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Annular Khovanov homology and augmented links

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Given an annular link L, there is a corresponding augmented link $\widetilde{L}$ in S^3 obtained by adding a meridian unknot component to L. We construct a spectral sequence with the second page isomorphic to the annular Khovanov homology of L that converges to the reduced Khovanov homology of $\widetilde{L}$. As an application, we classify all the links with the minimal rank of annular Khovanov homology. We also give a proof that annular Khovanov homology detects unlinks.

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1 Introduction

Khovanov [9] defined an invariant for links which assigns a bigraded abelian group $\mathrm{Kh}(L)$ to each link $L \subset S^3$. It is a categorification of the Jones polynomial in the sense that it replaces terms in the Jones polynomial by graded abelian groups. Since then, many related invariants have been studied, including Lee's deformation invariant [13] and Rasmussen's s–invariant [15], Khovanov's reduced version [10], the tangle invariant of Bar-Natan [5] and Khovanov–Rozansky homology [11].

Several spectral sequences that reveal the relationship between Khovanov homology theories and Floer theories have been established. The first one is due to Ozsváth and Szabó [14], which builds a connection between the reduced Khovanov homology of the mirror of a link L and the Heegaard Floer homology of the branched double cover of S^3 over L. Kronheimer and Mrowka [12] constructed a spectral sequence with the E_1 term isomorphic to Khovanov homology and converging to a version of singular instanton Floer homology.

Asaeda, Przytycki and Sikora [2] constructed Khovanov-type invariants for links in $\Sigma \times I$, where Σ is a surface. When $\Sigma = A$ is an annulus (sometimes it is convenient to view A as a punctured disk), the resulting invariant is called the *annular Khovanov homology*. Roberts [16] constructed a spectral sequence from annular Khovanov homology to Heegaard Floer homology. Grigsby, Licata and Wehrli [7] studied the analogue of Rasmussen's s–invariant in the annular setting. Xie [18] introduced annular instanton Floer homology for annular links as an analogue of the annular Khovanov homology, and they are also related by a spectral sequence, which can be used to distinguish braids from other tangles; see [18] and Xie and Zhang [19].

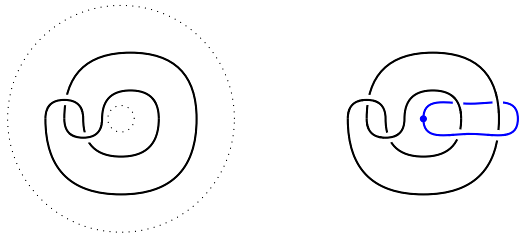

Figure 1: An annular link and its augmentation.

The relationship between annular Khovanov homology and the original Khovanov homology has been previously studied. There is a natural spectral sequence between them given by ignoring the punctured point [16, Lemma 2.3]. Stoffregen and Zhang [17] established a spectral sequence relating the (annular) Khovanov homologies of periodic knots and their quotients.

Considering the augmentation of links is an alternative approach to preserve the information about the punctured point.

Definition 1.1 Let $L \subset A \times I$ be an annular link. The *augmentation* of L is a pointed link $(\widetilde{L}, p) \subset \mathbb{R}^3$ obtained as follows. We view the thickened annulus $A \times I$ as a solid torus in $\mathbb{R}^3$, and $\widetilde{L}$ is given by the union of L and a meridian circle of $A \times I$ (sometimes we call it an *augmenting circle*). The basepoint p is chosen on the augmenting circle.

Under this convention, Xie [18, Section 4.3] showed that the annular instanton Floer homology $\mathrm{AHI}(L)$ is isomorphic to $\mathrm{I}^\natural(\widetilde{L})$, the reduced singular instanton Floer homology of the augmented link. We will prove the following theorem as an analogue of Xie's result on the Khovanov side. To avoid sign issues, all the coefficient rings will be $\mathbb{Z}/2\mathbb{Z}$ unless otherwise specified.

Theorem 1.2 *Let $L \subset A \times I$ be an annular link and let $(\widetilde{L}, p) \subset S^3$ be the corresponding augmented link of L. Then there is a spectral sequence with the E_2 term isomorphic to the annular Khovanov homology $\mathrm{AKh}(L)$ and it converges to the reduced Khovanov homology $\mathrm{Khr}(\widetilde{L}, p)$.*

We immediately obtain the following rank inequality:

Corollary 1.3 *Given an annular link L and its augmentation $\widetilde{L}$, we have*

$$\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) \geq \operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L}, p).$$

Question 1.4 *For what links L is $\mathrm{AKh}(L)$ isomorphic to $\mathrm{Khr}(\widetilde{L}, p)$?*

Theorem 1.2 provides an alternative way to prove some detection results by referring to the parallel consequences in reduced Khovanov homology. For a link L with n components, it is well known that $\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(L, p) \geq 2^{n-1}$. Hence, by the previous corollary, for an annular link L, we have

$$\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) \geq \operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L}, p) \geq 2^n.$$

On the other hand, links of minimal rank in $A \times I$ can be classified following Xie and Zhang [20]. Before stating the result, we first explain the notation. Recall that a *forest* is a graph (not necessarily connected) without cycles. Given a forest G, its corresponding link L_G is defined by assigning to each vertex of G an unknot component and linking two unknots in the way of Hopf links whenever their corresponding vertices are adjacent. For annular links, we need to assign which vertex corresponds to a nontrivial circle. We say such vertices are *annular* for convenience.

