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We investigate the minimal symplectic fillings of small Seifert 3–manifolds with a canonical contact structure. As a result, we list all minimal symplectic fillings using curve configurations for small Seifert 3–manifolds satisfying certain conditions. Furthermore, we also demonstrate that every such a minimal symplectic filling is obtained by a sequence of rational blowdowns from the minimal resolution of the corresponding weighted homogeneous complex surface singularity.

1 Introduction

One of the fundamental problems in symplectic 4–manifold topology is to classify symplectic fillings of certain 3–manifolds equipped with a natural contact structure. Among them, researchers have long studied symplectic fillings of the link of a normal complex surface singularity. Note that the link of a normal surface singularity carries a canonical contact structure also known as the Milnor fillable contact structure. For example, P Lisca [8], M Bhupal and K Ono [1], and H Park, J Park, D Shin and G Urzúa [12] completely classified all minimal symplectic fillings of lens spaces and certain small Seifert 3–manifolds coming from the link of quotient surface singularities. L Starkston [16] also investigated minimal symplectic fillings of the link of some weighted homogeneous surface singularities.

On the one hand, topologists working on 4–manifold topology are also interested in finding a surgical interpretation for symplectic fillings of a given 3–manifold. More specifically, one may ask whether there is any surgical description of those fillings. In fact, it has been known that *rational blowdown* surgery, introduced by R Fintushel and R Stern [5] and generalized by the second author [14] and A Stipsicz, Z Szabó and J Wahl [18], is a powerful tool to answer this question. For example, for the link of
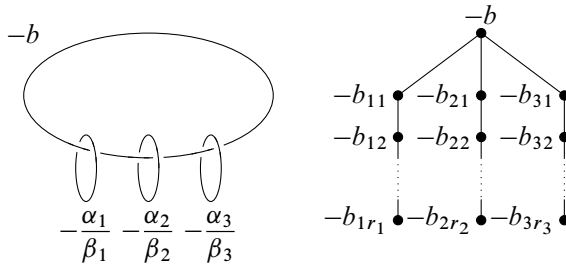
Figure 1: Surgery diagram of Y and its associated plumbing graph Γ.

quotient surface singularities equipped with a canonical contact structure, it was proven that every minimal symplectic filling is obtained by a sequence of rational blowdowns from the minimal resolution of the singularity; see Bhupal and Ozbagci [2], Choi and Park [4]. On the other hand, L Starkston [17] showed that there are symplectic fillings of some Seifert 3–manifolds that cannot be obtained by a sequence of rational blowdowns from the minimal resolution of the singularity. Note that Seifert 3–manifolds can be viewed as the link of weighted homogeneous surface singularities. Hence, it is an intriguing question as to which Seifert 3–manifolds have a rational blowdown interpretation for their minimal symplectic fillings.

In this paper, we investigate the minimal symplectic fillings of small Seifert 3–manifolds over the 2–sphere satisfying certain conditions. By a *small* Seifert (fibered) 3–manifold, we assume that it admits at most 3 singular fibers when it is considered as an S^1–fibration over the 2–sphere. In general, a Seifert 3–manifold as an S^1–fibration over a Riemann surface can have any number of singular fibers. We denote a small Seifert 3–manifold Y by $Y(-b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ whose surgery diagram is given in Figure 1 and which is also given as a boundary of a plumbing 4–manifold of disk bundles of 2–spheres according to the graph Γ in Figure 1. The integers $b_{ij} \geq 2$ in Figure 1 are uniquely determined by the continued fraction

$$\frac{\alpha_i}{\beta_i} = [b_{i1}, b_{i2}, \ldots, b_{ir_i}] = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\ldots - \cfrac{1}{b_{ir_i}}}}.$$

If the intersection matrix of a plumbing graph Γ is negative definite, which is always true for $b \geq 3$, then there is a canonical contact structure on Y induced from a symplectic structure of the plumbing 4–manifold, where each vertex corresponds to a symplectic 2–sphere and each edge represents an orthogonal intersection between the symplectic 2–spheres; see Gay and Stipsicz [7]. Furthermore, the canonical contact structure

on Y is contactomorphic to the contact structure defined by the complex tangency of a complex structure on the link of the corresponding singularity, which is called the *Milnor fillable* contact structure; see Park and Stipsicz [13].

This paper aims to classify all possible list of minimal symplectic fillings of small Seifert 3–manifolds satisfying certain conditions, and to prove that every such a minimal symplectic filling is obtained by a sequence of rational blowdowns from the minimal resolution of the corresponding weighted homogeneous surface singularity, as it is true for a quotient surface singularity. Our strategy is as follows. For a given minimal symplectic filling W of $Y(-b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ with $b \geq 4$, we glue W with a concave cap K to get a closed symplectic 4–manifold X. Then, since the concave cap K always contains an embedded $(+1)$ 2–sphere corresponding the central vertex, X is a rational symplectic 4–manifold by McDuff [9]. Furthermore, the adjunction formula and intersection data impose a constraint on the homological data of K in $X \cong \mathbb{CP}^2 \sharp N \overline{\mathbb{CP}^2}$. Under blowdowns along all exceptional 2–spheres away from the $(+1)$ 2–sphere in $X \cong \mathbb{CP}^2 \sharp N \overline{\mathbb{CP}^2}$, the concave cap K becomes a neighborhood of symplectic 2–spheres which are isotopic to b number of complex lines through symplectic 2–spheres in $\mathbb{CP}^2$; see Starkston [16; 17] for details. Since the symplectic deformation type of $W \cong X \setminus K$ is determined by the isotopy class of a symplectic embedding of K within a fixed homological embedding, we investigate a symplectic embedding of K using a *curve configuration* corresponding to W, which consists of strands representing irreducible components of K and exceptional 2–spheres intersecting them (Definition 3.1 and Figure 5). Since the curve configuration corresponding to W determines a symplectic embedding of K, we can recover all minimal symplectic fillings by investigating all possible curve configurations of Y. Sometimes, we find a certain chain of symplectic 2–spheres lying in W, which can be rationally blowing down, from the homological data of K. Note that by rationally blowing down the chain of symplectic 2–spheres lying in W, we obtain another minimal symplectic W' from W. In this case, we keep track of changes in the homological data of K so that we get a curve configuration of W' from that of W. Finally, by analyzing the effect of rational blowdown surgery on the curve configuration of minimal symplectic fillings, we obtain our main result.

Theorem 1.1 *For a small Seifert 3–manifold $Y(-b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ with its canonical contact structure and $b \geq 4$, all minimal symplectic fillings of Y are listed explicitly by curve configurations. Furthermore, they are also obtained by a sequence of rational blowdowns from the minimal resolution of the corresponding weighted homogeneous surface singularity.*

Remark L Starkston [16] originally described a general scheme for how to obtain minimal symplectic fillings from given homological data of an embedding of the cap and got some results in special cases. L Starkston [17] also showed that the isotopy type of a symplectic line arrangement is uniquely (up to deformation equivalence) determined by its intersection data in the cases that multi-intersection points of a symplectic line arrangement satisfy some mild conditions, which contain the cases appearing in Proposition 3.4. Thus, by combining Propositions 3.3 and 3.4 with Starkston's result [17, Proposition 4.2], we conclude that there exists at most one minimal symplectic filling for each possible curve configuration. Then we prove in Section 4 that every such a curve configuration gives the corresponding minimal symplectic filling, which implies the first statement in Theorem 1.1 above.

2 Preliminaries

2.1 Weighted homogeneous surface singularities and Seifert 3–manifolds

We briefly recall some basics of weighted homogeneous surface singularities and Seifert 3–manifolds; see [10] for details. Suppose that $(w_0, \ldots, w_n)$ are nonzero rational numbers. A polynomial $f(z_0, \ldots, z_n)$ is called *weighted homogeneous* of type $(w_0, \ldots, w_n)$ if it can be expressed as a linear combination of monomials $z_0^{i_0} \cdots z_n^{i_n}$ for which

$$\frac{i_0}{w_0} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1.$$

Equivalently, there exist nonzero integers $(q_0, \ldots, q_n)$ and a positive integer d satisfying $f(t^{q_0} z_0, \ldots t^{q_n} z_n) = t^d f(z_0, \ldots, z_n)$. Then, a weighted homogeneous surface singularity $(X, 0)$ is a normal surface singularity that is defined as the zero loci of weighted homogeneous polynomials of the same type. Note that there is a natural $\mathbb{C}^*$–action given by

$$t \cdot (z_0, \ldots, z_n) = (t^{q_0} z_0, \ldots, t^{q_n} z_n),$$

with a single fixed point $0 \in X$. This $\mathbb{C}^*$–action induces a fixed-point-free $S^1 \subset \mathbb{C}^*$ action on the link $L := X \cap \partial B$ of the singularity, where B is a small ball centered at the origin. Hence, the link L is a Seifert fibered 3–manifold over a genus g Riemann surface, denoted by $Y(-b; g; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_k, \beta_k))$ for some integers b, α_i and β_i with $0 < \beta_i < \alpha_i$ and $(\alpha_i, \beta_i) = 1$. Note that k is the number of singular fibers, and there is an associated star-shaped plumbing graph Γ: the central vertex has genus g and weight $-b$, and each vertex in k arms has genus 0 and weight $-b_{ij}$ uniquely determined by the continued fraction

$$\frac{\alpha_i}{\beta_i} = [b_{i1}, b_{i2}, \dots, b_{ir_i}] = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\dots - \cfrac{1}{b_{ir_i}}}}$$

with $b_{ij} \geq 2$. For example, Figure 1 shows the case of $g = 0$ and $k = 3$, which is called a *small* Seifert (fibered) 3–manifold. By P Orlik and P Wagreich [11], it is well known that the plumbing graph Γ is a dual graph of the minimal resolution of $(X, 0)$. Conversely, if the intersection matrix of Γ is negative definite, there is a weighted homogeneous surface singularity whose dual graph of the minimal resolution is Γ; see [15]. Note that a Seifert 3–manifold Y, as a boundary of a plumbed 4–manifold according to Γ, inherits a canonical contact structure providing that each vertex represents a symplectic 2–sphere, all intersections between them are orthogonal, and the intersection matrix of Γ is negative definite; see [7]. Furthermore, if the Seifert 3–manifold Y can be viewed as the link L of a weighted homogeneous surface singularity, then the canonical contact structure above is contactomorphic to the *Milnor fillable* contact structure, which is given by $TL \cap JTL$; see [13].

2.2 Rational blowdowns and symplectic fillings

Rational blowdown surgery, first introduced by R Fintushel and R Stern [5], is one of the most powerful cut-and-paste techniques. It replaces a certain linear plumbing C_p of disk bundles over a 2–sphere whose boundary is a lens space $L(p^2, p - 1)$ with a rational homology 4–ball B_p which has the same boundary. Later, Fintushel and Stern's rational blowdown surgery was generalized by J Park [14] using a configuration $C_{p,q}$ obtained by linear plumbing disk bundles over a 2–sphere according to the dual
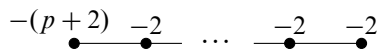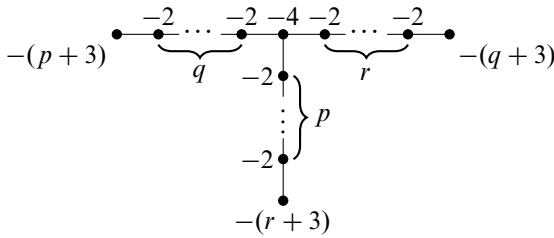

Figure 2: Linear plumbing C_p.

