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Partial Torelli groups and homological stability

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We prove a homological stability theorem for the subgroup of the mapping class group acting as the identity on some fixed portion of the first homology group of the surface. We also prove a similar theorem for the subgroup of the mapping class group preserving a fixed map from the fundamental group to a finite group, which can be viewed as a mapping class group version of a theorem of Ellenberg, Venkatesh and Westerland about braid groups. These results require studying various simplicial complexes formed by subsurfaces of the surface, generalizing work of Hatcher and Vogtmann.

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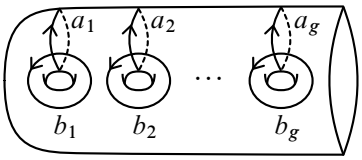
1 Introduction

Let Σ_g^b be an oriented genus-g surface with b boundary components. The *mapping class group* $\mathrm{Mod}(\Sigma_g^b)$ is the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g^b that fix $\partial\Sigma_g^b$ pointwise. Harer [11] proved that $\mathrm{Mod}(\Sigma_g^b)$ satisfies homological stability. More precisely, an orientation-preserving embedding $\Sigma_g^b \hookrightarrow \Sigma_{g'}^{b'}$ induces a map $\mathrm{Mod}(\Sigma_g^b) \to \mathrm{Mod}(\Sigma_{g'}^{b'})$ that extends mapping classes by the identity, and Harer's theorem says that the induced map $\mathrm{H}_k(\mathrm{Mod}(\Sigma_g^b)) \to \mathrm{H}_k(\mathrm{Mod}(\Sigma_{g'}^{b'}))$ is an isomorphism if $g \gg k$.

Torelli The group $\mathrm{Mod}(\Sigma_g^b)$ acts on $\mathrm{H}_1(\Sigma_g^b)$. For $b \leq 1$, the algebraic intersection pairing on $\mathrm{H}_1(\Sigma_g^b)$ is a $\mathrm{Mod}(\Sigma_g^b)$–invariant symplectic form. We thus get a map $\mathrm{Mod}(\Sigma_g^b) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}(\Sigma_g^b)$ is the *Torelli group*. The group $\mathcal{I}(\Sigma_g^b)$ is not homologically stable; indeed, Johnson [16] showed that $\mathrm{H}_1(\mathcal{I}(\Sigma_g^b))$ does not stabilize. Church and Farb's work on representation stability [4] connects this to the $\mathrm{Sp}_{2g}(\mathbb{Z})$–action on $\mathrm{H}_k(\mathcal{I}(\Sigma_g^b))$ induced by the conjugation action of $\mathrm{Mod}(\Sigma_g^b)$. Much recent work on $\mathrm{H}_k(\mathcal{I}(\Sigma_g^b))$ focuses on this action; see Boldsen and Hauge Dollerup [2], Kassabov and Putman [18] and Miller, Patzt and Wilson [21].

Partial Torelli We show that homological stability can be restored by enlarging the Torelli group to the group acting trivially on some fixed portion of homology. As an illustration of our results, we begin by describing them in a very special case. Fix a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ for $\mathrm{H}_1(\Sigma_g^1)$ in the usual way:



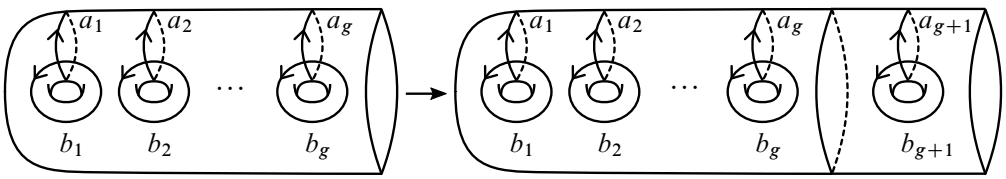
For $0 \le h \le g$, define $\mathcal{I}(\Sigma_g^1, h)$ to be the subgroup of $\mathrm{Mod}(\Sigma_g^1)$ fixing all elements of $\{a_1, b_1, \dots, a_h, b_h\}$. These groups interpolate between $\mathrm{Mod}(\Sigma_g^1)$ and $\mathcal{I}(\Sigma_g^1)$ in the sense that

$$\mathcal{I}(\Sigma_g^1) = \mathcal{I}(\Sigma_g^1, g) \subset \mathcal{I}(\Sigma_g^1, g-1) \subset \mathcal{I}(\Sigma_g^1, g-2) \subset \cdots \subset \mathcal{I}(\Sigma_g^1, 0) = \mathrm{Mod}(\Sigma_g^1).$$

They were introduced by Bestvina, Bux and Margalit [1]; see especially [1, Conjecture 1.2]. For a fixed $h \ge 1$, we have an increasing chain of groups

$$\mathcal{I}(\Sigma_h^1, h) \subset \mathcal{I}(\Sigma_{h+1}^1, h) \subset \mathcal{I}(\Sigma_{h+2}^1, h) \subset \cdots, \tag{1-1}$$

where $\mathcal{I}(\Sigma_g^1, h)$ is embedded in $\mathcal{I}(\Sigma_{g+1}^1, h)$ via



Our main theorem shows that (1-1) satisfies homological stability: for $h, k \ge 1$, we have

$$\mathrm{H}_k(\mathcal{I}(\Sigma_g^1, h)) \cong \mathrm{H}_k(\mathcal{I}(\Sigma_{g+1}^1, h))$$

for $g \ge (2h+2)k + (4h+2)$.

Homology markings To state our more general result, we need the notion of a homology marking. Let A be a finitely generated abelian group. An *A–homology marking* on Σ_g^1 is a homomorphism $\mu\colon \mathrm{H}_1(\Sigma_g^1) \to A$. Associated to this is a *partial Torelli group*

$$\mathcal{I}(\Sigma_g^1, \mu) = \{f \in \mathrm{Mod}(\Sigma_g^1) \mid \mu(f(x)) = \mu(x) \text{ for all } x \in \mathrm{H}_1(\Sigma_g^1)\}.$$

Example 1.1 If $A = \mathrm{H}_1(\Sigma_g^1)$ and $\mu = \mathrm{id}$, then $\mathcal{I}(\Sigma_g^1, \mu) = \mathcal{I}(\Sigma_g^1)$.

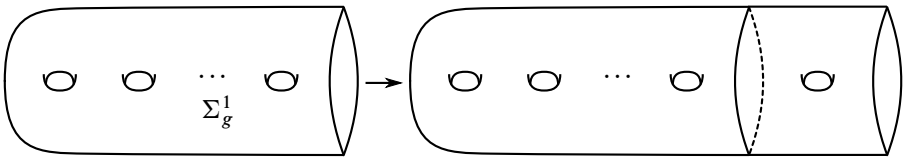
Example 1.2 If $A = \mathrm{H}_1(\Sigma_g^1; \mathbb{Z}/\ell)$ and $\mu\colon \mathrm{H}_1(\Sigma_g^1) \to A$ is the projection, then $\mathcal{I}(\Sigma_g^1, \mu)$ is the *level-ℓ subgroup* of $\mathrm{Mod}(\Sigma_g^1)$, ie the kernel of the action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathrm{H}_1(\Sigma_g^1; \mathbb{Z}/\ell)$.

Example 1.3 Let A be a *symplectic subspace* of $\mathrm{H}_1(\Sigma_g^1)$, ie a subspace with $\mathrm{H}_1(\Sigma_g^1) = A \oplus A^\perp$, where $\perp$ is defined via the intersection form. Such an A is of the form $A \cong \mathbb{Z}^{2h}$ for some $h \geq 0$, called the *genus* of A. If $\mu\colon \mathrm{H}_1(\Sigma_g^1) \to A$ is the projection, then

$$\mathcal{I}(\Sigma_g^1, \mu) = \{f \in \mathrm{Mod}(\Sigma_g^1) \mid f(x) = x \text{ for all } x \in A\}.$$

If A has genus h, then $\mathcal{I}(\Sigma_g^1, \mu) \cong \mathcal{I}(\Sigma_g^1, h)$.

Stability Our first main theorem is a homological stability theorem for the groups $\mathcal{I}(\Sigma_g^1, \mu)$. Define the *stabilization to* Σ_{g+1}^1 of an A–homology marking μ on Σ_g^1 to be the following A–homology marking μ' on Σ_{g+1}^1. Embed Σ_g^1 in Σ_{g+1}^1 just like we did above:



This identifies $\mathrm{H}_1(\Sigma_g^1)$ with a symplectic subspace of $\mathrm{H}_1(\Sigma_{g+1}^1)$, so $\mathrm{H}_1(\Sigma_{g+1}^1) = \mathrm{H}_1(\Sigma_g^1) \oplus \mathrm{H}_1(\Sigma_g^1)^\perp$. Let $\mu'\colon \mathrm{H}_1(\Sigma_{g+1}^1) \to A$ be the composition

$$\mathrm{H}_1(\Sigma_{g+1}^1) = \mathrm{H}_1(\Sigma_g^1) \oplus \mathrm{H}_1(\Sigma_g^1)^\perp \to \mathrm{H}_1(\Sigma_g^1) \xrightarrow{\mu} A,$$

where the first arrow is the orthogonal projection. The map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_{g+1}^1)$ induced by the above embedding restricts to a map $\mathcal{I}(\Sigma_g^1, \mu) \to \mathcal{I}(\Sigma_{g+1}^1, \mu')$, called the *stabilization map*. For a finitely generated abelian group A, let $\mathrm{rk}(A)$ denote the minimum size of a generating set[1] for A. Our main theorem is as follows:

Theorem A *Let A be a finitely generated abelian group, μ an A–homology marking on Σ_g^1, and μ' its stabilization to Σ_{g+1}^1. The map $\mathrm{H}_k(\mathcal{I}(\Sigma_g^1, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma_{g+1}^1, \mu'))$ induced by the stabilization map $\mathcal{I}(\Sigma_g^1, \mu) \to \mathcal{I}(\Sigma_{g+1}^1, \mu')$ is an isomorphism if $g \geq (\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+2)$ and a surjection if $g = (\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+1)$.*

[1] Equivalently, $\mathrm{rk}(A)$ is the maximum $n \geq 0$ such that A is the direct sum of n cyclic subgroups. There are several different commonly used definitions of the rank of an abelian group, and we emphasize that our $\mathrm{rk}(A)$ is *not* the maximum n such that A contains a subgroup isomorphic to $\mathbb{Z}^n$. In particular, $\mathrm{rk}(\mathbb{Z}/\ell) = 1$ for $\ell \geq 2$, and $\mathrm{rk}(A) = 0$ if and only if $A = 0$.

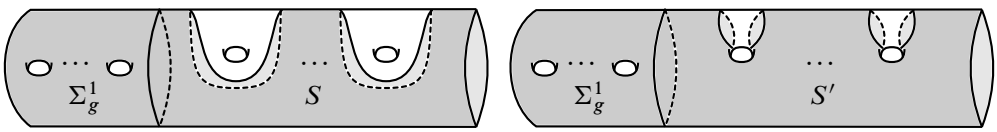
Closed surface trouble Harer's stability theorem implies that the map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g)$ arising from gluing a disc to $\partial\Sigma_g^1$ induces an isomorphism on H_k for $g \gg k$. One might expect a similar result to hold for the partial Torelli groups. Unfortunately, this is completely false. In Section 7, we will prove that it fails even for H_1 for A–homology markings satisfying a mild nondegeneracy condition called *symplectic nondegeneracy*. One special case of this is the following. For $1 \leq h \leq g$, define $\mathcal{I}(\Sigma_g, h)$ just like $\mathcal{I}(\Sigma_g^1, h)$, so we have a surjection $\mathcal{I}(\Sigma_g^1, h) \to \mathcal{I}(\Sigma_g, h)$.

Theorem B *For $h \leq g$ with $g \geq 3$ and $h \geq 2$, the map $\mathrm{H}_1(\mathcal{I}(\Sigma_g^1, h)) \to \mathrm{H}_1(\mathcal{I}(\Sigma_g, h))$ is not an isomorphism.*

The proof uses an extension of the Johnson homomorphism to the partial Torelli groups that was constructed by Broaddus, Farb and Putman [3].

Multiple boundary components In addition to Theorem A, which concerns surfaces with one boundary component, we have a theorem for surfaces with multiple boundary components. The correct statement here is a bit subtle, since the phenomenon underlying Theorem B also obstructs many obvious kinds of generalizations. The purpose of having a generalization like this is to understand how the partial Torelli groups restrict to subsurfaces, which turns out to be fundamental in the author's forthcoming work on the cohomology of the moduli space of curves with level structures [23]. Here is an example of the kind of result we prove; in fact, this is precisely the special case needed in [23].

Example 1.4 Consider an A–homology marking μ on Σ_g^1. For some $h \geq 1$, let μ' be its stabilization to Σ_{g+h}^1. Consider the following subsurfaces $S \cong \Sigma_g^{1+h}$ and $S' \cong \Sigma_g^{1+2h}$ of Σ_{g+h}^1:



Both S and S' include the entire shaded subsurface (including Σ_g^1). The inclusions $S \hookrightarrow \Sigma_{g+h}^1$ and $S' \hookrightarrow \Sigma_{g+h}^1$ induce homomorphisms $\phi\colon \mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma_{g+h}^1)$ and $\psi\colon \mathrm{Mod}(S') \to \mathrm{Mod}(\Sigma_{g+h}^1)$. Then define $\mathcal{I}(S, \mu') = \phi^{-1}(\mathcal{I}(\Sigma_{g+h}^1, \mu'))$ and $\mathcal{I}(S', \mu') = \psi^{-1}(\mathcal{I}(\Sigma_{g+h}^1, \mu'))$. Be warned: while it turns out that $\mathcal{I}(S, \mu')$ can be defined using the action of $\mathrm{Mod}(S)$ on $\mathrm{H}_1(S)$, the group $\mathcal{I}(S', \mu')$ cannot be defined using only $\mathrm{H}_1(S')$. Then our theorem will show that the map

$$\mathrm{H}_k(\mathcal{I}(S, \mu')) \to \mathrm{H}_k(\mathcal{I}(S', \mu'))$$

is an isomorphism if the genus of S (namely g) is at least $(\mathrm{rk}(A)+2)k+(2\,\mathrm{rk}(A)+2)$. However, except in degenerate cases, the maps

$$\mathrm{H}_1(\mathcal{I}(\Sigma_g^1,\mu))\to\mathrm{H}_1(\mathcal{I}(S,\mu')) \quad\text{and}\quad \mathrm{H}_1(\mathcal{I}(S,\mu'))\to\mathrm{H}_1(\mathcal{I}(\Sigma_{g+h}^1,\mu'))$$

are never isomorphisms, no matter how large g is.

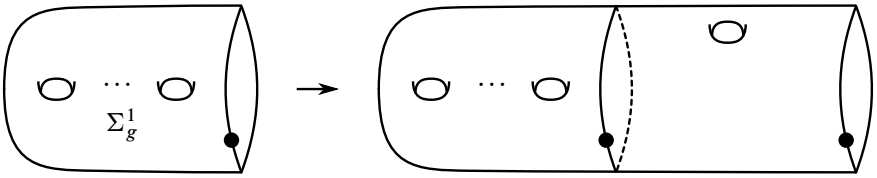
In the above example, we defined the partial Torelli groups on surfaces with multiple boundary components in an ad hoc way. Correctly formulating our theorem requires a more intrinsic definition, and we define a category of "homology-marked surfaces" with multiple boundary components that is inspired by the author's work on the Torelli group on surfaces with multiple boundary components in [22].

Nonabelian markings We also have a theorem for nonabelian markings, whose definition is as follows.[2] Fix a basepoint $*\in\partial\Sigma_g^1$. For a group Λ, a Λ–*marking* on Σ_g^1 is a group homomorphism $\mu\colon\pi_1(\Sigma_g^1,*)\to\Lambda$. If Λ is abelian, then this is equivalent to a Λ–homology marking on Σ_g^1. Given a Λ–marking $\mu\colon\pi_1(\Sigma_g^1,*)\to\Lambda$, define the associated partial Torelli group via the formula

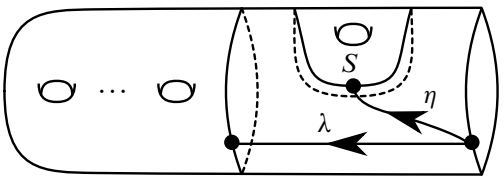
$$\mathcal{I}(\Sigma_g^1,\mu)=\{f\in\mathrm{Mod}(\Sigma_g^1)\mid\mu(f(x))=\mu(x)\text{ for all }x\in\pi_1(\Sigma_g^1,*)\}.$$

Again, this reduces to our previous definition if Λ is abelian.

Nonabelian stabilization Let μ be a Λ–marking on Σ_g^1. Due to basepoint issues, stabilizing μ to Σ_{g+1}^1 is a little more complicated than the case of homology markings. Let $*\in\partial\Sigma_g^1$ and $*'\in\partial\Sigma_{g+1}^1$ be the basepoints. Embed Σ_g^1 into Σ_{g+1}^1 as in



Let λ, η and $S\cong\Sigma_1^1$ be as in



[2] I am not sure who first defined this concept. Related things appear eg in work of Dunfield and Thurston [6] and Ivanov [13].

Letting $*'' \in \partial S$ be the basepoint of S as above, the paths λ and η induce injective homomorphisms

$$\pi_1(\Sigma_g^1, *) \hookrightarrow \pi_1(\Sigma_{g+1}^1, *') \quad \text{and} \quad \pi_1(S, *'') \hookrightarrow \pi_1(\Sigma_{g+1}^1, *')$$

taking $x \in \pi_1(\Sigma_g^1, *)$ to $\lambda \cdot x \cdot \lambda^{-1} \in \pi_1(\Sigma_{g+1}^1, *')$ and $y \in \pi_1(S, *'')$ to $\eta \cdot y \cdot \eta^{-1} \in \pi_1(\Sigma_{g+1}^1, *')$. Identifying $\pi_1(\Sigma_g^1, *)$ and $\pi_1(S, *'')$ with the corresponding subgroups of $\pi_1(\Sigma_{g+1}^1, *')$, we have a free product decomposition

$$\pi_1(\Sigma_{g+1}^1, *') = \pi_1(\Sigma_g^1, *) \star \pi_1(S, *'').$$

Then define the stabilization $\mu' \colon \pi_1(\Sigma_{g+1}^1, *') \to \Lambda$ of $\mu \colon \pi_1(\Sigma_g^1, *) \to \Lambda$ to be the composition

$$\pi_1(\Sigma_{g+1}^1, *') = \pi_1(\Sigma_g^1, *) \star \pi_1(S, *'') \to \pi_1(\Sigma_g^1, *) \xrightarrow{\mu} \Lambda,$$

where the first arrow quotients out by the normal closure of $\pi_1(S, *'')$. As in the abelian setting, the map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_{g+1}^1)$ induced by our embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g+1}^1$ restricts to a map $\mathcal{I}(\Sigma_g^1, \mu) \to \mathcal{I}(\Sigma_{g+1}^1, \mu')$, which we will call the *stabilization map*.

Nonabelian stability Our main theorem about this is as follows. It can be viewed as an analogue for the mapping class group of a theorem of Ellenberg, Venkatesh and Westerland [7, Theorem 6.1] concerning braid groups and Hurwitz spaces.

Theorem C *Let Λ be a finite group, μ a Λ–marking on Σ_g^1, and μ' its stabilization to Σ_{g+1}^1. The map $\mathrm{H}_k(\mathcal{I}(\Sigma_g^1, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma_{g+1}^1, \mu'))$ induced by the stabilization map $\mathcal{I}(\Sigma_g^1, \mu) \to \mathcal{I}(\Sigma_{g+1}^1, \mu')$ is an isomorphism if $g \geq (|\Lambda| + 2)k + (2|\Lambda| + 2)$ and a surjection if $g = (|\Lambda| + 2)k + (2|\Lambda| + 1)$.*

Remark 1.5 Ellenberg,Venkatesh and Westerland's main application in [7] of their stability result concerns point-counting in Hurwitz spaces via the Weil conjectures. Unfortunately, the vast amount of unknown unstable cohomology precludes such applications here.

Remark 1.6 If Λ is a finite abelian group, then Theorems A and C give a similar kind of stability, but the bounds in Theorem A are much stronger.

Remark 1.7 Because of basepoint issues, stating a version of Theorem C on surfaces with multiple boundary components would be rather technical, and unlike for Theorem A we do not know any potential applications of such a result. We thus do not pursue this kind of generalization of Theorem C.

Proof techniques There is an enormous literature on homological stability theorems, starting with unpublished work of Quillen on $\mathrm{GL}_n(\mathbb{F}_p)$. A standard proof technique has emerged that first appeared in its modern formulation in work of van der Kallen [17]. Consider a sequence of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \tag{1-2}$$

that we want to prove enjoys homological stability, ie $\mathrm{H}_k(G_{n-1}) \cong \mathrm{H}_k(G_n)$ for $n \gg k$. To compute $\mathrm{H}_k(G_n)$, we would need a contractible simplicial complex on which G_n acts freely. Since we are only interested in the low-degree homology groups, we can weaken contractibility to high connectivity. The key insight for homological stability is that since we only want to compare $\mathrm{H}_k(G_n)$ with the homology of previous groups in (1-2), what we want is not a free action but one whose stabilizer subgroups are related to the previous groups.

Machine There are many variants on the above machine. For proving homological stability for the groups G_n in (1-2), the easiest version requires simplicial complexes X_n upon which G_n acts with the following three properties:

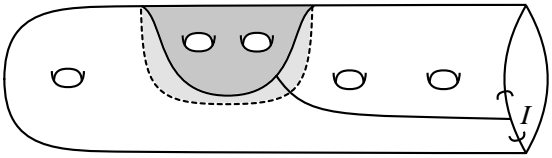
- The connectivity of X_n goes to ∞ as $n \mapsto \infty$.
- For $0 \leq k \leq n-1$, the G_n–stabilizer of a k–simplex of X_n is conjugate to G_{n-k-1}.
- The group G_n acts transitively on the k–simplices of X_n for all $k \geq 0$.

Some additional technical hypotheses are needed; we will review these in Section 3.1. Hatcher and Vogtmann [12] constructed such X_n for the mapping class group. Our proof of Theorem A is inspired by their work, so we start by describing a variant of it.

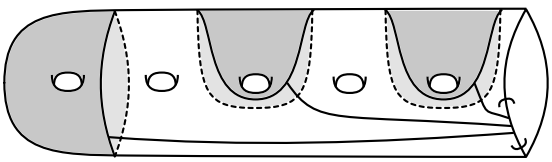
Subsurface complex For $h \geq 1$, the *complex of genus-h subsurfaces* of Σ_g^b, denoted by $\mathcal{S}_h(\Sigma_g^b)$, is the simplicial complex whose k–simplices are sets $\{\iota_0, \ldots, \iota_k\}$ of isotopy classes of orientation-preserving embeddings $\iota_i \colon \Sigma_h^1 \to \Sigma_g^b$ that can be isotoped so that, for $0 \leq i < j \leq k$, the subsurfaces $\iota_i(\Sigma_h^1)$ and $\iota_j(\Sigma_h^1)$ are disjoint. The group $\mathrm{Mod}(\Sigma_g^b)$ acts on $\mathcal{S}_h(\Sigma_g^b)$. However, it turns out that this is not quite the right complex for homological stability.

Tethered subsurfaces Let $\tau(\Sigma_h^1)$ be the result of gluing the interval $[0,1]$ to Σ_h^1 by identifying $1 \in [0,1]$ with a point of $\partial\Sigma_h^1$. The subset $[0,1] \subset \tau(\Sigma_h^1)$ is the *tether* and $0 \in [0,1] \subset \tau(\Sigma_h^1)$ the *initial point* of the tether. Let $I \subset \partial\Sigma_g^b$ be a finite disjoint union of open intervals. An *I–tethered genus-h subsurface* of Σ_g^b is an embedding

$\iota\colon \tau(\Sigma_h^1) \to \Sigma_g^b$ taking the initial point of the tether to a point of I whose restriction to Σ_h^1 preserves the orientation. For instance, here is an I–tethered genus-2 subsurface:



Tethered subsurface complex The *complex of I–tethered genus-h subsurfaces* of Σ_g^b, denoted by $\mathcal{TS}_h(\Sigma_g^b, I)$, is the simplicial complex whose k–simplices are collections $\{\iota_0, \dots, \iota_k\}$ of isotopy classes of I–tethered genus-h subsurfaces of Σ_g^b that can be realized disjointly. These isotopies are allowed to move the images of the initial points of the tethers within I, so the tethers can be slid past each other and made disjoint. For instance, here is a 2–simplex in $\mathcal{TS}_1(\Sigma_5^1, I)$:



High connectivity The complexes $\mathcal{S}_1(\Sigma_g^b)$ and $\mathcal{TS}_1(\Sigma_g^b, I)$ are closely related to complexes that were introduced by Hatcher and Vogtmann [12], and it follows easily from their work that they are $\frac{1}{2}(g-3)$–connected (see Putman and Sam [24, proof of Theorem 6.25] for details). We generalize this as follows:

Theorem D *Consider $g \geq h \geq 1$ and $b \geq 0$.*

- *The complex $\mathcal{S}_h(\Sigma_g^b)$ is $(g-(2h+1))/(h+1)$–connected.*
- *Assume that $b \geq 1$, and let $I \subset \partial\Sigma_g^b$ be a finite disjoint union of open intervals. The complex $\mathcal{TS}_h(\Sigma_g^b, I)$ is $(g-(2h+1))/(h+1)$–connected.*

Remark 1.8 Our convention is that a space is (-1)–connected if it is nonempty. Using this convention, the genus bounds for (-1)–connectivity and 0–connectivity in Theorem D are sharp. We do not know whether they are sharp for higher connectivity.

Remark 1.9 Hatcher and Vogtmann's proof in [12] that $\mathcal{S}_1(\Sigma_g^b)$ and $\mathcal{TS}_1(\Sigma_g^b, I)$ are $\frac{1}{2}(g-3)$–connected is closely connected to their proof that the separating curve complex is $\frac{1}{2}(g-3)$–connected. Looijenga [20] later showed that the separating curve complex is $(g-3)$–connected. Unfortunately, his techniques do not appear to give an improvement to Theorem D.

Remark 1.10 In applications to homological stability, we will only use complexes made out of genus-1 subsurfaces. However, the more general result of Theorem D will be needed for the proof even of the $h = 1$ case of Theorem E below.

Mod stability Consider the groups

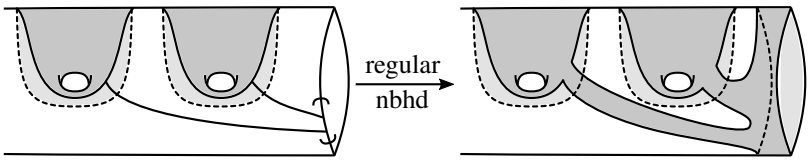
$$\mathrm{Mod}(\Sigma_1^1) \subset \mathrm{Mod}(\Sigma_2^1) \subset \mathrm{Mod}(\Sigma_3^1) \subset \cdots. \tag{1-3}$$

Let $I \subset \partial\Sigma_g^1$ be an open interval. The group $\mathrm{Mod}(\Sigma_g^1)$ acts on $\mathcal{TS}_1(\Sigma_g^1, I)$, and this complex has all three properties needed by the machine to prove homological stability for (1-3):

- As we said above, $\mathcal{TS}_1(\Sigma_g^1, I)$ is $\frac{1}{2}(g-3)$–connected.
- The $\mathrm{Mod}(\Sigma_g^1)$–stabilizer of a k–simplex $\{\iota_0, \ldots, \iota_k\}$ of $\mathcal{TS}_1(\Sigma_g^1, I)$ is the mapping class group of the complement of a regular neighborhood of

$$\partial\Sigma_g^1 \cup \iota_0(\tau(\Sigma_1^1)) \cup \cdots \cup \iota_k(\tau(\Sigma_1^1)).$$

See here:



This complement is homeomorphic to Σ_{g-k-1}^1, so this stabilizer is isomorphic to $\mathrm{Mod}(\Sigma_{g-k-1}^1)$. All such subsurface mapping class groups are conjugate; this follows from the change of coordinates principle of Farb and Margalit [8, Section 1.3.2].

- Another application of the change of coordinates principle shows that $\mathrm{Mod}(\Sigma_g^1)$ acts transitively on the k–simplices of $\mathcal{TS}_1(\Sigma_g^1, I)$.

Partial Torelli problem A first idea for proving homological stability for the partial Torelli groups $\mathcal{I}(\Sigma_g^1, \mu)$ is to consider their actions on $\mathcal{TS}_1(\Sigma_g^1, I)$. Unfortunately, this does not work. The fundamental problem is that $\mathcal{I}(\Sigma_g^1, \mu)$ does not act transitively on the k–simplices of $\mathcal{TS}_1(\Sigma_g^1, I)$; indeed, it does not even act transitively on the vertices. For A–homology markings μ, the issue is that, for an I–tethered torus $\iota\colon \tau(\Sigma_1^1) \to \Sigma_g^1$ and $f \in \mathcal{I}(\Sigma_g^1, \mu)$, the compositions

$$\mathrm{H}_1(\Sigma_1^1) \cong \mathrm{H}_1(\tau(\Sigma_1^1)) \xrightarrow{\iota_*} \mathrm{H}_1(\Sigma_g^1) \xrightarrow{\mu} A, \quad \mathrm{H}_1(\Sigma_1^1) \cong \mathrm{H}_1(\tau(\Sigma_1^1)) \xrightarrow{(f\circ\iota)_*} \mathrm{H}_1(\Sigma_g^1) \xrightarrow{\mu} A$$

will be the same, but the functions $\mu \circ \iota_*\colon \mathrm{H}_1(\tau(\Sigma_1^1)) \to A$ need not be the same for different tethered tori. A similar issue arises in the nonabelian setting. To fix this, we use a subcomplex of $\mathcal{TS}_1(\Sigma_g^1, I)$ that is adapted to μ.

