Algebraic & Geometric Topology

Volume 23 (2023)

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Algebraic & Geometric Topology **23**:8 (2023) 3805–3834
DOI: 10.2140/agt.2023.23.3805
Published: 5 November 2023

Cusps and commensurability classes of hyperbolic 4–manifolds

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There are six orientable compact flat 3–manifolds that can occur as cusp cross-sections of hyperbolic 4–manifolds. We provide criteria for exactly when a given commensurability class of arithmetic hyperbolic 4–manifolds contains a representative with a given cusp type. In particular, for three of the six cusp types, we provide infinitely many examples of commensurability classes that contain no manifolds with cusps of the given type; no such examples were previously known for any cusp type.

57M50; 11E20, 11F06, 16H05, 57K50

1 Introduction

Let $M = \mathbb{H}^n/\Gamma$ be a finite-volume noncompact hyperbolic n–manifold. A cusp of M is homeomorphic to $B \times \mathbb{R}^+$, where B is a compact flat $(n-1)$–manifold. If M is orientable, then B must be orientable. In [13], Long and Reid proved that every compact flat $(n-1)$–manifold, up to homeomorphism, must occur as a cusp cross-section of a hyperbolic n–orbifold; this result was upgraded from n–orbifolds to n–manifolds by McReynolds in [15]. Long and Reid [13] give a constructive algorithm which, given a compact flat $(n-1)$–manifold, outputs an arithmetic hyperbolic n–orbifold with a cusp with the specified cross-section. We discuss this algorithm in more detail in Section 5.

For ease of notation, we may refer to a cusp with cross-section B as a *cusp of type* B, as the cross-section of a cusp determines its homeomorphism class. We may also refer to a homeomorphism class of cusps, or "cusp type", by its cross-section. See Section 4 for a description of the six possible cusp types for hyperbolic 4–manifolds, and the names used below.

The above results tell us that each compact flat $(n-1)$–manifold occurs as a cusp of some hyperbolic n–manifold, but little is known about which conditions give rise to each cusp type. To investigate the occurrence of cusp types further, it makes sense to look at compact flat 3–manifolds in finite-volume hyperbolic 4–manifolds, as this is the lowest dimension in which multiple orientable cusp types can occur. It is well known that the 3–torus occurs as a cusp in every commensurability class of cusped hyperbolic 4–manifolds. Indeed, in every commensurability class of cusped hyperbolic 4–manifolds, manifolds where all cusp types are the 3–torus occur; see McReynolds, Reid, and Stover [16]. A striking result by Kolpakov and Martelli [10] showed that there exist one-cusped hyperbolic 4–manifolds having cusp type the 3–torus. Furthermore, Kolpakov and Slavich [11] showed that the $\frac{1}{2}$–twist also occurs as the cusp type of a one-cusped hyperbolic 4–manifold. On the other hand, the $\frac{1}{3}$–twist and $\frac{1}{6}$–twist have been obstructed from occurring as cusps of one-cusped manifolds; see Long and Reid [12]. Although it is as yet unknown whether the Hantzsche–Wendt manifold occurs as a cusp type of a one-cusped hyperbolic 4–manifold, it was shown by Ferrari, Kolpakov, and Slavich [9] that there exists a finite-volume hyperbolic 4–manifold where all cusp types are the Hantzsche–Wendt manifold. We also note that the isometry classes within each homeomorphism class that occur geometrically as cusps of hyperbolic 4–manifolds are dense in the moduli space of any compact flat 3–manifold; see Nimershiem [20].

We provide the first known examples of commensurability classes that avoid three cusp types. In fact, we provide infinitely many such examples, obtaining the result below. Furthermore, given any commensurability class C of cusped arithmetic hyperbolic 4–manifolds and any cusp type B, we give conditions on when C contains a manifold with a cusp of type B in Theorem 5.1. Notably, three cusp types occur in every such class. We refer to Section 2 for terminology used in Theorems 1.1 and 1.2.

Theorem 1.1 *Every commensurability class of arithmetic hyperbolic 4–manifolds contains manifolds with the 3–torus, the $\frac{1}{2}$–twist, and the Hantzsche–Wendt manifold as cusp types. There exist infinitely many commensurability classes C of hyperbolic 4–manifolds such that no manifold in C has a cusp of type $\frac{1}{3}$–twist. The same holds for cusps of type $\frac{1}{4}$–twist and $\frac{1}{6}$–twist.*

Additionally, we can use "inbreeding" of arithmetic hyperbolic 4–manifolds (see Agol [1]) to construct some nonarithmetic manifolds that avoid some cusp types, up to commensurability.

Theorem 1.2 *There exist infinitely many commensurability classes of finite-volume cusped nonarithmetic hyperbolic 4–manifolds that avoid each of the following cusp types: the $\frac{1}{3}$–twist, the $\frac{1}{4}$–twist, and the $\frac{1}{6}$–twist.*

We briefly review the organization of the paper. In Sections 2, 3, and 4, we provide preliminary information about quadratic forms, quaternion algebras, arithmetic hyperbolic manifolds, and the six orientable compact flat 3–manifolds that are the cusp types of orientable hyperbolic 4–manifolds. In Sections 5 and 6, we prove Theorem 1.1 and generalize it to give complete conditions on when a given commensurability class contains a manifold with a cusp of given type. In Section 7, we use this result to show that there are some commensurability classes of hyperbolic 5–manifolds that avoid some compact flat 4–manifold cusp types, and explain why we can't make the same argument in higher dimensions. In Section 8, we show that there are commensurability classes of nonarithmetic hyperbolic manifolds in both 4 and 5 dimensions that avoid certain cusp types as well, proving Theorem 1.2.

Acknowledgements The author wishes to thank his PhD advisor Alan Reid for his guidance and useful discussions. This paper is partially supported by NSF grant DMS-1745670.

2 Quadratic forms and quaternion algebras

2.1 Quadratic forms

Definition 2.1 (quadratic form) A *quadratic form* over a field K is a homogeneous polynomial of degree 2 with coefficients in K.

A quadratic form $q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ in n variables is said to have *rank* n, and can be written as an $n \times n$ symmetric matrix Q such that $q(x) = x^t Q x$. This can be accomplished by defining the entries by $Q_{ii} = a_{ii}$ and $Q_{ij} = \frac{1}{2} a_{ij}$ when $i \neq j$.

For any quadratic form q of rank n and ring R, we can define the orthogonal group $O(q, R)$ to be the group of all invertible $n \times n$ matrices A with entries in R such that $q(x) = q(Ax)$ for any $x \in R^n$. We can similarly define the special orthogonal group $\mathrm{SO}(q, R)$ to be the subgroup of $O(q, R)$ of matrices with determinant 1. Note that $\mathrm{SO}(q, \mathbb{R})$ is a Lie group, and thus has an identity component $\mathrm{SO}_0(q, \mathbb{R})$. Then, for any subring $R \subset \mathbb{R}$, we define $\mathrm{SO}_0(q, R) = \mathrm{SO}_0(q, \mathbb{R}) \cap \mathrm{SO}(q, R)$. Our focus is quadratic forms over $\mathbb{Q}$ and the corresponding groups $\mathrm{SO}_0(q, \mathbb{Z})$.

Definition 2.2 (rational equivalence) Quadratic forms given by symmetric matrices $Q_1, Q_2 \in \mathrm{GL}(n, \mathbb{Q})$ are *rationally equivalent* (or *equivalent over* $\mathbb{Q}$) if there exists $T \in \mathrm{GL}(n, \mathbb{Q})$ such that $T^t Q_1 T = Q_2$.

All quadratic forms over $\mathbb{Q}$ are rationally equivalent to a *diagonal* quadratic form, by which we mean a quadratic form whose corresponding matrix is diagonal. Thus, when working with a rational equivalence class of quadratic forms, we will always choose a diagonal representative. For ease of notation, we will denote diagonal quadratic forms $q(x) = \sum_{i=1}^n a_i x_i^2$ by writing their coefficients $\langle a_1, \dots, a_n \rangle$. Here all quadratic forms will be nondegenerate; that is, $a_i \neq 0$ for all i.

There is another relevant notion of equivalence, which is closely related to rational equivalence [18]:

Definition 2.3 (projective equivalence) Quadratic forms q_1 and q_2 are *projectively equivalent over* $\mathbb{Q}$, or just "projectively equivalent", if there are nonzero integers a and b such that aq_1 and bq_2 are rationally equivalent.

Let q_1 and q_2 be quadratic forms of odd rank with the same signature and discriminant. We can check for projective equivalence by scaling q_1 and q_2 so they have the same discriminant, and then checking for rational equivalence.

A complete set of invariants for diagonal quadratic forms up to rational equivalence is given by the signature, discriminant, and the Hasse–Witt invariants over all primes p. A quadratic form $q = \langle a_1, \dots, a_n \rangle$ of *signature* (a, b) has a positive coefficients and b negative coefficients. The *discriminant* $d \in \mathbb{Q}/(\mathbb{Q}^\times)^2$ is given by $d = \prod_{i=1}^n a_i$; note that it is defined only up to multiplication by squares. The Hasse–Witt invariants are a little harder to define, and contain the bulk of the number-theoretic information. For integers a and b and prime p, we first define the Hilbert symbol

$$(a, b)_p = \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a solution in } \mathbb{Q}_p, \\ -1 & \text{otherwise.} \end{cases}$$

Here $\mathbb{Q}_p$ denotes the p–adic field at p, or $\mathbb{R}$ if $p = \infty$.

Definition 2.4 (Hasse–Witt invariant) For a diagonal quadratic form $q = \langle a_1, \dots, a_n \rangle$ over $\mathbb{Q}$ and a prime p, possibly ∞, the *Hasse–Witt invariant of q at p* is given by

$$\epsilon_p(q) = \prod_{1 \le i < j \le n} (a_i, a_j)_p.$$

Every Hasse–Witt invariant must have value 1 or -1. There is a closed-form equation that allows us to easily compute a Hilbert symbol; thus, a Hasse–Witt invariant is easy to compute as well. Let $a = p^\alpha u$ and $b = p^\beta v$ with u and v both relatively prime to p in $\mathbb{Z}$. Then for $p > 2$

$$(a,b)_p = (-1)^{\alpha\beta\tau(p)}\left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha,$$

and, for $p = 2$,

$$(a,b)_p = (-1)^{\tau(u)\tau(v)+\alpha\omega(v)+\beta\omega(u)}.$$

Here we use the Legendre symbol and the functions $\tau(x) = \frac{1}{2}(x-1)$ and $\omega(x) = \frac{1}{8}(x^2-1)$, both of which only need to be defined modulo 2 [25, Chapter III, Theorem 1].

We can see from these equations that $(a,b)_p$ can only be -1 if either a or b is divisible by p an odd number of times. This means that $\epsilon_p(q) = 1$ for all primes p that don't occur as a factor of a coefficient of q. In particular, for any given quadratic form q, $\epsilon_p(q) = 1$ for all but finitely many values of p.

Additionally, Hilbert's reciprocity law states that the Hilbert symbols satisfy the identity $\prod_p (a,b)_p = 1$, where the product is taken over all places p of $\mathbb{Q}$, including $p = \infty$ [25, Chapter III, Theorem 3]. From this, we deduce the identity $\prod_p \epsilon_p(q) = 1$ for any quadratic form q. Since $(a,b)_\infty$ depends on the existence of a nonzero solution to $z^2 = ax^2 + by^2$ over the field $\mathbb{Q}_\infty = \mathbb{R}$, we know $(a,b)_\infty = -1$ if and only if both a and b are negative. We'll be working mostly with quadratic forms of signature $(4,1)$, so in this case $\prod_{1\le i<j\le n}(a_i,a_j)_\infty = 1$, as no pair (a_i,a_j) of distinct coefficients are both negative. As a result, the identity $\prod_p \epsilon_p(q) = 1$ holds when we consider only finite places p for quadratic forms of signature $(4,1)$.

