-a)}^{\pm}(\gamma_{1,l}^{\mp})^a).$$

The transfer of every neutral gathered block is 0, *and we realize the transfer of a Hopf monomial as the transfer product of the transfer of its constituent gathered blocks.*

Proof The statement for generators is a direct consequence of their definition at the cochain level. The general claim for Hopf monomials in A_D follows directly from Lemma 5.29. □

Proposition 5.31 *$\rho(\gamma_{k,n}) = \gamma^+_{k,n} + \gamma^-_{k,n}$ for all $n, k \geq 1$. Moreover, $\rho(\delta_m) = \delta^0_{m:0}$ for $n \geq 2$ and $\rho(\delta_1) = 0$. More generally, for every Hopf monomial $x \in \mathcal{M}_B$, $\rho(x)$ can be computed as follows. If $x = b_{l,\underline{t}}$ is a gathered block with profile $\underline{t} = (t_0, \dots, t_n)$, we have that*

$$\rho(x) = \begin{cases} x^0 & \text{if } n = 0, \\ \sum_{a=0}^{t_1} \binom{t_1}{a} b^+_{l,(t_0,a)} (\gamma^-_{1,l})^{t_1 - a} & \text{if } n = 1, \\ b^+_{l,\underline{t}} + b^-_{l,\underline{t}} & \text{if } n \geq 2. \end{cases}$$

The restriction of a Hopf monomial x with a constituent gathered block in $\widetilde{\mathcal{B}}^0$ is x^0. We calculate the restriction of a Hopf monomial $x \in \widetilde{\mathcal{B}}^c$ as follows. First, replace every constituent gathered block in x with the sum of the positively or neutrally charged elements of its restriction. Then, write the resulting linear combination as a sum of Hopf monomials in A_D. Finally, add to that the negatively charged counterpart of every positively charged Hopf monomials appearing in the sum.

Proof Using the cochain-level representative of $\gamma_{k,n}$ introduced in Definition 5.1, we immediately see that its restriction is represented in $\mathrm{FN}'^*_{W_{D_n 2^k}}$ by the sum of two elements obtained by providing this cochain with the two possible orientations. These elements correspond to cochain representatives of $\gamma^+_{k,n}$ and $\gamma^-_{k,n}$ via the cochain equivalence φ of Lemma 4.1. The formulas for δ_m are a consequence of the generators' definition in A_D and relation (4) of Proposition 5.13. □

We conclude this section with a short description of the Gysin sequence of the double cover $W_{D_n} \to W_{B_n}$. In [9], Giusti and Sinha adopt the analysis of a similar Gysin exact sequence as the starting point to compute the cohomology of the alternating groups as an almost-Hopf ring. While we retrieve that as a byproduct of our algebraic description, we stress that Giusti and Sinha's approach could be used in our framework as an alternative method to deduce relations in A'_D. Indeed, a direct consequence of the following proposition is that $\mathcal{M}_D = \mathcal{B}^0 \sqcup \mathcal{B}^+ \sqcup \mathcal{B}^-$ is the polarized basis arising from a Gysin decomposition in the sense of [9].

Proposition 5.32 (cf [9, Section 3]) *The restriction $\rho\colon A_B \to A'_D$ and the transfer $\mathrm{tr}\colon A'_D \to A_D$ fit into the Gysin sequence*

$$\cdots \xrightarrow{\partial_{k-1}} H^k(W_{B_n}; \mathbb{F}_2) \xrightarrow{\rho_k} H^k(W_{D_n}; \mathbb{F}_2) \xrightarrow{\mathrm{tr}_k} H^k(W_{B_n}; \mathbb{F}_2) \xrightarrow{\partial_k} H^{k+1}(W_{B_n}; \mathbb{F}_2) \xrightarrow{\rho_{k+1}} \cdots,$$

where ∂ is the multiplication with $\delta_1 \odot 1_{n-1}$. It can be described on skyline diagrams by the operation of replacing each column corresponding to δ_k^m with the diagram corresponding to $\delta_1^{m+1} \odot \delta_{k-1}^m$.

Proof By a general fact, the connecting homomorphism ∂ is the multiplication with the Euler class e of the double covering. In the case $n = 1$, this covering is isomorphic to the universal double covering $S^\infty \to \mathbb{P}^\infty(\mathbb{R})$, and its Euler class is δ_1. For bigger n, the Euler class is $\delta_1 \odot 1_{n-1}$ because it is the only class in the right degree that restricts to δ_1.

$\mathrm{tr} \circ \rho = 0$ because we are working modulo 2. Therefore the transfer of a neutral gathered block is 0. If $b = b^{\pm}_{l,\underline{t}}$ is a charged gathered block, then the restriction of $\mathrm{tr}(b)$ must be $b + \iota(b)$, and the multiplication with $\delta_1 \odot 1_{n-1}$ must kill $\mathrm{tr}(b)$. These two conditions force $\mathrm{tr}(b) = b_{l,\underline{t}}$. Since tr preserves the transfer product $\odot$, the formula for a general Hopf monomial follows. □

6 Restriction to elementary abelian subgroups

We recall here some theorems from Swenson's thesis [18], which constitute the formal framework in which we will calculate the cohomology of W_{B_n} and W_{D_n}. We will then exploit these theorems to determine the restriction of our generators in A_B and A_D to elementary abelian 2–subgroups. This yields the restriction of all the cohomology of the groups W_{B_n} and W_{D_n} to maximal elementary abelian subgroups, because the structural morphisms of our almost-Hopf rings behave in a predictable way: cup products and coproducts are preserved by such restriction, while the relation with transfer product is determined via double cosets formulas, as stated in Adem and Milgram's book [1, Section II.6].

6.1 Quillen's theorem for finite reflection groups

The relevance of these restriction maps is encompassed by a result of Quillen [15; 16], which we state here. Let G be a finite group and $\mathcal{F}$ a family of subgroups. Let $\theta_g : H^*(K; \mathbb{F}_p) \to H^*(gKg^{-1}; \mathbb{F}_p)$ be the conjugation homomorphism. Define

$$\mathcal{F}^n = \{\{f_K\}_{K \in \mathcal{F}},\, f_K \in H^n(K; \mathbb{F}_p) \mid g^{-1}Kg \subseteq K' \Longrightarrow f_K = \theta_g^*(f_{K'})|_K \text{ for all } K, K'\}.$$

Alternatively, we can consider $\mathcal{F}$ as a category in which

$$\mathrm{Hom}(K, K') = \{g \mid g^{-1}Kg \subseteq K'\}.$$

Then $\mathcal{F}^n$ is the inverse limit of the functor H^n from $\mathcal{F}$ into the category of $\mathbb{F}_p$–vector spaces. In other words, $\mathcal{F}^*$ consists of collections of cohomology classes of groups in $\mathcal{F}$ that are stable under restrictions and conjugation by elements of G. Observe that $\mathcal{F}^* = \bigoplus_n \mathcal{F}^n$ has a natural ring structure.

Theorem 6.1 [15, Theorem 6.2, page 564] *Let G be a finite group. Let $\mathcal{F}^*$ be as before. The map $q_G : H^*(G;\mathbb{F}_p) \to \mathcal{F}^*$ given by $q_G(f) = \{f|_K\}_K$ is a well-defined ring homomorphism. Moreover, if $\mathcal{F}$ is the family of elementary abelian p–subgroups, then the kernel and cokernel of q_G are nilpotent.*

Hence elementary abelian p–subgroups give much information on the $\mathbb{F}_p$–cohomology of a group. In the case of a finite reflection group, an even stronger property holds.