Theorem 1.5 *Let L be an n–component annular link. Then $\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) = 2^n$ if and only if L is a forest of unknots such that each connected component of the corresponding graph of L contains at most one annular vertex.*

We say an annular link U is an *unlink* if it has a link diagram D without any crossing. Notice that our definition given here is slightly different to [18]. The following corollary is a generalization of Theorem 3.1 of Baldwin and Grigsby [3] and Corollary 1.4 of Xie and Zhang [19], where the unlinks are required to have all the components trivial or nontrivial:

Corollary 1.6 *Let L be an annular link with n components and let U be an annular unlink with n components (which might be trivial or nontrivial). Assume that*

$$\mathrm{AKh}(L) \cong \mathrm{AKh}(U)$$

as bigraded (by homological and annular gradings) abelian groups. Then L is isotopic to U.

The paper is organized as follows. In Section 2 we review the construction and properties of Khovanov homology. After some preparation in Section 3, we prove Theorem 1.2 in the last section and discuss its applications.

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2 Review of Khovanov homology theories

In this section, we review the construction and properties of the reduced version and the annular version of Khovanov homology.

2.1 Reduced Khovanov homology

The reduced version of Khovanov homology is defined in [10] as a categorification of the (normalized) Jones polynomial. We first recall the definition of the original Khovanov homology.

Figure 2: Two types of smoothings.

For a link diagram D with n crossings, denote the number of right-handed (resp. left-handed) crossings of D by n_+ (resp. n_-). For a crossing of D, we can use the 0–smoothing or 1–smoothing to resolve it, as shown in Figure 2. Fix an order of crossings; we can then use vectors $\boldsymbol{v} \in \{0,1\}^n$ to encode resolutions of D. Denote the resolution indicated by $\boldsymbol{v}$ by $D_{\boldsymbol{v}}$, and let $|\boldsymbol{v}|$ be the number of 1–smoothings in $D_{\boldsymbol{v}}$. Two resolutions that only differ on one smoothing of crossings are related by a cobordism. The resolutions of D are disjoint unions of circles, and the cobordisms are the merging or splitting of circles.

In original Khovanov homology, we apply a (1+1)D TQFT to the resolution cube to obtain a chain complex by assigning to each circle a graded free abelian group $V := \mathbb{Z}/2\mathbb{Z}\{v_+, v_-\}$. The resulting complex has two gradings: the homological one and the quantum one, and the latter is specified by $q\deg v_\pm = \pm 1$. Following [4], we denote the shifts on these two gradings by $[\bullet]$ and $\{\bullet\}$, respectively. We then take a shift on the quantum grading of chain groups by $|\boldsymbol{v}|$ to ensure the differential preserves the quantum grading and a global shift $[-n_-]\{n_+ - 2n_-\}$ to ensure invariance under Reidemeister moves. We finally take cohomology on the chain complex $(\mathrm{CKh}(L), d)$ to obtain $\mathrm{Kh}(L)$.

Remark 2.1 The gradings of Khovanov homology can be read as follows [8]. For a diagram D and a state S associated to a fixed resolution, let $w(D) = n_+ - n_-$ be the writhe number of D, $\sigma(S)$ be the difference between the numbers of 1–resolutions and 0–resolutions of S, and $\tau(S)$ be the difference between the numbers of v_+ and v_- appearing in S. Then the homological grading and the quantum grading of an element are given by

$$h\deg = \tfrac{1}{2}(\sigma(S) + w(D)), \quad q\deg = \tfrac{1}{2}(\sigma(S) + 2\tau(S) + 3w(D)),$$

respectively.

To define the reduced version of Khovanov homology, as in other reduced theories, we need to choose a basepoint p on the link L. Every resolution of L has exactly one circle containing p, and the generators that take v_- (with the q–grading omitted) on this pointed circle span a subcomplex $\mathrm{CKhr}(L,p) \subset \mathrm{CKh}(L)$. The *reduced Khovanov homology* $\mathrm{Khr}(L,p)$ is then defined by the cohomology of $\mathrm{CKhr}(L,p)$. The basepoint is sometime omitted in the notation if it is clear from the context (eg when we are considering an augmented link). As an example, for the Hopf link H with a positive linking number, we have

$$\mathrm{Khr}(H,p) = (\mathbb{Z}/2\mathbb{Z})^{(0,1)} \oplus (\mathbb{Z}/2\mathbb{Z})^{(2,5)}.$$

In general, the following proposition describes the effect on Khovanov homology of making a connected sum with a Hopf link. Here our statement is slightly different to the original description because of the different grading conventions. See [1, Remark 1.6] and Remark 2.1.