Remark 1.11 The stabilizers are also wrong, but fixing the transitivity will also fix this.

Vanishing surfaces For an A–homology marking μ on Σ_g^1, define $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ to be the full subcomplex of $\mathcal{TS}_h(\Sigma_g^1, I)$ spanned by vertices ι such that the composition

$$\mathrm{H}_1(\tau(\Sigma_h^1)) \xrightarrow{\iota_*} \mathrm{H}_1(\Sigma_g^1) \xrightarrow{\mu} A$$

is the zero map. We will show that $\mathcal{I}(\Sigma_g^1, \mu)$ acts transitively on the k–simplices of $\mathcal{TS}_1(\Sigma_g^1, I, \mu)$, at least for k not too large. However, there is a problem: a priori the subcomplex $\mathcal{TS}_1(\Sigma_g^1, I, \mu)$ of $\mathcal{TS}_1(\Sigma_g^1, I)$ might not be highly connected. Our third main theorem says that in fact it is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected. More generally, we prove the following:

Theorem E *Let A be a finitely generated abelian group, let μ be an A–homology marking on Σ_g^1, and let $I \subset \partial\Sigma_g^1$ be a finite disjoint union of open intervals. Then the complex $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ is $\big(g-(2\,\mathrm{rk}(A)+2h+1)\big)/(\mathrm{rk}(A)+h+1)$–connected.*

We also prove a similar theorem in the nonabelian setting.

Outline We start in Section 2 by proving Theorem D. We then prove Theorems A, C, and E in Section 3. Next, in Section 4 we define a category of homology-marked surfaces with multiple boundary components. In Section 5 we use our category to state and prove Theorem F, which generalizes Theorem A to surfaces with multiple boundary components. This proof depends on a stabilization result which is proved in Section 6. We close with Section 7, which proves Theorem B.

Conventions Throughout this paper, A denotes a fixed finitely generated abelian group and Λ is a fixed finite group. We also fix a basepoint $* \in \partial\Sigma_g^1$.

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2 The complex of subsurfaces

This section is devoted to the proof of Theorem D, which asserts that $\mathcal{S}_h(\Sigma_g^b)$ and $\mathcal{TS}_h(\Sigma_g^b, I)$ are highly connected. There are three parts: Section 2.1 contains a technical result about fibers of maps, Section 2.2 discusses "link arguments", and Section 2.3 proves Theorem D.

2.1 Fibers of maps

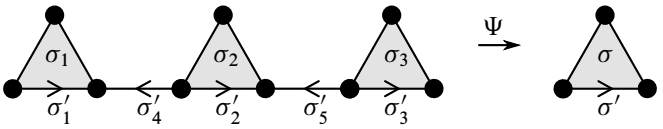
Our proofs will require a technical tool:

Homotopy theory conventions A space X is said to be n–connected if, for $k \leq n$, all maps $S^k \to X$ extend to maps $D^{k+1} \to X$. Since $S^{-1} = \varnothing$ and D^0 is a single point, a space is (-1)–connected precisely when it is nonempty. A map $\psi : X \to Y$ of spaces is an n–homotopy equivalence if, for all $0 \leq k \leq n$, the induced map $[S^k, X] \to [S^k, Y]$ on unbased homotopy classes of maps out of S^k is a bijection. This is equivalent to saying that the induced map on π_k is a bijection for each choice of basepoint.

Relative fibers If $\psi : X \to Y$ is a map of simplicial complexes, σ is a simplex of Y, and σ' is a face of σ, then denote by $\mathrm{Fib}_\psi(\sigma', \sigma)$ the subcomplex of X consisting of all simplices η' of X such that

- $\psi(\eta')$ is a face of σ', and
- there exists a simplex η of X such that η' is a face of η and $\psi(\eta) = \sigma$.

For instance, consider the following map, where ψ takes each 1–simplex σ_i' to σ' (with the specified orientation) and each 2–simplex σ_i to σ:



The relative fiber $\mathrm{Fib}_\psi(\sigma', \sigma)$ then consists of σ_1' and σ_2' and σ_3' (but not σ_4' or σ_5').

Fiber lemma With these definitions, we have the following lemma:

Lemma 2.1 *Let $\psi : X \to Y$ be a map of simplicial complexes. For some $n \geq 0$, assume the space $\mathrm{Fib}_\psi(\sigma', \sigma)$ is n–connected for all simplices σ of Y and all faces σ' of σ. Then ψ is an n–homotopy equivalence.*

Proof Replacing Y by its $(n+1)$–skeleton Y_{n+1} and X by $\psi^{-1}(Y_{n+1})$, we can assume that Y is finite dimensional. The proof will be by induction on $m = \dim(Y)$. The base case $m = 0$ is trivial, since in that case Y is a discrete set of points and our assumptions imply that the fiber over each of these points is n–connected. Assume now that $m \geq 1$. The key step in the proof is the following claim:

Claim *Assume that Y is the union of a subcomplex Y' and an m–simplex σ with $\sigma \cap Y' = \partial\sigma$. Define $X' = \psi^{-1}(Y')$, and assume that $\psi : X \to Y$ restricts to an n–homotopy equivalence $\psi' : X' \to Y'$. Then ψ is an n–homotopy equivalence.*

Proof Let $X'' = \mathrm{Fib}_\psi(\sigma, \sigma)$. In other words, X'' consists of all simplices of X mapping surjectively onto σ, along with their faces. We thus have $X = X' \cup X''$. By assumption, X'' is n–connected, which implies that ψ restricts to an n–homotopy equivalence $\psi'' \colon X'' \to \sigma$. Define $Z = X' \cap X''$. The map ψ restricts to a map $\psi_Z \colon Z \to \partial\sigma$.

We now come to the key observation: the space Z is precisely the subcomplex of X consisting of the union of the subcomplexes $\mathrm{Fib}_\psi(\sigma', \sigma)$ as σ' ranges over all simplices of $\partial\sigma$. Moreover, for all simplices σ' of $\partial\sigma$ and all faces σ'' of σ', we have $\mathrm{Fib}_{\psi_Z}(\sigma'', \sigma') = \mathrm{Fib}_\psi(\sigma'', \sigma)$, and thus by assumption $\mathrm{Fib}_{\psi_Z}(\sigma'', \sigma')$ is n–connected. We can therefore apply our inductive hypothesis to see that $\psi_Z \colon Z \to \partial\sigma \cong S^{m-1}$ is an n–homotopy equivalence.

Summing up, we have $X = X' \cup X''$ and $Y = Y' \cup \sigma$. The map ψ restricts to n–homotopy equivalences

$$\psi' \colon X' \to Y', \quad \psi'' \colon X'' \to \sigma, \quad \text{and} \quad \psi_Z \colon X' \cap X'' = Z \to \partial\sigma = Y' \cap \sigma,$$

and induces a map between the Mayer–Vietoris exact sequences associated to the decompositions $X = X' \cup X''$ and $Y = Y' \cup \sigma$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}_k(X' \cap X'') & \longrightarrow & \mathrm{H}_k(X') \oplus \mathrm{H}_k(X'') & \longrightarrow & \mathrm{H}_k(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{H}_k(Y' \cap \sigma) & \longrightarrow & \mathrm{H}_k(Y') \oplus \mathrm{H}_k(\sigma) & \longrightarrow & \mathrm{H}_k(Y) & \longrightarrow & \cdots
\end{array}$$

Other than the maps $\mathrm{H}_k(X) \to \mathrm{H}_k(Y)$, the vertical maps in this commutative diagram are isomorphisms for $k \leq n$, so by the five lemma the maps $\mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ are also isomorphisms for $k \leq n$. This implies in particular that the map $X \to Y$ is 0–connected, and thus induces a bijection between path-components. If $n \geq 1$, then a similar argument on each path component using the Seifert–van Kampen theorem shows that the map $\psi \colon X \to Y$ induces an isomorphism on π_1 for each choice of basepoint. This allows us to identify local coefficient systems on Y with local coefficient systems on X, and for each local coefficient system A on Y we can run the above Mayer–Vietoris argument on homology with coefficients in A to prove that the map $\psi \colon \mathrm{H}_k(X; A) \to \mathrm{H}_k(Y; A)$ is an isomorphism for $k \leq n$. Applying the nonsimply connected version of Whitehead's theorem [5, Theorem 6.71], we deduce that the map $X \to Y$ is an n–homotopy equivalence, as desired. □

Repeatedly applying this claim, we see that the lemma holds for m–dimensional Y with finitely many m–simplices.

The general case reduces to the case where Y has finitely many m–simplices as follows. Consider some $0 \le k \le n$. Our goal is to prove that the map $[S^k, X] \to [S^k, Y]$ induced by ψ is a bijection. The proofs that it is injective and surjective are similar compactness arguments, so we give the details for surjectivity and leave injectivity to the reader.

Consider a map $f\colon S^k \to Y$. By compactness, $f(S^k)$ lies in a subcomplex of Y' of Y containing the $(m-1)$–skeleton and finitely many m–simplices. Letting $X' = \psi^{-1}(Y')$, we know that the map $[S^k, X'] \to [S^k, Y']$ is a bijection, so there exists some $\tilde{f}\colon S^k \to X'$ such that $\psi \circ \tilde{f}\colon S^k \to Y'$ is homotopic to f. It follows that the map $[S^k, X] \to [S^k, Y]$ induced by ψ is surjective, as desired. □

Corollary 2.2 *Let $\psi\colon X \to Y$ be a map of simplicial complexes. For some $n \ge 0$, assume*

- *Y is n–connected, and*
- *all $(n+1)$–simplices of Y are in the image of ψ, and*
- *for all simplices σ of Y whose dimension is at most n and all faces σ' of σ, the space $\mathrm{Fib}_\psi(\sigma', \sigma)$ is n–connected.*

Then X is n–connected.

Proof Let Y' be the n–skeleton of Y and $X' = \psi^{-1}(Y')$, so X' contains the n–skeleton of X. Let $\psi'\colon X' \to Y'$ be the restriction of ψ to X'. Our assumptions allow us to apply Lemma 2.1 to ψ', so ψ' is an n–homotopy equivalence. Since Y is n–connected the space Y' is $(n-1)$–connected, so this implies that X' and thus X are $(n-1)$–connected. We also know that the induced map $\psi'\colon \pi_n(X') \to \pi_n(Y')$ is an isomorphism. Since Y is n–connected, attaching the $(n+1)$–simplices of Y to Y' kills $\pi_n(Y')$. By assumption, for each of these $(n+1)$–simplices σ of Y there is an $(n+1)$–simplex $\tilde{\sigma}$ of X such that $\psi(\tilde{\sigma}) = \sigma$. It follows that the element of $\pi_n(Y')$ represented by $\partial\sigma \to Y'$ lifts to the element of $\pi_n(X')$ represented by $\partial\tilde{\sigma} \to X'$. We conclude that attaching to X' the $(n+1)$–simplices of X that do not already lie in X' kills $\pi_n(X')$, which implies that $\pi_n(X) = 0$, as desired. □

2.2 Link arguments

Let X be a simplicial complex and let $Y \subset X$ be a subcomplex. This section is devoted to a result of Hatcher and Vogtmann [12] that gives conditions under which the pair (X, Y) is n–connected, ie $\pi_k(X, Y) = 0$ for $0 \le k \le n$. The idea is to identify a collection $\mathcal{B}$ of "bad simplices" of X that characterize Y in the sense that a simplex lies in Y precisely when none of its faces lie in $\mathcal{B}$. We then have to understand the local

topology of Y around a simplex of $\mathcal{B}$. To that end, if $\mathcal{B}$ is a collection of simplices of X and $\sigma \in \mathcal{B}$, then define $G(X, \sigma, \mathcal{B})$ to be the subcomplex of X consisting of simplices σ' satisfying:

- The join $\sigma * \sigma'$ is a simplex of X, ie σ' is a simplex in the link of σ.
- If σ'' is a face of $\sigma * \sigma'$ such that $\sigma'' \in \mathcal{B}$, then $\sigma'' \subset \sigma$.

Hatcher and Vogtmann's result is then as follows.

Proposition 2.3 [12, Proposition 2.1] *Let Y be a subcomplex of a simplicial complex X and assume that there exists a collection $\mathcal{B}$ of simplices of X satisfying, for some $n \geq 0$:*

(i) *A simplex of X lies in Y if and only if none of its faces lie in $\mathcal{B}$.*

(ii) *If $\sigma_1, \sigma_2 \in \mathcal{B}$ are such that $\sigma_1 \cup \sigma_2$ is a simplex of X, then $\sigma_1 \cup \sigma_2 \in \mathcal{B}$. Here σ_1 and σ_2 might share vertices, so $\sigma_1 \cup \sigma_2$ might not be the join $\sigma_1 * \sigma_2$.*

(iii) *For all k–dimensional $\sigma \in \mathcal{B}$, the complex $G(X, \sigma, \mathcal{B})$ is $(n-k-1)$–connected.*

Then the pair (X, Y) is n–connected.

As an illustration of how Proposition 2.3 might be used, we use it to prove the following result (which will in fact be how we use that proposition in all but two cases).

Corollary 2.4 *Let X be a simplicial complex and let $Y, Y' \subset X$ be disjoint full subcomplexes such that every vertex of X lies in either Y or Y'. For some $n \geq 0$, assume that for all k–dimensional simplices σ of Y' the intersection of Y with the link of σ is $(n-k-1)$–connected. Then the pair (X, Y) is n–connected.*

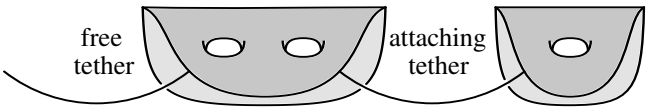
Proof We will verify the hypotheses of Proposition 2.3 for the set $\mathcal{B}$ of all simplices of Y'. Since Y is a full subcomplex of X and all vertices of X lie in either Y or Y', a simplex of X lies in Y if and only if none of its vertices lie in Y'. Hypothesis (i) follows. Hypothesis (ii) is immediate from the fact that Y' is a full subcomplex of X. As for hypothesis (iii), it is immediate from the definitions that, for a simplex $\sigma \in \mathcal{B}$, the complex $G(X, \sigma, \mathcal{B})$ is precisely the intersection of the link of σ with Y. □

2.3 Subsurface complexes

Proof of Theorem D The proofs for $\mathcal{S}_h(\Sigma_g^b)$ and $\mathcal{TS}_h(\Sigma_g^b, I)$ are similar. Keeping track of the tethers introduces a few complications, so we will give the details for $\mathcal{TS}_h(\Sigma_g^b, I)$ and leave $\mathcal{S}_h(\Sigma_g^b)$ to the reader.

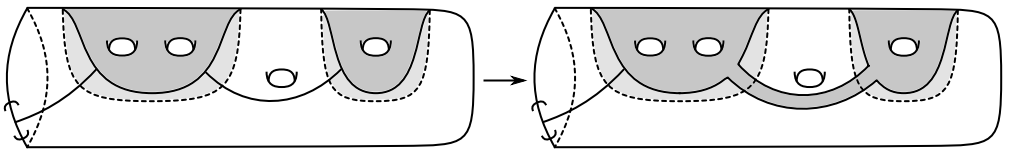
The proof that $\mathcal{TS}_h(\Sigma_g^b, I)$ is $(g-(2h+1))/(h+1)$–connected will be by induction on h. The base case $h = 1$ is [24, Theorem 6.25], which we remark shows how to derive it from a closely related result of Hatcher and Vogtmann [12]. For the inductive step, assume that $\mathcal{TS}_h(\Sigma_g^b, I)$ is $(g-(2h+1))/(h+1)$–connected. We will prove that $\mathcal{TS}_{h+1}(\Sigma_g^b, I)$ is $(g-(2h+3))/(h+2)$–connected.

Let $\tau(\Sigma_h^1, \Sigma_1^1)$ be the space obtained from $\tau(\Sigma_h^1) \sqcup \Sigma_1^1$ by gluing in an interval $[0, 1]$ with 0 being attached to a point of $\partial\Sigma_h^1$ different from the attaching point of the tether in $\tau(\Sigma_h^1)$ and 1 being attached to a point of $\partial\Sigma_1^1$:



The tether in $\tau(\Sigma_h^1)$ will be called the *free tether* and the interval connecting $\tau(\Sigma_h^1)$ to Σ_1^1 will be called the *attaching tether*. The points 0 of the two tethers will be called their *initial points* and the points 1 will be called their *endpoints*.

Given an embedding $\tau(\Sigma_h^1, \Sigma_1^1) \to \Sigma_g^b$ taking the initial point of the free tether to a point of I, thickening up the attaching tether gives an I–tethered Σ_{h+1}^1:



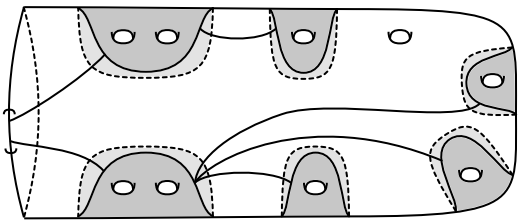
In fact, there is a bijection between isotopy classes of orientation-preserving I–tethered Σ_{h+1}^1 in Σ_g^b and isotopy classes of embeddings $\tau(\Sigma_h^1, \Sigma_1^1) \to \Sigma_g^b$ whose restrictions to Σ_h^1 and Σ_1^1 preserve the orientation and which take the initial point of the free tether to a point of I. For short, we will call these *orientation-preserving I–tethered* $\tau(\Sigma_h^1, \Sigma_1^1)$ in Σ_g^b. We remark that this is slightly awkward terminology, since the free tether is part of $\tau(\Sigma_h^1, \Sigma_1^1)$, while on the other hand we previously talked about I–tethered Σ_{h+1}^1 with the tether implicit. By the above, we can regard $\mathcal{TS}_{h+1}(\Sigma_g^b, I)$ as being the simplicial complex whose k–simplices are collections $\{\iota_0, \dots, \iota_k\}$ of isotopy classes of orientation-preserving I–tethered $\tau(\Sigma_h^1, \Sigma_1^1)$ in Σ_g^b that can be realized so that their images are disjoint.

We now define an auxiliary space. Let X be the simplicial complex whose k–simplices are collections $\{\iota_0, \dots, \iota_k\}$ of isotopy classes of orientation-preserving I–tethered $\tau(\Sigma_h^1, \Sigma_1^1)$ in Σ_g^b that can be realized so that, for all distinct $0 \le i, j \le k$:

- Either $\iota_i|_{\tau(\Sigma_h^1)} = \iota_j|_{\tau(\Sigma_h^1)}$, or the images under ι_i and ι_j of $\tau(\Sigma_h^1)$ are disjoint.

- If $\iota_i|_{\tau(\Sigma_h^1)} = \iota_j|_{\tau(\Sigma_h^1)}$, then the images under ι_i and ι_j of Σ_1^1 together with the attaching tether are disjoint, except for the initial point of the attaching tether.
- If the images under ι_i and ι_j of $\tau(\Sigma_h^1)$ are disjoint, then the images under ι_i and ι_j of $\tau(\Sigma_h^1, \Sigma_1^1)$ are disjoint.

For instance, here is a 3–simplex of X for $g = 9$, $b = 1$ and $h = 2$:



We have $\mathcal{TS}_{h+1}(\Sigma_g^b, I) \subset X$. The next claim says that X enjoys the connectivity property we are trying to prove for $\mathcal{TS}_{h+1}(\Sigma_g^b, I)$:

Claim *X is $(g-(2h+3))/(h+2)$–connected.*

Proof Let $\psi : X \to \mathcal{TS}_h(\Sigma_g^b, I)$ be the map that takes a vertex $\iota : \tau(\Sigma_h^1, \Sigma_1^1) \to \Sigma_g^b$ of X to the vertex $\iota|_{\tau(\Sigma_h^1)} : \tau(\Sigma_h^1) \to \Sigma_g^b$ of $\mathcal{TS}_h(\Sigma_g^b, I)$. We prove that $\psi : X \to \mathcal{TS}_h(\Sigma_g^b, I)$ satisfies the conditions of Corollary 2.2 for $n = (g-(2h+3))/(h+2)$. Once we have done this, Corollary 2.2 will show that X is n–connected, as desired.

The first condition is that $\mathcal{TS}_h(\Sigma_g^b, I)$ is n–connected. In fact, our inductive hypothesis says that it is $(g-(2h+1))/(h+1)$–connected, which is even stronger.

The second condition says that all $(n+1)$–simplices of $\mathcal{TS}_h(\Sigma_g^b, I)$ are in the image of ψ. The map ψ is $\mathrm{Mod}(\Sigma_g^b)$–equivariant, and by the change of coordinates principle from [8, Section 1.3.2] the actions of $\mathrm{Mod}(\Sigma_g^b)$ on $\mathcal{TS}_{h+1}(\Sigma_g^b, I)$ and $\mathcal{TS}_h(\Sigma_g^b, I)$ are transitive on k–simplices for all k. To prove the second condition, therefore, it is enough to show that $\mathcal{TS}_{h+1}(\Sigma_g^b, I) \subset X$ contains an $(n+1)$–simplex. Such a simplex contains $n+2$ disjoint copies of $\tau(\Sigma_h^1, \Sigma_1^1)$. Since

$$\begin{aligned}(n+2)(h+1) &= \left(\frac{g-(2h+3)}{h+2}+2\right)(h+1) = \left(\frac{g-(2h+3)}{h+2}\right)(h+1)+2(h+1)\\ &< (g-(2h+3))+2(h+1) = g-1 < g,\end{aligned}$$

there is enough room on Σ_g^b to find these $n+2$ disjoint copies of $\tau(\Sigma_h^1, \Sigma_1^1)$.

The final condition says that, for all simplices σ of $\mathcal{TS}_h(\Sigma_g^b, I)$ whose dimension is at most n and all faces σ' of σ, the space $\mathrm{Fib}_\psi(\sigma', \sigma)$, defined right before Lemma 2.1,

is n–connected. Recall that $\mathrm{Fib}_\psi(\sigma', \sigma)$ is the subcomplex of X consisting of all simplices η' of X such that

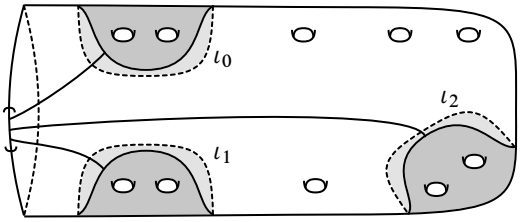
(i) $\psi(\eta')$ is a face of σ', and

(ii) there exists a simplex η of X such that η' is a face of η and $\psi(\eta) = \sigma$.

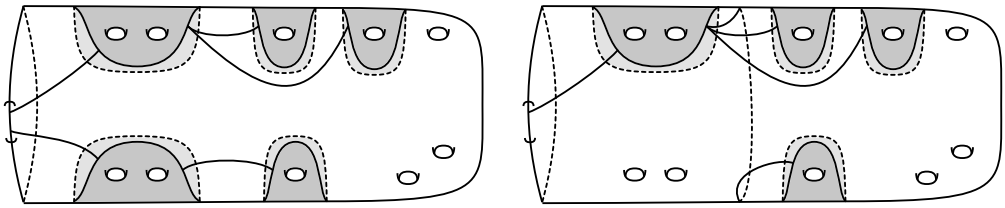
Write

$$\sigma' = \{\iota_0, \dots, \iota_{m'}\} \quad \text{and} \quad \sigma = \{\iota_0, \dots, \iota_{m'}, \dots, \iota_m\},$$

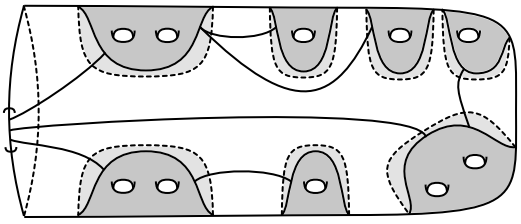
so $0 \le m' \le m \le n$. We will illustrate all our constructions here with the following running example, where $\sigma' = \{\iota_0, \iota_1\}$ and $\sigma = \{\iota_0, \iota_1, \iota_2\}$:



The left side of the following depicts a 2–simplex of X lying in $\mathrm{Fib}_\psi(\sigma', \sigma)$ and the right side depicts a 2–simplex of X that does not lie in $\mathrm{Fib}_\psi(\sigma', \sigma)$:



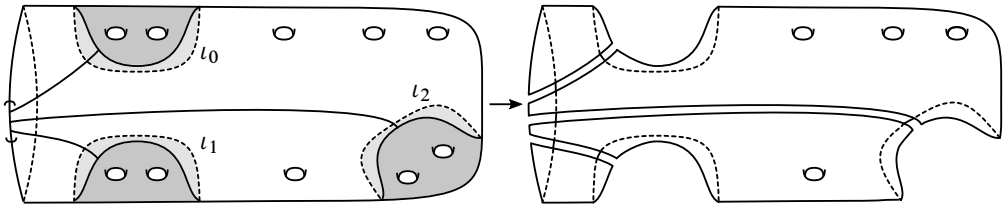
The issue with the simplex η' on the right is that there is not enough genus remaining on the surface to find a simplex η of X satisfying condition (ii) above. For the simplex η' on the left, the desired simplex η is as follows:



To understand the connectivity of $\mathrm{Fib}_\psi(\sigma', \sigma)$, we must relate it to a complex we already understand. Let Σ be the surface obtained by first removing the interior of

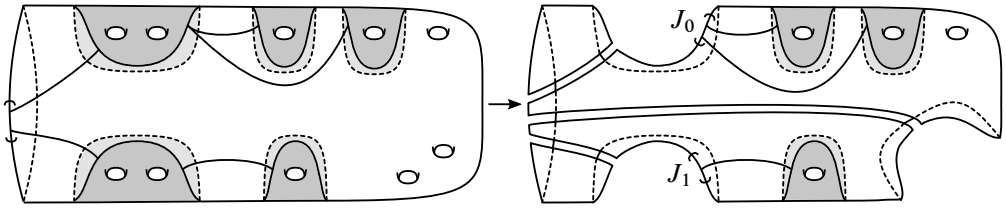
$$\iota_0(\Sigma_h^1) \cup \cdots \cup \iota_m(\Sigma_h^1)$$

from Σ_g^b and then cutting open the result along the images of the tethers. In our running example, Σ is obtained as follows:



We thus have $\Sigma \cong \Sigma_{g-(m+1)h}^b$. For $0 \le i \le m'$, let $J_i \subset \partial\Sigma$ be an open interval in $\iota_i(\partial\Sigma_h^1)$ containing the image of the point on $\partial\Sigma_h^1$ to which the attaching tether is attached when forming $\tau(\Sigma_h^1, \Sigma_1^1)$. Set $J = J_1 \cup \cdots \cup J_{m'}$.

The complex $\mathrm{Fib}_\psi(\sigma', \sigma)$ is isomorphic to a subcomplex $\mathcal{TS}_1'(\Sigma, J)$ of $\mathcal{TS}_1(\Sigma, J)$. In our running example, the simplex of $\mathrm{Fib}_\psi(\sigma', \sigma)$ on the left-hand side corresponds to the simplex of $\mathcal{TS}_1(\Sigma, J)$ on the right-hand side:



The different tethers in a simplex of $\mathrm{Fib}_\psi(\sigma', \sigma) \subset X$ that meet at a point of $\iota_i(\partial\Sigma_h^1)$ are "spread out" in J_i so as to be disjoint. The reason that $\mathrm{Fib}_\psi(\sigma', \sigma)$ is only isomorphic to a subcomplex $\mathcal{TS}_1'(\Sigma, J)$ of $\mathcal{TS}_1(\Sigma, J)$ and not the whole thing is that it only corresponds to simplices where there is enough genus remaining to ensure condition (ii) above holds.

Recall that m' and m satisfy $0 \le m' \le m \le n$. As we noted in the first paragraph, the connectivity of $\mathcal{TS}_1(\Sigma, J)$ is at least

$$\tfrac{1}{2}(g - (m+1)h - 3) \ge \tfrac{1}{2}(g - (n+1)h - 3).$$

To prove that connectivity of $\mathcal{TS}_1'(\Sigma, J)$ is at least $n = (g - (2h+3))/(h+2)$, it is enough to prove that

- $\frac{1}{2}(g - (n+1)h - 3) \ge n$, and
- $\mathcal{TS}_1'(\Sigma, J)$ contains the $(n+1)$–skeleton of $\mathcal{TS}_1(\Sigma, J)$.

For the first fact, we calculate as follows:

$$\frac{1}{2}(g-(n+1)h-3)=\frac{1}{2}\left(g-\left(\frac{g-(2h+3)}{h+2}+1\right)h-3\right)=\frac{g+\frac{1}{2}h^2-h-3}{h+2}\geq\frac{g-2h-3}{h+2}.$$

Here the final inequality follows from $\frac{1}{2}h^2-h\geq -2h$, which holds for $h\geq 0$.

For the second, consider a simplex $\{j_0,\dots,j_\ell\}$ of the $(n+1)$–skeleton of $\mathcal{TS}_1(\Sigma,J)$. Let Σ' be the surface obtained by first removing the interior of

$$j_0(\Sigma_1^1)\cup\cdots\cup j_\ell(\Sigma_1^1)$$

from $\Sigma\cong\Sigma^b_{g-(m+1)h}$ and then cutting open the result along the images of the tethers. It follows that

$$\Sigma'\cong\Sigma^b_{g-(m+1)h-(\ell+1)}.$$

In the worst case, where the corresponding simplex η' of X maps to a vertex of σ', we need at least m genus remaining to complete η' to a simplex mapping to σ. In other words, what we must prove is that

$$g-(m+1)h-(\ell+1)\geq m.$$

Since our simplex lies in the $(n+1)$–skeleton, we have $\ell\leq n+1$. Also, $m\leq n$. It follows that it is enough to prove that

$$g-(n+1)h-(n+2)\geq n.$$

Rearranging this, we get

$$\frac{g-h-2}{h+2}\geq n.$$

This follows from the fact that $n=(g-(2h+3))/(h+2)$. □

We now use this to prove the desired connectivity property for $\mathcal{TS}_{h+1}(\Sigma^b_g,I)$.