Theorem 6.2 [18, Theorem 11, page 2] *If G is a finite reflection group and $\mathcal{F}$ is the family of elementary abelian p–subgroups of G, then q_G is an isomorphism.*

6.2 Restriction from A_B

For the reasons explained in the previous subsection, Swenson has calculated the elementary abelian 2–subgroups of W_{B_n}. Before stating his result, we need to recall the structure of elementary abelian 2–groups of the symmetric group Σ_n on n objects. The relevant calculations are reviewed in [1]. Σ_n admits a transitive elementary abelian 2–subgroup if and only if $n = 2^k$. In this case, all these subgroups are conjugated in Σ_n to the image V_k of the homomorphism $\rho_k : \mathbb{F}_2^k \hookrightarrow \Sigma_{2^k}$ given by the regular action of $\mathbb{F}_2^k$ on itself. More generally, a maximal elementary abelian 2–subgroup of Σ_n is conjugated to a direct product

$$V_{k_1} \times \cdots \times V_{k_r} \hookrightarrow \Sigma_{2^{k_1}} \times \cdots \times \Sigma_{2^{k_r}} \hookrightarrow \Sigma_{2^{k_1}+\cdots+2^{k_r}}.$$

Hence, conjugacy classes of maximal elementary abelian 2–subgroups in Σ_n are parametrized by partitions π of n such that every element of π is an integral power of 2 and the multiplicity of $1 = 2^0$ in π is at most 1.

To further simplify notation, we borrow from Swenson's thesis the following definition.

Definition 6.3 [18] Let $n \in \mathbb{N}$. We say that a partition π of n is *admissible* if it consists only of parts that are integral powers of 2.

The main results about elementary abelian 2–subgroups in W_{B_n} is the following:

Proposition 6.4 [18, page 22] *Let $A_1, A_2 \leq W_{B_n}$ be maximal elementary abelian 2–subgroups. Then*

- *$\widetilde{A}_i = A_i \cap \Sigma_n \leq \Sigma_n$ is conjugated to a subgroup of the form $V_{k_1} \times \cdots \times V_{k_r}$, with $k_i \geq 0$ for all i;*
- *A_1 and A_2 are conjugated in W_{B_n} if and only if $\widetilde{A}_1$ and $\widetilde{A}_2$ are conjugated in Σ_n.*

In particular, conjugacy classes of maximal elementary abelian 2–subgroups in W_{B_n} are parametrized by admissible partitions π. Moreover, if we denote by A_π the subgroup corresponding to a partition π, we have that $A_{(2^k)} = V_k \times C_k$, where $C_k \cong \mathbb{F}_2$ is the center of $W_{B_{2^k}}$ and, more generally, if m_i is the multiplicity of 2^i in a partition π, then A_π is isomorphic to the direct product $\prod_i A_{(2^i)}^{m_i}$. Let $d_{2^i-1}, \ldots, d_{2^i-2^{i-1}}$ be the Dickson invariants in $H^*(V_i; \mathbb{F}_2) \hookrightarrow H^*(A_{(2^i)}; \mathbb{F}_2)$ and define

$$f_{2^i} = \prod_{y \in H^1(V_i; \mathbb{F}_2)} (x + y),$$

where $x \in H^1(A_{2^i}; \mathbb{F}_2)$ is the linear dual to the nontrivial element in the C_i–factor of A_{2^i}. There is a natural isomorphism

$$[H^*(A_\pi; \mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)} \cong \bigotimes_i (\mathbb{F}_2[f_{2^i}, d_{2^i-1}, \ldots, d_{2^i-2^{i-1}}]^{\otimes m_i})^{\Sigma_{m_i}}.$$

We can calculate the restriction of our generating classes $\gamma_{k,n}$ and δ_n to these abelian subgroups. The calculation for $\gamma_{k,n}$ has been essentially carried out by Giusti, Salvatore and Sinha [7]. We state here the result.

Proposition 6.5 [7, Corollary 7.6, page 189] *Let $l, n \geq 1$. Let π be a partition of $n2^l$ consisting of powers of 2, $\pi = (2^{k_1}, \ldots, 2^{k_r})$. Then*

$$\gamma_{l,n}|_{A_\pi} = \begin{cases} \bigotimes_{i=1}^r d_{2^{k_i}-2^{k_i-l}} & \text{if } k_i \geq l \text{ for all } 1 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.6 *Let $n \geq 0$. Let $\pi = (2^{k_1}, \ldots, 2^{k_r})$ be an admissible partition. The restriction of δ_n to the cohomology of the maximal elementary abelian 2–subgroup A_π is equal to $\bigotimes_{i=1}^r f_{2^{k_i}}$. Moreover, δ_n is the unique class in $H^n(W_{B_n}; \mathbb{F}_2)$ that has this property for every π.*

Proof We observe that the restrictions of a cohomology class to A_π for all the partitions π of n determine its restriction to every elementary abelian 2–subgroup (not necessarily maximal). Hence, by Theorem 6.2, a class that satisfies the condition in the statement for every π is necessarily unique.

Let $U_n = \mathbb{R}^n$ be the reflection representation of W_{B_n}. Recall that, if $n = 2^k$ and $\pi = (2^k)$, then $A_\pi = V_k \times C_k$, where $C_k = \langle t \rangle$ is a cyclic group of order 2, the center of W_{B_n}, and $V_k = \langle v_1, \dots, v_k \rangle \leq \Sigma_{2^k}$ is the subgroup defined above. $H^*(A_\pi; \mathbb{F}_2)$ is polynomial on degree 1 elements $x, y_1, \dots, y_k$, the linear duals to $t, v_1, \dots, v_k$, respectively. Given $a \in A_\pi \setminus \{1\}$, let ε_a, sgn_a, and $\mathbb{R}\langle a \rangle$ be the 1–dimensional trivial representation, the signum representation, and the regular representation of $\langle a \rangle \cong \mathbb{F}_2$, respectively. We first observe that, since t acts on U_n as the multiplication by -1, $U_n|_{A_\pi} \cong \mathrm{sgn}_t \otimes U_n|_{V_k}$. Moreover, the inclusion of V_k in Σ_{2^k} is given by the regular representation; hence

$$U_n|_{V_k} \cong \bigotimes_{i=1}^{k} \mathbb{R}\langle v_i \rangle \cong \bigoplus_{S \subseteq \{1,\dots,k\}} \bigotimes_{i=1}^{k} U_{S,i},$$

where $U_{S,i}$ is equal to sgn_{v_i} if $i \in S$, and to ε_{v_i} if $i \notin S$. Thus, with the notation used before in this document, the Stiefel–Whitney class of $U_n|_{A_\pi}$ is

$$\prod_{S \subseteq \{1,\dots,n\}} \left(1 + x + \sum_{i \in S} y_i\right).$$

Its n–dimensional part is exactly f_{2^k}. Hence, the thesis for $\pi = (2^k)$ follows from the naturality of the characteristic classes and Proposition 5.3

In the case of a general admissible partition $\pi = (2^{k_1}, \dots, 2^{k_r})$, the proposition follows from the fact that $A_\pi \cong \prod_{i=1}^r A_{(2^{k_i})}$ and $U_n|_{A_\pi} \cong \bigoplus_{i=1}^r U_{2^{k_i}}|_{A_{(2^{k_i})}}$. □

To complete the calculation of the restriction morphisms from A_B to maximal elementary abelian 2–subgroups, we need to describe how such maps behave with the structural morphisms of A_B. Restrictions preserve cup products, and, regarding the coproduct, there is nothing to say because every maximal elementary abelian subgroup of $W_{B_n} \times W_{B_m}$ is itself a maximal elementary abelian subgroup of $W_{B_{n+m}}$. The only nontrivial behavior occurs with the transfer product. We describe it in the following proposition.

Proposition 6.7 *Let $x, y \in A_B$ be in positive components n and m respectively. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ be an admissible partition of $n+m$. For all $I \subseteq \{1, \dots, r\}$, write $I = \{i_1, \dots, i_s\}$ with $i_1 < \cdots < i_s$ and let $\pi_I = (2^{k_{i_1}}, \dots, 2^{k_{i_s}})$. Then*

$$(x \odot y)|_{A_\pi} = \sum_{I,J} \tau_{I,J}(x|_{A_{\pi_I}} \otimes y|_{A_{\pi_J}}),$$

where the sum runs over all partitions $\{1,\dots,r\} = I \sqcup J$ of $\{1,\dots,r\}$ into two subsets such that $\sum_{i\in I} 2^{k_i} = n$ (and, consequently, $\sum_{j\in J} 2^{k_j} = m$), and

$$\tau_{I,J}\colon H^*(A_{\pi_I};\mathbb{F}_2)\otimes H^*(A_{\pi_J};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is the obvious permutation of tensor factors.

