The persistent topology of optimal transport based metric thickenings

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A metric thickening of a given metric space X is any metric space admitting an isometric embedding of X. Thickenings have found use in applications of topology to data analysis, where one may approximate the shape of a dataset via the persistent homology of an increasing sequence of spaces. We introduce two new families of metric thickenings, the p–Vietoris–Rips and p–Čech metric thickenings for all $1 \le p \le \infty$, which include all probability measures on X whose p–diameter or p–radius is bounded from above, equipped with an optimal transport metric. The p–diameter (resp. p–radius) of a measure is a certain ℓ_p relaxation of the usual notion of diameter (resp. radius) of a subset of a metric space. These families recover the previously studied Vietoris–Rips and Čech metric thickenings when $p = \infty$. As our main contribution, we prove a stability theorem for the persistent homology of p–Vietoris–Rips and p–Čech metric thickenings, which is novel even in the case $p = \infty$. In the specific case $p = 2$, we prove a Hausmann-type theorem for thickenings of manifolds, and we derive the complete list of homotopy types of the 2–Vietoris–Rips thickenings of the n–sphere as the scale increases.

1 Introduction

Geometric simplicial complexes, such as Vietoris–Rips or Čech complexes, are one of the cornerstones of topological data analysis. One can approximate the shape of a dataset X by building a growing sequence of Vietoris–Rips complexes with X as the underlying set, and then computing persistent homology. The shape of the data, as measured by persistence, is reflective of important patterns within; see Carlsson [25].

The popularity of Vietoris–Rips complexes relies on at least three facts. First, Vietoris–Rips filtrations and their persistent homology signatures are computable; see Bauer [12]. Second, Vietoris–Rips persistent homology is stable (see Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [27] and Chazal, de Silva and Oudot [30]), meaning that the topological data analysis pipeline is robust to certain types of noise. Third, Vietoris–Rips complexes are topologically faithful at low scale parameters: one can use them to recover the homotopy types (see Latschev [50]) or homology groups (see Chazal and Oudot [28]) of an unknown underlying space, when given only a finite noisy sampling.

At higher scale parameters, we mostly do not know how Vietoris–Rips complexes behave. This is despite the fact that one of the key insights of persistent homology is to allow the scale parameter to vary from small to large, tracking the lifetimes of features as the scale increases. Our practice is ahead of our theory in this regard: data science practitioners are building Vietoris–Rips complexes with scale parameters larger than those for which the reconstruction results of [28; 50] apply.

Stability implies that, as more and more data points are sampled from some "true" underlying space M, the Vietoris–Rips persistent homology of the dataset X converges to the Vietoris–Rips persistent homology of M. The simplest possible case is when the dataset X is sampled from a manifold M, and so we cannot fully understand the Vietoris–Rips persistent homology of data without also understanding the Vietoris–Rips persistent homology of manifolds. However, not much is known about Vietoris–Rips complexes of manifolds, except at small scales; see Hausmann [42]. Even the Vietoris–Rips persistent homology of the n–sphere $\mathbb{S}^n$ is almost entirely unknown.

Two potential obstacles for understanding the homotopy types of Vietoris–Rips complexes of a manifold M are that

(i) the natural inclusion $M \hookrightarrow \mathrm{VR}(M; r)$ is not continuous, and

(ii) we do not yet have a full Morse theory for Vietoris–Rips complexes of manifolds.

There are by now several strategies for handling these two obstacles. One strategy is to remain in the setting of Vietoris–Rips simplicial complexes. Obstacle (i) is then unavoidable. Regarding obstacle (ii), Bestvina–Brady Morse theory has only been successfully applied at low scale parameters, allowing Zaremsky [75] to prove that $\mathrm{VR}(\mathbb{S}^n; r)$ recovers the homotopy type of $\mathbb{S}^n$ for r small enough, but not to derive new homotopy types that appear as r increases. Simplicial techniques have been considered for a long time, but even successes such as an understanding of the homotopy types of the Vietoris–Rips

complexes of the circle at all scales (See Adamaszek and Adams [1]) are not accompanied by a broader Morse theory (though some techniques feel Morse-theoretic).

A second strategy is very recent. Lim, Mémoli and Okutan [52] and Okutan [62] show that the Vietoris–Rips simplicial complex filtration is equivalent to thickenings of the Kuratowski embedding into $L^\infty(M)$ or any other *injective* metric space. In particular, the two filtrations have the same persistent homology. This overcomes obstacle (i): the inclusion of a metric space into a thickening of its Kuratowski embedding is continuous, and indeed an isometry onto its image. This connection has created new opportunities, such as the Morse theoretic techniques employed by Katz [45; 46; 47; 48]. This Morse theory allows one to prove the first new homotopy types that occur for Vietoris–Rips simplicial complexes of the circle and 2–sphere, but have not yet inspired progress for larger scales, or for spheres above dimension two.

A third strategy is to consider Vietoris–Rips metric thickenings, which rely on optimal transport and Wasserstein distances; see Adamaszek, Adams and Frick [2]. We refer to these spaces as the ∞–metric thickenings, for reasons that will become clear in the following paragraph. Such thickenings were invented in order to enable Morse-theoretic proofs of the homotopy types of Vietoris–Rips type spaces. The first new homotopy type of the ∞–Vietoris–Rips metric thickening of the n–sphere is known [2], but only for a single (nonpersistent) scale parameter. It was previously only conjectured that the ∞–Vietoris–Rips metric thickenings have the same persistent homology as the more classical Vietoris–Rips simplicial complexes [2, Conjecture 6.12]; one of our contributions is to answer this conjecture in the affirmative. Mirth [61] considers a Morse theory in Wasserstein space, which is inspired in part by applications to ∞–Vietoris–Rips metric thickenings, but which does not apply as-is to these thickenings as the ∞–diameter functional is not "λ–convex"; see Santambrogio [69].

We introduce a generalization: the p–Vietoris–Rips metric thickening for any $1 \le p \le \infty$. Let X be an arbitrary metric space. For $1 \le p \le \infty$, the p–Vietoris–Rips metric thickening at scale parameter $r > 0$ contains all probability measures on X whose p–diameter is less than r. The p–Vietoris–Rips metric thickening will be equipped with the topology induced from the weak topology on $\mathcal{P}_X$. When X is bounded, the weak topology is generated by an optimal transport based metric; see Corollary A.2. For p finite, the p–diameter of a probability measure α on the metric space X is defined as

$$\mathrm{diam}_p(\alpha) := \left(\iint_{X \times X} d_X^p(x, x') \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}},$$

and $\mathrm{diam}_\infty(\alpha)$ is defined to be the diameter of the support of α.

The p–Vietoris–Rips metric thickenings at scale r form a metric bifiltration of X that is covariant in r and contravariant in p. Indeed, we have an inclusion map $\mathrm{VR}_p(X; r) \hookrightarrow \mathrm{VR}_{p'}(X; r')$ for $r \le r'$ and $p \ge p'$; see Table 1:

$$
\begin{array}{ccc}
\mathrm{VR}_{p'}(X; r) & \longhookrightarrow & \mathrm{VR}_{p'}(X; r') \\
\big\uparrow & & \big\uparrow \\
\mathrm{VR}_p(X; r) & \longhookrightarrow & \mathrm{VR}_p(X; r')
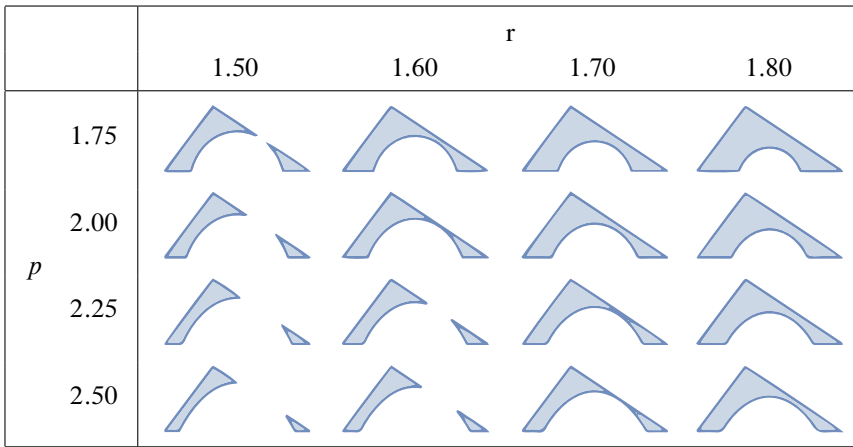\end{array}
$$

Table 1: The p–Vietoris–Rips bifiltration $\mathrm{VR}_p(X;r)$ for X a metric space of three points in $\mathbb{R}^2$, with $\mathcal{P}_X$ visualized as the convex hull of X in $\mathbb{R}^2$. Note that $\mathrm{VR}_p(X;r) \subseteq \mathrm{VR}_{p'}(X;r')$ for $r \leq r'$ and $p \geq p'$.

As one of our main contributions, we prove that the p–Vietoris–Rips metric thickening is stable. This means that if two totally bounded metric spaces X and Y are close in the Gromov–Hausdorff distance, then their filtrations $\mathrm{VR}_p(X;\cdot)$ and $\mathrm{VR}_p(Y;\cdot)$ are close in the homotopy type distance. This was previously unknown even in the case $p = \infty$ (see [2, Conjecture 6.14]); we prove stability for all $1 \leq p \leq \infty$. As a consequence, it follows that the (undecorated) persistent homology diagrams for the Vietoris–Rips simplicial complexes $\mathrm{VR}(X;r)$ and for the $p = \infty$ Vietoris–Rips metric thickenings $\mathrm{VR}_\infty(X;\cdot)$ are identical. In other words, the persistent homology barcodes for $\mathrm{VR}(X,\cdot)$ and $\mathrm{VR}_\infty(X;\cdot)$ are identical, up to replacing closed interval endpoints with open endpoints, or vice versa. This answers [2, Conjecture 6.12] in the affirmative. Another consequence of stability is that the p–metric thickenings give the same persistence diagrams whether one considers all Radon probability measures, or instead the restricted setting of only measures with finite support.

The proof of stability for metric thickenings is more intricate than the proof of stability for the corresponding simplicial complexes (see for instance Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [27], Chazal, de Silva and Oudot [30] and Mémoli [57]). Whereas simplicial complexes can be compared via simplicial maps, the direct analogues of simplicial maps on metric thickenings are not necessarily continuous. Thus, new techniques are required to construct continuous maps between metric thickenings. Our technique, relying on partitions of unity, allows us to continuously approximate measures on one metric space X by measures on another metric space Y (where this approximation depends on the Gromov–Hausdorff distance between X and Y), thereby allowing us to construct the desired interleavings.

We furthermore introduce the p–Čech metric thickenings, again for $1 \leq p \leq \infty$, and prove analogues of all of the above results. The p–Čech metric thickening at scale parameter $r > 0$ contains all probability measures supported on X whose p–radius is less than r; see Table 2.

Table 2: The p–Čech bifiltration $\check{C}_p(X;r)$ for X a metric space of three points in $\mathbb{R}^2$, with $\mathcal{P}_X$ visualized as the convex hull of X in $\mathbb{R}^2$. Note that $\check{C}_p(X;r) \subseteq \check{C}_{p'}(X;r')$ for $r \leq r'$ and $p \geq p'$.

We also deduce the complete spectrum of homotopy types of 2–Vietoris–Rips and 2–Čech metric thickenings of the n–sphere, equipped with the Euclidean metric ℓ_2: $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ first attains the homotopy type of $\mathbb{S}^n$ for scales $r \leq \sqrt{2}$, and then for all scales $r > \sqrt{2}$ the space is contractible. By contrast, the Vietoris–Rips simplicial complexes or the ∞–Vietoris–Rips metric thickenings of the n–sphere (with either the Euclidean or the geodesic metric) are only known for a bounded range of scales, including only a single change in homotopy type (see Adamaszek, Adams and Frick [2, Section 5]), even though infinitely many changes in homotopy type are conjectured (see Adams, Bush and Frick [4, Question 8.1]). See however Lim, Mémoli and Okutan [52, Corollary 7.18] for results for round spheres with the ℓ_∞ metric.

