The deformation space of nonorientable hyperbolic 3–manifolds

JUAN LUIS DURÁN BATALLA
JOAN PORTI

The deformation space of nonorientable hyperbolic 3–manifolds

JUAN LUIS DURÁN BATALLA

JOAN PORTI

We consider nonorientable hyperbolic 3–manifolds of finite volume M^3. When M^3 has an ideal triangulation Δ, we compute the deformation space of the pair (M^3, Δ) (its Neumann–Zagier parameter space). We also determine the variety of representations of $\pi_1(M^3)$ in $\mathrm{Isom}(\mathbb{H}^3)$ in a neighborhood of the holonomy. As a consequence, when some ends are nonorientable, there are deformations from the variety of representations that cannot be realized as deformations of the pair (M^3, Δ). We also discuss the metric completion of these structures and we illustrate the results on the Gieseking manifold.

57K32; 57K35, 57Q99

1 Introduction

Let M^3 be a complete noncompact hyperbolic three-manifold of finite volume. Assume first that M^3 is orientable. Assume also that M^3 has a geometric ideal triangulation Δ, defined by Neumann and Zagier [16]. Following Thurston's construction for the figure eight knot exterior in [18], Neumann and Zagier defined in [16] a deformation space of the pair (M^3, Δ) by considering the set of parameters of the ideal simplices of Δ subject to compatibility equations. We denote the Neumann–Zagier parameter space by $\mathrm{Def}(M^3, \Delta)$. It is proved in [16] that it is homeomorphic to an open subset of $\mathbb{C}^l$, where l is the number of ends of M^3.

Another approach to deformations is based on $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$, the variety of conjugacy classes of representations of $\pi_1(M^3)$ in $\mathrm{Isom}(\mathbb{H}^3)$. It is proved, for instance by Kapovich in [14], that a neighborhood of the holonomy of M^3 is bianalytic to an open subset of $\mathbb{C}^l$.

Both approaches to deformations can be used to prove the hyperbolic Dehn filling theorem (even if it is still an open question whether an orientable M^3 admits a geometric ideal triangulation). Among other things, one has to take into account that $\mathrm{Def}(M^3, \Delta)$ is a 2^l to 1 branched covering of the neighborhood in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. When M^3 is orientable, both approaches yield the same deformation space.

We investigate the nonorientable setting, that is, M^3 is a connected nonorientable hyperbolic 3–manifold of finite volume. When it has an ideal triangulation Δ, we define a deformation space of the pair $\mathrm{Def}(M^3, \Delta)$ à la Neumann and Zagier. Here is our main result (for simplicity, we assume that M^3 has a single end, which is nonorientable):

Theorem 1.1 *Let M^3 be a complete nonorientable hyperbolic* 3–*manifold of finite volume with a single end*, *which is nonorientable.*

(a) *If M^3 admits a geometric ideal triangulation Δ, then* $\mathrm{Def}(M^3, \Delta) \cong (-1, 1)$, *where the parameters $\pm t \in (-1, 1)$ correspond to the same structure.*

(b) *A neighborhood of the holonomy in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to an interval $(-1, 1)$.*

Furthermore, *the holonomy map* $\mathrm{Def}(M^3, \Delta) \to \mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ *folds the interval $(-1, 1)$ at 0 and its image is the half-open interval $[0, 1)$, where 0 corresponds to the complete structure.*

The version of this theorem with *several cusps* is Theorem 3.11.

For M^3 as in the theorem, structures in the subinterval $[0, 1) \subset (-1, 1)$ in the variety of representations are realized by $\mathrm{Def}(M^3, \Delta)$, but structures in $(-1, 0)$ are not. This corresponds to two different kinds of representations of the Klein bottle, which we call type I when realized, and II when not. These are described in Section 4.

Deformations of the complete structure are noncomplete, and therefore, for a deformation of the holonomy, the hyperbolic structure is not unique. Deformations of type I can be realized by ideal triangulations. Hence there is a natural choice of structure, and we prove in Theorem 5.15 that its metric completion consist in adding a singular geodesic so that it is the core of a solid Klein bottle and it has a singularity of cone angle in a neighborhood of zero. For deformations of type II, we prove that there is a natural choice of structure (radial), and the metric completion consists in adding a singular interval, also in Theorem 5.15. This singular interval is the soul of a twisted disc orbibundle over an interval with mirror boundary. Topologically, a neighborhood of this interval is the disc sum of two cones on a projective plane. Metrically, it is a conifold, with cone angle at the interior of the interval in a neighborhood of zero.

The paper is organized as follows. In Section 2 we describe $\mathrm{Def}(M^3, \Delta)$ and in Section 3 we describe $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. Section 4 is devoted to representations of the Klein bottle. Metric completions are described in Section 5, and finally, in Section 6, we describe in detail the deformation space(s) of the Gieseking manifold.

2 Deformation space from ideal triangulations

Before discussing nonorientable manifolds, we recall first the orientable case. The first example was constructed by Thurston in his notes [18, Chapter 4] for the figure eight knot exterior, and the general case was constructed by Neumann and Zagier in [16]. We point the reader to these references for the upcoming exposition.

From the point of view of a triangulation, the deformation of the hyperbolic structure on a manifold with a given *geometric ideal* triangulation is the space of parameters of ideal tetrahedra, subject to compatibility equations.

A *geometric ideal* tetrahedron is a geodesic tetrahedron of $\mathbb{H}^3$ with all of its vertices in the ideal sphere $\partial_\infty\mathbb{H}^3$. We say that a hyperbolic 3–manifold *admits a geometric ideal triangulation* if it is the union of such tetrahedra, along the geodesic faces. Though it has been established in many cases, it is still an open problem to decide whether every orientable hyperbolic three-manifold of finite volume admits a geometric ideal triangulation.

Given an ideal tetrahedron in $\mathbb{H}^3$, up to isometry, we may assume that its ideal vertices in $\partial_\infty\mathbb{H}^3 \cong \mathbb{C}\cup\{\infty\}$ are 0, 1, ∞ and $z \in \mathbb{C}$. The idea of Thurston is to equip the (unoriented) edge between 0 and ∞ with the complex number z, called the *edge invariant*. The edge invariant determines the isometry class of the tetrahedron, and for different edges the corresponding invariants satisfy some relations, called *tetrahedron relations*:

- Opposite edges have the same invariant.

- Given three edges with a common endpoint and invariants z_1, z_2 and z_3, indexed following the right-hand rule towards the common ideal vertex, they are related to z_1 by $z_2 = 1/(1 - z_1)$ and $z_3 = (z_1 - 1)/z_1$.

Let M^3 be a possibly nonorientable complete hyperbolic 3–manifold of finite volume, which admits a geometric ideal triangulation $\Delta = \{A_1, \ldots, A_n\}$. As we have stated before, up to (oriented) isometry the hyperbolic structure of each tetrahedron can be determined by a single edge invariant, thus the usual parametrization of the triangulation goes as follows: fix an edge e_i in each tetrahedron A_i, and consider its edge invariant z_i. Hence, the hyperbolic structure of M^3 can be parametrized by n parameters (one for each tetrahedron) and we will denote the parameters of the complete triangulation by $\{z_1^0, \ldots, z_n^0\}$. The *deformation space of M^3 with respect to* Δ, $\mathrm{Def}(M^3, \Delta)$, is defined as the set of parameters $\{z_1, \ldots, z_n\}$ in a small enough neighborhood of the complete structure for which the gluing bestows a hyperbolic structure on M^3. However, we find that the equations defining the deformation space are easier to work with if we use $3n$ parameters (one for each edge after taking into account the duplicity in opposite edges) and ask them to satisfy the second tetrahedron relation too.

When M^3 is orientable, in order for the gluing to be geometric, it is necessary and sufficient that around each edge cycle $[e] = \{e_{i_1, j_1}, \ldots, e_{i_n, j_n}\}$ the following two compatibility conditions are satisfied:

(1) $$\prod_{l=1}^{n} z(e_{i_l, j_l}) = 1,$$

(2) $$\sum_{l=1}^{n} \arg(z(e_{i_l, j_l})) = 2\pi.$$

Geometrically, if we try to realize in $\mathbb{H}^3$ the tetrahedra around the edge cycle $[e]$, (1) means that the triangulation must "close up", and (2) means that the angle around $[e]$ must be precisely 2π (instead of a multiple). The parameters of the complete hyperbolic structure are denoted by $\{z^0(e_{1,1}), \ldots, z^0(e_{n,3})\}$. In a small enough neighborhood of $\{z^0(e_{1,1}), \ldots, z^0(e_{n,3})\}$, fulfillment of (1) implies (2). We end the overview of the orientable case with the theorem we want to extend to the nonorientable case:

Theorem 2.1 (Neumann and Zagier [16]) *Let M^3 be connected oriented hyperbolic of finite volume with l cusps. Then $\mathrm{Def}(M^3, \Delta)$ is biholomorphic to an open set of $\mathbb{C}^l$.*

When we deal with nonorientable manifolds, again the problem of the gluing being geometric lives within a neighborhood of the edges. The compatibility equations in this case carry the same geometric meaning as in (1) and (2), while accounting for the possible change of orientation of the tetrahedra.

Proposition 2.2 *Let M^3 be a nonorientable manifold that is triangulated by a finite number of ideal tetrahedra A_i, which bestows a hyperbolic structure around the edge cycle $[e] = \{e_{i_1,j_1}, \ldots, e_{i_n,j_n}\}$ if and only if the following compatibility equations are satisfied:*

$$(3) \qquad \prod_{l=1}^{n} \frac{z(e_{i_l,j_l})^{\epsilon_l}}{z(e_{i_l,j_l})^{1-\epsilon_l}} = 1,$$

$$(4) \qquad \sum_{l=1}^{n} \arg(z(e_{i_l,j_l})) = 2\pi.$$

Here $z(e_{i_l,j_l})$ is the edge invariant of e_{i_l,j_l}, and $\epsilon_l = 0, 1$ in such a way that, in the gluing around the edge cycle $[e]$, a coherent orientation of the tetrahedra is obtained by gluing a copy of A_{i_l} with its orientation reversed if $\epsilon_l = 0$ (or preserved if $\epsilon_l = 1$), and with the initial condition that the orientation of the tetrahedron A_{i_1} is kept as given.

Proof When we follow a cycle of side identifications around an edge, we can always reorient the tetrahedra (maybe more than once) so that the gluing is done by orientable isometries. The compatibility equations for the orientable case can be then applied and hence, for the neighborhood of the edge cycle to inherit a hyperbolic structure, (1) must be satisfied, with the corresponding edge invariants.

Now, let us consider an edge $e_{i,j} \in A_i$ with parameter $z(e_{i_j})$. To see how the edge invariant changes under a nonorientable isometry, we can assume that A_i has vertices 0, 1, $z(e_{i,j})$ and ∞ in the upper space model, and consider the isometry c, the Poincaré extension of complex conjugation in $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$. Then, the edge invariant of $c(e_{i,j}) \in c(A_i)$ is $1/\overline{z(e_{i,j})}$.

Thus, the proposition follows with ease after changing the orientation of some tetrahedra. $\square$

Definition 2.3 Let M^3 be a connected complete nonorientable hyperbolic 3–manifold of finite volume. Let Δ be an ideal triangulation of M^3. The *deformation space of M^3 related to the triangulation Δ* is the

set $\mathrm{Def}(M^3, \Delta)$ consisting of those $(z_{1,1}, \ldots, z_{n,3}) \in U \cap \mathbb{C}^{3n}$ satisfying the compatibility equations (3) and (4) and the tetrahedron relations, where U is a small enough neighborhood of the parameters $(z^0_{i,j})$ of the complete structure.

Let M^3_+ be the orientation covering of M^3. The ideal triangulation on M^3, Δ, can be lifted to an ideal triangulation Δ_+ on M^3_+. There is an orientation-reversing homeomorphism ι acting on M^3_+ such that $M^3 = M^3_+/\iota$ and $\iota^2 = \mathrm{Id}$. The triangulation on M^3_+ is constructed in the usual way: for every tetrahedron A_i we take another tetrahedron with the opposite orientation, $\iota(A_i)$, and glue them so that the orientation is coherent. For every edge $e_{i,j} \in A_i$, let $z(e_{i,j})$ or $z_{i,j}$ denote its edge invariant. Analogously, $w(\iota(e_{i,j}))$ or $w_{i,j}$ will denote the edge invariant of $\iota(e_{i,j}) \in \iota(A_i)$.

Remark 2.4 The compatibility equations (3) and (4) around $[e] \in M^3$ are precisely the (orientable) compatibility equations in any lift of $[e]$ to the orientation covering.