Proof We begin by assuming that $r = 1$; thus $\pi = (2^k)$ for some k and $n+m = 2^k$. Then, since A_π acts transitively on $\{1,\dots,2^k\}$, no conjugate of A_π in $W_{B_{n+m}}$ is contained in $W_{B_n}\times W_{B_m}$. Given that A_π is abelian, the classically known property stated in [1, Proposition 5.6, page 69] implies that the transfer map

$$H^*(A_\pi\cap\sigma(W_{B_n}\times W_{B_m})\sigma^{-1};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is identically zero. Eilenberg's double coset formula then guarantees that the composition of the restriction with the transfer product

$$H^*(W_{B_n};\mathbb{F}_2)\otimes H^*(W_{B_m};\mathbb{F}_2)\xrightarrow{\odot} H^*(W_{B_{n+m}};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is zero. Thus $(x\odot y)|_{A_{(2^k)}} = 0$.

In the general case, the restriction of $x\odot y$ to this subgroup factors through the r–fold coproduct. By the calculations above, addends in this coproduct for which a factor is a nontrivial transfer product restrict to 0. Since $\odot$ and Δ form a bialgebra structure on A_B, the other addends have the desired form. □

6.3 Restriction from A_D and proof of relations

We can adapt the argument to calculate the restriction to elementary abelian subgroups of generators also in the D_n case. First, we state the analog of Proposition 6.4. Recall that a partition π of n is admissible if and only if it consists of parts that are powers of 2.

Theorem 6.8 [18, Theorem 5.4.3, page 40] *Let π be an admissible partition of n. Let m_1 and m_2 be the multiplicities of 1 and 2 in π. We write $\pi = (1)^{m_1}\cup(2)^{m_2}\cup\pi'$. Let $A_\pi\le W_{B_n}$ the maximal elementary abelian 2–subgroup corresponding to π and let $\widehat{A}_\pi = A_\pi\cap W_{D_n}$. Then $\widehat{A}_\pi$ is maximal as an elementary abelian subgroup of W_{D_n} if and only if $m_1\neq 2$. Moreover*:

- *If $m_1 > 0$, then $\widehat{A}_\pi = \ker\big(\sum\colon \mathbb{F}_2^{m_1}\to\mathbb{F}_2\big)\times A_{(2)^{m_2}\cup\pi'}$. If $e_1,\dots,e_{m_1}$ are the elementary symmetric functions in $H^*(\mathbb{F}_2^{m_1};\mathbb{F}_2) = H^*(A_{(1)^{m_1}};\mathbb{F}_2)$, we define $\bar{e}_i =$*

$e_i + e_1 e_{i-1}$ *if* $2 \le i < m$ *and* $\bar{e}_m = e_1 e_{m-1}$. *There is an isomorphism between the invariant subalgebra* $[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)}$ *and*

$$\mathbb{F}_2[\bar{e}_2, \dots, \bar{e}_m] \otimes [H^*(A_{(2)^{m_2} \cup \pi'}; \mathbb{F}_2)]^{N_{W_{B_{n-m_1}}}(A_{(2)^{m_2} \cup \pi'})}.$$

Moreover, the cohomological restriction from $A_{(1)^m}$ *to* $\widehat{A}_{(1)^m}$ *is given by* $e_1 \mapsto 0$ *and* $e_i \mapsto \bar{e}_i$ *if* $2 \le i \le m$.

- *If* $m_1 = 0$ *and* $m_2 > 0$, *then* $\widehat{A}_\pi = A_\pi$. *Identifying* $H^*(A_{(2)}; \mathbb{F}_2)^{\otimes m_2}$ *with* $\bigotimes_{i=1}^{m_2} \mathbb{F}_2[x_i, y_i]$, *we can define*

$$h_{m_2} = \sum_{\substack{S \subseteq \{1,\dots,m_2\} \\ |S| = 2l}} \prod_{i \in S} (x_i + y_i) \prod_{j \notin S} x_j.$$

Then $[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)}$ *is the free* $[H^*(A_\pi; \mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}$*–module with basis* $\{1, h_{m_2} \otimes 1_{H^*(A_{\pi'}; \mathbb{F}_2)}\}$.

- *If* $m_1 = m_2 = 0$, *then* $\widehat{A}_\pi = A_\pi$ *and* $N_{W_{D_n}}(A_\pi) = N_{W_{B_n}}(A_\pi)$; *hence*

$$[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)} = [H^*(A_\pi; \mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}.$$

Moreover, if $m_1 \neq 0$ *or* $m_2 \neq 0$, *then* A_π *is* W_{B_n}*–conjugate to* A' *if and only if* $\widehat{A}_\pi$ *is* W_{D_n}*–conjugate to* $A' \cap W_{D_n}$. *Conversely, if* $m_1 = m_2 = 0$, *then the* W_{B_n}*–conjugacy class of* A_π *contains exactly two* W_{D_n}*–conjugacy classes of elementary abelian 2–subgroups.*

We now determine the restriction of our generators to the elementary abelian subgroups.

Proposition 6.9 *Let* $n = 2^k m$, *for some* $k, m \ge 1$. *Let* π *be an admissible partition of* n. *Let* m_1 *and* m_2 *be the multiplicities of* 1 *and* 2 *in* π. *Then*

(1) *for every* $k \ge 1$, *if* $m_1 = m_2 = 0$, *then* $\gamma^+_{k,m}|_{A_\pi} = \gamma_{k,m}|_{A_\pi}$, $\gamma^+_{k,m}|_{A^{s_0}_\pi} = 0$, $\gamma^-_{k,m}|_{A_\pi} = 0$, *and* $\gamma^-_{k,m}|_{A^{s_0}_\pi} = \gamma_{k,m}|_{A^{s_0}_\pi}$;

(2) *for every* $k \ge 2$, *if* $m_1 \neq 0$ *or* $m_2 \neq 0$, *or for* $k = 1$ *if* $m_1 \neq 0$, *then* $\gamma^\pm_{k,m}|_{\widehat{A}_\pi} = 0$;

(3) *if* $m_1 = 0$ *but* $m_2 \neq 0$, *ie* $\pi = (2)^{m_2} \sqcup \pi'$, *then the restriction of* $\gamma^+_{1,m}$ (*resp.* $\gamma^-_{1,m}$) *to* $\widehat{A}_\pi = A_{(2)^{m_2}} \times A_{\pi'}$ *is* $h_{m_2} \otimes \gamma_{1,m-m_2}|_{A_{\pi'}}$ (*resp.* $(d_1^{\otimes m_2} + h_{m_2}) \otimes \gamma_{1,m-m_2}|_{A_{\pi'}}$);

(4) *if* $\pi = (1)^{m_1} \sqcup \pi'$, *then the restriction of* $\delta^0_{k;m}$ *to* $\widehat{A}_\pi = \widehat{A}_{(1)^{m_1}} \times A_{\pi'}$ *is*

$$1 \otimes (\delta_k \odot 1_{W_{B_{m-m_1}}})|_{A_{\pi'}} + \sum_{i=2}^{k} \bar{e}_i \otimes (\delta_{k-i} \odot 1_{W_{B_{m-m_1+i}}})|_{A_{\pi'}},$$

with the convention that $1_{W_{B_r}} = 0$ *when* $r < 0$.

Proof If π has more than 1 element and is different from $(1,1,\dots,1)$, then the restriction to $\widehat{A}_\pi$ or $\widehat{A}^{s_0}_\pi$ factors through the coproduct. Thus, by applying the coproduct formulas of Proposition 5.12, we can inductively reduce to these two cases.