One of our main motivations for introducing the p–Vietoris–Rips and p–Čech metric thickenings is to enable effective Morse theories on these types of spaces. The p–variance for Čech metric thickenings is a minimum of linear functionals, and therefore fits in the framework of Morse theory for min-type functions; see Baryshnikov, Bubenik and Kahle [8], Bryzgalova [21], Gershkovich and Rubinstein [38] and Matov [54]. On the Vietoris–Rips side, we remark that gradient flows of functionals on Wasserstein space can be defined when the functional is "λ–convex"; see Mirth [61] and Santambrogio [69]. Though the ∞–diameter is not λ–convex, we hope that the p–diameter functional for $p < \infty$ may be λ–convex in certain settings.

Organization   In Section 2 we describe background material and set notation. We define the p–Vietoris–Rips and p–Čech metric thickenings in Section 3, and consider their basic properties in Section 4. In Section 5 we prove stability. We consider Hausmann-type theorems in Section 6, and deduce the 2–Vietoris–Rips metric thickenings of Euclidean spheres in Section 7. In Section 8 we bound the length of intervals in p–Vietoris–Rips and p–Čech metric thickenings using a generalization of the spread of a metric space, called the p–spread.

We conclude the paper by providing some discussion in Section 9.

In Section A.1 we explain how the q–Wasserstein distance metrizes the weak topology for $1 \leq q < \infty$. We describe connections to min-type Morse theories in Section A.2. In Section A.3 we show that p–Čech thickenings of finite metric spaces are homotopy equivalent to simplicial complexes. We prove the persistent homology diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of a family of discrete metric spaces in Section A.4, and we describe the 0–dimensional persistent homology of the p–Vietoris–Rips and p–Čech metric thickenings of an arbitrary metric space in Section A.5. We consider crushings in Section A.6. In Section A.7, we show that the main properties we prove for the (intrinsic) p–Čech metric thickening also hold for the ambient p–Čech metric thickening.

Acknowledgments

2 Background

This section introduces background material and notation.

Metric spaces and the Gromov–Hausdorff distance   Let (X, d_X) be a metric space. For any $x \in X$, we let $B(x; r) := \{y \in X \mid d_X(x, y) < r\}$ denote the open ball centered at x of radius r.

Given a metric space (X, d_X), the *diameter* of a nonempty subset $A \subset X$ is $\mathrm{diam}(A) := \sup_{a,a' \in A} d_X(a, a')$, whereas its *radius* is $\mathrm{rad}(A) := \inf_{x \in X} \sup_{a \in A} d_X(x, a)$. Note that, in general,

$$\tfrac{1}{2}\,\mathrm{diam}(A) \leq \mathrm{rad}(A) \leq \mathrm{diam}(A).$$

Definition 2.1 (uniform discrete metric space)   For any natural number n, we use Z_n to denote the metric space consisting of n points with all interpoint distances equal to 1.

Definition 2.2 (ε–net)   Let X be a metric space. A subset $U \subset X$ is called an ε–*net* of X if, for any point $x \in X$, there is a point $u \in U$ with $d_X(x, u) < \varepsilon$.

Let X and Y be metric spaces. The *distortion* of an arbitrary function $\varphi \colon X \to Y$ is

$$\mathrm{dis}(\varphi) := \sup_{x,x' \in X} |d_X(x, x') - d_Y(\varphi(x), \varphi(x'))|.$$

The *codistortion* of a pair of arbitrary functions $\varphi \colon X \to Y$ and $\psi \colon Y \to X$ is

$$\mathrm{codis}(\varphi, \psi) := \sup_{x \in X, y \in Y} |d_X(x, \psi(y)) - d_Y(\varphi(x), y)|.$$

We will use the following expression for the Gromov–Hausdorff distance between X and Y [43]:

(1) $$d_{\mathrm{GH}}(X, Y) = \tfrac{1}{2} \inf_{\varphi, \psi} \max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi, \psi)).$$

Remark 2.3 If $\mathrm{codis}(\varphi, \psi) < \eta$, then for all $(x, y) \in X \times Y$,

$$|d_X(x, \psi(y)) - d_Y(\varphi(x), y)| < \eta.$$

In particular, by letting $y = \varphi(x)$ above, this means that

$$d_X(x, \psi \circ \varphi(x)) < \eta \quad \text{for all } x \in X.$$

See [23, Chapter 7] for more details about the Gromov–Hausdorff distance.

Simplicial complexes Two of the most commonly used methods for producing filtrations in applied topology are the Vietoris–Rips and Čech simplicial complexes, defined as follows. Let X be a metric space, and let $r \geq 0$. The *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has X as its vertex set, and contains a finite subset $[x_0, \ldots, x_k]$ as a simplex when $\mathrm{diam}([x_0, \ldots, x_k]) := \max_{i,j} d_X(x_i, x_j) < r$. The *Čech simplicial complex* $\check{\mathrm{C}}(X; r)$ has X as its vertex set, and contains a finite subset $[x_0, \ldots, x_k]$ as a simplex when $\bigcap_{i=0}^{k} B(x_i; r) \neq \varnothing$.

Probability measures and Wasserstein distances Our main reference for measure theory elements is [15] (albeit we use slightly different notation). Given a metric space (X, d_X), by $\mathcal{P}_X$ we denote the set of all *Radon* probability measures on X. We equip $\mathcal{P}_X$ with the weak topology: a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \in \mathcal{P}_X$ is said to converge weakly to $\alpha \in \mathcal{P}_X$ if, for all bounded, continuous functions $\varphi \colon X \to \mathbb{R}$, we have $\lim_{n \to \infty} \int_X \varphi(x) \, \alpha_n(dx) = \int_X \varphi(x) \, \alpha(dx)$. The *support* $\mathrm{supp}(\alpha)$ of a probability measure $\alpha \in \mathcal{P}_X$ is the largest closed set C such that every open set which has nonempty intersection with C has positive measure. If $\mathrm{supp}(\alpha)$ consists of a finite set of points, then α is called *finitely supported* and can be written as $\alpha = \sum_{i \in I} a_i \, \delta_{x_i}$, where I is finite, $a_i \geq 0$ for all i, $\sum_{i \in I} a_i = 1$, and each δ_{x_i} is a Dirac delta measure at x_i. Let $\mathcal{P}_X^{\mathrm{fin}}$ denote the set of all finitely supported Radon probability measures on X.

Given another metric space Y and a measurable map $f \colon X \to Y$, the *pushforward* map $f_\sharp \colon \mathcal{P}_X \to \mathcal{P}_Y$ induced by f is defined by $f_\sharp(\alpha)(B) = \alpha(f^{-1}(B))$ for every Borel set $B \subset Y$. In the case of finitely supported probability measures, we have the explicit formula $f_\sharp(\sum_{i \in I} a_i \, \delta_{x_i}) = \sum_{i \in I} a_i \, \delta_{f(x_i)}$, so $f_\sharp$ restricts to a function $\mathcal{P}_X^{\mathrm{fin}} \to \mathcal{P}_Y^{\mathrm{fin}}$. If f is a continuous map, then $f_\sharp$ is a continuous map between $\mathcal{P}_X$ and $\mathcal{P}_Y$ in the weak topology; see Chapter 5 in [15]. In the finitely supported case, the restriction of $f_\sharp$ is a continuous map from $\mathcal{P}_X^{\mathrm{fin}}$ to $\mathcal{P}_Y^{\mathrm{fin}}$.

Given $\alpha, \beta \in \mathcal{P}_X$, a *coupling* between them is any probability measure μ on $X \times X$ with marginals α and β, meaning that $(\pi_1)_\sharp \mu = \alpha$ and $(\pi_2)_\sharp \mu = \beta$, where $\pi_i \colon X \times X \to X$ is the projection map defined by $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$. By $\mathrm{Cpl}(\alpha, \beta)$ we denote the set of all couplings between α and β. Notice that $\mathrm{Cpl}(\alpha, \beta)$ is always nonempty as the product measure $\alpha \otimes \beta$ is in $\mathrm{Cpl}(\alpha, \beta)$.

Given $q \in [1, \infty)$, let $\mathcal{P}_{q,X}$ be the subset of $\mathcal{P}_X$ consisting of Radon probability measures with finite moments of order q, that is, measures α with $\int_X d_X^q(x, x_0) \, \alpha(dx) < \infty$ for some, and thus any, $x_0 \in X$. Note that when X is a bounded metric space, $\mathcal{P}_{q,X} = \mathcal{P}_X$. We can equip $\mathcal{P}_{q,X}$ with the q–*Wasserstein distance* (or *Kantorovich q–metric*). In this setting, the q–Wasserstein distance is given by

$$(2) \qquad d_{\mathrm{W},q}^X(\alpha, \beta) := \inf_{\mu \in \mathrm{Cpl}(\alpha, \beta)} \left(\iint_{X \times X} d_X^q(x, x') \, \mu(dx \times dx') \right)^{\frac{1}{q}},$$

and it can be shown that $d_{\mathrm{W},q}^X$ defines a metric on $\mathcal{P}_{q,X}$; see Chapter 3.3 in [15]. If $q = \infty$, then again for any metric space (X, d_X) we define the ∞–*Wasserstein distance*

$$(3) \qquad d_{\mathrm{W},\infty}^X(\alpha, \beta) := \inf_{\mu \in \mathrm{Cpl}(\alpha, \beta)} \sup_{(x, x') \in \mathrm{supp}(\mu)} d_X(x, x')$$

for $\alpha, \beta \in \mathcal{P}_{\infty,X}$, the set of measures with bounded support. The ∞–Wasserstein distance $d_{\mathrm{W},\infty}^X$ is clearly symmetric, and the triangle inequality comes from a similar gluing trick as in Chapter 3.3 of [15]. Thus $d_{\mathrm{W},\infty}^X$ is a pseudometric that is bounded below by a metric — for example $d_{\mathrm{W},1}^X$. Therefore $d_{\mathrm{W},\infty}^X$ is also a metric. For general $q \leq q'$, it follows from Hölder's inequality that $d_{\mathrm{W},q}^X \leq d_{\mathrm{W},q'}^X$.

It is known that on a bounded metric space, for any $q \in [1, \infty)$, the q–Wasserstein metric generates the weak topology. For a summary of the result, see Section A.1. On the other hand, the ∞–Wasserstein metric $d_{\mathrm{W},\infty}^X$ generates a finer topology than the weak topology in general.

Let $f, g \colon X \to \mathbb{R}$ be measurable functions over the metric space X, and let $p \in [1, \infty)$. We will frequently use the Minkowski inequality, which states that

$$\left(\int_X |f(x) + g(x)|^p \, \alpha(dx) \right)^{\frac{1}{p}} \leq \left(\int_X |f|^p(x) \, \alpha(dx) \right)^{\frac{1}{p}} + \left(\int_X |g|^p(x) \, \alpha(dx) \right)^{\frac{1}{p}}.$$

The following proposition shows that we may construct linear homotopies in spaces of probability measures, analogous to linear homotopies in Euclidean spaces. Linear homotopies will play an important role in Section 5.