Claim *$\mathcal{TS}_{h+1}(\Sigma^b_g,I)$ is $(g-(2h+3))/(h+2)$–connected.*

Proof We prove that $\mathcal{TS}_{h+1}(\Sigma^b_g,I)$ is n–connected for $-1\leq n\leq(g-(2h+3))/(h+2)$ by induction on n. The base case $n=-1$ simply asserts that $\mathcal{TS}_{h+1}(\Sigma^b_g,I)$ is nonempty when $(g-(2h+3))/(h+2)\geq -1$. This condition is equivalent to $g\geq h+1$, in which case $\mathcal{TS}_{h+1}(\Sigma^b_g,I)\neq\varnothing$ is obvious.

Assume now that $0\leq n\leq(g-(2h+3))/(h+2)$ and that, for all surfaces $\Sigma^{b'}_{g'}$ and all finite disjoint unions of open intervals $I'\subset\partial\Sigma^{b'}_{g'}$, the space $\mathcal{TS}_{h+1}(\Sigma^{b'}_{g'},I')$ is n'–connected for $n'=\min\{n-1,(g'-(2h+3))/(h+2)\}$. We must prove that $Y:=\mathcal{TS}_{h+1}(\Sigma^b_g,I)$ is n–connected.

We know that X is n–connected, so to prove that its subcomplex Y is n–connected it is enough to prove that the pair (X, Y) is $(n+1)$–connected. We do this using Proposition 2.3. For this, we must identify a set $\mathcal{B}$ of "bad simplices" of X and verify the three hypotheses of the proposition. Define $\mathcal{B}$ to be the set of all simplices σ of X such that, for all vertices v of σ, there exists another vertex v' of σ such that the edge $\{v, v'\}$ of σ does not lie in $Y = \mathcal{TS}_{h+1}(\Sigma_g^b, I)$.

We now verify the hypotheses of Proposition 2.3. The first two are easy:

(i) A simplex of X lies in $Y = \mathcal{TS}_{h+1}(\Sigma_g^b, I)$ if and only if none of its faces lie in $\mathcal{B}$, which is obvious.

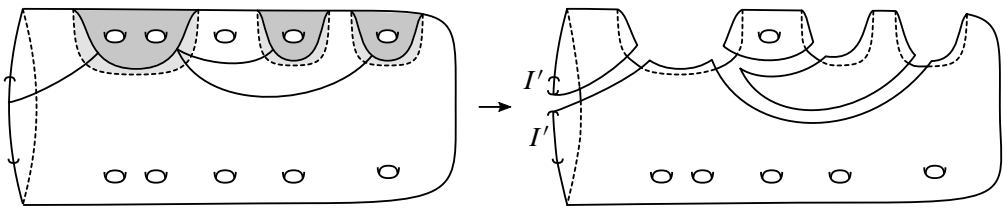
(ii) If $\sigma_1, \sigma_2 \in \mathcal{B}$ are such that $\sigma_1 \cup \sigma_2$ is a simplex of X, then $\sigma_1 \cup \sigma_2 \in \mathcal{B}$, which again is obvious.

The only thing left to check is (iii), which says that, for all k–dimensional $\sigma \in \mathcal{B}$, the complex $G(X, \sigma, \mathcal{B})$ has connectivity at least $(n+1)-k-1 = n-k$.

Write $\sigma = \{\iota_0, \dots, \iota_k\}$. Let Σ' be the surface obtained by first removing the interiors of

$$\iota_0(\Sigma_h^1 \sqcup \Sigma_1^1) \cup \cdots \cup \iota_k(\Sigma_h^1 \sqcup \Sigma_1^1)$$

from Σ_g^b and then cutting open the result along the images of the free and attaching tethers:



The surface Σ' is connected, and when cutting along the free and attaching tethers the open set $I \subset \partial\Sigma_g^b$ is divided into a finer collection I' of open segments (as in the above example). Examining its definition in Section 2.2, we see that

$$G(X, \sigma, \mathcal{B}) \cong \mathcal{TS}_{h+1}(\Sigma', I').$$

We must prove that $\mathcal{TS}_{h+1}(\Sigma', I')$ is $(n-k)$–connected. Let g' be the genus of Σ'. Since $k \geq 1$, we have $n-k < n$, so our inductive hypothesis will say that $\mathcal{TS}_{h+1}(\Sigma', I')$ is $(n-k)$–connected if we can prove that $n-k \leq (g'-(2h+3))/(h+2)$.

This requires estimating g'. The most naive such estimate of g' is

$$g' \geq g - (k+1)(h+1).$$

This is a poor estimate since it does not use the fact that $\sigma \in \mathcal{B}$, which implies that every genus-h surface contributing to this estimate is at least double-counted. Taking this into account, we see that in fact

$$g' \geq g - \tfrac{1}{2}(k+1)h - (k+1) = g - \tfrac{1}{2}(k+1)(h+2).$$

This implies that

$$\frac{g' - (2h+3)}{h+2} \geq \frac{g - (2h+3)}{h+2} - \frac{\frac{1}{2}(k+1)(h+2)}{h+2} \geq n - \tfrac{1}{2}(k+1) \geq n - k,$$

where the final inequality uses the fact that $k \geq 1$. □

This completes the proof of Theorem D. □

3 Stability for surfaces with one boundary component

In this section we prove Theorems A and C. The outline is as follows. In Section 3.1, we discuss the homological stability machine. In Sections 3.2 and 3.3 we prove a number of preliminary results needed to apply this machine. Our proof of Theorem E (and its nonabelian analogue) is in Section 3.3.2. Finally, in Section 3.4 we prove Theorems A and C.

3.1 The stability machine

We now introduce the standard homological stability machine. This is discussed in many places, but the account in [12, Section 1] is particularly convenient for our purposes. We remark that our results could also be proved using the framework of [19] (which generalizes [25]), but since it would not simplify our proofs we decided not to use that framework.

Semisimplicial sets The natural setting for the machine is that of semisimplicial sets, whose definition we now briefly recall. For more details see [9], which calls them Δ–sets. Let Δ be the category with objects the sets $[k] = \{0, \dots, k\}$ for $k \geq 0$ and whose morphisms $[k] \to [\ell]$ are the strictly increasing functions. A *semisimplicial set* is a contravariant functor X from Δ to the category of sets. The *k–simplices* of X are the image X_k of $[k] \in \Delta$. The maps $X_\ell \to X_k$ corresponding to the Δ–morphisms $[k] \to [\ell]$ are called the *face maps*.

Geometric properties A semisimplicial set X has a geometric realization $|X|$ obtained by taking standard k–simplices for each element of X_k and then gluing these simplices

together using the face maps. Whenever we talk about topological properties of a semisimplicial set, we are referring to its geometric realization. An action of a group G on a semisimplicial set X consists of actions of G on each X_n that commute with the face maps. This induces an action of G on $|X|$.

The machine The version of the homological stability machine we need is as follows. In it, the indexing is chosen so that the complex X_1 upon which G_1 acts is connected.

Theorem 3.1 *Let*

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

be an increasing sequence of groups. For each $n \geq 1$, let X_n be a semisimplicial set upon which G_n acts. Assume, for some $c \geq 2$, that the following hold:

(1) *The space X_n is $(n-1)/c$–connected.*

(2) *For all $0 \leq i < n$, the group G_{n-i-1} is the G_n–stabilizer of some i–simplex of X_n.*

(3) *For all $0 \leq i < n$, the group G_n acts transitively on the i–simplices of X_n.*

(4) *For all $n \geq c+1$ and all 1–simplices e of X_n whose boundary consists of vertices v and v', there exists some $\lambda \in G_n$ such that $\lambda(v) = v'$ and such that λ commutes with all elements of $(G_n)_e$.*

Then, for $k \geq 1$, the map $\mathrm{H}_k(G_{n-1}) \to \mathrm{H}_k(G_n)$ is an isomorphism for $n \geq ck+1$ and a surjection for $n = ck$.

Proof This is proved exactly like [12, Theorem 1.1]. □

3.2 Stabilizing and destabilizing markings

We next discuss the process of stabilizing and destabilizing markings. Recall that A is a fixed finitely generated abelian group and Λ is a fixed finite group.

Stabilizing and destabilizing, abelian If μ is an A–homology marking on Σ_g^1 and $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ is an embedding, then we can define the stabilization to $\Sigma_{g'}^1$ of μ just like we did in the introduction. Namely, $\mathrm{H}_1(\Sigma_g)$ can be identified with a symplectic subspace of $\mathrm{H}_1(\Sigma_{g'})$, so

$$\mathrm{H}_1(\Sigma_{g'}) = \mathrm{H}_1(\Sigma_g^1) \oplus \mathrm{H}_1(\Sigma_g^1)^{\perp},$$

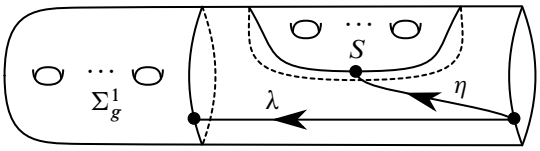
where the $\perp$ is with respect to the algebraic intersection pairing. Define the *stabilization* $\mu'\colon \mathrm{H}_1(\Sigma^1_{g'}) \to A$ of μ to be the composition

$$\mathrm{H}_1(\Sigma^1_{g'}) = \mathrm{H}_1(\Sigma^1_g) \oplus \mathrm{H}_1(\Sigma^1_g)^\perp \to \mathrm{H}_1(\Sigma^1_g) \xrightarrow{\mu} A,$$

where the first arrow is the orthogonal projection. We also say μ is a *destabilization* of μ'.

Stabilizing and destabilizing, nonabelian Now let μ be a Λ–marking on Σ^1_g and $\Sigma^1_g \hookrightarrow \Sigma^1_{g'}$ be an embedding. Defining the stabilization of μ to $\Sigma^1_{g'}$ is subtle since there is not a canonical[3] way to stabilize. We thus need to make some auxiliary choices.

Let $* \in \partial\Sigma^1_g$ and $*' \in \partial\Sigma^1_{g'}$ be the basepoints. Let S be a subsurface of $\Sigma^1_{g'} \setminus \mathrm{Int}(\Sigma^1_g)$ with $S \cong \Sigma^1_{g'-g}$. Choose a basepoint $*'' \in \partial S$, and let λ and η be embedded paths in $\Sigma^1_{g'} \setminus \mathrm{Int}(\Sigma^1_g \cup S)$ connecting $*$ to $*'$ and $*''$, respectively. Assume that λ and η are disjoint aside from their initial points:



The paths λ and η induce injective homomorphisms

$$\pi_1(\Sigma^1_g, *) \hookrightarrow \pi_1(\Sigma^1_{g'}, *') \quad \text{and} \quad \pi_1(S, *'') \hookrightarrow \pi_1(\Sigma^1_{g'}, *')$$

taking $x \in \pi_1(\Sigma^1_g, *)$ to $\lambda \cdot x \cdot \lambda^{-1} \in \pi_1(\Sigma^1_{g'}, *')$ and $y \in \pi_1(S, *'')$ to $\eta \cdot y \cdot \eta^{-1} \in \pi_1(\Sigma^1_{g'}, *')$. Identifying $\pi_1(\Sigma^1_g, *)$ and $\pi_1(S, *'')$ with the corresponding subgroups of $\pi_1(\Sigma^1_{g'}, *')$, we have a free product decomposition

$$\pi_1(\Sigma^1_{g'}, *') = \pi_1(\Sigma^1_g, *) \star \pi_1(S, *'').$$

Define $\mu'\colon \pi_1(\Sigma^1_{g'}, *') \to \Lambda$ to be the composition

$$\pi_1(\Sigma^1_{g'}, *') = \pi_1(\Sigma^1_g, *) \star \pi_1(S, *'') \to \pi_1(\Sigma^1_g, *) \xrightarrow{\mu} \Lambda,$$

where the first arrow quotients out by the normal closure of $\pi_1(S, *'')$.

A different choice of η would change the subgroup $\pi_1(S, *'')$ of $\pi_1(\Sigma^1_{g'}, *')$ to a conjugate subgroup, so would not change μ'. It follows that μ' only depends on the pair (S, λ), and we will call μ' the (S, λ)–*stabilization* of μ to $\Sigma^1_{g'}$. If we do not want to specify (S, λ) we will just say that μ' is a stabilization of μ, but be warned that different choices of (S, λ) will lead to different stabilizations. We will also say that μ is a *destabilization* of μ' with *destabilization data* (S, λ).

[3] In the introduction, we made a very specific choice when we stabilized a Λ–marking on Σ^1_g to Σ^1_{g+1}.

Remark 3.2 The choice of S is more important than the choice of λ. Indeed, changing λ would have the effect of conjugating μ' by an element of Λ. This would not affect the associated partial Torelli group $\mathcal{I}(\Sigma^1_{g'}, \mu')$.

Maps between partial Torelli groups Let μ be either an A–homology marking or a Λ–marking on Σ^1_g, let $\Sigma^1_g \hookrightarrow \Sigma^1_{g'}$ be an embedding, and let μ' be a stabilization of μ to $\Sigma^1_{g'}$. The embedding $\Sigma^1_g \hookrightarrow \Sigma^1_{g'}$ induces an injective map $\mathrm{Mod}(\Sigma^1_g) \to \mathrm{Mod}(\Sigma^1_{g'})$ on mapping class groups, and from our definitions it is clear that this restricts to a map $\mathcal{I}(\Sigma^1_g, \mu) \to \mathcal{I}(\Sigma^1_{g'}, \mu')$ between the associated partial Torelli groups. In fact:

Lemma 3.3 *Suppose that μ is either an A–homology marking or a Λ–marking on Σ^1_g, that $\Sigma^1_g \hookrightarrow \Sigma^1_{g'}$ is an embedding, and that μ' is a stabilization of μ to $\Sigma^1_{g'}$. Let $\iota\colon \mathrm{Mod}(\Sigma^1_g) \to \mathrm{Mod}(\Sigma^1_{g'})$ be the map induced by $\Sigma^1_g \hookrightarrow \Sigma^1_{g'}$. Then*

$$\mathcal{I}(\Sigma^1_g, \mu) = \{\phi \in \mathrm{Mod}(\Sigma^1_g) \mid \iota(\phi) \in \mathcal{I}(\Sigma^1_{g'}, \mu')\}.$$

Proof This is immediate. □

Vanishing surfaces Recall that the rank $\mathrm{rk}(A)$ of the finitely generated abelian group A is the minimum size of a generating set for A. Consider a subsurface S of Σ^1_g. For an A–homology marking μ on Σ^1_g, we say that μ *vanishes on* S if μ vanishes on the image of $\mathrm{H}_1(S)$ in $\mathrm{H}_1(\Sigma^1_g)$. Similarly, for a Λ–marking μ, we say that μ *vanishes on* S if $\mu(x) = 1$ for all $x \in \pi_1(\Sigma^1_g, *)$ that are freely homotopic to a loop in S. Here $* \in \partial\Sigma^1_g$ is our fixed basepoint.

Proposition 3.4 *Consider some $g, h \geq 1$.*

- *Let μ be an A–homology marking on Σ^1_g, and assume that $g \geq \mathrm{rk}(A) + h$. Then there exists an embedding $S \hookrightarrow \Sigma^1_g$ with $S \cong \Sigma^1_h$ such that μ vanishes on S.*
- *Let μ be a Λ–marking on Σ^1_g, and assume that $g \geq |\Lambda| + h$. Then there exists an embedding $S \hookrightarrow \Sigma^1_g$ with $S \cong \Sigma^1_h$ such that μ vanishes on S.*

Proof of Proposition 3.4 for A–homology markings Consider a symplectic subspace U of $\mathrm{H}_1(\Sigma^1_g)$, ie a subgroup such that $\mathrm{H}_1(\Sigma^1_g) = U \oplus U^\perp$, where the $\perp$ is with respect to the algebraic intersection pairing. Such a U is of the form $U \cong \mathbb{Z}^{2k}$ for an integer $k \geq 0$, called the *genus* of U. Every genus-h symplectic subspace U of $\mathrm{H}_1(\Sigma^1_g)$ can be written as $U = \mathrm{H}_1(S)$ for some subsurface S of Σ^1_g satisfying $S \cong \Sigma^1_h$; see eg [15, Lemma 9]. The proposition is thus equivalent to the purely algebraic Lemma 3.5. □

Lemma 3.5 *Let $V \cong \mathbb{Z}^{2g}$ be a free abelian group equipped with a symplectic form $\omega(-,-)$ and let $\mu\colon V \to A$ be a group homomorphism. Assume that $g \geq \mathrm{rk}(A)+h$ for some $h \geq 1$. There then exists a genus-h symplectic subspace U of V such that $\mu|_U = 0$.*

Proof Without loss of generality, μ is surjective and $A \neq 0$. Also, increasing h if necessary, we can assume that $g = \mathrm{rk}(A) + h$. We will prove the "dual" statement that there exists a genus-$\mathrm{rk}(A)$ symplectic subspace W of V such that $\mu|_{W^\perp} = 0$. The desired U is then $U = W^\perp$. The proof will be by induction on $\mathrm{rk}(A)$. The base case is $\mathrm{rk}(A) = 1$, so A is cyclic. We can factor μ as

$$V \xrightarrow{\tilde{\mu}} \!\!\!\!\rightarrow \mathbb{Z} \to A.$$

By definition, $\omega(-,-)$ identifies V with its dual, $\mathrm{Hom}(V,\mathbb{Z})$. There thus exists some $a \in V$ such that $\tilde{\mu}(x) = \omega(a,x)$ for all $x \in V$. Pick $b \in V$ with $\omega(a,b) = 1$ and let $W = \langle a, b\rangle$. Then W is a genus-1 symplectic subspace and

$$W^\perp \subset \ker(\omega(a,-)) = \ker(\tilde{\mu}) \subset \ker(\mu),$$

as desired.

Now assume that $\mathrm{rk}(A) > 1$ and that the lemma is true for all smaller ranks. We can then find a short exact sequence

$$0 \to A' \to A \xrightarrow{\phi} A'' \to 0$$

such that $0 < \mathrm{rk}(A') < \mathrm{rk}(A)$ and $\mathrm{rk}(A'') + \mathrm{rk}(A') = \mathrm{rk}(A)$. By our inductive hypothesis, there exists a genus-$\mathrm{rk}(A'')$ symplectic subspace W'' of V such that $(\phi\circ\mu)|_{(W'')^\perp} = 0$. Set $V' = (W'')^\perp$, so V' is a symplectic subspace of V and the image of $\mu' := \mu|_{V'}$ lies in A'. Our inductive hypothesis implies that there is a genus-$\mathrm{rk}(A')$ symplectic subspace W' of V' such that $\mu'|_{(W')^\perp} = 0$. Setting $W = W' \oplus W''$, we have that W is a genus $\mathrm{rk}(A') + \mathrm{rk}(A'') = \mathrm{rk}(A)$ symplectic subspace of V such that $\mu|_{W^\perp} = 0$, as desired. □

Proof of Proposition 3.4 for Λ–markings The proposition is a small variant of a result of Dunfield and Thurston [6, Proposition 6.16] — the only difference is that their result is for closed surfaces, while we need to deal with Σ_g^1. However, the exact same proof works, so we omit the details. □

Deeply destabilizing Proposition 3.4 has the following corollary:

Corollary 3.6 *Consider some $g' \geq 1$.*

- *Let μ' be an A–homology marking on $\Sigma_{g'}^1$. Assume that $g' > \mathrm{rk}(A)$, and let $g = \mathrm{rk}(A)$. Then there exists an embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ and an A–homology marking μ on Σ_g^1 such that μ is a destabilization of μ'.*

- *Let μ' be a Λ–marking on $\Sigma_{g'}^1$. Assume that $g' > |\Lambda|$, and let $g = |\Lambda|$. Then there exists an embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ and a Λ–marking μ on Σ_g^1 such that μ is a destabilization of μ'.*

Proof The proofs for A–homology markings and Λ–markings are similar, so we give the details for Λ–markings (which are slightly more complicated). Let $*' \in \partial\Sigma_{g'}^1$ be the basepoint. By Proposition 3.4, we can find a subsurface $S \hookrightarrow \Sigma_{g'}^1$ with $S \cong \Sigma_{g'-g}^1$ such that μ' vanishes on S. Pick

- an embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ that is disjoint from S, as well as a basepoint $* \in \partial\Sigma_g^1$,
- an embedded path λ in $\Sigma_{g'}^1 \setminus \mathrm{Int}(\Sigma_g^1 \cup S)$ connecting $*'$ to $*$.

Define $\mu\colon \pi_1(\Sigma_g^1, *) \to \Lambda$ via the formula

$$\mu(x) = \mu'(\lambda \cdot x \cdot \lambda^{-1}) \quad \text{for } x \in \pi_1(\Sigma_g^1, *).$$

It is immediate from the definitions that μ' is the (S, λ)–stabilization of μ to $\Sigma_{g'}^1$. □

3.3 Vanishing surfaces

This section constructs the semisimplicial sets we need to apply Theorem 3.1 to the partial Torelli groups.

3.3.1 Vanishing surfaces: definition and basic properties We define the complexes separately for A–homology markings and Λ–markings.

Vanishing subsurfaces, abelian We start by recalling the definition of the complex of vanishing subsurfaces for a homology marking from the introduction. Let μ be an A–homology marking on Σ_g^1. Then define $\mathcal{S}_h(\Sigma_g^1, \mu)$ to be the full subcomplex of $\mathcal{S}_h(\Sigma_g^1)$ spanned by vertices $\iota\colon \Sigma_h^1 \to \Sigma_g^1$ such that μ vanishes on Σ_h^1 in the sense of Section 3.2. The group $\mathcal{I}(\Sigma_g^1, \mu)$ acts on $\mathcal{S}_h(\Sigma_g^1, \mu)$. Similarly, if $I \subset \partial\Sigma_g^1$ is a finite disjoint union of open intervals, then define $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ to be the full subcomplex of $\mathcal{TS}_h(\Sigma_g^1, I)$ spanned by vertices $\iota\colon \tau(\Sigma_h^1) \to \Sigma_g^1$ whose restriction to Σ_h^1 is a vertex of $\mathcal{S}_h(\Sigma_g^1, \mu)$. Again, the group $\mathcal{I}(\Sigma_g^1, \mu)$ acts on $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$.

Vanishing subsurfaces, nonabelian Let μ be a Λ–marking on Σ_g^1. Then define $\mathcal{S}_h(\Sigma_g^1, \mu)$ to be the full subcomplex of $\mathcal{S}_h(\Sigma_g^1)$ spanned by vertices $\iota\colon \Sigma_h^1 \to \Sigma_g^1$ such that μ vanishes on Σ_h^1 in the sense of Section 3.2. The group $\mathcal{I}(\Sigma_g^1, \mu)$ acts on $\mathcal{S}_h(\Sigma_g^1, \mu)$. Similarly, if $I \subset \partial\Sigma_g^1$ is a finite disjoint union of open intervals, then

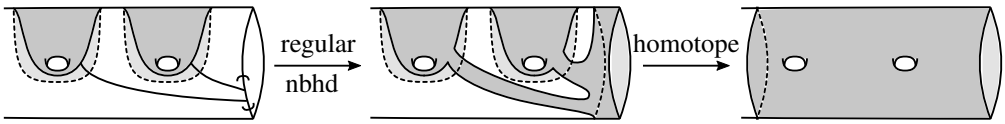
define $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ to be the full subcomplex of $\mathcal{TS}_h(\Sigma_g^1, I)$ spanned by vertices $\iota\colon \tau(\Sigma_h^1) \to \Sigma_g^1$ whose restriction to Σ_h^1 is a vertex of $\mathcal{S}_h(\Sigma_g^1, \mu)$. Again, the group $\mathcal{I}(\Sigma_g^1, \mu)$ acts on $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$.

Semisimplicial In the rest of this section, let μ be either an A–homology marking or a Λ–marking on Σ_g^1 and let $I \subset \partial\Sigma_g^1$ be a single interval. We claim then that $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ is naturally a semisimplicial set. The key point here is that its simplices $\{\iota_0, \dots, \iota_k\}$ possess a natural ordering based on the order their tethers leave I.

Stabilizers The $\mathrm{Mod}(\Sigma_g^1)$–stabilizers of simplices of $\mathcal{S}_h(\Sigma_g^1)$ are poorly behaved. The issue is that mapping classes can permute their vertices arbitrarily (which is not possible for $\mathcal{TS}_h(\Sigma_g^1, I)$ since mapping classes must preserve the order in which the tethers leave I). This prevents their stabilizers from being mapping class groups of subsurfaces. For $\mathcal{TS}_h(\Sigma_g^1, I)$, however, this issue does not occur, and the $\mathrm{Mod}(\Sigma_g^1)$–stabilizer of a simplex $\{\iota_0, \dots, \iota_k\}$ of $\mathcal{TS}_h(\Sigma_g^1, I)$ equals $\mathrm{Mod}(\Sigma)$, where Σ is the complement of an open regular neighborhood of

$$\partial\Sigma_g^1 \cup \iota_0(\tau(\Sigma_h^1)) \cup \cdots \cup \iota_k(\tau(\Sigma_h^1)).$$

We will call the complement of this open neighborhood the *stabilizer subsurface* of the simplex. See here, where the stabilizer subsurface is the complement of the shaded region:

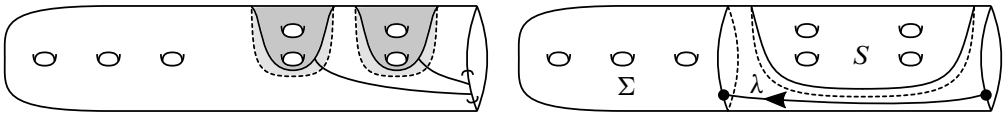


The $\mathcal{I}(\Sigma_g^1, \mu)$ version of this is the following lemma:

Lemma 3.7 *Let μ be either an A–homology marking or a Λ–marking on Σ_g^1, let $I \subset \partial\Sigma_g^1$ be an open interval, and let σ be a k–simplex of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$. Let $\Sigma \cong \Sigma_{g-(k+1)h}^1$ be the stabilizer subsurface of σ. Then there exists a marking μ_0 of the same type as μ (either an A–homology marking or a Λ–marking) on Σ such that μ_0 is a destabilization of μ and such that the $\mathcal{I}(\Sigma_g^1, \mu)$–stabilizer of σ is $\mathcal{I}(\Sigma, \mu_0)$.*

Proof The proofs for A–homology markings and Λ–markings are similar, so we will give the details for Λ–markings. Let $* \in \partial\Sigma_g^1$ and $*_0 \in \partial\Sigma$ be basepoints. Write $\sigma = \{\iota_0, \dots, \iota_k\}$. For $0 \le i \le k$, let $S_i = \iota_i(\Sigma_h^1)$. Let S be a subsurface of $\Sigma_g^1 \setminus \mathrm{Int}(\Sigma)$ such that S contains each S_i and $S \cong \Sigma_{(k+1)h}^1$, and let λ be an embedded path in

$\Sigma_g^1 \setminus \mathrm{Int}(\Sigma \cup S)$ connecting $*$ to $*_0$:



Define $\mu_0 \colon \pi_1(\Sigma, *_0) \to \Lambda$ via the formula

$$\mu_0(x) = \mu(\lambda \cdot x \cdot \lambda^{-1}) \quad \text{for } x \in \pi_1(\Sigma, *_0).$$

It follows from the definitions that μ is the (S, λ)–stabilization of μ_0. Since the $\mathrm{Mod}(\Sigma_g^1)$–stabilizer of σ is $\mathrm{Mod}(\Sigma)$, it follows that the $\mathcal{I}(\Sigma_g^1, \mu)$–stabilizer of σ is $\mathrm{Mod}(\Sigma) \cap \mathcal{I}(\Sigma_g^1, \mu)$, which by Lemma 3.3 is $\mathcal{I}(\Sigma, \mu_0)$. □

3.3.2 Vanishing surfaces: high connectivity The following theorem subsumes Theorem E:

Theorem 3.8 *Fix $g \geq h \geq 1$ and let $I \subset \partial\Sigma_g^1$ be a finite disjoint union of open intervals.*

- *Let μ be an A–homology marking on Σ_g^1. The complexes $\mathcal{S}_h(\Sigma_g^1, \mu)$ and $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ are both $\big(g-(2\,\mathrm{rk}(A)+2h+1)\big)/(\mathrm{rk}(A)+h+1)$–connected.*
- *Let μ be a Λ–marking on Σ_g^1. The complexes $\mathcal{S}_h(\Sigma_g^1, \mu)$ and $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ are both $(g-(2|\Lambda|+2h+1))/(|\Lambda|+h+1)$–connected.*

Proof The proofs for A–homology markings and Λ–markings are identical, so we will give the details for Λ–markings. Also, the proofs that $\mathcal{S}_h(\Sigma_g^b, \mu)$ and $\mathcal{TS}_h(\Sigma_g^b, I, \mu)$ are $(g-(2|\Lambda|+2h+1))/(|\Lambda|+h+1)$–connected are similar. Keeping track of the tethers introduces a few complications, so we will give the details for $\mathcal{TS}_h(\Sigma_g^b, I, \mu)$ and leave $\mathcal{S}_h(\Sigma_g^b, \mu)$ to the reader.

We start by defining an auxiliary space. Let X be the simplicial complex whose vertices are the union of the vertices of the complexes $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ and $\mathcal{TS}_{|\Lambda|+h}(\Sigma_g^1, I)$ and whose simplices are collections $\{\iota_0, \ldots, \iota_k\}$ of vertices that can be isotoped so that their images are disjoint. Both $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ and $\mathcal{TS}_{|\Lambda|+h}(\Sigma_g^1, I)$ are thus full subcomplexes of X.