The orientation-reversing homeomorphism acts on $\mathrm{Def}(M^3_+, \Delta_+)$ by pulling back (equivalently, pushing forward) the associated hyperbolic metric on each tetrahedron. Combinatorially, the action is described in the following lemma:

Lemma 2.5 *Let* $M^3 = M^3_+/\iota$, *where* ι *is an orientation-reversing homeomorphism. Let* M^3 *admit an ideal triangulation* Δ. *Then* ι *acts on* $\mathrm{Def}(M^3_+, \Delta_+)$ *as*

$$(5) \qquad \iota_*((z_{i,j}, w_{i,j})) = \left(\frac{1}{\overline{w}_{i,j}}, \frac{1}{\overline{z}_{i,j}} \right).$$

Proof The proof follows easily from the fact that ι permutes the edges and, for $e_{i,j} \in A_i$ with invariant $z(e_{i,j})$, the edge invariant of $c(e_{i,j}) \in c(A_i)$ is $1/\overline{z(e_{i,j})}$, where c is the Poincaré extension of complex conjugation. $\square$

Remark 2.6 Metrics on tetrahedra are considered up to isotopy.

Corollary 2.7 *The map defined by* $(z_{i,j}) \in \mathrm{Def}(M^3, \Delta) \mapsto (z_{i,j}, 1/\overline{z}_{i,j}) \in \mathrm{Def}(M^3_+, \Delta_+)^{\iota}$ *is a real analytic isomorphism.*

Proof This follows from Remark 2.4 and Lemma 2.5. $\square$

Our goal is to use Corollary 2.7 and Theorem 2.1 to identify the deformation space of M^3 with the fixed points under an action on $\mathbb{C}^k$. Let us suppose for the time being that M^3 has only one cusp which is nonorientable. The section of this cusp must be a Klein bottle. In order to define the biholomorphism through generalized Dehn filling coefficients, we must first fix a longitude–meridian pair in the peripheral torus in the orientation covering M^3_+. As we will see, there is a canonical choice. Afterwards, following Thurston, we will compute the derivative of the holonomy, hol', and translate the action of ι over there, and finally to the generalized Dehn filling coefficients.
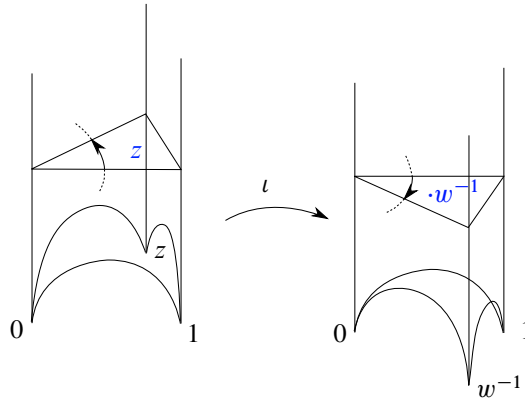
Figure 1: Change under the action of ι.

Fixing a longitude–meridian pair Letting K^2 be the Klein bottle, its fundamental group admits a presentation

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements a^2 and b in the orientation covering T^2 are generators of $\pi_1(T^2)$ and are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves). From now on, we will choose as longitude–meridian pair the elements:

$$l := a^2, \quad m := b.$$

Definition 2.8 The previous generators of $\pi_1(T^2)$ are called *distinguished* elements.

Lemma 2.9 *Let $[\alpha] \in \pi_1(T)$, and let ι be the involution in the orientation covering M_+^3, that is, $M^3 \cong M_+^3/\iota$. We also denote by ι the restriction of ι to the peripheral torus T. If*

$$(6) \qquad \mathrm{hol}'(\alpha) = \prod_{r \in I} z(e_{i_r, j_r})^{\epsilon_r} \prod_{s \in J} w(\iota(e_{i_s, j_s}))^{\epsilon_s},$$

where $\epsilon_r, \epsilon_s \in \{\pm 1\}$, then

$$(7) \qquad \mathrm{hol}'(\iota(\alpha)) = \prod_{r \in I} w(\iota(e_{i_r, j_r}))^{-\epsilon_r} \prod_{s \in J} z(e_{i_s, j_s})^{-\epsilon_s}.$$

When we compute the derivative of the holonomy of an element, $\mathrm{hol}'(\gamma)$, we assume that $\mathrm{hol}(\gamma)$ fixes ∞.

Proof We use Thurston's method for computing the holonomy through the developing of triangles in $\mathbb{C}$; see [18]. Thus, the factor that each piece of path adds to the derivative of the holonomy changes, as in Figure 1, under the action of ι. □

Proposition 2.10 *For the chosen longitude–meridian pair, the action of ι on $\mathrm{Im}(\mathrm{hol}') \subset \mathbb{C}^2$ is*

$$(8) \qquad \iota_*(L, M) = (\overline{L}, \overline{M}^{-1}),$$

where $L = \mathrm{hol}'(l)$, $M = \mathrm{hol}'(m)$.

Proof The action of ι on the longitude–meridian pair is $\iota_*(l) = l$ and $\iota_*(m) = m^{-1}$. Hence, the previous lemma implies that the derivative holonomy of the longitude and the meridian has the following features:

$$\text{hol}'(m) = \prod_{r \in I} z(e_{i_r, j_r})^{\epsilon_r} \prod_{s \in J} w(\iota(e_{i_s, j_s}))^{\epsilon_s} = \prod_{r \in I} w(\iota(e_{i_r, j_r}))^{\epsilon_r} \prod_{s \in J} z(e_{i_s, j_s})^{\epsilon_s},$$

$$\text{hol}'(l) = \prod_{r \in I} (z(e_{i_r, j_r}) w(\iota(e_{i_r, j_r}))^{-1})^{\epsilon_r}. \qquad \square$$

Remark 2.11 Following the notation of Proposition 2.10, $(L, M) \in (\mathbb{C}^2)^{\iota}$ if and only if $L \in \mathbb{R}$ and $|M| = 1$.

Let us denote by $u := \log \text{hol}'(l)$ and $v := \log \text{hol}'(m)$ the *generalized Dehn coefficients*. These are the solutions in $\mathbb{R}^2 \cup \{\infty\}$ to *Thurston's equation*

$$(9) \qquad\qquad\qquad\qquad\qquad pu + qv = 2\pi i.$$

Indeed, Neumann and Zagier's Theorem 2.1 (see also [18]) states that, for M^3 orientable, the map $(z_{i,j}) \in \text{Def}(M^3, \Delta) \mapsto (p_k, q_k)$ is a biholomorphism and the image is a neighborhood of $(\infty, \ldots, \infty) \in \bar{\mathbb{C}}^l$, where l is the number of cusps of M^3.

Proposition 2.12 *The action of ι on $(p, q) \in U \cap \mathbb{R}^2 \cup \{\infty\}$, where (p, q) are the generalized Dehn coefficients, is*

$$(10) \qquad\qquad\qquad\qquad\qquad \iota_*(p, q) = (-p, q).$$

Proof The action of ι can be translated through the logarithm to (u, v) from the action on the holonomy (8) as $\iota_*(u, v) = (\bar{u}, -\bar{v})$. Then, to find the action on generalized Dehn coefficients, we have to solve Thurston's equation (9) with $\bar{u}$ and $-\bar{v}$, that is,

$$(11) \qquad\qquad\qquad\qquad\qquad p'\bar{u} - q'\bar{v} = 2\pi i,$$

where $\iota_*(p, q) = (p', q') \in \mathbb{R}^2 \cup \{\infty\}$. It is straightforward to check that $(p', q') = (-p, q)$ is the solution. $\qquad\qquad \square$

Corollary 2.13 *The fixed points under ι, which are in correspondence with $\text{Def}(M^3, \Delta)$, are those whose generalized Dehn filling coefficients are of type $(0, q)$.*

Theorem 2.14 *Let M^3 be a connected complete nonorientable hyperbolic 3–manifold of finite volume. Let M^3 have k nonorientable cusps and l orientable ones, and let it admit an ideal triangulation Δ. Then $\text{Def}(M^3, \Delta)$ is real bianalytic to an open set of $\mathbb{R}^{k+2l}$.*

Proof We have already proved the theorem for $k = 1$ and $l = 0$.

Let $k = 0$ and $l = 1$. Any peripheral torus on M^3 is lifted to two peripheral tori, T_1 and T_2, on M_+^3. Here ι acts by permutation. More precisely, we can fix any longitude–meridian pair in one, $l_1, m_1 \in \pi_1(T_1)$,

and choose the longitude–meridian pair in the second torus as $l_2 := \iota_*(l_1), m_2 := \iota_*(m_1) \in \pi_1(T_2)$. The same arguments as in Proposition 2.10 show that $\iota_*(p_1, q_1, p_2, q_2) = -(p_2, q_2, p_1, q_1)$, and hence the fixed points have generalized coefficients $(p, q, -p, -q)$, $p, q \in \mathbb{R}$.

Finally, in general the action of ι on $\mathrm{Im}(\mathrm{hol}') \subset \mathbb{C}^{k+2l}$ can be understood as a product of $k + l$ actions $\iota_1 \times \cdots \times \iota_l$, the first k, ι_i for $i = 1, \ldots, k$, acting on $\mathbb{C}$ as in the case for a Klein bottle cusp, and the subsequent l, ι_j for $j = k + 1, \ldots, k + l$, acting on $\mathbb{C}^2$ as in the case for a peripheral torus.   □

3 Varieties of representations

The group of isometries of hyperbolic space is denoted by G, and we will use the well-known isomorphisms

$$G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PO}(3, 1) \cong \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$$

in order to identify elements of G with elements of $\mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$. The group G has two connected components, according to whether the isometries preserve or reverse the orientation.

For a finitely generated group Γ, the variety of representations of Γ in G is denoted by

$$\mathrm{hom}(\Gamma, G).$$

As G is algebraic, it has a natural structure of an algebraic set (see Johnson and Millson [13]), but we consider only its topological structure. We are interested in the set of conjugacy classes of representations

$$\mathcal{R}(\Gamma, G) = \mathrm{hom}(\Gamma, G)/G.$$

When M^3 is hyperbolic, we write $\Gamma = \pi_1(M^3)$. The holonomy of M^3

$$\mathrm{hol} \colon \Gamma \to G$$

is well defined up to conjugacy, and hence $[\mathrm{hol}] \in \mathcal{R}(\Gamma, G)$. To understand deformations, we analyze a neighborhood of the holonomy in $\mathcal{R}(\Gamma, G)$. The main result of this section is:

Theorem 3.1 *Let M^3 be a hyperbolic manifold of finite volume. Assume that it has k nonorientable cusps and l orientable cusps. Then there exists a neighborhood of $[\mathrm{hol}]$ in $\mathcal{R}(\Gamma, G)$ homeomorphic to $\mathbb{R}^{k+2l}$.*

When M^3 is orientable this result is well known (see for instance Boileau and Porti [4] or Kapovich [14]), and hence we assume that M^3 is nonorientable. We will prove a more precise result in Theorem 3.10, as for our purposes it is relevant to describe local coordinates in terms of the geometry of holonomy structures at the ends.

Before starting the proof, we need a lemma on varieties of representations. The projection to the quotient $\pi \colon \mathrm{hom}(\Gamma, G) \to \mathcal{R}(\Gamma, G)$ can have quite bad properties. For instance, even if $\mathrm{hom}(\Gamma, G)$ is Hausdorff, in general $\mathcal{R}(\Gamma, G)$ is not. But, in a neighborhood of the holonomy:

Lemma 3.2 *There exists a neighborhood* $V \subset \mathcal{R}(\Gamma, G)$ *of* [hol] *such that*:

(a) *If* $[\rho] = [\rho'] \in V$, *then the matrix* $A \in G$ *satisfying* $A\rho(\gamma)A^{-1} = \rho'(\gamma)$ *for all* $\gamma \in \Gamma$ *is* **unique**.

(b) V *is Hausdorff and the projection* $\pi : \pi^{-1}(V) \to V$ *is open.*

(c) *If* $[\rho] \in V$, *then for all* $\gamma \in \Gamma$, $\rho(\gamma)$ *preserves the orientation of* $\mathbb{H}^3$ *if and only if* γ *is represented by a loop that preserves the orientation of* M^3.

Assertions (a) and (b) are proved, for instance, by Johnson and Millson in [13]. They define the property of *good* representation, that is, open in $\mathcal{R}(\Gamma, G)$; it implies assertions (a) and (b), and it is satisfied by the conjugacy class of the holonomy. Assertion (c) is clear by continuity and the decomposition of G into two components, according to the orientation.

To describe the neighborhood of the holonomy in $\mathcal{R}(\Gamma, G)$ we use the orientation covering.