2.2 Quaternion algebras

Definition 2.5 (quaternion algebra) A *quaternion algebra* over a field F with $\mathrm{char}(F) \neq 2$ is an algebra consisting of elements $w+xi+yj+zij$, with $w,x,y,z \in F$, equipped with relations $i^2 = a$, $j^2 = b$, and $ij = -ji$ for some fixed $a,b \in F$. We write this as $((a,b)/F)$.

Alternatively, a quaternion algebra Q over F is any central simple algebra of dimension 4 over F. Every such $Q = ((a,b)/F)$ has a *norm form*, given by $N(w+xi+yj+zij) = w^2 - ax^2 - by^2 + abz^2$, which is compatible with multiplication in Q.

The *pure quaternions* Q_0 of Q are the elements $w + xi + yj + zij$ with $w = 0$. Restricted to the pure quaternions, the norm form of Q (or, for short, the norm form of Q_0) becomes $N(xi + yj + zij) = -ax^2 - by^2 + abz^2$. Note that any quadratic form of rank 3 and discriminant 1 is rationally equivalent to such a form. To see this, observe that if $\langle a, b, c\rangle$ has discriminant 1 then $c = ab$ up to multiplication by a square. In particular, the quadratic form $\langle a, b, ab\rangle$ coincides with the norm form of $((-a, -b)/\mathbb{Q})_0$. We will make use of quadratic forms of signature $(4, 1)$ that are the direct sum of a positive definite norm form of some Q_0 and $\langle 1, -1\rangle$.

Definition 2.6 (quaternion type) A *quadratic form of quaternion type* is a quadratic form $q = \langle a, b, ab, 1, -1\rangle$ for some positive $a, b \in \mathbb{Z}$.

Lemma 2.7 *Every quadratic form q over $\mathbb{Q}$ of signature* $(4, 1)$ *is projectively equivalent to a quadratic form q' of quaternion type.*

In order to prove this lemma we'll need to use Conway's p–excesses, as described in [7, Chapter 15]. These will not appear in the rest of the paper, so readers not interested in the proof of this lemma may ignore these definitions.

Definition 2.8 (p–excess of rank-1 quadratic form) Let $p \neq 2$ be a prime, possibly ∞, and let $q = \langle a\rangle$ be a rank-1 quadratic form such that $a = p^k u$ with u relatively prime to p. If $p = \infty$, then let p^k be the sign of a and u its magnitude. Then we define the p–excess of q to be

$$e_p(q) \equiv \begin{cases} p^k + 3 \pmod 8 & \text{if } k \text{ is odd and } u \text{ is a quadratic nonresidue modulo p,} \\ p^k - 1 \pmod 8 & \text{otherwise.} \end{cases}$$

If $p = 2$, then

$$e_p(q) \equiv \begin{cases} -u - 3 \pmod 8 & \text{if k is odd and } u \equiv 3, 5 \pmod 8, \\ -u + 1 \pmod 8 & \text{otherwise.} \end{cases}$$

Definition 2.9 (p–excess of arbitrary quadratic form) Let p be a prime, possibly ∞, and let $q = \langle a_1, \dots, a_n\rangle$ be a diagonal quadratic form. Then we define the p–excess of q to be

$$e_p(q) \equiv \sum_{i=1}^{n} e_p(\langle a_i\rangle) \pmod 8.$$

The most notable properties of the p–excesses are that they are additive under direct sum of quadratic forms, and that they are invariant under rational equivalence. In fact, p–excesses are part of a complete invariant of quadratic forms up to rational equivalence, together with the signature and, in the case of forms of even rank, the discriminant

[7, Section 15.5.1, Theorem 3]. We can also extract the Hasse–Witt invariants of a quadratic form q from the discriminant d and p–excesses $e_p(q)$ [7, Section 15.5.3]:

$$\epsilon_p(q) = \begin{cases} 1 & \text{if } e_p(q) = e_p(\langle d, 1, \dots, 1\rangle), \\ -1 & \text{otherwise.} \end{cases}$$

To prove Lemma 2.7, we'll use the additivity of e_p to construct a rank-3 form q_3 of discriminant 1 such that $q_3 \oplus \langle 1, -1\rangle$ has certain desired Hasse–Witt invariants. We'll also use the following lemma, which can be found in greater generality in [25, Chapter IV, Proposition 7].

Lemma 2.10 *Let d, r, s, and n be integers, and ϵ_p be 1 or -1 for each prime p, including ∞. Then there exists a rank-n quadratic form q of discriminant d, signature (r, s), and Hasse–Witt invariants ϵ_p if and only if the following conditions are satisfied*:

(1) $\epsilon_p = 1$ *for almost all* p *and* $\prod \epsilon_p = 1$ *over all primes* p.
(2) $\epsilon_p = 1$ *if* $n = 1$, *or if* $n = 2$ *and the image of* d *in* $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ *is* -1.
(3) $r, s \geq 0$ *and* $n = r + s$.
(4) *The sign of* d *is equal to* $(-1)^s$.
(5) $\epsilon_\infty = (-1)^{s(s-1)/2}$.

Proof of Lemma 2.7 We can scale q to ensure it has discriminant -1 by multiplying the entire form by $-d$, where d is its discriminant. This will multiply the product of the terms by $-d^5$, and thus we'll obtain the new discriminant $-d^6 \equiv -1$. Note that scaling a form does not change its projective equivalence class.

Now, compute the p–excesses $e_p(q)$ and set $e'_p = e_p(q) - e_p(\langle 1, -1\rangle)$. By definition $e_p(\langle a_1, a_2, \dots, a_n\rangle) = \sum_{i=1}^n e_p(a_i)$, so if we can find a quadratic form q_3 of signature $(3, 0)$, discriminant 1, and p–excesses equal to e'_p, then $q' = q_3 \oplus \langle 1, -1\rangle$ will have p–excesses equal to those of q and discriminant -1. Since it will also have signature $(4, 1)$, q' will be rationally equivalent to q.

It suffices, then, to show that q_3 exists. Lemma 2.10 gives five conditions on the Hasse–Witt invariants, signature, and discriminant under which a quadratic form must exist. Conditions (2)–(4) hold trivially for q_3, either because they only apply to forms of rank 2 or less, or because they merely require that the signature is valid and agrees with the sign of the discriminant. This is true because the desired signature of q_3 is $(3, 0)$ and the discriminant is 1.

Conditions (1) and (5) concern the desired Hasse–Witt invariants ϵ'_p of q_3, which can be determined from the desired discriminant 1 and desired p–excesses e'_p. We will show that $\epsilon'_p = \epsilon_p(q)$ for all p, and thus that conditions (1) and (5) are satisfied.

Recall that $\epsilon'_p = 1$ if $e'_p = e_p(\langle d(q_3), 1, 1\rangle) = e_p(\langle 1, 1, 1\rangle)$, and -1 otherwise. We can similarly compute the Hasse–Witt invariants of q to be $\epsilon_p(q) = 1$ if and only if $e_p(q) = e_p(\langle -1, 1, 1, 1, 1\rangle)$. By construction, $e'_p = e_p(q) - e_p(\langle 1, -1\rangle)$. Then note that $e_p(\langle 1, 1, 1\rangle) = e_p(\langle -1, 1, 1, 1, 1\rangle) - e_p(\langle 1, -1\rangle)$ by additivity of p–excesses. Thus, $\epsilon'_p = \epsilon_p(q)$ for all p.

In particular, for any quadratic form q, $\epsilon_p(q) = 1$ for all but finitely many p, and $\prod \epsilon_p(q) = 1$ over all primes p. These same properties must hold for ϵ'_p, so condition (1) holds. Similarly, $\epsilon_\infty(q) = 1$ since q has signature $(4, 1)$, so $\epsilon'_\infty = 1$ as well, satisfying condition (5). Now we can apply Lemma 2.10 to deduce that a valid quadratic form q_3 exists with signature $(3, 0)$, discriminant 1, and Hasse–Witt invariants $\epsilon_p(q_3) = \epsilon'_p$.

As stated above, we can take the form $q' = q_3 \oplus \langle 1, -1\rangle$, which is rationally equivalent to q, has discriminant -1, and is of the form $\langle a, b, c, 1, -1\rangle$, where $q_3 = \langle a, b, c\rangle$. □

On the pure quaternions of any quaternion algebra, we can define the orthogonal group

$$O(N, Q_0) = \{f : Q_0 \to Q_0 \mid f \text{ is linear and } N(f(x)) = N(x) \text{ for all } x \in Q_0\}$$

as the set of linear transformations on Q_0 that preserve the norm form. These transformations can be described as conjugation by the units Q^* of Q. This is the intuition behind the following theorem from [14, Section 2.4]:

Theorem 2.11 *Let $Q = ((-a, -b)/\mathbb{Q})$ and $q = \langle a, b, ab\rangle$. Then $\mathrm{SO}(q, \mathbb{Q})$ is isomorphic to $Q^*/Z(Q^*)$, where $Z(G)$ denotes the center of G.*

There are three more theorems from [14] that are used in our argument. We state them here, along with a relevant definition, and remark that it will be important to obstruct certain torsion from occurring in $Q^*/Z(Q^*)$.

Definition 2.12 (ramification) A prime p ramifies a quaternion algebra Q over $\mathbb{Q}$ if $Q \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to the unique division algebra of dimension 4 over $\mathbb{Q}_p$. Otherwise, $Q \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to the algebra of 2×2 matrices $M_2(\mathbb{Q}_p)$, and we say Q splits over p.

Theorem 2.13 [14, Lemma 12.5.6] *Let ξ_n for $n > 2$ be a primitive n^{th} root of unity, and Q be a quaternion algebra over $\mathbb{Q}$. Then $Q^*/Z(Q^*)$ contains an element of order n if and only if $\xi_n + \xi_n^{-1} \in \mathbb{Q}$ and $\mathbb{Q}(\xi_n)$ embeds in Q.*

Theorem 2.14 [14, Theorem 7.3.3] *Given a quaternion algebra Q over $\mathbb{Q}$ and a quadratic extension L of $\mathbb{Q}$, then L embeds in Q if and only if, for each prime p that ramifies Q, p does not split in L.*

Theorem 2.15 [14, Theorem 2.6.6] *Let $p \neq 2, \infty$ be a prime in $\mathbb{Q}$. Consider the quaternion algebra $Q = ((a,b)/\mathbb{Q})$, with both a and b squarefree.*

(1) *If p does not divide a or b, then p does not ramify Q.*
(2) *If p divides a but not b, then p ramifies Q if and only if b is a quadratic nonresidue modulo p.*
(3) *If p divides both a and b, then p ramifies Q if and only if $-a^{-1}b$ is a quadratic nonresidue modulo p.*

3 Arithmetic hyperbolic manifolds

3.1 Hyperbolic manifolds

Let $q = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ be a quadratic form of signature $(n,1)$. We define hyperbolic space using the hyperboloid model $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid q(x) = -1, x_{n+1} > 0\}$, equipped with the metric derived from the inner product

$$x \circ y = \sqrt{x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}},$$

so that $(x \circ x)^2 = q(x)$. A hyperplane in $\mathbb{H}^n$ is an intersection of a subspace $V \subset \mathbb{R}^{n+1}$ with $\mathbb{H}^n$, and $\mathbb{H}^n$ has a boundary $\partial\mathbb{H}^n$ consisting of 1–dimensional subspaces of *light-like vectors* $y \in \mathbb{R}^{n+1}$ such that $q(y) = 0$. The isometries of $\mathbb{H}^n$ must preserve q, and in fact $\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO}_0(q, \mathbb{R})$.