We now prove that X enjoys the connectivity property we are trying to prove for $\mathcal{TS}_h(\Sigma_g^b, I, \mu)$:

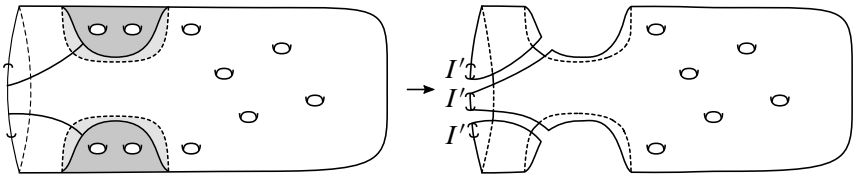
Claim *The space X is $(g-(2|\Lambda|+2h+1))/(|\Lambda|+h+1)$–connected.*

Proof Set $n = (g - (2|\Lambda| + 2h + 1))/(|\Lambda| + h + 1)$, $Y = \mathcal{TS}_{|\Lambda|+h}(\Sigma_g^1, I)$ and $Y' = \mathcal{TS}_h(\Sigma_g^1, I, \mu)$. Theorem D says that Y is n–connected, so it is enough to prove that the pair (X, Y) is n–connected. To do this, we apply Corollary 2.4. This requires, letting σ be a k–dimensional simplex of $Y' = \mathcal{TS}_h(\Sigma_g^1, I, \mu)$ and L be the link of σ in X, showing that $L \cap Y$ is $(n-k-1)$–connected.

Write $\sigma = \{\iota_0, \dots, \iota_k\}$. Let Σ' be the surface obtained by first removing the interiors of

$$\iota_0(\Sigma_h^1) \cup \cdots \cup \iota_k(\Sigma_h^1)$$

from Σ_g^1 and then cutting open the result along the images of the tethers:



The surface Σ' is connected, and when cutting along the tethers the open set $I \subset \partial\Sigma_g^1$ is divided into a finer collection I' of open segments (as in the above example). Then

$$L \cap Y \cong \mathcal{TS}_{|\Lambda|+h}(\Sigma', I'),$$

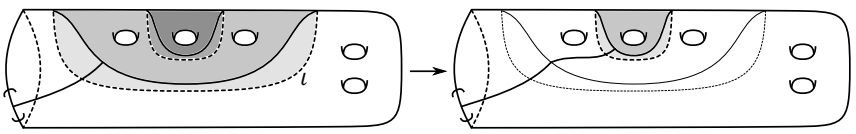
so we must prove that $\mathcal{TS}_{|\Lambda|+h}(\Sigma', I')$ is $(n-k-1)$–connected. Letting g' be the genus of Σ', Theorem D says that $\mathcal{TS}_{|\Lambda|+h}(\Sigma', I')$ is $(g'-(2|\Lambda|+2h+1))/(|\Lambda|+h+1)$–connected, so what we must prove is that

$$n - k - 1 \leq \frac{g' - (2|\Lambda| + 2h + 1)}{|\Lambda| + h + 1}.$$

Examining the construction of Σ', we see that $g' = g - (k+1)h$. We now calculate that

$$\frac{g' - (2|\Lambda| + 2h + 1)}{|\Lambda| + h + 1} = \frac{g - (2|\Lambda| + 2h + 1)}{|\Lambda| + h + 1} - \frac{(k+1)h}{|\Lambda| + h + 1} \geq n - (k+1). \qquad \square$$

To complete the proof, it is enough to construct a retraction $r\colon X \to \mathcal{TS}_h(\Sigma_g^1, I, \mu)$. For a vertex ι of X, we define $r(\iota)$ as follows. If ι is a vertex of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$, then $r(\iota) = \iota$. If instead ι is a vertex of $\mathcal{TS}_{|\Lambda|+h}(\Sigma_g^1, I)$, then Proposition 3.4 implies that we can find a subsurface $\Sigma_h^1 \hookrightarrow \iota(\Sigma_{|\Lambda|+h})$ such that μ vanishes on Σ_h^1. Define $r(\iota)$ to be the vertex of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ obtained by adjoining the tether of ι and an arbitrary arc in $\iota(\Sigma_{|\Lambda|+h}^1)$ connecting the boundary to Σ_h^1:



Of course, $r(\iota)$ depends on various choices, but we simply make an arbitrary choice. It is clear that this extends over the simplices of X to give a retract $r : X \to \mathcal{TS}_h(\Sigma_g^1, I, \mu)$. □

3.3.3 Vanishing surfaces: transitivity The last fact about the complex of vanishing surfaces we will need is as follows:

Lemma 3.9 *Fix $g \geq h \geq 1$ and let $I \subset \partial\Sigma_g^1$ be an open interval.*

- *Let μ be an A–homology marking on Σ_g^1. The group $\mathcal{I}(\Sigma_g^1, \mu)$ acts transitively on the k–simplices of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ if $g \geq 2h + 2\,\mathrm{rk}(A) + 1 + kh$.*
- *Let μ be a Λ–marking on Σ_g^1. The group $\mathcal{I}(\Sigma_g^1, \mu)$ acts transitively on the k–simplices of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ if $g \geq 2h + 2|\Lambda| + 1 + kh$.*

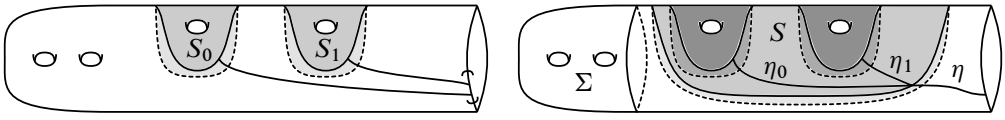
Proof The proofs for A–homology markings and Λ–markings are identical, so we will give the details for Λ–markings. The proof will be by induction on k. We start with the base case $k = 0$.

Claim *If $g \geq 2h + 2|\Lambda| + 1$, then $\mathcal{I}(\Sigma_g^1, \mu)$ acts transitively on the 0–simplices of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$.*

Proof In this case, Theorem 3.8 says that $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ is connected, so it is enough to prove that if ι_0 and ι_1 are vertices of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ that are connected by an edge, then there exists some $f \in \mathcal{I}(\Sigma_g^1, \mu)$ taking ι_0 to ι_1. Let Σ be the stabilizer subsurface of the edge $\{\iota_0, \iota_1\}$, and let $S_0 = \iota_0(\Sigma_h^1)$ and $S_1 = \iota_1(\Sigma_h^1)$. Let S, η, η_0 and η_1 be as follows:

- S is a subsurface of $\Sigma_g^1 \setminus \mathrm{Int}(\Sigma)$ containing S_0 and S_1 and satisfying $S \cong \Sigma_{2h}^1$.
- η is an embedded path in $\Sigma_g^1 \setminus \mathrm{Int}(\Sigma \cup S)$ connecting a point of I to a basepoint of S lying in ∂S.
- For $i = 0, 1$, we have that η_i is an embedded arc in $S \setminus \mathrm{Int}(S_0 \cup S_1)$ connecting the basepoint in ∂S to a basepoint in S_i lying in ∂S_i.
- For $i = 0, 1$, the path $\eta \cdot \eta_i$ is isotopic to the tether of ι_i while keeping its initial point in I and its terminal point fixed.

See here:[4]



[4] In this figure η_0 and η_1 are disjoint, aside from their initial points. This can always be achieved, but is not needed for our proof.

It follows that there exists a Λ–marking μ_0 on Σ and some λ such that μ is the (S,λ)–stabilization of μ_0.

Using the change of coordinates principle from [8, Section 1.3.2], we find $F \in \mathrm{Mod}(S)$ taking $S_0 \cup \eta_0$ to something isotopic to $S_1 \cup \eta_1$. This isotopy will fix the common initial point of η_0 and η_1. Let $f \in \mathrm{Mod}(\Sigma_g^1)$ be the image of F under the map $\mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma_g^1)$. Since f is supported on S, we have $f \in \mathcal{I}(\Sigma_g^1, \mu)$, and by construction we have $f(\iota_0) = \iota_1$. □

Now assume that $k > 0$ and that the theorem is true for simplices of dimension $k-1$. For some $g \geq 2h + 2|\Lambda| + 1 + kh$, let μ be a Λ–marking on Σ_g^1 and $I \subset \partial\Sigma_g^1$ be an open interval. Consider k–simplices σ and σ' of $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$. Enumerate these simplices using the natural ordering discussed above:

$$\sigma = \{\iota_0, \ldots, \iota_k\} \quad \text{and} \quad \sigma' = \{\iota_0', \ldots, \iota_k'\}. \tag{3-1}$$

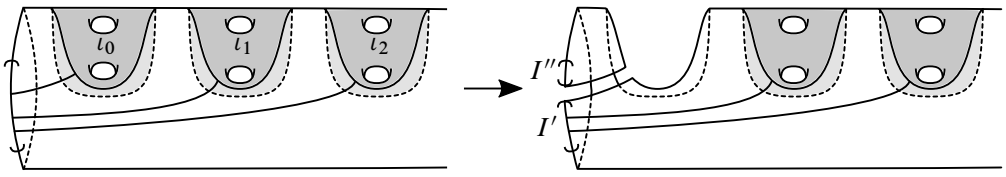
We want to find some $f \in \mathcal{I}(\Sigma_g^1, \mu)$ such that $f(\sigma) = \sigma'$. By the base case $k = 0$, there exists some $f_0 \in \mathcal{I}(\Sigma_g^1, \mu)$ such that $f(\iota_0) = \iota_0'$. Replacing σ by $f(\sigma)$, we can assume that $\iota_0 = \iota_0'$.

Define

$$\sigma_1 = \{\iota_1, \ldots, \iota_k\} \quad \text{and} \quad \sigma_1' = \{\iota_1', \ldots, \iota_k'\}.$$

Both σ_1 and σ_1' are $(k-1)$–simplices in the link of the vertex ι_0, and our goal is to find an element f_1 in the $\mathcal{I}(\Sigma_g^1, \mu)$–stabilizer of ι_0 such that $f_1(\sigma_1) = \sigma_1'$.

Let Σ' be the stabilizer subsurface of ι_0 and let μ' be the Λ–marking on Σ' given by Lemma 3.7, so the $\mathcal{I}(\Sigma_g^1, \mu)$–stabilizer of ι_0 is $\mathcal{I}(\Sigma', \mu')$. The surface Σ' can be constructed by removing the interior of $\iota_0(\Sigma_h^1)$ and then cutting open the result along the tether:



We thus have $\Sigma' \cong \Sigma_{g-h}^1$. Cutting along the tether divides the interval $I \subset \partial\Sigma_g^1$ into two disjoint intervals $I', I'' \subset \partial\Sigma'$, and the link of ι_0 in $\mathcal{TS}_h(\Sigma_g^1, I, \mu)$ can be identified with $\mathcal{TS}_h(\Sigma', I' \sqcup I'', \mu')$. Identifying σ_1 and σ_1' with simplices in $\mathcal{TS}_h(\Sigma', I' \sqcup I'', \mu')$, the key observation is that, since we enumerated the simplices in (3-1) using the order

coming from I, we have (possibly flipping I' and I'') that $\sigma_1, \sigma_1' \subset \mathcal{TS}_h(\Sigma', I', \mu')$. Since $\Sigma' \cong \Sigma^1_{g-h}$ and

$$g - h \geq (2h + 2|\Lambda| + 1 + kh) - h = 2h + 2|\Lambda| + 1 + (k-1)h,$$

we can apply our inductive hypothesis and find some $f_1 \in \mathcal{I}(\Sigma', \mu')$ with $f_1(\sigma_1) = \sigma_1'$, as desired. □

3.4 Proof of stability for surfaces with one boundary component

Proof of Theorems A and C The proofs of the two theorems are identical, so we will give the details for Theorem C. We start by recalling the statement and introducing some notation. Let Λ be a nontrivial finite group, let μ be a Λ–marking on Σ^1_g, and let μ' be the stabilization of μ to Σ^1_{g+1} in the sense of the introduction.[5] Setting

$$c = |\Lambda| + 2 \quad \text{and} \quad d = 2|\Lambda| + 2,$$

we want to prove that the map $\mathrm{H}_k(\mathcal{I}(\Sigma^1_g, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma^1_{g+1}, \mu'))$ induced by the stabilization map $\mathcal{I}(\Sigma^1_g, \mu) \to \mathcal{I}(\Sigma^1_{g+1}, \mu')$ is an isomorphism if $g \geq ck + d$ and a surjection if $g = ck + d - 1$. We will prove this using Theorem 3.1. This requires fitting $\mathcal{I}(\Sigma^1_g, \mu) \hookrightarrow \mathcal{I}(\Sigma^1_{g+1}, \mu')$ into an increasing sequence of group $\{G_n\}$ and constructing appropriate simplicial complexes.

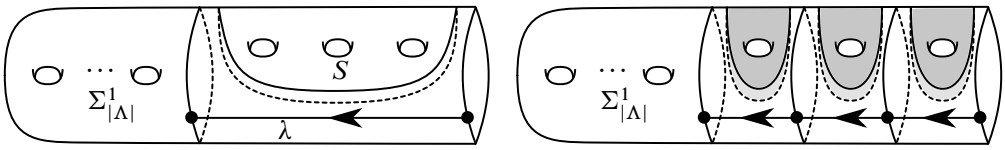
Corollary 3.6 says that there exists an embedding $\Sigma^1_{|\Lambda|} \hookrightarrow \Sigma^1_g$ and a Λ–marking $\mu_{|\Lambda|}$ on $\Sigma^1_{|\Lambda|}$ such that $\mu_{|\Lambda|}$ is a destabilization of μ. The embedding $\Sigma^1_{|\Lambda|} \hookrightarrow \Sigma^1_g$ can be factored into a sequence of embeddings

$$\Sigma^1_{|\Lambda|} \hookrightarrow \Sigma^1_{|\Lambda|+1} \hookrightarrow \cdots \hookrightarrow \Sigma^1_g,$$

which can then be continued to

$$\Sigma^1_{|\Lambda|} \hookrightarrow \Sigma^1_{|\Lambda|+1} \hookrightarrow \cdots \hookrightarrow \Sigma^1_g \hookrightarrow \Sigma^1_{g+1} \hookrightarrow \Sigma^1_{g+2} \hookrightarrow \cdots.$$

As in the following figure, we can break up the destabilization data (S, λ) for the destabilization $\mu_{|\Lambda|}$ of μ into stabilization data (S_h, λ_h) for $|\Lambda| + 1 \leq h \leq g$, where (S_h, λ_h) allows us to stabilize from Σ^1_{h-1} to Σ^1_h:



[5] This uses a specific choice of stabilization data (S, λ).

Starting with $\mu_{|\Lambda|}$, for $h+1 \le |\Lambda| \le g$ inductively let μ_h be the (S_h, λ_h)–stabilization of μ_{h-1} on Σ^1_{h-1} to μ_h on Σ^1_h. By construction, $\mu_g = \mu$. Continue stabilizing (now using the choice of stabilization data from the introduction) to define μ_h on Σ^1_h for $h \ge g+1$, so $\mu_{g+1} = \mu'$.

We have thus fit our partial Torelli groups into an increasing sequence of groups

$$\mathcal{I}(\Sigma^1_{|\Lambda|}, \mu_{|\Lambda|}) \subset \mathcal{I}(\Sigma^1_{|\Lambda|+1}, \mu_{|\Lambda|+1}) \subset \mathcal{I}(\Sigma^1_{|\Lambda|+2}, \mu_{|\Lambda|+2}) \subset \cdots.$$

For $h \ge |\Lambda|$, let $I_h \subset \partial\Sigma^1_h$ be an open interval. Theorem 3.8 says that $\mathcal{TS}_1(\Sigma^1_h, I_h, \mu_h)$ is $(h-(d+1))/c$–connected, where c and d are as defined in the first paragraph.

For $n \ge 0$, let

$$G_n = \mathcal{I}(\Sigma_{d+n}, \mu_{d+n}) \quad \text{and} \quad X_n = \mathcal{TS}_1(\Sigma_{d+n}, I_{d+n}, \mu_{d+n}).$$

For this to make sense, we must have $d+n \ge |\Lambda|$, which follows from

$$d+n = 2|\Lambda| + 2 + n \ge |\Lambda|.$$

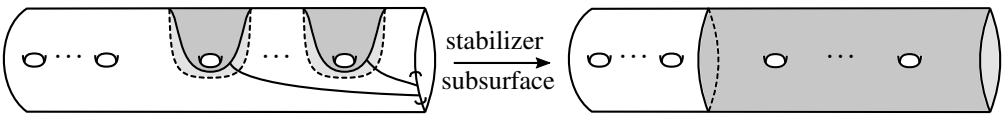
We thus have an increasing sequence of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

with G_n acting on X_n. The indexing convention here is chosen so that X_1 is 0–connected, and more generally so that X_n is $(n-1)/c$–connected, as in Theorem 3.1. Our goal is to prove that the map $\mathrm{H}_k(G_{n-1}) \to \mathrm{H}_k(G_n)$ is an isomorphism for $n \ge ck+1$ and a surjection for $n = ck$, which will follow from Theorem 3.1 once we check its conditions.

- The first is that X_n is $(n-1)/c$–connected, which follows from Theorem 3.8.

- The second is that, for $0 \le i < n$, the group G_{n-i-1} is the G_n–stabilizer of some i–simplex of X_n, which follows from Lemma 3.7 via the following picture:



- The third is that, for all $0 \le i < n$, the group G_n acts transitively on the i–simplices of X_n, which follows from Lemma 3.9. For transitivity on the i–simplices this lemma requires that the genus $g = d+n$ used to define $G_n = \mathcal{I}(\Sigma_{d+n}, \mu_{d+n})$ satisfies $g \ge 3 + 2|\Lambda| + i$, which follows from the fact that

$$d+n = (2|\Lambda|+2) + n \ge (2|\Lambda|+2) + (i+1) = 3 + 2|\Lambda| + i.$$

• The fourth is that, for all $n \geq c+1$ and all 1–simplices e of X_n whose boundary consists of vertices v and v', there exists some $\lambda \in G_n$ such that $\lambda(v) = v'$ and such that λ commutes with all elements of $(G_n)_e$. This actually does not require the condition $n \geq c+1$. Let Σ be the stabilizer subsurface of e, so by Lemma 3.7 the stabilizer $(G_n)_e$ consists of mapping classes supported on Σ. The surface $\Sigma^1_{d+n} \setminus \mathrm{Int}(\Sigma)$ is diffeomorphic to Σ^2_2 (as in the picture above), and in particular is connected. The change of coordinates principle from [8, Section 1.3.2] implies that we can find a mapping class λ supported on $\Sigma^1_{d+n} \setminus \mathrm{Int}(\Sigma)$ taking the tethered torus v to v'. This λ clearly lies in G_n and commutes with $(G_n)_e$. □

4 Homology-marked partitioned surfaces

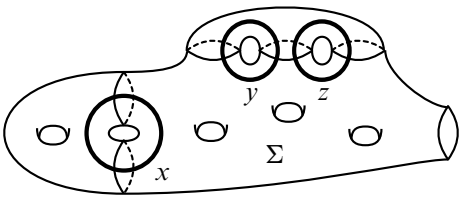
We now turn to partial Torelli groups on surfaces with multiple boundary components. Unfortunately, this introduces genuine difficulties in the proofs, so quite a bit more technical setup is needed. This section contains the categorical framework we will need to even state our result.

Let `Surf` be the category whose objects are compact connected oriented surfaces with boundary and whose morphisms are orientation-preserving embeddings. There is a functor from `Surf` to groups taking $\Sigma \in \mathtt{Surf}$ to $\mathrm{Mod}(\Sigma)$ and a morphism $\Sigma \hookrightarrow \Sigma'$ to the map $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\Sigma')$ that extends mapping classes by the identity. In this section, we augment `Surf` to construct a new category `PSurf` on which we can define partial Torelli groups. This is done in two steps: in Section 4.1 we define the category `PSurf` along with a "partitioned homology functor", and in Section 4.2 we discuss homology markings and construct their associated partial Torelli groups.

4.1 The category `PSurf`

We start with the partitioned surface category, which was introduced in [22].

Motivation This category captures aspects of the homology of a larger surface in which our surface is embedded. For instance, consider the following embedding of a genus-3 surface Σ with six boundary components into Σ^1_7:



For $f \in \mathrm{Mod}(\Sigma)$, the action of f on $\mathrm{H}_1(\Sigma)$ does not determine the action of f on $\mathrm{H}_1(\Sigma_7^1)$. The issue is that we also need to know the action of f on $[x], [y], [z] \in \mathrm{H}_1(\Sigma_7^1)$. The portions of these homology classes that live on Σ are arcs connecting boundary components, so we must consider relative homology groups that incorporate such arcs. However, we do not want to allow all arcs connecting boundary components, since some of these cannot be completed to loops in the larger ambient surface.

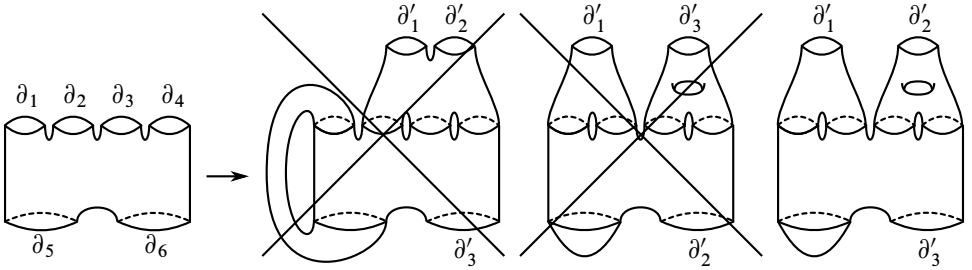
Category To that end, we define a category `PSurf` whose objects are pairs $(\Sigma, \mathcal{P})$:

- Σ is a compact connected oriented surface with boundary.
- $\mathcal{P}$ is a partition of the components of $\partial\Sigma$.

The partition $\mathcal{P}$ tells us which boundary components are allowed to be connected by arcs. The morphisms in `PSurf` from $(\Sigma, \mathcal{P})$ to $(\Sigma', \mathcal{P}')$ are orientation-preserving embeddings $\Sigma \hookrightarrow \Sigma'$ compatible with the partitions $\mathcal{P}$ and $\mathcal{P}'$ in the following sense. For a component S of $\Sigma' \setminus \mathrm{Int}(\Sigma)$, let B_S (resp. B'_S) denote the set of components of $\partial\Sigma$ (resp. $\partial\Sigma'$) that lie in S. In the degenerate case where $S \cong S^1$ (so S is a component of $\partial\Sigma$ and $\partial\Sigma'$), we have $B_S = B'_S$. Our compatibility requirements are then that

- each B_S is a subset of some $p \in \mathcal{P}$, and
- for all $p' \in \mathcal{P}'$ and all $\partial'_1, \partial'_2 \in p'$ such that $\partial'_1 \in B'_{S_1}$ and $\partial'_2 \in B'_{S_2}$ with $S_1 \neq S_2$, there exists some $p \in \mathcal{P}$ such that $B_{S_1} \cup B_{S_2} \subset p$.

Example 4.1 Let $\Sigma = \Sigma_0^6$, $\mathcal{P} = \{\{\partial_1, \partial_2, \partial_3, \partial_4\}, \{\partial_5, \partial_6\}\}$, $\Sigma' = \Sigma_3^3$ and $\mathcal{P}' = \{\{\partial'_1, \partial'_2\}, \{\partial'_3\}\}$. Here are two embeddings $(\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ that are not `PSurf`–morphisms and one that is:



We remark that the difference between the second and third embedding is the labeling of the boundary components.

Partitioned homology Consider some $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. Say that components ∂_1 and ∂_2 of $\partial\Sigma$ are *$\mathcal{P}$–adjacent* if there exists some $p \in \mathcal{P}$ with $\partial_1, \partial_2 \in p$. Define $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ to be the subgroup of the relative homology group $\mathrm{H}_1(\Sigma, \partial\Sigma)$ spanned

by the homology classes of oriented closed curves and arcs connecting $\mathcal{P}$–adjacent boundary components. The group $\mathrm{Mod}(\Sigma)$ acts on $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$.

Remark 4.2 This is slightly different from the partitioned homology group defined in [22], which was *not* functorial. The Torelli groups defined via the above homology groups are thus different from those in [22].

Functoriality The assignment

$$(\Sigma, \mathcal{P}) \mapsto \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$$

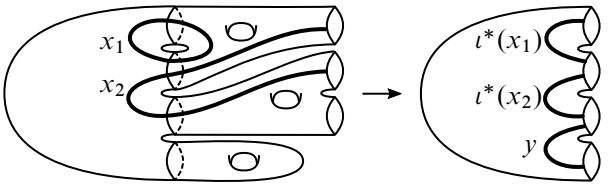
is a contravariant functor from `PSurf` to abelian groups. To see this, consider a `PSurf`–morphism $\iota\colon (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$. Identify Σ with its image under ι. We then have maps

$$\mathrm{H}_1(\Sigma', \partial\Sigma') \to \mathrm{H}_1(\Sigma', \Sigma' \setminus \mathrm{Int}(\Sigma)) \xrightarrow{\cong} \mathrm{H}_1(\Sigma, \partial\Sigma),$$

where the second map is the excision isomorphism. From the definition of a `PSurf`–morphism, it follows immediately that this composition restricts to a map

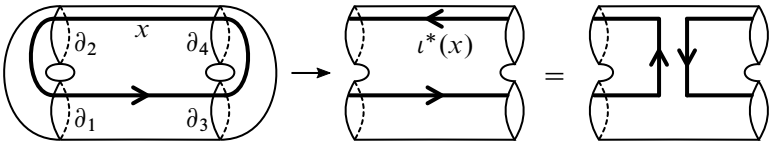
$$\iota^*\colon \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma).$$

Example 4.3 Let $\Sigma = \Sigma_0^4$ and $\Sigma' = \Sigma_4^3$. Let $\mathcal{P}$ (resp. $\mathcal{P}'$) be the partition of the components of $\partial\Sigma$ (resp. $\partial\Sigma'$) consisting of a single partition element containing all the boundary components. The following picture shows a `PSurf`–morphism $\iota\colon (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ along with $x_1, x_2 \in \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma')$ and $\iota^*(x_1), \iota^*(x_2) \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$:



To simplify the picture we do not indicate the orientations of the curves/arcs. The above picture also shows an element $y \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ that is not in the image of ι^*.

Example 4.4 Let $\Sigma = \Sigma_0^4$ and $\Sigma' = \Sigma_2$. Let $\{\partial_1, \dots, \partial_4\}$ be the boundary components of Σ, and let $\mathcal{P} = \{\{\partial_1, \partial_2\}, \{\partial_3, \partial_4\}\}$ and $\mathcal{P}' = \varnothing$. The following picture shows a `PSurf`–morphism $\iota\colon (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ along with $x \in \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma')$ and $\iota^*(x) \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$:



As it is initially drawn, $\iota^*(x)$ does not appear to be in $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ since it is a pair of arcs connecting boundary components that are not $\mathcal{P}$–adjacent; however, as the figure shows, this pair of arcs is homologous to a pair of arcs connecting boundary components that are $\mathcal{P}$–adjacent.

Action on partitioned homology The mapping class group is a covariant functor from `Surf` to groups, while the partitioned homology group is a contravariant functor from `PSurf` to abelian groups. They are related by the following "push-pull" formula:

Lemma 4.5 *Let $\iota\colon(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a `PSurf`–morphism, $\iota_*\colon \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\Sigma')$ be the induced map on mapping class groups and $\iota^*\colon \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ be the induced map on partitioned homology groups. Then*

$$\iota^*(\iota_*(f)(x')) = f(\iota^*(x')) \quad \text{for } f \in \mathrm{Mod}(\Sigma) \text{ and } x' \in \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma').$$

Proof This is obvious. □

4.2 Homology markings on `PSurf`

Recall that A is a fixed finitely generated abelian group.

Markings and partial Torelli groups Consider $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. An *$A$–homology marking* on $(\Sigma, \mathcal{P})$ is a homomorphism $\mu\colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to A$. The associated *partial Torelli group* is

$$\mathcal{I}(\Sigma, \mathcal{P}, \mu) = \{f \in \mathrm{Mod}(\Sigma) \mid \mu(f(x)) = \mu(x) \text{ for all } x \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)\}.$$

Stabilizations If $\iota\colon(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ is a `PSurf`–morphism and μ is an A–homology marking on $(\Sigma, \mathcal{P})$, then the *stabilization* of μ to $(\Sigma', \mathcal{P}')$ is the composition

$$\mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \xrightarrow{\iota^*} \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \xrightarrow{\mu} A.$$

Lemma 4.6 *Let $\iota\colon(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a `PSurf`–morphism, μ be an A–homology marking on $(\Sigma, \mathcal{P})$, μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$, and $\iota_*\colon \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\Sigma')$ be the induced map. Then $\iota_*(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \subset \mathcal{I}(\Sigma', \mathcal{P}', \mu')$.*

Proof Let $\iota^*\colon \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ be the induced map. For $f \in \mathcal{I}(\Sigma, \mathcal{P}, \mu)$ and $x' \in \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma')$, we have

$$\mu'(\iota_*(f)(x')) = \mu\big(\iota^*(\iota_*(f)(x'))\big) = \mu\big(f(\iota^*(x'))\big) = \mu(\iota^*(x')) = \mu'(x').$$

Here the second equality follows from Lemma 4.5 and the third from the fact that $f \in \mathcal{I}(\Sigma, \mathcal{P}, \mu)$. The lemma follows. □

5 Stability for surfaces with multiple boundary components

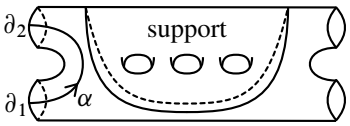
In this section, we state our stability theorem for the partial Torelli groups on surfaces with multiple boundary components and reduce this theorem to a result that will be proved in the next section using the homological stability machine. The statement of our result is in Section 5.1 and the reductions are in Sections 5.2, 5.3 and 5.4.

5.1 Statement of result

To get around the issues with closed surfaces underlying Theorem B from Section 1, we will need to impose some conditions on our stabilization maps.