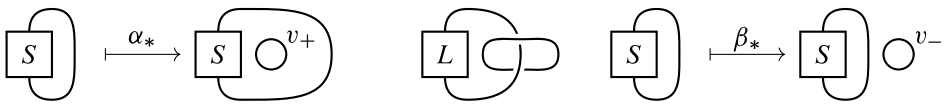

Figure 3: The maps α_* and β_*.

Proposition 2.2 [1, Theorem 6.1] *Let L be a pointed link and let H be the Hopf link with a positive linking number. Then we have a short exact sequence*

$$0 \to \mathrm{Khr}^{i-1,j-2}(L) \xrightarrow{\alpha_*} \mathrm{Khr}^{i+1,j+3}(L \# H) \xrightarrow{\beta_*} \mathrm{Khr}^{i+1,j+2}(L) \to 0.$$

Here α_* and β_* are given on a state S as in Figure 3.

2.2 Annular Khovanov homology

The annular version of Khovanov homology can be viewed as a special case of the link homology for links in thickened surfaces defined in [2]. Let A be an annulus. The annular Khovanov homology assigns a triply graded abelian group $\mathrm{AKh}(L)$ for each annular link $L \subset A \times I$. We follow the process and notation of [18].

Let D be a link diagram of L and define $n, n_\pm, \boldsymbol{v}, D_{\boldsymbol{v}}, V$ as in the previous subsection. In the annular case, there might be two types of circles in a resolution: circles that bound disks and circles with nontrivial homologies. We call the first type of circles *trivial* and the second ones *nontrivial*. To obtain the chain groups, we assign V to trivial circles and assign $W := \mathbb{Z}/2\mathbb{Z}\{w_+, w_-\}$ to nontrivial circles. The differentials are specified by the map corresponding to the merging or splitting of circles, as follows:

- Two trivial circles merge into a trivial circle, or one trivial circle splits into two trivial circles. In these cases, the maps are given the same as in Khovanov's original TQFT.
- One trivial circle and one nontrivial circle merge into a nontrivial circle. In this case, the maps are given by
$$v_+ \otimes w_\pm \mapsto w_\pm, \quad v_- \otimes w_\pm \mapsto 0.$$
- One nontrivial circle splits into a trivial circle and a nontrivial circle. In this case, the maps are given by
$$w_\pm \mapsto v_- \otimes w_\pm.$$
- Two nontrivial circles merge into a trivial circle. In this case, the maps are given by
$$w_\pm \otimes w_\pm \mapsto 0, \quad w_\pm \otimes w_\mp \mapsto v_-.$$
- One trivial circle splits into two nontrivial circles. In this case, the maps are given by
$$v_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+, \quad v_- \mapsto 0.$$

The homological and quantum grading are given the same as the original case with the additional request that $q \deg w_\pm = \pm 1$. After appropriate shifts, the differential is still filtered of degree $(1, 0)$.

There is the third grading on the chain complex, the so-called *annular grading* or f–grading, which is specified by $f\deg v_\pm = 0$ and $f\deg w_\pm = \pm 1$. The differential preserves the f–grading and hence it descends onto the cohomology groups $\mathrm{AKh}(L)$, the annular Khovanov homology.

Theorem 2.3 [2] *The annular Khovanov homology* $\mathrm{AKh}(L)$ *is an invariant of annular links in the sense that it is independent of the choice of link diagrams and the order of crossings.*

We conclude this section with some additional remarks.

Remark 2.4 Sometimes we write $\mathrm{AKh}(L,m)$ to indicate the f–degree m summand of $\mathrm{AKh}(L)$. If L is contained in a ball $B^3 \subset A \times I$, then $\mathrm{AKh}(L)$ is supported on $f = 0$ and $\mathrm{AKh}(L) \cong \mathrm{Kh}(L)$. Both the reduced Khovanov homology and the annular Khovanov homology are functorial [8]. That is, a link cobordism $\rho\colon L_1 \to L_2$ between links (resp. annular links) induces a (filtered) map between Khovanov homology groups

$$\mathrm{Khr}(\rho)\colon \mathrm{Khr}(L_1) \to \mathrm{Khr}(L_2) \quad (\text{resp. } \mathrm{AKh}(\rho)\colon \mathrm{AKh}(L_1) \to \mathrm{AKh}(L_2)).$$

3 The unlink case

In this section, we construct an isomorphism between the annular Khovanov homology of an annular unlink and the reduced Khovanov homology of its augmentation. We show that such an isomorphism is compatible with the group homomorphisms induced by the cobordism maps.

3.1 Homology groups

Denote the annular unlink with n nontrivial unknot components by U_n and let $\widetilde{U}_n$ be its augmentation, which corresponds to the graph shown in Figure 4 in the language of [20], as described before Theorem 1.5.

The obvious diagram of U_n contains n disjoint nontrivial circles. In this section, we will use this diagram to calculate homology groups. We assign the numbers 1 to n from the innermost nontrivial circle to the outermost one. Applying Proposition 2.2 inductively on unknot components, we can calculate the Poincaré polynomial of $\mathrm{Khr}(\widetilde{U}_n)$ as

$$P(\widetilde{U}_n) = (tq^3)^n (tq^2 + t^{-1}q^{-2})^n.$$

Here the homological and quantum gradings are indicated by t and q, respectively.