Figure 3: Plumbing graph $\Gamma_{p,q,r}$.

resolution graph of $L(p^2, pq-1)$, which also bounds a rational homology 4–ball $B_{p,q}$. In the case of a symplectic 4–manifold (X, ω), rational blowdown surgery can be performed in the symplectic category: if all 2–spheres in the plumbing graph are symplectically embedded and their intersections are ω–orthogonal, then the surgered 4–manifold $X_{p,q} = (X - C_{p,q}) \cup B_{p,q}$ also admits a symplectic structure induced from the symplectic structure of X; see [19; 20]. In fact, the rational homology 4–ball $B_{p,q}$ admits a symplectic structure compatible with the canonical contact structure on the boundary $L(p^2, pq - 1)$. More generally, in addition to the linear plumbing of 2–spheres, there is a plumbing of 2–spheres according to star-shaped plumbing graphs with 3 or 4 legs admitting a symplectic rational homology 4–ball; see [18; 3]. That is, the corresponding Seifert 3–manifold $Y(-b, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, or $Y(-b, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4))$ with a canonical contact structure, has a minimal symplectic filling whose rational homology is isomorphic to that of the 4–ball [6]. For example, a plumbing graph $\Gamma_{p,q,r}$ in Figure 3 can be rationally blown down. We will use this later in the proof of the main theorem.

As rational blowdown surgery does not affect the symplectic structure near the boundary, if there is a plumbing of disk bundles over symplectically embedded 2–spheres that can be rationally blown down, then one can obtain another symplectic filling by replacing the plumbing with a rational homology 4–ball. In the case of the link of quotient surface singularities, it was proven [2; 4] that every minimal symplectic filling is obtained by a sequence of rational blowdowns from the minimal resolution of the singularity, which is diffeomorphic to a plumbing of disk bundles over symplectically embedded 2–spheres. First, they constructed a genus-0 or genus-1 Lefschetz fibration X on each minimal symplectic filling of the link of a quotient surface singularity. Suppose that w_1 and w_2 are two words consisting of right-handed Dehn twists along curves in a generic fiber, which represent the same element in the mapping class group of the generic fiber. If the monodromy factorization of X is given by $w_1 \cdot w'$, one can construct another Lefschetz

fibration X' whose monodromy factorization is given by $w_2 \cdot w'$. The operation of replacing w_1 with w_2 is called a *monodromy substitution*. Next, they showed that the monodromy factorization of each minimal symplectic filling of the link of a quotient surface singularity is obtained by a sequence of monodromy substitutions from that of the minimal resolution. Furthermore, these monodromy substitutions can be interpreted as rational blowdown surgeries topologically. Note that all rational blowdown surgeries mentioned here are linear: a certain linear chain $C_{p,q}$ of 2–spheres is replaced with a rational homology 4–ball.

2.3 Minimal symplectic fillings of a small Seifert 3–manifold

In this subsection, we briefly review Starkston's results [16; 17] for minimal symplectic fillings of a small Seifert fibered 3–manifold $Y(-b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ with $b \geq 4$. The condition $b \geq 4$ on the weight (equivalently, degree) of a central vertex of the plumbing graph Γ ensures that one can always choose a concave cap K, which is also star-shaped, with a $(+1)$ central 2–sphere and $b - 4$ arms, each of which consists of a single (-1) 2–sphere as in Figure 4. Here $[a_{i1}, a_{i2}, \ldots, a_{in_i}]$ denotes a dual continued fraction of $[b_{i1}, b_{i2}, \ldots, b_{ir_i}]$; that is, $\alpha_i/(\alpha_i - \beta_i) = [a_{i1}, a_{i2}, \ldots, a_{in_i}]$ while $\alpha_i/\beta_i = [b_{i1}, b_{i2}, \ldots, b_{ir_i}]$.

For a given minimal symplectic filling W of Y, we glue W and K along Y to get a closed symplectic 4–manifold X. Then, the existence of a $(+1)$ 2–sphere implies that X is a rational symplectic 4–manifold and, after a finite number of blowdowns, X becomes $\mathbb{CP}^2$ and the $(+1)$ 2–sphere in K becomes a complex line $\mathbb{CP}^1 \subset \mathbb{CP}^2$; see McDuff [9] for details. Under these circumstances, it is natural to ask the following question: What is the image of K in $\mathbb{CP}^2$ under blowdowns? In the case that K is linear, which means that the corresponding Y is a lens space, Lisca showed that the image of K is two symplectic 2–spheres in $\mathbb{CP}^2$, each of which is homologous to $\mathbb{CP}^1 \subset \mathbb{CP}^2$.
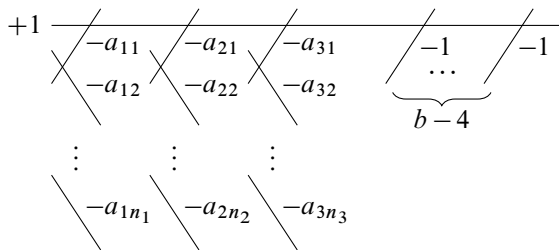

Figure 4: Concave cap K.

By analyzing the proof of Lisca's result [8, Theorem 4.2], Starkston [16] showed that the image of K is b symplectic 2–spheres in $\mathbb{CP}^2$, each of which is homologous to $\mathbb{CP}^1 \subset \mathbb{CP}^2$. For the complete classification of minimal symplectic fillings of Y, one needs to classify the isotopy classes of these b symplectic 2–spheres, which are called *symplectic line arrangements*. Since all these spheres are J–holomorphic for some J tamed by the standard Kähler form of $\mathbb{CP}^2$ and are homologous to $\mathbb{CP}^1 \subset \mathbb{CP}^2$, they intersect each other at a single point for each pair of 2–spheres. Note that these intersection points need not be all distinct. These intersection data of a symplectic line arrangement are determined by the homological data of K, which also have constraints from the adjunction formula. In [17], Starkston showed that symplectic line arrangements with certain types of intersections are isotopic to *complex line arrangements*, that is, the corresponding b symplectic 2–spheres are isotopic (through symplectic spheres) to b complex lines in $\mathbb{CP}^2$. For example, Starkston classified minimal symplectic fillings by an explicit computation of all possible homological embeddings of K for some families of Seifert fibered spaces; see [16, Sections 3 and 4.4; 17, Section 5].

3 Strategy for the main theorem

As we saw in the previous section, for each minimal symplectic filling W of Y, we obtain a rational symplectic 4–manifold X which is symplectomorphic to $\mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$ for some integer N by gluing K to W along Y. Conversely, given an embedding of a concave cap K into $\mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$, we obtain a symplectic filling W of Y by taking a complement of K in $\mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$. So the classification of minimal symplectic fillings of Y is equivalent to the classification of the embeddings of K into $\mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$ for some N. Hence, in order to investigate minimal symplectic fillings W of Y, we first introduce two notions, *homological data* and *curve configuration* of the corresponding embedding of K, which are defined as follows.

Definition 3.1 Suppose that W is a minimal symplectic filling of a small Seifert 3–manifold Y equipped with a concave cap K. Then we have an embedding of K into a rational symplectic 4–manifold $X \cong \mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$ such that the $(+1)$ 2–sphere in K is identified with $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Let l be a homology class represented by a complex line $\mathbb{CP}^1$ in $\mathbb{CP}^2$ and e_i be homology classes of exceptional spheres coming from blowups. Then $\{l, e_1, \dots, e_N\}$ becomes a basis for $H_2(X; \mathbb{Z})$, so that the homology class of each irreducible component of K can be expressed in terms of this basis, which we call the *homological data of K* for W.

Note that K is symplectically embedded in $X \cong \mathbb{CP}^2 \sharp N \overline{\mathbb{CP}^2}$ and each irreducible component of K can be assumed to be J–holomorphic for some J tamed by standard Kähler form on X. Then there is a sequence of blowdowns from X to $\mathbb{CP}^2$ and we can find a J–holomorphic exceptional sphere Σ_i whose homology class is e_i disjoint from the central $(+1)$ 2–sphere of K at each stage of blowing down. Because of the J–holomorphic condition and homological restrictions from the adjunction formula together with intersection data of K, the exceptional sphere Σ_i intersects positively at most once with the image of an irreducible component of K or is one of the image of irreducible components of K; see [8, Proposition 4.4]. In particular, for each image C_j of irreducible components of K, the intersection number between e_i and $[C_j]$ lies in $\{-1, 0, 1\}$.

As mentioned in the previous section, we finally get a symplectic line arrangement in $\mathbb{CP}^2$ which consists of J–holomorphic 2–spheres, each of which is the image of the first component of each arm under blowdowns. The intersection data of the symplectic line arrangement are determined by the homological data of K, so that it can be represented as a configuration of strands: each strand represents a J–holomorphic 2–sphere of a symplectic line arrangement in $\mathbb{CP}^2$, while the intersection of two strands represents a geometric intersection of two 2–spheres. Then, starting from the configuration of the symplectic line arrangement, we can draw a configuration C of strands with degrees by blowups according to the homological data of K until we get K in the configuration. Here the degree of each strand in C means a self-intersection number of the strand. To be more precise, when we blow up a point p on a strand in a configuration, we introduce a new strand with degree -1 to the point p so that we resolve intersection of strands at p and we decrease the degree of the strands containing p by one. Hence the configuration C, which represents the total transform of a symplectic line arrangement, contains strands representing irreducible components of K and exceptional (-1) 2–spheres intersecting with the irreducible components. We say that two configurations C_1 and C_2 for W are *equivalent* if there is a bijective map between (-1) strands preserving intersections with the irreducible components of K.

Definition 3.2 If there are no strands with degree less than or equal to -2 in C except for irreducible components of K, we call the configuration C the *curve configuration* of a minimal symplectic filling W.

Remark A curve configuration C of W consists of strands representing irreducible components of K and exceptional 2–spheres intersecting the irreducible components
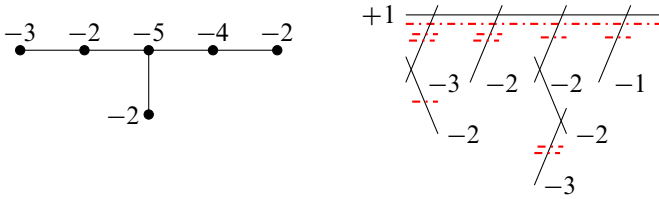
Figure 5: Plumbing graph Γ and curve configuration for corresponding concave cap K.

of K. We denote the exceptional 2–spheres by dash-dotted strands. See Figure 5 for example.

Remark We often use the terminology *configuration of strands* when we deal with an intermediate configuration between a symplectic line arrangement and a curve configuration, or a configuration containing K but with strands with degree less than or equal to -2 other than irreducible components of K.