3.1 Orientation covering and the involution on representations

As mentioned, we assume M^3 is nonorientable. Let

$$M_+^3 \to M^3$$

denote the orientation covering, with fundamental group $\Gamma_+ = \pi_1(M_+^3)$. In particular, we have a short exact sequence:

$$1 \to \Gamma_+ \to \Gamma \to \mathbb{Z}_2 \to 1.$$

Definition 3.3 For $\zeta \in \Gamma \setminus \Gamma_+$, define the group automorphism

$$\sigma_* : \Gamma_+ \to \Gamma_+, \quad \gamma \mapsto \zeta\gamma\zeta^{-1}.$$

The automorphism σ_* depends on the choice of $\zeta \in \Gamma \setminus \Gamma_+$: automorphisms corresponding to different choices of ζ differ by composition (or precomposition) with an inner automorphism of Γ_+. Furthermore, σ_*^2 is an inner automorphism because $\zeta^2 \in \Gamma_+$. This automorphism σ_* is the map induced by the deck transformation of the orientation covering $M_+^3 \to M^3$.

The map induced by σ_* in the variety of representations is denoted by

$$\sigma^* : \mathcal{R}(\Gamma, G) \to \mathcal{R}(\Gamma, G), \quad [\rho] \mapsto [\rho \circ \sigma_*],$$

and σ^* does not depend on the choice of ζ because σ_* is well defined up to inner automorphism. Furthermore σ^* is an involution, $(\sigma^*)^2 = \mathrm{Id}$.

Consider the restriction map

$$\mathrm{res} : \mathcal{R}(\Gamma, G) \to \mathcal{R}(\Gamma_+, G)$$

that maps the conjugacy class of a representation of Γ to the conjugacy class of its restriction to Γ_+.

Lemma 3.4 *There exist $U \subset \mathcal{R}(\Gamma, G)$ a neighborhood of* [hol] *and $V \subset \mathcal{R}(\Gamma_+, G)$ a neighborhood of* res([hol]) *such that*

$$\mathrm{res} \colon U \xrightarrow{\cong} \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

is a homeomorphism.

Proof We show first that $\mathrm{res}(\mathcal{R}(\Gamma, G)) \subset \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}$. If $\rho_+ = \mathrm{res}(\rho)$, then for all $\gamma \in \Gamma_+$,

$$\sigma^*(\rho_+)(\gamma) = \rho_+(\sigma_*(\gamma)) = \rho_+(\zeta \gamma \zeta^{-1}) = \rho(\zeta)\rho_+(\gamma)\rho(\zeta)^{-1}.$$

Hence $\sigma^*([\mathrm{res}(\rho)]) = [\mathrm{res}(\rho)]$.

Next, given $[\rho_+] \in \mathcal{R}(\Gamma_+, G)$ satisfying $\sigma^*([\rho_+]) = [\rho_+]$, by construction there exists $A \in G$ that conjugates ρ_+ and $\rho_+ \circ \sigma_*$. We chose the neighborhood V so that Lemma 3.2 applies, and hence such an $A \in G$ is unique. From uniqueness (of A and A^2), it follows easily that, if $\zeta \in \Gamma \setminus \Gamma_+$ is the element such that σ_* is conjugation by ζ, then, by choosing $\rho(\zeta) = A$, ρ_+ extends to $\rho \colon \Gamma \to G$. Hence

$$\mathrm{res}(\mathcal{R}(\Gamma, G)) = \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}.$$

Let $U = \mathrm{res}^{-1}(V)$. With this choice of U and V,

$$\mathrm{res} \colon U \to \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

is a continuous bijection.

Finally we establish continuity of res^{-1} using a slice. The existence of a slice $S \subset \mathcal{R}(\Gamma_+, G)$ at res(hol) is proved by Johnson and Millson in [13, Theorem 1.2], who point to Borel and Wallach [5, IX.5.3] for a definition of slice. From the properties of the slice, and as the stabilizer of $\mathrm{hol}|_{\Gamma_+}$ is trivial, the natural map $G \times S \to \mathcal{R}(\Gamma_+, G)$, that maps $(g, s) \in G \times S$ to gsg^{-1}, yields a homeomorphism between $G \times S$ and a neighborhood of the orbit of res(hol), and the projection induces a homeomorphism $S \cong V$. It follows from the product structure that the $A \in G$ that conjugates ρ_+ and $\rho_+ \circ \sigma_*$ is continuous on ρ_+, so the extension of ρ_+ to a representation of the whole Γ is continuous on ρ_+. Then continuity of res^{-1} follows by composing the homeomorphism $V \cong S$ (restricted to the fixed point set of σ^*) with the extension from Γ_+ to Γ, and projecting to $U \subset \mathcal{R}(\Gamma, G)$.   □

As Γ_+ preserves the orientation, next we use the complex structure of the identity component $G_0 = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$.

3.2 Representations in PSL(2, ℂ)

The holonomy of the orientation covering M_+^3 is contained in $\mathrm{PSL}(2, \mathbb{C})$, and it is well defined up to the action of $G = \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ by conjugation. If we furthermore choose an orientation on M_+^3, then the holonomy is unique up to the action by conjugacy of $G_0 = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, and complex conjugation corresponds to changing the orientation. We call the conjugacy class in $\mathrm{PSL}(2, \mathbb{C})$ of the holonomy of M_+^3 the *oriented holonomy*.

We consider
$$\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})) = \hom(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})) / \mathrm{PSL}(2,\mathbb{C}).$$

Its local structure is well known:

Theorem 3.5 *A neighborhood of the oriented holonomy of M_+^3 in $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ has a natural structure of a $\mathbb{C}$–analytic variety defined over $\mathbb{R}$.*

The fact that it is $\mathbb{C}$–analytic follows, for instance, from [13] or [14]. In Theorem 3.9 we precisely describe $\mathbb{C}$–analytic coordinates; for the moment this is sufficient for our purposes.

Lemma 3.6 *Let hol_+ be the oriented holonomy of M_+^3. Then*
$$[\mathrm{hol}_+] \neq [\overline{\mathrm{hol}}_+] \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})).$$

Namely, the oriented holonomy and its complex conjugate are not conjugate by a matrix in $\mathrm{PSL}(2,\mathbb{C})$.

Proof For contradiction, assume that hol_+ and $\overline{\mathrm{hol}}_+$ are conjugate by a matrix in $\mathrm{PSL}(2,\mathbb{C})$, so there exists an orientation-preserving isometry $A \in \mathrm{PSL}(2,\mathbb{C})$ such that
$$A\,\mathrm{hol}_+(\gamma)A^{-1} = \overline{\mathrm{hol}_+(\gamma)} \quad \text{for all } \gamma \in \Gamma_+.$$

Consider the orientation-reversing isometry $B = c \circ A$, where c is the isometry with Möbius transformation complex conjugation, $z \mapsto \bar{z}$. The previous equation is equivalent to

(12) $$B\,\mathrm{hol}_+(\gamma)B^{-1} = \mathrm{hol}_+(\gamma) \quad \text{for all } \gamma \in \Gamma_+.$$

Brouwer's fixed point theorem yields that the fixed point set of B in the ball compactification $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ is nonempty:
$$\mathrm{Fix}(B) = \{x \in \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \mid B(x) = x\} \neq \varnothing.$$

By (12), $\mathrm{hol}_+(\Gamma_+)$ preserves $\mathrm{Fix}(B)$. Thus, by minimality of the limit set of a Kleinian group, since $\mathrm{Fix}(B) \neq \varnothing$ is closed and $\mathrm{hol}_+(\Gamma_+)$–invariant, it contains the whole ideal boundary: $\partial_\infty \mathbb{H}^3 \subset \mathrm{Fix}(B)$. Hence B is the identity, contradicting that B reverses the orientation. $\square$
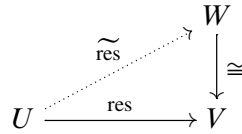
From Lemma 3.6 and Theorem 3.5, we obtain:

Corollary 3.7 *There exists a neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ of the conjugacy class of the oriented holonomy of M_+ that is disjoint from its complex conjugate:*
$$\overline{W} \cap W = \varnothing.$$

By choosing the neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ sufficiently small, we may assume that its projection to $\mathcal{R}(\Gamma_+, G)$ is contained in V, as in Lemma 3.4. The neighborhood V can also be chosen smaller, to be equal to the projection of W, as this map is open. Namely the neighborhoods can be chosen

so that $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C})) \to \mathcal{R}(\Gamma_+, G)$ restricts to a homeomorphism between W (or $\overline{W}$) and V. In particular, we can lift to W the restriction map from U to V:

$$
\begin{array}{ccc}
 & & W \\
 & \overset{\widetilde{\mathrm{res}}}{\nearrow} & \downarrow \cong \\
U & \xrightarrow{\quad \mathrm{res} \quad} & V
\end{array}
$$

Lemma 3.8 *For $U \subset \mathcal{R}(\Gamma, G)$ and $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$ as above, the lift of the restriction map yields a homeomorphism*

$$\widetilde{\mathrm{res}} \colon U \xrightarrow{\cong} \{[\rho] \in W \mid [\rho \circ \sigma_*] = [\bar{\rho}]\}.$$

This lemma has same proof as Lemma 3.4, just taking into account that $\rho(\zeta) \in G$ reverses the orientation, for $[\rho] \in U$ and $\zeta \in \Gamma \setminus \Gamma_+$.

3.3 Local coordinates

Here we give the local coordinates of Theorem 3.5 and prove a stronger version of Theorem 3.1.

For $\gamma \in \Gamma_+$ and $[\rho] \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$, as defined by Culler and Shalen in [7],

$$(13) \qquad\qquad\qquad I_\gamma([\rho]) = (\mathrm{trace}(\rho(\gamma)))^2 - 4.$$

Thus I_γ is a function from $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$ to $\mathbb{C}$, which plays a role in the generalization of Theorem 3.1.

Theorem 3.9 *Let M_+^3 be as above, and assume that it has n cusps. Chose $\gamma_1, \ldots, \gamma_n \in \Gamma_+$ a nontrivial element for each peripheral subgroup. Then, for a neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$ of the oriented holonomy,*

$$(I_{\gamma_1}, \ldots, I_{\gamma_n}) \colon W \to \mathbb{C}^n$$

defines a bianalytic map between W and a neighborhood of the origin.

This theorem holds for any orientable hyperbolic manifold of finite volume, though we only use it for the orientation covering. Again, see [4; 14] for a proof. As explained in these references, this is the algebraic part of the proof of Thurston's hyperbolic Dehn filling theorem using varieties of representations.

For a Klein bottle K^2, in Definition 2.8 we considered the presentation of its fundamental group

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements a^2 and b are called *distinguished* elements. Recall that, in terms of paths, those are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves).

Here we prove the following generalization of Theorem 3.1:

Theorem 3.10 *Let M^3 be a nonorientable manifold of finite volume with k nonorientable cusps and l orientable cusps. For each horospherical Klein bottle K_i^2, chose $\gamma_i \in \pi_1(K_i^2)$ distinguished for $i = 1, \ldots, k$. For each horospherical torus T_j^2, chose a nontrivial $\mu_j \in \pi_1(T_j^2)$ for $j = 1, \ldots, l$.*

There exists a neighborhood $U \subset \mathcal{R}(\Gamma, G)$ of the holonomy of M^3 such that the map

$$(I_{\gamma_1}, \ldots, I_{\gamma_k}, I_{\mu_1}, \ldots, I_{\mu_l}) \circ \widetilde{\mathrm{res}} \colon U \to \mathbb{R}^k \times \mathbb{C}^l$$

defines a homeomorphism between U and a neighborhood of the origin in $\mathbb{R}^k \times \mathbb{C}^l$.

Proof Let $M_+^3 \to M^3$ be the orientation covering. By construction, by the choice of distinguished elements in the peripheral Klein bottles, $\gamma_i \in \Gamma_+$. Furthermore, as the peripheral tori are orientable, $\mu_j \in \Gamma_+$. Hence

$$\{\gamma_1, \ldots, \gamma_k, \mu_1, \ldots, \mu_l, \sigma_*(\mu_1), \ldots, \sigma_*(\mu_l)\}$$

gives a nontrivial element for each peripheral subgroup of Γ_+, where σ_* is the group automorphism from Definition 3.3. We apply Theorem 3.9, which gives that

$$I = (I_{\gamma_1}, \ldots, I_{\gamma_k}, I_{\mu_1}, \ldots, I_{\mu_l}, I_{\sigma_*(\mu_1)}, \ldots, I_{\sigma_*(\mu_l)}) \colon W \to \mathbb{C}^{k+2l}$$

is a bianalytic map with a neighborhood of the origin. Furthermore, as $\sigma_*(\gamma_i) = \gamma_i^{\pm 1}$ and $(\sigma^*)^2 = \mathrm{Id}$,

$$I \circ \sigma^* \circ I^{-1}(x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_l) = (x_1, \ldots, x_k, z_1, \ldots, z_l, y_1, \ldots, y_l).$$

In addition, by construction I commutes with complex conjugation. Hence, by Lemma 3.8, the image $(I \circ \widetilde{\mathrm{res}})(U)$ is the subset of a neighborhood of the origin in $\mathbb{C}^{k+2l}$ defined by

$$\begin{cases} x_i = \bar{x}_i & \text{for all } i = 1, \ldots, k, \\ z_j = \bar{y}_j & \text{for all } j = 1, \ldots, l. \end{cases}$$

Finally, by combining Theorem 3.9 and Lemma 3.8, the map $I \circ \widetilde{\mathrm{res}}$ is a homeomorphism between U and its image. $\qquad\square$

We now state the generalization of Theorem 1.1 to several cusps. Here $D(1) \subset \mathbb{C}$ denotes a disk of radius 1.