Proposition 2.4 *Let Z be a metric space (or more generally a first-countable space), let X be a metric space, and let $f, g \colon Z \to \mathcal{P}_X$ be continuous. Then the linear homotopy $H \colon Z \times [0, 1] \to \mathcal{P}_X$ given by $H(z, t) = (1 - t) f(z) + t g(z)$ is continuous.*

Proof It suffices to show sequential continuity, because $Z \times I$ is metrizable (or more generally first-countable) and the weak topology on the set $\mathcal{P}_X$ is also metrizable; see [15, Theorem 3.1.4] and also Section A.1. If $\{(z_n, t_n)\}_n$ converges to (z, t) in $Z \times I$, then to show weak convergence of the image in $\mathcal{P}_X$, let $\gamma \colon X \to \mathbb{R}$ be any bounded, continuous function. We have

$$\lim_{n \to \infty} \int_X \gamma(x) H(z_n, t_n)(dx) = \lim_{n \to \infty} \left((1 - t_n) \int_X \gamma(x) f(z_n)(dx) + t_n \int_X \gamma(x) g(z_n)(dx) \right)$$

$$= (1 - t) \int_X \gamma(x) f(z)(dx) + t \int_X \gamma(x) g(z)(dx) = \int_X \gamma(x) H(z, t)(dx),$$

where the second equality uses the fact that $f(z_n)$ and $g(z_n)$ are weakly convergent, since f and g are continuous. Therefore $H(z_n, t_n)$ converges weakly to $H(z, t)$, so H is continuous. $\qquad\square$

Fréchet means For each $p \in [1, \infty]$, let the *p–Fréchet function of* $\alpha \in \mathcal{P}_X$, namely $F_{\alpha, p} : X \to \mathbb{R} \cup \{\infty\}$, be defined by

$$F_{\alpha, p}(x) := \begin{cases} \left(\int_X d_X^p(z, x) \, \alpha(dz) \right)^{1/p} & \text{for } p < \infty, \\ \sup_{z \in \mathrm{supp}(\alpha)} d_X(x, z) & \text{for } p = \infty. \end{cases}$$

Note that $F_{\alpha, p}(x) = d_{\mathrm{W}, p}^X(\delta_x, \alpha)$. A point $x \in X$ that minimizes $F_{\alpha, p}(x)$ is called a *Fréchet mean* of α; in general, Fréchet means need not be unique. See [44] for some of the basic properties of Fréchet means.

Metric thickenings with $p = \infty$ Let X be a bounded metric space. The Vietoris–Rips and Čech metric thickenings were introduced in [2] with the notation $\mathrm{VR}^m(X; r)$ and $\check{\mathrm{C}}^m(X; r)$, where the superscript m denoted "metric." We instead denote these spaces by $\mathrm{VR}_\infty^{\mathrm{fin}}(X; r)$ and $\check{\mathrm{C}}_\infty^{\mathrm{fin}}(X; r)$, since one of our main contributions will be to introduce the generalizations $\mathrm{VR}_p(X; r)$ and $\check{\mathrm{C}}_p(X; r)$ — and their finitely supported variants $\mathrm{VR}_p^{\mathrm{fin}}(X; r)$ and $\check{\mathrm{C}}_p^{\mathrm{fin}}(X; r)$ — for any $1 \le p \le \infty$.

The *Vietoris–Rips metric thickening* $\mathrm{VR}_\infty^{\mathrm{fin}}(X; r)$ is the space of all finitely supported probability measures of the form $\sum_{i=0}^k a_i \delta_{x_i}$ such that $\mathrm{diam}(\{x_0, \dots, x_k\}) < r$, equipped with the q–Wasserstein metric for some $1 \le q < \infty$. The choice of $q \in [1, \infty)$ does not affect the homeomorphism type by Corollary A.2. The *Čech metric thickening* $\check{\mathrm{C}}_\infty^{\mathrm{fin}}(X; r)$ is the space of all finite probability measures of the form $\sum_{i=0}^k a_i \delta_{x_i}$ such that $\bigcap_{i=0}^k B(x_i; r) \ne \varnothing$, equipped with the q–Wasserstein metric for some $1 \le q < \infty$.

Comparisons We give a brief survey of the various advantages and disadvantages of using simplicial complexes and $p = \infty$ metric thickenings.

The Vietoris–Rips and Čech simplicial complexes were developed first, and they enjoy the benefits of simplicial and combinatorial techniques. For this reason, these complexes (and related complexes) have been used in (co)homology theories for metric spaces, and in discrete versions of homotopy theories; see [7; 13; 19; 20; 26; 32; 33; 59; 64; 66; 67; 71]. The persistent homology of Vietoris–Rips complexes on top of finite metric spaces can be efficiently computed [12]. Even when built on top of infinite metric spaces, much is known about the theory of Vietoris–Rips simplicial complexes using Hausmann's theorem [42], Latschev's theorem [50], the stability of Vietoris–Rips persistent homology [27; 30; 31], Bestvina–Brady Morse theory [75] and the Vietoris–Rips complexes of the circle [1].

Despite this rich array of simplicial techniques, there are some key disadvantages to Vietoris–Rips simplicial complexes. If X is not a discrete metric space, then the inclusion from X into $\mathrm{VR}(X; r)$ for any $r \ge 0$ is not continuous, since the vertex set of a simplicial complex is equipped with the discrete topology. Another disadvantage is that even though we start with a metric space X, the Vietoris–Rips simplicial complex $\mathrm{VR}(X; r)$ may not be metrizable. Indeed, a simplicial complex is metrizable if and only if it is locally finite [68, Proposition 4.2.16(2)], that is, if and only if each vertex is contained in only

a finite number of simplices. So when X is infinite, $\mathrm{VR}(X; r)$ is often not metrizable, ie its topology cannot be induced by any metric. Though Vietoris–Rips complexes accept metric spaces as input, they do not remain in this same category, and may produce as output topological spaces that cannot be equipped with any metric structure.

By contrast, the metric thickening $\mathrm{VR}^{\mathrm{fin}}_\infty(X; r)$ is always a metric space that admits an isometric embedding $X \hookrightarrow \mathrm{VR}^{\mathrm{fin}}_\infty(X; r)$ [2]. Furthermore, metric thickenings allow for nicer proofs of Hausmann's theorem. For M a Riemannian manifold and $r > 0$ sufficiently small depending on the curvature of M, Hausmann produces a map $T \colon \mathrm{VR}(M; r) \to M$ from the simplicial complex to the manifold that is not canonical in the sense that it depends on a total order of all points in the manifold. Also, the inclusion map $M \hookrightarrow \mathrm{VR}(M; r)$ is not continuous, and therefore cannot be a homotopy inverse for T. Nevertheless, Hausmann is able to prove T is a homotopy equivalence without constructing an explicit inverse. By contrast, in the context of metric thickenings, one can produce a canonical map $\mathrm{VR}^{\mathrm{fin}}_\infty(M; r) \to M$ by mapping a measure to its Fréchet mean (whenever r is small enough that measures of diameter less than r have unique Fréchet means). The (now continuous) inclusion $M \hookrightarrow \mathrm{VR}^{\mathrm{fin}}_\infty(M; r)$ can be shown to be a homotopy inverse via linear homotopies [2, Theorem 4.2].

One of our main contributions is showing how these various spaces relate to each other, especially when it comes to persistent homology. This allows one to work either simplicially, geometrically, or with measures — whichever perspective is most convenient for the task at hand.

Homology and persistent homology For each integer $k \geq 0$, let H_k denote the singular homology functor from the category Top of topological spaces to the category Vec of vector spaces and linear transformations. We use coefficients in a fixed field, so that homology groups are furthermore vector spaces. For background on persistent homology, we refer the reader to [34; 35; 76].

In applications of topology, such as topological data analysis [25], one often models a dataset not as a single space X, but instead as an increasing sequence of spaces. We refer to an increasing sequence of spaces, that is, a functor from the poset $(\mathbb{R}, \leq)$ to Top, as a *filtration*. If X is a metric space, then a common filtration is the Vietoris–Rips simplicial complex filtration $\mathrm{VR}(X; \cdot)$. We will introduce relaxed versions, the p–Vietoris–Rips metric thickening filtrations $\mathrm{VR}_p(X; \cdot)$; see Section 3.

By applying homology (with coefficients in a field) to a filtration, we obtain a functor from the poset $(\mathbb{R}, \leq)$ to Vec, ie a *persistence module*. We will use symbols like V, W and so on to denote persistence modules. Following [29], a persistence module is Q–*tame* if, for any $s < t$, the structure map $V(s) \to V(t)$ is of finite rank. In [29], it is shown that a Q–tame persistence module V can be converted into a *persistence diagram*,[1] $\mathrm{dgm}(V)$, which is a multiset in the extended open half-plane consisting of pairs $p = (b, d)$ for $-\infty \leq b < d \leq +\infty$. The persistence diagrams can be compared via the bottleneck distance d_{B}, which is defined as follows. Given persistence diagrams D_1 and D_2, a subset $M \subset D_1 \times D_2$ is said to be a *partial matching* between D_1 and D_2 if:

---

[1]Specifically, we consider *undecorated* persistence diagrams, as described in [29].

- For every point p in D_1, there is at most one point q in D_2 such that $(p, q) \in M$. (If there is no such q, we then say p is unmatched.)

- For every point q in D_2, there is at most one point p in D_1 such that $(p, q) \in M$. (If there is no such p, we then say q is unmatched.)

The *bottleneck distance* d_{B} between two persistence diagrams D_1 and D_2 is

$$d_{\mathrm{B}}(D_1, D_2) := \inf_M \max \left\{ \sup_{(p,q) \in M} \| p - q \|_\infty, \sup_{s \in D_1 \sqcup D_2 \text{ unmatched}} \left| \tfrac{1}{2}(s_b - s_d) \right| \right\},$$

where s is an element in D_1 or D_2, s_b is the birth time of s, s_d is the death time of s, and M varies among all possible partial matchings.

Interleavings Let $\mathcal{C}$ be a category. We call any functor from the poset $(\mathbb{R}, \leq)$ to $\mathcal{C}$ an $\mathbb{R}$–*space*. Such a functor gives a *structure map* $X(s) \to X(t)$ for any $s \leq t$. For X an $\mathbb{R}$–space, the δ–*shift* of X is the functor $X^\delta \colon \mathbb{R} \to C$ with $X^\delta(t) = X(t + \delta)$ for all $t \in \mathbb{R}$. We have a natural transformation id_X^δ from X to X^δ which maps $X(t)$ to $X^\delta(t)$ using the structure maps from X. For two $\mathbb{R}$–spaces X and Y, we say they are δ–*interleaved* if there are natural transformations $F \colon X \to Y^\delta$ and $G \colon Y \to X^\delta$ such that $F \circ G = \mathrm{id}_X^{2\delta}$ and $G \circ F = \mathrm{id}_Y^{2\delta}$. Now we define a pseudodistance between $\mathbb{R}$–spaces, which we call the *interleaving distance* $d_{\mathrm{I}}^{\mathcal{C}}$, as follows:

$$d_{\mathrm{I}}^{\mathcal{C}}(X, Y) := \inf\{\delta \mid X \text{ and } Y \text{ are } \delta\text{–interleaved}\}.$$

We note that $d_{\mathrm{I}}^{\mathrm{Vec}}$ is the interleaving distance for persistence modules. The isometry theorem of [29; 51] states that $d_{\mathrm{I}}^{\mathrm{Vec}}(V, W) = d_{\mathrm{B}}(\mathrm{dgm}(V), \mathrm{dgm}(W))$ for Q–tame persistence modules V and W.