Proposition 3.3 *For given homological data of K for W, there is a unique curve configuration C up to equivalence.*

Proof Since each strand in a curve configuration C represents a J–holomorphic 2–sphere for some J tamed by standard Kähler form on $X \cong \mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$, all intersections between the strands represent positive geometric intersections between the corresponding J–holomorphic 2–spheres. Note that there is at most one intersection point between any two strands due to homological restrictions. Furthermore, if e_i is a homology class of an exceptional 2–sphere satisfying $e_i \cdot [C_j] \in \{0, 1\}$ for any irreducible component C_j of K, then there is a (-1) strand L_i in C whose homology class is e_i; otherwise, there is a blowup on the strand L_i so that the proper transform of L_i becomes an irreducible component C_j of K whose intersection with e_i is -1, contradicting the assumption. Hence there is a (-1) strand L_i representing a J–holomorphic exceptional sphere Σ_i whose homology class is e_i in C if and only if $e_i \cdot [C_j] \in \{0, 1\}$ for any irreducible component C_j of K.

Let C and C' be two curve configurations for a fixed homological data of K for W. Then, the numbers of (-1) strands in C and C' are equal to the number of e_i satisfying the condition $e_i \cdot [C_j] \in \{0, 1\}$ for any irreducible component C_j of K. Hence we can construct a desired bijection between the (-1) strands by finding correspondence between such e_i and (-1) strands in two curve configurations.   □
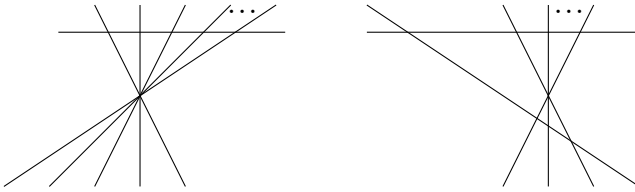
Figure 6: Symplectic line arrangements.

Now, we investigate minimal symplectic fillings of a given small Seifert 3–manifold Y by analyzing all the possible curve configurations. For this, we first determine all possible symplectic line arrangements.

Proposition 3.4 *For minimal symplectic fillings of a small Seifert fibered 3–manifold $Y(-b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ with $b \geq 4$, there are only two possible intersection relations of symplectic line arrangements, which can be drawn as in Figure 6.*

Proof Since Y is a small Seifert 3–manifolds with $b \geq 4$, we can always choose a concave cap K with a $(+1)$ central 2–sphere and $b - 4$ arms, each of which consists of a single (-1) 2–sphere as in Figure 4. Furthermore, since the blowdowns are disjoint from the central 2–sphere in K, each of $b - 1$ arms in K descends to a single $(+1)$ J–holomorphic 2–sphere intersecting at a distinct point with an image of the central 2–sphere of K under the blowdowns. Let $C_1, C_2, \ldots, C_{b-4}$ be the images of $b - 4$ many (-1) 2–spheres in K under the blowdowns. Then they should have a common intersection point in $\mathbb{CP}^2$: otherwise, we have distinct two points p and q on some C_i such that C_i intersects C_j and C_k at p and q, respectively. Let r be an intersection point of C_j and C_k. Then any J–holomorphic 2–sphere coming from an arm of K other than $C_1, \ldots C_{b-4}$ must pass two of p, q and r, which is a contradiction.

If $b \geq 6$, a similar argument shows that there is at most one J–holomorphic 2–sphere coming from an arm of K intersecting at a different point from the common intersection point p with C_i, which proves the proposition.
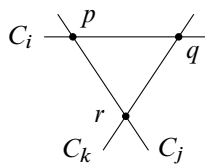

Figure 7: Configuration for C_i, C_j and C_k.

In the case of $b \leq 5$, we can easily check that Figure 6 gives all possible symplectic line arrangements: if $b = 5$, then there is only one C_1 coming from (-1) 2–sphere from K. Recall that there are at most two intersection points on C_1. If there is only one intersection point on C_1, then we get the left-hand figure in Figure 6. If there are two intersection points p and q on C_1, then two of three J–holomorphic 2–spheres coming from the arms of K other than C_1 pass p, and the other passes q (or vice versa), so that we get the right-hand figure in Figure 6. For $b = 4$ case, we have only three strands in a figure for a symplectic line arrangement except the strand from $(+1)$ 2–sphere so that we have only two possibilities.   □

Next, for the complete classification of minimal symplectic fillings of Y, we need to consider the isotopy classes of embeddings of K with a fixed homological data in $X \cong \mathbb{CP}^2 \sharp N\overline{\mathbb{CP}^2}$. By blowing down J–holomorphic 2–spheres, it descends to isotopic types of corresponding symplectic line arrangement in $\mathbb{CP}^2$. By [17, Propositions 4.1 and 4.2], two symplectic line arrangements in Figure 6 are actually isotopic to complex line arrangements through symplectic configurations, which means that there is a unique minimal symplectic filling up to symplectic deformation equivalence for each possible choice of homological data of K. Since a choice of homological data of K gives a unique curve configuration C up to equivalence by Proposition 3.3, we analyze minimal symplectic fillings of a small Seifert 3–manifold Y by considering all possible curve configurations obtained from the complex line arrangements in Figure 6.

As previously mentioned, in the case of quotient surface singularities that include all lens spaces and some small Seifert 3–manifolds, every minimal symplectic filling is obtained by linear rational blowdown surgeries from the minimal resolution of the corresponding singularity. However, this is not true anymore for small Seifert 3–manifolds in general. For example, a rational homology 4–ball of $\Gamma_{p,q,r}$ in Figure 3 might not be obtained by linear rational blowdown surgeries. Nevertheless, many cases such as $b \geq 5$ are in fact obtained by linear rational blowdowns from their minimal resolutions. For the case of $b = 4$, one might need 3–legged rational blowdown surgeries to get a minimal symplectic filling. Hence, it is natural to prove the two cases $b \geq 5$ and $b = 4$ separately.

3.1 The case $b \geq 5$

We consider all possible curve configurations coming from two complex line arrangements in Figure 6 which can be divided into three types. First, we need to blow up all intersection points in the line arrangements so that we get two configurations as
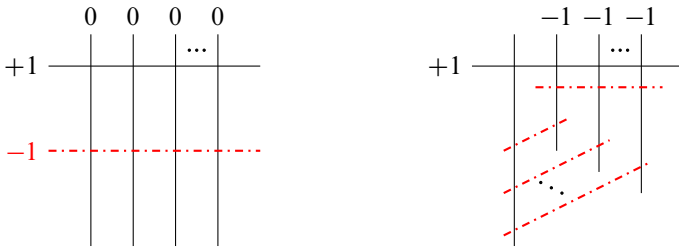
Figure 8: Blowups of the line arrangements.

in Figure 8. There are two possibilities for a strand representing exceptional sphere in intermediate configurations coming from blowups: to blow up some intersection points, or not. Once we blow up an intersection point on a strand representing an exceptional sphere Σ, which means the proper transform of Σ becomes an irreducible component of K, we should blow up all the intersection points except one intersection point because each strand intersecting the strand for Σ become irreducible components of distinct arms in K. We can also blow up the last intersection point we did not blow up to get another curve configuration, but it is not necessary in general.

If we do blow up an intersection point on the dash-dotted strand of the left-hand side of Figure 8, we get the configuration on the left-hand side of Figure 9. When we start with two configurations in Figure 9, we can assume without loss of generality that the first three arms become *essential arms* in K, which consist of strands with degree less than or equal to -2. Since the degree of the other arms is already -1, we can only blow up e_1 and e_2 among dotted exceptional strands. In conclusion, we can divide all the possible curve configurations into following three types:

- **Type A** Curve configurations obtained from Figure 8, left, without blowing up the exceptional strand.
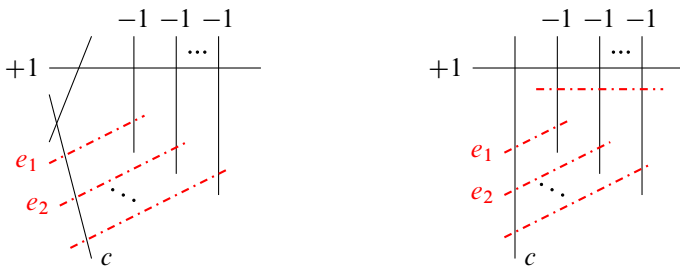
Figure 9: Two configurations.

- **Type B**   Curve configurations obtained from Figure 9, left or right, by blowing up at most one e_i, with $1 \leq i \leq 2$.

- **Type C**   Curve configurations obtained from Figure 9, left or right, by blowing up both e_1 and e_2.

3.2   The case $b = 4$

We divide all curve configurations for $b = 4$ into the following two cases:

- Curve configurations of types A, B or, C as in the $b \geq 5$ case.

- **Type D**   Curve configurations obtained from Figure 9, right, by blowing up all exceptional (-1) strands.

Then, since we can deal with the first case using the same argument in the $b \geq 5$ case, it suffices to prove the type D case whose corresponding curve configurations come from some configurations $C_{p,q,r}$ in Figure 22, which are obtained from the right-hand figure in Figure 9; see Section 4.4 for details. The main difference between the $b = 4$ case and the $b \geq 5$ case is that one can use all three exceptional 2–spheres to get a concave cap K for $b = 4$, while one can use only e_1 and e_2 for $b \geq 5$ from the right-hand figure in Figure 9.

4   Proof of main theorem

In this section, for a given possible curve configuration C, we show that there is a sequence of rational blowdowns from the minimal resolution $\widetilde{M}$ to the minimal symplectic filling W of Y corresponding to C. Since any minimal symplectic filling of a lens space is obtained by a sequence of rational blowdowns from a linear plumbing which is the minimal resolution corresponding to the lens space [2], it suffices to construct a sequence of curve configurations $C = C_0, C_1, \ldots, C_n$ such that each minimal symplectic filling W_i corresponding to C_i is obtained from W_{i+1} by replacing a certain linear plumbing L_i with its minimal symplectic filling. Here C_n denotes a curve configuration for the minimal resolution $\widetilde{M}$. As previously mentioned, since our possible symplectic line arrangements are isotopic to complex line arrangements, it suffices to work in complex category with a symplectic form ω coming from the standard Kähler form on $\mathbb{CP}^2$. In order to show that there is a symplectic embedding of L_i in W_{i+1}, we construct a configuration C'_{i+1} of strands, which is not a curve

configuration for W_{i+1}, from a complex line arrangement by blowups with the same homological data of K for W_{i+1} so that we have L_i disjoint from K in C'_{i+1}. Since we work in complex category, each strand in C'_{i+1} can be considered as a complex rational curve in a rational surface X while the intersections between strands represent positive geometric intersections between the corresponding rational curves. This observation implies that L_i is symplectically embedded in W_{i+1}.

Now we introduce the notion of *standard blowups*, which frequently appears in the construction of W_i from W_{i+1}. Let K and K' be two star-shaped plumbing graphs having the same number of arms together with a $(+1)$ central vertex, and let $-a_{ij}$ for $1 \leq j \leq n_i$ and $-a'_{ij}$ for $1 \leq j \leq n'_i$ be the weights (equivalently, degrees) of the jth vertex in the ith arm of K and K', respectively. We say $K' \leq K$ if $n'_i \leq n_i$ and $a'_{ij} \leq a_{ij}$ for any i and j except for $a'_{in'_i} < a_{in'_i}$ in the case of $n'_i < n_i$. The condition $K' \leq K$ guarantees that we can obtain a configuration of strands representing K by blowups from a configuration representing K' in the following way: we blow up nonintersection points of the last component of each ith arm in K' consecutively until we get n_i components, and then we blow up each component at nonintersection points to get the right weights.