We begin by assuming that $\pi = (2^n)$ has only one element, and we prove the first statement. If $k \geq 1$ and $n \geq 2$, the restriction of $\gamma^{\pm}_{k,2^l}$ to A_π $(n = k + l)$ must be $N_{W_{D_{2^n}}}(A_\pi)$–invariant. Hence, for degree reasons, it can be 0 or $d_{2^n-2^l}$. Since $i_+^*(\gamma^+_{k,2^l}) = \gamma_{k,2^l}$ (resp. $i_+^*(\gamma^-_{k,2^l}) = 0$) by Proposition 5.27, its restriction to $A_\pi \cap \Sigma_{2^n}$ must be the Dickson invariant of degree $2^n - 2^l$ (resp. 0). This forces

$$\gamma^+_{k,2^l}|_{A_\pi} = d_{2^n-2^l} = \gamma_{k,2^l}|_{A_\pi}$$

(resp. $\gamma^-_{k,2^l}|_{A_\pi} = 0$). By essentially the same argument, considering i_- instead of i_+, we determine the restrictions to $A^{s_0}_\pi$, proving the first point.

Claim (2) is immediate from the fact that, if $k \geq 1$, there are no nonzero elements in $H^*(\widehat{A}_{(1)^{2^k n}}; \mathbb{F}_2)^{N_{W_{D_{2^k n}}}(\widehat{A}_{(1)^{2^k n}})}$ in the same degree of $\gamma^{\pm}_{k,n}$, and that if $k \geq 2$ the coproduct of $\gamma^{\pm}_{k,n}$ has no element in component 2.

To prove (3) when $\pi = (2)$, we notice that $A_{(2)} = W_{D_2}$ and $\gamma^+_{1,1}$ can be identified with h_1, while $\gamma^-_{1,1,}$ with $d_1 + h_1$.

By the coproduct formula for $\delta^0_{k:m}$, we have $\delta^0_{k:m}|_{A^{k+m}_{(1)}} = e_k$. Thus, the last statement for $\pi = (1,\dots,1)$ follows directly by combining Proposition 6.6 and Theorem 6.8. □

As in A_B, the behavior of the restriction to maximal elementary abelian 2–subgroups with the cup product and coproduct is straightforward. We describe the relation between such restriction maps and the transfer product in the following proposition, which is the counterpart of Proposition 6.7.

Proposition 6.10 *Let $x, y \in A_D$ be elements in positive component n and m respectively. Let $\pi = (2^{k_1},\dots,2^{k_r})$ be an admissible partition of $n+m$. For all $I \subseteq \{1,\dots,r\}$, write $I = \{i_1,\dots,i_s\}$ with $i_1 < \cdots < i_s$ and let $\pi_I = (2^{k_{i_1}},\dots,2^{k_{i_s}})$. Then*

$$(x \odot y)|_{\widehat{A}_\pi} = \sum_{I,J} \tau_{I,J}(x|_{\widehat{A}_{\pi_I}} \otimes y|_{\widehat{A}_{\pi_J}} + \iota(x)|_{\widehat{A}_{\pi_I}} \otimes \iota(y)|_{\widehat{A}_{\pi_J}}),$$

where the sum runs over all partitions $\{1,\dots,r\} = I \sqcup J$ of $\{1,\dots,r\}$ into two subsets such that $\sum_{i\in I} 2^{k_i} = n$ (and, consequently, $\sum_{j\in J} 2^{k_j} = m$) and at least one between I and J does not contains any $l \in \{1,\dots,r\}$ such that $k_l = 0$, and where

$\tau_{I,J}\colon H^*(\widehat{A}_{\pi_I};\mathbb{F}_2)\otimes H^*(\widehat{A}_{\pi_J};\mathbb{F}_2)\to H^*(\widehat{A}_\pi;\mathbb{F}_2)$ *is the obvious permutation of tensor factors. Moreover, if* $1\notin\pi$ *and* $2\notin\pi$,

$$(x\odot y)|_{\widehat{A}_\pi^{s_0}} = c_{s_0}^*\sum_{I,J}\tau_{I,J}(x|_{\widehat{A}_{\pi_I}}\otimes\iota(y)|_{\widehat{A}_{\pi_J}}+\iota(x)|_{\widehat{A}_{\pi_I}}\otimes y|_{\widehat{A}_{\pi_J}}),$$

where $I, J, \tau_{I,J}$ *are as above, and* $c_{s_0}^*\colon H^*(\widehat{A}_\pi;\mathbb{F}_2)\to H^*(\widehat{A}_\pi^{s_0};\mathbb{F}_2)$ *is induced by the conjugation with* s_0.

Proof We cannot repeat the proof of Proposition 6.7 because in A'_D the transfer product and the coproduct do not form a bialgebra. Therefore, we argue by considering Eilenberg's double coset formula associated with the two subgroups $W_{D_n}\times W_{D_m}$ and $\widehat{A}_\pi$ of $W_{D_{n+m}}$. We preliminarily fix some notation. Let P_π be the partition of the set $\{1,\dots,n+m\}$ given by

$$P_\pi=\left\{\{1,\dots,2^{k_1}\},\{2^{k_1}+1,\dots,2^{k_1}+2^{k_2}\},\dots,\left\{\sum_{l=1}^{r-1}2^{k_l}+1,\dots,n+m\right\}\right\}.$$

Moreover, let $P_{j,\pi}=\{\sum_{l=1}^{j-1}2^{k_l}+1,\dots,\sum_{l=1}^{j}2^{k_l}\}$.

Assume that $1\notin\pi$. A set of representatives for $W_{D_{n+m}}/(W_{D_n}\times W_{D_m})$ is the set $\mathrm{Sh}(n,m)\cdot\{1,t\}$, where $\mathrm{Sh}(n.m)\subseteq\Sigma_{n+m}\hookrightarrow W_{D_{n+m}}$ is the set of (n,m)–shuffles, and $t=s_0\times s_0\in W_{B_n}\times W_{B_m}$. Note that $\widehat{A}_\pi\subseteq(\sigma t^\varepsilon)(W_{D_n}\times W_{D_m})(\sigma t^\varepsilon)^{-1}$ if and only if $\sigma(\{1,\dots,n\})$ is a union of parts of P_π. Since $\widehat{A}_\pi$ is abelian, these provide the only nonzero terms in the summation of the double coset formula. Moreover, by inspecting the image of $\{1,\dots,n\}\subseteq\{1,\dots,n+m\}$ under the signed permutation action of $W_{D_{n+m}}\subseteq W_{B_{n+m}}$, we see that if σt^ε and $\sigma' t^{\varepsilon'}$ are two coset representatives satisfying this condition, then $\widehat{A}_\pi\sigma t^\varepsilon(W_{D_n}\times W_{D_m})=\widehat{A}_\pi\sigma' t^{\varepsilon'}(W_{D_n}\times W_{D_m})$ if and only if $\sigma=\sigma'$ and $\varepsilon=\varepsilon'$.

Consequently, the double coset formula allows us to rewrite $\rho^{W_{D_{n+m}}}_{\widehat{A}_\pi}(x\odot y)$ as the sum

$$\sum_{\substack{I\subseteq\{1,\dots,r\}\\ \sum_{i\in I}2^{k_i}=n}}(c_{\sigma_I}^*\rho^{W_{D_n}\times W_{D_m}}_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi_J}}(x\otimes y)\otimes+c_{\sigma_I}^*(c_{s_0}^*\otimes c_{s_0}^*)\rho^{W_{D_n}\times W_{D_m}}_{\widehat{A}^{s_0}_{\pi_I}\otimes\widehat{A}^{s_0}_{\pi_J}}),$$

where σ_I is the unique (n,m)–shuffle satisfying $\sigma(\{1,\dots,n\})=\bigcup_{i\in I}P_{i,\pi}$ and $J=\{1,\dots,r\}\setminus I$. The statement follows by observing that $c_{\sigma_I}^*=\tau_{I,J}$ and

$$c_{s_0}^*\rho^{W_{D_l}}_{\widehat{A}^{s_0}_{\pi'}}=\rho^{W_{D_l}}_{\widehat{A}_{\pi'}}\iota$$

for all $l\geq 1$ and an π' admissible partition of l.