Observe that we can perform this construction with any form q' of signature $(n,1)$ in place of q. The resulting space $\mathbb{H}^n_{q'}$ is isometric to $\mathbb{H}^n$, although both are different subsets of $\mathbb{R}^{n+1}$ and points in $\mathbb{Q}^{n+1}$ in one model may not correspond to points in $\mathbb{Q}^{n+1}$ in the other. Thus, $\mathrm{Isom}^+(\mathbb{H}^n)$ is isomorphic to $\mathrm{Isom}^+(\mathbb{H}^n_{q'})$. In particular, there is a linear transformation A that maps any $\mathbb{H}^n_{q'}$ to $\mathbb{H}^n$ isometrically, so any isometry $\gamma \in \mathrm{Isom}^+(\mathbb{H}^n_{q'})$ can be said to sit in $\mathrm{Isom}^+(\mathbb{H}^n)$ as $A\gamma A^{-1}$. We will sometimes abuse notation and refer to any $\mathbb{H}^n_{q'}$ as $\mathbb{H}^n$ when it is clear which quadratic form is being used.

We will use the notion of hyperplanes P sitting *rationally* inside $\mathbb{H}^n_q$. By this, we mean P is the intersection of $\mathbb{H}^n_q$ with a subspace $V \subset \mathbb{R}^{n+1}$ determined by a system of equations with rational coefficients. This notion depends on our choice of q, which in our case will always have coefficients in $\mathbb{Z}$.

A hyperbolic n–manifold is a quotient $\mathbb{H}^n/\,\Gamma$ of hyperbolic n–space by a discrete, torsion-free group Γ acting on $\mathbb{H}^n$ via isometries. If Γ is not torsion-free, a hyperbolic orbifold results instead. A cusp of a finite-volume hyperbolic n–manifold or orbifold is a subset of the manifold homeomorphic to $B \times \mathbb{R}^+$ for some cross-section B. Cusps result from the parabolic elements of Γ that fix a single point y of $\partial\mathbb{H}^n$. Specifically, since $\mathrm{Stab}_\Gamma(y)$ acts on a horosphere centered at y, which has a flat geometry, the cross-section of the corresponding cusp is given by $B = \mathbb{E}^{n-1}/\,\mathrm{Stab}_\Gamma(y)$. We consider only finite-volume hyperbolic manifolds, so B is compact. Furthermore, if $\mathbb{H}^n/\,\Gamma$ is orientable then so is B. For more information on cusps of hyperbolic manifolds and the *thick–thin decomposition* we refer the reader to [23, Chapter 12].

Definition 3.1 (commensurability) Two subgroups Γ_1 and Γ_2 of a group Γ are *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. Two hyperbolic orbifolds $\mathbb{H}^n/\,\Gamma_1$ and $\mathbb{H}^n/\,\Gamma_2$ are *commensurable* if $\gamma\Gamma_1\gamma^{-1}$ and Γ_2 are commensurable in $\mathrm{Isom}(\mathbb{H}^n)$ for some $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$.

Note that two orbifolds are commensurable if and only if they share a finite cover.

3.2 Arithmetic manifolds

Since we are working solely with cusped hyperbolic manifolds, all arithmetic hyperbolic manifolds in this paper are of *simplest type*. This allows us to use a simpler definition of arithmetic hyperbolic manifolds than the more involved general definition. This is stated, for example, in [19, Proposition 6.4.2] with the condition $n \neq 3, 7$, although this condition is unnecessary.

Definition 3.2 (arithmetic hyperbolic orbifold/arithmetic group) Let M be a finite-volume cusped hyperbolic n–orbifold with $\pi_1(M) = \Gamma < \mathrm{Isom}(\mathbb{H}^n)$. Then M is arithmetic if there exists a quadratic form q of signature $(n, 1)$ such that $A^{-1}\Gamma A < \mathrm{Isom}(\mathbb{H}^n_q)$ is commensurable to $\mathrm{SO}_0(q, \mathbb{Z})$, where A is the linear transformation that maps $\mathbb{H}^n_q$ to $\mathbb{H}^n$ isometrically. We say Γ is arithmetic under the same condition, that is, when Γ is conjugate to a subgroup of $\mathrm{Isom}^+(\mathbb{H}^n_q)$ commensurable to $\mathrm{SO}_0(q, \mathbb{Z})$.

A hyperbolic arithmetic n–manifold is a hyperbolic arithmetic n–orbifold that is also a hyperbolic manifold. Henceforth we may refer to the arithmetic orbifold $\mathbb{H}^n_q/\mathrm{SO}_0(q, \mathbb{Z})$ as $\mathbb{H}^n/\mathrm{SO}_0(q, \mathbb{Z})$ using this particular embedding, without ambiguity.

To any cusped arithmetic hyperbolic n–orbifold M we can associate the (nonunique) quadratic form q from the definition. There are easily checkable conditions on quadratic forms q_1 and q_2 that determine whether $\Gamma_1 = \mathrm{SO}_0(q_1, \mathbb{Z})$ and $\Gamma_2 = \mathrm{SO}_0(q_2, \mathbb{Z})$ are commensurable as subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, identifying both $\mathrm{Isom}(\mathbb{H}^n_{q_1})$ and $\mathrm{Isom}(\mathbb{H}^n_{q_2})$ with $\mathrm{Isom}(\mathbb{H}^n)$, and are thus associated to the same orbifolds.

Proposition 3.3 [18, Theorem 1] *Let M_1 and M_2 be arithmetic hyperbolic orbifolds with associated quadratic forms q_1 and q_2, respectively. Then M_1 and M_2 are commensurable if and only if q_1 and q_2 are projectively equivalent.*

One way to determine whether two quadratic forms q_1 and q_2 of signature $(4, 1)$ are projectively equivalent is to scale both so they have the same discriminant, and then compare Hasse–Witt invariants. In particular, since such forms have odd rank, if q_i has discriminant $-d_i$ then the form $d_i q_i$ must have discriminant -1. Thus, we can deal with rational equivalence rather than projective equivalence by associating to a commensurability class of arithmetic hyperbolic 4–manifolds a (nonunique) quadratic form q of discriminant -1. Furthermore, by Lemma 2.7 we can take q to be of quaternion type. We summarize this discussion:

Corollary 3.4 *Every commensurability class C of cusped arithmetic hyperbolic 4–orbifolds has an associated quadratic form q of quaternion type such that $\mathbb{H}^4/\mathrm{SO}_0(q, \mathbb{Z})$ lies in C.*

3.3 Systoles

Definition 3.5 (systole length) The systole length of a manifold M is the minimal length of a closed geodesic in M.

The arithmetic n–manifolds we deal with have a minimum bound on the systole length. The following proposition is an application of Corollary 1.3 or 1.8 from [8], depending on whether n is even or odd:

Proposition 3.6 *There is a lower bound on the systole length of a cusped arithmetic hyperbolic 4–manifold.*

We will use this fact to show that certain finite-volume hyperbolic n–manifolds are nonarithmetic.

4 Compact flat 3–manifolds

Recall from Section 3.1 that finite-volume cusped hyperbolic n–manifolds $M = \mathbb{H}^n/\Gamma$ have compact flat $(n-1)$–manifolds B for the cross-sections of their cusps, and if M

M	$\pi_1(M)$	$\mathrm{Hol}(\pi_1(M))$
3–torus	$\mathbb{Z}^3 = \langle t_1, t_2, t_3 \mid t_i t_j = t_j t_i \rangle$	**1**
$\frac{1}{2}$–twist	$\langle \alpha, t_1, t_2, t_3 \mid t_i t_j = t_j t_i, \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}, \alpha t_3 \alpha^{-1} = t_3^{-1} \rangle$	$\mathbb{Z}/2\mathbb{Z}$
H–W	$\langle x, y, z \mid xy^2x^{-1}y^2 = 1, yx^2y^{-1}x^2 = 1, xyz = 1 \rangle$ [4]	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\frac{1}{3}$–twist	$\langle \alpha, t_1, t_2, t_3 \mid t_i t_j = t_j t_i, \alpha^3 = t_1, \alpha t_2 \alpha^{-1} = t_3, \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1} \rangle$	$\mathbb{Z}/3\mathbb{Z}$
$\frac{1}{4}$–twist	$\langle \alpha, t_1, t_2, t_3 \mid t_i t_j = t_j t_i, \alpha^4 = t_1, \alpha t_2 \alpha^{-1} = t_3, \alpha t_3 \alpha^{-1} = t_2^{-1} \rangle$	$\mathbb{Z}/4\mathbb{Z}$
$\frac{1}{6}$–twist	$\langle \alpha, t_1, t_2, t_3 \mid t_i t_j = t_j t_i, \alpha^6 = t_1, \alpha t_2 \alpha^{-1} = t_3, \alpha t_3 \alpha^{-1} = t_2^{-1} t_3 \rangle$	$\mathbb{Z}/6\mathbb{Z}$

Table 1: The six orientable compact flat 3–manifolds [21].

is orientable then so is B. Considering only orientable manifolds, this means that hyperbolic 2– and 3–manifolds only have one type of cusp cross-section each: S^1 and T^2, respectively. However, there are six orientable compact flat 3–manifolds up to homeomorphism, which means there are six possible cusp cross-sections for an orientable finite-volume hyperbolic 4–manifold. We give a brief description of each in Table 1 and Figure 1.

In the images depicting the fundamental domains, a face without a label is paired with its opposite face via translation, and labeled faces are paired in such a way that the labels align. Note that all but the Hantzsche–Wendt manifold differ from the 3–torus by at most a twist on one of the face pairings. All six flat manifolds are commensurable, and are in fact finitely covered by the 3–torus.

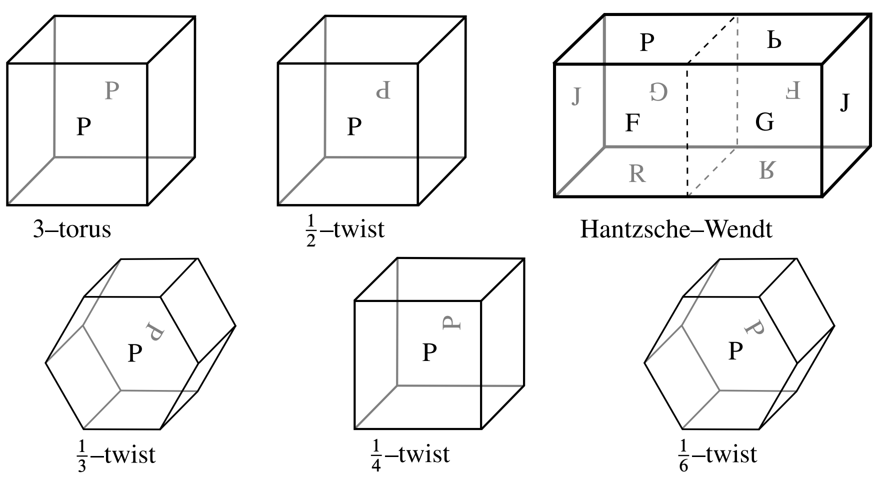

Figure 1: The fundamental domains for the manifolds in Table 1.