Definition 4.1 Let C' be a configuration of strands obtained from a complex line arrangement by blowups containing a star-shaped plumbing graph K' with a homological data. If $K' \leq K$ and the degree of all strands in $C' \setminus K'$ is -1, then we can obtain a curve configuration $\widetilde{C}'$ from C' by blowing up at nonintersection points only. In this case, we say that the curve configuration $\widetilde{C}'$ is obtained by *standard blowups* from C'.

Remark With given homological data of K' in C', the standard blowups induce a unique homological data of K for $\widetilde{C}'$: Let e be a homology class of an exceptional sphere coming from blowing-ups from C' to $\widetilde{C}'$. Since we blow up nonintersection points, e appears in at most two $[C^{i_1}_{j_1}]$ and $[C^{i_2}_{j_2}]$, where C^i_j denotes the jth component in the ith arm of K. Moreover, if e appears in two $[C^{i_1}_{j_1}]$ and $[C^{i_2}_{j_2}]$, then $i_1 = i_2 = i$ and $j_2 = j_1 + 1$ with $e \cdot [C^i_{j_1}] = 1$ and $e \cdot [C^i_{j_1+1}] = -1$.

For a given star-shaped plumbing graph $K' \leq K$, in general if $n'_i < n_i$ for some i, where n'_i and n_i are the number of components in i^{th} arm of K' and K respectively, there are possibly other ways of blowing up to get the ith arm of K from that of K'. Let C' be a configuration of strands containing $K' \leq K$ as in Definition 4.1. Assume furthermore that $n'_i < n_i$ for some i. Let $\widetilde{C}'$ be a curve configuration obtained from C' by standard blowups. Then we get the following three fundamental lemmas.
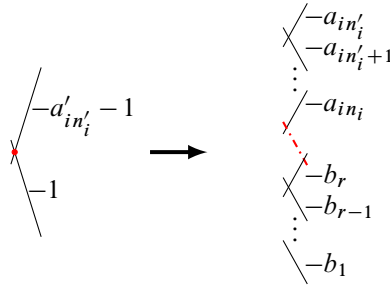
Figure 10: Finding an embedding of L.

Lemma 4.2 *Let C be a curve configuration for K, and let W be the minimal symplectic filling of Y corresponding to C. Suppose C' is a configuration for $K' \leq K$ such that the standard blowups $\widetilde{C}'$ of C' differs from C only in the components C_j^i for $n_i' \leq j \leq n_i$. Let $\widetilde{W}$ denote the minimal symplectic filling of Y corresponding to $\widetilde{C}'$. Then there is a symplectically embedded linear plumbing L of 2–spheres determined by $[b_1, b_2, \ldots, b_r]$ in $\widetilde{W}$ such that W is obtained by $\widetilde{W}$ by replacing the plumbing L with some minimal filling W_L of the lens space boundary of the linear plumbing L. Further, $[b_1, b_2, \ldots, b_r]$ is the dual of $[(a_{in_i'} - a_{in_i'}'), a_{in_i'+1}, a_{in_i'+2}, \ldots, a_{in_i}]$, where $-a_{ij}$ and $-a_{ij}'$ are the weights of the jth component in the ith arm of K and K', respectively.*

Proof We can assume that $a_{in_i'} - a_{in_i'}' \geq 2$ because the way of blowing up from the ith arm of K' to that of K remains the same when we replace K' with K'', where K'' is obtained from K' by blowing up the last component of the ith arm.

First we show that there is a symplectic linear embedding L in $\widetilde{W}$. Let S be a configuration of strands containing K obtained as follows. We blow up the last component in the ith arm of K' in C' at a nonintersection point so that we have two consecutive strands of degree $-a_{in_i'}' - 1$ and -1. Since the continued fraction $[b_1, b_2, \ldots, b_r]$ is dual to $[(a_{in_i'} - a_{in_i'}'), a_{in_i'+1}, a_{in_i'+2}, \ldots a_{in_i}]$ by the definition of L, we obtain a linear chain of strands containing the rest of the ith arm in K and L from the two strands by blowing up consecutively at intersection points as in Figure 10, so that there is an embedding L in the complement of K in a rational surface X. Furthermore, since we started from the same homological data of K' in C' and since a blowup for C' to S either increases the number of components or decreases the degree of an irreducible component of K, the homological data of K for both $\widetilde{C}'$ and S are the same, so that there is a symplectic embedding L in $\widetilde{W}$.

Before we examine the effect of replacing L with its minimal symplectic filling W_L, we briefly review the classification of minimal symplectic fillings of lens space, which can be found in [1] and [8]. For notational convenience, we denote a linear plumbing graph and a lens space determined the plumbing graph by the same L. For a lens space L given by $[b_1, b_2, \ldots, b_r]$, we can choose a concave cap K_L of the form

$$\underset{\bullet}{+1} \quad \underset{\bullet}{-a_1+1} \quad \underset{\bullet}{-a_2} \quad \cdots\cdots \quad \underset{\bullet}{-a_n}$$

where $[a_1, a_2, \ldots, a_n]$ is a dual continued fraction of $[b_1, b_2, \ldots, b_r]$. Suppose $X_L \cong \mathbb{CP}^2 \,\sharp\, N_0 \overline{\mathbb{CP}^2}$ is a rational symplectic 4–manifold obtained by gluing two plumbings according to L and K_L whose second homology class is generated by $\{l\} \cup E = \{E_1, \ldots, E_{N_0}\}$. Then, for a given minimal symplectic filling W_L of L, we get a rational symplectic 4–manifold $X_{W_L} \cong \mathbb{CP}^2 \,\sharp\, N \overline{\mathbb{CP}^2}$ by gluing W_L and K_L and the image of K_L under blowing down is isotopic to two complex lines in $\mathbb{CP}^2$, which means that a minimal symplectic filling of L is determined by a choice of homological data of K_L in $\mathbb{CP}^2 \,\sharp\, N \overline{\mathbb{CP}^2}$ for some N. Hence, we draw a curve configuration C_{W_L} for W_L starting from a configuration of two $(+1)$ strands in $\mathbb{CP}^2$ by blowing-ups with only one $(+1)$ strand. This observation shows that the effect of replacing L in X_L with W_L is the following: We have another rational symplectic 4–manifold $X_{W_L} \cong \mathbb{CP}^2 \,\sharp\, N \overline{\mathbb{CP}^2}$ and the second homology classes in the complement of L are changed so that

$$l \to l \qquad \text{and} \qquad [L_i]^E \to [L_i]^e \qquad \text{for } 1 \le i \le n,$$

where $[L_i]^E$ and $[L_i]^e$ are homology classes of irreducible components of K_L in terms of $\{l\} \cup E = \{E_1, \ldots, E_{N_0}\}$ and $\{l\} \cup e = \{e_1, \ldots, e_N\}$ respectively.

Let $[C_j^i]^C$ and $[C_j^i]^{C'}$ be homology classes of C_j^i in C and C' respectively. Note that C is a curve configuration completed from the last $(-a'_{i\,n'_i})$ strand in the ith arm of K' by blowups without using any other strand in C'. If we blow up in the same ways starting with a single $(+1)$ strand instead of $(-a'_{i\,n'_i})$ strand, we get a curve configuration C_{W_L} containing K_L. Hence there is a minimal symplectic filling W_L of L whose homological data of K_L in $X_{W_L} (= W_L \cup K_L) \cong \mathbb{CP}^2 \,\sharp\, N \overline{\mathbb{CP}^2}$ are given by $[L_j] = [C^i_{n'_i + j - 1}]^C$ except for $[L_0] = l$ and $[L_1] = l + [C^i_{n'_i}]^C - [C^i_{n'_i}]^{C'}$, where $e = \{e_1, \ldots, e_N\}$ is homology classes of exceptional spheres coming from the blowups from C' to C.

Finally, we show that the minimal symplectic filling W corresponding to C is given by $(\widetilde{W} \setminus L) \cup W_L$. Suppose X' is a rational symplectic 4–manifold obtained by

blowups from a complex line arrangement so that it contains C'. We take a small Darboux neighborhood B' of a disk D in $C^i_{n'_i}$ of K' so that B' is disjoint from any other irreducible components of K'. Now we arrange all the blowups from C' to C inside B' and let B be blowups of B'. Then we have a symplectic embedding of K in $X = (X' \setminus B') \cup B$ and homological data of K that agrees with C. Furthermore, $B \setminus K$ is symplectic deformation equivalent to W_L: consider two complex lines in $\mathbb{CP}^2$ and a symplectic embedding of B' such that the image of D in $C^i_{n'_i}$ is a disk in one complex line and B' is disjoint from the other complex line. By the construction of B, there is a symplectic embedding of K_L in $(\mathbb{CP}^2 \setminus B') \cup B$, where the first component of K_L is the complex line in $(\mathbb{CP}^2 \setminus B')$ and the complement of K_L in $(\mathbb{CP}^2 \setminus B') \cup B$ is symplectic deformation equivalent to W_L. Since the complement of a neighborhood of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ is a ball, $B \setminus K = B \setminus K_L$ is also symplectic deformation equivalent to W_L. Note that $K = (K \cap (X' \setminus B')) \cup (K \cap B) = (K' \setminus B') \cup (K \cap B)$. Hence

$$W = X \setminus K = ((X' \setminus B') \setminus K) \cup (B \setminus K) \cong (X' \setminus (K' \cup B')) \cup W_L.$$

By a similar argument, $\widetilde{W} \cong (X' \setminus (K' \cup B')) \cup L$, so that W is obtained from $\widetilde{W}$ by replacing L by W_L. $\square$

Assume furthermore that there is a (-1) curve intersecting both $C^i_{n'_i}$ and another irreducible component C^k_l of K' in C'. Then there is a slight modification of the Lemma 4.2, involving two arms of K.

Lemma 4.3 *Suppose that there is a (-1) curve E intersecting $C^i_{n'_i}$ and C^k_l of K' in C' with $a'_{kl} < a_{kl}$. If the standard blowup $\widetilde{C}'$ of C' differs from C only in C^k_l and components C^i_j for $n'_i \leq j \leq n_i$, then there is a symplectically embedded linear plumbing $L \subset W$, described in Figure 11, such that W is obtained by $\widetilde{W}$ by replacing the plumbing L with some minimal filling W_L. Furthermore, $[b_1, b_2, \ldots, b_r]$ is the dual of $[(a_{in'_i} - a'_{in'_i}) + 1, a_{in'_i+1}, a_{in'_i+2}, \ldots, a_{in_i}]$, where $-a_{ij}$ and $-a'_{ij}$ are the weights of the jth component in the ith arm of K and K', respectively.*

Proof The proof is similar to that of Lemma 4.2 except for blowups at two intersection points of E in C' to find an embedding L. That is, we construct a configuration S of strands containing K, as in Figure 12, whose homological data is equal to that of $\widetilde{C}'$, so that there is a symplectic embedding of L in $\widetilde{W}$.