Theorem 3.11 *Let M^3 be a complete nonorientable hyperbolic* 3*–manifold of finite volume with k nonorientable cusps and l orientable cusps.*

(a) *If M^3 admits a geometric ideal triangulation Δ, then $\mathrm{Def}(M^3, \Delta) \cong (-1, 1)^k \times D(1)^l$. The parameters $(\pm t_1, \ldots, \pm l_k, \pm u_1, \ldots, \pm u_l) \in (-1, 1)^k \times D(1)^l$ correspond to the same structure.*

(b) *A neighborhood of the holonomy in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to $(-1, 1)^k \times D(1)^l$.*

Furthermore, the holonomy map $\mathrm{Def}(M^3, \Delta) \to \mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is written, in coordinates, as

$$(-1, 1)^k \times D(1)^l \to (-1, 1)^k \times D(1)^l,$$

$$(t_1, \ldots, t_k, v_1, \ldots, v_l) \mapsto (t_1^2, \ldots, t_k^2, v_1^2, \ldots, v_l^2).$$

Namely, each interval $(-1, 1)$ is folded along 0 and has image $[0, 1)$, and disks $D(1)$ are mapped to disks by a $2 \colon 1$ branched covering.

Proof Assertion (a) is Theorem 2.14, and assertion (b) is Theorem 3.10. To describe the holonomy map in coordinates, for each cusp (orientable or not) choose an orientation-preserving peripheral element m and let v be the logarithm of the holonomy of m, defined as in (9), in a neighborhood of the origin

in $\mathbb{C}$ (with $v \in i\mathbb{R}$ in the nonorientable case). In particular v is a component of the local coordinates of $\mathrm{Def}(M^3, \Delta)$. Furthermore, the holonomy of m is conjugate to

$$\pm \begin{pmatrix} e^{v/2} & 1 \\ 0 & e^{-v/2} \end{pmatrix}.$$

So it has trace $\pm 2 \cosh\left(\frac{1}{2}v\right)$, which is a component of the local coordinates of $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. Then the assertion follows from applying a suitable coordinate change. $\qquad\square$

4 Representations of the Klein bottle

Let $\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ be a presentation of the fundamental group of the Klein bottle, and $G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$. The variety of representations $\hom(\pi_1(K^2), G)$ is identified as

$$\hom(\pi_1(K^2), G) \cong \{A, B \in G \mid ABA^{-1} = B^{-1}\}.$$

Topologically we can expect to have at least four (possibly empty) connected components according to the orientable nature of A and B. We are interested in studying one of them.

Definition 4.1 A representation $\rho \in \hom(\pi_1(K^2), G)$ is said to *preserve the orientation type* if, for every $\gamma \in \pi_1(K^2)$, $\rho(\gamma)$ is an orientation-preserving isometry if and only if γ is represented by an orientation-preserving loop of K^2. We denote this subspace of representations by

$$\hom_+(\pi_1(K^2), G).$$

Let $T^2 \to K^2$ be the orientation covering. The restriction map on the varieties of representations (without quotienting by conjugation) is

$$\mathrm{res}\colon \hom(\pi_1(K^2), G) \to \hom(\pi_1(T^2), \mathrm{PSL}(2, \mathbb{C})).$$

Theorem 4.2 *Let $\rho \in \hom_+(\pi_1(K^2), G)$ preserve the orientation type and let $\rho(b) \neq \mathrm{Id}$. By writing $A = \rho(a)$ and $B = \rho(b)$ as Möbius transformations, up to conjugation one of the following holds:*

(a) $A(z) = \bar{z} + 1$ *and* $B(z) = z + \tau i$, *with* $\tau \in \mathbb{R}_{>0}$.

(a′) $A(z) = \bar{z}$ *and* $B(z) = z + \tau i$, *with* $\tau \in \mathbb{R}_{>0}$.

(b) $A(z) = e^l \bar{z}$ *and* $B(z) = e^{\alpha i} z$, *with* $l \in \mathbb{R}_{\geq 0}$ *and* $\alpha \in (0, \pi]$.

(c) $A(z) = e^{\alpha i}/\bar{z}$ *and* $B(z) = e^l z$, *with* $l \in \mathbb{R}_{>0}$ *and* $\alpha \in [0, \pi]$.

Proof Let $G^0 = \mathrm{PSL}(2, \mathbb{C}) \lhd G$ be the connected component of the identity. The variety of representations $\hom(\pi_1(T^2), G^0)/G^0$ is well known. A representation $[\rho_0]$ in this variety is the class of a parabolic representation with $\rho_0(l)(z) = z + 1$, $\rho_0(m)(z) = z + \tau$ and $\tau \in \mathbb{C}$, a parabolic degenerated one with $\rho_0(l)(z) = z$, $\rho_0(m)(z) = z + \tau$ and $\tau \in \mathbb{C}$, or a hyperbolic one with $\rho_0(l)(z) = \lambda z$, $\rho_0(m)(z) = \mu z$ and $\lambda, \mu \in \mathbb{C}$, where $\pi_1(T^2) = \langle l, m \mid lm = ml \rangle$.

For $\rho_0 = \mathrm{res}(\rho)$, let $A = \rho(a)$ and $B = \rho(b)$, where a and b are generators of $\pi_1(K^2)$, and $L = \rho(l)$ and $M = \rho(m)$. Then

$$(A^2, B) = (L, M) \quad \text{(restriction of a representation to the torus)},$$

$$ABA^{-1} = B^{-1} \qquad \text{(Klein bottle relation)}.$$

In fact, in order for ρ_0 to be a restriction, there must be A and B satisfying the previous conditions. We prove the theorem using these equations.

If $[\rho]$ is in the parabolic case, by hypothesis $\tau \neq 0$. Then the solution is unique, $A(z) = \bar{z} + 1$, $B(z) = z + \tau i$ and $\tau \in \mathbb{R} \setminus \{0\}$, and hence $L(z) = z + 2$, $M(z) = z + \tau i$. Similarly, for the degenerated parabolic case, $A(z) = \bar{z}$, $B(z) = z + \tau i$ and $\tau \in \mathbb{R} \setminus \{0\}$.

On the other hand, for $[\rho]$ hyperbolic, either L corresponds to a real dilation and M to a rotation, or the other way around. In the case $L(z) = e^{2l} z$, $M(z) = e^{\alpha i} z$, $l \in \mathbb{R}$ and $\alpha \in (-\pi, \pi]$, the representation can be written as the restriction of several representations of the Klein bottle, but all of them are conjugated to $A(z) = e^l \bar{z}$ and $B(z) = e^{\alpha i} z$. A similar situation happens when $L(z) = e^{2\alpha i} z$, $M(z) = e^l z$, $l \in \mathbb{R}$ and $\alpha \in (-\pi, \pi]$, obtaining $A(z) = e^{\alpha i}/\bar{z}$ and $B(z) = e^l z$. However, in the last case we should note down that, for every such representation $[\rho]$, we get two nonconjugated representations $[\rho_1]$ and $[\rho_2]$ such that $[\rho_0] = \mathrm{res}([\rho_1]) = \mathrm{res}([\rho_2])$, where they differ in that $A_1(z) = e^{\alpha i}/\bar{z}$ and $A_2(z) = e^{(\alpha+\pi)i}/\bar{z} = -e^{\alpha i}/\bar{z}$.

Thus, we obtain a classification of representations in $\hom(\pi_1(K^2), G)/G^0$. To get the classification quotienting by the whole group $\hom(\pi_1(K^2), G)/G$, we only have to see how complex conjugation c acts by conjugation on each representation. In (a) and (a′), c maps $z + \tau i$ to $z - \tau i$, in (b) $e^{\alpha i} z o$ to $e^{-\alpha i} z$, and in (c) $e^{\alpha i}/\bar{z}$ to $e^{-\alpha i}/\bar{z}$. The choice $\alpha > 0, l > 0$ in (b) and (c) is obtained by taking into account that $[\rho] = [\rho^{-1}]$.　□

Definition 4.3 According to the cases in [Theorem 4.2](), a representation $\rho \in \hom_+(\pi_1(K^2), G)$ is called

- *parabolic nondegenerate* in case (a) and *parabolic degenerate* in case (a′),
- *type I* in case (b), and
- *type II* in case (c).

Further, types I and II are called nondegenerate if $l \neq 0$ or $\alpha \neq 0$, respectively, and degenerate otherwise.

Remark 4.4 The holonomy of a nonorientable cusp restricts to a representation of the Klein bottle that preserves the orientation type and is parabolic nondegenerate.

Furthermore, deformations of this representation still preserve the orientation type and are nondegenerate (possibly of type I or II), by continuity.

For $\gamma \in \pi_1(T^2) \lhd \pi_1(K^2)$, recall from [(13)]() that

$$I_\gamma \colon \hom(\pi_1(K^2), G) \to \mathbb{C}, \quad \rho \mapsto (\mathrm{trace}_{\mathrm{PSL}(2,\mathbb{C})}(\rho(\gamma)))^2 - 4,$$

where $\mathrm{trace}_{\mathrm{PSL}(2,\mathbb{C})}$ means the trace as a matrix in $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 4.5 *Let $\rho \in \hom(\pi_1(K^2), G)$ preserve the orientation type and $\rho(b) \neq \mathrm{Id}$.*

- *If ρ is parabolic, then $I_\gamma(\rho) = 0$ for all $\gamma \in \pi_1(T^2)$.*
- *If ρ is of type I, then $I_{a^2}(\rho) \geq 0$ and $I_b(\rho) < 0$.*
- *If ρ is of type II, then $I_{a^2}(\rho) \leq 0$ and $I_b(\rho) > 0$.*

Proof This is a straightforward computation from Theorem 4.2. □

Corollary 4.6 (a) *The holonomy of a representation in $\mathrm{Def}(M, \Delta)$ is of type I.*

(b) *Representations in a neighborhood of $[\mathrm{hol}]$ in $\mathcal{R}(M^3, G)$ are of both type I and II.*

(c) *In particular, the holonomy map $\mathrm{Def}(M, \Delta) \to \mathcal{R}(M^3, G)$ is not surjective in a neighborhood of the holonomy.*

Proof Assertion (a) follows from Remark 2.11 and (b) from Theorem 3.10, both using Lemma 4.5. □

5 Metric completion

As we deform noncompact manifolds, the deformations into noncomplete manifolds are not unique (eg one can consider proper open subsets of a noncomplete manifold). We are not discussing the different issues related to this nonuniqueness, just the existence of a deformation into a metric that can be complete as a conifold (see below).

The main result of this section is Theorem 5.15. In the orientable case, the metric completion after deforming an orientable cusp is a singular space with a singularity called of *Dehn type* (this includes nonsingular manifolds); see Hodgson's thesis [11] and Boileau and Porti [4, Appendix B]. In the nonorientable case, the singularity is more specific, a so-called conifold.

5.1 Conifolds and cylindrical coordinates

A *conifold* is a metric length space locally isometric to the metric cone of constant curvature on a spherical conifold of dimension one less; see for instance [3]. When, as topological space, a conifold is homeomorphic to a manifold, it is called a *cone manifold*, but in general it is only a pseudomanifold. In dimension two, conifolds are also cone manifolds, but in dimension three there may be points with a neighborhood homeomorphic to the cone on a projective plane P^2.

We are interested in three local models of singular spaces as conifolds:

- The first is the hyperbolic cone over a round sphere S^2. This corresponds to a point with a nonsingular hyperbolic metric.
- The second is the hyperbolic cone over $S^2(\alpha, \alpha)$, the sphere with two cone points of angle α, that is the spherical suspension of a circle of perimeter α. This corresponds to a singular axis of angle α.
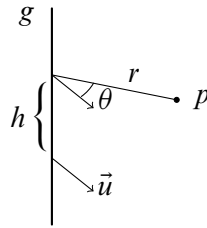
Figure 2: Cylindrical coordinates.