We will use the following lemma in our proofs in Section 5 on stability:

Lemma 2.5 *If a persistence module P can be approximated arbitrarily well in the interleaving distance by Q–tame persistence modules, then P is also Q–tame.*

Proof For any $s < t$, let $\varepsilon = t - s$, so there is a Q–tame persistence module P_ε such that $d_{\mathrm{I}}(P, P_\varepsilon) < \tfrac{1}{3}\varepsilon$. Then using the maps of an $\tfrac{1}{3}\varepsilon$–interleaving between P and P_ε, the structure map $P(s) \to P(t)$ can be factored as follows:

$$
\begin{array}{ccc}
P(s) & \xrightarrow{\hspace{5cm}} & P(t) \\
\searrow & & \nearrow \\
& P_\varepsilon\left(s + \tfrac{1}{3}\varepsilon\right) \longrightarrow P_\varepsilon\left(s + \tfrac{2}{3}\varepsilon\right) &
\end{array}
$$

As P_ε is a Q–tame module, the rank of $P_\varepsilon\left(s + \tfrac{1}{3}\varepsilon\right) \to P_\varepsilon\left(s + \tfrac{2}{3}\varepsilon\right)$ is finite, and hence so is the rank of $P(s) \to P(t)$. Since s and t are arbitrary, we get the Q–tameness of P. $\qquad\square$

The homotopy type distance We next recall the definition of the homotopy type distance d_{HT} from [37]. The following is a small generalization of [37, Definition 2.2] in that we do not require the maps φ_X and φ_Y to be continuous.

Definition 2.6 Let (X, φ_X) and (Y, φ_Y) be two topological spaces with real-valued functions on them. For any $\delta \geq 0$, a δ–*map* between (X, φ_X) and (Y, φ_Y) is a continuous map $\Phi \colon X \to Y$ such that $\varphi_Y \circ \Phi(x) \leq \varphi_X(x) + \delta$ for any $x \in X$. For any two δ–maps $\Phi_0 \colon X \to Y$ and $\Phi_1 \colon X \to Y$, a δ–*homotopy* between Φ_0 and Φ_1 with respect to the pair (φ_X, φ_Y) is a continuous map $H \colon X \times [0, 1] \to Y$ such that

(i) $\Phi_0 \equiv H(\cdot, 0)$,

(ii) $\Phi_1 \equiv H(\cdot, 1)$,

(iii) $H(\cdot, t)$ is a δ–map with respect to the pair (φ_X, φ_Y) for every $t \in [0, 1]$.

Definition 2.7 [37, Definitions 2.5 and 2.6] For every $\delta \geq 0$ and for any two pairs (X, φ_X) and (Y, φ_Y), we say (X, φ_X) and (Y, φ_Y) are δ–*homotopy equivalent* if there exist δ–maps $\Phi \colon X \to Y$ and $\Psi \colon Y \to X$, with respect to (φ_X, φ_Y) and (φ_Y, φ_X), respectively, such that

- the map $\Psi \circ \Phi \colon X \to X$ is 2δ–homotopic to id_X with respect to (φ_X, φ_X), and

- the map $\Phi \circ \Psi \colon Y \to Y$ is 2δ–homotopic to id_Y with respect to (φ_Y, φ_Y).

The d_{HT}–*distance* between (X, φ_X) and (Y, φ_Y) is

$$d_{\mathrm{HT}}((X, \varphi_X), (Y, \varphi_Y)) := \inf\{\delta \geq 0 \mid (X, \varphi_X) \text{ and } (Y, \varphi_Y) \text{ are } \delta\text{–homotopy equivalent}\}.$$

If (X, φ_X) and (Y, φ_Y) are not δ–homotopy equivalent for any δ, then we declare

$$d_{\mathrm{HT}}((X, \varphi_X), (Y, \varphi_Y)) = \infty.$$

Proposition 2.8 [37, Proposition 2.10] *The d_{HT} distance is an extended pseudometric on the set of pairs of topological spaces and real-valued functions.*

A pair (X, φ_X), where X is a topological space and $\varphi_X \colon X \to \mathbb{R}$ is a real-valued function, induces an $\mathbb{R}$–space $[X, \varphi_X; \cdot]$ given by the sublevel set filtration, and defined by

$$[X, \varphi_X; r] := \varphi_X^{-1}((-\infty, r)) \quad \text{for } r \in \mathbb{R}$$

and

$$[X, \varphi_X; \cdot] := \{\varphi_X^{-1}((-\infty, r)) \subseteq \varphi_X^{-1}((-\infty, r'))\}_{r \leq r'}.$$

The following theorem shows that the interleaving distance of the persistent homology of the sublevel set filtrations is bounded by the d_{HT} distance of the respective pairs. The theorem is a slight generalization of [37, Lemma 3.1] in which we do not require the continuity of φ_X and φ_Y; we omit its (identical) proof.

Lemma 2.9 [37, Lemma 3.1] *Let (X, φ_X) and (Y, φ_Y) be two pairs. Then for any integer $k \geq 0$,*

$$d_{\mathrm{I}}^{\mathrm{Vec}}(H_k \circ [X, \varphi_X; \cdot], H_k \circ [Y, \varphi_Y; \cdot]) \leq d_{\mathrm{HT}}((X, \varphi_X), (Y, \varphi_Y)).$$

3 The *p*–relaxation of metric thickenings

We now describe the construction of the *p*–relaxations of metric thickenings.

3.1 The relaxed diameter and radius functionals

Throughout this section, (X, d_X) will denote a bounded metric space. Recall that $\mathcal{P}_X$ is the set of all Radon probability measures on X, equipped with the weak topology. We now introduce, for each $p \in [1, \infty]$, the *p–diameter* of a measure α in $\mathcal{P}_X$, where the ∞–diameter of α is precisely the diameter of its support $\mathrm{supp}\,(\alpha)$ (as a subset of X). Consider, for each $p \in [1, \infty]$, the *p–diameter* map $\mathrm{diam}_p \colon \mathcal{P}_X \to \mathbb{R}_{\geq 0}$ given by

$$\mathrm{diam}_p(\alpha) = \begin{cases} \left(\iint_{X \times X} d_X^p(x, x') \, \alpha(dx) \, \alpha(dx') \right)^{1/p} & \text{for } p < \infty, \\ \mathrm{diam}(\mathrm{supp}(\alpha)) & \text{for } p = \infty. \end{cases}$$

We remark that the *p*–diameter has been studied and critiqued as a measure of diversity in population biology [63; 65], and it has also been considered in relation to the Gromov–Wasserstein distance [55]. Similarly, define the *p–radius* map $\mathrm{rad}_p \colon \mathcal{P}_X \to \mathbb{R}_{\geq 0}$ via

$$\mathrm{rad}_p(\alpha) = \begin{cases} \inf_{x \in X} \left(\int_X d_X^p(x, x') \, \alpha(dx') \right)^{1/p} & \text{for } p < \infty, \\ \mathrm{rad}(\mathrm{supp}(\alpha)) & \text{for } p = \infty. \end{cases}$$

Note that the *p*–radius of a measure is simply the pth root of its *p*–variance.

We observe that

$$\mathrm{diam}_p(\alpha) = \left(\int_X F_{\alpha, p}^p(x) \, \alpha(dx) \right)^{\frac{1}{p}} = \left(\int_X (d_{\mathrm{W}, p}^X(\alpha, \delta_x))^p \, \alpha(dx) \right)^{\frac{1}{p}}$$

and

$$\mathrm{rad}_p(\alpha) = \inf_{x \in X} F_{\alpha, p}(x) = \inf_{x \in X} d_{\mathrm{W}, p}^X(\delta_x, \alpha).$$

Proposition 3.1 *The functions* $\mathrm{diam}_p, \mathrm{rad}_p \colon \mathcal{P}_X \to \mathbb{R}$ *satisfy*

$$\mathrm{rad}_p(\alpha) \leq \mathrm{diam}_p(\alpha) \leq 2\,\mathrm{rad}_p(\alpha).$$

Proof To see that $\mathrm{rad}_p(\alpha) \leq \mathrm{diam}_p(\alpha)$, note that

$$\mathrm{diam}_p(\alpha) = \left(\iint_{X \times X} (d_X(x, x'))^p \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}} = \left(\int_X \left(\int_X d_X^p(x, x') \, \alpha(dx) \right) \alpha(dx') \right)^{\frac{1}{p}}$$

$$\geq \left(\inf_{x' \in X} \left(\int_X d_X^p(x, x') \, \alpha(dx) \right) \right)^{\frac{1}{p}} = \inf_{x' \in X} \left(\int_X d_X^p(x, x') \, \alpha(dx) \right)^{\frac{1}{p}} = \mathrm{rad}_p(\alpha).$$

To see that $\mathrm{diam}_p(\alpha) \leq 2\,\mathrm{rad}_p(\alpha)$, for any $\varepsilon > 0$ there is a point $z \in X$ with

$$\left(\int_X d_X^p(x, z) \, \alpha(dx) \right)^{\frac{1}{p}} \leq \mathrm{rad}_p(\alpha) + \varepsilon.$$

Then we have

$$\text{diam}_p(\alpha) = \left(\iint_{X \times X} (d_X(x, x'))^p \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}} \leq \left(\iint_{X \times X} (d_X(x, z) + d_X(z, x'))^p \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}}$$

$$\leq \left(\iint_{X \times X} (d_X(x, z))^p \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}} + \left(\iint_{X \times X} (d_X(z, x'))^p \, \alpha(dx) \, \alpha(dx') \right)^{\frac{1}{p}}$$

$$\leq 2 \, \text{rad}_p(\alpha) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\text{diam}_p(\alpha) \leq 2 \, \text{rad}_p(\alpha)$. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.2 The tightness of the bound $\text{rad}_p(\alpha) \leq \text{diam}_p(\alpha)$ can be seen from the calculation for the uniform measure on Z_n, the metric space on a set of size n with all interpoint distances equal to 1. To see the (asymptotic) tightness of $\text{diam}_p(\alpha) \leq 2 \, \text{rad}_p(\alpha)$, we consider the metric space $Z_n \cup \{O\}$, where the newly introduced "center" O has distance $\frac{1}{2}$ to every other point. Then for the measure $\alpha := \sum_{z \in Z_n} (1/n) \delta_z$, we have $\text{rad}_p(\alpha) = \frac{1}{2}$ and $\text{diam}_p(\alpha) = ((n-1)/n)^{1/p}$. We obtain asymptotic tightness by letting n go to infinity.

3.2 The relaxed Vietoris–Rips and Čech metric thickenings

Definition 3.3 (*p*–Vietoris–Rips filtration) For each $r > 0$ and $p \in [1, \infty]$, let the *p–Vietoris–Rips metric thickening at scale* r be

$$\text{VR}_p(X; r) := \{ \alpha \in \mathcal{P}_X \mid \text{diam}_p(\alpha) < r \}.$$

We regard $\text{VR}_p(X; r)$ as a topological space by endowing it with the subspace topology from $\mathcal{P}_X$ (ie the weak topology). By convention, when $r \leq 0$, we will let $\text{VR}_p(X; r) = \varnothing$. By

$$\text{VR}_p(X; \cdot) := \{ \text{VR}_p(X; r) \xrightarrow{v_{r,r'}^X} \text{VR}_p(X; r') \}_{r \leq r'}$$

we will denote the filtration thus induced.

We use $\text{VR}_p^{\text{fin}}(X; r)$ and $\text{VR}_p^{\text{fin}}(X; \cdot)$ to denote the finitely supported variants obtained by replacing $\mathcal{P}_X$ in the above definition with $\mathcal{P}_X^{\text{fin}}$.

Note that in the specific case $p = \infty$, the ∞–diameter of a measure is simply the diameter of its support. For a bounded metric space X, the weak topology is generated by the 1–Wasserstein metric, so the definition of $\text{VR}_p^{\text{fin}}(X; r)$ generalizes [2, Definition 3.1], which is the specific case $p = \infty$.

Remark 3.4 ($\text{VR}_p(X; \cdot)$ as a softening of $\text{VR}_\infty(X, \cdot)$) The definition of $\text{VR}_p(X; \cdot)$ can be extended to the whole range $p \in [0, \infty]$ as follows. First note that $\text{diam}_p(\alpha)$ can still be defined as above for $p \in (0, 1)$, and by $\text{diam}_0(\alpha) = 0$ when $p = 0$. Furthermore, if $\alpha = \sum_{i \in I} a_i \delta_{x_i}$, then

$$\lim_{p \downarrow 0} \text{diam}_p(\alpha) = \prod_{i, j \in I} (d_X(x_i, x_j))^{a_i a_j},$$

which equals 0 since the product contains terms with $i = j$.