The case of $\widehat{A}^{s_0}_{\pi}$ where $1,2 \notin \pi$ is done similarly. If $1 \in \pi$, the same argument holds, but if there exists $i \in I$ and $j \in J$ such that $k_i = k_j = 0$, then, interpreting the elements of $W_{D_{n+m}}$ as signed permutations, $(p_i, -p_i)(p_j, -p_j)$ belongs to $\widehat{A}_{\pi}$ but not to $(\sigma_I t^{\varepsilon})(W_{D_n} \times W_{D_m})(\sigma_I t^{\varepsilon})^{-1}$, where $P_{i,\pi} = \{p_i\}$ and $P_{j,\pi} = \{p_j\}$. Thus, we need to restrict the summation only to partitions $\{1, \dots, r\} = I \sqcup J$ in which all the occurrences of 1 in π belong to the same part. □

This result provides a way to detect the charge of a Hopf monomial via restriction to maximal elementary abelian 2–subgroups. We first fix preliminary notation.

Definition 6.11 With the notation of Theorem 6.8, write $H^*(\widehat{A}_{(2)}; \mathbb{F}_2) = \mathbb{F}_2[x, y]$ and let $z = x + y$. Let $H^+_{A_{(2)}}$ (resp. $H^-_{A_{(2)}}$) be the vector subspace generated by elements of the form $x^a z^b$ where $a > b$ (resp. $b > a$). If π is an admissible partition of n, write $\pi = (1)^{m_1} \cup (2)^{m_2} \cup \pi'$ where $1 \notin \pi'$ and $2 \notin \pi'$. For $S \subseteq \{1, \dots, m_2\}$, we define $H_{i,S} = H^+_{A_{(2)}}$ if $i \notin S$ and $H^-_{A_{(2)}}$ if $i \in S$. Then we define

$$H^+_{\widehat{A}_{\pi}} = \begin{cases} 0 & \text{if } m_1 > 0, \\ \bigoplus_{S \subset \{1,\dots,m_2\}, |S|=2k} \bigotimes_{i=1}^{m_2} H_{i,S} \otimes H^*(A_{\pi'}; \mathbb{F}_2) & \text{if } m_1 = 0, \end{cases}$$

$$H^-_{\widehat{A}_{\pi}} = \begin{cases} 0 & \text{if } m_1 > 0, \\ \bigoplus_{S \subset \{1,\dots,m_2\}, |S|=2k+1} \bigotimes_{i=1}^{m_2} H_{i,S} \otimes H^*(A_{\pi'}; \mathbb{F}_2) & \text{if } m_1 = 0. \end{cases}$$

Moreover, if $m_1 = m_2 = 0$, we define $H^+_{A^{s_0}_{\pi}} = 0$ and $H^-_{A^{s_0}_{\pi}} = H^*(A^{s_0}_{\pi}; \mathbb{F}_2)$.

Proposition 6.12 *Referring to Definition 6.11, for every maximal elementary abelian 2–subgroup $A = \widehat{A}_{\pi}$ or $A = A^{s_0}_{\pi}$ of W_{D_n}, the restriction of a positively (resp. negatively) charged Hopf monomial in $\mathcal{M}_D \cap H^*(W_{D_n}; \mathbb{F}_2)$ to the cohomology of $A_{(\pi)}$ belongs to H^+_A (resp. H^-_A).*

Proof Every positively charged gathered block b restricts to an element of H^+_A. Nontrivial computations arise only if $b = (\delta^0_{2m:0})^r (\gamma^+_{1,m})^s$ with $r \geq 0$ and $s > 0$ and $A = A_{(2)^m}$. In this case, with the notation of Theorem 6.8, we observe that

$$h_m^{2^k} = \sum_{\substack{S \subseteq \{1,\dots,m\} \\ |S|=2l}} \prod_{i \in S} z_i^{2^k} \prod_{j \notin S} x_j^{2^k},$$

where $z_i = x_i + y_i$. Thus $h_m^{2^k} \in H^+_{(2)^m}$. Let 2^k be the biggest power of 2 smaller than s. Then $h_m^{s-2^k}$ is a sum of pure tensors of the form $w_1 \otimes \cdots \otimes w_m$, where w_i is a monomial in x_i and z_i with total degree smaller than 2^k. Therefore, $h^s_m = h^{2^k}_m h^{s-2^k}_m$

still belongs to $H^+_{(2)^m}$. The restriction of b to $A_{(2)^m}$ is equal to $\bigotimes_{i=1}^m (x_i z_i)^r h_m^s$, which belongs to $H^+_{(2)^m}$ because multiplication by $\bigotimes_{i=1}^m x_i z_i$ preserves $H^+_{(2)^m}$.

We see the corresponding statement for negatively charged gathered blocks by noting that conjugation with s_0 exchanges H_A^+ and $H_{A^{s_0}}^-$.

In general, a positively (resp. negatively) charged Hopf monomial x is a transfer product of gathered blocks, all positively charged (resp. all positively charged except one). Consequently, Proposition 6.10 yields the statement for x. □

We can finally complete our relations for A'_D by providing the proofs of the two leftover propositions of Section 5.2.

Proof of Proposition 5.14 Let b be a positively charged gathered block in A_D and $x \in A_D$. From Lemma 5.10 and the definition of Δ' we deduce that $\Delta(b) = \Delta'(b) + (\iota \otimes \iota)\Delta'(b)$, and that $\Delta(\iota(b)) = (\mathrm{id} \otimes \iota + \iota \otimes \mathrm{id})\Delta'(b)$. During this proof, we assume, by convention, that $x|_{A_\pi} = 0$ when $x \in H^*(W_{D_n}; \mathbb{F}_2)$ and π is not an admissible partition of n. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ and $\pi' = (2^{h_1}, \dots, 2^{h_s})$ be admissible partitions of some integers. From Proposition 6.10, we deduce that

$$
\begin{aligned}
&[(\odot\otimes\odot)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta'\otimes\Delta)(b\otimes x)]|_{\widehat{A}_\pi\times\widehat{A}_{\pi'}}\\
&= \sum_{\substack{I\sqcup J=\{1,\dots,r\}\\ I'\sqcup J'=\{1,\dots,s\}}} \tau_{I,I',J,J'}\big(\Delta'(b)\otimes\Delta(x)+(\mathrm{id}\otimes\iota)\Delta'(b)\otimes(\mathrm{id}\otimes\iota)\Delta(x)\\
&\qquad\qquad +(\iota\otimes\mathrm{id})\Delta'(b)\otimes(\iota\otimes\mathrm{id})\Delta(x)\\
&\qquad\qquad +(\iota\otimes\iota)\Delta'(b)\otimes(\iota\otimes\iota)\Delta(x)\big)|_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi'_{I'}}\times\widehat{A}_{\pi_J}\times\widehat{A}_{\pi'_{J'}}}\\
&= \sum_{\substack{I\sqcup J=\{1,\dots,r\}\\ I'\sqcup J'=\{1,\dots,s\}}} \tau_{I,I',J,J'}\big((\mathrm{id}+\iota\otimes\iota)\Delta'(b)\otimes\Delta(x)\\
&\qquad\qquad +(\mathrm{id}+\iota\otimes\iota)\Delta'(\iota(b))\otimes\Delta(\iota(x))\big)|_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi'_{I'}}\times\widehat{A}_{\pi_J}\times\widehat{A}_{\pi'_{J'}}}\\
&= \sum_{\bar{I}\sqcup\bar{J}=\{1,\dots,r+s\}} (b\otimes x+\iota(b)\otimes\iota(x))|_{\widehat{A}_{(\pi\sqcup\pi')_{\bar{I}}}\times\widehat{A}_{(\pi\sqcup\pi')_{\bar{J}}}}\\
&= [\Delta(b\odot x)]|_{\widehat{A}_\pi\times\widehat{A}_{\pi'}}.
\end{aligned}
$$

In these equalities we used the identities of Lemma 5.10 to perform the substitutions $(\iota \otimes \iota)\Delta(x) = \Delta(x)$ and $(\mathrm{id} \otimes \iota)\Delta(x) = (\iota \otimes \mathrm{id})\Delta(x) = \Delta(\iota(x))$; $\pi \sqcup \pi'$ is assumed to be $(2^{k_1}, \dots, 2^{k_r}, 2^{h_1}, \dots, 2^{h_s})$; $I = \bar{I} \cap \{1, \dots, r\}$ and $J = \bar{J} \cap \{1, \dots, r\}$, while I' and J' are $\bar{I} \cap \{r+1, \dots, r+s\}$ and $\bar{J} \cap \{r+1, \dots, r+s\}$ suitably shifted. The sum should be over all I, J, I' and J' such that at least one between I and J does not

contain an l such that $k_l = 0$ and at least one between I' and J' does not contain an l such that $h_l = 0$. However, since the restriction of positively charged gathered blocks is zero on elementary abelian 2–subgroups corresponding to admissible partitions containing 1, we can restrict the sum only to the terms for which $k_i \neq 0$ for all $i \in I$ and $h_i \neq 0$ for all $i' \in I'$. This condition is equivalent to $\bar{I}$ not containing 1, and we can, once again, restrict the last sum only to these terms and get the last equality. □

Proof of Proposition 5.13 Using Proposition 6.9, the newly proved Proposition 5.14, Proposition 6.10, and the fact that cup products commute with restrictions, we check that the desired identity hold when restricted to maximal elementary abelian 2–subgroups. Then Theorem 6.2 yields the relations in A_D. □

7 Proofs of the main theorems

We devote this section to the proofs of the presentation theorems for A_B and A_D. They will be proved by comparing restrictions to elementary abelian 2–subgroups and exploiting Theorem 6.2. We will separate two technical lemmas from the proofs for the sake of clarity of exposition.