- The third is the hyperbolic cone over $P^2(\alpha)$, the projective plane with a cone point of angle α. This is the quotient of the previous one by a metric involution, which is the antipodal map on each concentric sphere.

Next we describe metrically those local models by using cylindrical coordinates in the hyperbolic space. These coordinates are defined from a geodesic line g in $\mathbb{H}^3$, and we fix a point in the unit normal bundle to g, ie a vector $\vec{u}$ of norm 1 and perpendicular to g. Cylindrical coordinates give a diffeomorphism

$$\mathbb{H}^3 \setminus g \xrightarrow{\cong} (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}, \quad p \mapsto (r, \theta, h),$$

where r is the distance between g and p, θ is the angle parameter (the angle between the parallel transport of $\vec{u}$ and the tangent vector to the orthogonal geodesic from g to p) and h is the arc parameter of g, the signed distance between the base point of $\vec{u}$ and the orthogonal projection from p to g; see Figure 2.

In the upper half-space model of $\mathbb{H}^3$, if g is the geodesic from 0 and ∞, then there exists a choice of coordinates (a choice of $\vec{u}$) such that the projection from g to the ideal boundary $\partial_\infty \mathbb{H}^3$ maps a point with cylindrical coordinates (r, θ, h) to $e^{h+i\theta} \in \mathbb{C}$; see Figure 3. A different choice of $\vec{u}$ would yield instead $\lambda e^{h+i\theta} \in \mathbb{C}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

The hyperbolic metric on $\mathbb{H}^3$ in these coordinates is

$$dr^2 + \sinh^2(r)\, d\theta^2 + \cosh^2(r)\, dh^2.$$

More precisely, $\mathbb{H}^3$ is the metric completion of $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ with this metric.
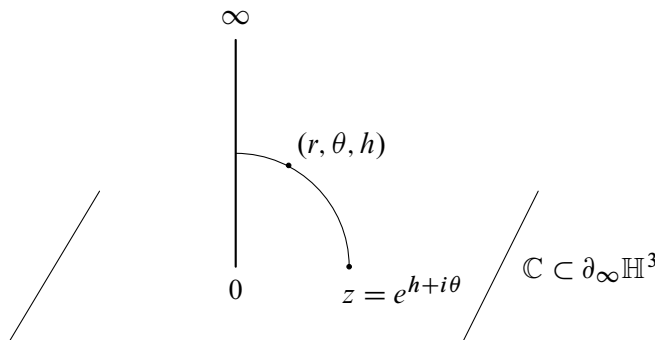
Figure 3: Orthogonal projection to $\partial_\infty \mathbb{H}^3$ with g the geodesic with ideal endpoints 0 and ∞.

Definition 5.1 For $\alpha \in (0, 2\pi)$, $\mathbb{H}^3(\alpha)$ is the metric completion of $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ for the metric

$$ds^2 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 \sinh^2(r)\, d\theta^2 + \cosh^2(r)\, dh^2.$$

The metric space $\mathbb{H}^3(\alpha)$ may be visualized by taking a sector in $\mathbb{H}^3$ of angle α and identifying its sides by a rotation. Alternatively, with the change of coordinates $\tilde{\theta} = \alpha/(2\pi)\theta$, the metric completion of $(0, +\infty) \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}$ is $\mathbb{H}^3(\alpha)$, for the metric $dr^2 + \sinh^2(r)\, d\tilde{\theta}^2 + \cosh^2(r)\, dh^2$.

Remark 5.2 The metric models are

- $\mathbb{H}^3$ for the nonsingular case (the cone on the round sphere),

- $\mathbb{H}^3(\alpha)$ for the singular axis (the cone on $S^2(\alpha, \alpha)$),

- the quotient

$$\mathbb{H}^3(\alpha)/(r, \theta, h) \sim (r, -\theta, -h)$$

  for the cone on $P^2(\alpha)$.

5.2 Conifolds bounded by a Klein bottle

We keep the notation of Section 5.1, with cylindrical coordinates. Before discussing conifolds bounded by a Klein bottle, we describe a cone manifold bounded by a torus.

Definition 5.3 A *solid torus with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric action of $\mathbb{Z}$ generated by

$$(r, \theta, h) \mapsto (r, \theta + \tau, h + L)$$

for $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ and $L > 0$.

The space $\mathbb{H}^3(\alpha)/\sim$ is a solid torus of infinite radius with singular soul of cone angle α, length of the singularity $L > 0$ and torsion parameter $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ (the rotation angle induced by parallel transport along the singular geodesic is $\alpha/(2\pi)\tau \in \mathbb{R}/\alpha\mathbb{Z}$).

By considering the metric neighborhood of radius $r_0 > 0$ on the singular soul, we get a compact solid torus, bounded by a 2–torus. This compact solid torus depicts a tubular neighborhood of a component of the singular locus of a cone manifold (compare Hodgson and Kerckhoff [12] and Hodgson's thesis [11]).

We describe two conifolds bounded by a Klein bottle that are a quotient of this solid torus by an involution.

Definition 5.4 A *solid Klein bottle with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric action of $\mathbb{Z}$ generated by

$$(r, \theta, h) \mapsto (r, -\theta, h + L)$$

for $L > 0$.
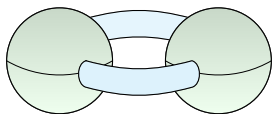
Figure 4: A solid torus as two 3–balls joined by two 1–handles.

The space $\mathbb{H}^3(\alpha)/\sim$ is a solid Klein bottle of infinite radius with singular soul of cone angle α, and length of the singularity $L > 0$. We may consider a metric tubular neighborhood of radius r_0, bounded by a Klein bottle. Its orientation cover is a solid torus with singular soul, cone angle α, length of the singularity $2L$ and torsion parameter $\tau = 0$.

Definition 5.5 The *disc orbibundle with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric involutions

$$(r, \theta, h) \mapsto (r, \theta + \pi, -h) \quad \text{and} \quad (r, \theta, h) \mapsto (r, \theta + \pi, 2L - h)$$

for $L > 0$.

To describe this space, it is useful first to look at the action on the preserved geodesic, corresponding to $r = 0$. These involutions map $h \in \mathbb{R}$ to $-h$ and to $2L - h$, respectively. Thus it is the action of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ on a line generated by two reflections. Its orientation-preserving subgroup is $\mathbb{Z}$ acting by translations on $\mathbb{R}$. Thus $\mathbb{R}/\mathbb{Z}$ is a circle, and $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$ is an orbifold. The solid torus is a disc bundle over the circle, and our space is an orbifold-bundle over $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$ with fiber a disc.

This space is the quotient of an involution on the solid torus. View the solid torus as two 3–balls joined by two 1–handles; see Figure 4. On each 3–ball, apply the antipodal involution (on each concentric sphere of given radius), and extend this involution by permuting the 1–handles. The quotient of each ball is the (topological) cone on P^2, and hence our space is the result of joining two cones on P^2 by a 1–handle. Its boundary is the connected sum $P^2 \# P^2 \cong K^2$.

The singular locus of the disc orbibundle $\mathbb{H}^3(\alpha)/\sim$ is an interval (the underlying space of the orbifold bundle) of length L. The interior points of the singular locus have cone angle α, and the boundary points of the interval are precisely the points where it is not a topological manifold.

Again $\mathbb{H}^3(\alpha)/\sim$ has radius ∞, and the metric tubular neighborhood of radius r of the singularity is bounded by a Klein bottle. It is the quotient of a solid torus of length $2L$ and torsion parameter $\tau = 0$ by an isometric involution with two fixed points (thus, as an orbifold, its orientation orbicovering is a solid torus).

Remark 5.6 The boundaries of both the solid Klein bottle and the disc orbibundle are Klein bottles. In both cases the holonomy preserves the orientation type, but the type of the presentation as in Definition 4.3 is different:

 (a) The holonomy of the boundary of a solid Klein bottle with singular soul is a representation of type I.

 (b) The holonomy of the boundary of a disc orbibundle over a singular interval is of type II.

For a nonorientable cusp, the holonomy of the peripheral torus is either parabolic nondegenerate, of type I or of type II, also nondegenerate (Remark 4.4). The aim of next section is to prove that the deformations can be defined so that the metric completion is either a solid Klein bottle with singular soul or a disc orbibundle with singular soul, according to the type. This is the content of Theorem 5.15, which we prove at the end of the section.

5.3 The radial structure

Let M^3 be a noncompact hyperbolic 3–manifold of finite volume. We deform its holonomy representation and accordingly we deform its hyperbolic metric. Nonetheless, incomplete metrics are not unique, so here we give a statement about the existence of a maximal structure, which corresponds to the one completed in Theorem 5.15.

Let $[\rho] \in \mathcal{R}(\pi_1(M^3), G)$ be a deformation of its complete structure. There is some nuance in associating to $[\rho]$ a hyperbolic structure which is made explicit by Canary, Epstein and Green in [6]. Here, the authors conclude that deformations with a given holonomy representation are related by an isotopy of the inclusion of M^3 in some fixed thickening $(M^3)^*$, where a thickening is just another hyperbolic 3–manifold containing ours.

We will start by making clear what we mean by a maximal structure.

Definition 5.7 Let M be a manifold with an analytic (G, X)–structure. We say that M^* is an *isotopic thickening of M* if it is a thickening and there is a isotopy i' of the inclusion $i : M \hookrightarrow M^*$ such that $i'(M) = M^*$.

Given two isotopic thickenings of M, we say that $M_1^* \leq M_2^*$ if there is a (G, X) isomorphism from M_1^* to some subset of M_2^* extending the identity on M. Hence, we say that an isotopic thickening is maximal if it is maximal with respect the partial order relation we have just defined.

In general, it is not clear whether maximal isotopic thickenings exist, nor under which circumstances they do exist. However, we will construct in our situation an explicit maximal thickening.

Lemma 5.8 Let $\mathrm{inj}_{M^3}(x)$ denote the injectivity radius at a point $x \in M^3$. Then a necessary condition for a nontrivial thickening of M^3 to exist is that there must exist a sequence $\{x_n\} \subset M^3$ with $\mathrm{inj}_{M^3}(x_n) \to 0$.

Proof Let us suppose a thickening $(M^3)^*$ exists. Then take a point $x \in \partial((M^3)^* \setminus M^3)$. Any sequence $\{x_n\} \subset M^3$ such that $x_n \to x$ satisfies $\mathrm{inj}_{M^3}(x_n) \to 0$.   □

The purpose of Lemma 5.8 is twofold. First, it gives a condition for a thickening to be maximal (in the sense of the partial order relation we just defined), and second, it shows where a manifold could possibly be thickened. Taking into account a thick–thin decomposition of the manifold, the thickening can only be done in the deformed cusps.

Each cusp of M^3 is diffeomorphic to either $T^2 \times [0, \infty)$ or $K^2 \times [0, \infty)$. Let us consider a proper product compact subset $K^2 \times [0, \lambda]$ or $T^2 \times [0, \lambda]$ of an end, for some $\lambda > 0$, and let us denote by D_ρ the developing map of a structure with holonomy ρ in the equivalence class $[\rho] \in \mathcal{R}(\Gamma, G)$.

Lemma 5.9 *The image of the proper product subset under the developing map, $D_\rho(\widetilde{K}^2 \times [0, \lambda])$ or $D_\rho(\widetilde{T}^2 \times [0, \lambda])$, lies within two tubular neighborhoods of a geodesic $\gamma \in M^3$, that is, in $N_{\epsilon_2}(\gamma) \setminus N_{\epsilon_1}(\gamma)$, where $N_\epsilon(\gamma) = \{x \in \mathbb{H}^3 \mid d(x, \gamma) < \epsilon\}$. Moreover, for every geodesic ray exiting orthogonally from γ, the intersection of the ray with $D_\rho(\widetilde{K}^2 \times [0, \lambda])$ is nonempty and transverse to any section $D_\rho(\widetilde{K}^2 \times \{\mu\})$, $\mu \in [0, \lambda]$, and analogously for an orientable end.*

Proof We use a modified argument of Thurston (see his notes [18, Chapters 4 and 5]) to prove the lemma for a nonorientable end (the same idea goes for an orientable one). The original argument of Thurston shows that, in an ideal triangulated manifold, the image of the universal cover of the end under the developing map is the whole tubular neighborhood except the geodesic. Let $[\rho_0]$ be the parabolic representation corresponding to the complete structure. Then $D_{\rho_0}(\widetilde{K}^2 \times [0, \lambda])$ is the region between two horospheres centered at an ideal point $p_\infty \in \partial_\infty \mathbb{H}^3$. Let $K \subset \widetilde{K}^2 \times [0, \lambda]$ denote a fundamental domain of $K^2 \times [0, \lambda]$. The domain K can be taken so that $D_{\rho_0}(\overline{K})$ is a rectangular prism between two horospheres.