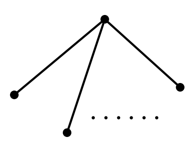

Figure 4: The tree corresponding to $\widetilde{U}_n$.

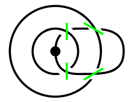

Figure 5: The symmetric resolution (10) of $\widetilde{U}_2$.

Each original component of $\widetilde{U}_n$ has two crossings with the meridian circle. There are 2^n resolutions such that every pair of crossings is resolved by the same smoothing. We say such resolutions are *symmetric* and encode them by 0–1 sequences of length n, as illustrated in Figure 5. Notice that a symmetric resolution always has n (unpointed) components. We denote the cobordism of changing one crossing (on the k^{th} strand) from 0–smoothing to 1–smoothing by $(\cdots\bullet\cdots)$ (here the mark $\bullet$ is on the k^{th} digit).

We can now describe the generators of $\mathrm{Khr}(\widetilde{U}_n)$ explicitly.

Proposition 3.1 *For each symmetric resolution $v \in \{0,1\}^n$, we can choose an element e_v lying in the chain group corresponding to this resolution. The collection of the e_v descends to a generating set of* $\mathrm{Khr}(\widetilde{U}_n)$.

Proof We prove the proposition by induction. There is nothing to say for $n = 0$. For $n = 1$, one can easily check that $e_{(1)} = v_+$ and $e_{(0)} = v_-$ gives a generating set of $\mathrm{Khr}(\widetilde{U}_1)$. In general, by applying Proposition 2.2 to $L = \widetilde{U}_{n-1}$ and $L \# H = \widetilde{U}_n$, we obtain a short exact sequence

$$0 \to \mathrm{Khr}^{i-1,j-2}(\widetilde{U}_{n-1}) \xrightarrow{\alpha_*} \mathrm{Khr}^{i+1,j+3}(\widetilde{U}_n) \xrightarrow{\beta_*} \mathrm{Khr}^{i+1,j+2}(\widetilde{U}_{n-1}) \to 0. \tag{3.2}$$

Here α_* and β_* come from the corresponding maps on the chain level.

Let $v = (v_1, v_2, \dots, v_n) \in \{0,1\}^n$ and let $v' = (v_1, \dots, v_{n-1})$. The sequence v' corresponds to a symmetric resolution $R'_{v'}$ of $\widetilde{U}_{n-1}$. If $v_n = 1$, we just need to take

$$e_v = \alpha_*(e_{v'}) = e_{v'} \otimes v_+.$$

If $v_n = 0$ and $e_{v'} = A \otimes v_+ + B \otimes v_-$, where $v_\pm$ are associated to the $(n-1)^{\text{st}}$ circle, we take

$$e_v = e_{v'} \otimes v_- + A \otimes v_- \otimes v_+.$$

It remains to show that e_v is a cycle. Assuming this, then we have $\beta_*(e_v) = e_{v'}$, and the conclusion follows from the short exact sequence (3.2) and the inductive hypothesis. Notice that the cobordism $(v', \bullet)$ always corresponds to a merging (rather than a splitting) of circles, and the construction ensures that $\mathrm{Khr}((v', \bullet))(e_v) = 0$. We show that other cobordisms also vanish by discussing the value of v_{n-1} (see Figure 6).

Notice that the cobordism map vanishes on A and B if the change is on the i^{th} strand $(1 \le i \le n-2)$. Hence, if $v_{n-1} = 1$, then there is no possibly nonvanishing cobordism map. Now assume that $v_{n-1} = 0$ and let $v'' = (v_1, \dots, v_{n-2})$,

$$e_{v''} = A_1 \otimes v_+ + B_1 \otimes v_-.$$

Figure 6: Possible resolutions with $v_n = 0$.

Then

$$e_{v'} = (A_1 \otimes v_+ + B_1 \otimes v_-) \otimes v_- + A_1 \otimes v_- \otimes v_+,$$
$$e_v = ((A_1 \otimes v_+ + B_1 \otimes v_-) \otimes v_- + A_1 \otimes v_- \otimes v_+) \otimes v_- + A_1 \otimes v_- \otimes v_- \otimes v_+,$$

and hence $\mathrm{Khr}((v'', \bullet, 1))(e_v) = 0$. □

We now construct an explicit identification between $\mathrm{AKh}(U_n)$ and $\mathrm{Khr}(\widetilde{U}_n)$. On the level of homology, this is quite easy: The Poincaré polynomial of $\mathrm{AKh}(U_n)$ is given by

$$P(U_n) = (fq + f^{-1}q^{-1})^n.$$

Here the f–grading is indicated by f. The substitution $f \mapsto tq$ gives an isomorphism between $\mathrm{AKh}(U_n)$ and $\mathrm{Khr}(\widetilde{U}_n)$ (up to shifting). More concretely, the generator

$$w = w_\pm^{(1)} \otimes w_\pm^{(2)} \otimes \cdots \otimes w_\pm^{(n)} \in \mathrm{AKh}(U_n)$$

is identified with the generator corresponding to the symmetric resolution of label $(v_1, v_2, \dots, v_n)$, where $v_i = 1$ if and only if $w_+^{(i)}$ appears in w ($i = 1, 2, \dots, n$), as in Proposition 3.1.