We first provide a proof for our structure theorem for A_B.

Lemma 7.1 *Let $k > 0$. The kernel of the restriction map*

$$H^*(A_{(2^k)}; \mathbb{F}_2)^{N_{W_{B_{2^k}}}(A_{(2^k)})} \to H^*(A_{(2^k)} \cap A_{(2^{k-1},2^{k-1})}; \mathbb{F}_2)$$

is the ideal generated by d_{2^k-1}.

Proof From Swenson's description of A_π, stated as in Proposition 6.4, we can identify $A_{(2^k)}$ with the image of the diagonal embedding $\mathrm{id}\times d : \Sigma_2 \times V_{k-1} \to \Sigma_2 \wr V_{k-1} \to W_{B_{2^k}}$. Its intersection with the product $A_{(2^{k-1},2^{k-1})} = V_{k-1} \times V_{k-1}$ is identified with the subgroup $V_{k-1} \subseteq \Sigma_2 \times V_{k-1}$, embedded diagonally in $W_{B_{2^k}}$.

The restriction to this subgroup maps f_{2^k} to $(f_{2^{k-1}})^2$, $d_{2^k-2^l}$ to $(d_{2^{k-1}-2^{l-1}})^2$ if $l > 0$, and d_{2^k-1} to 0. This is known, but we sketch a proof for completeness. If we chose bases $\{x, y_1, \dots, y_k\}$ of $H^1(A_{(2^k)}; \mathbb{F}_2)$ and $\{x, y_1, \dots, y_{k-1}\}$ of $H^1(A_{(2^{k-1})}; \mathbb{F}_2)$ as in Section 6.2, the restriction is given by $x \mapsto x$, $y_i \mapsto y_i$ if $1 \leq i < k$ and $y_k \mapsto 0$. The polynomial $F_k(t) = \prod_{v \in H^1(V_k;\mathbb{F}_2)}(t + v)$ in $H^*(V_k; \mathbb{F}_2)[t]$ restricts to $(F_{k-1}(t))^2$. Since $f_{2^k} = F_k(x)$, we deduce the formula for f_{2^k}. The identities for $d_{2^k-2^l}$ are obtained from this by using the classical identity $F_k(t) = \sum_{i=0}^{k} t^{2^i} d_{2^k-2^i}$. □

Proof of Theorem 5.9 Let A'_B be the Hopf ring generated by $\gamma_{k,m}$ and δ_m with the desired relations. Since the relations mentioned above hold in A_B, there exists an obvious morphism $\varphi\colon A'_B \to A_B$.

We need to fix a total ordering $\leq$ on the set $\mathcal{P}_n$ of admissible partitions of n such that, for all $\pi, \pi' \in \mathcal{P}_n$, $\pi' > \pi$ if π' is a refinement of π. In other words, $\leq$ extends the partial ordering given by refinement. Let b be a nontrivial gathered block in A_B. There exist unique nonnegative integers n and m such that $b = \prod_{i=1}^{n} \gamma_{i,2^{n-i}m}^{a_i} \delta_{2^n m}^{a_0}$ with $a_n \neq 0$. We consider the partition of $2^n m$ $\pi_b = (2^n, \dots, 2^n)$. Given $x = b_1 \odot \cdots \odot b_r \in \mathcal{M}_B$, let $\pi_x = \bigsqcup_{i=1}^{r} \pi_{b_i}$. As a consequence of Propositions 6.5, 6.6 and 6.7, $x|_{A_\pi} \neq 0$ implies that $\pi_x > \pi$. Explicitly, if $b = \prod_{i=1}^{n} \gamma_{i,2^{n-i}m}^{a_i} \delta_{2^n m}^{a_0}$,

$$b|_{A_{\pi_b}} = \left(f_{2^n}^{a_0} \prod_{i=1}^{n} d_{2^n - 2^{n-i}}^{a_i} \right)^{\otimes m}.$$

For any $x = b_1 \odot \cdots \odot b_r \in \mathcal{M}_B$, $x|_{A_{\pi_n}}$ is the symmetrization of $\bigotimes_{i=1}^{r} b_i|_{A_{\pi_{b_i}}}$. Given a partition π, let $\mathcal{M}_\pi$ be the set of elements $x \in \mathcal{M}$ such that $\pi_x = \pi$.

We first prove that φ is injective. We proceed by contradiction, and we assume that there exists a nontrivial sum $\sum_i x_i$ of elements of $\mathcal{M}_B$ that is 0 when restricted to every elementary abelian 2–subgroup. Let π be maximal among the set of partitions $\{\pi_{x_i}\}_i$. Since, by the explicit calculation above, the restrictions of the elements of $\mathcal{M}_\pi$ to A_π are linearly independent, this gives a contradiction.

To prove surjectivity, it is sufficient, by Theorem 6.2, to prove that an element α of the Quillen group $\mathcal{F}^*_{W_{B_n}}$ can be written as the image via $q_{W_{B_n}}$ of a linear combination of elements of $\mathcal{M}_B$. Note that such an α is determined by its values α_π on the maximal abelian 2–subgroups A_π. Let $\bar{\pi}_\alpha = \max\{\pi \in \mathcal{P}_n \mid \alpha_\pi \neq 0\}$ with respect to the chosen linear ordering. We write $\bar{\pi}_\alpha = (2^{k_1}, \dots, 2^{k_r})$. We proceed by induction on $\bar{\pi}_\alpha$. $\alpha_{\bar{\pi}_\alpha}$ must be invariant with respect to the action of the normalizer $N_{W_{B_n}}(A_{\bar{\pi}_\alpha})$. By Swenson's description of these invariant subalgebras stated in Proposition 6.4, it is a sum of elements $\sum_i c_{i,1} \otimes \cdots \otimes c_{i,r}$, with

$$c_{i,j} = \prod_{l=1}^{k_j - 1} d_{2^{k_j} - 2^{k_j - l}}^{a_{i,j,l}} f_{2^{k_j}}^{a_{i,j,0}} \in H^*(A_{(2^{k_j})}; \mathbb{F}_2).$$

We must have $a_{i,j,k_j} \neq 0$ for all i and j. Otherwise, we can define a partition π' obtained from $\bar{\pi}_\alpha$ by substituting 2^{k_j} with two parts both equal to $2^{k_j - 1}$ and observe that, by Lemma 7.1, we must have $\alpha_{\bar{\pi}_\alpha}|_{A_{\bar{\pi}_\alpha} \cap A_{\pi'}} \neq 0$. Thus $\alpha_{\pi'} \neq 0$ and this would

contradict the maximality of $\bar{\pi}_\alpha$. By our calculations above, since $\alpha_{\bar{\pi}_\alpha}$ must be invariant by permutations of tensor factors, this condition guarantees the existence of an element x in the linear span of $\mathcal{M}_{\bar{\pi}_\alpha}$ such that $x|_{A_{\bar{\pi}_\alpha}} = \alpha_{\bar{\pi}_\alpha}$. This reduces the statement to $\alpha' = \alpha + q_{W_{B_n}}(x)$ for which, by construction, $\bar{\pi}_{\alpha'} < \bar{\pi}_\alpha$, and completes the induction argument. □

We now focus on the presentation of A_D.