Every isometry of Euclidean 3–space $\mathbb{E}^3$ is an affine transformation $v \mapsto Av + w$ for some $A \in \mathrm{SO}(3)$. For a group $G < \mathrm{Isom}(\mathbb{E}^3)$, the holonomy of G is given by

$$\mathrm{Hol}(G) = \{A \in \mathrm{SO}(3) \mid (v \mapsto Av + w) \in G \text{ for some } w \in \mathbb{R}^3\}.$$

$\mathrm{Hol}(G)$ is independent of the faithful representation of G into $\mathrm{Isom}(\mathbb{E}^3)$.

5 Classes with a given cusp

One goal of the next two sections is to prove Theorem 1.1. In fact, we generalize Theorem 1.1 to a full description of exactly when a commensurability class of cusped arithmetic hyperbolic 4–manifolds contains a manifold with a given cusp type.

Theorem 5.1 *Let C be a commensurability class of cusped arithmetic hyperbolic 4–manifolds, with associated quadratic form q, scaled so that the discriminant of q is -1. Then:*

- *C must contain a manifold with a 3–torus cusp, a manifold with a $\frac{1}{2}$–twist cusp, and a manifold with a Hantzsche–Wendt cusp.*
- *C contains a manifold with a $\frac{1}{4}$–twist cusp if and only if $\epsilon_p(q) = 1$ for all $p \equiv 1 \pmod 4$.*
- *C contains a manifold with a $\frac{1}{3}$–twist cusp if and only if $\epsilon_p(q) = 1$ for all $p \equiv 1 \pmod 3$. C contains a manifold with a $\frac{1}{6}$–twist cusp under the same condition.*

In this section, we prove the positive portion of the theorem, namely that C does indeed contain certain cusp types.

Proposition 5.2 *Let C be a commensurability class of arithmetic hyperbolic 4–manifolds, with associated quadratic form q of discriminant -1. Then:*

- *C must contain a manifold with a 3–torus cusp, a manifold with a $\frac{1}{2}$–twist cusp, and a manifold with a Hantzsche–Wendt cusp.*
- *If $\epsilon_p(q) = 1$ for all $p \equiv 1 \pmod 4$, then C contains a manifold with a $\frac{1}{4}$–twist cusp.*
- *If $\epsilon_p(q) = 1$ for all $p \equiv 1 \pmod 3$, then C contains a manifold with a $\frac{1}{3}$–twist cusp and a manifold with a $\frac{1}{6}$–twist cusp.*

Our primary tool for showing that a commensurability class must contain a given cusp type is the algorithm given by Long and Reid [13]. Given a compact flat n–manifold B, this algorithm yields an arithmetic hyperbolic $(n+1)$–orbifold with a cusp of type B. We can always find an $(n+1)$–manifold with a cusp of type B covering this orbifold by [15].

Given a cusp type B of dimension n, the algorithm works as follows. Consider the holonomy group of $\pi_1(B)$. We can find a faithful representation of $\mathrm{Hol}(\pi_1(B))$ into $\mathrm{GL}(n,\mathbb{Z})$, which yields an embedding $\mathrm{Hol}(\pi_1(B)) \subset \mathrm{GL}(n,\mathbb{Z})$. Further, we can choose a signature-$(n,0)$ quadratic form q_n that is invariant under $\mathrm{Hol}(\pi_1(B))$ by considering an arbitrary signature-$(n,0)$ quadratic form r and taking the average of all the quadratic forms $r \circ A$ over $A \in \mathrm{Hol}(\pi_1(B))$, since $\mathrm{Hol}(\pi_1(B))$ is finite. Then, using linear algebra, the algorithm extends the representation into $\mathrm{GL}(n+2,\mathbb{Z})$ in such a way that $\mathrm{Hol}(\pi_1(B))$ leaves a quadratic form q' rationally equivalent to $q_n \oplus \langle 1,-1\rangle$ invariant. As a result, we see that some cover of $\mathbb{H}^{n+1}/\mathrm{SO}_0(q',\mathbb{Z})$ must contain a cusp of type B, and is commensurable to $\mathbb{H}^{n+1}/\mathrm{SO}_0(q_n \oplus \langle 1,-1\rangle,\mathbb{Z})$.

By investigating properties of quadratic forms q_n invariant under $\mathrm{Hol}(\pi_1(B))$, we characterize the commensurability classes of arithmetic hyperbolic manifolds that can be output by this algorithm. Since we're working with flat 3–manifolds and hyperbolic 4–manifolds, we apply the algorithm with $n=3$.

Proof of Proposition 5.2 Given the commensurability class C, we can choose a quadratic form $q = \langle x, y, xy, 1, -1\rangle$ of quaternion type such that $\mathbb{H}^4/\mathrm{SO}_0(q,\mathbb{Z}) \in C$ by Lemma 2.7. Note that q has discriminant -1. We can compute the Hasse–Witt invariants $\epsilon_p(q)$.

First let B be the 3–torus, $\frac{1}{2}$–twist, or Hantzsche–Wendt manifold. These have holonomy groups of 1, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, respectively. Each holonomy group has a representation into $\mathrm{GL}(3,\mathbb{R})$ consisting solely of diagonal matrices with ± 1 along the diagonal. In particular, these representations fix any quadratic form $\langle a, b, ab\rangle$ of rank 3. Thus, we can apply the Long–Reid algorithm to find a representation of the corresponding Bieberbach group into $\mathrm{SO}_0(\langle a, b, ab, k, -k\rangle, \mathbb{Z})$. Set $a = x$ and $b = y$. Then $\langle a, b, ab, k, -k\rangle$ is rationally equivalent to $\langle a, b, ab, 1, -1\rangle = \langle x, y, xy, 1, -1\rangle$. This yields an orbifold commensurable to $\mathbb{H}^4/\mathrm{SO}_0(q,\mathbb{Z})$ that has the desired cusp type. By [15], there is also a manifold with the desired cusp type.

Next, consider the $\frac{1}{4}$–twist cusp. This flat manifold has holonomy group $\mathbb{Z}/4\mathbb{Z}$, and has a representation ρ into $\mathrm{SL}(3,\mathbb{Z})$ mapping its generator g_4 to

$$\rho(g_4) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This holonomy preserves any quadratic form $q_3 = \langle a, a, b\rangle$, so the Long–Reid algorithm finds a representation of B into $\mathrm{SO}_0(\langle a, a, b, k, -k\rangle, \mathbb{Z})$, which is commensurable to $\mathrm{SO}_0(\langle ab, ab, 1, 1, -1\rangle, \mathbb{Z})$. The Hasse–Witt invariant at p of the form $q' = \langle ab, ab, 1, 1, -1\rangle$ is equal to the Hilbert symbol $(ab, ab)_p$. Let $ab = up^{\alpha}$, where u is an integer not divisible by p. By definition, for $p > 2$ and $\tau(p) = \frac{1}{2}(p-1)$,

$$(ab, ab)_p = (-1)^{\tau(p)\alpha\alpha}\left(\frac{u}{p}\right)^{\alpha}\left(\frac{u}{p}\right)^{\alpha} = (-1)^{\tau(p)\alpha}.$$

Note that $\tau(p)$ is even if $p \equiv 1 \pmod 4$ and odd if $p \equiv 3 \pmod 4$.

So if $p \equiv 1 \pmod 4$, we always have $\epsilon_p(q') = (ab, ab)_p = 1$. But if $p \equiv 3 \pmod 4$, then $\epsilon_p(q') = -1$ if and only if p divides ab an odd number of times. Given the finite set of primes $p_i > 2$ such that $\epsilon_p(q) = -1$, as long as there is no p_i such that $p_i \equiv 1 \pmod 4$, we can now ensure that there is a quadratic form $q'' = \left\langle \prod p_i, \prod p_i, 1, 1, -1\right\rangle$ such that $\epsilon_p(q'') = \epsilon_p(q)$. Note that the identity $\prod \epsilon_p(q) = 1$ ensures that $\epsilon_2(q'') = \epsilon_2(q)$ as well. Thus q'' and q both have the same Hasse–Witt invariants, as well as discriminant -1 and signature $(4, 1)$. Hence q'' is rationally equivalent to q and, taking $ab = \prod p_i$, we see that $\mathbb{H}^4/\mathrm{SO}_0(q'', \mathbb{Z})$ must have a finite cover with a $\frac{1}{4}$–twist cusp. Thus we can construct a manifold in C with a $\frac{1}{4}$–twist cusp.

The arguments for the $\frac{1}{3}$–twist and the $\frac{1}{6}$–twist cusps are similar. The holonomy groups $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ have representations ρ_3 and ρ_6 into $\mathrm{SL}(3, \mathbb{Z})$ mapping the respective generators g_3 and g_6 as

$$\rho_3(g_3) = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_6(g_6) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Under this representation, both holonomy groups preserve quadratic forms of the form $q'(x) = 4ax_1^2 + 4ax_2^2 - 4ax_1x_2 + 3bx_3^2$. With some effort, we can show that this form is projectively equivalent to $q'' = \langle ab, 3ab, 3, 1, -1\rangle$. We then compute that $\epsilon_p(q'') = (ab, 3ab)_p(3, 3)_p$. The second Hilbert symbol $(3, 3)_p$ is equal to -1 at $p = 2, 3$ and equal to 1 everywhere else. To compute the first Hilbert symbol $(ab, 3ab)_p$, we consider the case $p = 3$ separately from $p \neq 2, 3$. (We'll ignore $p = 2$ for now since the identity $\prod \epsilon_p(q) = \prod \epsilon_p(q'') = 1$ will ensure that $\epsilon_2(q'') = \epsilon_2(q)$ if all other Hasse–Witt invariants are equal.)

For $p = 3$, suppose $ab = 3^{\alpha}u$ where u is not divisible by 3. Then $3ab = 3^{\alpha+1}u$, so

$$(ab, 3ab)_3 = (-1)^{\alpha(\alpha+1)\tau(3)}\left(\frac{u}{3}\right)^{\alpha}\left(\frac{u}{3}\right)^{\alpha+1} = \left(\frac{u}{3}\right).$$

Thus $(ab, 3ab)_3 = 1$ if $u \equiv 1 \pmod 3$ and -1 if $u \equiv 2 \pmod 3$.