We want to deform $D_{\rho_0}(\overline{K})$ as we deform ρ_0 to ρ. We do that by deforming the horosphere centered at p_∞ to surfaces equidistant to the geodesics γ_ρ invariant by the holonomy of the peripheral subgroup $\rho(\pi_1(K^2))$. The deformation of the horosphere to equidistant surfaces is described in [18, Section 4.4] in the half-space model of $\mathbb{H}^3$; see also Benedetti and Petronio [2, Section E.6.iv]. Alternatively, we can view the deformation of the horosphere to the equidistant surfaces as follows. Considering $\mathbb{Z}^2 < \pi_1(K^2)$ the orientation-preserving subgroup of index 2, $\rho(\mathbb{Z}^2)$ is contained in a unique one-complex parameter subgroup $U_\rho \subset \mathrm{PSL}(2, \mathbb{C})$ — ie U_ρ is the exponential image of a $\mathbb{C}$–line in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. This U_ρ depends continuously on ρ, and given $x \in \mathbb{H}^3$ the orbit $U_\rho(x) = \{g(x) \mid g \in U_\rho\}$ is a surface containing x such that when $\rho = \rho_0$, $U_\rho(x)$ is a horosphere centered at p_∞ and when $\rho \neq \rho_0$, $U_\rho(x)$ is a surface equidistant to the geodesic γ_ρ. Using this construction, the image of the domain $D_{\rho_0}(\overline{K})$ deforms to $D_\rho(\overline{K})$ with the required properties by following the equidistant surfaces for the factor $\widetilde{K}^2$ and the geodesics orthogonal to these surfaces for the factor $[0, \lambda]$. $\square$

Definition 5.10 The geodesic of Lemma 5.9 is called the *soul* of the end.

Remark 5.11 The face of the section of the proper product subset of the cusp $K^2 \times [0, \lambda]$ or $T^2 \times [0, \lambda]$ that is glued to the thick part of the manifold is the section of the cusp which is further away from the geodesic. Hence, we will only consider thickenings "towards" the soul.

Let x be a point in a cusp of the manifold and consider the image under the developing map $y = D_\rho(\tilde{x})$ of any lift $\tilde{x}$. There is only one geodesic segment in $\mathbb{H}^3$ such that $\gamma(0) = y$ and that goes towards the soul orthogonally. In cylindrical coordinates, if $y = (r, \theta, h)$, the image of the geodesic consists of $\{(t, \theta, h) \mid t \in [0, r]\}$. Let us denote by γ_x the corresponding geodesic in M^3.

Theorem 5.12 *There exists a maximal thickening M^* of a half-open product $M = K^2 \times [0, \lambda)$ or $T^2 \times [0, \mu)$. It is characterized by the following property: for every point $x \in M$, the geodesic γ_x can be extended in M^* so that $D_\rho(\tilde{\gamma}_x)$ is the geodesic whose cylindrical coordinates with respect to the soul are $\{(t, \theta, h) \mid t \in (0, r]\}$.*

Proof Given a cusp section $S := K^2$ or T^2, a product subset of the end $K := S \times [0, \lambda]$, a fixed fundamental domain K_0 of K and a small neighborhood $N(K_0)$ of K_0, the set

$$T := \{t \in \mathrm{Deck}(\widetilde{K}/K) \mid tN(K_0) \cap N(K_0) \neq \varnothing\}$$

is finite, where $\mathrm{Deck}(\widetilde{K}/K)$ denotes the group of covering transformations of the universal cover. Hence, we can suppose that $D_{\rho|(T\overline{K}_0)}$ is an embedding.

Let $\mathcal{U}$ be an open cover of K by simply connected charts. For each U, take a lift $U_0 \in \widetilde{\mathcal{U}}$ such that $U_0 \cap K_0 \neq \varnothing$ and consider $D_\rho(U_0)$. Given such a lift U_0, the other possible lifts that could have nonempty intersection with K_0 are tU, for $t \in T$. Furthermore, we can always assume that the chart U coincides with the image of U_0 under the developing map $D_\rho(U_0)$. Thus, we can identify

$$K \cong \left(\bigcup_{U \in \mathcal{U}} D_\rho(U_0) \right) \Big/ \sim,$$

where the equivalence relation is by the action of $\mathrm{hol}(t)$, for $t \in T$.

Each $U \in \mathcal{U}$ can be thickened by identifying U with $D_\rho(U_0)$ and considering, in cylindrical coordinates, the set of rays $R(U) = \{(t, \theta, h) \in \mathbb{H}^3 \setminus \{\mathrm{soul}\} \mid \text{there exists } (t_0, \theta, h) \in U, t < t_0\}$. Given two lifts of two thickened charts $R(U_1)$ and $R(U_2)$ with nonempty intersection with K_0, we glue them together at the points corresponding to $\mathrm{hol}(t)(R(U_1)) \cap R(U_2)$, where $t \in T$. This defines a thickening of the cusp K^*.

We have yet to show that this is isotopic to the original (half-open) product subset. Let us consider the section $S \times \{0\}$ of the cusp; the radial geodesics γ_x for $x \in S \times \{0\}$ define a foliation of K^* of finite length. Moreover, due to Lemma 5.9, the foliation is transversal to $S \times \{0\}$. By reparametrizing the foliation and considering its flow, we obtain a trivialization of the cusp, $K^* \cong S \times [0, \mu)$. Similarly, $K^* \setminus K$ is also a product. This let us construct an isotopy from K^* to K.

For this thickening, clearly $\gamma_x \subset K^*$ can be extended so that $D_\rho(\tilde{\gamma}_x) = \{(t, \theta, h) \mid t \in (0, r]\}$. By taking geodesics γ_x to geodesics through the developing map, it is clear our thickening can be mapped into every other thickening satisfying this property. Furthermore, if we consider the thickenings to be isotopic, we obtain an embedding.

Regarding the maximality, we will differentiate between an orientable end and a nonorientable one. The general idea will be the same: for another isotopic thickening $(K)^{**}$ to include ours, the developing map should map some open set V into a ball W around a point y_0 in the soul, which will lead to a contradiction.

If K is nonorientable, let us denote the distinguishable generators of $\pi_1(K^2)$ by a and b, with $aba^{-1} = b^{-1}$. If $[\rho]$ is type I, y_0 is fixed by $\rho(b)$. Let $y \in W \setminus \{\mathrm{soul}\}$ and $x \in V$ be its preimage. W is invariant by $\rho(b)$
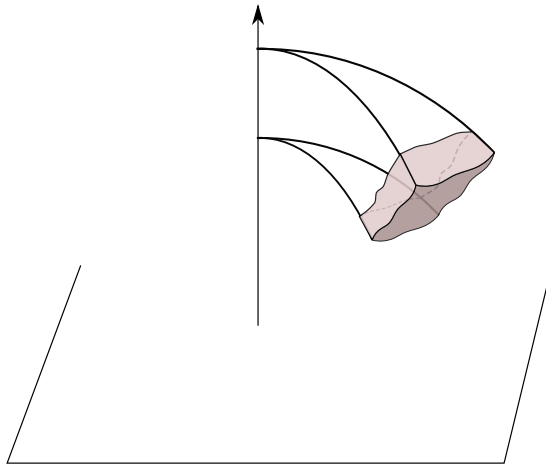
Figure 5: The radial thickening.

and, in addition, both x and $b \cdot x$ belong to V. Now take the geodesic $\gamma : I \mapsto \widetilde{(K)^{**}}$ from x to x_0 which corresponds to the geodesic from y to y_0. By equivariance and continuity, $x_0 = \lim \gamma(t) = \lim b\gamma(t) = bx_0$. This contradicts b being a covering transformation. If $[\rho]$ is type II, the previous argument with a^2 holds.

If K is orientable, we will follow the same arguments leading to the completion of the cusp (for more details see, for instance, [2]). the deformation $[\rho]$ is characterized in terms of its generalized Dehn filling coefficients $\pm(p, q)$. The cases $p = 0$ or $q = 0$ are solved as in the nonorientable cusp, so we have the two usual cases, $p/q \in \mathbb{Q}$ or $p/q \in \mathbb{I}$. For $p/q \in \mathbb{Q}$, there exists $k > 0$ such that $k(p, q) \in \mathbb{Q}^2$ and $(kp)a + (kq)b$ is a trivial loop in the new thickening. If $p/q \in \mathbb{I}$, then y_0 is dense in $\{\text{soul}\} \cap V$, which is a contradiction. □

Definition 5.13 We call the previous thickening the *radial thickening* of the cusp.

Remark 5.14 If the manifold M^3 admits an ideal triangulation, the canonical structure coming from the triangulation is precisely the radial thickening of the cusp.

Theorem 5.15 *For a deformation of the holonomy M^3, the corresponding deformation of the metric can be chosen so that on a nonorientable end*:

- *It is a cusp (a metrically complete end) if the peripheral holonomy is parabolic.*
- *The metric completion is a solid Klein bottle with singular soul if the peripheral holonomy is of type I.*
- *The metric completion is a disc orbibundle with singular soul if it is of type II.*

Furthermore, the cone angle α and the length L of the singular locus are described by the peripheral boundary, so that those parameters start from $\alpha = L = 0$ for the complete structure and grow continuously when deforming in either direction.

Proof The proof uses the orientation covering and equivariance. More precisely, the deformation is constructed in the complete case for the orientation covering and it can be made equivariant. The holonomy of a torus restricted from a Klein bottle is either parabolic or the holonomy of a solid torus with singular soul (and $\tau = 0$). In particular, the holonomy of a Klein bottle is parabolic if and only if its restriction to the orientable covering is parabolic. Furthermore, by using the description in cylindrical coordinates (and using Figure 3) and as $\tau = 0$, the solid torus is equivariant by the action of $\pi_2(K^2)/\pi_1(T^2) \cong \mathbb{Z}_2$. □

6 Example: the Gieseking manifold

We use the Gieseking manifold to illustrate our results. In particular the difference between deformation spaces obtained from ideal triangulations and from the variety of representations.

The Gieseking manifold M is a nonorientable hyperbolic 3–manifold with finite volume and one cusp, with horospherical section a Klein bottle. It has an ideal triangulation with a single tetrahedron. The orientation cover of the Gieseking manifold is the figure eight knot exterior, and the ideal triangulation with one simplex lifts to Thurston's ideal triangulation with two ideal simplices; see Thurston's notes [18].

This manifold M was constructed by Gieseking in his thesis in 1912; here we follow the description of Magnus in [15], using the notation of Alperin, Dicks and Porti [1]. Start with the regular ideal vertex Δ in $\mathbb{H}^3$, with vertices $\{0, 1, \infty, \frac{1}{2}(1 - i\sqrt{3})\}$; see Figure 6. The side identifications are the nonorientable isometries defined by the Möbius transformations

$$U(z) = \frac{1}{1 + \frac{1}{2}(1 + i\sqrt{3})\bar{z}} \quad \text{and} \quad V(z) = -\frac{1}{2}(1 + i\sqrt{3})\bar{z} + 1.$$

The identifications of the faces are defined by their action on vertices:

$$U : \left(\tfrac{1}{2}(1 - i\sqrt{3}), 0, \infty\right) \mapsto \left(\tfrac{1}{2}(1 - i\sqrt{3}), 1, 0\right) \quad \text{and} \quad V : (1, 0, \infty) \mapsto \left(\tfrac{1}{2}(1 - i\sqrt{3}), 1, \infty\right).$$
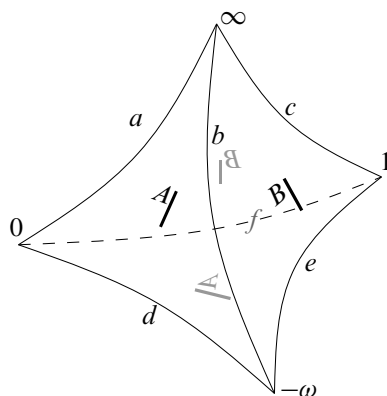
Figure 6: Gieseking manifold with labeled edges.

By applying Poincaré's fundamental theorem,

$$\pi_1(M) \cong \langle U, V \mid VU = U^2 V^2 \rangle. \tag{14}$$

The relation $VU = U^2 V^2$ corresponds to a cycle of length six around the edge.

6.1 The deformation space Def(M, Δ)

We compute the deformation space of the triangulation with a single tetrahedron, as in Section 2.