Next, by viewing L as a two-legged plumbing graph with a degree $(-b_1 - 1)$ of a central vertex, we get a concave cap K_L as in Figure 11: starting from the zero section
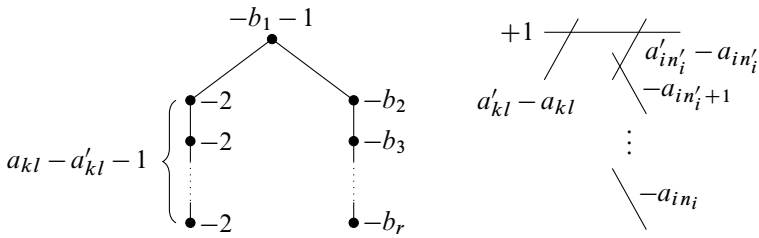
Figure 11: A plumbing graph of L and its concave cap K_L.

and infinity section together with two generic fibers of $\mathbb{F}_{b_1-1}$, we construct arms corresponding to $[-2, \ldots, -2]$ and $[-b_2, \ldots, -b_r]$. Then, by consecutive blowups at intersection points of the proper transform of zero section and the arm corresponding to $[-b_2, \ldots, -b_r]$, we get a concave cap K_L for L. As before, for a given minimal symplectic filling of L, we get a rational symplectic 4–manifold by gluing K_L along L and the image of K_L in $\mathbb{CP}^2$ under blowing down is three complex lines in $\mathbb{CP}^2$ intersecting generically, implying that any curve configuration for K_L is obtained from blowing up at an intersection point between two complex lines in $\mathbb{CP}^2$. Therefore, using blowup data from C' to C (Figure 13), we get a minimal symplectic filling W_L of L.

Suppose that X' is a rational symplectic 4–manifold containing C'', obtained from C' by blowing down E, and let B' be a small Darboux neighborhood of the intersection point coming from the blowing down. Then a similar argument as in Lemma 4.2 above shows that the minimal symplectic filling W corresponding to C is obtained from $\widetilde{W}$ by replacing L with W_L. □
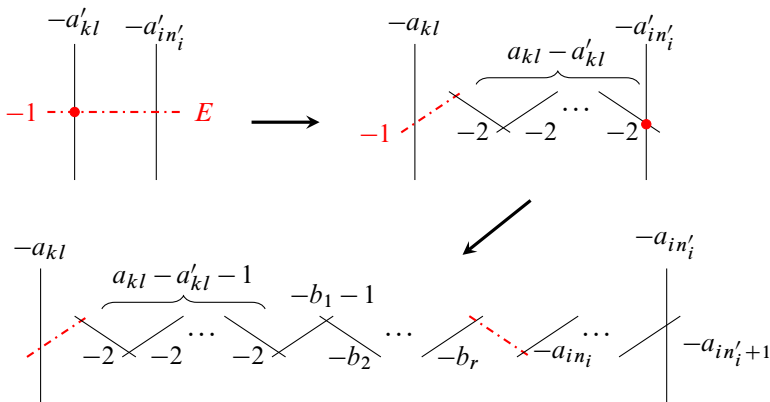
Figure 12: Embedding of L to $\widetilde{W}$.
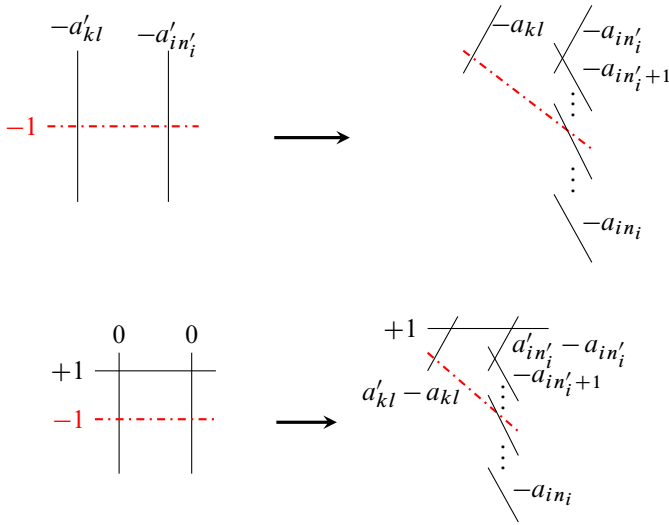
Figure 13: Top: blowups from C' to C. Bottom: curve configuration for W_L.

Assume that C' is a curve configuration containing $K' \leq K$ corresponding to a minimal symplectic filling W' of another small Seifert 3–manifold Y' and $\widetilde{C}'$ is a curve configuration obtained from C' by standard blowups. Then we can describe a minimal symplectic filling $\widetilde{W}$ of Y corresponding to $\widetilde{C}'$ explicitly.

Lemma 4.4 *Under the assumption above, there is a symplectically embedded plumbing of 2–spheres Γ' in the minimal resolution $\widetilde{M}$ such that a minimal symplectic filling $\widetilde{W}$ of Y corresponding to $\widetilde{C}'$ is obtained from $\widetilde{M}$ by replacing Γ' with W'.*

Proof Let K_0 be a plumbing graph determined by black strands in the left-hand side of Figure 8. Clearly, $K_0 \leq K$ so that there is a curve configuration $C_{\widetilde{M}}$ obtained by standard blowups from Figure 8, left. We first show that the curve configuration $C_{\widetilde{M}}$ corresponds to the minimal resolution $\widetilde{M}$. Recall that a concave cap K in Figure 4 can be found in [18; 16]: Starting from the zero and infinity sections with $b-1$ generic fibers of a Hirzebruch surface $\mathbb{F}_1$ which can be drawn as the left-hand side in Figure 8, we blow up intersection points of generic fibers and the infinity section so that we have a $(-b)$ rational curve which corresponds to the central vertex of the minimal resolution graph Γ. Then, we obtain a linear chain of strands containing both i^{th} arm of K and Γ from two (-1) strands by blowups, as in Figure 14, bottom. As a result, we have a configuration $S_{\widetilde{M}}$ containing both Γ and K disjointly, so that the complement of K in a rational surface X_Y is the minimal resolution $\widetilde{M}$ and K is a concave cap for Y. By
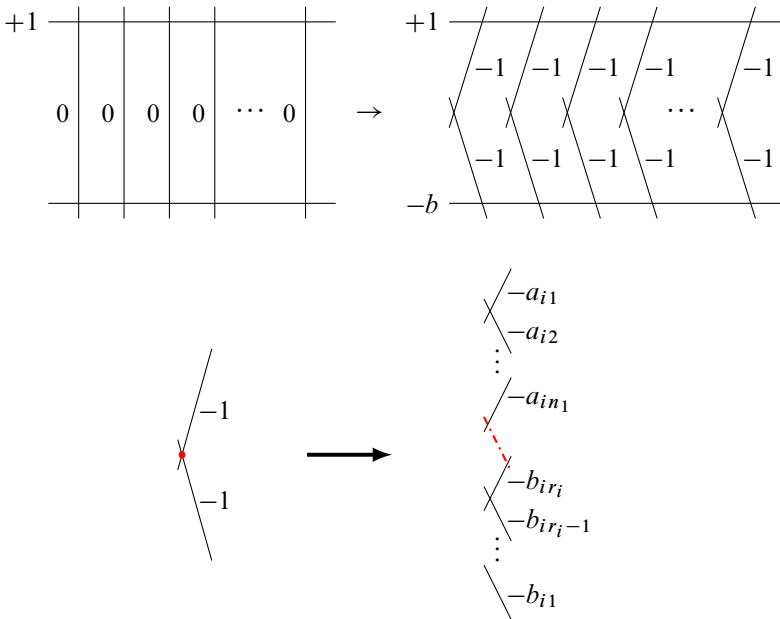
Figure 14: Top: blowing up Hirzebruch surface $\mathbb{F}_1$. Bottom: construction of each arm in K and Γ.

using the same argument as in the proof of Lemma 4.2 above, we conclude that $C_{\tilde{M}}$ is a curve configuration for $\tilde{M}$.

In the same way, we could get a configuration $S_{\Gamma'}$ of strands containing both K' and a plumbing graph Γ' so that the complement of K' in the resulting rational symplectic 4–manifold $X_{Y'} \cong \mathbb{CP}^2 \sharp M \overline{\mathbb{CP}^2}$ is a plumbing of 2–spheres according to Γ'. Note that $M-1$ is the number of blowups in the standard blowups from the left-hand side of Figure 8 to K'. Since $K' \leq K$, we obtain a configuration $S'_{\tilde{M}}$ of strands containing Γ' and K disjointly from $S_{\Gamma'}$ by standard blowups at nonintersection points in the last component of each ith arm of K'. Let $X = X_{Y'} \sharp N \overline{\mathbb{CP}^2}$ be a resulting rational symplectic 4–manifold. Then $X \cong X_Y = \tilde{M} \cup K$, since the number of blowups in the standard blowups from the left-hand side of Figure 8 to K is equal to the sum of numbers of blowups for the left-hand side of Figure 8 to K', and K' to K. Furthermore, the homological data of K in $S'_{\tilde{M}}$ is also equal to that of $C_{\tilde{M}}$. Hence a plumbing graph Γ' is symplectically embedded in $\tilde{M}$.

If there is a sequence of blowups from a configuration of strands representing K' to K, then we have a corresponding symplectic cobordism Z from Y' to Y because the total transform of K' is still a concave cap for Y' while K is a concave cap for Y.

In particular, if C' is a curve configuration for a minimal symplectic filling W', then we get a curve configuration C for a minimal filling W by the sequence of blowups from K' to K and $W = W' \cup Z$. In the case of standard blowups from K' to K, we can deduce from the construction of $S'_{\widetilde{M}}$ that the corresponding cobordism is equal to $\widetilde{M} \setminus \Gamma'$. Hence we have $\widetilde{W} = W' \cup (\widetilde{M} \setminus \Gamma')$, so that $\widetilde{W}$ is obtained from $\widetilde{M}$ by replacing Γ' with W'.   □

4.1 Proof for type A

For a curve configuration C of type A, we want to show that the corresponding minimal symplectic filling W is obtained from the minimal resolution $\widetilde{M}$ by replacing each arm in the resolution graph Γ with its minimal symplectic filling. Since we already know in the proof of Lemma 4.4 above that a curve configuration $C_{\widetilde{M}}$, which is obtained from the left-hand side of Figure 8 by standard blowups, corresponds to $\widetilde{M}$ by repeatedly applying Lemma 4.2 with K' as in the left-hand side of Figure 8 so that the corresponding L is one of three arms in Γ, we conclude that all minimal symplectic fillings corresponding to a curve configuration C of type A are obtained by a sequence of rational blowdowns from the minimal resolution $\widetilde{M}$.

The following example illustrates this case.