For $p \neq 3$, let $ab = p^\alpha u$ where u is not divisible by p, so that $3ab = p^\alpha(3u)$. Then

$$(ab, 3ab)_p = (-1)^{\alpha\alpha\tau(p)}\left(\frac{u}{p}\right)^\alpha \left(\frac{3u}{p}\right)^\alpha = \left((-1)^{\tau(p)}\left(\frac{3}{p}\right)\right)^\alpha.$$

Combining $(-1)^{\tau(p)}$ with quadratic reciprocity, we can see that, for $p > 2$, $(ab, 3ab)_p = 1^\alpha$ if $p \equiv 1 \pmod 3$ and $(-1)^\alpha$ if $p \equiv 2 \pmod 3$. Consider the finite set of primes $p_i > 2$ such that $\epsilon_p(q) = -1$. As long as there is no p_i such that $p_i \equiv 1 \pmod 3$, we can take $ab = \prod p_i$ over all $p_i \equiv 2 \pmod 3$. Additionally, we can multiply ab by 2 if necessary to set $ab \equiv 1$ or $2 \pmod 3$ to obtain the desired value of $\epsilon_3(q'')$. Now $\epsilon_p(q) = \epsilon_p(q'')$ for all $p > 2$ and, as before, $\epsilon_2(q) = \epsilon_2(q'')$ due to the identity $\prod \epsilon_p(q) = \prod \epsilon_p(q'') = 1$. Now q'' is rationally equivalent to q, and from $\mathrm{SO}_0(q'', \mathbb{Z})$ we can construct a manifold in C with a $\frac{1}{3}$–twist or $\frac{1}{6}$–twist cusp, as desired. □

Remark 5.3 In addition to the six orientable compact flat 3–manifolds, there are four nonorientable ones: two double-covered by the 3–torus and two double-covered by the $\frac{1}{2}$–twist. A thorough description of these manifolds can be found in [6]. Notably, all of them have holonomies generated by orthogonal reflections. In particular, this means each fundamental group has a holonomy representation into $\mathrm{GL}(3, \mathbb{R})$ with image consisting of diagonal matrices with ± 1 along the diagonal. Thus, for the same reasons as the 3–torus, $\frac{1}{2}$–twist, and Hantzsche–Wendt manifold, all four nonorientable compact flat 3–manifolds occur as a cusp cross-section in every commensurability class of arithmetic hyperbolic 4–manifolds.

6 Classes without a given cusp

The goal of this section is to prove the negative part of Theorem 5.1, that is, to obstruct some cusp types from occurring in some commensurability classes of hyperbolic 4–manifolds. This obstruction will yield infinitely many commensurability classes that avoid each of the $\frac{1}{3}$–twist, $\frac{1}{6}$–twist, and $\frac{1}{4}$–twist.

Proposition 6.1 *Let C be a commensurability class of arithmetic hyperbolic 4–manifolds, with associated quadratic form q with discriminant -1. Then:*

- *If $\epsilon_p(q) \neq 1$ for some $p \equiv 1 \pmod 4$, then C does not contain a manifold with a $\frac{1}{4}$–twist cusp.*
- *If $\epsilon_p(q) \neq 1$ for some $p \equiv 1 \pmod 3$, then C contains neither a manifold with a $\frac{1}{3}$–twist cusp, nor a manifold with a $\frac{1}{6}$–twist cusp.*

Proof By Lemma 2.7, we can take q to be of quaternion form. Thus, without loss of generality, we can set $q = \langle a, b, ab, 1, -1\rangle$ for some positive integers a and b.

Let B be the cusp type that we want to obstruct, and let $\Delta = \pi_1(B)$. We will show that it suffices to obstruct the existence of an injective homomorphism $\Delta \to \mathrm{SO}_0(q, \mathbb{Q})$.

For the sake of contradiction, suppose C contains a manifold M with the cusp type in question. This yields an embedding $\Delta \to \pi_1(M) = \Gamma$. Because Γ is an arithmetic lattice in $\mathrm{SO}(4,1)$, we know that Γ lies in the $\mathbb{Q}$–points of some quadratic form q' [3]. Because $M \in C$, q and q' are projectively equivalent. Thus by Proposition 3.3, there exists a matrix $F \in \mathrm{GL}(5, \mathbb{Q})$ such that $F\Delta F^{-1}$ is commensurable with $\mathrm{SO}_0(q, \mathbb{Z})$ and embeds into $\mathrm{SO}_0(q, \mathbb{Q})$. Note that Δ acts on a horosphere centered at some point y in $\partial\mathbb{H}^4$.

Since y is fixed by isometries that lie in $\mathrm{SO}_0(q, \mathbb{Z})$, we can take y itself to lie in $\mathbb{Q}^5$. Additionally, since $\mathrm{SO}_0(q, \mathbb{Q})$ acts transitively on the rational points of $\partial\mathbb{H}^4_q$, we can choose y to be $(0,0,0,1,1)$ without loss of generality. Specifically, we can conjugate the image of Δ by some matrix $A' \in \mathrm{SO}_0(q, \mathbb{Q})$ such that $A'y = y_0 = (0,0,0,1,1)$ to get a new rational representation of Δ acting on a horosphere H centered at y_0.

Let $q_3 = \langle a, b, ab\rangle$ be the quadratic form such that $q_3 \oplus \langle 1, -1\rangle = q$. Given any affine transformation $\varphi \in \mathrm{Isom}(\mathbb{E}^3)$, we can write the isometry as $\varphi(v) = Av + w$, with $A \in \mathrm{SO}_0(q_3, \mathbb{R})$. Then we can map φ to an action $\rho(\varphi)$ on H by taking ρ to be induced by an isometry from $\mathbb{E}^3$ to H. Imitating [22], we can write ρ as

$$\rho(\varphi)\colon v \mapsto \begin{bmatrix} A & w & -w \\ f(w)^t A & 1 + \frac{1}{2}q_3(w) & -\frac{1}{2}q_3(w) \\ f(w)^t A & \frac{1}{2}q_3(w) & 1 - \frac{1}{2}q_3(w) \end{bmatrix} v.$$

Here f is the linear function $f(x) = (ax_1, bx_2, abx_3)^t$ such that $f(x)^t x = q_3(x)$ for any $x \in \mathbb{R}^3$. Since one can recover A from the top left and $-w$ from the top right of $\rho(\varphi)$, we see that ρ must be injective. One can check through manual calculation that ρ is a homomorphism, and that all elements in $\rho(\mathrm{Isom}(\mathbb{E}^3))$ preserve both q and y_0, and thus act on H. All isometries of H must be of the form $\rho(\varphi)$ above for some $\varphi \in \mathrm{Isom}(\mathbb{E}^3)$, so in particular, every element of $\rho(\Delta)$ has this form.

If Δ is the fundamental group of the $\frac{1}{3}$–twist, $\frac{1}{4}$–twist, or $\frac{1}{6}$–twist cusp, it has holonomy group $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/6\mathbb{Z}$, respectively. The holonomy is represented by the matrix A above, so in order to embed Δ into $\mathrm{SO}_0(q, \mathbb{Q})$ there must exist an isometry φ with A that is 3–torsion or 4–torsion. Since A is a submatrix of $\rho(\varphi)$, which has rational entries, it must have rational entries. Thus, if we can obstruct 3–torsion or 4–torsion from $\mathrm{SO}_0(q_3, \mathbb{Q})$, then we can obstruct the existence of an embedding $\Delta \to \mathrm{SO}_0(q, \mathbb{Q})$.

Now consider the quaternion algebra $Q = ((-a,-b)/\mathbb{Q})$. The norm form of Q_0 is given by $ax_1^2 + bx_2^2 + abx_3^2 = q_3(x)$, so by Theorem 2.11, $\mathrm{SO}(q_3,\mathbb{Q})$ is isomorphic to $Q^*/Z(Q^*)$. Thus, if we obstruct torsion of some degree from appearing in $Q^*/Z(Q^*)$, then we obstruct it from $\mathrm{SO}_0(q_3,\mathbb{Q}) < \mathrm{SO}(q_3,\mathbb{Q})$ as well.

Now we apply Theorem 2.13. For $n = 3$ and $n = 4$, clearly $\xi_n + \xi_n^{-1} \in \mathbb{Q}$. So there are no order-$n$ elements of $Q^*/Z(Q^*)$ if and only if $\mathbb{Q}(\xi_n)$ does not embed in Q. Furthermore, by Theorem 2.14, the field $\mathbb{Q}(\xi_n)$ embeds in Q if and only if $\mathbb{Q}(\xi_n) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field for each $p \in \mathrm{Ram}(Q)$. The latter occurs exactly when p does not split in $\mathbb{Q}(\xi_n)$. Thus, in order to obstruct n–torsion, we wish to show there is some $p \in \mathrm{Ram}(Q)$ such that p splits in $\mathbb{Q}(\xi_n)$.

To check this condition, we must first determine when $p \in \mathrm{Ram}(Q)$. If neither $-a$ nor $-b$ is divisible by p an odd number of times, then p does not ramify by Theorem 2.15(1). Note that if both $-a$ and $-b$ are divisible by p an odd number of times, then ab is not. Since a, b, and ab are interchangeable when constructing Q, in this case we can pass to $Q' = ((-a,-ab)/\mathbb{Q})$ to ensure that p divides only one of $-a$ and $-b$ an odd number of times. Without loss of generality, say p divides $-a$ but not $-b$. Then, by Theorem 2.15(2), p ramifies if and only if b is a nonsquare modulo p.

We claim that p ramifies over Q exactly when the Hasse–Witt invariant $\epsilon_p(q)$ equals -1. Using the definitions of the Hasse–Witt invariant and the Hilbert symbol, we can expand $\epsilon_p(q)$. Let $a = p^\alpha j$ and $b = p^\beta k$ with j and k relatively prime to p. Then $ab = p^{\alpha+\beta} jk$, so

$$\begin{aligned}
\epsilon_p(q) &= \epsilon_p(\langle a,b,ab,1,-1\rangle) = (a,b)_p(ab,ab)_p \\
&= \left[(-1)^{\alpha\beta\tau(p)}\left(\frac{j}{p}\right)^{\beta}\left(\frac{k}{p}\right)^{\alpha}\right]\left[(-1)^{(\alpha+\beta)(\alpha+\beta)\tau(p)}\left(\frac{jk}{p}\right)^{\alpha+\beta}\left(\frac{jk}{p}\right)^{\alpha+\beta}\right] \\
&= (-1)^{\tau(p)(\alpha\beta+\alpha+\beta)}\left(\frac{j}{p}\right)^{\beta}\left(\frac{k}{p}\right)^{\alpha}.
\end{aligned}$$

If both α and β are even, then $\epsilon_p(q) = (-1)^0(j/p)^0(k/p)^0 = 1$. As shown above, p does not ramify over Q in this case.

If both α and β are odd, then we can choose to use $Q' = ((-a,-ab)/\mathbb{Q})$ as before. So, unless both α and β are even, without loss of generality we can assume α is odd and β is even. Then $\epsilon_p(q) = (-1)^{\tau(p)}(k/p)$. Note that $(-1/p)$ is 1 when $p \equiv 1 \pmod 4$ and -1 when $p \equiv 3 \pmod 4$, so $(-1)^{\tau(p)} = (-1/p)$. Thus, since $b = p^\beta k$ with β even, we have $\epsilon_p(q) = (-1/p)(k/p) = (-k/p) = (-b/p)$. We already showed that

p ramifies over Q exactly when $-b$ is a nonsquare modulo p in this case, which is equivalent to the condition $\epsilon_p(q) = (-b/p) = -1$. Now, in all cases, p ramifies over Q exactly when $\epsilon_p(q) = -1$.

Next, we investigate when p splits in $\mathbb{Q}(\xi_n)$. When $n = 3$ or 6 we have $\mathbb{Q}(\xi_n) = \mathbb{Q}(\sqrt{-3})$, and if $n = 4$ we have $\mathbb{Q}(\xi_n) = \mathbb{Q}(\sqrt{-1})$. It is well known that p splits in $\mathbb{Q}(\sqrt{a})$ if and only if a is a quadratic residue modulo p, so p splits in $\mathbb{Q}(\sqrt{-3})$ exactly when $p \equiv 1 \pmod 3$ and in $\mathbb{Q}(\sqrt{-1})$ exactly when $p \equiv 1 \pmod 4$.