Support If μ is an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$, we say that μ is *supported on a genus-h symplectic subsurface* if there exists a $\mathtt{PSurf}$–morphism $(\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ with $\Sigma' \cong \Sigma_h^1$ and an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ such that μ is the stabilization of μ' to $(\Sigma, \mathcal{P})$. If there exists some $h \geq 1$ such that μ is supported on a genus-h symplectic subsurface, then we will simply say that μ is *supported on a symplectic subsurface*.

Remark 5.1 Not all A–homology markings are supported on a symplectic subsurface. Indeed, letting ∂_1 and ∂_2 be $\mathcal{P}$–adjacent boundary components of Σ, this condition implies that we can find an arc α connecting ∂_1 to ∂_2 such that $\mu([\alpha]) = 0$; see here:



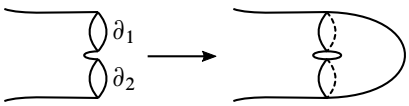
It is easy to construct A–homology markings not satisfying this property; for instance, let $A = \widetilde{\mathrm{H}}_0(\partial\Sigma)$ and let $\mu\colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to A$ be the restriction to $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ of the boundary map $\mathrm{H}_1(\Sigma, \partial\Sigma) \to \widetilde{\mathrm{H}}_0(\partial\Sigma)$. We will later show that this is the only obstruction; see Lemma 6.2 below.

Partition bijectivity Consider a $\mathtt{PSurf}$–morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$. Identify Σ with its image in Σ'. We will call this morphism *partition-bijective* if the following holds for all $p \in \mathcal{P}$:

- Let S be the union of the components of $\Sigma' \setminus \mathrm{Int}(\Sigma)$ that contain a boundary component in p. Then there exists a unique $p' \in \mathcal{P}'$ such that p' consists of the components of $S \cap \partial\Sigma'$.

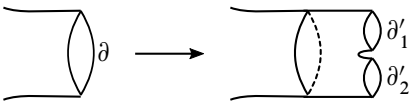
This condition implies in particular that S contains components of $\partial\Sigma'$. It rules out two kinds of morphisms:

- The first is morphisms where, for some $p \in \mathcal{P}$, the union of the components of $\Sigma' \setminus \mathrm{Int}(\Sigma)$ that contain a boundary component in p contains no components of $\partial\Sigma'$. See here:



Here $p = \{\partial_1, \partial_2\}$.

- The second is morphisms where a single $p \in \mathcal{P}$ "splits" into multiple elements of $\mathcal{P}'$ like this:



Here $p = \{\partial\}$ and $\mathcal{P}'$ contains both $\{\partial'_1\}$ and $\{\partial'_2\}$.

Main theorem With this definition, we have the following theorem:

Theorem F *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a partition-bijective $\mathtt{PSurf}$–morphism and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Then the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism if the genus of Σ is at least $(\mathrm{rk}(A) + 2)k + (2\,\mathrm{rk}(A) + 2)$.*

Remark 5.2 Theorem F does not assert that the map is a surjection if the genus of Σ is at least $(\mathrm{rk}(A) + 2)k + (2\,\mathrm{rk}(A) + 1)$. We do not know if this is true — while an appropriate surjectivity statement will follow from our invocation of the homological stability machine, this will only cover certain special kinds of morphisms $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$, the "double boundary stabilizations", and the general case will be reduced to these special morphisms in a fairly involved way.

Counterexamples We do not know whether or not the condition in Theorem F that μ be supported on a symplectic subsurface is necessary. However, the condition that the morphism be partition-bijective is necessary. Indeed, in Section 7 we will prove the following theorem. The condition of being *symplectically nondegenerate* in it will be defined in that section; it is satisfied by most interesting homology markings.

Theorem 5.3 *Let μ be a symplectically nondegenerate A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$*

be a non-partition-bijective PSurf*–morphism and let* μ' *be the stabilization of* μ *to* $(\Sigma', \mathcal{P}')$*. Assume that the genus of* Σ *is at least* $3\,\mathrm{rk}(A) + 4$*. Then the induced map* $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ *is not an isomorphism.*

Remark 5.4 The map is frequently not an isomorphism, even when the genus of Σ is smaller. We use the genus assumption in Theorem 5.3 so we can apply Theorem F to change Σ and Σ' so as to put ourselves in a situation where the phenomenon underlying Theorem B occurs.

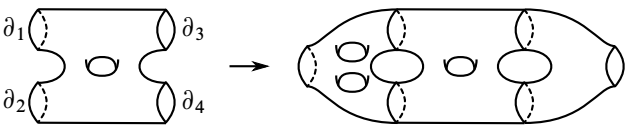
5.2 Reduction I: open cappings

In this section, we reduce Theorem F to certain kinds of PSurf–morphisms called open cappings, whose definition is below.

Open cappings An *open capping* is a PSurf–morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ such that the following holds for all $p \in \mathcal{P}$:

- Let S be the union of the components of $\Sigma' \setminus \mathrm{Int}(\Sigma)$ that contain a boundary component in p. Then S is connected and $S \cap \partial\Sigma'$ consists of a single component.

Unraveling the definition of a PSurf–morphism, this implies that $\mathcal{P}'$ is the *discrete partition*, that is, the partition $\mathcal{P}' = \{\{\partial'\} \mid \partial' \text{ is a component of } \partial\Sigma'\}$. See the following example, where $\mathcal{P} = \{\{\partial_1, \partial_2\}, \{\partial_3, \partial_4\}\}$:



By definition, an open capping is partition-bijective.

Remark 5.5 In [22], a capping is defined similarly to an open capping, but where Σ' is closed and ∂S is simply an element of $\mathcal{P}$.

Reduction The following is a special case of Theorem F:

Proposition 5.6 *Let* μ *be an* A*–homology marking on* $(\Sigma, \mathcal{P}) \in$ PSurf *that is supported on a symplectic subsurface. Let* $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ *be an open capping and let* μ' *be the stabilization of* μ *to* $(\Sigma', \mathcal{P}')$*. Then the induced map*

$$\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$$

is an isomorphism if the genus of Σ *is at least* $(\mathrm{rk}(A) + 2)k + (2\,\mathrm{rk}(A) + 2)$*.*

The proof of Proposition 5.6 begins in Section 5.3. First, we use it to deduce Theorem F:

Proof of Theorem F, assuming Proposition 5.6 We start by recalling the statement of the theorem. Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a partition-bijective $\mathtt{PSurf}$–morphism and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Assume that the genus of Σ is at least $(\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+2)$. Our goal is to prove that the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism.

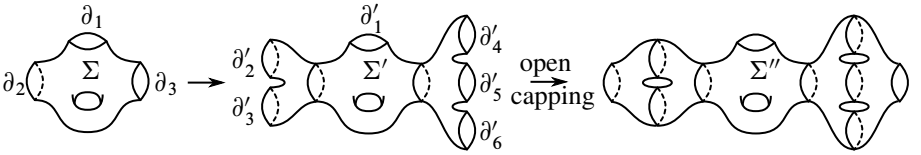
Identify Σ with its image in Σ'. The proof has two cases. Recall that the discrete partition of the boundary components of a surface S is $\{\{\partial\} \mid \partial \text{ is a component of } \partial S\}$.

Case 1 *$\mathcal{P}$ is the discrete partition of Σ.*

Let $S_1, \dots, S_b$ be the components of $\Sigma' \setminus \mathrm{Int}(\Sigma)$. For each $1 \le i \le b$, let B_{S_i} be the components of $\partial\Sigma$ that are contained in S_i and let B'_{S_i} be the components of $\partial\Sigma'$ that are contained in S_i. Then:

- Since $\mathcal{P}$ is the discrete partition, each B_{S_i} is a one-element set containing a single boundary component of Σ, and $\mathcal{P} = \{B_{S_1}, \dots, B_{S_b}\}$.
- Since the morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ is partition-bijective, each B'_{S_i} is a nonempty set of boundary components of Σ', and $\mathcal{P}' = \{B'_{S_1}, \dots, B'_{S_b}\}$.

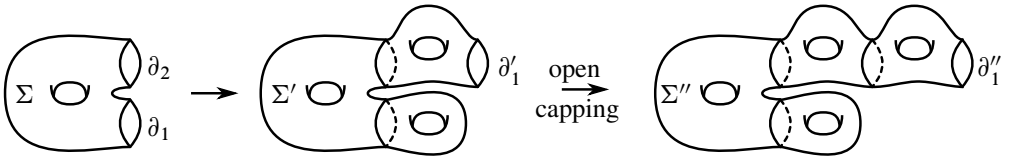
See the following figure, where $\Sigma \cong \Sigma_1^3$ with the discrete partition $\mathcal{P} = \{\{\partial_1\}, \{\partial_2\}, \{\partial_3\}\}$ and $\Sigma' \cong \Sigma_1^6$ with the partition $\mathcal{P}' = \{\{\partial'_1\}, \{\partial'_2, \partial'_3\}, \{\partial'_4, \partial'_5, \partial'_6\}\}$:



As in that figure, let $(\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$ be an open capping and let μ'' be the stabilization of μ' to $(\Sigma'', \mathcal{P}'')$. It follows from the above that the composition

$$(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$$

is also an open capping. We remark that this can fail if $\mathcal{P}$ is not the discrete partition. For instance, consider the morphisms $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ and $(\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$ in the following figure, where $\mathcal{P} = \{\{\partial_1, \partial_2\}\}$, $\mathcal{P}' = \{\{\partial'_1\}\}$ and $\mathcal{P}'' = \{\{\partial''_1\}\}$:



We have maps

$$\mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu')) \to \mathrm{H}_k(\mathcal{I}(\Sigma'',\mathcal{P}'',\mu'')).$$

Proposition 5.6 implies that

$$\mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma'',\mathcal{P}'',\mu'')), \quad \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu')) \to \mathrm{H}_k(\mathcal{I}(\Sigma'',\mathcal{P}'',\mu''))$$

are isomorphisms. We conclude that the map

$$\mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu'))$$

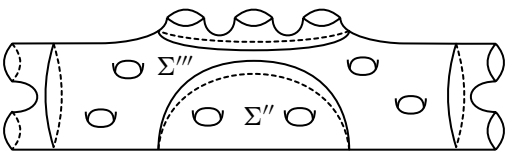
is an isomorphism, as desired.

Case 2 *$\mathcal{P}$ is not the discrete partition of $\partial\Sigma$.*

Since μ is supported on a symplectic subsurface, we can find a `PSurf`–morphism $(\Sigma'',\mathcal{P}'') \to (\Sigma,\mathcal{P})$ with $\Sigma'' \cong \Sigma_h^1$ and an A–homology marking μ'' on $(\Sigma'',\mathcal{P}'')$ such that μ is the stabilization of μ'' to $(\Sigma,\mathcal{P})$. We can factor $(\Sigma'',\mathcal{P}'') \to (\Sigma,\mathcal{P})$ as

$$(\Sigma'',\mathcal{P}'') \to (\Sigma''',\mathcal{P}''') \to (\Sigma,\mathcal{P})$$

so that Σ''' has the same genus as Σ, $\mathcal{P}'''$ is the discrete partition of $\partial\Sigma'''$, and $(\Sigma''',\mathcal{P}''') \to (\Sigma,\mathcal{P})$ is partition-bijective; see here:



In this example, $\mathcal{P}$ consists of three sets of boundary components (the ones on the left, right, and top). Let μ''' be the stabilization of μ'' to $(\Sigma''',\mathcal{P}''')$. We have maps

$$\mathrm{H}_k(\mathcal{I}(\Sigma''',\mathcal{P}''',\mu''')) \to \mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu')).$$

Case 1 implies that the maps

$$\mathrm{H}_k(\mathcal{I}(\Sigma''',\mathcal{P}''',\mu''')) \to \mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)), \quad \mathrm{H}_k(\mathcal{I}(\Sigma''',\mathcal{P}''',\mu''')) \to \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu'))$$

are isomorphisms. We conclude that the map

$$\mathrm{H}_k(\mathcal{I}(\Sigma,\mathcal{P},\mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma',\mathcal{P}',\mu'))$$

is an isomorphism, as desired. □

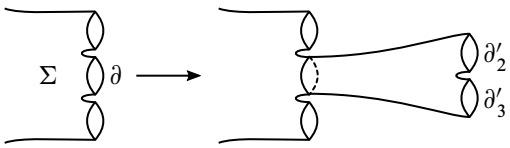
5.3 Reduction II: boundary stabilizations

In this section we reduce Proposition 5.6 to showing that certain kinds of PSurf–morphisms, called increasing boundary stabilizations and decreasing boundary stabilizations, induce isomorphisms on homology.

Increasing boundary stabilization Let $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. An *increasing boundary stabilization* of $(\Sigma, \mathcal{P})$ is a PSurf–morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ constructed as follows. Let ∂ be a component of $\partial\Sigma$ and let $p \in \mathcal{P}$ be the partition element with $\partial \in p$. Also, let $\partial\Sigma_0^3 = \{\partial'_1, \partial'_2, \partial'_3\}$.

- Σ' is obtained by attaching Σ_0^3 to Σ by gluing $\partial'_1 \subset \Sigma_0^3$ to $\partial \subset \Sigma$.
- $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing p with $p' = (p \setminus \{\partial\}) \cup \{\partial'_2, \partial'_3\}$.

See here:



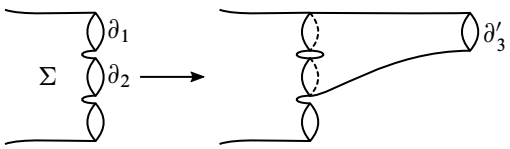
In Section 5.4, we will prove the following.

Proposition 5.7 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be an increasing boundary stabilization and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Then the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism if the genus of Σ is at least $(\mathrm{rk}(A) + 2)k + (2\,\mathrm{rk}(A) + 2)$.*

Decreasing boundary stabilization Let $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. A *decreasing boundary stabilization* of $(\Sigma, \mathcal{P})$ is a PSurf–morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ constructed as follows. Let ∂_1 and ∂_2 be distinct components of $\partial\Sigma$ that both lie in some $p \in \mathcal{P}$, and let $\partial\Sigma_0^3 = \{\partial'_1, \partial'_2, \partial'_3\}$.

- Σ' is obtained by attaching Σ_0^3 to Σ by gluing ∂'_1 and ∂'_2 to ∂_1 and ∂_2, respectively.
- $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing p with $p' = (p \setminus \{\partial_1, \partial_2\}) \cup \{\partial'_3\}$.

See here:

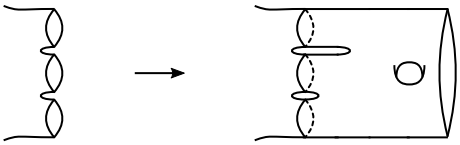


In Section 5.4, we will prove the following:

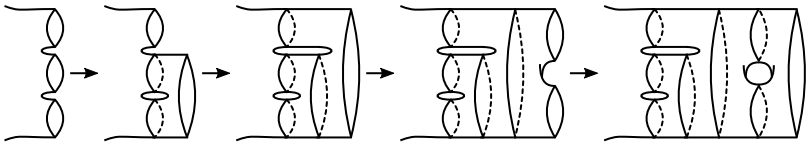
Proposition 5.8 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a decreasing boundary stabilization and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Then the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism if the genus of Σ is at least $(\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+2)$.*

Deriving Proposition 5.6 As we said above, we will prove Propositions 5.7 and 5.8 in Section 5.4. Here we will explain how to use them to prove Proposition 5.6.

Proof of Proposition 5.6, assuming Propositions 5.7 and 5.8 It is geometrically clear that an open capping $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ can be factored as a composition of increasing boundary stabilizations and decreasing boundary stabilizations. For instance,



can be factored as



The proposition follows. □

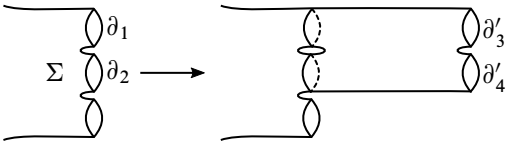
5.4 Reduction III: double boundary stabilizations

In this section, we adapt a beautiful idea of Hatcher and Vogtmann [12] to show how to reduce our two different boundary stabilizations (increasing and decreasing) to a single kind of stabilization called a double boundary stabilization.

Double boundary stabilization Let $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. A *double boundary stabilization* of $(\Sigma, \mathcal{P})$ is a $\mathtt{PSurf}$–morphism $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ constructed as follows. Let ∂_1 and ∂_2 be components of $\partial\Sigma$ that lie in a single element $p \in \mathcal{P}$. Also, let $\partial\Sigma_0^4 = \{\partial_1', \partial_2', \partial_3', \partial_4'\}$.

- Σ' is obtained by attaching $\Sigma_{0,4}$ to Σ by gluing ∂_1' and ∂_2' to ∂_1 and ∂_2, respectively.
- $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing p with $p' = (p \setminus \{\partial_1, \partial_2\}) \cup \{\partial_3', \partial_4'\}$.

See here:



In Section 6, we will use the homological stability machine to prove the following:

Proposition 5.9 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a double boundary stabilization and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Then the induced map*

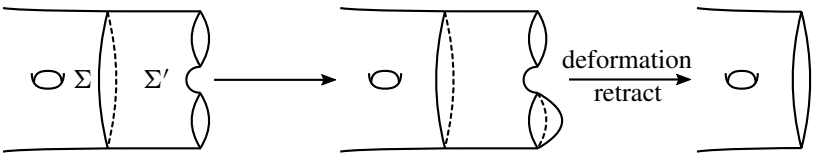
$$\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$$

is an isomorphism if the genus of Σ is at least $(\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+2)$ and a surjection if the genus of Σ is $(\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+1)$.

Deriving Propositions 5.7 and 5.8 As we said above, we will prove Proposition 5.9 in Section 6. Here we will explain how to use it to prove Propositions 5.7 and 5.8.

Proof of Proposition 5.7, assuming Proposition 5.9 We start by recalling the statement. Consider an increasing boundary stabilization $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$. Let μ be an A–homology marking on $(\Sigma, \mathcal{P})$ that is supported on a symplectic subsurface and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Assume that the genus of Σ is at least $(\mathrm{rk}(A)+2)k + (2\,\mathrm{rk}(A)+2)$. We must prove that the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism.

The first observation is that the map $\mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathcal{I}(\Sigma', \mathcal{P}', \mu')$ is split injective via a splitting map $\mathcal{I}(\Sigma', \mathcal{P}', \mu') \to \mathcal{I}(\Sigma, \mathcal{P}, \mu)$ induced by gluing a disc to one of the two components of $\partial\Sigma' \setminus \partial\Sigma$:



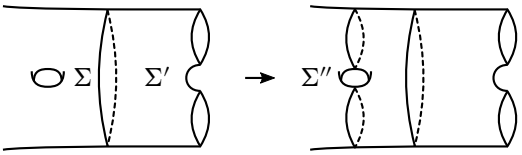
The map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is thus injective, so it is enough to prove that it is surjective.

Combining the fact that μ is supported on a symplectic subsurface with Corollary 3.6, we see that μ is in fact supported on a symplectic subsurface of genus at most $\mathrm{rk}(A)$. Since the genus of Σ is greater than $\mathrm{rk}(A)$, this implies that we can find a decreasing boundary

stabilization $(\Sigma'', \mathcal{P}'') \to (\Sigma, \mathcal{P})$ and an A–homology marking μ'' on $(\Sigma'', \mathcal{P}'')$ that is supported on a symplectic subsurface such that μ is the stabilization of μ'' to $(\Sigma, \mathcal{P})$ and such that the composition

$$(\Sigma'', \mathcal{P}'') \to (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$$

is a double boundary stabilization; see here:



The genus of Σ'' is one less than that of Σ, and so is at least $(\mathrm{rk}(A)+2)k+(2\,\mathrm{rk}(A)+1)$. We can thus apply Proposition 5.9 to deduce that the composition

$$\mathrm{H}_k(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu'')) \to \mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$$

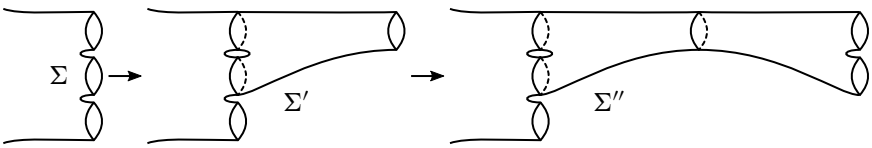
is surjective, and thus that the map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is surjective, as desired. □

Proof of Proposition 5.8, assuming Proposition 5.9 We start by recalling the statement. Consider a decreasing boundary stabilization $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$. Let μ be an A–homology marking on $(\Sigma, \mathcal{P})$ that is supported on a symplectic subsurface and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Assume that the genus of Σ is at least $(\mathrm{rk}(A) + 2)k + (2\,\mathrm{rk}(A) + 2)$. We must prove that the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism.

Let ∂' be the component of $\partial\Sigma'$ that is not a component of $\partial\Sigma$. As in the following picture, we can construct an increasing boundary stabilization $(\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$ that attaches a 3–holed torus to ∂' such that the composition

$$(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$$

is a double boundary stabilization:



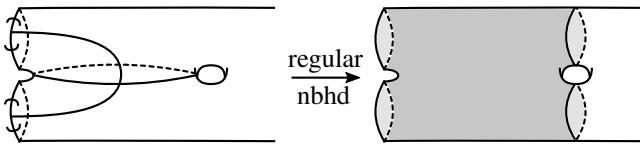
Let μ'' be the stabilization of μ' to $(\Sigma'', \mathcal{P}'')$. We then have maps

$$\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu')) \to \mathrm{H}_k(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu'')). \tag{5-1}$$

Proposition 5.9 implies that the composition (5-1) is an isomorphism, and Proposition 5.7 implies that $\mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu')) \to \mathrm{H}_k(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu''))$ is an isomorphism. We conclude that $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism, as desired. □

6 Double boundary stabilization

Adapting an argument due to Hatcher and Vogtmann [12], we will prove Proposition 5.9 by studying a complex of "order-preserving double-tethered loops" whose vertex-stabilizers yield double boundary stabilizations:



We will require that the homology classes of both the loop and the "double-tether" arc vanish under the homology marking. Getting the loop to vanish will be an easy variant on the argument we used for vanishing surfaces in Section 3.3.2, but getting the double-tether to vanish is harder and will require new ideas. We will build up the complex in three stages (tethered vanishing loops, then double-tethered vanishing loops, and then finally order-preserving double-tethered vanishing loops) in Section 6.3–6.7. These five sections are preceded by two technical sections: Section 6.1 gives a necessary and sufficient condition for an A–homology marking to be supported on a symplectic subsurface, and Section 6.2 is about destabilizing A–homology markings. After all this is complete, we prove Proposition 5.9 in Section 6.8.

6.1 Identifying markings supported on a symplectic subsurface

Consider some $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. In this section, we give a necessary and sufficient condition for an A–homology marking on $(\Sigma, \mathcal{P})$ to be supported on a symplectic subsurface. This requires some preliminary definitions (which will also be used later).

Intersection map Let q be a finite set of oriented simple closed curves on Σ and let $\mathbb{Z}[q]$ be the set of formal $\mathbb{Z}$–linear combinations of elements of q. Define the *q–intersection map* to be the map $\mathfrak{i}_q \colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to \mathbb{Z}[q]$ defined as follows. Let

$$\omega_\Sigma \colon \mathrm{H}_1(\Sigma, \partial\Sigma) \times \mathrm{H}_1(\Sigma) \to \mathbb{Z}$$

be the algebraic intersection pairing. For $x \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$, we then set

$$\mathfrak{i}_q(x) = \sum_{\gamma \in q} \omega_\Sigma(x, [\gamma]) \cdot \gamma.$$

Total boundary map For a set q as above, define

$$\widetilde{\mathbb{Z}}[q] = \left\{ \sum_{\gamma \in q} c_\gamma \cdot \gamma \in \mathbb{Z}[q] \;\middle|\; \sum_{\gamma \in q} c_\gamma = 0 \right\}.$$

Consider $p \in \mathcal{P}$. Each boundary component $\partial \in p$ is a simple closed curve on Σ, and the orientation on Σ induces an orientation on ∂ such that $\mathrm{Int}(\Sigma)$ lies to the left of ∂. We thus have the map $\mathfrak{i}_p \colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to \mathbb{Z}[p]$. Since $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ is generated by the homology classes of oriented loops and arcs connecting $\mathcal{P}$–adjacent boundary components, the image of $\mathfrak{i}_p$ is $\widetilde{\mathbb{Z}}[p]$. Define

$$\widetilde{\mathbb{Z}}_{\mathcal{P}} = \bigoplus_{p \in \mathcal{P}} \widetilde{\mathbb{Z}}[p].$$

The *total boundary map* of $(\Sigma, \mathcal{P})$ is the map $\mathfrak{i}_{\mathcal{P}} \colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to \widetilde{\mathbb{Z}}_{\mathcal{P}}$ obtained by taking the direct sum of all the $\mathfrak{i}_p$ for $p \in \mathcal{P}$.

Remark 6.1 Each $\widetilde{\mathbb{Z}}[p]$ naturally lies in $\widetilde{\mathrm{H}}_0(\partial\Sigma)$, and the total boundary map can be identified with the restriction to $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ of the usual boundary map $\mathrm{H}_1(\Sigma, \partial\Sigma) \to \widetilde{\mathrm{H}}_0(\partial\Sigma)$.

Symplectic support Now consider an A–homology marking μ on $(\Sigma, \mathcal{P})$. Back in Remark 5.1, we observed that a necessary condition for μ to be supported on a symplectic subsurface is that $\mathfrak{i}_{\mathcal{P}}(\ker(\mu)) = \widetilde{\mathbb{Z}}_{\mathcal{P}}$. The following lemma says that this condition is also sufficient:

Lemma 6.2 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. Then μ is supported on a symplectic subsurface if and only if $\mathfrak{i}_{\mathcal{P}}(\ker(\mu)) = \widetilde{\mathbb{Z}}_{\mathcal{P}}$.*

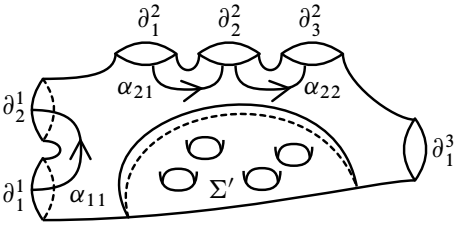
Proof The nontrivial direction is that if $\mathfrak{i}_{\mathcal{P}}(\ker(\mu)) = \widetilde{\mathbb{Z}}_{\mathcal{P}}$, then μ is supported on a symplectic subsurface, so that is what we prove. Write

$$\mathcal{P} = \{\{\partial_1^1, \dots, \partial_{k_1}^1\}, \{\partial_1^2, \dots, \partial_{k_2}^2\}, \dots, \{\partial_1^n, \dots, \partial_{k_n}^n\}\}.$$

Below we will prove that, for all $1 \le i \le n$ and $1 \le j < k_i$, we can find embedded arcs α_{ij} satisfying

- α_{ij} connects ∂_j^i to ∂_{j+1}^i,
- the α_{ij} are pairwise disjoint, and
- $\mu([\alpha_{ij}]) = 0$ for all i and j.

Letting g be the genus of Σ, we can then find a subsurface Σ' of Σ that is homeomorphic to Σ_g^1 such that Σ' is disjoint from $\partial\Sigma$ and the α_{ij}; see here:



Let $\mathcal{P}' = \{\partial\Sigma'\}$, so $(\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ is a `PSurf`–morphism. It is easy to see that we can find an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ such that μ is the stabilization of μ' to $(\Sigma, \mathcal{P})$; see Lemma 6.3 for a more general result that implies this. The lemma follows.

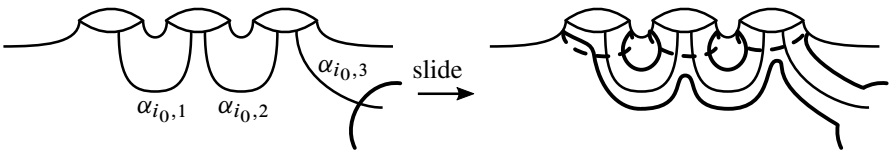
It remains to find the α_{ij}. The assumptions in the lemma imply that, for $1 \le i \le n$ and $1 \le j < k_i$, we can find arcs α_{ij} (not necessarily embedded or pairwise disjoint) satisfying

- α_{ij} connects ∂_j^i to ∂_{j+1}^i, and
- $\mu([\alpha_{ij}]) = 0$ for all i and j.

Homotoping the α_{ij}, we can assume that their endpoints are disjoint from each other, their interiors lie in the interior of Σ, and all intersections and self-intersections are transverse. Choose these α_{ij} so as to minimize the number of intersections and self-intersections. We claim that the α_{ij} are then all embedded and pairwise disjoint from each other. Assume otherwise. Let α_{i_0,j_0} be the first element of the ordered list

$$\alpha_{11}, \alpha_{12}, \dots, \alpha_{1,k_1-1}, \alpha_{21}, \dots, \alpha_{2,k_2-1}, \alpha_{31}, \dots, \alpha_{n,k_n-1}$$

that intersects either itself or one of the other α_{ij}. As in the following picture, we can then "slide" the first intersection of α_{i_0,j_0} off of the union of the $\partial_j^{i_0}$ and $\alpha_{i_0,j}$ with $j \le j_0$:



Since the homology classes of all the ∂_j^i are in the kernel of μ, this does not change the value of any of the $\mu([\alpha_{ij}])$, but it does eliminate one of the intersections, contradicting the minimality of this number. □

6.2 Destabilizing homology-marked partitioned surfaces

Consider $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. This section is devoted to "destabilizing" A–homology markings on $(\Sigma, \mathcal{P})$ to subsurfaces.