Example 4.1 Let Y be a small Seifert 3–manifold whose associated plumbing graph and concave cap are shown in Figure 15, top. Then, there are two curve configurations of type A as in Figure 15, bottom. Of course, there exist other curve configurations of type B and C for minimal symplectic fillings of Y, which will be treated in Example 4.2 and Example 4.3 later. Note that each dash-dotted strand represents an exceptional 2–sphere, that is, a 2–sphere with self-intersection -1. We omit the degree of irreducible components of the concave cap for the sake of convenience in the figure. The bottom left curve configuration in Figure 15 is obtained by standard blowups from that of Figure 8, which means that the corresponding minimal filling is the minimal resolution. Note that only the third arm in the plumbing graph Γ has a nontrivial minimal symplectic filling that is obtained by rationally blowing down the (-4) 2–sphere. Using Lisca's description of the minimal symplectic fillings of lens spaces, we obtain the bottom right curve configuration in Figure 15, which represents a minimal symplectic filling obtained from the minimal resolution by rationally blowing down the (-4) 2–sphere in the third arm.
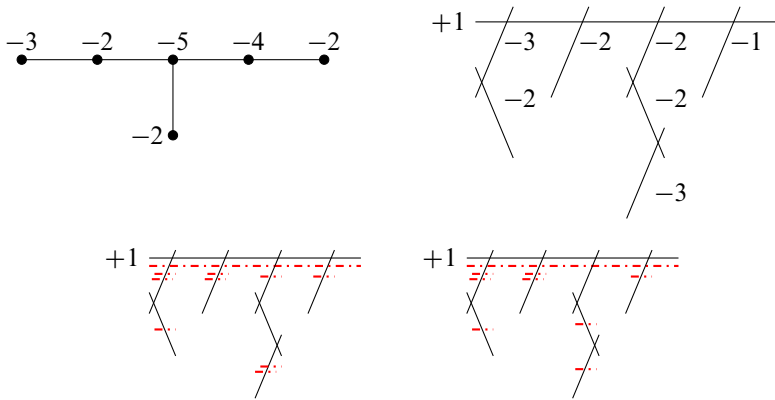
Figure 15: Top: plumbing graph Γ, and its concave cap K. Bottom: two curve configurations in Example 4.1.

4.2 Proof for type B

For a curve configuration C of type B, we want to show that the corresponding minimal symplectic filling W is obtained from the minimal resolution $\tilde{M}$ by replacing disjoint subgraphs in the resolution graph Γ with their minimal symplectic filling. By reindexing if needed, we assume that the first and the second arms of the configurations in Figure 9 become the first and the second arm of K in C, respectively, and the proper transform of e_2 is not an irreducible component of K. Since we do not use e_2 during the blowups, we can get the first and the second arm of K, so that the homological data for the irreducible components in these arms agrees with that of C, from the configurations in Figure 9 leaving the third single (-1) arm unchanged. Hence we arrange the order of blowups from a configuration in Figure 9 to C so that we have an intermediate configuration C' of strands containing $K' \leq K$ as in Figure 16. Note that the degree of strands in $C' \setminus K'$ is all -1. If we choose a linear plumbing graph

$$L' = \overset{-b_{1r_1}}{\bullet} \cdots \overset{-b_{11}}{\bullet} \overset{-b}{\bullet} \overset{-b_{21}}{\bullet} \cdots \overset{-b_{2r_2}}{\bullet}$$

to be a subgraph of Γ as a two-legged plumbing graph with the $(-b)$ central vertex, then K' gives a concave cap of L' and C' is a curve configuration for a minimal symplectic filling $W_{L'}$ of L'.

Let C_1 be a curve configuration obtained by standard blowups from C'. Then, by Lemma 4.4, the curve configuration C_1 corresponds to a minimal symplectic filling W_1, which is obtained from the minimal resolution $\tilde{M}$ by replacing L' with $W_{L'}$. Furthermore, since $[a_{31}, a_{32}, \ldots, a_{3n_3}] = [2, \ldots, 2, c_1 + 1, c_2, \ldots, c_k]$, where $[c_1, c_2, \ldots, c_k]$
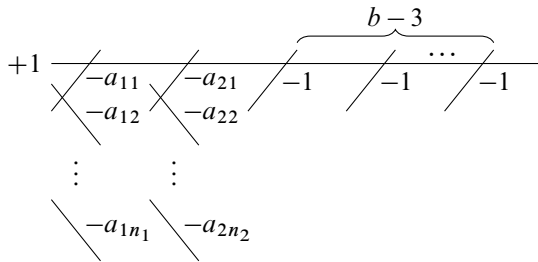
Figure 16: Concave cap K' for linear subgraph of Γ.

is the dual of $[b_{32}, b_{33}, \ldots, b_{3r_3}]$, by Lemma 4.2 with L as a linear chain determined by $[b_{32}, b_{33}, \ldots, b_{3r_3}]$, we conclude that the minimal symplectic filling W corresponding to C is obtained from W_1 by replacing L with its minimal symplectic filling. Hence the desired minimal symplectic filling W is obtained from $\widetilde{M}$ by replacing disjoint linear subgraphs

$$\underset{-b_{1r_1}}{\bullet} \cdots \underset{-b_{11}}{\bullet} \underset{-b}{\bullet} \underset{-b_{21}}{\bullet} \cdots \underset{-b_{2r_2}}{\bullet} \quad \text{and} \quad \underset{-b_{32}}{\bullet} \underset{-b_{33}}{\bullet} \cdots \underset{-b_{3r_3}}{\bullet}$$

of Γ with their minimal symplectic fillings, so that there is a sequence of rational blowdowns from $\widetilde{M}$ to W.

The following example illustrates the curve configurations of type B.

Example 4.2 We again consider a small Seifert 3–manifold Y used in Example 4.1. Since the left-hand configuration without exceptional 2–spheres in Figure 17 gives a concave cap of a lens space determined by a subgraph

$$\underset{-3}{\bullet} \underset{-2}{\bullet} \underset{-5}{\bullet} \underset{-2}{\bullet}$$

of Γ, it gives a minimal symplectic filling W_L of the lens space $L(39, 16)$. Then, by blowups at points lying on the third arm different from the intersection point with the exceptional curve e, we get an embedding of a concave cap K of Y as in the right-hand curve configuration C_1 of Figure 17, which gives a minimal symplectic filling W_1 of Y. Furthermore, since there is a unique minimal symplectic filling of lens space $L(2, 1)$ corresponding to the (-2) 2–sphere in the third arm of Γ, W_1 is obtained from the minimal symplectic filling W_L. In fact, there are three more minimal symplectic fillings of Y which are of type B — see Figure 17 for the corresponding curve configurations. Note that the curve configuration C_1 for W_1 in Figure 17 comes from the right-hand configuration in Figure 9 and the curve c becomes a component of the first arm of K in
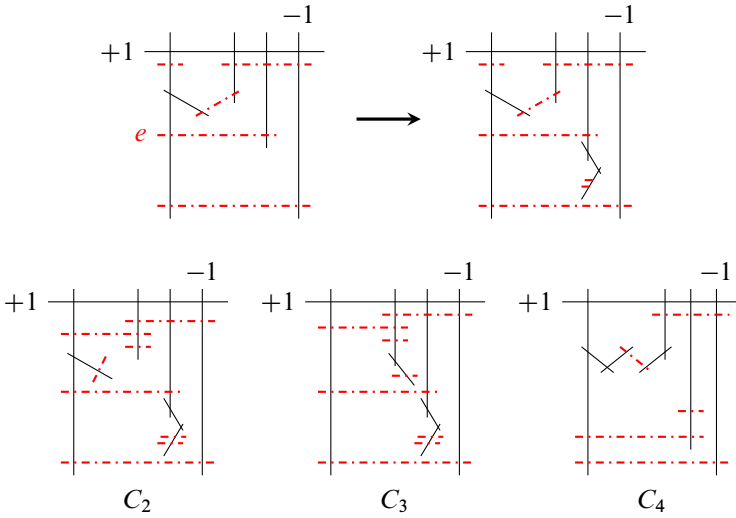
Figure 17: Top: curve configuration C_1 for W_1. Bottom: curve configurations for other symplectic fillings of Y.

the top right of Figure 15. Similarly, the curve configuration C_i for W_i ($2 \leq i \leq 4$) is also obtained from the right-hand configuration in Figure 9. One can easily check that each W_i is obtained from the minimal resolution of Y by a linear rational blowdown surgery: explicitly, W_2, W_3 and W_4 are obtained by rationally blowing down along subgraphs

$$\overset{-2}{\bullet}\!\!-\!\!\overset{-5}{\bullet}, \quad \overset{-5}{\bullet}\!\!-\!\!\overset{-2}{\bullet} \quad \text{and} \quad \overset{-3}{\bullet}\!\!-\!\!\overset{-2}{\bullet}\!\!-\!\!\overset{-5}{\bullet}\!\!-\!\!\overset{-4}{\bullet}\!\!-\!\!\overset{-2}{\bullet}$$

in Γ, respectively. And W_1 is also obtained by rationally blowing down along

$$\overset{-3}{\bullet}\!\!-\!\!\overset{-5}{\bullet}\!\!-\!\!\overset{-2}{\bullet},$$

which is embedded in another plumbing

$$\overset{-3}{\bullet}\!\!-\!\!\overset{-2}{\bullet}\!\!-\!\!\overset{-5}{\bullet}\!\!-\!\!\overset{-2}{\bullet}.$$

4.3 Proof for type C

For a minimal symplectic filling W corresponding to a curve configuration C of type C, we want to find a curve configuration C_1 of type B such that there is a symplectically embedded linear chain L of 2–spheres (that is not visible in Γ) in W_1 corresponding to C_1 so that W is obtained from W_1 by replacing L with its minimal symplectic filling W_L.
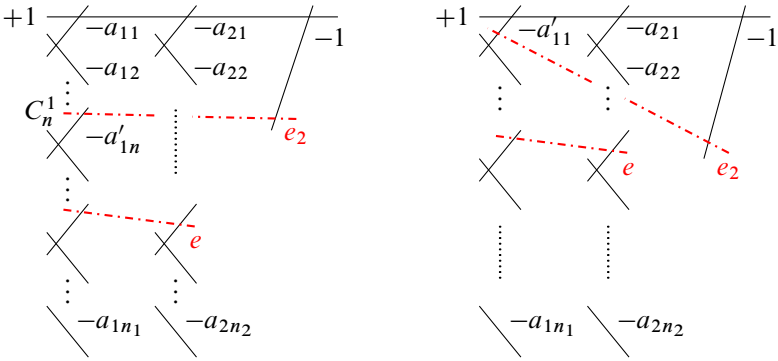
Figure 18: Part of intermediate configuration C'.

By reindexing if needed, we may assume that the first and the second arm of configurations in Figure 9 become that of K respectively, and the proper transform of e_2 becomes an irreducible component in the third arm of K after blowups. For convenience, we omit in figures all exceptional (-1) strands that intersect only one irreducible component of the corresponding concave cap K.