Now, suppose there is some prime p such that $\epsilon_p(q) = -1$ and $p \equiv 1 \pmod 4$. Then, p ramifies over Q and p splits in $\mathbb{Q}(\xi_n)$. Thus, as stated above, $\mathrm{SO}_0(q_3, \mathbb{Q}) \cong Q^*/Z(Q^*)$ has no 4–torsion. As a result, the $\frac{1}{4}$–twist group B cannot possibly embed into $\mathrm{SO}_0(q, \mathbb{Q})$, so there is no $\frac{1}{4}$–twist cusp in the associated commensurability class of hyperbolic 4–manifolds. In fact, this same argument suffices to show there are no $\frac{1}{4}$–twist cusps in the class of orbifolds, either.

By similar logic, we can also see that if there is a prime p such that $\epsilon_p(q) = -1$ and $p \equiv 1 \pmod 3$, then there is no 3–torsion in $\mathrm{SO}_0(q_3, \mathbb{Q}) \cong Q^*/Z(Q^*)$. Thus the commensurability class of hyperbolic 4–manifolds (or orbifolds) associated to q must avoid $\frac{1}{3}$–twist cusps and $\frac{1}{6}$–twist cusps. □

Between Propositions 5.2 and 6.1, we've exhausted all possible commensurability classes for each cusp type. This suffices to prove Theorem 5.1. Theorem 1.1 follows.

Example 6.2 For $q_6 = \langle 1, 1, 7, 7, -1 \rangle$ the commensurability class of $\mathbb{H}^4/\mathrm{SO}_0(q_6, \mathbb{Z})$ avoids the $\frac{1}{3}$–twist and $\frac{1}{6}$–twist, since $\epsilon_7(q_6) = -1$ and $7 \equiv 1 \pmod 3$.

Example 6.3 For $q_4 = \langle 1, 2, 5, 10, -1 \rangle$ the commensurability class of $\mathbb{H}^4/\mathrm{SO}_0(q_4, \mathbb{Z})$ avoids the $\frac{1}{4}$–twist, since $\epsilon_5(q_4) = -1$ and $5 \equiv 1 \pmod 4$.

7 Obstructions in higher dimensions

Using Theorem 5.1, we can prove a version of Theorem 1.1 one dimension higher. Namely, some commensurability classes of hyperbolic 5–manifolds avoid some cusp types associated to flat 4–manifolds. Our strategy will be to show that an arithmetic hyperbolic 5–manifold with cusp $B \times S^1$ must contain a 4–dimensional totally geodesic submanifold with cusp B, and then manipulate Hasse–Witt invariants to show that, sometimes, no such submanifold can contain B as a cusp.

Proposition 7.1 *Let B be either the $\frac{1}{3}$–twist, $\frac{1}{4}$–twist, or $\frac{1}{6}$–twist. Then any arithmetic hyperbolic 5–manifold M with $B \times S^1$ as a cusp cross-section contains an immersed finite-volume totally geodesic submanifold W of codimension 1 with B as a cusp cross-section.*

Proof Let Γ be the fundamental group of M. As M is arithmetic, it is commensurable to some orbifold $\mathbb{H}^5/\mathrm{SO}_0(q,\mathbb{Z})$. Let y be a light-like vector in $\mathbb{H}^5_q$ that lies above the $B \times S^1$ cusp under the universal covering map of M.

The parabolic elements of Γ that fix y act on a horosphere E centered at y which is isomorphic to $\mathbb{E}^4$. Without loss of generality, we can take E to be the horosphere passing through $(0,0,0,0,0,1)$ by conjugating by an element of $\mathrm{SO}_0(q,\mathbb{Q})$. Note that $\mathrm{Stab}_\Gamma(y)$ is isomorphic to $\pi_1(B \times S^1) = \pi_1(B) \times \mathbb{Z}$, which acts on $\mathbb{E}^3 \times \mathbb{E}^1$. We can choose a flat subspace $P' \subset E$ of dimension 3 such that $H = \mathrm{Stab}_\Gamma(y) \cap \mathrm{Stab}_\Gamma(P')$ is isomorphic to $\pi_1(B)$. Let γ_1, γ_2, and γ_3 be three translations that generate the translation subgroup of H.

Unlike in $\mathrm{Isom}^+(\mathbb{H}^4)$, we can't assume that each γ_i lies in $\mathrm{SO}(q,\mathbb{Q})$ for $i = 1,2,3$. However, we can argue as follows. The γ_i act by translation on E, and so are parabolic translations. One can check, by applying ρ from Proposition 6.1 to any translation $v \mapsto Iv + w$, that this means each γ_i must be unipotent as an element of $\mathrm{SO}_0(q,\mathbb{R})$. For each γ_i, there is some positive integer k such that γ_i^k lies in $\mathrm{SO}_0(q,\mathbb{Z})$, since Γ is commensurable to $\mathrm{SO}_0(q,\mathbb{Z})$. Hence, the field of coefficients of γ_i^k, denoted by $F(\gamma_i^k)$, is $\mathbb{Q}$. This allows us to argue that $F(\gamma_i) = \mathbb{Q}$, and so $\gamma_i \in \mathrm{SO}_0(q,\mathbb{Q})$. The justification of the previous sentence is somewhat technical, so we defer it to Lemma 7.6.

The three translations γ_i act on the three-dimensional subspace $P' \subset E$. Since each $\gamma_i \in \mathrm{SO}_0(q,\mathbb{Q})$, P' must sit rationally in $E \subset \mathbb{H}^5_q$. To see this, pick any rational point in E, say $O = (0,0,0,0,1)$, and notice that $\gamma_i(O) \in \mathbb{Q}^5$ for all i.

The four points O, $\gamma_1(O)$, $\gamma_2(O)$, and $\gamma_3(O)$, together with y a rational line in $\partial\mathbb{H}^5_q \subset \mathbb{R}^6$, determine a four-dimensional hyperplane P which must also sit rationally in $\mathbb{H}^5_q$. Hence, after an appropriate change of basis over $\mathbb{Q}$, the quadratic form q restricts to a rank-5 form f on the 5–dimensional subspace $V \subset \mathbb{R}^6$ containing P. Then since P consists of exactly the points in V satisfying $f(x) = q(x) = -1$ and $x_6 > 0$, P sits in V as $\mathbb{H}^4_f$. In particular, this means $\mathrm{Isom}^+(P) = \mathrm{SO}_0(f,\mathbb{R})$, so $\mathrm{Isom}^+(P) \cap \mathrm{SO}_0(q,\mathbb{Z}) = \mathrm{SO}_0(f,\mathbb{Z})$. Note that this group is commensurable to $\mathrm{Isom}^+(P) \cap \Gamma$, as $\mathrm{SO}_0(q,\mathbb{Z})$ is commensurable to Γ. Thus $\mathrm{Isom}^+(P) \cap \Gamma$ is arithmetic and its action on P has finite covolume. Furthermore, $\mathrm{Isom}^+(P) \cap \mathrm{Stab}_\Gamma(y) = \mathrm{Isom}^+(P') \cap \mathrm{Stab}_\Gamma(y) = H$, so $W = P/(\mathrm{Isom}^+(P) \cap \Gamma)$ has a cusp at y with cross-section B. Now, W is an immersed finite-volume totally geodesic submanifold of M with cusp B. $\square$

This completes the first half of the proof. For the second, using Hasse–Witt invariants we prove that we should not find any totally geodesic 4–manifolds in our 5–manifold class with a cusp of type B, yielding a contradiction. The next step, then, is to find the Hasse–Witt invariants associated to such submanifolds.

Proposition 7.2 *Let q be a quadratic form of signature $(5,1)$, discriminant -1, and Hasse–Witt invariants $\epsilon_p(q)$, and let M be a hyperbolic 5–manifold commensurable to $\mathbb{H}^5/\mathrm{SO}_0(q,\mathbb{Z})$. Then any immersed finite-volume totally geodesic 4–dimensional submanifold $W \subset M$ must be commensurable to $\mathbb{H}^4/\mathrm{SO}_0(f,\mathbb{Z})$, where f is a quadratic form of signature $(4,1)$, discriminant -1, and Hasse–Witt invariants $\epsilon_p(f) = \epsilon_p(q)$.*

Proof Since M is arithmetic, W is also arithmetic [17, Theorem 3.2]. Thus, we know W is commensurable to $\mathbb{H}^4/\mathrm{SO}_0(f,\mathbb{Z})$ for some quadratic form f of signature $(4,1)$, which we can scale to ensure discriminant -1. All that remains to be shown is that $\epsilon_p(f) = \epsilon_p(q)$ at all primes p.

Let $f = \langle a, b, c, d, -abcd\rangle$ over a quadratic space with basis $\{v_1, \dots, v_5\}$. Since W is an arithmetic manifold commensurable to $\mathbb{H}^4/\mathrm{SO}_0(f,\mathbb{Z})$, we know $\pi_1(W) < \mathrm{SO}_0(f,\mathbb{Q})$ [3]. In particular, $\pi_1(W)$ acts on $\mathbb{H}^5$ in such a way that it preserves f and a 4–dimensional hyperplane P. Taking a vector w transverse to P and adding it to the basis above, we have a basis $\{v_1, \dots, v_5, w\}$ upon which we can define our quadratic form q. Though q may not be diagonal, we can use the Gram–Schmidt process to find a basis which makes q diagonal. And, since q restricted to span($\{v_1, \dots, v_5\}$) is already diagonal, the only basis element that is affected is w. Thus, since q has signature $(5,1)$, it can be written as a diagonal form $\langle a, b, c, d, -abcd, e\rangle$ for some positive $e \in \mathbb{Z}$. Since we started with the assumption that the discriminant of q is -1, we can conclude $e = 1$.

It is now easy to show that the Hasse–Witt invariants of $f = \langle a, b, c, d, -abcd\rangle$ are equal to the Hasse–Witt invariants of $q = \langle a, b, c, d, -abcd, 1\rangle$. Since any Hilbert symbol $(1,x)_p$ equals 1,

$$\begin{aligned}\epsilon_p(q) &= (a,b)_p(a,c)_p(a,d)_p(a,-abcd)_p(b,c)_p(b,d)_p(b,-abcd)_p(c,d)_p \\ &\qquad \cdot (c,-abcd)_p(d,-abcd)_p(1,a)_p(1,b)_p(1,c)_p(1,d)_p(1,-abcd)_p \\ &= (a,b)_p(a,c)_p(a,d)_p(a,-abcd)_p(b,c)_p(b,d)_p(b,-abcd)_p(c,d)_p \\ &\qquad \cdot (c,-abcd)_p(d,-abcd)_p \\ &= \epsilon_p(f).\end{aligned}$$

$\square$

Theorem 7.3 *Let B be either the $\frac{1}{3}$–twist, $\frac{1}{4}$–twist, or $\frac{1}{6}$–twist. Then there exist infinitely many commensurability classes of arithmetic hyperbolic 5–manifolds that contain no manifolds with cusp cross-section given by $B \times S^1$.*

Proof Consider any quadratic form q of signature $(5,1)$ and discriminant -1. We claim that if $\epsilon_p(q) = -1$ for any $p \equiv 1 \pmod 3$ then the commensurability class C of $\mathbb{H}^5/\mathrm{SO}_0(q,\mathbb{Z})$ cannot contain $B \times S^1$ for B the $\frac{1}{3}$–twist or the $\frac{1}{6}$–twist, and if $\epsilon_p(q) = -1$ for any $p \equiv 1 \pmod 4$ then this commensurability class cannot contain $B \times S^1$ for B the $\frac{1}{4}$–twist.