Existence Let μ be an A–homology marking on $(\Sigma, \mathcal{P})$ and let $(\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ be a $\mathtt{PSurf}$–morphism. One obvious necessary condition for there to exist an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ whose stabilization to $(\Sigma, \mathcal{P})$ is μ is that μ must vanish on elements of $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ supported on $\Sigma \setminus \Sigma'$. This condition is also sufficient:

Lemma 6.3 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $(\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ be a $\mathtt{PSurf}$–morphism. Then there exists an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ whose stabilization to $(\Sigma, \mathcal{P})$ is μ if and only if $\mu(x) = 0$ for all $x \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ supported on $\Sigma \setminus \Sigma'$.*

Proof The nontrivial assertion here is that, if $\mu(x) = 0$ for all $x \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ supported on $\Sigma \setminus \Sigma'$, then μ' exists, so this is what we prove. Let $\iota\colon (\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ be the inclusion. We want to show that $\mu\colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to A$ factors through

$$\iota^*\colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma').$$

The cokernel of ι^* is obviously free abelian. It is thus enough to prove that μ vanishes on $\ker(\iota^*)$. To do this, we will show that $\ker(\iota^*)$ is generated by elements supported on $\Sigma \setminus \Sigma'$. The map ι^* is the restriction to $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ of the composition

$$\mathrm{H}_1(\Sigma, \partial\Sigma) \xrightarrow{f} \mathrm{H}_1(\Sigma, \Sigma \setminus \mathrm{Int}(\Sigma')) \xrightarrow{\cong} \mathrm{H}_1(\Sigma', \partial\Sigma').$$

It is thus enough to show that all elements of $\ker(f)$ are supported on $\Sigma \setminus \Sigma'$. The long exact sequence in homology for the triple $(\Sigma, \Sigma \setminus \mathrm{Int}(\Sigma'), \partial\Sigma)$ implies that $\ker(f)$ is generated by the image of

$$\mathrm{H}_1(\Sigma \setminus \mathrm{Int}(\Sigma'), \partial\Sigma) \to \mathrm{H}_1(\Sigma, \partial\Sigma).$$

The desired result follows. □

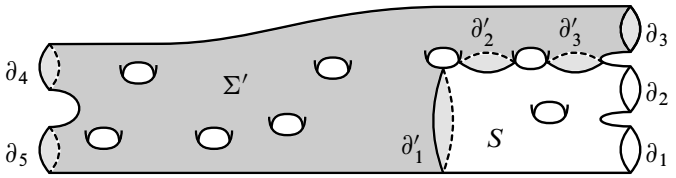
$\mathcal{P}$–simple subsurfaces We now study when destabilizations of markings supported on symplectic subsurfaces are supported on symplectic subsurfaces. Rather than prove the most general result possible, we will focus on the case of *$\mathcal{P}$–simple subsurfaces* of Σ, which are subsurfaces Σ' satisfying (see Example 6.4 below):

- Σ' is connected.
- The closure S of $\Sigma \setminus \Sigma'$ is connected.

- The set of components of ∂S can be partitioned into two disjoint nonempty subsets q and q' as follows:
  - The elements of q' all lie in the interior of Σ, and are thus components of $\partial\Sigma'$. These will be called the *interior boundary components*.
  - The elements of q are components of $\partial\Sigma \setminus \partial\Sigma'$ lying in a single $p \in \mathcal{P}$. These will be called the *exterior boundary components*.

Given a $\mathcal{P}$–simple subsurface Σ' of Σ, the *induced partition* $\mathcal{P}'$ of the components of $\partial\Sigma'$ is obtained from $\mathcal{P}$ by replacing p with $(p \setminus q) \cup q'$, where p, q and q' are as above. The map $(\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ is clearly a $\mathtt{PSurf}$–morphism.

Example 6.4 Let $\Sigma = \Sigma_8^5$ and $\mathcal{P} = \{\{\partial_1, \partial_2, \partial_3\}, \{\partial_4, \partial_5\}\}$. Consider the following shaded subsurface Σ' of Σ:



The subsurface Σ' is a $\mathcal{P}$–simple subsurface with interior boundary components $q' = \{\partial'_1, \partial'_2, \partial'_3\}$, exterior boundary components $q = \{\partial_1, \partial_2\}$, and induced partition $\mathcal{P}' = \{\{\partial'_1, \partial'_2, \partial'_3, \partial_3\}, \{\partial_4, \partial_5\}\}$.

Closed markings and intersection maps We now introduce some notation needed to state our result. Let $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let μ be an A–homology marking on $(\Sigma, \mathcal{P})$. The associated *closed marking* on $(\Sigma, \mathcal{P})$ is the map $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ defined via the composition

$$\mathrm{H}_1(\Sigma) \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \xrightarrow{\mu} A.$$

Also, for a finite set q of oriented simple closed curves on Σ, define the *closed q–intersection map* to be the map $\hat{\mathrm{i}}_q\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}[q]$ defined via the composition

$$\mathrm{H}_1(\Sigma) \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \xrightarrow{\mathrm{i}_q} \mathbb{Z}[q].$$

If the elements of q are disjoint and their union bounds a subsurface on one side (with respect to the orientations on the curves of q), then the image of $\hat{\mathrm{i}}_q$ lies in $\widetilde{\mathbb{Z}}[q]$.

Destabilizing and symplectic support With the above notation, we have the following lemma:

Lemma 6.5 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P})$ that is supported on a symplectic subsurface. Let Σ' be a $\mathcal{P}$–simple subsurface of Σ with induced partition $\mathcal{P}'$ and let μ' be an A–homology marking on $(\Sigma', \mathcal{P}')$ whose stabilization to $(\Sigma, \mathcal{P})$ is μ. Assume the following*:

- *Let q' be the interior boundary components of Σ' and let $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ. Orient each $\partial' \in q'$ so that Σ' lies to its left. Then $\hat{\mathrm{i}}_{q'}(\ker(\hat{\mu})) = \widetilde{\mathbb{Z}}[q']$.*

Then μ' is supported on a symplectic subsurface.

Proof By Lemma 6.2, we must prove that the map

$$\mathrm{i}_{\mathcal{P}'}\colon \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \to \widetilde{\mathbb{Z}}_{\mathcal{P}'}$$

takes $\ker(\mu')$ onto $\widetilde{\mathbb{Z}}_{\mathcal{P}'}$. Below we will prove two facts:

- $\widetilde{\mathbb{Z}}[q'] \subset \mathrm{i}_{\mathcal{P}'}(\ker(\mu'))$.
- Letting $\iota\colon (\Sigma', \mathcal{P}') \to (\Sigma, \mathcal{P})$ be the inclusion and $\iota^*\colon \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \to \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma')$ be the induced map, there exists a surjection $\beta\colon \widetilde{\mathbb{Z}}_{\mathcal{P}} \twoheadrightarrow \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q']$ such that the diagram

(6-1)
$$\begin{array}{ccccc} \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) & & \xrightarrow{\quad\mathrm{i}_{\mathcal{P}}\quad} & & \widetilde{\mathbb{Z}}_{\mathcal{P}} \\ \downarrow{\scriptstyle \iota^*} & & & & \downarrow{\scriptstyle \beta} \\ \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') & \xrightarrow{\mathrm{i}_{\mathcal{P}'}} & \widetilde{\mathbb{Z}}_{\mathcal{P}'} & \xrightarrow{\pi} & \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q'] \end{array}$$

commutes.

Assume these, for the moment. Since $\widetilde{\mathbb{Z}}[q'] \subset \mathrm{i}_{\mathcal{P}'}(\ker(\mu'))$, to prove $\mathrm{i}_{\mathcal{P}'}(\ker(\mu')) = \widetilde{\mathbb{Z}}_{\mathcal{P}'}$ it is enough to prove that $\pi\big(\mathrm{i}_{\mathcal{P}'}(\ker(\mu'))\big) = \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q']$. Since μ is supported on a symplectic subsurface, Lemma 6.2 says that $\mathrm{i}_{\mathcal{P}}(\ker(\mu)) = \widetilde{\mathbb{Z}}_{\mathcal{P}}$, so

$$\pi\big(\mathrm{i}_{\mathcal{P}'}\big(\iota^*(\ker(\mu))\big)\big) = \beta\big(\mathrm{i}_{\mathcal{P}}(\ker(\mu))\big) = \beta(\widetilde{\mathbb{Z}}_{\mathcal{P}}) = \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q']. \tag{6-2}$$

Since μ is the stabilization of μ' to $(\Sigma, \mathcal{P})$, by definition we have $\mu = \mu' \circ \iota^*$, so $\iota^*(\ker(\mu)) \subset \ker(\mu')$. Plugging this into (6-2), we get that

$$\pi\big(\mathrm{i}_{\mathcal{P}'}(\ker(\mu'))\big) = \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q'],$$

as desired.

It remains to prove the above two facts. We start with the first. Since elements of $\mathrm{H}_1(\Sigma)$ can be represented by cycles that are disjoint from all components of $\partial\Sigma$, the image of the composition

$$\mathrm{H}_1(\Sigma) \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma) \xrightarrow{\iota^*} \mathrm{H}_1^{\mathcal{P}'}(\Sigma', \partial\Sigma') \xrightarrow{\mathrm{i}_{\mathcal{P}'}} \widetilde{\mathbb{Z}}_{\mathcal{P}'}$$

must lie in $\widetilde{\mathbb{Z}}[q'] \subset \widetilde{\mathbb{Z}}_{\mathcal{P}'}$. From its definition, it is clear that this composition in fact equals $\hat{\mathrm{i}}_{q'}$. Our hypothesis about $\hat{\mathrm{i}}_{q'}$ thus implies that

$$\widetilde{\mathbb{Z}}[q'] \subset \mathrm{i}_{\mathcal{P}'}\big(\iota^*(\ker(\mu))\big) \subset \mathrm{i}_{\mathcal{P}'}(\ker(\mu')),$$

as desired. Here we are using the fact (already observed in the previous paragraph) that $\iota^*(\ker(\mu)) \subset \ker(\mu')$.

We now construct $\beta\colon \widetilde{\mathbb{Z}}_{\mathcal{P}} \twoheadrightarrow \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q']$. Let q be the exterior boundary components of Σ'. Write $\mathcal{P} = \{p_1, \dots, p_k\}$ with $q \subset p_1$. Setting $p_1' = (p_1 \setminus q) \cup q'$, we then have $\mathcal{P}' = \{p_1', p_2, \dots, p_k\}$. Thus

$$\widetilde{\mathbb{Z}}_{\mathcal{P}} = \widetilde{\mathbb{Z}}[p_1] \oplus \bigoplus_{i=2}^{k} \widetilde{\mathbb{Z}}[p_i] \quad \text{and} \quad \widetilde{\mathbb{Z}}_{\mathcal{P}'}/\widetilde{\mathbb{Z}}[q'] = \widetilde{\mathbb{Z}}[p_1']/\widetilde{\mathbb{Z}}[q'] \oplus \bigoplus_{i=2}^{k} \widetilde{\mathbb{Z}}[p_i].$$

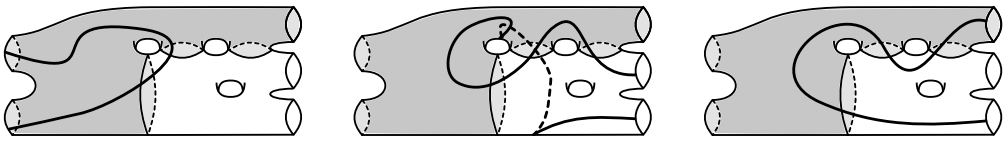
On the $\widetilde{\mathbb{Z}}[p_i]$ summand for $2 \le i \le k$, the map β is the identity. On the $\widetilde{\mathbb{Z}}[p_1]$ summand, the map β is the restriction to $\widetilde{\mathbb{Z}}[p_1]$ of the map

$$\mathbb{Z}[p_1] = \mathbb{Z}[p_1 \setminus q] \oplus \mathbb{Z}[q] \to \mathbb{Z}[p_1 \setminus q] \oplus \mathbb{Z}[q']/\widetilde{\mathbb{Z}}[q'] = \widetilde{\mathbb{Z}}[p_1']/\widetilde{\mathbb{Z}}[q']$$

that is the identity on $\mathbb{Z}[p_1 \setminus q]$ and takes every element of q to the generator of $\mathbb{Z}[q']/\widetilde{\mathbb{Z}}[q'] \cong \mathbb{Z}$. This map β is clearly a surjection. The fact that (6-1) commutes follows from the fact that an arc in Σ from a component ∂_1 of $\partial\Sigma$ to a component ∂_2 of $\partial\Sigma$ with ∂_1 and ∂_2 both lying in some p_i has the following algebraic intersection number with the union of the components of q:

- 0 if $i \ge 2$, if $i = 1$ and $\partial_1, \partial_2 \in p_1 \setminus q$, or if $i = 1$ and $\partial_1, \partial_2 \in q$,
- 1 if $i = 1$, $\partial_1 \in p_1 \setminus q$ and $\partial_2 \in q$,
- -1 if $i = 1$, $\partial_2 \in q$ and $\partial_1 \in p_1 \setminus q$.

The reason for this is that each time the arc crosses from Σ' to S it adds $+1$ to its total intersection with q, while each time it crosses from S to Σ' it adds -1 to its total intersection with q. See the following figure, where Σ is shaded and S is unshaded:



The lemma follows. □

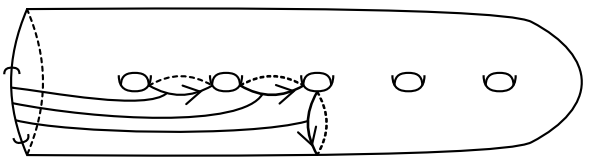
6.3 The complex of tethered vanishing loops

We now begin our long trek to the complex of order-preserving double-tethered vanishing loops, starting with the complex of tethered vanishing loops. The definition takes several steps.

Tethered loops Define $\tau(S^1)$ to be the result of gluing $1 \in [0,1]$ to a point of S^1. The subset $[0,1] \in \tau(S^1)$ is the *tether* and $0 \in [0,1] \subset \tau(S^1)$ is the *initial point* of the tether. For a surface $\Sigma \in \mathtt{Surf}$ and a finite disjoint union of open intervals $I \subset \partial\Sigma$, an *I–tethered loop* in Σ is an embedding $\iota\colon \tau(S^1) \to \Sigma$ such that

- ι takes the initial point of the tether to a point of I, and
- orienting $\iota(S^1)$ using the natural orientation of S^1, the image $\iota([0,1])$ of the tether approaches $\iota(S^1)$ from its right.

Complex of tethered loops For a surface $\Sigma \in \mathtt{Surf}$ and a finite disjoint union of open intervals $I \subset \partial\Sigma$, the *complex of I–tethered loops* on Σ, denoted by $\mathcal{TL}(\Sigma, I)$, is the simplicial complex whose k–simplices are collections $\{\iota_0, \dots, \iota_k\}$ of isotopy classes of I–tethered loops on Σ that can be realized so as to be disjoint and not separate Σ:

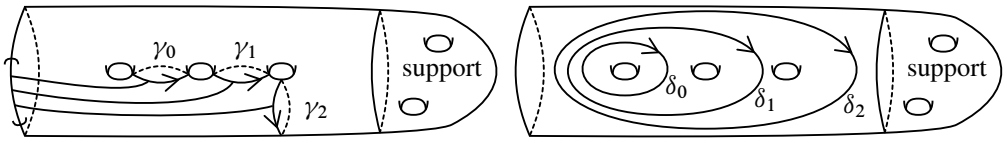


This complex was introduced by Hatcher and Vogtmann [12], who proved that if Σ has genus g then $\mathcal{TL}(\Sigma, I)$ is $\frac{1}{2}(g-3)$–connected; see [12, Proposition 5.1].

Complex of tethered vanishing loops Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $I \subset \partial\Sigma$ be a finite disjoint union of open intervals. Define $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ to be the subcomplex of $\mathcal{TL}(\Sigma, I)$ consisting of k–simplices $\{\iota_0, \dots, \iota_k\}$ satisfying the following conditions. For $0 \le i \le k$, let γ_i be the oriented loop $(\iota_i)|_{S^1}$. Set $\Gamma = \{\gamma_0, \dots, \gamma_k\}$. As in Section 6.2, let $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ and let $\hat{\mathrm{i}}_\Gamma\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}[\Gamma]$ be the closed Γ–intersection map. We then require that $\hat{\mu}([\gamma_i]) = 0$ for all $0 \le i \le k$ and that $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma]$.

Remark 6.6 This last condition might seem a little unmotivated, but is needed to ensure that the stabilizer of our simplex is supported on a symplectic subsurface (at least

in favorable situations). It clearly always holds when μ is supported on a symplectic subsurface that is disjoint from the images of all the ι_i. This is best illustrated by an example:



If $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$ and $\delta_0, \delta_1, \delta_2$ are as shown, then $\hat{\mu}([\delta_i]) = 0$ and $\hat{\mathrm{i}}_\Gamma([\delta_i]) = \gamma_i$ for $0 \le i \le 2$, which implies that $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma]$.

High connectivity Our main topological theorem about $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ is as follows:

Theorem 6.7 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$, $I \subset \partial\Sigma$ be a finite disjoint union of open intervals, and g be the genus of Σ. Then $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected.*

Proof The proof is very similar to that of Theorem 3.8. We start by defining an auxiliary space. Let X be the simplicial complex whose vertices are the union of the vertices of the spaces $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ and $\mathcal{TS}_{\mathrm{rk}(A)+1}(\Sigma, I)$ and whose simplices are collections σ of vertices such that:

- The vertices in σ (which are embeddings of either $\tau(S^1)$ or $\tau(\Sigma^1_{\mathrm{rk}(A)+1})$ into Σ) can be homotoped so that their images are disjoint and do not separate Σ.
- Let $\sigma' \subset \sigma$ be the subset consisting of vertices of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$. Then σ' is a simplex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$.

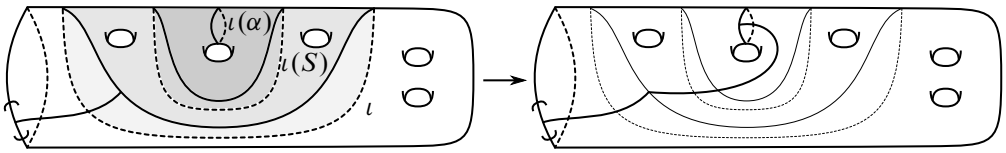
Both $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ and $\mathcal{TS}_{\mathrm{rk}(A)+1}(\Sigma, I)$ are subcomplexes of X.

The subcomplex $\mathcal{TS}_{\mathrm{rk}(A)+1}(\Sigma, I)$ of X is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected by Theorem D. An argument using Corollary 2.4 identical to the one in the proof of Theorem 3.8 shows that this implies that X is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected. As in the proof of Theorem 3.8, this implies that it is enough to construct a retraction $r\colon X \to \mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$.

For a vertex ι of X, we define $r(\iota)$ as follows. If ι is a vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, then $r(\iota) = \iota$. If instead ι is a vertex of $\mathcal{TS}_{\mathrm{rk}(A)+1}(\Sigma, I)$, then we do the following. Let $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ. Define $\mu'\colon \mathrm{H}_1(\Sigma^1_{\mathrm{rk}(A)+1}) \to A$ to be the composition

$$\mathrm{H}_1(\Sigma^1_{\mathrm{rk}(A)+1}) \cong \mathrm{H}_1(\tau(\Sigma^1_{\mathrm{rk}(A)+1})) \xrightarrow{\iota_*} \mathrm{H}_1(\Sigma) \xrightarrow{\hat{\mu}} A.$$

Proposition 3.4 implies that there exists a subsurface $S \subset \Sigma^1_{\mathrm{rk}(A)+1}$ with $S \cong \Sigma^1_1$ and $\mu'|_{\mathrm{H}_1(S)} = 0$. Let α be a nonseparating oriented simple closed curve in S. Define $r(\iota)$ to be the vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ obtained by adjoining the tether of ι and an arbitrary arc in $\iota(\Sigma^1_{\mathrm{rk}(A)+1})$ to $\iota(\alpha)$; see here:



To see that this is actually a vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, observe that by construction

$$\hat{\mu}([\iota(\alpha)]) = 0, \quad \iota_*(\mathrm{H}_1(S)) \subset \ker(\hat{\mu}) \quad \text{and} \quad \hat{\mathrm{i}}_{\{\iota(\alpha)\}}\big(\iota_*(\mathrm{H}_1(S))\big) = \iota(\alpha).$$

Of course, $r(\iota)$ depends on various choices, but we simply make an arbitrary choice.

To complete the proof, we must show that r extends over the simplices of X. Let σ be a simplex of X. Enumerate the vertices of σ as $\{\iota_0, \ldots, \iota_k, \iota'_0, \ldots, \iota'_\ell\}$, where the ι_i are vertices of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ and the ι'_j are vertices of $\mathcal{TS}_{\mathrm{rk}(A)+1}(\Sigma, I)$. We must prove that

$$r(\sigma) = \{\iota_0, \ldots, \iota_k, r(\iota'_0), \ldots, r(\iota'_\ell)\}$$

is a simplex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$. The images of the vertices in $r(\sigma)$ can clearly be homotoped so as to be disjoint, so the only thing we must prove is the following. For $0 \le i \le k$ and $0 \le j \le \ell$, let $\gamma_i = \iota_i|_{S^1}$ and $\gamma'_j = \iota'_j|_{S^1}$. Setting $\Gamma = \{\gamma_0, \ldots, \gamma_k, \gamma'_0, \ldots, \gamma'_\ell\}$, we have to show that $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma]$. Setting $\Gamma_1 = \{\gamma_0, \ldots, \gamma_k\}$ and $\Gamma_2 = \{\gamma'_0, \ldots, \gamma'_\ell\}$, we will show that Γ_1 and Γ_2 are both contained in $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu}))$.

We start with Γ_2. By construction, for $0 \le j \le \ell$ there exists a subsurface S_j of Σ with $S_j \cong \Sigma^1_1$ such that

- $\gamma'_j \subset S_j$, and
- the S_j are disjoint from each other and from all the γ_i, and
- regarding $\mathrm{H}_1(S_j)$ as a subgroup of $\mathrm{H}_1(\Sigma)$, we have $\mathrm{H}_1(S_j) \subset \ker(\hat{\mu})$.

Since $\hat{\mathrm{i}}_\Gamma(\mathrm{H}_1(S_j)) = \gamma'_j$, we have $\gamma'_j \in \hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu}))$, as desired.

It remains to show that $\Gamma_1 \subset \hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu}))$. Since $\{\iota_0, \ldots, \iota_k\}$ is a simplex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, by definition we have $\hat{\mathrm{i}}_{\Gamma_1}(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma_1]$. For some $0 \le i \le k$, let $x \in \ker(\hat{\mu})$ be such that $\hat{\mathrm{i}}_{\Gamma_1}(x) = \gamma_i$. We then have $\hat{\mathrm{i}}_\Gamma(x) = \gamma_i + z$ with $z \in \mathbb{Z}[\Gamma_2]$. Since we already showed that $\mathbb{Z}[\Gamma_2] \subset \hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu}))$, we conclude that $\gamma_i \in \hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu}))$, as desired. □

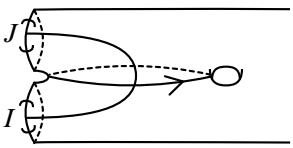
6.4 The complex of double-tethered vanishing loops

The definition of the complex of double-tethered vanishing loops takes several steps.

Double-tethered loops Define $\tau^2(S^1)$ to be the result of gluing $1 \in [0,2]$ to S^1. We will call $[0,2] \subset \tau^2(S^1)$ the *double tether*; the point $0 \in [0,2]$ is the double tether's *initial point* and $2 \in [0,2]$ is its *terminal point*. For a surface $\Sigma \in \mathtt{Surf}$ and finite disjoint unions of open intervals $I, J \subset \partial\Sigma$ with $I \cap J = \varnothing$, an (I,J)*–double-tethered loop* in Σ is an embedding $\iota\colon \tau^2(S^1) \to \Sigma$ where

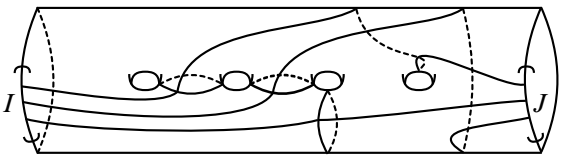
- ι takes the initial point of the double tether to a point of I and the terminal point of the double tether to a point of J, and
- orienting $\iota(S^1)$ using the natural orientation of S^1, the image $\iota([0,1])$ approaches $\iota(S^1)$ from its right and the image $\iota([1,2])$ leaves $\iota(S^1)$ from its left.

See here:



We remark that right now we allow boundary components that contain components of both I and J, but later when we discuss double-tethered vanishing loops our hypotheses will exclude this possibility.

Complex of double-tethered loops For a surface $\Sigma \in \mathtt{Surf}$ and finite disjoint unions of open intervals $I, J \subset \partial\Sigma$ with $I \cap J = \varnothing$, the *complex of* (I,J)*–double-tethered loops* on Σ, denoted by $\mathcal{DTL}(\Sigma, I, J)$, is the simplicial complex whose k–simplices are collections $\{\iota_0, \dots, \iota_k\}$ of isotopy classes of (I,J)–double-tethered loops on Σ that can be realized so as to be disjoint and not separate Σ. See here:



This complex was introduced by Hatcher and Vogtmann [12], who proved that if Σ has genus g then, like $\mathcal{TL}(\Sigma, I)$, it is $\frac{1}{2}(g-3)$–connected; see [12, Proposition 5.2].

$\mathcal{P}$–adjacency Consider $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $I, J \subset \partial\Sigma$ be finite disjoint unions of open intervals with $I \cap J = \varnothing$. Recall that components ∂ and ∂' of $\partial\Sigma$ are said to be $\mathcal{P}$–adjacent if there exists some $p \in \mathcal{P}$ such that $\partial, \partial' \in p$. We will say that I and J are

$\mathcal{P}$–adjacent if, for all components ∂_I and ∂_J of $\partial\Sigma$ such that ∂_I contains a component of I and ∂_J contains a component of J, the components ∂_I and ∂_J are distinct and $\mathcal{P}$–adjacent.

Complex of double-tethered vanishing loops Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent disjoint unions of open intervals with $I \cap J = \varnothing$. In particular, there are no boundary components of Σ containing components of both I and J. Define $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ to be the subcomplex of $\mathcal{DTL}(\Sigma, I, J)$ consisting of k–simplices $\{\iota_0, \dots, \iota_k\}$ satisfying the following conditions. Let $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ.

- For $0 \le i \le k$, let γ_i be the oriented loop $(\iota_i)|_{S^1}$ and α_i be the oriented arc $(\iota_i)|_{[0,2]}$. We then require that $\hat{\mu}([\iota_i]) = 0$ and $\mu([\alpha_i]) = 0$. This second condition makes sense since I and J are $\mathcal{P}$–adjacent.
- Set $\Gamma = \{\gamma_0, \dots, \gamma_k\}$. We then require that $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma]$.

Identifying $\tau(S^1)$ with the union of $[0, 1]$ and S^1 in $\tau^2(S^1)$, these conditions imply that $\{(\iota_0)|_{\tau(S^1)}, \dots, (\iota_k)|_{\tau(S^1)}\}$ is a simplex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$.

6.5 The complex of mixed-tethered vanishing loops

Our main theorem about the complex of double-tethered vanishing loops says that it is highly connected. We will prove this in Section 6.6 below. This section is devoted to an intermediate complex that will play a technical role in that proof.

Complex of mixed-tethered vanishing loops Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent disjoint unions of open intervals with $I \cap J = \varnothing$. Let $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ. Define $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$ to be the simplicial complex whose k–simplices are sets $\{\iota_0, \dots, \iota_k\}$, where each ι_i is the isotopy class of either an I–tethered loop or an (I, J)–double-tethered loop and where the following conditions are satisfied:

- The ι_i can be realized so that their images are disjoint and do not separate Σ.
- For $0 \le i \le k$, let γ_i be the oriented loop $(\iota_i)|_{S^1}$. We then require that $\hat{\mu}([\gamma_i]) = 0$.
- For $0 \le i \le k$ such that ι_i is an (I, J)–double-tethered loop, let α_i be the oriented arc $(\iota_i)|_{[0,2]}$. We then require that $\mu([\alpha_i]) = 0$.
- Set $\Gamma = \{\gamma_0, \dots, \gamma_k\}$. We then require that $\hat{\mathrm{i}}_\Gamma(\ker(\hat{\mu})) = \mathbb{Z}[\Gamma]$.

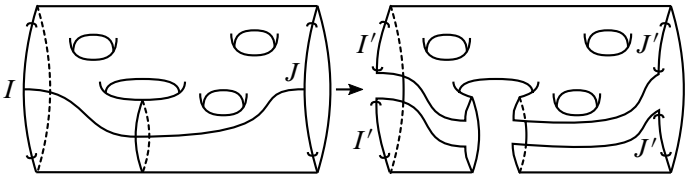
These conditions ensure that both $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ and $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ are full subcomplexes of $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$.

Links Our first task will be to identify links in $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$.

Lemma 6.8 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. Let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent finite disjoint unions of open intervals with $I \cap J = \varnothing$. Finally, let σ be a k–simplex of $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$. Then there exists some $(\Sigma', \mathcal{P}') \in \mathtt{PSurf}$, an A–homology marking μ' on $(\Sigma', \mathcal{P}')$, and $\mathcal{P}'$–adjacent finite disjoint unions of open intervals $I', J' \subset \partial\Sigma'$ with $I' \cap J' = \varnothing$ such that:*

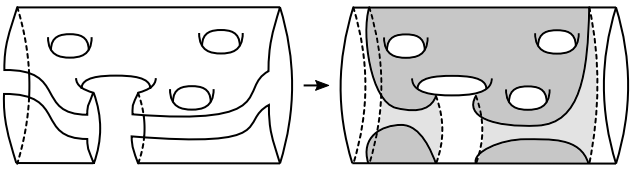
- *The link of σ is isomorphic to $\mathcal{MTL}(\Sigma', I', J', \mathcal{P}', \mu')$. Moreover, the intersections of the link of σ with $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$ and $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ are $\mathcal{TL}(\Sigma', I', \mathcal{P}', \mu')$ and $\mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$, respectively.*
- *If Σ is a genus-g surface, then Σ' is a genus-$(g-k-1)$ surface.*
- *If μ is supported on a symplectic subsurface, then so is μ'.*

Proof It is enough to deal with the case where σ has dimension 0; the general case will then follow by applying the dimension 0 case repeatedly. We thus can assume that $\sigma = \{\iota\}$, where ι is either an I–tethered loop or an (I, J)–double-tethered loop. The two cases are similar, so we will give the details for when ι is an (I, J)–double-tethered loop. Let Σ' be the result of cutting Σ open along the image of ι:



We remark that the fact that I and J are $\mathcal{P}$–adjacent implies that the initial and terminal points of the double tether are on distinct boundary components.