Now, by blowups at intersection points of e_1 consecutively, we get the first and the second arm so that the homological data for the irreducible components in these arms agrees with that of C except for one irreducible component, say C_n^1, of the first arm of K leaving the third single (-1) arm unchanged. Note that there is only one exceptional strand e connecting the first and the second arm as in Figure 18 because we blow up at intersection points of e_1 to get the first and the second arm of K. Hence we can arrange a sequence of blowups from a configuration in Figure 9 to a curve configuration C of type C so that we have an intermediate configuration C' of strands as in Figure 18: The left-hand/right-hand figures comes from the left-hand/right-hand figures in Figure 9, respectively. For simplicity, we only explain a curve configuration coming from the left-hand side in Figure 9. In contrast to the type B case, we have a $(-a'_{1n})$ strand with $a_{1n} > a'_{1n}$ in C' because we need to blow up at the intersection point of e_2 and c in Figure 9, which becomes the $(-a_{1n})$ strand in the curve configuration C at the top of Figure 19. Let C_1 be a curve configuration obtained from C' by standard blowups and W_1 be a minimal symplectic filling of Y corresponding to C_1. Then, by Lemma 4.3, there is a symplectic embedding L in W_1 so that W is obtained from W_1 by replacing L with its minimal symplectic filling W_L, where L is a plumbing graph at the bottom of Figure 19. Since a curve configuration C_1 for W_1 is of type B, there is a sequence of rational blowdowns from $\widetilde{M}$ to W as desired.
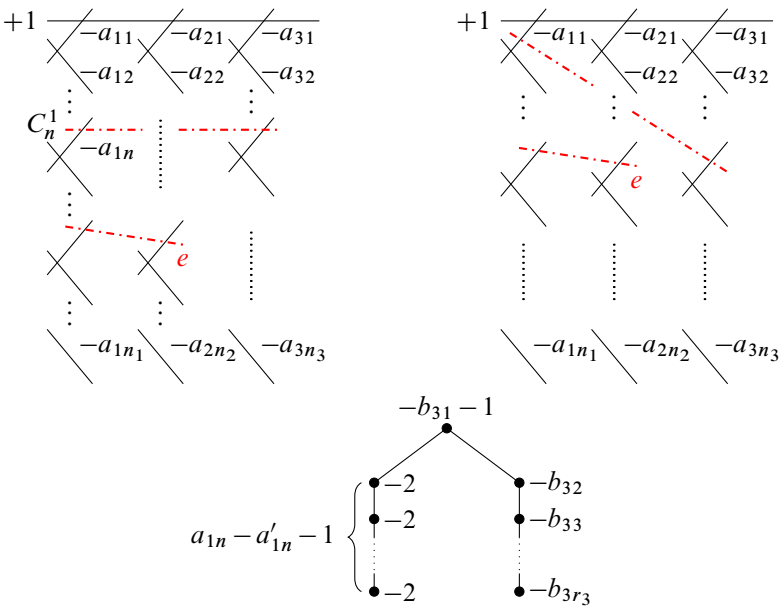
Figure 19: Top: part of curve configuration C for W. Bottom: a plumbing graph of L.

The following example illustrates this case.

Example 4.3 We consider a minimal symplectic filling W_5 of Y in Example 4.1, represented by a curve configuration C_5 in Figure 20. The curve configuration C_5 is obtained from the right-hand configuration in Figure 9, and the proper transforms of e_1 and e_2 are irreducible components of the concave cap K. Thus, as in the proof, we can find an intermediate configuration C' between the right-hand configuration in Figure 9 and C_5. Then it is easy to check that the homological data for standard blowups $\tilde{C}'$ of C' and that of C_1 is equal; see Figures 17 and 20. From the proof of Lemma 4.3, we
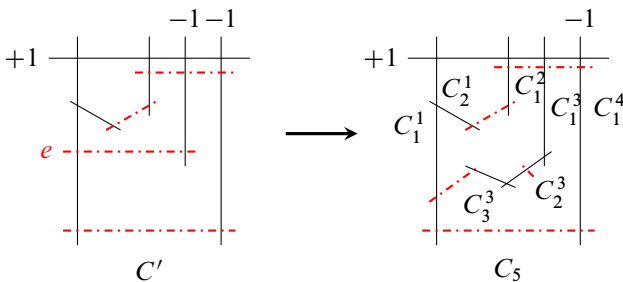
Figure 20: Curve configuration C_5 for symplectic filling W_5 of Y.

can explicitly check that there is a symplectic embedding of

$$L_1 = \overset{-5 \quad -2}{\bullet\!\!-\!\!\!-\!\!\bullet}$$

to W_1 in Example 4.2, and W_5 is obtained by rationally blowing it down: let C_i^j be the ith component of the jth arm in K. Then the homological data of K for W_1 in $X = W_1 \cup K \cong \mathbb{CP}^2 \,\sharp\, 10\overline{\mathbb{CP}^2}$ is given by

$$[C_0] = l,$$
$$[C_1^1] = l - e_2 - e_3 - e_4 - e_5, \quad [C_2^1] = e_2 - e_6,$$
$$[C_1^2] = l - e_1 - e_2 - e_6,$$
$$[C_1^3] = l - e_1 - e_3 - e_7, \qquad [C_2^3] = e_7 - e_8, \qquad [C_3^3] = e_8 - e_9 - e_{10},$$
$$[C_1^4] = l - e_1 - e_4,$$

where C_0 is the central $(+1)$ 2–sphere of K, l is the homology class representing the complex line in $\mathbb{CP}^2$, and e_i is the homology class of each exceptional 2–sphere. As in the proof of Lemma 4.3, we can find a symplectic embedding of

$$L = \overset{-5 \quad -2}{\bullet\!\!-\!\!\!-\!\!\bullet}$$

to $W_1 \subset X$ whose homological data is given by $e_3 - e_5 - e_7 - e_8 - e_9$ and $e_9 - e_{10}$; refer to Figure 21, top. There are two minimal symplectic fillings of L whose corresponding curve configurations are as in Figure 21, bottom. Note that the first figure represents a linear plumbing while the second figure represents a rational homology 4–ball.

Hence, if we rationally blow down L from $X_L = L \cup K_L \cong \mathbb{CP}^2 \,\sharp\, 6\overline{\mathbb{CP}^2}$, then we get a new rational symplectic 4–manifold $X_L' \cong \mathbb{CP}^2 \,\sharp\, 4\overline{\mathbb{CP}^2}$, and the homological data of K_L changes as follows:

$$l \to l,$$
$$l - e_1 - e_2 \to l - E_1 - E_2,$$
$$e_2 - e_3 \to E_2 - E_3,$$
$$e_3 - e_4 \to E_3 - E_4,$$
$$e_4 - e_5 - e_6 \to E_1 - E_2 - E_3.$$

Here e_i and E_i denote the homology classes of exceptional spheres in X_L and X_L'. Note that the homological data of L in X_L is given by $e_1 - e_2 - e_3 - e_4 - e_5$ and
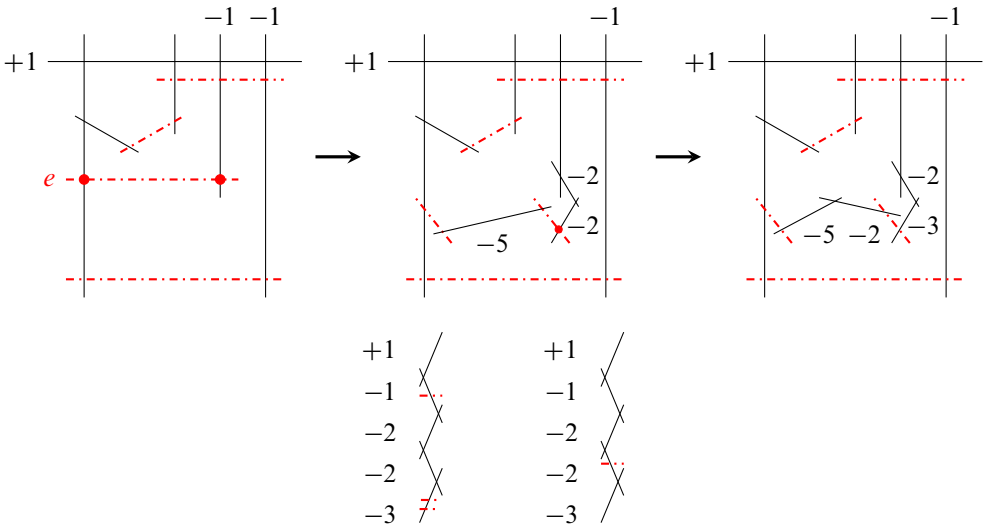
Figure 21: Top: embedding of L_1 in W_1. Bottom: two curve configurations for Y_L.

$e_5 - e_6$. Therefore, if we see X as $X_L \,\sharp\, 4\overline{\mathbb{CP}^2}$, we get $X' \cong \mathbb{CP}^2 \,\sharp\, 8\overline{\mathbb{CP}^2}$ by rationally blowing down L from X, and the homological data of K_L is changed by

$$l \to l,$$
$$l - e_3 - e_5 \to l - E_1 - E_2,$$
$$e_5 - e_7 \to E_2 - E_3,$$
$$e_7 - e_8 \to E_3 - E_4,$$
$$e_8 - e_9 - e_{10} \to E_1 - E_2 - E_3,$$

where e_1, e_2, e_4, e_6 and E_1, E_2, E_3, E_4 represent the standard exceptional 2–spheres in $X' \cong \mathbb{CP}^2 \,\sharp\, 8\overline{\mathbb{CP}^2}$. Therefore, the new homological data for the concave cap K, which give the right-hand curve configuration in Figure 20, are as follows:

$$[C_0] = l,$$
$$[C_1^1] = l - e_2 - e_4 - E_1 - E_2, \quad [C_2^1] = e_2 - e_6,$$
$$[C_1^2] = l - e_1 - e_2 - e_6,$$
$$[C_1^3] = l - e_1 - E_1 - E_3, \qquad [C_2^3] = E_3 - E_4, \quad [C_3^3] = E_1 - E_2 - E_3,$$
$$[C_1^4] = l - e_1 - e_4.$$

Remark We investigated all possible curve configurations for a small Seifert 3–manifold Y with $b \geq 5$ in the proof of the main theorem. As a result, we can find all minimal symplectic fillings of Y via corresponding curve configurations. For example, a complete list of minimal symplectic fillings of Y in Example 4.1 are given by Examples 4.1–4.3.

4.4 Proof for type D

We start to prove this case for a curve configuration coming from $C_{0,0,0}$. Note that $C_{0,0,0}$ itself is a curve configuration containing $K_{0,0,0}$ corresponding to rational homology ball filling of $\Gamma_{0,0,0}$ in Figure 3. By repeatedly blowing up at intersection points between exceptional strands and the first component of each arm, we can get a curve configuration $C_{p,q,r}$ containing $K_{p,q,r}$ corresponding to a rational homology ball filling of $\Gamma_{p,q,r}$ as in Figure 22. For notational convenience, we denote three exceptional strands in each $C_{p,q,r}$ by the same e_i with $i \in \mathbb{Z}_3$ so that e_i intersects the last component of ith arm and the first component of $(i+1)^{\text{st}}$ arm of $K_{p,q,r}$. Let $C_{-1,-1,-1}$ be the right-hand figure of Figure 9 and $C_{p,q,-1}$ be a configuration of strands obtained from $C_{p,q,0}$ by blowing down e_2 in Figure 22. Then $C_{p,q,-1}$ contains $K_{p,q,-1}$, which is the proper transform of $K_{p,q,0}$ under blowing down.