The effect of adding a trivial unknot component to U_n is just taking two copies of the original homology groups with generators tensoring with $v_\pm$, respectively, by the Künneth formula.

Now we discuss the grading shifts. Let U be an annular unlink with m trivial unknot components and n nontrivial unknot components. Let

$$w = v_\pm^{(1)} \otimes \cdots \otimes v_\pm^{(m)} \otimes w_\pm^{(1)} \otimes w_\pm^{(2)} \otimes \cdots \otimes w_\pm^{(n)}$$

be a generator of $\mathrm{AKh}^{i,j,k}(U)$ and $\Phi_U(w) \in \mathrm{Khr}^{i',j'}(\widetilde{U})$ be the generator corresponding to w. Assume that w_+ (resp. w_-) appears t_+ (resp. t_-) times in w. Then $k = t_+ - t_-$ and $n = t_+ + t_-$. The homological grading i' increases by $2t_+ = k + n$, and the quantum grading j' increases by $2n + 2t_+ = k + 3n$.

We summarize the consequence of this subsection in the following form:

Theorem 3.3 *Let U be an annular unlink with n nontrivial unknot components, and let $\widetilde{U}$ be its augmentation. Then there is an isomorphism Φ_U between the annular Khovanov homology of U and the reduced Khovanov homology of $\widetilde{U}$. More precisely, we have an isomorphism*

$$\Phi_U : \mathrm{AKh}^{i,j,k}(U) \to \mathrm{Khr}^{i+k+n,\, j+k+3n}(\widetilde{U}).$$

The correspondence of generators is as given above. □

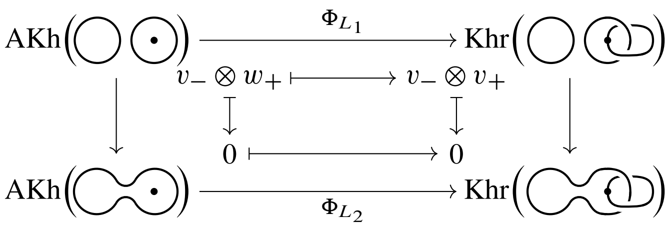

Figure 7: Case (a).

3.2 Functoriality

A cobordism between annular links naturally induces a cobordism between their augmentations. In this subsection, we show that the isomorphism Φ_L defined in Theorem 3.3 is compatible with cobordisms. For our purpose (see the next section), we don't need to deal with the Reidemeister moves on the diagram of L, and we concentrate on Morse moves, ie the merging and splitting of circles. We first verify the compatibility with only related circles and then consider the effect of adding other unlink components. There are four cases we need to discuss:

(a) one trivial circle and one nontrivial circle merge into a nontrivial circle;

(b) one nontrivial circle splits into a trivial circle and a nontrivial circle;

(c) two nontrivial circles merge into a trivial circle;

(d) one trivial circle splits into two nontrivial circles.

Since the homomorphisms induced by cobordisms are well defined [8], we may choose specific link diagrams to calculate them. Cases (a) and (b) are simple diagram chasing. Figure 7 illustrates this process.

In cases (c) and (d), we need to check the diagrams in Figure 8 commute.

Denote the upper and the lower links in the leftmost column of Figure 8 by L_3 and L_4, respectively. We have

$$\operatorname{Khr}(\widetilde{L}_3) \cong (\mathbb{Z}/2\mathbb{Z})^{(0,2)} \oplus ((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{(4,10)},$$
$$\operatorname{Khr}(\widetilde{L}_4) \cong (\mathbb{Z}/2\mathbb{Z})^{(0,1)} \oplus (\mathbb{Z}/2\mathbb{Z})^{(0,-1)}.$$

We first check case (c). Notice that $w(D)$, $\sigma(S)$ and $\tau(S)$ decrease by 4, 0 and 1 from $\widetilde{L}_3$ to $\widetilde{L}_4$, respectively. Hence, the cobordism map $\operatorname{Khr}(\widetilde{L}_3) \to \operatorname{Khr}(\widetilde{L}_4)$ is of degree $(-2,-7)$ by Remark 2.1, and

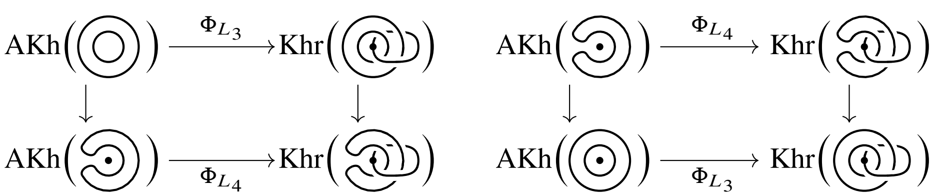

Figure 8: Cases (c) and (d).