Lemma 7.2 *Let $\mathcal{M}_2 \subseteq \mathcal{M}_D$ be the set of Hopf monomials in A'_D whose constituent gathered blocks are all of the form $(\delta^0_{2k:0})^r(\gamma^{\pm}_{1,k})^s$ with $r \geq 0$ and $s > 0$, or of the form $(\delta^0_{2:0})^a$ with $a \geq 0$. Then, for all $m \geq 0$, $\mathcal{M}_2 \cap H^*(W_{D_{2^m}}; \mathbb{F}_2)$ restricts to a linearly independent set in $H^*(\widehat{A}_{(2)^m}; \mathbb{F}_2)$. Moreover, the image of $\mathcal{M}_2$ in the cohomology of $\widehat{A}_{(2)^m}$ generates the kernel of the restriction*

$$\rho_{2,1}\colon H^*(\widehat{A}_{(2)^m}; \mathbb{F}_2)^{N_{W_{D_{2^m}}}(\widehat{A}_{(2)^m})} \to H^*(\widehat{A}_{(1)^4\cup(2)^{m-2}} \cap \widehat{A}_{(2,2)}; \mathbb{F}_2).$$

Proof Note that, due to Theorem 6.8 and Proposition 6.9, the Hopf monomials in $\mathcal{M}_2 \cap H^*(W_{D_2}; \mathbb{F}_2)$ restrict to linearly independent elements in $H^*(\widehat{A}_{(2)}; \mathbb{F}_2)$. Therefore, to prove the linear independence claim for $m > 1$, it is enough to check that the restrictions of the elements of $\mathcal{M}_2 \cap H^*(W_{D_{2^m}}; \mathbb{F}_2)$ to $H^*(W^m_{D_2}; \mathbb{F}_2)$ (which is a component $\Delta_{(2)^m}$ of the coproduct) are linearly independent. Let $\mathcal{F}$ be the weight filtration on A'_D provided by Definition 5.20. It is enough to prove that this set is linearly independent when working in the associated graded spaces $\mathrm{gr}_{\mathcal{F}}(A'_D)$ and $\mathrm{gr}_{\mathcal{F}}(H^*(W^m_{D_2}; \mathbb{F}_2))$. In this setting, the image of a gathered block $b^+_{l,\underline{t}} \in \mathcal{M}_2$ (resp. $b^-_{l,\underline{t}} \in \mathcal{M}_2$) under $\mathrm{gr}_{\mathcal{F}}(\Delta_{(2)^m})$ is $\sum_{\varepsilon_1,\dots,\varepsilon_l} \bigotimes_{i=1}^m b^{\varepsilon_i}_{1,\underline{t}}$, where the sum is over all l–tuples $(\varepsilon_1, \dots, \varepsilon_l)$ with $\varepsilon_i \in \{+,-\}$ and the cardinality of the set $\{i : 1 \leq i \leq l, \varepsilon_i = -\}$ is even (resp. odd). Combining this with Proposition 5.14, we check the claim directly.

By Propositions 6.9 and 6.10, every element of $\mathcal{M}_2$ restricts to 0 on $\widehat{A}_\pi$ whenever $1 \in \pi$. Therefore, it is contained in the kernel of $\rho_{2,1}$. We now prove the opposite inclusion. With the notation of Theorem 6.8, we write

$$H^*(A_{(2)^m}; \mathbb{F}_2)^{N_{W_{D_{2^m}}}(A_{(2)^m})} = (\mathbb{F}_2[f_2, d_1]^{\otimes m})^{\Sigma_m}\{1, h_m\}.$$

We note that $A_{(2)^m} \cap \widehat{A}_{(1)^4\cup(2)^{m-2}} = A_{(2)^m} \cap A_{(1)^4\cup(2)^{m-2}}$. Moreover, $h_m = \gamma^+_{1,m}|_{A_{(2)^m}}$ is 0 when restricted to $A_{(2)^m} \cap \widehat{A}_{(1)^4\cup(2)^{m-2}}$. Therefore, Lemma 7.1 implies that $\ker(\rho_{2,1})$ is the ideal generated by h_m and $d_1^{\otimes m} = \rho_{2,1}(\gamma^+_{1,m} + \gamma^-_{1,m})|_{A_{(2)^m}}$. Finally, the generators belong to the image of $\mathcal{M}_2$, the linear subspace generated by $\mathcal{M}_2$ is a $\cdot$–subalgebra by our formulas in A'_D, and restriction maps preserve cup products. □

Proof of Theorem 5.15 Let A''_D be the almost Hopf ring generated by elements of the form $\delta^0_{n:m}$, $\gamma^{\pm}_{k,m}$, and 1^- with the desired relations. Let $\varphi\colon A''_D \to A'_D$ be the obvious morphism. We also consider the $\mathbb{F}_2$–vector space A'''_D with basis $\mathcal{M}_D$. By our relations for A'_D, $\mathcal{M}_D$ generates A''_D. Thus, there is a surjective linear map $\varphi'\colon A'''_D \to A''_D$. To prove that φ' is an isomorphism it is enough to prove that $\varphi'' = \varphi'\varphi$ is. Since in component 0 this is obvious, we can consider only positive components and replace A'_D with A_D. For technical reasons, we consider the set $\mathcal{M}'_D$, which differ from $\mathcal{M}_D$ by replacing neutral gathered blocks with elements of the form $\rho\big(\prod_{i=2}^{n}(\delta_i \odot 1_{n-i})^{k_i}\big)$ for $k_2, \dots, k_n \geq 0$. As shown in Lemma 5.18, this corresponds, at the level of A'''_D, to performing a change of basis. Hence, it does not affect the argument.

We adapt the argument used in the proof of Theorem 5.9. We define π_x for $x \in \mathcal{M}'_D$ as we did for A_B, with the only difference that gathered blocks of the form $b = (\delta^0_{2:0})^m$ have $\pi_b = (2)$, because $(1,1)$ does not define a maximal elementary abelian subgroup in W_{D_2}. It is still true that $x|_{\widehat{A}_\pi} = 0$ unless π_x is a refinement of π. We extend refinement of admissible partitions to a total ordering $\leq$, and we use the same argument by induction on $\leq$ adopted for A_B. Our choice of the new basis $\mathcal{M}'_D$ makes evident that for all admissible partitions π the set $\mathcal{M}'_\pi = \{x \in \mathcal{M}'_D \mid \pi_x = \pi\}$ restricts to a linearly independent set in the cohomology of $\widehat{A}_\pi$ when $1 \in \pi$, and Lemma 7.2 guarantees that this is true if $1 \notin \pi$ and $2 \in \pi$. Hence, the injectivity part works verbatim. We need to adapt the surjectivity argument for admissible partitions π such that $1 \notin \pi$ and $2 \in \pi$ (in all other cases, nothing changes). In these cases, we use Lemma 7.2 instead of Lemma 7.1 to carry on the proof. □

8 Steenrod algebra action

This section is devoted to the calculation of the Steenrod algebra action on A_B and A_D. We first observe that, since the coproducts and transfer products are induced by (stable) maps, they satisfy a Cartan formula with respect to Steenrod squares. In other words, A_B and A_D are almost-Hopf rings over the Steenrod algebra. Thus it is sufficient to determine the action of the Steenrod squares on the generators δ_{2^n}, $\gamma_{k,2^n}$, $\delta^0_{n:m}$ and $\gamma^{\pm}_{k,2^n}$.

Definition 8.1 [7] We define the following notions:

- The *height* (ht) of a gathered block in A_B or A_D is the number of generators that are cup-multiplied to obtain it, and the height of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ is $\max_{i=1}^{r} \mathrm{ht}(b_i)$.

- The *effective scale* (effsc) of a gathered block in the cohomology of W_{B_n} (resp. W_{D_n}) is the least l such that $n/2^l$ is an integer and its restriction to $W_{B_{2^l}}^{n/2^l}$ (resp. $W_{D_{2^l}}^{n/2^l}$) is nonzero, and the effective scale of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ as $\min_{i=1}^r \operatorname{effsc}(b_i)$.
- A *full-width* monomial is a Hopf monomial in A_B (resp. A_D) of which no constituent block is of the form $1_{W_{B_n}}$ (resp. $1_{W_{D_n}}$).