By isotoping Σ' into the interior of Σ, we can regard Σ' as a $\mathcal{P}$–simple subsurface of Σ:



Let $\mathcal{P}'$ be the induced partition of the components of $\partial\Sigma'$. By Lemma 6.3, there exists an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ such that μ is the stabilization of μ' to $(\Sigma, \mathcal{P})$.

As is clear from the above figure, when forming Σ' the sets I and J are divided into finer collections I' and J' of open intervals in $\partial\Sigma'$ such that the link of σ is isomorphic to $\mathcal{MTL}(\Sigma', I', J', \mathcal{P}', \mu')$. By construction, Σ' has genus $g-1$. The only thing that

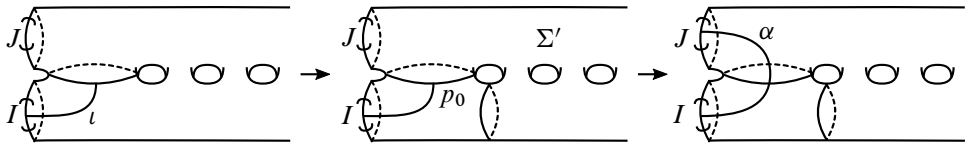
remains to be proved is that if μ is supported on a symplectic subsurface, then so is μ'. Letting $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ be the closed marking associated to μ and q be the interior boundary components of Σ' (as in the definition of a $\mathcal{P}$–simple subsurface in Section 6.2), Lemma 6.5 says that it is enough to prove that $\hat{\mathrm{i}}_q(\ker(\hat{\mu})) = \widetilde{\mathbb{Z}}[q]$.

Let $\gamma = \iota|_{S^1}$. Since ι is a vertex of $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$, there exists some $x \in \ker(\hat{\mu})$ such that $\hat{\mathrm{i}}_\gamma(x) = \gamma$. By construction, we have $q = \{\gamma_1, \gamma_2\}$, where γ_1 (resp. γ_2) is obtained by band-summing γ with a component of $\partial\Sigma$ containing a component of I (resp. J). The orientations on the γ_i are such that γ_1 is homologous in $\mathrm{H}_1(\Sigma, \partial\Sigma)$ to γ and γ_2 is homologous to $-\gamma$. It follows that $\hat{\mathrm{i}}_q(x) = \gamma_1 - \gamma_2$, which generates $\widetilde{\mathbb{Z}}[q]$. The lemma follows. □

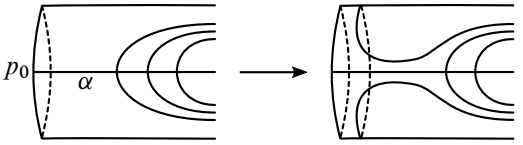
Completing a tethered loop to a double-tethered loop As a first application of Lemma 6.8 (or, rather, its proof), we prove the following:

Lemma 6.9 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent finite disjoint unions of open intervals with $I \cap J = \varnothing$. Then, for all vertices ι of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, there exists a vertex $\hat{\iota}$ of $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ such that $\hat{\iota}|_{\tau(S^1)} = \iota$.*

Proof Let $(\Sigma', \mathcal{P}')$ and I', J' and μ' be the output of applying Lemma 6.8 to the 0–simplex $\{\iota\}$ of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu) \subset \mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$. The A–homology marking μ' on $(\Sigma', \mathcal{P}')$ is thus supported on a symplectic subsurface. As in the following figure, it is enough to find an embedded arc α in Σ' connecting the endpoint p_0 of the tether of ι to a point of J such that $\mu'([\alpha]) = 0$:



Since μ' is supported on a symplectic subsurface, Lemma 6.2 implies that there exists an immersed arc α (not necessarily embedded) connecting p_0 to a point of J such that $\mu'([\alpha]) = 0$. Choose α so as to have the fewest possible self-intersections. Then α is embedded; indeed, if it has a self-intersection, then as in the following figure we can "comb" its first self-intersection over the component of $\partial\Sigma'$ containing p_0:



This has the effect of removing a self-intersection from α, but since μ' vanishes on all components of $\partial\Sigma'$ it does not change the fact that $\mu'([\alpha]) = 0$. The lemma follows. □

High connectivity We close this section by proving that $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is highly connected.

Theorem 6.10 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. Let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent finite disjoint unions of open intervals with $I \cap J = \varnothing$ and let g be the genus of Σ. Then $\mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected.*

Proof Set $n = \big(g - (2\,\mathrm{rk}(A) + 3)\big)/(\mathrm{rk}(A) + 2)$ and

$$X = \mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu), \quad Y = \mathcal{TL}(\Sigma, I, J, \mathcal{P}, \mu) \quad \text{and} \quad Y' = \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu).$$

Theorem 6.7 says that Y is n–connected, so it is enough to prove that the pair (X, Y) is n–connected. To do this, we will apply Corollary 2.4. This requires showing the following. Let σ be a k–dimensional simplex of Y' and let L be the link of σ in X. Then we must show that $L \cap Y$ is $(n-k-1)$–connected.

Lemma 6.8 says that $L \cap Y \cong \mathcal{TL}(\Sigma', I', J', \mathcal{P}', \mu')$, where Σ', I', J', $\mathcal{P}'$ and μ' are as follows:

- $(\Sigma', \mathcal{P}') \in \mathtt{PSurf}$ with Σ' a genus-$(g-k-1)$ surface.
- μ' is an A–homology marking on $(\Sigma', \mathcal{P}')$.
- $I', J' \subset \partial\Sigma'$ are $\mathcal{P}'$–adjacent finite disjoint unions of open intervals satisfying $I' \cap J' = \varnothing$.

Theorem 6.7 thus says that $L \cap Y$ is n'–connected for

$$n' = \frac{g' - (2\,\mathrm{rk}(A) + 3)}{\mathrm{rk}(A) + 2} = \frac{g - (2\,\mathrm{rk}(A) + 3)}{\mathrm{rk}(A) + 2} - \frac{k+1}{\mathrm{rk}(A) + 2} \geq n - k - 1. \qquad \square$$

6.6 High connectivity of the complex of double-tethered vanishing loops

In this section, we finally prove that the complex of double-tethered vanishing loops is highly connected:

Theorem 6.11 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent finite disjoint unions of open intervals with $I \cap J = \varnothing$ and let g be the genus of Σ. Then $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected.*

The proof of Theorem 6.11 requires the following lemma. Say that a simplicial map $f: M \to X$ between simplicial complexes is *locally injective* if $f|_\sigma$ is injective for all simplices σ of M.

Lemma 6.12 *Let M be a compact n–dimensional manifold (possibly with boundary) equipped with a combinatorial triangulation, let X be a simplicial complex, and let $f: M \to X$ be a simplicial map. Assume*

- *$f|_{\partial M}$ is locally injective,*
- *for all simplices σ of X, the link of σ in X is $(n-\dim(\sigma)-2)$–connected.*

Then after possibly subdividing simplices of M lying in its interior, f is homotopic through maps fixing ∂M to a simplicial map $f': M \to X$ that is locally injective.

Proof We remark that the proof of this is very similar to Hatcher and Vogtmann's proof of Proposition 2.3 above, though it seems hard to deduce it from that proposition. This result is also related to [10, Theorem 2.4].

The proof will be by induction on n. The base case $n = 0$ is trivial, so assume that $n > 0$ and that the result is true for all smaller dimensions. Call a simplex σ of M a *noninjective simplex* if, for all vertices v of σ, there exists a vertex v' of σ with $v \neq v'$ but $f(v) = f(v')$. If M has no noninjective simplices, then we are done. Assume, therefore, that M has noninjective simplices, and let σ be a noninjective simplex of M whose dimension is as large as possible. Since no simplices of ∂M are noninjective, the simplex σ does not lie in ∂M. Letting $L \subset M$ be the link of σ, this implies that $L \cong S^{n-\dim(\sigma)-1}$. Letting L' be the link of $f(\sigma)$ in X, the maximality of the dimension of σ implies two things:

- $f(L) \subset L'$.
- The restriction of f to L is locally injective.

Our assumptions imply that L' has connectivity at least

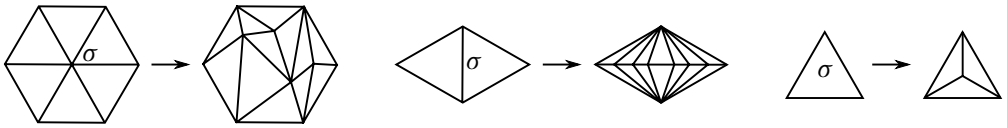
$$n - \dim(f(\sigma)) - 2 \geq n - (\dim(\sigma) - 1) - 2 = n - \dim(\sigma) - 1.$$

Here we are using the fact that $f|_\sigma$ is not injective. We can thus extend $f|_L$ to a map

$$F: D^{n-\dim(\sigma)} \to L'$$

that is simplicial with respect to some combinatorial triangulation of $D^{n-\dim(\sigma)}$ that restricts to $L \cong S^{n-\dim(\sigma)-1}$ on $\partial D^{n-\dim(\sigma)}$. Since $\dim(\sigma) \geq 1$ and $F|_{\partial D^{n-\dim(\sigma)}} = f|_L$ is locally injective, we can apply our inductive hypothesis to F and ensure that F is locally injective. The star S of σ is isomorphic to the join $\sigma * L$. Subdividing M

and homotoping f, we can replace $S \subset D^{n-\dim(\sigma)}$ with $\partial\sigma * D^{n-\dim(\sigma)}$ and $f|_S$ with $f|_{\partial\sigma} * F$. Here are pictures of this operation for $n = 2$ and $\dim(\sigma) \in \{0, 1, 2\}$; on the left-hand side is S, and on the right-hand side is $\partial\sigma * D^{n-\dim(\sigma)}$:



In doing this, we have eliminated the noninjective simplex σ without introducing any new noninjective simplices. Repeating this over and over again, we can eliminate all noninjective simplices, and we are done. □

Proof of Theorem 6.11 We will prove by induction on n that $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is n–connected for $-1 \le n \le \big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$. The base case simply asserts that $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is nonempty when $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2) \ge -1$. In this case, Theorem 6.7 asserts that $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu) \neq \varnothing$, and thus Lemma 6.9 implies that $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu) \neq \varnothing$, as desired.

Assume now that $0 \le n \le \big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$ and that all complexes $\mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$ as in the theorem are n'–connected for

$$n' = \min\left\{n-1, \frac{g'-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2}\right\},$$

where g' is the genus of Σ'. We must prove that $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is n–connected.

Set $X = \mathcal{MTL}(\Sigma, I, J, \mathcal{P}, \mu)$. The complex $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ that we want to show is n–connected is a subcomplex of X, and Theorem 6.10 says that the connectivity of X is at least

$$\frac{g-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2} \ge n.$$

Define Y to be the subcomplex of X consisting of simplices containing at most one vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, so

$$\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu) \subsetneq Y \subsetneq X.$$

The first step is as follows.

Claim 1 *The complex Y is n–connected.*

Proof We know that X is n–connected, so to prove that its subcomplex Y is n–connected it is enough to prove that the pair (X, Y) is $(n+1)$–connected. We will do this using Proposition 2.3. For this, we must identify a set $\mathcal{B}$ of "bad simplices" of

X and verify the three hypotheses of the proposition. Define $\mathcal{B}$ to be the set of all simplices of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu) \subset X$ whose dimension is at least 1.

We now verify the hypotheses of Proposition 2.3. The first two are easy:

(i) A simplex of X lies in Y if and only if none of its faces lie in $\mathcal{B}$, which is obvious.

(ii) If $\sigma_1, \sigma_2 \in \mathcal{B}$ are such that $\sigma_1 \cup \sigma_2$ is a simplex of X, then $\sigma_1 \cup \sigma_2 \in \mathcal{B}$, which again is obvious.

The only thing left to check is (iii), which says that for all k–dimensional $\sigma \in \mathcal{B}$, the complex $G(X, \sigma, \mathcal{B})$ has connectivity at least $(n+1)-k-1 = n-k$.

Let L be the link of σ in X. Examining its definition in Section 2.2, we see that

$$G(X, \sigma, \mathcal{B}) \cong L \cap \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu).$$

Lemma 6.8 says that $L \cap \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu) \cong \mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$, where Σ', I', J', $\mathcal{P}'$ and μ' are as follows:

- $(\Sigma', \mathcal{P}') \in \mathtt{PSurf}$ with Σ' a genus $g' = g-k-1$ surface.
- μ' is an A–homology marking on $(\Sigma', \mathcal{P}')$ that is supported on a symplectic subsurface.
- $I', J' \subset \partial\Sigma'$ are $\mathcal{P}'$–adjacent finite disjoint unions of open intervals satisfying $I' \cap J' = \varnothing$.

Our goal is thus to show that $\mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$ is $(n-k)$–connected. Our inductive hypothesis shows that $\mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$ is n'–connected for

$$\begin{aligned} n' &= \min\left\{n-1, \frac{g'-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2}\right\} = \min\left\{n-1, \frac{g-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2} - \frac{k+1}{\mathrm{rk}(A)+2}\right\} \\ &\geq \min\left\{n-1, n-\tfrac{1}{2}(k+1)\right\} \geq n-k. \end{aligned}$$

Here we are using the fact that, by the definition of $\mathcal{B}$, we have $k \geq 1$, and thus $k \geq \frac{1}{2}(k+1)$. □

This allows us to fill n–spheres in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ with $(n+1)$–discs in Y. We will modify these $(n+1)$–discs so that they lie in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$. For technical reasons, we will need our spheres and discs to be locally injective. That this is possible is the content of the following two steps.

Claim 2 *Equip the n–sphere S^n with a combinatorial triangulation and let $f\colon S^n \to \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ be a simplicial map. Then after possibly subdividing S^n, the map f is homotopic to a locally injective simplicial map.*

Proof By Lemma 6.12, this will follow if we can show that, for all k–simplices σ of $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$, the link L of σ is $(n-k-2)$–connected. Applying Lemma 6.8, we see that $L \cong \mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$, where Σ', I', J', $\mathcal{P}'$ and μ' are as follows:

- $(\Sigma', \mathcal{P}') \in \mathtt{PSurf}$ with Σ' a genus $g' = g-k-1$ surface.
- μ' is an A–homology marking on $(\Sigma', \mathcal{P}')$ that is supported on a symplectic subsurface.
- $I', J' \subset \partial\Sigma'$ are $\mathcal{P}'$–adjacent finite disjoint unions of open intervals satisfying $I' \cap J' = \varnothing$.

Our inductive hypothesis thus says that $L \cong \mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu')$ is n'–connected for

$$n' = \min\left\{n-1, \frac{g'-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2}\right\} = \min\left\{n-1, \frac{g-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2} - \frac{k+1}{\mathrm{rk}(A)+2}\right\}$$
$$\geq \min\left\{n-1, n-\frac{k+1}{\mathrm{rk}(A)+2}\right\} \geq n-k-2,$$

as desired. □

Claim 3 *Equip the n–sphere S^n with a combinatorial triangulation and let $f\colon S^n \to Y$ be a locally injective simplicial map that extends to a simplicial map of a combinatorial triangulation of D^{n+1}. Then there exists a combinatorial triangulation of D^{n+1} that restricts to our given triangulation on $\partial D^{n+1} = S^n$ and a locally injective simplicial map $F\colon D^{n+1} \to Y$ such that $F|_{\partial D^{n+1}} = f$.*

Proof By Lemma 6.12, this will follow if we can show that, for all k–simplices σ of Y, the link L of σ is $(n-k-1)$–connected. As temporary notation, write $Y(\Sigma, I, J, \mathcal{P}, \mu)$ for Y. By Lemma 6.8, we have either

$$L \cong \mathcal{DTL}(\Sigma', I', J', \mathcal{P}', \mu') \quad \text{or} \quad L \cong Y(\Sigma', I', J', \mathcal{P}', \mu'),$$

depending on whether or not σ contains a vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$. Here Σ', I', J', $\mathcal{P}'$ and μ' are as follows:

- $(\Sigma', \mathcal{P}') \in \mathtt{PSurf}$ with Σ' a genus $g' = g-k-1$ surface.
- μ' is an A–homology marking on $(\Sigma', \mathcal{P}')$ supported on a symplectic subsurface.
- $I', J' \subset \partial\Sigma'$ are $\mathcal{P}'$–adjacent finite disjoint unions of open intervals satisfying $I' \cap J' = \varnothing$.

Applying either our inductive hypothesis or Claim 1, we see that L is n'–connected for

$$n' = \min\left\{n-1, \frac{g'-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2}\right\} = \min\left\{n-1, \frac{g-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2} - \frac{k+1}{\mathrm{rk}(A)+2}\right\}$$
$$\geq \min\left\{n-1, n-\frac{k+1}{\mathrm{rk}(A)+2}\right\} \geq n-k-1,$$

as desired. □

We now finally turn to proving that $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is n–connected. Our inductive hypothesis says that it is $(n-1)$–connected, so it is enough to prove that every continuous map $f\colon S^n \to \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ can be extended to a continuous map $F\colon D^{n+1} \to \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$. Using simplicial approximation, we can assume that f is simplicial with respect to a combinatorial triangulation of S^n. Next, using Claim 2 we can ensure that f is locally injective. The complex $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is a subcomplex of Y and Claim 1 says that Y is n–connected, so we can extend f to a continuous map $F\colon D^{n+1} \to Y$, which by the relative version of simplicial approximation we can ensure is simplicial with respect to a combinatorial triangulation of D^{n+1} that restricts to our given triangulation on S^n. Finally, applying Claim 3 we can ensure that F is locally injective.

If F does not map any vertices of D^{n+1} to $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, then the image of F lies in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ and we are done. Assume, therefore, that x is a vertex of D^{n+1} such that $F(x)$ is a vertex $\iota\colon \tau(S^1) \to \Sigma$ of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$. Let $L \subset D^{n+1}$ be the link of x and let $\mathcal{L} \subset Y$ be the link of $\iota = F(x)$. Since F is locally injective, we have $F(L) \subset \mathcal{L}$. Also, since simplices of Y can contain at most one vertex of $\mathcal{TL}(\Sigma, I, \mathcal{P}, \mu)$, we have $\mathcal{L} \subset \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$.

By Lemma 6.9, we can find a vertex $\hat{\iota}\colon \tau^2(S^1) \to \Sigma$ of $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ such that $\hat{\iota}|_{\tau(S^1)} = \iota$. Let $\widehat{\mathcal{L}}$ be the link of $\hat{\iota}$ in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$, so $\widehat{\mathcal{L}} \subset \mathcal{L}$. As we said above, we have $F(L) \subset \mathcal{L}$. If $F(L) \subset \widehat{\mathcal{L}}$, then we could redefine F to take x to $\hat{\iota}$ instead of ι. Repeating this process would modify F so that its image would lie in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$, and we would be done.

Unfortunately, it might not be the case that $F(L) \subset \widehat{\mathcal{L}}$. We will therefore have to perform a more complicated modification to F. Since $L \subset D^{n+1}$ is the link of the vertex x and x does not lie in ∂D^{n+1}, we have $L \cong S^n$. Recall that

$$F(L) \subset \mathcal{L} \subset \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu).$$

Among all simplicial maps

$$G\colon L \to \mathcal{L} \subset \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$$

that are homotopic to $F|_L$ through maps $S^n \to \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$, pick the one that minimizes the total number of intersections between the image of $\hat{\iota}\colon \tau^2(S^1) \to \Sigma$ and the images of $G(y)\colon \tau^2(S^1) \to \Sigma$ as y ranges over the vertices of L.

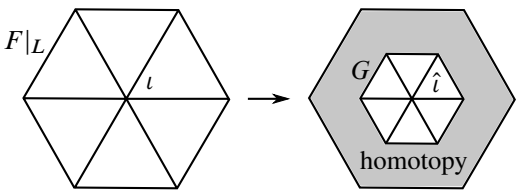
Below in Claim 4 we will prove that with this choice there are in fact no such intersections, and thus the image $G(L)$ lies in the link $\widehat{\mathcal{L}}$ of $\hat{\iota}$ in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$. Letting $*$ denote the join, we can then replace the restriction of F to the subset

$$x * L \cong \mathrm{pt} * S^n \cong D^{n+1}$$

of D^{n+1} with the following two pieces:

- The first is an annular region that is a combinatorial triangulation of $S^n \times [0, 1]$, both of whose boundary components are L. On this region, F maps to a homotopy from $F|_L$ to G.
- The second is the cone $x * L \cong \mathrm{pt} * S^n \cong D^{n+1}$, on which F is defined to equal G on L and to take x to $\hat{\iota}$.

See the following figure, where the shaded region is the homotopy from $F|_L$ to G:



This redefines F so that $F(x) = \hat{\iota}$ without introducing any other vertices mapping to vertices of $\mathcal{TL}(\Sigma, I, J, \mathcal{P}, \mu)$, completing the proof.

It remains to prove the aforementioned claim about $G\colon L \to \mathcal{L} \subset \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$.

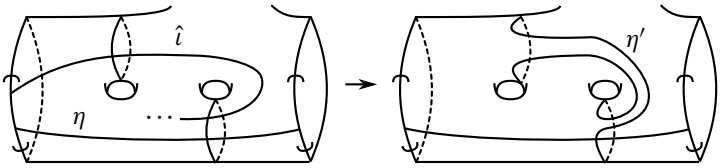
Claim 4 *For all vertices y of L, we can choose a representative of $G(y)\colon \tau^2(S^1) \to \Sigma$ whose image is disjoint from the image of $\hat{\iota}\colon \tau^2(S^1) \to \Sigma$.*

Proof Assume otherwise. Since the image of G lies in the link $\mathcal{L}$ of ι, we can choose representatives of the $G(y)$ for $y \in L$ that are disjoint from the image of $\iota\colon \tau(S^1) \to \Sigma$. Pick these representatives so that their intersections with the image of $\hat{\iota}|_{[1,2]}\colon [1, 2] \to \Sigma$ are transverse and all distinct. Let y be the vertex of L such that the

image of $\eta := G(y)\colon \tau^2(S^1) \to \Sigma$ intersects the image of $\hat{\iota}|_{[1,2]}\colon [1,2] \to \Sigma$ in the first of these intersection points (enumerated from $\hat{\iota}(1)$ to $\hat{\iota}(2)$).

The argument is slightly different depending on whether this intersection point is contained in the image under $\eta\colon \tau^2(S^1) \to \Sigma$ of $[0,1]$, S^1 or $[1,2]$. We will give the details for when this intersection point is contained in $\eta(S^1)$; the other cases are similar.

As in the following figure, let $\eta'\colon \tau^2(S^1) \to \Sigma$ be the result of "sliding" the intersection point of η in question across $\iota(S^1)$ via the initial segment of $\iota([1,2])$:

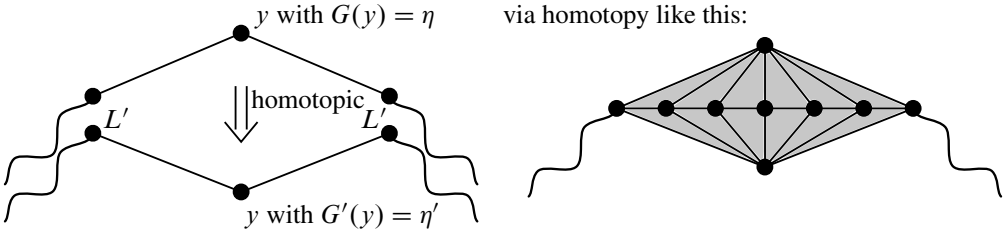


The image of η' intersects the image of $\hat{\iota}$ in one fewer place than the image of η. Define

$$G'\colon L \to \mathcal{L} \subset \mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$$

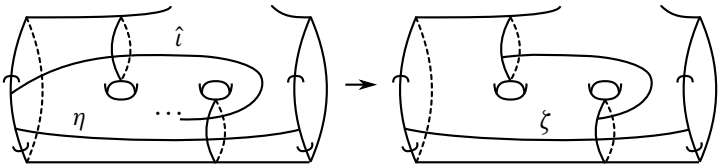
to be the map which equals G except at the vertex y, where $G'(y) = \eta'$ instead of η. It is easy to see that G' is indeed a simplicial map. Since the image of η' intersects the image of $\hat{\iota}$ in one fewer place than the image of η, to derive a contradiction to the minimality of the total number of these intersections it is enough to prove that G and G' are homotopic through maps landing in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$.

Define $L' \cong S^{n-1}$ to be the link of y in S^n, define $\mathcal{L}_\eta$ to be the link of η in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$, and define $\mathcal{L}_{\eta'}$ to be the link of η' in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$. We have $G|_{L'} = G'|_{L'}$, and the image $G(L') = G'(L')$ lies in $\mathcal{L}_\eta \cap \mathcal{L}_{\eta'}$. Below we will prove that the map $G|_{L'}\colon L' \to \mathcal{L}_\eta \cap \mathcal{L}_{\eta'}$ can be homotoped to a constant map. This will imply that G and G' are homotopic through maps lying in $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ via a homotopy like the one in this figure:



This figure depicts the case $n = 1$; pictured is a fragment of $L \cong S^1$, along with the vertex y and $L' \cong S^0$.

Since $L' \cong S^{n-1}$, to prove that the map $G|_{L'}\colon L' \to \mathcal{L}_\eta \cap \mathcal{L}_{\eta'}$ can be homotoped to a constant map, it is enough to prove that $\mathcal{L}_\eta \cap \mathcal{L}_{\eta'}$ is $(n-1)$–connected. Define ζ to be the union of $\eta(\tau^2(S^1))$, $\iota(S^1)$, and the portion of the arc of $\iota([1,2])$ connecting $\iota(0) \in \iota(S^1)$ to a point of $\eta(S^1)$; see here:



The images of both η and η' are contained in a regular neighborhood of ζ. Let Σ' be the surface obtained by cutting open Σ along ζ. The surface Σ' thus has genus $g' = g-2$. Moreover, an argument identical to that in the proof of Lemma 6.8 shows that there exist a partition $\mathcal{P}'$ of the components of $\partial\Sigma'$, an A–homology marking μ' on $(\Sigma', \mathcal{P}')$, and $\mathcal{P}'$–adjacent finite disjoint unions of open intervals $I', J' \subset \partial\Sigma'$ with $I' \cap J' = \varnothing$ such that

- $\mathcal{L}_\eta \cap \mathcal{L}_{\eta'} \cong \mathcal{MTL}(\Sigma', I', J', \mathcal{P}', \mu')$,
- μ' is supported on a symplectic subsurface.

Our inductive hypothesis thus says that $\mathcal{L}_\eta \cap \mathcal{L}_{\eta'} \cong \mathcal{MTL}(\Sigma', I', J', \mathcal{P}', \mu')$ is n'–connected for

$$n' = \min\left\{n-1, \frac{g'-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2}\right\} = \min\left\{n-1, \frac{g-(2\,\mathrm{rk}(A)+3)}{\mathrm{rk}(A)+2} - \frac{2}{\mathrm{rk}(A)+2}\right\}$$
$$\geq \min\left\{n-1, n-\frac{2}{\mathrm{rk}(A)+2}\right\} = n-1,$$

as desired. □

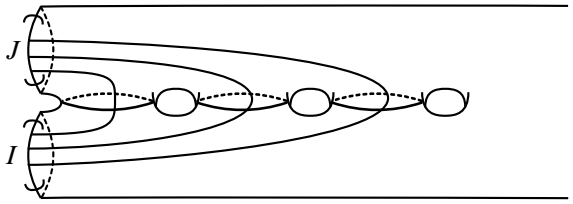
This completes the proof of Theorem 6.11. □

6.7 The complex of order-preserving double-tethered vanishing loops

We finally come to the complex of order-preserving double-tethered vanishing loops.

Complex of order-preserving double-tethered loops Let $\Sigma \in \mathtt{Surf}$ be a surface and let $I, J \subset \partial\Sigma$ be disjoint open intervals. Orient I so that Σ lies on its right and J so that Σ lies on its left. These two orientations induce two natural orderings on simplices of $\mathcal{DTL}(\Sigma, I, J)$. The *complex of order-preserving* (I, J)*–double-tethered*

loops, denoted by $\mathcal{ODTL}(\Sigma, I, J)$, is the subcomplex of $\mathcal{DTL}(\Sigma, I, J)$ consisting of simplices such that these two orderings agree. Here is an example of such a simplex:



The complex $\mathcal{ODTL}(\Sigma, I, J)$ was introduced by Hatcher and Vogtmann [12], who proved that if Σ has genus g then (like $\mathcal{TL}(\Sigma, I)$ and $\mathcal{DTL}(\Sigma, I, J)$) it is $\frac{1}{2}(g-3)$–connected; see [12, Proposition 5.3].

Complex of order-preserving double-tethered vanishing loops Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let $I, J \subset \partial\Sigma$ be disjoint $\mathcal{P}$–adjacent open intervals in $\partial\Sigma$. Define the complex $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ to be the intersection of $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ with $\mathcal{ODTL}(\Sigma, I, J)$. The orientations on I and J endow $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ with a natural ordering on its simplices, and thus with the structure of a semisimplicial set.