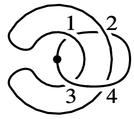

Figure 9: The labeling of crossings on $\widetilde{L}_4$.

the only possibly nontrivial map is

$$((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2} \to (\mathbb{Z}/2\mathbb{Z})^{(0,-1)},$$

which corresponds to the merging map in the leftmost column of Figure 8,

$$w_+ \otimes w_-, w_- \otimes w_+ \mapsto v_-.$$

By the algorithm given in Theorem 3.3, $w_- \otimes w_+$ and $w_+ \otimes w_-$ correspond to $v_- \otimes v_+$ (associated to the symmetric resolution (01)) and $v_+ \otimes v_- + v_- \otimes v_+$ (associated to the symmetric resolution (10)), respectively. Their images are $v_- \otimes v_+ \otimes v_-$ and v_-, respectively. It suffices to show they are nonvanishing and cohomologous.

To write down the differentials

$$d^{(-1,-1)}\colon \mathrm{CKhr}^{(-1,-1)}(\widetilde{L}_4) \to \mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4) \quad \text{and} \quad d^{(0,-1)}\colon \mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4) \to \mathrm{CKhr}^{(1,-1)}(\widetilde{L}_4)$$

in matrix form, we need to fix orders of bases of the chain groups as follows. Assign the crossing numbers 1 to 4 as in Figure 9. We first take the lexicographical order on the resolutions (ie take the states associated to the resolution (1100) first, then (1010), etc). Most resolutions correspond to exactly one state in these chain groups, except resolutions (0100), (0001) and (0101). For them, we give the order by where the unique v_+ appears (from top to bottom). Under this convention, we denote the bases of chain groups $\mathrm{CKhr}^{(-1,-1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4)$ by e_i $(1 \le i \le 6)$ and f_j $(1 \le j \le 8)$, respectively. We have

$$d^{(-1,-1)} = \begin{pmatrix} 1&1&0&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&1&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&1 \end{pmatrix} \quad \text{and} \quad d^{(0,-1)} = \begin{pmatrix} 1&0&1&0&1&0&0&0\\ 0&0&0&1&0&0&1&1 \end{pmatrix}.$$

The elements v_- and $v_- \otimes v_+ \otimes v_-$ correspond to the vectors f_2 and f_6, respectively. It is easy to see that $f_2 - f_6 = d^{(-1,-1)}(e_1 + e_2 + e_5)$ and $f_2 \notin \operatorname{Im} d^{(-1,-1)}$. This finishes the verification in case (c).

The verification in case (d) is essentially the same. The only possibly nontrivial map in the rightmost column of Figure 8 is

$$(\mathbb{Z}/2\mathbb{Z})^{(0,1)} \to ((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2},$$

which corresponds to the splitting map

$$v_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+$$

in the third column of Figure 8. We give orders for the bases of $\mathrm{CKhr}^{(-1,1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(0,1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(1,6)}(\widetilde{L}_3)$ and $\mathrm{CKhr}^{(2,6)}(\widetilde{L}_3)$ as in case (c). The only exception is that, for the resolution (0101) in $\mathrm{CKhr}^{(0,1)}(\widetilde{L}_4)$, we give the order according to the position of the unique v_- (from top to bottom). Under this convention, we have

$$d^{(0,1)} = \begin{pmatrix} 1&1&0&1&0&0&0&0\\ 0&1&1&0&0&0&0&1\\ 1&0&1&0&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&1&0&1&0&1\\ 0&0&0&0&1&0&0&1 \end{pmatrix} \quad \text{and} \quad d^{(-1,1)} = \begin{pmatrix} 1&0&0&1&0&1&1&0\\ 0&0&1&0&1&1&0&1 \end{pmatrix}^T.$$

The generator of $\mathrm{Khr}^{(0,1)}(\widetilde{L}_4)$ can be represented by $v^{(0,1)} = (1,1,0,0,1,1,1,1)^T$, and we have

$$\mathrm{Khr}(\widetilde{L}_4) \to \mathrm{Khr}(\widetilde{L}_3)\colon v^{(0,1)} \mapsto (1,0,1,0,0,1,0,1)^T.$$

The boundary subgroup of degree $(2,6)$ is spanned by the image of

$$d^{(1,6)} = \begin{pmatrix} 1&1&0&0\\ 1&0&1&0\\ 0&0&0&0\\ 1&0&0&1\\ 0&1&1&0\\ 0&1&0&1\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix}.$$

Therefore, under the map $\mathrm{Khr}(\widetilde{L}_4) \to \mathrm{Khr}(\widetilde{L}_3)$, we have

$$v^{(0,1)} \mapsto \Phi_{L_3}(w_+ \otimes w_- + w_- \otimes w_+) + d^{(1,6)}(1,0,0,1)^T.$$

This completes the verification in case (d).