At any rate, by the standard generalized means inequality [22], we have $\mathrm{diam}_{p'}(\alpha) \leq \mathrm{diam}_p(\alpha)$ for any $p' \leq p$ in the range $[0, \infty]$. So, for fixed $r > 0$, not only does $\mathrm{VR}_p(X; r)$ become larger and larger as p decreases, but also for $p = 0$, $\mathrm{VR}_0(X; r)$ contains *all* Radon probability measures on X and thus has trivial reduced homology.

Definition 3.5 (*p*–Čech filtration) For each $r > 0$ and $p \in [1, \infty]$, let the *p–Čech metric thickening at scale r* be

$$\check{C}_p(X; r) := \{\alpha \in \mathcal{P}_X \mid \mathrm{rad}_p(\alpha) < r\}.$$

We regard $\check{C}_p(r; X)$ as a topological space by endowing it with the subspace topology from $\mathcal{P}_X$ (ie the weak topology). By convention, when $r \leq 0$, we will let $\check{C}_p(X; r) = \varnothing$. By

$$\check{C}_p(X; \cdot) := \{\check{C}_p(X; r) \xrightarrow{\ \nu^X_{r,r'}\ } \check{C}_p(X; r')\}_{r \leq r'}$$

we will denote the filtration thus induced.[2]

Remark 3.6 Let α be a measure in $\mathcal{P}_X$. As $\mathrm{rad}_p(\alpha) = \inf_{x \in X} d^X_{\mathrm{W}, p}(\delta_x, \alpha)$, we have $\mathrm{rad}_p(\alpha) < r$ if and only if there is some $x \in X$ such that $d^X_{\mathrm{W}, p}(\delta_x, \alpha) < r$. Therefore $\check{C}_p(X; r)$ is exactly the union over all $x \in X$ of the balls $B(\delta_x; r)$, with respect to the *p*–Wasserstein metric, centered at points δ_x in the isometric image of X in $\mathcal{P}_X$.

We use $\check{C}_p^{\mathrm{fin}}(X; r)$ and $\check{C}_p^{\mathrm{fin}}(X; \cdot)$ to denote the finitely supported variants, obtained by replacing $\mathcal{P}_X$ in the above definition with $\mathcal{P}_X^{\mathrm{fin}}$.

Note that in the specific case $p = \infty$, the ∞–radius of a measure is simply the radius of its support. For a bounded metric space X, the weak topology is generated by the 1–Wasserstein metric, so the definition of $\check{C}_p^{\mathrm{fin}}(X; r)$ generalizes [2], which considers the specific case $p = \infty$.

Though the above definitions are given with the $<$ convention, we remark that they instead could have been given with the $\leq$ convention, namely $\mathrm{diam}_p(\alpha) \leq r$ or $\mathrm{rad}_p(\alpha) \leq r$. We restrict attention to the $<$ convention, even though many of the statements we give are also true with the $\leq$ convention.

The next proposition shows that *p*–Vietoris–Rips and *p*–Čech metric thickenings are nested in the same way that Vietoris–Rips and Čech simplicial complexes are.

Proposition 3.7 *Let* X *be a bounded metric space. Then, for any* $r > 0$,

$$\mathrm{VR}_p(X; r) \subseteq \check{C}_p(X; r) \subseteq \mathrm{VR}_p(X; 2r).$$

Proof This follows immediately from Proposition 3.1, which implies that, for any $\alpha \in \mathcal{P}_X$,

$$\mathrm{diam}_p(\alpha) \geq \mathrm{rad}_p(\alpha) \geq \tfrac{1}{2} \mathrm{diam}_p(\alpha). \qquad \square$$

---

[2]We will use $\nu^X_{r,r'}$ to denote the inclusion maps in both Čech and Vietoris–Rips thickenings; this will not lead to confusion in this paper.

If X and Z are metric spaces with $X \subseteq Z$, and if the metric on Z is an extension of that on X, then following Gromov [41, Section 1.B] we say that Z is an *r–metric thickening* of X if, for all $z \in Z$, there is some $x \in X$ with $d_Z(x, z) \le r$.

Proposition 3.8 *Let X be a bounded metric space. When equipped with the q–Wasserstein metric for $1 \le q \le p$, we have that $\check{C}_p(X; r)$ and $\mathrm{VR}_p(X; r)$ are each r–metric thickenings of X for all $r > 0$.*

Proof We use the isometric embedding $X \to \check{C}_p(X; r)$ given by $x \mapsto \delta_x$. Let $\alpha \in \check{C}_p(X; r)$. Hence there exists some $x \in X$ with $r > d_{\mathrm{W},p}^X(\delta_x, \alpha) \ge d_{\mathrm{W},q}^X(\delta_x, \alpha)$, which shows that $\check{C}_p(X; r)$, equipped with the q–Wasserstein metric, is an r–metric thickening of X.

The Vietoris–Rips case follows immediately since $\mathrm{VR}_p(X; r) \subseteq \check{C}_p(X; r)$ by Proposition 3.7. □

The next remark follows the perspective introduced in [52], which shows that the filling radius is related to the persistent homology of the Vietoris–Rips simplicial complex filtration $\mathrm{VR}(X, \cdot)$.

Remark 3.9 Let X be a closed connected n–dimensional manifold so that the *fundamental class* of X is well defined. Let Z be an r–metric thickening of X for some $r > 0$. It is shown in [40, Page 8, Another Corollary] that the map

$$H_n(X) \hookrightarrow H_n(Z)$$

induced from the inclusion $\iota \colon X \to Z$ is an injection whenever r is less than a scalar geometric invariant $\mathrm{FillRad}(X)$, called the *filling radius* of X. Hence Proposition 3.8 implies that, for all $p \in [1, \infty]$, the persistence diagram in dimension n of either the filtration $\mathrm{VR}_p(X; \cdot)$ or $\check{C}_p(X; \cdot)$ contains an interval with left endpoint equal to zero and length at least $\mathrm{FillRad}(X)$. As proved in [45], the filling radius of the n–sphere (with its geodesic metric) is $\mathrm{FillRad}(\mathbb{S}^n) = \frac{1}{2} \arccos(-1/(n+1))$. Therefore, when $X = \mathbb{S}^n$, the n–dimensional persistence diagram of either $\mathrm{VR}_p(\mathbb{S}^n; \cdot)$ or $\check{C}_p(\mathbb{S}^n; \cdot)$ contains an interval starting at zero which is no shorter than $\frac{1}{2} \arccos(-1/(n+1))$.

Lemma 3.10 *For all $r > 0$ and all $p, p' \in [1, \infty]$ with $p \ge p'$, one has*

$$\mathrm{VR}_p(X; r) \subseteq \mathrm{VR}_{p'}(X; r) \quad \text{and} \quad \check{C}_p(X; r) \subseteq \check{C}_{p'}(X; r).$$

Proof This comes from applying the Hölder inequality on diam_p and rad_p. □

Example 3.11 We recall that Z_{n+1} is the metric space consisting of $n + 1$ points with all interpoint distances equal to 1. For any natural number n, the Vietoris–Rips or Čech simplicial complex filtrations of Z_{n+1} do not produce any nondiagonal point in their persistence diagram, except in homological dimension zero. However, for any $1 < p < \infty$, both the p–Vietoris–Rips and p–Čech filtrations of Z_{n+1} will contain nondiagonal points in their persistence diagrams for homological degrees from zero to $n - 1$. More specifically, both the p–Vietoris–Rips and p–Čech filtrations are homotopy equivalent to a filtration
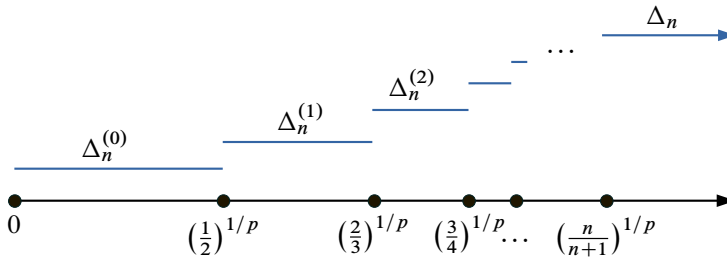
Figure 1: Homotopy types of both $\mathrm{VR}_p(Z_{n+1}; \cdot)$ and $\check{\mathrm{C}}_p(Z_{n+1}; \cdot)$.

of an n–simplex Δ_n by its k–skeleta, $\Delta_n^{(k)}$. We prove the following result in Section A.4. For r in the interval $((k/(k+1))^{1/p}, ((k+1)/(k+2))^{1/p}]$ with $0 \le k \le n-1$, we have

$$\mathrm{VR}_p(Z_{n+1}; r) \simeq \check{\mathrm{C}}_p(Z_{n+1}; r) \simeq \Delta_n^{(k)},$$

and when $r > (n/(n+1))^{1/p}$, both $\mathrm{VR}_p(Z_{n+1}; r)$ and $\check{\mathrm{C}}_p(Z_{n+1}; r)$ become the n–simplex Δ_n, which is contractible.

From this we get the persistence diagrams

$$\mathrm{dgm}_{k,p}^{\mathrm{VR}}(Z_{n+1}) = \mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(Z_{n+1}) = \begin{cases} \left(0, \left(\tfrac{1}{2}\right)^{1/p}\right)^{\otimes n} \oplus (0, \infty) & \text{if } k = 0, \\ ((k/(k+1))^{1/p}, ((k+1)/(k+2))^{1/p})^{\otimes \binom{n}{k+1}} & \text{if } 1 \le k \le n-1, \\ \varnothing & \text{if } k \ge n. \end{cases}$$

Note that, from the definition of $\mathrm{VR}_\infty(Z_{n+1}; \cdot)$, we have $\mathrm{dgm}_{k,\infty}^{\mathrm{VR}} = \varnothing$ for $k \ge 1$, and furthermore that $\lim_{p \uparrow \infty} \mathrm{dgm}_{k,p}^{\mathrm{VR}}(Z_{n+1}) = \mathrm{dgm}_{k,\infty}^{\mathrm{VR}}(Z_{n+1})$ for each $k \ge 0$. An analogous result holds for $\check{\mathrm{C}}_p(Z_{n+1}; \cdot)$.

3.3 A more general setting: controlled invariants

We now generalize the relaxed p–Vietoris–Rips and p–Čech filtrations.

Definition 3.12 (controlled invariants on $\mathcal{P}_X$) Let i be a functional that associates to each bounded metric space X a function $\mathrm{i}^X \colon \mathcal{P}_X \to \mathbb{R}$. We say, for some $L > 0$, that i is an L–*controlled* invariant if the following conditions are satisfied:

(i) **Stability under pushforward** For any map f between finite metric spaces X and Y and any α in $\mathcal{P}_X$,

$$\mathrm{i}^Y(f_\sharp(\alpha)) \le \mathrm{i}^X(\alpha) + L \operatorname{dis}(f).$$

(ii) **Stability with respect to $d_{\mathrm{W},\infty}^X$** For any bounded metric space X and any α and β in $\mathcal{P}_X$,

$$|\mathrm{i}^X(\alpha) - \mathrm{i}^X(\beta)| \le 2L \, d_{\mathrm{W},\infty}^X(\alpha, \beta).$$

In the next section, we will prove that both diam_p and rad_p induce controlled invariants.