For any ideal tetrahedron in $\mathbb{H}_3$, we set its ideal vertices as 0, 1, ∞ and $-\omega$, where ω is in $\mathbb{C}_+$, the upper half-space of $\mathbb{C}$. The role played by $-\omega$ will be that of $\frac{1}{2}(1 - i\sqrt{3})$ in the complete structure. For any such ω it is possible to glue the faces of the tetrahedron in the same pattern as in the Gieseking manifold via two orientation-reversing hyperbolic isometries, which we will likewise call U and V.

For the gluing to follow the same pattern, it must map

$$U: (-\omega, 0, \infty) \to (-\omega, 1, 0) \quad \text{and} \quad V: (1, 0, \infty) \to (-\omega, 1, \infty).$$

The orientation-reversing isometries U and V satisfying this are

$$U(z) = \frac{1}{((1+\omega)/|\omega|^2)\bar{z} + 1} \quad \text{and} \quad V(z) = -(1+\omega)\bar{z} + 1.$$

Although it is always possible to glue the faces in the same pattern as in the Gieseking manifold, the gluing will not always have a hyperbolic structure.

Let us label the edges as in Figure 6. For the topological manifold to be geometric, we only have to check that the pairing is proper; see [17]. In this case, the only condition which we need to satisfy is that the isometry that goes through the only edge cycle is the identity. This is given by

$$a \xrightarrow{V} c \xrightarrow{V} b \xrightarrow{U} d \xrightarrow{U} e \xrightarrow{V^{-1}} f \xrightarrow{U^{-1}} a,$$

and therefore we will have a hyperbolic structure if and only if $U^{-1}V^{-1}U^2V^2 = \mathrm{Id}$. Doing this computation, we obtain the equation

$$|\omega(1+\omega)| = 1. \tag{15}$$

Let us show that this equation matches the one obtained from Definition 2.3. If we denote by $z(a)$ the edge invariant of a and analogously for the rest of the edges, we have that the equation describing the deformation space of the manifold in terms of this triangulation is

$$\frac{z(a)z(b)z(e)}{z(c)z(d)z(f)} = 1.$$

Writing down all of the edge invariants in terms of $z(a)$ by means of the tetrahedron relations results in the equation

$$\frac{z(a)^2 \overline{z(a)}^2}{(1 - z(a))(1 - \overline{z(a)})} = \frac{|z(a)|^4}{|1 - z(a)|^2} = 1. \tag{16}$$
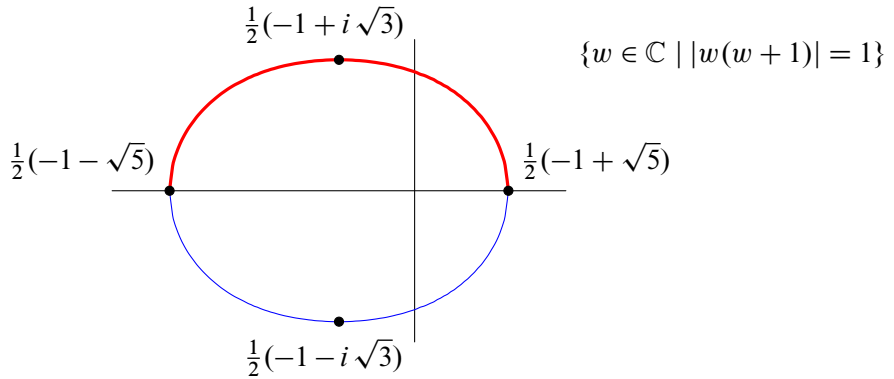
Figure 7: The set of solutions of the compatibility equations and $\mathrm{Def}(M, \Delta)$ (the top half).

If we substitute $z(a) = -1/\omega$, we obtain

$$\frac{1}{\omega\bar{\omega}(\omega + 1)(\bar{\omega} + 1)} = 1,$$

which is equivalent to (15).

Remark 6.1 The set $\{w \in \mathbb{C} \mid |w(1 + w)| = 1\}$ is homeomorphic to S^1, and the deformation space $\{w \in \mathbb{C} \mid |w(1 + w)| = 1 \text{ and } \mathrm{Im}(w) > 0\}$ is homeomorphic to an open interval; see Figure 7.

We justify the remark and Figure 7. Firstly, to prove that the set of algebraic solutions is homeomorphic to a circle, we write the defining equation $|w(1 + w)| = 1$ as

$$\left|\left(w + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right| = 1.$$

Thus $\left(w + \tfrac{1}{2}\right)^2$ lies in the circle of center $\tfrac{1}{4}$ and radius 1. As this circle separates 0 from ∞, the equation defines a connected covering of degree two of the circle. Secondly, the set of algebraic solutions is invariant by the involutions $w \mapsto \bar{w}$ and $w \mapsto -1 - w$, and hence symmetric with respect to the real line and the line defined by the set of points with real part equal to $-\tfrac{1}{2}$. Furthermore, it intersects the real line at $w = \tfrac{1}{2}(-1 \pm \sqrt{5})$ and the line with real part $-\tfrac{1}{2}$ at $\tfrac{1}{2}(-1 \pm i\sqrt{3})$.

Let us construct the link of the cusp. We denote the link of each cusp point as in Figure 8, left, and glue them to obtain the link as in Figure 8, right, which is a Klein bottle.

Now we take two tetrahedra and construct the orientation covering of M (the figure eight knot exterior). For the first tetrahedron, we will define $z_1 := z(a)$, and define z_2 and z_3 so that they follow the cyclic order described in the tetrahedron relations. Afterwards, in the second tetrahedron, we denote by w_i the edge invariant of the corresponding edge after applying an orientation-reversing isometry to the tetrahedron, that is, $w_i = 1/\bar{z}_i$.

We consider the link of the orientation covering. The derivatives of the holonomy of the two loops in the link of the orientation covering l_1 and l_2, depicted in Figure 9, left (which are free homotopic), are $w_1/z_1 = 1/|w_1|^2$ and $w_3/z_3 = 1/|w_3|^2$. For the manifold to be complete we need $\mathrm{hol}'(l_i) = 1$ for $i = 1, 2$,
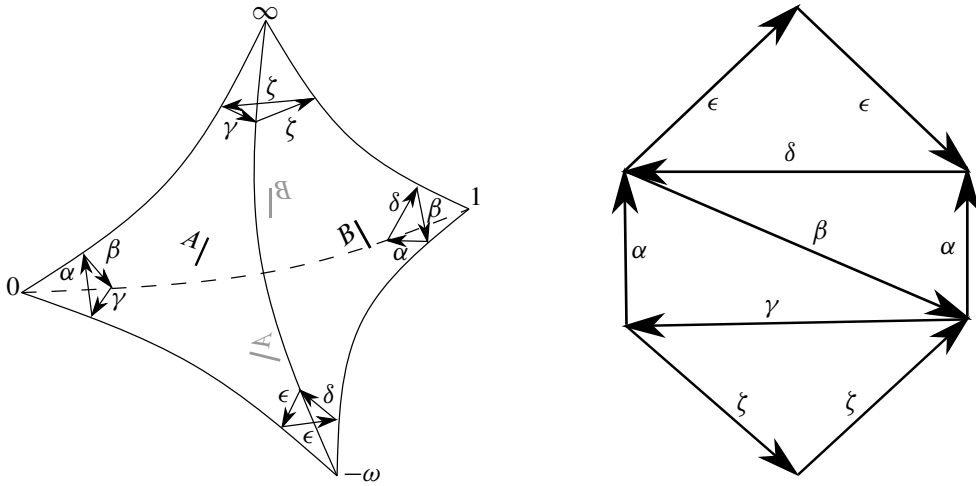
Figure 8: Left: Gieseking manifold with link. Right: link of the cusp point.

which happens if and only if $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. This corresponds to the regular ideal tetrahedron, which, as expected, is the manifold originally given by Gieseking. Notice that the upper loop (the one going through the side ϵ) can be taken as a *distinguished* longitude. A suitable meridian is drawn in Figure 9, right.

Let us check that both the longitude and the meridian satisfy the conditions we stated for their holonomy in Remark 2.11, that is, $\mathrm{hol}'(l) \in \mathbb{R}$ and $|\mathrm{hol}'(m)| = 1$. We have already shown it for the longitude. Regarding the meridian,

$$\mathrm{hol}'(m) = \frac{z_2 z_3 w_2 w_3}{w_2 z_1 z_2 w_1} = \frac{z_1}{z_3}\frac{w_1}{w_3} = \frac{z_1}{\bar{z}_1}\frac{\bar{z}_3}{z_3},$$

therefore $|\mathrm{hol}'(m)| = 1$. This leads to the result that the generalized Dehn filling coefficients of a lifted structure have the form $(0, q)$, after an appropriate choice of longitude–meridian pair.

The last result could also have been obtained from Thurston's triangulation. By rotating the tetrahedra, our triangulation could be related with his, and the parameters identified. We can then check that, in his choice of longitude and meridian, the holonomy has the same features if the structure is a lift from the Gieseking manifold.

6.2 The Gieseking manifold as a punctured torus bundle

The Gieseking manifold M is fibered over the circle with fiber a punctured torus $T^2 \setminus \{*\}$. We use this structure to compute the variety of representations. The monodromy of the fibration is an automorphism

$$\phi : T^2 \setminus \{*\} \to T^2 \setminus \{*\}.$$

The map ϕ is the restriction of a map of the compact torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ that lifts to the linear map of $\mathbb{R}^2$ with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
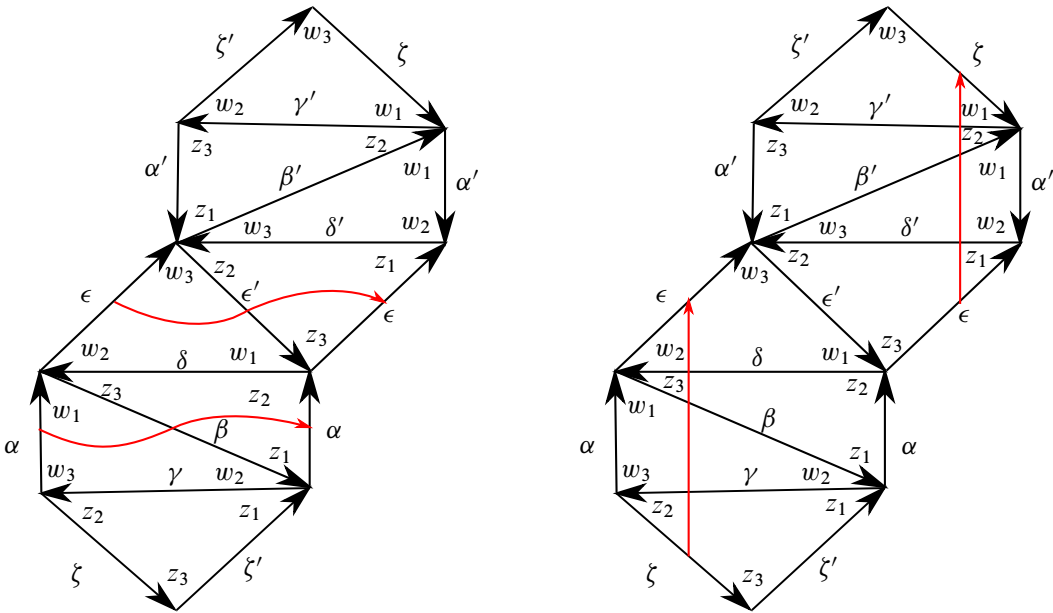
Figure 9: Left: two free homotopic loops. Right: meridian in the link of the cover.

This matrix also describes the action on the first homology group $H_1(T^2 \setminus \{*\}, \mathbb{Z}) \cong \mathbb{Z}^2$. The map ϕ is orientation reversing (the matrix has determinant -1) and ϕ^2 is the monodromy of the orientation covering of M, the figure eight knot exterior.

The fibration induces a presentation of the fundamental group of M

$$\pi_1(M) \cong \langle r, s, t \mid trt^{-1} = \phi(r), \ tst^{-1} = \phi(s) \rangle,$$

where $\langle r, s \mid \rangle = \pi_1(T^2 \setminus \{*\}) \cong F_2$, and

$$\phi_* \colon F_2 \to F_2, \qquad r \mapsto s, \quad s \mapsto rs,$$

is the algebraic monodromy, the map induced by ϕ on the fundamental group.

The relationship with the presentation (14) of $\pi_1(M)$ from the triangulation is given by

$$r = UV, \quad s = VU, \quad t = U^{-1}.$$

Furthermore, a peripheral group is given by $\langle rsr^{-1}s^{-1}, t \rangle$, which is the group of the Klein bottle.