It remains to consider the effect of adding a new unlink component to the cobordism. The case of adding a trivial unknot component is trivial and we assume that the additional unknot component is nontrivial. Let L_1 and L_2 be two annular unlinks and let $\rho\colon L_1 \to L_2$ be a cobordism obtained by a Morse move. We have

$$\mathrm{AKh}(\rho \amalg \mathrm{id}) = \mathrm{AKh}(\rho) \otimes \mathrm{id}_U;$$

here $U = U_1$ is the nontrivial annular unknot. Take $S \in \mathrm{AKh}(L_1)$ and let $T = \mathrm{AKh}(\rho)(S)$. By Proposition 2.2 and Theorem 3.3, the following diagram commutes:

$$\begin{tikzcd} S \otimes w_+ \arrow[r, mapsto, "\Phi_{L_1}"] \arrow[d, mapsto, "\mathrm{AKh}(\rho)"'] & \Phi_{L_1}(S) \otimes v_+ \arrow[d, mapsto, "\mathrm{Khr}(\rho')"] \\ T \otimes w_+ \arrow[r, mapsto, "\Phi_{L_2}"'] & \Phi_{L_2}(S) \otimes v_+ \end{tikzcd}$$

Assume that $\Phi_{L_1}(S) = A \otimes v_+ + B \otimes v_-$ and $\Phi_{L_2}(T) = C \otimes v_+ + D \otimes v_-$. By Proposition 2.2 and Theorem 3.3, the diagram

$$\begin{tikzcd} S \otimes w_- \arrow[r, mapsto, "\Phi_{L_1}"] \arrow[d, mapsto, "\mathrm{AKh}(\rho)"'] & \Phi_{L_1}(S) \otimes v_- + A \otimes v_- \otimes v_+ \arrow[d, mapsto, "\mathrm{Khr}(\widetilde{\rho})"] \\ T \otimes w_- \arrow[r, mapsto, "\Phi_{L_2}"'] & \Phi_{L_2}(T) \otimes v_- + C \otimes v_- \otimes v_+ \end{tikzcd}$$

commutes, which completes the proof.

In summary, we have shown the following theorem. Roughly speaking, it gives a natural isomorphism between two cohomology theories on annular unlinks.

Theorem 3.4 *Let L_1 and L_2 be two annular unlinks and let $\rho\colon L_1 \to L_2$ be a cobordism obtained by composition of Morse moves. The cobordism ρ induces a cobordism $\widetilde{\rho}$ between the augmentations $\widetilde{L}_1$ and $\widetilde{L}_2$. Let Φ_{L_1} and Φ_{L_2} be the isomorphisms given in Theorem 3.3. Then the following diagram commutes*:

$$\begin{tikzcd} \mathrm{AKh}(L_1) \arrow[r, "\Phi_{L_1}"] \arrow[d, "\mathrm{AKh}(\rho)"'] & \mathrm{Khr}(\widetilde{L}_1) \arrow[d, "\mathrm{Khr}(\widetilde{\rho})"] \\ \mathrm{AKh}(L_2) \arrow[r, "\Phi_{L_2}"'] & \mathrm{Khr}(\widetilde{L}_2) \end{tikzcd}$$

□

4 The spectral sequence

In this section, we prove Theorem 1.2 and discuss some examples and applications. To prove Theorem 1.2, we choose a link diagram as shown in Figure 10. For convenience, we call the strands appearing in the right of the left diagram the *annular strands*.

Proof of Theorem 1.2 Fix a link diagram D as in Figure 10. Crossings of $\widetilde{L}$ can be classified into two types: crossings of the augmenting circle and the annular strands, and the original crossings of L. We encode the resolutions of the first type of crossings by 0–1 sequences $\boldsymbol{w}_1$ and the second type by $\boldsymbol{w}_2$. Then the chain complex $\mathrm{CKhr}(\widetilde{L})$ can be encoded by the concatenation $\boldsymbol{v} = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. Every summand of the

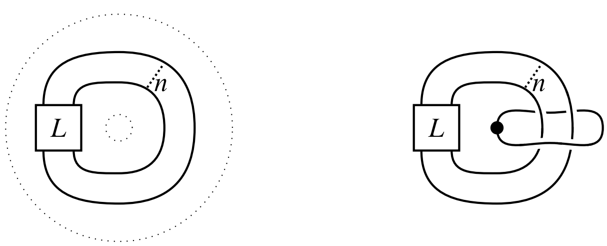

Figure 10: A standard link diagram and its augmentation.

differential comes from exactly one change of the smoothing, and the differential splits as $d = d_1 + d_2$, where d_i corresponds to the changes of smoothing on type i crossings. Denote the partial resolution of $\widetilde{L}$ on $\boldsymbol{w}_2$ by $\widetilde{L}_{\boldsymbol{w}_2}$, which is also the augmentation of the annular unlink $L_{\boldsymbol{w}_2}$ and hence there is no ambiguity.

The chain complex $\mathrm{CKhr}(\widetilde{L})$ is bigraded by $(|\boldsymbol{w}_1|, |\boldsymbol{w}_2|)$, and the differentials d_1 and d_2 have degrees $(1,0)$ and $(0,1)$, respectively. The spectral sequence of double complexes [6, Section III.7, Proposition 10] applies. The E_1 term is given by the cohomology of $(\mathrm{CKhr}(\widetilde{L}), d_1)$, which is a chain complex with chain groups $\mathrm{Khr}(\widetilde{L}_{\boldsymbol{w}_2})$ and differentials given by cobordisms. Since the link diagram is fixed, such cobordisms correspond to Morse moves. By Theorem 3.4, the E_1 term is isomorphic to the chain complex that calculates $\mathrm{AKh}(L)$, and hence the E_2 term is isomorphic to $\mathrm{AKh}(L)$. The spectral sequence converges to the cohomology of $(\mathrm{CKhr}(\widetilde{L}), d)$, ie $\mathrm{Khr}(\widetilde{L})$.