For any controlled invariant $\mathfrak{i}$, we will use the notation $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ to denote the sublevel set filtration induced by the pair $(\mathcal{P}_X, \mathfrak{i}^X)$. That is,

$$[\mathcal{P}_X, \mathfrak{i}^X; r] = (\mathfrak{i}^X)^{-1}((-\infty, r)) \text{ for } r \in \mathbb{R}$$

and

$$[\mathcal{P}_X, \mathfrak{i}^X; \cdot] = \{(\mathfrak{i}^X)^{-1}((-\infty, r)) \subseteq (\mathfrak{i}^X)^{-1}((-\infty, r'))\}_{r \leq r'}.$$

This construction generalizes both the definition of $\mathrm{VR}_p(X; \cdot)$ and that of $\check{\mathrm{C}}_p(X; \cdot)$.

Similarly, any controlled invariant $\mathfrak{i} \colon \mathcal{P}_X^{\mathrm{fin}} \to \mathbb{R}$ induces an analogous sublevel set filtration $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ of $\mathcal{P}_X^{\mathrm{fin}}$. This construction generalizes both the definition of $\mathrm{VR}_p^{\mathrm{fin}}(X; \cdot) = [\mathcal{P}_X, \mathrm{diam}_p^X; \cdot]$ and that of $\check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot) = [\mathcal{P}_X, \mathrm{rad}_p^X; \cdot]$.

We'll see later on in Corollary 5.3 that the filtrations $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ and $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ yield persistence modules which are 0–interleaved.

4 Basic properties

We prove some basic properties. These include the convexity of the Wasserstein distance, the fact that diam_p and rad_p are 1–controlled and the fact that, for X finite, the $p = \infty$ metric thickenings $\mathrm{VR}_\infty(X; r)$ and $\check{\mathrm{C}}_\infty(X; r)$ are homeomorphic to the simplicial complexes $\mathrm{VR}(X; r)$ and $\check{\mathrm{C}}(X; r)$, respectively. We begin with the property that nearby measures have nearby integrals. Throughout this section, X is a bounded metric space.

Lemma 4.1 *Let φ be an L–Lipschitz function on X. Then, for any α and β in $\mathcal{P}_X$ and any $q \in [1, \infty]$,*

$$\left| \left(\int_X |\varphi(x)|^q \, \alpha(dx) \right)^{\frac{1}{q}} - \left(\int_X |\varphi(x)|^q \, \beta(dx) \right)^{\frac{1}{q}} \right| \leq L \, d_{\mathrm{W},q}^X(\alpha, \beta).$$

Proof Let $\mu \in \mathrm{Cpl}(\alpha, \beta)$ be any coupling. Then

$$\left| \left(\int_X |\varphi(x)|^q \, \alpha(dx) \right)^{\frac{1}{q}} - \left(\int_X |\varphi(x)|^q \, \beta(dx) \right)^{\frac{1}{q}} \right|$$

$$= \left| \left(\iint_{X \times X} |\varphi(x)|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} - \left(\iint_{X \times X} |\varphi(x')|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} \right|$$

$$\leq \left(\iint_{X \times X} |\varphi(x) - \varphi(x')|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} \leq L \left(\iint_{X \times X} d_X^q(x, x') \, \mu(dx \times dx') \right)^{\frac{1}{q}}.$$

We obtain the claim by taking the infimum of the right-hand side with respect to the coupling μ. $\qquad\square$

We will often use the following convexity result of the Wasserstein distance; for a more general result, see [72, Theorem 4.8].

Lemma 4.2 *Suppose that $c_1, \ldots, c_n$ are nonnegative real numbers satisfying $\sum_{k=1}^{n} c_k = 1$ and that $\alpha_1, \ldots, \alpha_n, \alpha_1', \ldots, \alpha_n' \in \mathcal{P}_X$. Then for all $q \in [1, \infty)$, we have*

$$d_{\mathrm{W},q}^X\left(\sum_{k=1}^{n} c_k \alpha_k, \sum_{k=1}^{n} c_k \alpha_k'\right) \leq \left(\sum_{k=1}^{n} c_k \left(d_{\mathrm{W},q}^X(\alpha_k, \alpha_k')\right)^q\right)^{\frac{1}{q}}$$

and

$$d_{\mathrm{W},\infty}^X\left(\sum_{k=1}^{n} c_k \alpha_k, \sum_{k=1}^{n} c_k \alpha_k'\right) \leq \max_k d_{\mathrm{W},\infty}^X(\alpha_k, \alpha_k').$$

Proof Let $\varepsilon > 0$ and $q \in [1, \infty)$. For each k, suppose μ_k is a coupling between α_k and α_k' such that

$$\iint_{X \times X} d_X^q(x, x') \, \mu_k(dx \times dx') < (d_{\mathrm{W},q}^X(\alpha_k, \alpha_k') + \varepsilon)^q.$$

Then it can be checked that $\sum_{k=1}^{n} c_k \mu_k$ is a coupling between $\sum_{k=1}^{n} c_k \alpha_k$ and $\sum_{k=1}^{n} c_k \alpha_k'$. We have

$$d_{\mathrm{W},q}^X\left(\sum_{k=1}^{n} c_k \alpha_k, \sum_{k=1}^{n} c_k \alpha_k'\right) \leq \left(\iint_{X \times X} d_X^q(x, x') \sum_{k=1}^{n} c_k \, \mu_k(dx \times dx')\right)^{\frac{1}{q}}$$

$$< \left(\sum_{k=1}^{n} c_k (d_{\mathrm{W},q}^X(\alpha_k, \alpha_k') + \varepsilon)^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{n} c_k (d_{\mathrm{W},q}^X(\alpha_k, \alpha_k'))^q\right)^{\frac{1}{q}} + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, the claimed inequality holds. The case for $q = \infty$ can be checked separately using the same matching. $\square$

We define an invariant generalizing diam_p, and then prove that it is 1–controlled.

Definition 4.3 ($\mathrm{i}_{q,p}^X$ invariant) Let X be a bounded metric space. For any $p, q \in [1, \infty]$, we define the $\mathrm{i}_{q,p}$ *invariant*, which associates to each bounded metric space X a function $\mathrm{i}_{q,p}^X : \mathcal{P}_X \to \mathbb{R}$, by

$$\mathrm{i}_{q,p}^X(\alpha) = \left(\int_X (d_{\mathrm{W},q}^X(\alpha, \delta_x))^p \, \alpha(dx)\right)^{\frac{1}{p}}.$$

Note that $\mathrm{i}_{q,p}^X$ recovers diam_p^X when $p = q$.

Remark 4.4 In the construction of $\mathrm{i}_{q,p}^X$, one can get other interesting filtration functions by replacing the Wasserstein distance with other distances between measures. These include, for example, the Lévy–Prokhorov metric, the Fortet–Mourier metric and variants of the Kantorovich–Rubinstein metric. For definitions of these metrics, see [15, Section 3].

Lemma 4.5 *Let X and Y be bounded metric spaces, and let $f : X \to Y$ be a map. Then, for any element $\alpha \in \mathcal{P}_X$ and $p \in [1, \infty]$,*

$$\mathrm{i}_{q,p}^Y(f_\sharp(\alpha)) \leq \mathrm{i}_{q,p}^X(\alpha) + \mathrm{dis}(f).$$

Proof We compute

$$\mathfrak{i}_{q,p}^Y(f_\sharp(\alpha)) = \left(\int_Y (d_{W,q}^Y(f_\sharp(\alpha), \delta_y))^p f_\sharp(\alpha)(dy)\right)^{\frac{1}{p}} = \left(\int_X (d_{W,q}^Y(f_\sharp(\alpha), \delta_{f(x)}))^p \alpha(dx)\right)^{\frac{1}{p}}$$

$$= \left(\int_X \left(\int_Y d_Y^q(y, f(x)) f_\sharp(\alpha)(dy)\right)^{\frac{p}{q}} \alpha(dx)\right)^{\frac{1}{p}}$$

$$= \left(\int_X \left(\int_X d_Y^q(f(x'), f(x)) \alpha(dx')\right)^{\frac{p}{q}} \alpha(dx)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_X \left(\int_X (d_X(x', x) + \mathrm{dis}(f))^q \alpha(dx')\right)^{\frac{p}{q}} \alpha(dx)\right)^{\frac{1}{p}} \leq \mathfrak{i}_{q,p}^X(\alpha) + \mathrm{dis}(f). \qquad \square$$

Lemma 4.6 *Let X be a bounded metric space. Then, for any α, β in $\mathcal{P}_X$,*

$$|\mathfrak{i}_{q,p}^X(\alpha) - \mathfrak{i}_{q,p}^X(\beta)| \leq d_{W,q}^X(\alpha, \beta) + d_{W,p}^X(\alpha, \beta).$$

In particular, $|\mathfrak{i}_{q,p}^X(\alpha) - \mathfrak{i}_{q,p}^X(\beta)| \leq 2d_{W,\infty}^X(\alpha, \beta)$.

Proof We first note that

$$\mathfrak{i}_{q,p}^X(\alpha) = \left(\int_X (d_{W,q}^X(\alpha, \delta_x))^p \alpha(dx)\right)^{\frac{1}{p}} \leq \left(\int_X (d_{W,q}^X(\beta, \delta_x) + d_{W,q}^X(\alpha, \beta))^p \alpha(dx)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_X (d_{W,q}^X(\beta, \delta_x))^p \alpha(dx)\right)^{\frac{1}{p}} + d_{W,q}^X(\alpha, \beta).$$

We also have

$$\left|\left(\int_X (d_{W,q}^X(\beta, \delta_x))^p \alpha(dx)\right)^{\frac{1}{p}} - \mathfrak{i}_{q,p}(\beta)\right|$$

$$= \left|\left(\int_X (d_{W,q}^X(\beta, \delta_x))^p \alpha(dx)\right)^{\frac{1}{p}} - \left(\int_X (d_{W,q}^X(\beta, \delta_x))^p \beta(dx)\right)^{\frac{1}{p}}\right|$$

$$\leq d_{W,p}^X(\alpha, \beta),$$

where the last line follows from [Lemma 4.1](#) since $d_{W,q}^X(\beta, \delta_x)$ is a 1–Lipschitz function in x. Therefore,

$$\mathfrak{i}_{q,p}(\alpha) \leq \mathfrak{i}_{q,p}(\beta) + d_{W,q}^X(\alpha, \beta) + d_{W,p}^X(\alpha, \beta).$$

We then get the result by swapping the roles of α and β. $\qquad \square$

Remark 4.7 This establishes the continuity of $\mathfrak{i}_{q,p}$ for $q, p \in [1, \infty)$. If p or q equals infinity, then $\mathfrak{i}_{q,p}$ is not necessarily continuous in the weak topology since $d_{W,\infty}^X$ does not necessarily induce the same topology on $\mathcal{P}_X$ as $d_{W,p}^X$ does for $p < \infty$. Indeed, if $X = [0, 1]$ is the unit interval, then $((n-1)/n)\delta_0 + (1/n)\delta_1 \to \delta_0$ in $\mathcal{P}_X$ as $n \to \infty$, even though $\mathrm{diam}_\infty(((n-1)/n)\delta_0 + (1/n)\delta_1)$ is equal to 1 for all $n \geq 1$, and hence does not converge to $0 = \mathrm{diam}(\delta_0)$.

The above two lemmas imply that $\mathfrak{i}_{q,p}$ (and hence $\mathrm{diam}_p = \mathfrak{i}_{p.p}$) is a 1–controlled invariant. We next consider the analogous properties for rad_p.