Proposition 4.5 *For a curve configuration C coming from $C_{0,0,0}$, there is a curve configuration $C_{a,b,c}$ containing $K_{a,b,c}$ with $a, b, c \geq -1$ such that*

(i) *there is a sequence of blowups from $C_{a,b,c}$ to C,*

(ii) *there is either no blowup at e_i or blowups at both intersection points on e_i during the sequence of blowups,*

(iii) *there is no blowup at intersection points of $K_{a,b,c}$.*
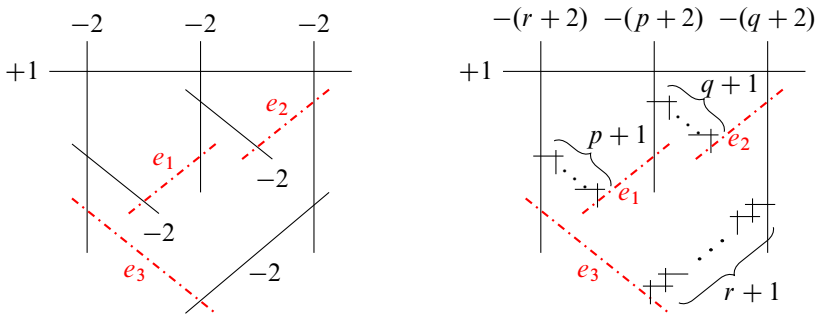
Figure 22: Curve configurations $C_{0,0,0}$ and $C_{p,q,r}$.
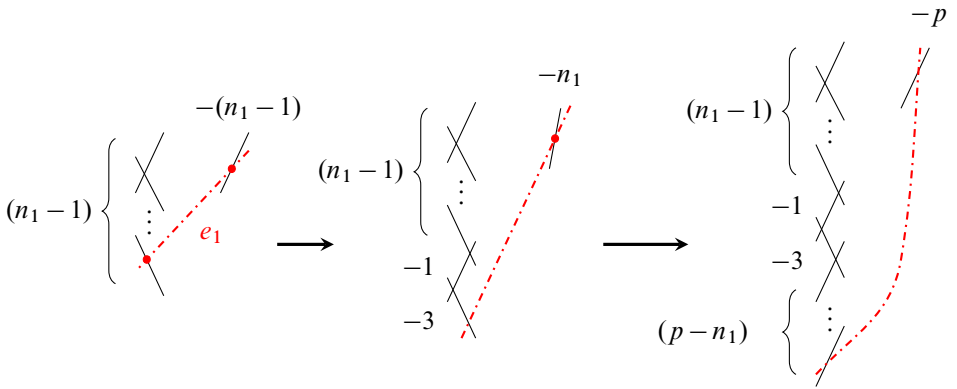
Figure 23: Part of the blowups from C_{n_1-3,n_2-3,n_3-3} to C'.

Proof Since there are no strands with degree ≤ -2 in C except for irreducible components of K, each irreducible component of $K_{0,0,0}$ in $C_{0,0,0}$ should become an irreducible component of K under blowups from $C_{0,0,0}$ to C. Hence, in order to get C from $C_{0,0,0}$ by blowups e_i, we should blow up at either two intersection points of e_i with arms or an intersection point of e_i with the $(i+1)^{\text{st}}$ arm only. Note that we get $C_{p,q,r}$ containing $K_{p,q,r}$ by blowups the latter case repeatedly. Hence, by rearranging the order of blowups from $C_{0,0,0}$ to a curve configuration C, we may assume that C is obtained from $C_{p,q,r}$ with $p, q, r \geq 0$ and there are no more blowups at an intersection point of e_i with the $(i+1)^{\text{st}}$ arm only. Since the configuration $C_{p,q,r}$ clearly satisfies conditions (i) and (ii), we are done if there is no blowup at intersection points of $K_{p,q,r}$ in $C_{p,q,r}$.

If there are blowups at intersection points of $K_{p,q,r}$ in $C_{p,q,r}$ to C, then we will find another $C_{a,b,c}$ with $a \geq p$, $b \geq q$ and $c \geq r$ satisfying conditions (i)–(iii) as follows. Let x_i be the first intersection point in the ith arm of $K_{p,q,r}$ among the intersection points to be blown up, and C' be a configuration of strands obtained by blowing up at x_i for $1 \leq i \leq 3$. For notational convenience, we denote exceptional strands in $C_{p,q,r}$ and the proper transform of e_i in C' by the same e_i. There is a unique (-1) exceptional strand in each ith arm of K' in C', which is the n_i^{th} component of the ith arm with $n_i \geq 2$, where K' is the total transform of $K_{p,q,r}$. Then we claim that there is a sequence of blowups from C_{n_1-3,n_2-3,n_3-3} to C': we blow up two intersection points of e_i simultaneously, and then we blow up at the intersection point between the exceptional (-1) strand and the first component of the $(i+1)^{\text{st}}$ arm consecutively to get C'; for example, see Figure 23 for the first arm. We see from the construction that a configuration C_{n_1-3,n_2-3,n_3-3} satisfies conditions (i) and (ii). Moreover, since x_i is
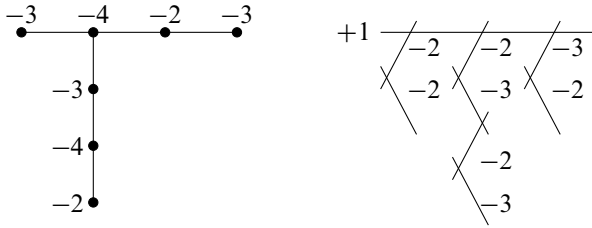
Figure 24: Plumbing graph Γ and its concave cap K.

the uppermost point among the intersection points to be blown up, there is no blowup at intersection points of K_{n_1-3,n_2-3,n_3-3} during the blowups from C_{n_1-3,n_2-3,n_3-3} to C. Therefore C_{n_1-3,n_2-3,n_3-3} is a desired curve configuration $C_{a,b,c}$. $\qquad\square$

Since $K_{a,b,c} \leq K$ (guaranteed by condition (ii) in Proposition 4.5), there is a curve configuration C_1 of Y obtained from $C_{a,b,c}$ by standard blowups. If one of a, b, c is -1, then the curve configuration C_1 is of type B or type C, so that there is a sequence of rational blowdowns from $\widetilde{M}$ to the minimal symplectic filling W_1 corresponding to C_1. If all $a, b, c \geq 0$, then W_1 is obtained from $\widetilde{M}$ by replacing $\Gamma_{a,b,c}$ with its rational homology ball filling by Lemma 4.4. On the other hand, conditions (ii) and (iii) in Proposition 4.5 guarantee that there is a sequence of rational blowdowns from W_1 to the minimal symplectic filling W corresponding to C by using Lemma 4.2 or Lemma 4.3 repeatedly.

We end this section by giving an example of minimal symplectic fillings involving 3–legged rational blowdown surgery.

Example 4.4 Let Y be a small Seifert 3–manifold whose minimal resolution graph Γ and concave cap K are given by Figure 24. We consider two minimal symplectic fillings W_1 and W_2 of Y whose respective curve configurations are given by Figure 25, top and bottom. Note that the curve configuration in Figure 25, top, is obtained from $C_{0,0,0}$ by standard blowups. Thus, as in the proof, W_1 is obtained from the minimal resolution by rationally blowing down $\Gamma_{0,0,0}$. Let us denote v_0 by a central vertex and v_i^j by ith vertex of the jth arm in Γ. Then, v_0, v_1^1, v_1^2 and $v_1^3 + v_2^3$ give a symplectic embedding of $\Gamma_{0,0,0}$ to the minimal resolution. A computation similar to that of Example 4.3 shows that there is a symplectic embedding L of

$$\overset{-5}{\bullet}\!\!-\!\!\overset{-2}{\bullet}$$

to W_1, and W_2 is obtained from W_1 by rationally blowing down L.
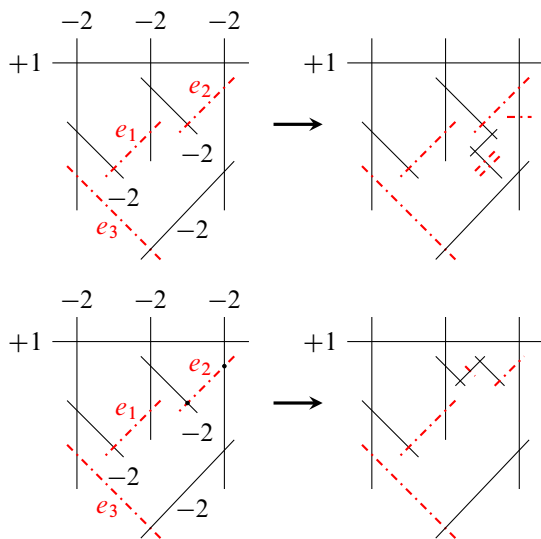
Figure 25: Top: curve configuration for W_1. Bottom: curve configuration for W_2.

References

[1] **M Bhupal**, **K Ono**, *Symplectic fillings of links of quotient surface singularities*, Nagoya Math. J. 207 (2012) 1–45 MR Zbl

[2] **M Bhupal**, **B Ozbagci**, *Symplectic fillings of lens spaces as Lefschetz fibrations*, J. Eur. Math. Soc. 18 (2016) 1515–1535 MR Zbl

[3] **M Bhupal**, **A I Stipsicz**, *Weighted homogeneous singularities and rational homology disk smoothings*, Amer. J. Math. 133 (2011) 1259–1297 MR Zbl

[4] **H Choi**, **J Park**, *A Lefschetz fibration on minimal symplectic fillings of a quotient surface singularity*, Math. Z. 295 (2020) 1183–1204 MR Zbl

[5] **R Fintushel**, **R J Stern**, *Rational blowdowns of smooth 4–manifolds*, J. Differential Geom. 46 (1997) 181–235 MR Zbl

[6] **D T Gay**, **A I Stipsicz**, *Symplectic rational blow-down along Seifert fibered 3–manifolds*, Int. Math. Res. Not. 2007 (2007) art. id. rnm084 MR Zbl

[7] **D T Gay**, **A I Stipsicz**, *Symplectic surgeries and normal surface singularities*, Algebr. Geom. Topol. 9 (2009) 2203–2223 MR Zbl

[8] **P Lisca**, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. 360 (2008) 765–799 MR Zbl

[9] **D McDuff**, *The structure of rational and ruled symplectic 4–manifolds*, J. Amer. Math. Soc. 3 (1990) 679–712 MR Zbl

[10] **P Orlik**, *Seifert manifolds*, Lecture Notes in Math. 291, Springer (1972) MR Zbl

[11] **P Orlik**, **P Wagreich**, *Isolated singularities of algebraic surfaces with C* action*, Ann. of Math. 93 (1971) 205–228 MR Zbl

[12] **H Park**, **J Park**, **D Shin**, **G Urzúa**, *Milnor fibers and symplectic fillings of quotient surface singularities*, Adv. Math. 329 (2018) 1156–1230 MR Zbl

[13] **H Park**, **A I Stipsicz**, *Smoothings of singularities and symplectic surgery*, J. Symplectic Geom. 12 (2014) 585–597 MR Zbl

[14] **J Park**, *Seiberg–Witten invariants of generalised rational blow-downs*, Bull. Austral. Math. Soc. 56 (1997) 363–384 MR Zbl

[15] **H Pinkham**, *Normal surface singularities with C* action*, Math. Ann. 227 (1977) 183–193 MR Zbl

[16] **L Starkston**, *Symplectic fillings of Seifert fibered spaces*, Trans. Amer. Math. Soc. 367 (2015) 5971–6016 MR Zbl

[17] **L Starkston**, *Comparing star surgery to rational blow-down*, J. Gökova Geom. Topol. GGT 10 (2016) 60–79 MR Zbl

[18] **A I Stipsicz**, **Z Szabó**, **J Wahl**, *Rational blowdowns and smoothings of surface singularities*, J. Topol. 1 (2008) 477–517 MR Zbl

[19] **M Symington**, *Symplectic rational blowdowns*, J. Differential Geom. 50 (1998) 505–518 MR Zbl

[20] **M Symington**, *Generalized symplectic rational blowdowns*, Algebr. Geom. Topol. 1 (2001) 503–518 MR Zbl

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