A Reidemeister move induces an isomorphism between the converging terms that is compatible with the filtration, and an isomorphism between the E_2 terms. The comparison theorem then applies and hence the spectral sequence is independent of the choice of the link diagram. □

Example 4.1 Consider the annular link L shown in Figure 1. The augmentation $\widetilde{L}$ is isotopic to the link L5a1 and $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) = 8 = \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L})$. Hence, the spectral sequence collapses at the E_2 term. This illustrates that the spectral sequence can collapse for links not isotopic to braid closures.

We can derive a finer rank inequality from Theorem 1.2.

Corollary 4.2 *Let L and $\widetilde{L}$ be as in Theorem 1.2. Let n_0 be the number of annular strands and n'_- be the number of left-handed crossings on the augmenting circle. Then*

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}^n(\widetilde{L}) \leq \sum_{n_a + f_a + n_0 - n'_- = n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}^{n_a}(L, f_a).$$

Proof Denote the gradings of $\mathrm{AKh}(L)$ by (n_a, q_a, f_a). Let $\tilde{n}_-$ be the number of left-handed crossings of $\widetilde{L}$. Then $\tilde{n}_- = n_- + n'_-$ and $n_a = |\boldsymbol{w}_2| - n_-$. Let n'_0 be the number of nontrivial unknot components of a specific partial resolution. Let $(n'_0)_+$ (resp. $(n'_0)_-$) be the number of 1–smoothings (resp. 0–smoothings). Then, by Theorem 3.4, we have

$$f_a = (n'_0)_+ - n'_0 = \tfrac{1}{2}((n'_0)_+ - (n'_0)_-) = |\boldsymbol{w}_1| - n_0$$

on the E_1 term. On the E_∞ term, we have $n = |\boldsymbol{w}_1| + |\boldsymbol{w}_2| - \tilde{n}_-$. Therefore, from Theorem 1.2, we obtain

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}^n(\widetilde{L}) &= \sum_{|\boldsymbol{w}_1| + |\boldsymbol{w}_2| - \tilde{n}_- = n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} E_\infty^{|\boldsymbol{w}_1|, |\boldsymbol{w}_2|} \\ &\leq \sum_{|\boldsymbol{w}_1| + |\boldsymbol{w}_2| - \tilde{n}_- = n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} E_2^{|\boldsymbol{w}_1|, |\boldsymbol{w}_2|} \\ &= \sum_{n_a + f_a + n_0 - n'_- = n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}^{n_a}(L, f_a). \end{aligned}$$

□

We now prove Theorem 1.5 and Corollary 1.6. The following simple observation is useful:

Lemma 4.3 *Let L be an annular link with a link diagram such that there is only one annular strand. View L as a link in S^3 and let p be a basepoint on this annular strand. Then* $\mathrm{AKh}(L)$ *is supported on $f = \pm 1$, and*

$$\mathrm{AKh}(L, \pm 1) \cong \mathrm{Khr}(L, p).$$

Proof There is exactly one nontrivial circle in each resolution of L, which is the circle containing p. Hence, the chain complex is supported on $f = \pm 1$. Furthermore, the subcomplexes of f–grading ± 1 are isomorphic to $\mathrm{CKhr}(L)$ by replacing the generators $w_\pm$ of the nontrivial circle by v_-, respectively. □

Proof of Theorem 1.5 Let G be a forest such that each connected component contains at most one annular vertex. Then L_G is a disjoint union of links with at most one annular strand. Then Lemma 4.3 applies and we have $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L_G) = 2^n$ by the Künneth formula.

Conversely, let L be an annular link with n components and

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) = 2^n.$$

Then Corollary 1.3 gives $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L}) = 2^n$. By [20, Theorem 1.2], $\widetilde{L}$ is a forest of unknots in S^3. Therefore, L is a forest of unknots in $A \times I$. Denote their corresponding forests by $\widetilde{G}$ and G, respectively. Notice that $\widetilde{G}$ is constructed from G by adding a vertex adjacent to all the annular vertices. Two annular vertices cannot lie in the same connected component of G since otherwise a cycle would occur in $\widetilde{G}$, which is absurd since $\widetilde{G}$ is a forest. □

Proof of Corollary 1.6 By Theorem 1.5, L is a forest of unknots in $A \times I$. Denote the corresponding forest by G. If G had an edge, then $\mathrm{AKh}(L)$ would not be supported on $t = 0$ as $\mathrm{AKh}(U)$ is (see the discussion in Section 3.1), which is a contradiction. Hence, every vertex is an independent connected component of G, ie L is an annular unlink. The number of nontrivial unknot components in L can be read from the Poincaré polynomial of L. Therefore, L is isotopic to U. □

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