Lemma 4.8 (rad$_p$ under a map with bounded distortion) *Let X and Y be bounded metric spaces, and let $f : X \to Y$ be a map. Then, for any $\alpha \in \mathcal{P}_X$ and $p \in [1, \infty]$,*

$$\operatorname{rad}_p(f_\sharp(\alpha)) \leq \operatorname{rad}_p(\alpha) + \operatorname{dis}(f).$$

Proof We only give the proof for the case $p < \infty$; the case $p = \infty$ is similar. We have

$$\operatorname{rad}_p(f_\sharp(\alpha)) = \inf_{y \in Y} \left(\int_Y d_Y^p(y, y') f_\sharp(\alpha)(dy') \right)^{\frac{1}{p}} \leq \inf_{y \in f(X)} \left(\int_Y d_Y^p(y, y') f_\sharp(\alpha)(dy') \right)^{\frac{1}{p}}$$

$$= \inf_{x \in X} \left(\int_Y d_Y^p(f(x), y') f_\sharp(\alpha)(dy') \right)^{\frac{1}{p}} = \inf_{x \in X} \left(\int_X d_Y^p(f(x), f(x')) \alpha(dx') \right)^{\frac{1}{p}}$$

$$\leq \inf_{x \in X} \left(\int_X (d_X(x, x') + \operatorname{dis}(f))^p \alpha(dx') \right)^{\frac{1}{p}} \leq \inf_{x \in X} \left(\int_X d_X^p(x, x') \alpha(dx') \right)^{\frac{1}{p}} + \operatorname{dis}(f)$$

$$= \operatorname{rad}_p(\alpha) + \operatorname{dis}(f). \qquad \square$$

Lemma 4.9 (stability of rad$_p$) *Let X be a bounded metric space. For any two probability measures $\alpha, \beta \in \mathcal{P}_X$ and for every $p \in [1, \infty]$,*

$$|\operatorname{rad}_p(\alpha) - \operatorname{rad}_p(\beta)| \leq d_{W,p}^X(\alpha, \beta).$$

Proof We compute

$$\operatorname{rad}_p(\alpha) = \inf_{x \in X} d_{W,p}^X(\alpha, \delta_x) \leq \inf_{x \in X} (d_{W,p}^X(\beta, \delta_x) + d_{W,p}^X(\beta, \alpha)) = \inf_{x \in X} d_{W,p}^X(\beta, \delta_x) + d_{W,p}^X(\beta, \alpha)$$

$$= \operatorname{rad}_p(\beta) + d_{W,p}^X(\beta, \alpha). \qquad \square$$

Remark 4.10 Lemma 4.9 establishes the continuity of rad$_p$ for $p \in [1, \infty)$, as these functions are 1–Lipschitz. Similarly to diam$_\infty$, we note that rad$_\infty$ is not necessarily continuous because the metric topology given by $d_{W,\infty}^X$ is not necessarily equal to the weak topology.

The above two lemmas imply that rad$_p$ is a 1–controlled invariant.

We end this section of basic properties by showing that $\mathcal{P}_X$ is homeomorphic to a simplex when X is finite. Hence for X finite, the $p = \infty$ metric thickenings $\operatorname{VR}_\infty(X; r)$ and $\check{\operatorname{C}}_\infty(X; r)$ are homeomorphic to the simplicial complexes $\operatorname{VR}(X; r)$ and $\check{\operatorname{C}}(X; r)$, respectively; see also [2, Corollary 6.4].

Lemma 4.11 *If X is a finite metric space with n points, then $\mathcal{P}_X$ is homeomorphic to the standard $(n-1)$–simplex.*

Proof Let $X = \{x_1, \dots, x_n\}$. The space $\mathcal{P}_X$ of probability measures is in bijection with the standard $(n-1)$–simplex $\Delta_{n-1} = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n y_i = 1 \text{ with } y_i \geq 0 \text{ for all } i\}$ via the function $f : \mathcal{P}_X \to \Delta_{n-1}$ defined by $f\left(\sum_{i=1}^n a_i \delta_{x_i}\right) = (a_1, \dots, a_n)$. Suppose we have a sequence $\{\alpha_k\}$ in $\mathcal{P}_X$ given by $\alpha_k = \sum_{i=1}^n a_{k,i} \delta_{x_i}$. By the definition of weak convergence, $\{\alpha_k\}$ converges to $\alpha = \sum_{i=1}^n a_i \delta_{x_i}$ in $\mathcal{P}_X$ if and only if $\int_X \phi(x) \alpha_k(dx)$ converges to $\int_X \phi(x) \alpha(dx)$ for all bounded and continuous functions $\phi : X \to \mathbb{R}$. These integrals are equal to $\sum_{i=1}^n a_{k,i} \phi(x_i)$ and $\sum_{i=1}^n a_i \phi(x_i)$, respectively, so $\{\alpha_k\}$

converges to α if and only if $\lim_{k\to\infty} a_{k,i} = a_i$ for each i. Therefore $\{\alpha_k\}$ converges to α in $\mathcal{P}_X$ if and only if $\{f(\alpha_k)\}$ converges to $f(\alpha)$ in Δ_{n-1}. □

In general, the $p = \infty$ metric thickenings $\mathrm{VR}_\infty(X; r)$ and $\check{\mathrm{C}}_\infty(X; r)$ of a finite metric space X are in bijection, as sets, with the geometric realizations of the usual Vietoris–Rips and Čech simplicial complexes on X, via the natural bijection $\sum_{i=1}^n a_i \delta_{x_i} \mapsto \sum_{i=1}^n a_i x_i$, where $\sum_{i=1}^n a_i x_i$ denotes the formal sum in the geometric realization. Therefore Lemma 4.11 implies the following; see also [2, Proposition 6.6].

Lemma 4.12 *For a finite metric space X and any $r > 0$, we have homeomorphisms $\mathrm{VR}_\infty(X; r) \cong \mathrm{VR}(X; r)$ and $\check{\mathrm{C}}_\infty(X; r) \cong \check{\mathrm{C}}(X; r)$.*

5 Stability

In this section we establish the stability of all the filtrations we have introduced, under reasonable assumptions on the metric spaces. The proof requires new techniques in order to construct maps that adequately compare two filtrations of metric thickenings. In the proof of stability for simplicial complexes [30], simplicial maps are used to construct maps between filtrations of simplicial complexes, and the shift in parameter can be bounded using the Gromov–Hausdorff distance. A naive attempt to apply this technique to filtrations of metric thickenings fails because there is no analogue of simplicial maps for metric thickenings. Indeed, if X and Y are metric spaces that are close in the Gromov–Hausdorff distance, then we get a map $f : X \to Y$ that need not be continuous, but which is of bounded distortion. This induces a *continuous* map $\mathrm{VR}(X; r) \to \mathrm{VR}(Y; r')$ between simplicial complexes, as long as r' is chosen to be large enough, and from there one can obtain interleavings. However, the analogous map $\mathrm{VR}_p(X; r) \to \mathrm{VR}_p(Y; r')$ between metric thickenings cannot be continuous if $f : X \to Y$ is not continuous, since there are natural isometric embeddings of X and Y into $\mathrm{VR}_p(X; r)$ and $\mathrm{VR}_p(Y; r')$; see Proposition 3.8. In essence, the fact that metric thickenings have a more well-behaved topology means that it is more difficult to construct interleavings between them.

To overcome this difficulty, we instead construct continuous functions between metric thickenings by distributing mass according to finite partitions of unity, subordinate to open coverings by δ–balls. This ensures that these maps distort distances in a controlled way, allowing for a controlled change in the invariants defining the filtration and thus providing an interleaving. Creating these maps and checking their properties will require many of the ideas from the previous sections.

In Section 5.1 we state our main results: Theorem A and its immediate consequence, Theorem B. We give two applications of stability. First, Section 5.2 applies the stability theorem in order to show that various persistence modules of interest are Q–tame, and therefore have persistence diagrams. Next, in Section 5.3 we apply stability to show the close relationship between ∞–metric thickenings and simplicial complexes, generalizing and answering in the affirmative [2, Conjecture 6.12], which states that these filtrations have identical persistence diagrams. We give the proof of the stability theorem, Theorem A, in Section 5.4.

5.1 Statement of the stability theorem

Let X and Y be totally bounded metric spaces, let $p \in [1, \infty]$ and let $k \geq 0$. We will see, for example, that we have the following stability bound:

$$(4) \qquad d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(X; \cdot), H_k \circ \mathrm{VR}_p(Y; \cdot)) \leq 2 d_{\mathrm{GH}}(X, Y).$$

This implies that if two metric spaces are close in the Gromov–Hausdorff distance, then their resulting filtrations and persistence modules are also close.

Example 5.1 Consider the metric space Z_{n+1} from Example 3.11, which has $n+1$ points all at interpoint distance 1 apart. Let $\zeta_{p,k}$ be the interleaving distance between persistent homology modules

$$\zeta_{p,k} := d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(Z_{n+1}; \cdot), H_k \circ \mathrm{VR}_p(Z_{m+1}; \cdot)) \quad \text{for } n \neq m.$$

Notice that, from Example 3.11, when $n \neq m$ and p is finite,

$$\zeta_{p,0} = \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{1/p} \quad \text{and} \quad \zeta_{p,1} = \tfrac{1}{2}\left(\left(\tfrac{2}{3}\right)^{1/p} - \left(\tfrac{1}{2}\right)^{1/p}\right).$$

However, when $p = \infty$, we have $\zeta_{\infty,0} = \tfrac{1}{2}$ and $\zeta_{\infty,k} = 0$ for all $k \geq 1$.

From these calculations we can make the following observations:

(i)   For p infinite, the only value of k for which $\zeta_{\infty,k} \neq 0$ is $k = 0$.

(ii) For p finite, $\zeta_{p,1} > 0$.

(iii) For p finite, $\sup_k \zeta_{\infty,k} = \tfrac{1}{2} > \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{1/p} = \sup_k \zeta_{p,k}$.

From items (i) and (ii) we can see that, whereas persistence diagrams for $k \geq 1$ of $p = \infty$ thickenings do not contain discriminative information for the Z_n spaces, in contrast, the analogous quantities for p finite do absorb useful information.

Since it is known (see [56, Example 4.1]) that $d_{\mathrm{GH}}(Z_{n+1}, Z_{m+1}) = \tfrac{1}{2}$ whenever $n \neq m$, item (iii) suggests that the lower bound for Gromov–Hausdorff given by (4) may not be tight for p finite. This phenomenon can actually be explained by the more general theorem below, which identifies a certain pseudometric between filtrations which mediates between the two terms appearing in (4).

Theorem A *Let $\mathfrak{i}$ be an L–controlled invariant. Then, for any two totally bounded metric spaces X and Y and any integer $k \geq 0$,*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}_Y, \mathfrak{i}^Y; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_Y, \mathfrak{i}^Y)) \leq 2L\, d_{\mathrm{GH}}(X, Y),$$

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}^Y; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X), (\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}^Y)) \leq 2L\, d_{\mathrm{GH}}(X, Y).$$

Note that there are instances when the quantity in the middle vanishes, yet the spaces X and Y are nonisometric; see Section A.6 for results about this in terms of the notion of *crushing* considered by Hausmann [42] and [2; 58].