We use this fibered structure to compute the variety of conjugacy classes of representations. Set

$$G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PO}(3,1) \cong \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2,$$

and let

$$\mathrm{hom}^{\mathrm{irr}}(\pi_1(M), G)$$

denote the space of *irreducible* representations (ie representations that have no invariant line in $\mathbb{C}^2$). As we are interested in deformations, we restrict to representations ρ that preserve the orientation type: $\rho(\gamma)$

is an orientation-preserving isometry if and only if $\gamma \in \pi_1(M)$ is represented by a loop that preserves the orientation of M for all $\gamma \in \pi_1(M)$. We denote the subspace of representations that preserve the orientation type by

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G).$$

Let

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$$

be their the space of their conjugacy classes.

Proposition 6.2 *We have a homeomorphism, via the trace of $\rho(s)$,*

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G \to (\{x \in \mathbb{C} \mid |x - 1| = 1 \text{ and } x \neq 2\})/{\sim}, \quad [\rho] \mapsto \mathrm{trace}(\rho(s)),$$

where $\sim$ is the relation given by complex conjugation.

In particular, $\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$ is homeomorphic to a half-open interval.

Proof Let $\rho \colon \pi_1(M) \to G$ be an irreducible representation. The fiber $T^2 \setminus \{*\}$ is orientable, so the restriction of ρ to the free group $\langle r, s \mid \rangle \cong F_2$ is contained in $\mathrm{PSL}(2, \mathbb{C})$. Furthermore, as $\langle r, s \mid \rangle$ is the commutator subgroup, we may assume that $\rho(\langle r, s \mid \rangle) \subset \mathrm{SL}(2, \mathbb{C})$; see Heusener–Porti [9].

We consider the variety of characters $X(F_2, \mathrm{SL}(2, \mathbb{C}))$ and the action of the algebraic monodromy ϕ_* on the variety of characters:

$$\phi^* \colon X(F_2, \mathrm{SL}(2, \mathbb{C})) \to X(F_2, \mathrm{SL}(2, \mathbb{C})), \quad \chi \mapsto \chi \circ \phi_*.$$

Lemma 6.3 *The restriction of $\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$ to $X(F_2, \mathrm{SL}(2, \mathbb{C}))$ is contained in*

$$\{\chi \in X(F_2, \mathrm{SL}(2, \mathbb{C})) \mid \phi^*(\chi) = \bar{\chi}\}.$$

Proof Let $\rho \in \hom^{\mathrm{irr}}(\pi_1(M), G)$. If we write $\rho(t) = A \circ c$ for $A \in \mathrm{PSL}(2, \mathbb{C})$ and c complex conjugation, from the relation

$$t\gamma t^{-1} = \phi_*(\gamma) \quad \text{for all } \gamma \in F_2$$

we get

$$A\overline{\rho(\gamma)}A^{-1} = \rho(\phi_*(\gamma)) \quad \text{for all } \gamma \in F_2.$$

Hence, if ρ_0 denotes the restriction of ρ to F_2, $\bar{\rho}_0$ and $\rho_0 \circ \phi_*$ are conjugate, so they have the same character and the lemma follows. □

[Lemma 6.3](#) motivates the following computation:

Lemma 6.4 *We have a homeomorphism*

$$\{\chi_\rho \in X(F_2, \mathrm{SL}(2, \mathbb{C})) \mid \phi^*(\chi_\rho) = \bar{\chi}_\rho\} \cong \{x \in \mathbb{C} \mid |x - 1| = 1\}$$

by setting $x = \mathrm{trace}(\rho(s)) = \chi_\rho(s)$.

Proof   First we describe coordinates for $X(F_2, \mathrm{SL}(2, \mathbb{C}))$. Let τ_r, τ_s and τ_{rs} denote the trace functions, ie $\tau_r(\chi_\rho) = \chi_\rho(r) = \mathrm{trace}(\rho(r))$, and similarly for s and rs. The Fricke–Klein theorem yields an isomorphism

$$(\tau_r, \tau_s, \tau_{rs}) \colon X(F^2, \mathrm{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$$

(see Goldman [8] for a proof). From the relations

$$\phi_*(r) = s, \quad \phi_*(s) = rs, \quad \phi_*(rs) = srs,$$

the equality $\phi^*(\chi_\rho) = \bar{\chi}_\rho$ is equivalent to

$$\bar{\tau}_r = \tau_s, \quad \bar{\tau}_s = \tau_{rs}, \quad \bar{\tau}_{rs} = \tau_{srs} = \tau_s \tau_{rs} - \tau_r.$$

In the expression for τ_{srs} we have used the relation $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$ for A, $B \in \mathrm{SL}(2, \mathbb{C})$. Taking $x = \tau_r = \tau_{rs}$ and $\tau_s = \bar{x}$, the defining equation is $x + \bar{x} = x\bar{x}$. Namely, the circle $|x - 1| = 1$.   □

To prove Proposition 6.2, we need to know which conjugacy classes of representations of F^2 are irreducible. By Culler and Shalen [7], a character χ_ρ in $X(F^2, \mathrm{SL}(2, \mathbb{C}))$ is reducible if and only if $\chi_\rho([r, s]) = \mathrm{tr}(\rho([r, s])) = 2$, and a straightforward computation shows that this happens in the circle $|x - 1| = 1$ precisely when $x = 2$. Now, let ρ be a representation of F^2 in $\mathrm{SL}(2, \mathbb{C})$ whose character χ_ρ satisfies $\phi^*(\chi_\rho) = \bar{\chi}_\rho$. Assume ρ is irreducible. Then $\rho \circ \phi_*$ and $\bar{\rho}$ are conjugate by a unique matrix $A \in \mathrm{PSL}(2, \mathbb{C})$,

$$Ac\rho(\gamma)cA^{-1} = A\overline{\rho(\gamma)}A^{-1} = \rho(\phi_*(\gamma)) \quad \text{for all } \gamma \in F_2,$$

where c means complex conjugation. Thus, defining $\rho(t) = A \circ c$ gives a unique way to extend ρ to $\pi_1(M)$.

When χ_ρ is reducible, $x = 2$ and the character χ_ρ is trivial. Then either ρ is trivial or parabolic. In any case, it is easy to check that all possible extensions to $\pi_1(M)$ yield reducible representations.   □

6.3   Comparing both ways of computing deformation spaces

We relate both ways of computing deformation spaces, via the ideal simplex and via the fibration:

Lemma 6.5   *Given a triangulated structure with parameter w as in* (15), *the parameter x of its holonomy as in Proposition 6.2 is*

$$x = 1 + w + |w|^2$$

(*or $x = 1 + \bar{w} + |w|^2$, because x is only defined up to complex conjugation*).

Proof   As $r = UV$, a straightforward computation yields

$$\rho(r) = \begin{pmatrix} 0 & |w|^2 \\ -1/|w|^2 & 1 + w + |w|^2 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Then the lemma follows from $x = \mathrm{trace}(\rho(r))$.   □

The fact that not all deformations are obtained from triangulations (Corollary 4.6) is illustrated in the following remark, whose proof is an elementary computation.
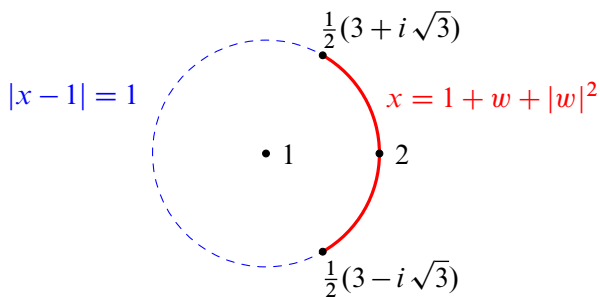
Figure 10: The image of $x = 1 + w + |w|^2$ in the circle $|x - 1| = 1$.

Remark 6.6 The image of the map

$$\{w \in \mathbb{C} \mid |w(1+w)| = 1\} \to \{x \in \mathbb{C} \mid |x - 1| = 1\}, \quad w \mapsto x = 1 + w + |w|^2,$$

is $\{|x - 1| = 1\} \cap \{\operatorname{Re}(x) \geq \frac{3}{2}\}$, ie the arc of a circle bounded by the image of the holonomy structure (and its complex conjugate); see Figure 10.

To be precise on the type of structures at the peripheral Klein bottle, we compute the trace of the peripheral element $[r, s]$ for each method and apply Lemma 4.5:

- We compute it from the variety of representations, ie from x. Using the notation of the proof of Proposition 6.2:

$$\tau_{[r,s]} = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 - 2 = (x + \bar{x})^2 - 3(x + \bar{x}) - 2 = (x + \bar{x})((x + \bar{x}) - 3) - 2.$$

 The complete hyperbolic structure corresponds to $x + \bar{x} = 3$; hence, by deforming x we may have either $\tau_{[r,s]} > -2$ or $\tau_{[r,s]} < -2$.

- Next we compute it from the ideal triangulation, ie from w. As $x = 1 + w + |w|^2$, we get

$$\tau_{[r,s]} = 2 \operatorname{Re}(w + w^2) \geq -2$$

 because $|w + w^2| = 1$.

Remark 6.7 Finally, the path of deformations of the Gieseking manifold lifts to a path of deformations of the figure eight knot exterior that is the same as the one considered by Hilden, Lozano and Montesinos in [10] by deforming polyhedra. The transition from type I to type II of the Gieseking manifold corresponds to the *spontaneous surgery* in [10].

References

[1] **R C Alperin**, **W Dicks**, **J Porti**, *The boundary of the Gieseking tree in hyperbolic three-space*, Topology Appl. 93 (1999) 219–259 MR Zbl

[2] **R Benedetti**, **C Petronio**, *Lectures on hyperbolic geometry*, Springer (1992) MR Zbl

[3]   **M Boileau**, **B Leeb**, **J Porti**, *Geometrization of 3–dimensional orbifolds*, Ann. of Math. 162 (2005) 195–290 MR Zbl

[4]   **M Boileau**, **J Porti**, *Geometrization of 3–orbifolds of cyclic type*, Astérisque 272, Soc. Math. France, Paris (2001) MR Zbl

[5]   **A Borel**, **N Wallach**, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd edition, Mathematical Surveys and Monographs 67, Amer. Math. Soc., Providence, RI (2000) MR Zbl

[6]   **R D Canary**, **D B A Epstein**, **P Green**, *Notes on notes of Thurston*, from "Analytical and geometric aspects of hyperbolic space" (D B A Epstein, editor), London Math. Soc. Lecture Note Ser. 111, Cambridge Univ. Press (1987) 3–92 MR Zbl

[7]   **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of 3–manifolds*, Ann. of Math. 117 (1983) 109–146 MR Zbl

[8]   **W M Goldman**, *Trace coordinates on Fricke spaces of some simple hyperbolic surfaces*, from "Handbook of Teichmüller theory, II" (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 13, Eur. Math. Soc., Zürich (2009) 611–684 MR Zbl

[9]   **M Heusener**, **J Porti**, *The variety of characters in* $\mathrm{PSL}_2(\mathbb{C})$, Bol. Soc. Mat. Mexicana 10 (2004) 221–237 MR Zbl

[10] **H Hilden**, **M T Lozano**, **J M Montesinos-Amilibia**, *On a remarkable polyhedron geometrizing the figure eight knot cone manifolds*, J. Math. Sci. Univ. Tokyo 2 (1995) 501–561 MR Zbl

[11] **C D Hodgson**, *Degeneration and regeneration of geometric structures on* 3–manifolds, PhD thesis, Princeton University (1986) MR Available at `https://www.proquest.com/docview/303436581`

[12] **C D Hodgson**, **S P Kerckhoff**, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom. 48 (1998) 1–59 MR Zbl

[13] **D Johnson**, **J J Millson**, *Deformation spaces associated to compact hyperbolic manifolds*, from "Discrete groups in geometry and analysis" (R Howe, editor), Progr. Math. 67, Birkhäuser, Boston, MA (1987) 48–106 MR Zbl

[14] **M Kapovich**, *Hyperbolic manifolds and discrete groups*, Progr. Math. 183, Birkhäuser, Boston, MA (2001) MR Zbl

[15] **W Magnus**, *Noneuclidean tesselations and their groups*, Pure and Applied Mathematics 61, Academic, New York (1974) MR Zbl

[16] **W D Neumann**, **D Zagier**, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985) 307–332 MR Zbl

[17] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, Graduate Texts in Math. 149, Springer (1994) MR Zbl

[18] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) Available at `https://url.msp.org/gt3m`

Departament de Matemàtiques, Universitat Autònoma de Barcelona

Barcelona, Spain

Departament de Matemàtiques, Universitat Autònoma de Barcelona, and Centre de Recerca Matemàtica

Barcelona, Spain

jduranba.mat@gmail.com,   joan.porti@uab.cat

https://mat.uab.cat/~porti/

ALGEBRAIC & GEOMETRIC TOPOLOGY