By Proposition 7.1, any manifold M in C with a $B \times S^1$ cusp must contain an immersed totally geodesic submanifold W with a B cusp. By Proposition 7.2, W must be commensurable to some $\mathbb{H}^4/\mathrm{SO}_0(q',\mathbb{Z})$ with $\epsilon_p(q') = \epsilon_p(q)$ for all primes p. But by Theorem 5.1, a manifold with these Hasse–Witt invariants cannot have a cusp with cross-section B. Thus we've reached a contradiction, and such an M cannot exist in C. □

It is tempting to apply this argument repeatedly to find commensurability classes in higher-dimensional hyperbolic manifolds that avoid certain cusp types. Unfortunately, this argument fails to work even in dimension 6, because Proposition 7.2 fails to generalize. Proposition 7.2 relies on the fact that we can rescale a quadratic form of rank 5 to control the discriminant. In rank 6, rescaling a quadratic form by k multiplies the discriminant by k^6, so the discriminant does not change in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

In fact, we can prove that repeatedly taking products of a compact flat manifold B with S^1 will eventually yield a manifold that occurs as a cusp cross-section in all arithmetic hyperbolic manifolds of the appropriate dimension. Thus, if we want to find cusp types with obstructions in higher dimensions, we'll have to use nontrivial high-dimensional flat manifolds.

Theorem 7.4 *Let B be a compact flat n–manifold. Then $B \times (S^1)^k$ occurs as a cusp cross-section in every commensurability class C of cusped arithmetic hyperbolic $(n+k+1)$–manifolds of simplest type for sufficiently high k.*

Proof First, we prove the result for $n+k+1$ even. When $n+k+1$ is even, any commensurability class C is associated with a quadratic form q of discriminant -1, since q has odd rank and we can scale q to control the discriminant.

Note that $B \times (S^1)^k$ has the same associated holonomy group as B. Since B is a flat manifold, the holonomy of its fundamental group $\mathrm{Hol}(\pi_1(B))$ must be finite. As such, $\mathrm{Hol}(\pi_1(B))$ must be a subgroup of a symmetric group S_m. Let q_m denote the quadratic form $\langle 1,\dots,1\rangle$ of rank m. The natural representation σ of S_m into permutation matrices in $\mathrm{GL}(m,\mathbb{Z})$ clearly preserves q_m. Restricting σ to $\mathrm{Hol}(\pi_1(B))$, we have a representation of $\mathrm{Hol}(\pi_1(B))$ that preserves q_m and must have entries in $\mathbb{Z}$. Let $q'_m = q_m \oplus \langle 1,-1\rangle$. We can use the Long–Reid algorithm [13] as in Proposition 5.2 to construct an orbifold with cusp cross-section $B\times (S^1)^k$ in the commensurability class of $\mathbb{H}^{n+k+1}/\mathrm{SO}_0(q'\oplus q'_m,\mathbb{Z})$ for any positive definite quadratic form q' of rank $n+k-m\ge 0$.

Now, if m is even, let $k=m-n+3$ so that $n+k+1=m+4$, and if m is odd, let $k=m-n+4$ so that $n+k+1=m+5$. This ensures $n+k+1$ is even. Consider the class C of $(n+k+1)$–manifolds with quadratic form q of discriminant -1. We can show that q must be rationally equivalent to a quadratic form $f=\langle a,b,c\rangle\oplus q'_m$ (or $f=\langle a,b,c,1\rangle\oplus q'_m$ if m is odd) by the same argument used to prove Lemma 2.7, with q'_m in the place of $\langle 1,-1\rangle$. Then $\mathbb{H}^{n+k+1}/\mathrm{SO}_0(f,\mathbb{Z})$ lies in C and is commensurable to a manifold with a cusp of type $B\times(S^1)^k$.

When $n+k+1$ is odd, we cannot control the discriminant of the quadratic form q associated to C. However, we can take a rank-$(n+k)$ subform q' of q such that $q=q'\oplus\langle x\rangle$ for some positive integer x. Then we can scale q' by y so that it has discriminant -1, and, as in the paragraph above, q' is rationally equivalent to $f=\langle a,b,c\rangle\oplus q'_m$ or $f=\langle a,b,c,1\rangle\oplus q'_m$. But now $yq=yq'\oplus\langle xy\rangle$ is rationally equivalent to $f\oplus\langle xy\rangle$, and we can conclude that $\mathbb{H}^{n+k+1}/\mathrm{SO}_0(f\oplus\langle xy\rangle)$ lies in C and is commensurable to a manifold with a cusp of type $B\times(S^1)^k$, as before. □

Corollary 7.5 *Every commensurability class C of cusped arithmetic hyperbolic 8–manifolds contains a manifold with a cusp of type $B\times(S^1)^3$, where B is any compact flat 3–manifold.*

Proof According to Theorem 5.1, every B occurs in the commensurability class of $\mathbb{H}^4/\mathrm{SO}(\langle 1,1,1,1,-1\rangle,\mathbb{Z})$. The result follows from the third paragraph of the proof of Theorem 7.4, using $m=3$. □

7.1 Fields of coefficients of unipotent matrices

In proving Proposition 7.1, we used the fact that, for a unipotent matrix M, the field of coefficients $F(M)$, defined to be the number field obtained by adjoining the entries of M to $\mathbb{Q}$, is unchanged under powers of M. We prove this result here:

Lemma 7.6 *For any unipotent matrix M and any positive integer k, $F(M) = F(M^k)$.*

Proof Because the entries of M^k are polynomial in the entries of M, $F(M^k) \subset F(M)$. This holds for any M, so in particular, $F(M^{ak}) \subset F(M^k)$ for any nonnegative integer a. We will show that M can be written as a linear combination over $\mathbb{Q}$ of matrices M^{ak}, and thus that each entry in M is polynomial in entries of M^k. This will suffice to show $F(M) \subset F(M^k)$.

By definition, a unipotent matrix M can be written as $M = I + T$, where T is a nilpotent matrix. There is a positive integer l such that $T^l = 0$. Now we can expand $M^k = (I+T)^k$ using binomial coefficients:

$$M^k = \sum_{i=0}^{k} \binom{k}{i} T^i = \sum_{i=0}^{l-1} \binom{k}{i} T^i.$$

Consider the vector space V over $\mathbb{Q}$ consisting of the matrices spanned by all T^i for nonnegative integers i. V must have dimension at most l, since only l of the T^i are nonzero. We will show that if $T^l = 0$ then the $l+1$ matrices M^{ak} for $a \in \{0, 1, \dots, l\}$ span V. Since $M \in V$, this will show that M is a linear combination of these M^{ak}.

Choose some $n \in \mathbb{Z}^+$, and consider the linear combination of matrices M^{ak}

$$\begin{aligned}
\sum_{a=0}^{n} (-1)^{n+a} \binom{n}{a} M^{ak} &= \sum_{a=0}^{n} (-1)^{n+a} \binom{n}{a} \left[\sum_{b=0}^{ak} \binom{ak}{b} T^b \right] \\
&= \sum_{a=0}^{n} \sum_{b=0}^{ak} (-1)^{n+a} \binom{n}{a} \binom{ak}{b} T^b \\
&= \sum_{b=0}^{nk} \sum_{a=\lceil b/k \rceil}^{n} (-1)^{n+a} \binom{n}{a} \binom{ak}{b} T^b \\
&= \sum_{b=0}^{nk} (-1)^n \left[\sum_{a=0}^{n} (-1)^a \binom{n}{a} \binom{ak}{b} \right] T^b.
\end{aligned}$$

Note that when we interchange the summations in line three, we see that a is indexed from $\lceil b/k \rceil$ to n. However, when $ak < b$, $\binom{ak}{b} = 0$ anyway, so we can start a at 0 in line four to get the same value.

The coefficient of T^b in this sum is given by $\sum_{a=0}^{n} (-1)^{n+a} \binom{n}{a} \binom{ak}{b}$. Note that for fixed b, $\binom{t}{b}$ is a degree-b polynomial in t, defined over all nonnegative integers t. When

$b < n$, the coefficient of T^b is 0; we apply Lemma 7.7, proven below, with $f(t) = \binom{t}{b}$ and $y = k$. Since the function $g_f^{n,k}(x)$ is uniformly 0, it is 0 at $x = 0$ in particular. Furthermore, when $b = n$, $\binom{t}{b}$ is a degree-n polynomial, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The coefficient of T^b then must be $a_n n!(-k)^n \neq 0$, by Lemma 7.7.

Now we can use induction on i to construct each T^i as a linear combination of M^{ak}. For the base case, consider $i = l-1$. Choose $n = l-1$, and in the above summation, T^b has coefficient 0 when $b < n = l-1$, and $T^b = 0$ when $b > n$ because $b \geq l$. Thus we've obtained a rational multiple of T^{l-1}, which we can rescale to write T^{l-1} as a linear combination of M^{ak}.

For the induction step, assume T^j can be written as such a linear combination for all $i < j \leq l-1$. Consider the linear combination above with $n = i$. Then, by Lemma 7.7, the coefficients of T^b are 0 for $b < i$, nonzero for $b = i$, and $T^b = 0$ for $b \geq l$. Since T^b can already be written as a linear combination for $i < b \leq l-1$ by the induction hypothesis we can subtract out the appropriate linear combinations to leave only a multiple of T^i.

This suffices to show that every T^i is a linear combination of M^{ak}, and thus $M = T^0 + T^1$ is some linear combination of matrices M^{ak}. Since $F(M^{ak}) \subset F(M^k)$ for all a and we have already proven $F(M^k) \subset F(M)$, we conclude $F(M) = F(M^k)$. □

Finally, we prove here the technical result that allowed us to conclude certain coefficients were zero or nonzero:

Lemma 7.7 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and fix $y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Let*

$$g_f^{n,y}(x) = \sum_{a=0}^{n} (-1)^a f(x+ay)\binom{n}{a}.$$

If f is a polynomial of degree less than n, then $g_f^{n,y} = 0$ uniformly. Furthermore, if $f(x) = x^n$, then $g_f^{n,y}$ is the constant function $n!(-y)^n$.

Proof First, we prove that $g_f^{n,y} = 0$ when f is a polynomial of degree less than n by induction on n. For the base case, consider $n = 1$. In order for f to be a polynomial of degree less than 1 it must be a constant function $f(x) = c$. Then

$$g_f^{1,y}(x) = \sum_{a=0}^{1} (-1)^a f(x+ay)\binom{1}{a} = f(x) - f(x+y) = c - c = 0.$$

Now assume the statement holds for $n-1$. We can compute

$$\begin{aligned} g_f^{n,y}(x) &= \sum_{a=0}^{n}(-1)^a f(x+ay)\binom{n}{a} \\ &= \sum_{a=1}^{n}(-1)^a f(x+ay)\binom{n-1}{a-1}+\sum_{a=0}^{n-1}(-1)^a f(x+ay)\binom{n-1}{a} \\ &= -\sum_{a=0}^{n-1}(-1)^a f(x+y+ay)\binom{n-1}{a}+\sum_{a=0}^{n-1}(-1)^a f(x+ay)\binom{n-1}{a} \\ &= -g_f^{n-1,y}(x+y)+g_f^{n-1,y}(x) = \int_{x+y}^{x}\frac{\partial}{\partial t}[g_f^{n-1,y}(t)]\,dt = \int_{x+y}^{x} g_{f'}^{n-1,y}(t)\,dt \end{aligned}$$

The second line above follows from the identity $\binom{n}{a}=\binom{n-1}{a-1}+\binom{n-1}{a}$. The final equality follows from the fact that $g_f^{n,y}$ is a particular linear combination of $f(x+ay)$, with fixed coefficients depending on n; concisely, g is linear in f. Since f is a polynomial of degree less than n, its derivative f' is a polynomial of degree less than $n-1$. Thus, $g_{f'}^{n-1,y}(t)=0$ everywhere by induction, and therefore $g_f^{n,y}=0$.