Theorem 8.2 (cf [7, Theorem 8.3, page 191]) *Let $k, n \geq 1$ and $i \geq 0$. Then, in A_B, the following formulas hold*:

- $\mathrm{Sq}^i(\gamma_{k,2^n})$ *is the sum of all the full-width monomials $x \in \mathcal{M}_B$ of degree $2^{n+k} - 2^n + i$ with $\operatorname{ht}(x) \leq 2$ and $\operatorname{effsc}(x) \geq k$ in which generators of the form δ_k do not appear.*
- $\mathrm{Sq}^i(\delta_{2^n})$ *is the sum of all the full-width monomials $x \in \mathcal{M}_B$ of degree $2^n + i$ with $\operatorname{ht}(x) \leq 2$ and $\operatorname{effsc}(x) \geq 1$ such that a generator of the form δ_k appears in every constituent gathered block of x.*

Proof The calculation for $\mathrm{Sq}^i(\gamma_{k,2^n})$ is an obvious consequence of [7, Theorem 8.3, page 191]. Regarding $\mathrm{Sq}^i(\delta_{2^n})$, since δ_{2^n} is the top-dimensional Stiefel–Whitney class of the reflection representation U_{2^n} by Proposition 5.3, by Wu's formula $\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n}$. Defining, by convention, $\gamma_{k,0} = 1$, let

$$u_i = \sum_{\substack{j_0,\dots,j_n \geq 0, \sum_{r=1}^{n-1} 2^r j_r + j_n + j_0 = 2^n \\ \sum_{r=1}^{n-1}(2^r-1)j_r + j_n = i}} \bigodot_{r=1}^{n-1} \gamma_{r,j_r} \odot \delta_{r_n} \odot 1_{W_{B_{j_0}}}.$$

We computed the restriction of $w_i(U_{2^n})$ to the maximal elementary abelian subgroups A_π in the proof of Proposition 6.6. It coincides with the restriction of u_i by our previous calculations based on Proposition 6.5. Thus,

$$\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n} = u_i\delta_{2^n},$$

and this class is exactly the sum of all the desired Hopf monomials x. □

Regarding the calculation of the Steenrod squares on the generators of A_D, We observe that the calculation for $\mathrm{Sq}^i(\delta_{n:m})$ is implicit in Theorem 8.2 since $\delta_{n:m} = \rho(\delta_n \odot 1_m)$ and ρ commute with Steenrod operations. Thus we only need to consider generators of the form $\gamma_{k,n}^{\pm}$.

Theorem 8.3 *Let $k, n \geq 1$ and $i \geq 0$. Then, in A_D, $\mathrm{Sq}^i(\gamma^+_{k,n})$ (resp. $\mathrm{Sq}^i(\gamma^-_{k,n})$) is the sum of all the full-width monomials $x \in \mathcal{B}^+$ (resp. $x \in \mathcal{B}^-$) of degree $2^{n+k} - 2^n + i$ with $\mathrm{ht}(x) \leq 2$ and $\mathrm{effsc}(x) \geq l$ in which generators of the form $\delta_{n:m}$ do not appear.*

Proof We recall that Definition 6.11 provides, for all maximal elementary abelian 2–subgroup $A \subseteq W_{D_{2n}}$, subspaces H^+_A and H^-_A of the cohomology of A. A direct calculation shows that $\mathrm{Sq}^i(h_n) \in H^+_{(2)^n}$. Since restrictions preserve the Steenrod squares, $\mathrm{Sq}^i(\gamma^+_{1,n})$ is mapped to an element of H^+_A for all maximal elementary abelian 2–subgroups $A \subseteq W_{D_{2n}}$ and all choices of i and n. Similarly, the restriction of $\mathrm{Sq}^i(\gamma^-_{1,n})$ to every such subgroup A lies in H^-_A. Let x^+ (resp. x^-) be the sum of all the positively (resp. negatively) charged Hopf monomials considered in the statement. By Proposition 6.12, the restriction of x^+ (resp. x^-) belongs to H^+_A (resp. H^-_A).

Moreover $\mathrm{Sq}^i(\gamma^+_{k,n}) + \mathrm{Sq}^i(\gamma^-_{k,n}) = \rho(\mathrm{Sq}^i(\gamma_{k,n}))$. Consequently, Theorem 8.2 implies that $\mathrm{Sq}^i(\gamma^+_{k,n}) + \mathrm{Sq}^i(\gamma^-_{k,n}) = x^+ + x^-$.

Since $H^+_A \cap H^-_A = 0$ for all A, the two facts above guarantee that $\mathrm{Sq}^i(\gamma^+_{k,n}) = x^+$ and $\mathrm{Sq}^i(\gamma^-_{k,n}) = x^-$. □

References

[1] **A Adem**, **R J Milgram**, *Cohomology of finite groups*, 2nd edition, Grundl. Math. Wissen. 309, Springer (2004) MR Zbl

[2] **A Björner**, **F Brenti**, *Combinatorics of Coxeter groups*, Graduate Texts in Math. 231, Springer (2005) MR Zbl

[3] **M W Davis**, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series 32, Princeton Univ. Press (2008) MR Zbl

[4] **C De Concini**, **M Salvetti**, *Cohomology of Coxeter groups and Artin groups*, Math. Res. Lett. 7 (2000) 213–232 MR Zbl

[5] **M Feshbach**, *The mod 2 cohomology rings of the symmetric groups and invariants*, Topology 41 (2002) 57–84 MR Zbl

[6] **G Friedman**, **A M Medina-Mardones**, **D Sinha**, *Foundations of geometric cohomology*: *From co-orientations to product structures*, preprint (2022) arXiv 2212.07482

[7] **C Giusti**, **P Salvatore**, **D Sinha**, *The mod-2 cohomology rings of symmetric groups*, J. Topol. 5 (2012) 169–198 MR Zbl

[8] **C Giusti**, **D Sinha**, *Fox–Neuwirth cell structures and the cohomology of symmetric groups*, from "Configuration spaces" (A Björner, F Cohen, C De Concini, C Procesi, M Salvetti, editors), CRM Series 14, Ed. Norm., Pisa (2012) 273–298 MR Zbl

[9] **C Giusti**, **D Sinha**, *Mod-two cohomology rings of alternating groups*, J. Reine Angew. Math. 772 (2021) 1–51 MR Zbl

[10] **L Guerra**, *Hopf ring structure on the* mod p *cohomology of symmetric groups*, Algebr. Geom. Topol. 17 (2017) 957–982 MR Zbl

[11] **R Hepworth**, *Homological stability for families of Coxeter groups*, Algebr. Geom. Topol. 16 (2016) 2779–2811 MR Zbl

[12] **N H V Hu'ng**, *The modulo* 2 *cohomology algebras of symmetric groups*, Japan. J. Math. 13 (1987) 169–208 MR Zbl

[13] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Mathematics Studies 76, Princeton Univ. Press (1974) MR Zbl

[14] **P Papi**, *Inversion tables and minimal left coset representatives for Weyl groups of classical type*, J. Pure Appl. Algebra 161 (2001) 219–234 MR Zbl

[15] **D Quillen**, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. 94 (1971) 549–572 MR Zbl

[16] **D Quillen**, **B B Venkov**, *Cohomology of finite groups and elementary abelian subgroups*, Topology 11 (1972) 317–318 MR Zbl

[17] **N P Strickland**, **P R Turner**, *Rational Morava E–theory and DS^0*, Topology 36 (1997) 137–151 MR Zbl

[18] **J A Swenson**, *The mod-*2 *cohomology of finite Coxeter groups*, PhD thesis, University of Minnesota (2006) Available at `https://www.proquest.com/docview/305305327`

[19] **V A Vassiliev**, *Complements of discriminants of smooth maps: Topology and applications*, Translations of Mathematical Monographs 98, Amer. Math. Soc., Providence, RI (1992) MR Zbl

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