High connectivity The following theorem asserts that $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ has the same connectivity that Theorem 6.11 says $\mathcal{DTL}(\Sigma, I, J, \mathcal{P}, \mu)$ enjoys.

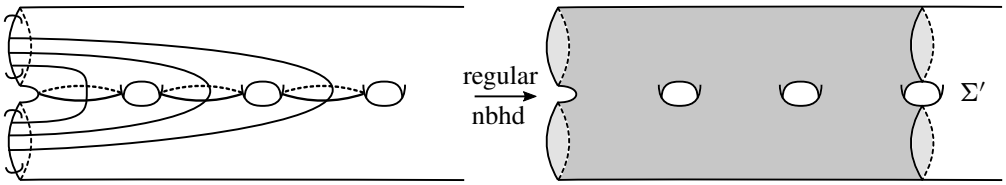
Theorem 6.13 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $I, J \subset \partial\Sigma$ be $\mathcal{P}$–adjacent disjoint open intervals and let g be the genus of Σ. Then $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is $\big(g-(2\,\mathrm{rk}(A)+3)\big)/(\mathrm{rk}(A)+2)$–connected.*

Proof In [12, Proposition 5.3], Hatcher and Vogtmann show how to derive the fact that $\mathcal{ODTL}(\Sigma, I, J)$ is $\frac{1}{2}(g-3)$–connected from the fact that $\mathcal{DTL}(\Sigma, I, J)$ is $\frac{1}{2}(g-3)$–connected. Their argument works word-for-word to prove this theorem. □

Stabilizers In the remainder of this section, we will be interested in the case where I and J are open intervals in distinct components ∂_I and ∂_J of $\partial\Sigma$ (much of what we say will also hold if $\partial_I = \partial_J$, but the pictures would be a bit different). The $\mathrm{Mod}(\Sigma)$–stabilizer of a simplex $\sigma = \{\iota_0, \ldots, \iota_k\}$ of $\mathcal{ODTL}(\Sigma, I, J)$ is the mapping class group of the complement Σ' of an open regular neighborhood of

$$\partial_I \cup \partial_J \cup \iota_0(\tau^2(S^1)) \cup \cdots \cup \iota_k(\tau^2(S^1)).$$

We will call this the *stabilizer subsurface* of σ. See here:



If ∂_I and ∂_J are $\mathcal{P}$–adjacent, then the surface Σ' is a $\mathcal{P}$–simple subsurface of Σ, and thus has an induced partition $\mathcal{P}'$. The following lemma records some of its properties if σ is a simplex of $\mathcal{ODTL}(\Sigma, I, \mathcal{P}, \mu)$ for an A–homology marking μ on $(\Sigma, \mathcal{P})$:

Lemma 6.14 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P})$ and let I and J be open intervals in distinct $\mathcal{P}$–adjacent components of $\partial\Sigma$. Let σ be a simplex of $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$, let Σ' be its stabilizer subsurface, and let $\mathcal{P}'$ be the induced partition of $\partial\Sigma'$. Then there exists an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ such that μ is the stabilization of μ'. Moreover, if μ is supported on a symplectic subsurface then so is μ'.*

Proof The proof is identical to that of Lemma 6.8. □

Transitivity The final fact we need about these complexes is as follows:

Lemma 6.15 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface and let I and J be open intervals in distinct $\mathcal{P}$–adjacent components of $\partial\Sigma$. The group $\mathcal{I}(\Sigma, \mathcal{P}, \mu)$ acts transitively on the k–simplices of $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ if the genus of Σ is at least $2\,\mathrm{rk}(A) + 3 + k$.*

Proof Just like in the proof of Lemma 3.9, this will be by induction on k. In fact, once we prove the base case $k = 0$, the inductive step is handled exactly like Lemma 3.9, so we will only give the details for $k = 0$.

Assume that the genus of Σ is at least $2\,\mathrm{rk}(A) + 3$. Theorem 6.13 then implies that $\mathcal{ODTL}(\Sigma, I, J, \mathcal{P}, \mu)$ is connected, so to prove that $\mathcal{I}(\Sigma, \mathcal{P}, \mu)$ acts transitively on its vertices it is enough to prove that if $\iota_0, \iota_1 \colon \tau^2(S^1) \to \Sigma$ are vertices that are joined by an edge, then there exists some $f \in \mathcal{I}(\Sigma, \mathcal{P}, \mu)$ such that $f(\iota_0) = \iota_1$. Let Σ' be the stabilizer subsurface of $\{\iota_0, \iota_1\}$ and let $\mathcal{P}'$ be the induced partition of $\partial\Sigma'$. By Lemma 6.14, there exists an A–homology marking μ' on $(\Sigma', \mathcal{P}')$ that is supported on a symplectic subsurface such that μ is the stabilization of μ' to $(\Sigma, \mathcal{P})$. Let $S \cong \Sigma_h^1$ be a subsurface of Σ' on which μ' is supported.

The change of coordinates principle from [8, Section 1.3.2] implies that there is a mapping class f' on $\Sigma' \setminus \mathrm{Int}(S)$ with $f'(\iota_0) = \iota_1$. Let $f \in \mathrm{Mod}(\Sigma)$ be the result of extending f' over S by the identity. Since μ is supported on S, we have $f \in \mathcal{I}(\Sigma, \mathcal{P}, \mu)$ and $f(\iota_0) = \iota_1$, as desired. □

6.8 The double boundary stabilization proof

We now prove Proposition 5.9.

Proof of Proposition 5.9 We start by recalling the statement and introducing some notation. Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a double boundary stabilization and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Setting

$$c = \mathrm{rk}(A) + 2 \quad \text{and} \quad d = 2\,\mathrm{rk}(A) + 2,$$

we want to prove that the induced map $\mathrm{H}_k(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_k(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is an isomorphism if the genus of Σ is at least $ck + d$ and a surjection if the genus of Σ is $ck + d - 1$. We will prove this using Theorem 3.1. This requires fitting $\mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathcal{I}(\Sigma', \mathcal{P}', \mu')$ into an increasing sequence of group $\{G_n\}$ and constructing appropriate simplicial complexes.

As notation, let $(S_g, \mathcal{P}_g) = (\Sigma, \mathcal{P})$, $\mu_g = \mu$, $(S_{g+1}, \mathcal{P}_{g+1}) = (\Sigma', \mathcal{P}')$ and $\mu_{g+1} = \mu'$. In a double boundary stabilization like $(S_g, \mathcal{P}_g) \to (S_{g+1}, \mathcal{P}_{g+1})$, two boundary components of Σ_0^4 are glued to two boundary components of S_g to form S_{g+1}. We will call the two boundary components of S_g to which Σ_0^4 is glued the *attaching components* and the two components of $\partial \Sigma_0^4 \cap \partial S_{g+1}$ the *new components*.

By assumption, μ_g is supported on a genus-h symplectic subsurface for some h, ie there exists a $\mathtt{PSurf}$–morphism $(T, \mathcal{P}_T) \to (S_g, \mathcal{P}_g)$ with $T \cong \Sigma_h^1$ and an A–homology marking μ_T on $(T, \mathcal{P}_T)$ such that μ_g is the stabilization of μ_T to $(S_g, \mathcal{P}_g)$. Applying Corollary 3.6 to μ_T, we can assume without loss of generality that $h \le \mathrm{rk}(A)$. We can then factor $(T, \mathcal{P}_T) \to (S_g, \mathcal{P}_g)$ into an increasing sequence of subsurfaces

$$(T, \mathcal{P}_T) \to (S_h, \mathcal{P}_h) \to (S_{h+1}, \mathcal{P}_{h+1}) \to \cdots \to (S_g, \mathcal{P}_g)$$

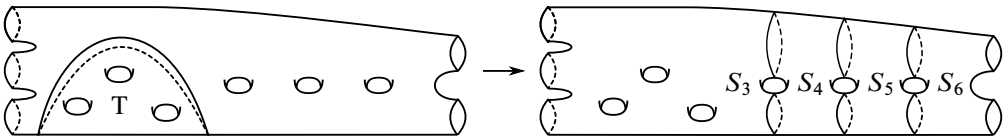
such that

(i) each S_r has genus r,

(ii) each $(S_r, \mathcal{P}_r) \to (S_{r+1}, \mathcal{P}_{r+1})$ is a double boundary stabilization, and

(iii) for $r > h$, the attaching components of $(S_r, \mathcal{P}_r) \to (S_{r+1}, \mathcal{P}_{r+1})$ equal the new components of $(S_{r-1}, \mathcal{P}_{r-1}) \to (S_r, \mathcal{P}_r)$.

See here:



This can then be continued indefinitely to form an increasing sequence of subsurfaces

$$(T, \mathcal{P}_T) \to (S_h, \mathcal{P}_h) \to \cdots \to (S_g, \mathcal{P}_g) \to (S_{g+1}, \mathcal{P}_{g+1}) \to (S_{g+2}, \mathcal{P}_{g+2}) \to \cdots$$

satisfying (i)–(iii). Here $(S_{g+1}, \mathcal{P}_{g+1})$ is as defined above. For $r \geq h$, let μ_r be the stabilization of μ_T to (S_r, μ_r). This agrees with our previous definitions of μ_g and μ_{g+1}.

We thus have an increasing sequence of groups

$$\mathcal{I}(S_h, \mathcal{P}_h, \mu_h) \subset \mathcal{I}(S_{h+1}, \mathcal{P}_{h+1}, \mu_{h+1}) \subset \mathcal{I}(S^1_{h+2}, \mathcal{P}_{h+2}, \mu_{h+2}) \subset \cdots.$$

For $r \geq h$, let $I_r, J_r \subset \partial S_r$ be open intervals in the two attaching components for $(S_r, \mathcal{P}_r) \to (S_{r+1}, \mathcal{P}_{r+1})$. According to Theorem 6.13, $\mathcal{ODTL}(S_r, I_r, J_r, \mathcal{P}_r, \mu_r)$ is $(r-(d+1))/c$–connected (where c and d are as defined in the first paragraph).

For $n \geq 0$, let

$$G_n = \mathcal{I}(S_{d+n}, \mathcal{P}_{d+n}, \mu_{d+n}) \quad \text{and} \quad X_n = \mathcal{ODTL}(S_{d+n}, I_{d+n}, J_{d+n}, \mathcal{P}_{d+n}, \mu_{d+n}).$$

For this to make sense, we must have $d + n \geq h$, which follows from

$$d + n = 2\,\mathrm{rk}(A) + 2 + n \geq \mathrm{rk}(A) \geq h.$$

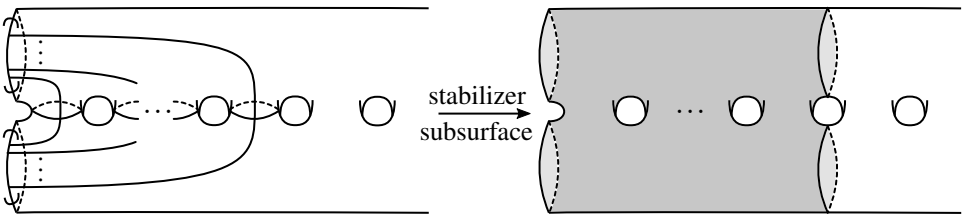
We thus have an increasing sequence of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

with G_n acting on X_n. The indexing convention here is chosen so that X_1 is 0–connected, and more generally so that X_n is $(n-1)/c$–connected, as in Theorem 3.1. Our goal is to prove that the map $\mathrm{H}_k(G_{n-1}) \to \mathrm{H}_k(G_n)$ is an isomorphism for $n \geq ck + 1$ and a surjection for $n = ck$, which will follow from Theorem 3.1 once we check its conditions:

(1) The first is that X_n is $(n-1)/c$–connected, which follows from Theorem 6.13.

(2) The second is that, for $0 \leq i < n$, the group G_{n-i-1} is the G_n–stabilizer of some i–simplex of X_n, which follows from Lemma 6.14 via the following picture:



(3) The third is that, for all $0 \leq i < n$, the group G_n acts transitively on the i–simplices of X_n, which follows from Lemma 6.15.

(4) The fourth is that, for all $n \geq c+1$ and all 1–simplices e of X_n whose boundary consists of vertices v and v', there exists some $\lambda \in G_n$ such that $\lambda(v) = v'$ and such that λ commutes with all elements of $(G_n)_e$. Let S' be the stabilizer subsurface of e, so by Lemma 6.14 the stabilizer $(G_n)_e$ consists of mapping classes supported on S'. The surface $S_{d+n} \setminus \mathrm{Int}(S')$ is diffeomorphic to Σ_1^4 (as in the picture above), and in particular is connected. The change of coordinates principle from [8, Section 1.3.2] implies that we can find a mapping class λ supported on $S_{d+n} \setminus \mathrm{Int}(S')$ taking the double-tethered loop v to v'. Lemma 6.14 implies that μ_{d+n} can be destabilized to an A–homology marking on S' (with respect to an appropriate partition) that is supported on a symplectic subsurface. This implies that λ lies in $G_n = \mathcal{I}(S_{d+n}, \mathcal{P}_{d+n}, \mu_{d+n})$ and commutes with $(G_n)_e$. □

7 Nonstability

This section concerns situations where homological stability does *not* occur. The highlights are the proofs of Theorems B and 5.3.

Disc-pushing subgroup Let $\Sigma \in \mathtt{Surf}$ be a surface and let ∂ be a component of $\partial\Sigma$. Let $\widehat{\Sigma}$ be the result of gluing a disc to ∂. The embedding $\Sigma \hookrightarrow \widehat{\Sigma}$ induces a homomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\widehat{\Sigma})$, which is easily seen to be surjective. Its kernel, denoted by $\mathrm{DP}(\partial)$, is the *disc-pushing subgroup* and is isomorphic to the fundamental group of the unit tangent bundle $U\widehat{\Sigma}$ of $\widehat{\Sigma}$; see [8, Section 4.2.5]. Elements of $\mathrm{DP}(\partial)$ "push" ∂ around paths in $\widehat{\Sigma}$ while allowing it to rotate.

Disc-pushing and partial Torelli If ∂ is the single component of $\partial\Sigma_g^1$, then $\mathrm{DP}(\partial) \subset \mathrm{Mod}(\Sigma_g^1)$ is contained in the Torelli group $\mathcal{I}(\Sigma_g^1)$, and thus is also contained in $\mathcal{I}(\Sigma_g^1, \mu)$ for any A–homology marking μ on Σ_g^1. he following lemma generalizes this to the partial Torelli groups on surfaces with multiple boundary components:

Lemma 7.1 *Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let ∂ be a component of $\partial\Sigma$ such that $\{\partial\} \in \mathcal{P}$. Then* $\mathrm{DP}(\partial) \subset \mathcal{I}(\Sigma, \mathcal{P}, \mu)$.

Proof Let $f \in \mathrm{DP}(\partial)$ and let $x \in \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$. It is enough to prove that $f(x) = x$. Let $\widehat{\Sigma}$ be the result of gluing a disc to ∂ and let $\widehat{\mathcal{P}} = \mathcal{P} \setminus \{\{\partial\}\}$. We thus have a $\mathtt{PSurf}$–morphism $\iota\colon (\Sigma, \mathcal{P}) \to (\widehat{\Sigma}, \widehat{\mathcal{P}})$. Since the homology classes of arcs connecting ∂ to other components of $\partial\Sigma$ do not contribute to $\mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$, the map $\iota^*\colon \mathrm{H}_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}, \partial\widehat{\Sigma}) \to \mathrm{H}_1^{\mathcal{P}}(\Sigma, \partial\Sigma)$ is a surjection (in fact, it is an isomorphism, but we will not need this). We can thus write $x = \iota^*(\hat{x})$ for some $\hat{x} \in \mathrm{H}_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}, \partial\widehat{\Sigma})$. Since

$$f \in \mathrm{DP}(\partial) = \ker(\mathrm{Mod}(\Sigma) \xrightarrow{\iota_*} \mathrm{Mod}(\widehat{\Sigma})),$$

we clearly have $\iota_*(f)(\hat{x}) = \hat{x}$, so Lemma 4.5 implies that

$$x = \iota^*(\hat{x}) = \iota^*(\iota_*(f)(\hat{x})) = f(\iota^*(\hat{x})) = f(x),$$

as desired. □

Johnson homomorphism Fix some $g \geq 2$ and let $H = \mathrm{H}_1(\Sigma_g^1)$. The Johnson homomorphism [14] is an important homomorphism $\tau\colon \mathcal{I}(\Sigma_g^1) \to \bigwedge^3 H$. Letting ∂ be the single component of $\partial\Sigma_g^1$, it interacts with the disc-pushing subgroup $\mathrm{DP}(\partial) \cong \pi_1(U\Sigma_g)$ in the following way. Let $\omega \in \bigwedge^2 H$ be the *symplectic element*, ie the element corresponding to the algebraic intersection pairing under the isomorphism

$$\left(\textstyle\bigwedge^2 H\right)^* \cong \bigwedge^2 H^* \cong \bigwedge^2 H,$$

where we identify H with its dual H^* via Poincaré duality. We then have an injection $H \hookrightarrow \bigwedge^3 H$ taking $h \in H$ to $h \wedge \omega$. The restriction of τ to $\mathrm{DP}(\partial)$ is the composition

$$\mathrm{DP}(\partial) \cong \pi_1(U\Sigma_g) \to \pi_1(\Sigma_g) \to H \xrightarrow{-\wedge\omega} \textstyle\bigwedge^3 H.$$

Symplectic nondegeneracy Let μ be an A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$. The *$\mu$–symplectic element* $\omega_\mu \in \bigwedge^2 A$ is as follows. Let H be the quotient of $\mathrm{H}_1(\Sigma)$ by the subgroup generated by the loops around the boundary components. Since H is the first homology group of the closed surface obtained by gluing discs to all components of $\partial\Sigma$, there is a symplectic element $\omega \in \bigwedge^2 H$. The closed marking $\hat{\mu}\colon \mathrm{H}_1(\Sigma) \to A$ factors through a homomorphism $H \to A$, and ω_μ is the image of $\omega \in \bigwedge^2 H$ under the induced map $\bigwedge^2 H \to \bigwedge^2 A$. We then have a map $A \to \bigwedge^3 A$ taking $a \in A$ to $a \wedge \omega_\mu$. We will say that μ is *symplectically nondegenerate* if this map is nonzero.

Example 7.2 Let V be a symplectic subspace of $\mathrm{H}_1(\Sigma_g^1)$, so $\mathrm{H}_1(\Sigma_g^1) = V \oplus V^\perp$, and let $\mu\colon \mathrm{H}_1(\Sigma_g^1) \to V$ be the orthogonal projection. We claim that μ is symplectically nondegenerate if and only if V has genus at least 2. Indeed, $\omega_\mu \in \bigwedge^2 V$ equals the

symplectic element arising from the symplectic form on V, and the map $V \to \bigwedge^3 V$ taking $v \in V$ to $v \wedge \omega_\mu$ is nonzero precisely when V has genus at least 2. We remark that if V has genus 0 or 1 then $\bigwedge^3 V = 0$, so the map $V \to \bigwedge^3 V$ is automatically the zero map.

Partial Johnson homomorphism The homomorphism given by the following lemma is a version of the Johnson homomorphism for the partial Torelli groups:

Lemma 7.3 *Let μ be an symplectically nondegenerate A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ and let ∂ be a component of $\partial\Sigma$ such that $\{\partial\} \in \mathcal{P}$ (and thus by Lemma 7.1 such that $\mathrm{DP}(\partial) \subset \mathcal{I}(\Sigma, \mathcal{P}, \mu)$). Then there exists a homomorphism $\tau\colon \mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathrm{H}_3(A)$ whose restriction to $\mathrm{DP}(\partial)$ is nontrivial.*

Remark 7.4 The target group $\mathrm{H}_3(A)$ contains $\bigwedge^3 A$, though sometimes it is a bit larger.

Proof of Lemma 7.3 Let Σ' be the result of gluing discs to all components of $\partial\Sigma$ except for ∂, let $\mathcal{P}' = \{\{\partial\}\}$, and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. From their definitions, the μ'–symplectic element $\omega_{\mu'} \in \bigwedge^2 A$ is the same as the μ–symplectic element $\omega_\mu \in \bigwedge^2 A$, so μ' is symplectically nondegenerate. In [3, Theorem 5.8], Broaddus, Farb and Putman construct a homomorphism

$$\tau'\colon \mathcal{I}(\Sigma', \mathcal{P}', \mu') \to \mathrm{H}_3(A).$$

We remark that their notation is a little different from ours — the group W in the statement of [3, Theorem 5.8] should be taken to be $W = \ker(\mu')$. Let $\mathrm{DP}'(\partial)$ be the disc-pushing subgroup of $\mathcal{I}(\Sigma', \mathcal{P}', \mu')$, let $\widehat{\Sigma}'$ be the result of gluing a disc to the component ∂ of $\partial\Sigma'$, and let $\hat{\mu}'\colon \mathrm{H}_1(\Sigma') \to A$ be the closed marking associated to μ'. One of the characteristic properties of τ' is that its restriction to $\mathrm{DP}'(\partial)$ is

$$\mathrm{DP}'(\partial) = \pi_1(U\widehat{\Sigma}') \to \pi_1(\widehat{\Sigma}') \to \mathrm{H}_1(\widehat{\Sigma}') = \mathrm{H}_1(\Sigma') \xrightarrow{\hat{\mu}'} A \xrightarrow{-\wedge\omega_{\mu'}} \textstyle\bigwedge^3 A \hookrightarrow \mathrm{H}_3(A).$$

In particular, since μ' is symplectically nondegenerate, the restriction of τ' to $\mathrm{DP}'(\partial)$ is nontrivial. Let $\tau\colon \mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathrm{H}_3(A)$ be the composition of τ' with the map $\mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathcal{I}(\Sigma', \mathcal{P}', \mu')$. The restriction of this latter map to $\mathrm{DP}(\partial)$ is a surjection $\mathrm{DP}(\partial) \to \mathrm{DP}'(\partial)$, so the restriction of τ to $\mathrm{DP}(\partial)$ is nontrivial, as desired. □

Closing up surfaces and nonstability In light of Example 7.2 above, the following theorem generalizes Theorem B:

Theorem 7.5 *Let μ be a symplectically nondegenerate A–homology marking on $(\Sigma, \mathcal{P})$, let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a $\mathtt{PSurf}$–morphism, and let μ' be the stabilization*

of μ to $(\Sigma', \mathcal{P}')$. Assume there is a component ∂ of $\partial\Sigma$ with $\{\partial\} \in \mathcal{P}$ whose image in Σ' bounds a disc. Then the map $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is not injective.

Proof Lemma 7.1 implies that $\mathrm{DP}(\partial) \subset \mathcal{I}(\Sigma, \mathcal{P}, \mu)$, and Lemma 7.3 implies that there exists a homomorphism from $\mathcal{I}(\Sigma, \mathcal{P}, \mu)$ to an abelian group whose restriction to $\mathrm{DP}(\partial)$ is nontrivial. Since

$$\mathrm{DP}(\partial) \subset \ker(\mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathcal{I}(\Sigma', \mathcal{P}', \mu')),$$

this implies that the induced map on abelianizations is not injective, as desired. □

General nonstability We now prove Theorem 5.3.

Proof of Theorem 5.3 We start by recalling what we must prove. Let μ be a symplectically nondegenerate A–homology marking on $(\Sigma, \mathcal{P}) \in \mathtt{PSurf}$ that is supported on a symplectic subsurface. Let $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ be a non-partition-bijective $\mathtt{PSurf}$–morphism and let μ' be the stabilization of μ to $(\Sigma', \mathcal{P}')$. Assume that the genus of Σ is at least $3\,\mathrm{rk}(A) + 4$. We must prove that the induced map $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is not an isomorphism. We will ultimately prove this by reducing it to Theorem 7.5 above.

Identify Σ with its image in Σ'. We start with the following reduction. Recall that, for a surface S, the discrete partition of the components of ∂S is

$$\{\{\partial\} \mid \partial \text{ is a component of } \partial S\}.$$

Claim *We can assume without loss of generality that $\mathcal{P}$ and $\mathcal{P}'$ are the discrete partitions of the components of $\partial\Sigma$ and $\partial\Sigma'$ and that the genera of Σ and Σ' are equal.*

Proof We do this in three steps:

- First, let $(\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$ be an open capping (see Section 5.2; this implies in particular that $\mathcal{P}''$ is the discrete partition of $\partial\Sigma''$) and let μ'' be the stabilization of μ' to $(\Sigma'', \mathcal{P}'')$. Since open cappings are partition-bijective, Theorem F implies that the map $\mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu')) \to \mathrm{H}_1(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu''))$ is an isomorphism. The composition

$$(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}') \to (\Sigma'', \mathcal{P}'')$$

is still not partition-bijective, so replacing $(\Sigma', \mathcal{P}')$ and μ' with $(\Sigma'', \mathcal{P}'')$ and μ'', we can assume without loss of generality that $\mathcal{P}'$ is the discrete partition of $\partial\Sigma'$.

- Next, just like in Case 2 of the proof of Theorem F in Section 5.2, we can use the fact that μ is supported on a symplectic subsurface to find a partition-bijective $\mathtt{PSurf}$–morphism $(\Sigma''', \mathcal{P}''') \to (\Sigma, \mathcal{P})$ and an A–homology marking μ''' on $(\Sigma''', \mathcal{P}''')$

such that μ is the stabilization of μ''' to $(\Sigma, \mathcal{P})$, such that $\mathcal{P}'''$ is the discrete partition of $\partial\Sigma'''$, and such that the genera of Σ''' and Σ are the same. Theorem F implies that the map $\mathrm{H}_1(\mathcal{I}(\Sigma''', \mathcal{P}''', \mu''')) \to \mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu))$ is an isomorphism. The composition

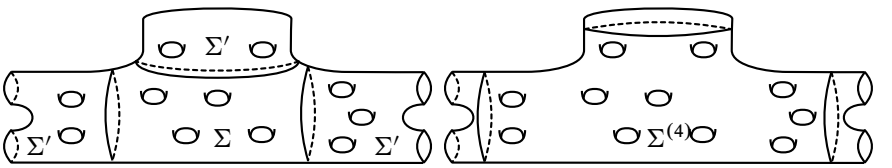
$$(\Sigma''', \mathcal{P}''') \to (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$$

is still not partition-bijective, so replacing $(\Sigma, \mathcal{P})$ and μ with $(\Sigma''', \mathcal{P}''')$ and μ''', we can assume without loss of generality that $\mathcal{P}$ is the discrete partition of $\partial\Sigma$.

- We have now ensured that $\mathcal{P}$ and $\mathcal{P}'$ are the discrete partitions, and it remains to show that we can ensure that the genera of Σ and Σ' are the same. As in the following picture, we can factor $(\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ into

$$(\Sigma, \mathcal{P}) \to (\Sigma^{(4)}, \mathcal{P}^{(4)}) \to (\Sigma', \mathcal{P}'),$$

where $(\Sigma, \mathcal{P}) \to (\Sigma^{(4)}, \mathcal{P}^{(4)})$ is partition-bijective, where $\mathcal{P}^{(4)}$ is the discrete partition of $\partial\Sigma^{(4)}$, and where the genera of $\Sigma^{(4)}$ and Σ' are the same:



Theorem F implies that the map $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P})) \to \mathrm{H}_1\left(\mathcal{I}(\Sigma^{(4)}, \mathcal{P}^{(4)})\right)$ is an isomorphism. Since the map $(\Sigma^{(4)}, \mathcal{P}^{(4)}) \to (\Sigma', \mathcal{P}')$ is still not partition-bijective, we can replace $(\Sigma, \mathcal{P})$ with $(\Sigma^{(4)}, \mathcal{P}^{(4)})$ and ensure that the genera of Σ and Σ' are the same. □

Since the genera of Σ and Σ' are the same, all components of $\overline{\Sigma' \setminus \Sigma}$ are genus-0 surfaces intersecting Σ in a single boundary component. If any of these components are discs, then Theorem 7.5 implies that the map $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is not injective, and we are done. We can thus assume that no components of $\overline{\Sigma' \setminus \Sigma}$ are discs. Furthermore, if any of these components are annuli, then we can deformation retract Σ' over them without changing anything; doing this, we can assume that none of them are annuli.

It follows that all the components of $\overline{\Sigma' \setminus \Sigma}$ are genus-0 surfaces with at least three boundary components intersecting Σ in a single boundary component. Let $\{\partial_1, \dots, \partial_k\}$ be a set of components of $\partial\Sigma'$ containing precisely one component in each component of $\overline{\Sigma' \setminus \Sigma}$. Let Σ'' be the result of gluing discs to all components of Σ' except for the ∂_i, let $\mathcal{P}''$ be the discrete partition of $\partial\Sigma''$ (so in particular $\{\partial_i\} \in \mathcal{P}''$ for all i), and let μ'' be the stabilization of μ' to $(\Sigma'', \mathcal{P}'')$. All components of $\overline{\Sigma'' \setminus \Sigma}$ are annuli, so Σ'' deformation retracts to Σ'.

From this, we see that the composition

$$\mathcal{I}(\Sigma, \mathcal{P}, \mu) \to \mathcal{I}(\Sigma', \mathcal{P}', \mu') \to \mathcal{I}(\Sigma'', \mathcal{P}'', \mu'')$$

is an isomorphism, and thus the composition

$$\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu')) \to \mathrm{H}_1(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu'')) \tag{7-1}$$

is also an isomorphism. Since $\mathcal{P}'$ is the discrete partition and at least one disc was glued to a component of $\partial\Sigma'$ when we formed Σ'', Theorem 7.5 implies that the map $\mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu')) \to \mathrm{H}_1(\mathcal{I}(\Sigma'', \mathcal{P}'', \mu''))$ is not injective. Since the composition (7-1) is an isomorphism, we conclude that the map $\mathrm{H}_1(\mathcal{I}(\Sigma, \mathcal{P}, \mu)) \to \mathrm{H}_1(\mathcal{I}(\Sigma', \mathcal{P}', \mu'))$ is not surjective, and we are done. □

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