Next, we prove that $g_f^{n,y}=n!(-y)^n$ for $f=x^n$ by induction on n. For the base case, consider $n=1$. Then

$$g_f^{1,y}(x)=\sum_{a=0}^{1}(-1)^a(x+ay)\binom{1}{a}=(x)-(x+y)=-y.$$

Now assume the statement holds for $n-1$. Let $f(x)=x^n$ and $h(x)=x^{n-1}$, so that $f'=nh$. Then

$$\begin{aligned} g_f^{n,y}(x) &= \int_{x+y}^{x} g_{f'}^{n-1,y}(t)\,dt = n\int_{x+y}^{x} g_h^{n-1,y}(t)\,dt \\ &= n\int_{x+y}^{x}(n-1)!(-y)^{n-1}\,dt = n!(-y)^n. \end{aligned}$$

We proved the first equality in the first part of this proof. The rest follows from the fact that $g_f^{n,y}$ is linear in f, and the induction hypothesis. □

8 Commensurability classes of nonarithmetic manifolds

We can turn arithmetic commensurability classes that avoid certain cusp types into nonarithmetic ones by "inbreeding" the arithmetic manifolds with themselves, in a manner introduced by Agol [1]. We mimic the argument in [1] to construct a manifold with arbitrarily short geodesic, which must be nonarithmetic by Proposition 3.6. Further, this nonarithmetic group is constructed in such a way that it still lies in the $\mathbb{Q}$–points of the original quadratic form, so we can conclude by the same argument as our proof of Proposition 6.1 that it avoids the same cusps. Since this construction can be performed on any of the infinitely many classes that avoid the $\frac{1}{3}$–twist, $\frac{1}{4}$–twist, and $\frac{1}{6}$–twist cusps, there are infinitely many nonarithmetic commensurability classes that avoid such cusps.

Proof of Theorem 1.2 Let q be a quadratic form such that the commensurability class of $\mathbb{H}^4/\mathrm{SO}_0(q,\mathbb{Z})$ does not contain any manifolds or orbifolds with a certain cusp B. Let M be any manifold in this commensurability class, and Γ its fundamental group. By [5, Theorem 4.2], there exist infinitely many closed totally geodesic hyperbolic 3–manifolds immersed in M. These 3–manifolds lift to copies of $\mathbb{H}^3$ in $\mathbb{H}^4$; pick one such copy and call it P. Since the immersed 3–manifold is compact, $H = \mathrm{Isom}(P) \cap \Gamma$ acts cocompactly on P.

By Margulis' commensurability criterion for arithmeticity [19, Theorem 16.3.3], since Γ is arithmetic, its commensurator $\mathrm{Comm}(\Gamma)$ contains $\mathrm{PO}(q,\mathbb{Q})$. Thus for any $\epsilon > 0$, we can choose $\gamma \in \mathrm{Comm}(\Gamma)$ such that $\gamma(P)$ is disjoint from P and the distance $d(P,\gamma(P))$ is less than $\frac{1}{2}\epsilon$. Since $\gamma \in \mathrm{Comm}(\Gamma)$, the stabilizer of $\gamma(P)$, namely $(\gamma H\gamma^{-1}) \cap \Gamma$, acts cocompactly on $\gamma(P)$. Then $H_\gamma = \mathrm{Isom}(\gamma(P)) \cap \Gamma$ must act cocompactly on $\gamma(P)$, since $(\gamma H\gamma^{-1}) \cap \Gamma < H_\gamma$.

Let g be the geodesic segment orthogonal to both P and $\gamma(P)$ intersecting P at p_1 and $\gamma(P)$ at p_2. Because H is discrete and residually finite, as a finitely generated linear group we can choose a finite-index subgroup $H_1 < H$ such that $d(p_1,h(p_1)) > 2\,\mathrm{arctanh}\big(\mathrm{sech}\big(\frac{1}{4}\epsilon\big)\big)$ for all nonidentity $h \in H_1$. Similarly, choose $H_2 < H_\gamma$ such that $d(p_2,h(p_2)) > 2\,\mathrm{arctanh}\big(\mathrm{sech}\big(\frac{1}{4}\epsilon\big)\big)$ for all nonidentity $h \in H_2$. Let $\Sigma_1 = P/H_1$ and $\Sigma_2 = \gamma(P)/H_2$. Let $E_i \subset \mathbb{H}^4$ be the Dirichlet domain of H_i centered at p_i.

Now, $U = \Sigma_1 \cup_{p_1} g \cup_{p_2} \Sigma_2$ is an embedded compact spine for $E_1 \cap E_2$, with one component of $\mathbb{H}^4 - P$ retracting to Σ_1, the opposite component of $\mathbb{H}^4 - \gamma(P)$ retracting to Σ_2, and the space in between P and $\gamma(P)$ retracting to g.

We claim $G := \langle H_1, H_2\rangle = H_1 * H_2$ and G is geometrically finite, and defer the proof to Lemma 8.1.

Then G is separable in Γ [2]. By Scott's separability criterion [24], for some finite index subgroup $\Gamma_1 < \Gamma$, U embeds in $\mathbb{H}^4/\Gamma_1$. Thus, Σ_1 and Σ_2 embed in $\mathbb{H}^4/\Gamma_1$. Now let $N = (\mathbb{H}^4/\Gamma_1) - (\Sigma_1 \cup \Sigma_2)$, and D be the double of N along its boundary.

D is a hyperbolic manifold, since N is a hyperbolic manifold with totally geodesic boundary. Note that the double of g is a closed geodesic of length bounded by ϵ, since g is perpendicular to Σ_1 and Σ_2. Through choice of ϵ, we can construct D so that it has a geodesic of arbitrarily small length. Thus, by Proposition 3.6, we can construct D to be nonarithmetic.

Next, we claim that $\pi_1(D) < \mathrm{SO}_0(q, \mathbb{Q})$. First, note that the universal cover of N is $\mathbb{H}^4$ with some half-spaces removed, with its group action given by Γ_1. By construction, $\Gamma_1 < \Gamma < \mathrm{SO}_0(q, \mathbb{Q})$. Thus, we can find a fundamental domain S for N such that all the face pairings of S lie in $\mathrm{SO}_0(q, \mathbb{Q})$. We can construct a fundamental domain for D by taking two copies of S glued together at one of the boundary faces F that lifts to Σ_1, and pairing the remaining boundary faces by mapping each to its counterpart in the other copy of S. We will show that $\pi_1(D) < \mathrm{SO}_0(q, \mathbb{Q})$ by showing that these face pairings, which generate $\pi_1(D)$, each lie in $\mathrm{SO}_0(q, \mathbb{Q})$.

By construction, the face pairings ϕ_i on the original copy of S must lie in $\mathrm{SO}_0(q, \mathbb{Q})$. The corresponding face pairings in the other copy of S are given by $r_P \phi_i r_P$, where r_P is reflection across P. Recall that P was constructed as a hyperplane perpendicular to some $v \in \mathbb{Q}^5$, so the reflection r_P across P lies in $\mathrm{SO}_0(q, \mathbb{Q})$. Thus each $r_P \phi_i r_P$ must also lie in $\mathrm{SO}_0(q, \mathbb{Q})$.

The remaining face pairings are the new ones formed from identifying boundary components of N. To pair a boundary component C with its corresponding mirror component, we can use the isometry $r_P r_F$, where r_F is the reflection across the hyperplane F containing C. Note that F must be the image of $\gamma(P)$ under some isometry $\alpha \in \pi_1(N)$, so $r_F = \alpha^{-1} r_{\gamma(P)} \alpha = \alpha^{-1} \gamma^{-1} r_P \gamma \alpha$. Since we chose γ to lie in $\mathrm{SO}_0(q, \mathbb{Q})$ and α must be an element of $\pi_1(N)$, r_F lies in $\mathrm{SO}_0(q, \mathbb{Q})$ as well. Now $r_F r_P$ lies in $\mathrm{SO}_0(q, \mathbb{Q})$, and thus every face pairing does as well. Therefore, $\pi_1(D)$ is generated by elements of $\mathrm{SO}_0(q, \mathbb{Q})$, and so $\pi_1(D) < \mathrm{SO}_0(q, \mathbb{Q})$.

Now, if we choose the quadratic form q in such a way that the commensurability class of $\mathbb{H}^4/\mathrm{SO}_0(q, \mathbb{Q})$ avoids cusps with cross-section B, then D cannot have cusps with cross-section B, using the same argument as in the proof of Proposition 6.1. In this way, we use Theorem 1.1 to construct infinitely many commensurability classes of nonarithmetic manifolds that avoid the $\frac{1}{3}$–twist, $\frac{1}{4}$–twist, and $\frac{1}{6}$–twist. □

The same proof can be applied to provide examples of commensurability classes of nonarithmetic hyperbolic 5–manifolds that avoid certain cusp types, with Theorem 7.3. We finish the proof by proving the claim we deferred:

Lemma 8.1 *Let H_1 and H_2 be as above. Then $G = \langle H_1, H_2\rangle$ is isomorphic to $H_1 * H_2$, and is geometrically finite.*

Proof As in the proof of Theorem 1.2, we let g be the geodesic segment connecting $p_1 \in P$ with $p_2 \in \gamma(P)$, meeting both planes perpendicularly. Let L be the 3–plane that perpendicularly bisects g, and consider the projections $\mathrm{pr}_1 : \mathbb{H}^4 \to P$ and $\mathrm{pr}_2 : \mathbb{H}^4 \to \gamma(P)$ that map each point in $\mathbb{H}^4$ to the closest point on the target 3–plane. Using hyperbolic geometry (see Theorem 3.5.10 in [23]), we can see that $\mathrm{pr}_i(L)$ is a disk with radius bounded by $\operatorname{arcsinh}\bigl(\operatorname{csch}\bigl(\frac{1}{4}\epsilon\bigr)\bigr) = \operatorname{arctanh}\bigl(\operatorname{sech}\bigl(\frac{1}{4}\epsilon\bigr)\bigr)$ centered at p_i. We defined H_1 so that $d(p_1, h(p_1)) > 2\operatorname{arctanh}\bigl(\operatorname{sech}\bigl(\frac{1}{4}\epsilon\bigr)\bigr)$ for all nonidentity $h \in H_1$, so $\mathrm{pr}_1(L)$ must lie inside E_1, the Dirichlet domain of H_1 centered at p_1. Thus, since $H_1 < \mathrm{Isom}(P)$, L must lie inside of E_1. Similarly, L lies in E_2 as well. Now L splits $\mathbb{H}^4$ into two parts, with ∂E_1 lying in the part with P, and ∂E_2 lying in the part with $\gamma(P)$. Thus $\partial E_1 \cap \partial E_2 = \varnothing$. Since E_1 and E_2 are each geometrically finite, $E_1 \cap E_2$, the fundamental domain of G, is geometrically finite too. Also, note that $E_1 \cap E_2 = E_1 \mathbin{\#} E_2$, with the two sets glued along L, so it's a fundamental domain of $H_1 * H_2$. We can conclude that G is geometrically finite and $G = H_1 * H_2$. □

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Received: 11 October 2021 Revised: 17 May 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY
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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 8 (pages 3417–3908) 2023