Corollary 5.2 *For any two totally bounded metric spaces X and Y, for any $q, p \in [1, \infty]$, and for any integer $k \geq 0$, we have*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}_{q,p}^X; \cdot], H_k \circ [\mathcal{P}_Y, \mathfrak{i}_{q,p}^Y; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}_{q,p}^X), (\mathcal{P}_Y, \mathfrak{i}_{q,p}^Y)) \leq 2d_{\mathrm{GH}}(X, Y),$$

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}_{q,p}^X; \cdot], H_k \circ [\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}_{q,p}^Y; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}_{q,p}^X), (\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}_{q,p}^Y)) \leq 2d_{\mathrm{GH}}(X, Y).$$

Proof From Lemmas 4.5 and 4.6 we know $\mathfrak{i}_{q,p}$ is a 1–controlled invariant; we then apply Theorem A. $\square$

Theorem B *Let X and Y be totally bounded metric spaces, let $p \in [1, \infty]$ and let $k \geq 0$ be an integer. Then the kth persistent homology of $\mathrm{VR}_p(X; \cdot)$ and $\mathrm{VR}_p(Y; \cdot)$ are ε–interleaved for any $\varepsilon \geq d_{\mathrm{GH}}(X, Y)$, and similarly for $\check{\mathrm{C}}_p(X; \cdot)$ and $\check{\mathrm{C}}_p(Y; \cdot)$:*

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(X; \cdot), H_k \circ \mathrm{VR}_p(Y; \cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{diam}_p^X), (\mathcal{P}_Y, \mathrm{diam}_p^Y)) \leq 2d_{\mathrm{GH}}(X, Y),$$

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(Y; \cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathrm{diam}_p^X), (\mathcal{P}_Y^{\mathrm{fin}}, \mathrm{diam}_p^Y)) \leq 2d_{\mathrm{GH}}(X, Y),$$

$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_p(X; \cdot), H_k \circ \check{\mathrm{C}}_p(Y; \cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{rad}_p^X), (\mathcal{P}_Y, \mathrm{rad}_p^Y)) \leq 2d_{\mathrm{GH}}(X, Y),$$

$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot), H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(Y; \cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathrm{rad}_p^X), (\mathcal{P}_Y^{\mathrm{fin}}, \mathrm{rad}_p^Y)) \leq 2d_{\mathrm{GH}}(X, Y).$$

Proof The p–Vietoris–Rips case follows from Corollary 5.2 by letting $q = p$, in which case $\mathfrak{i}_{p,p}^X = \mathrm{diam}_p^X$ (see Definition 4.3). The p–Čech case follows from Theorem A since rad_p is a 1–controlled invariant by Lemmas 4.8 and 4.9. $\square$

In the corollary below, we use Theorem A to show that a controlled invariant on all Radon probability measures produces a filtration at interleaving distance zero from that same invariant restricted to only finitely supported measures.

Corollary 5.3 *Let X be a totally bounded metric space, let $k \geq 0$ be an integer and let $\mathfrak{i}$ be an L–controlled invariant. Then*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]) = d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X)) = 0.$$

Proof For any $\varepsilon > 0$, let U be an ε–net in X. Then the stability theorem, Theorem A, shows

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_U, \mathfrak{i}^U)) \leq 2L\, d_{\mathrm{GH}}(X, U) \leq 2L\varepsilon, \quad d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X), (\mathcal{P}_U^{\mathrm{fin}}, \mathfrak{i}^U)) \leq 2L\, d_{\mathrm{GH}}(X, U) \leq 2L\varepsilon.$$

Note, $(\mathcal{P}_U, \mathfrak{i}^U) = (\mathcal{P}_U^{\mathrm{fin}}, \mathfrak{i}^U)$ since U is finite. Since d_{HT} satisfies the triangle inequality (Proposition 2.8),

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X)) \leq 4L\varepsilon.$$

By letting ε go to zero, we see $d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X)) = 0$. It then follows from Lemma 2.9 that $d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]) = 0$. $\square$

5.2 Consequence of stability: tameness

We next determine conditions under which a persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ associated with a controlled invariant $\mathfrak{i}$ will be Q–tame. In particular, we apply the results to the Vietoris–Rips and Čech metric thickenings.

Corollary 5.4 *Let $\mathfrak{i}$ be a controlled invariant and let $k \geq 0$ be an integer. Suppose the persistence module $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ (resp. $H_k \circ [\mathcal{P}_V^{\mathrm{fin}}, \mathfrak{i}^V; \cdot]$) is Q–tame for any finite metric space V. Then, for any totally bounded metric space X, the persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ (resp. $H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$) is Q–tame.*

Proof Since X is totally bounded, Theorem A implies that the persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ can be approximated arbitrarily well in the interleaving distance by the Q–tame persistence modules on finite ε–nets V_ε as ε goes to zero. Then the result follows from Lemma 2.5. □

Now, we give a sufficient condition for the Q–tameness of the persistence module $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ over a finite metric space V.

Corollary 5.5 *Suppose $\mathfrak{i}$ is a controlled invariant such that, for any finite metric space V, $\mathfrak{i}^V$ is a continuous function on $\mathcal{P}_V$. Then, for any totally bounded metric space X, the persistence modules $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ and $H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ are Q–tame for any integer $k \geq 0$.*

Proof According to Corollary 5.4, it suffices to check Q–tameness over finite metric spaces. If V is a finite metric space, we may identify $\mathcal{P}_V$ with a simplex by Lemma 4.11. Then $[\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ is a sublevel set filtration on a simplex, and since $\mathfrak{i}^V$ is continuous, [29, Theorem 2.22] shows $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ is Q–tame. □

Here we summarize the Q–tameness results related to relaxed Vietoris–Rips and Čech metric thickenings.

Corollary 5.6 *Let X be a totally bounded metric space, let $p \in [1, \infty]$ and let $k \geq 0$ be an integer. The persistence modules $H_k \circ \mathrm{VR}_p(X; \cdot)$, $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot)$ are Q–tame.*

Proof For $H_k \circ \mathrm{VR}_p(X; \cdot)$, when p is finite, Lemma 4.6 implies diam_p is a continuous function over any $\mathcal{P}_X$ where X is a bounded metric space. Then we get the result by applying Corollary 5.5. When $p = \infty$, Lemma 4.12 shows the persistent homology associated with diam_∞ is Q–tame over finite metric spaces. We then get the result by applying Corollary 5.4.

We obtain the case of $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$ from $H_k \circ \mathrm{VR}_p(X; \cdot)$ by using the interleaving distance result in Corollary 5.3 along with the Q–tame approximation result in Lemma 2.5.

For $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$, when p is finite, Lemma 4.9 implies rad_p is a continuous function over any $\mathcal{P}_X$ where X is a bounded metric space. The rest is similar to the Vietoris–Rips case. □

Remark 5.7 The proof that $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ is Q–tame can be made more direct at one step. For U finite, to see that $H_k \circ \check{\mathrm{C}}_p(U; \cdot)$ is Q–tame, one could appeal to Theorem F in Section A.3 instead of Theorem 2.22 of [29].

The Q–tame persistence modules given by this theorem allow us to discuss persistence diagrams, using the results of [29].

Corollary 5.8 *Let X be a totally bounded metric space, let $p \in [1, \infty]$ and let $k \geq 0$ be an integer. Then $H_k \circ \mathrm{VR}_p(X; \cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$ have the same persistence diagram, $\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X)$. Similarly, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot)$ have the same persistence diagram, $\mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(X)$.*

Proof Persistence diagrams are well defined for Q–tame persistence modules, so for any totally bounded metric space X, any $p \in [1, \infty]$ and any $k \geq 0$, by Corollary 5.6 we have persistence diagrams associated to $H_k \circ \mathrm{VR}_p(X; \cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$. From Corollary 5.3 we know that the interleaving distance between $H_k \circ \mathrm{VR}_p(X; \cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$ is zero, and so the isometry theorem [29, Theorem 4.11] implies that these persistence modules have the same (undecorated) persistence diagram, denoted by $\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X)$. The same proof also works for Čech metric thickenings.   □

Combining the isometry theorem [29, Theorem 4.11] with Theorem B and Corollary 5.6, we obtain the following:

Corollary 5.9 *If X and Y are totally bounded metric spaces, then for any $p \in [1, \infty]$ and integer $k \geq 0$,*

$$d_{\mathrm{B}}(\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X), \mathrm{dgm}_{k,p}^{\mathrm{VR}}(Y)) \leq 2d_{\mathrm{GH}}(X, Y) \quad and \quad d_{\mathrm{B}}(\mathrm{dgm}_{k,p}^{\mathrm{Č}}(X), \mathrm{dgm}_{k,p}^{\mathrm{Č}}(Y)) \leq 2d_{\mathrm{GH}}(X, Y).$$

5.3 Consequence of stability: connecting ∞–metric thickenings and simplicial complexes

We show how the ∞–Vietoris–Rips and ∞–Čech metric thickenings recover the persistent homology of the Vietoris–Rips and Čech simplicial complexes. Our Corollary 5.10 answers [2, Conjecture 6.12] in the affirmative.

We recall that $\mathrm{VR}_p(X; r)$ denotes the p–Vietoris–Rips metric thickening, that $\mathrm{VR}_p^{\mathrm{fin}}(X; r)$ denotes the p–Vietoris–Rips metric thickening for measures of finite support, and that $\mathrm{VR}(X; r)$ denotes the Vietoris–Rips simplicial complex. Theorem B shows, for any $p \in [1, \infty]$, any $\delta > 0$ and any finite δ–net U_δ of a totally bounded metric space X, that

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(U_\delta; \cdot)) \leq 2\delta.$$

For $p = \infty$, by Lemma 4.12, $H_k \circ \mathrm{VR}_\infty(U_\delta; \cdot) \cong H_k \circ \mathrm{VR}(U_\delta, \cdot)$, so from the above we have

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}(U_\delta, \cdot)) \leq 2\delta.$$

Now, by the triangle inequality for the interleaving distance, by the inequality above and by the Gromov–Hausdorff stability of $X \mapsto H_k \circ \mathrm{VR}(X, \cdot)$ [30; 27],

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}(X, \cdot))$$
$$\leq d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}(U_\delta, \cdot)) + d_{\mathrm{I}}(H_k \circ \mathrm{VR}(U_\delta, \cdot), H_k \circ \mathrm{VR}(X, \cdot))$$
$$\leq 4\delta.$$

Since this holds for any $\delta > 0$, we find $d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}(X, \cdot)) = 0$. This implies that the bottleneck distance between persistence diagrams is 0, and the (undecorated) diagrams are in fact equal (see for instance [29, Theorem 4.20]). We can apply Corollary 5.3 to get the same result for $\mathrm{VR}_\infty(X; \cdot)$ (measures with infinite support), and the same proof works in the Čech case. We state this as the following theorem:

Corollary 5.10 *For any totally bounded metric space X and any integer $k \geq 0$,*

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \mathrm{VR}(X, \cdot)) = 0, \quad d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty(X; \cdot), H_k \circ \mathrm{VR}(X, \cdot)) = 0,$$

$$\mathrm{dgm}_{k,\infty}^{\mathrm{VR}}(X) = \mathrm{dgm}_k^{\mathrm{VR}}(X),$$

$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_\infty^{\mathrm{fin}}(X; \cdot), H_k \circ \check{\mathrm{C}}(X; \cdot)) = 0, \quad d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_\infty(X; \cdot), H_k \circ \check{\mathrm{C}}(X; \cdot)) = 0,$$

$$\mathrm{dgm}_{k,\infty}^{\check{\mathrm{C}}}(X) = \mathrm{dgm}_k^{\check{\mathrm{C}}}(X).$$

Remark 5.11 Whereas the ∞–metric thickenings $\mathrm{VR}_\infty(X; \cdot)$ and the simplicial complexes $\mathrm{VR}(X, \cdot)$ yield persistence modules with an interleaving distance of 0, we are in the interesting landscape where the metric thickenings $\mathrm{VR}_p(X; \cdot)$ may yield something new and different for $p < \infty$. For example, in Section 7 we explore the new persistence modules that arise for p–metric thickenings of Euclidean spheres $(\mathbb{S}^n, \ell_2)$ in the case $p = 2$.

Example 5.12 Let $X = \mathbb{S}^1$ be the circle. The persistent homology diagrams of the simplicial complex filtrations $\mathrm{VR}(\mathbb{S}^1, \cdot)$ and $\check{\mathrm{C}}(\mathbb{S}^1; \cdot)$ are known from [1], and therefore Corollary 5.10 gives the persistent homology diagrams of $\mathrm{VR}_\infty(\mathbb{S}^1; \cdot)$ and $\check{\mathrm{C}}_\infty(\mathbb{S}^1; \cdot)$. However, although $\mathrm{VR}(\mathbb{S}^1; r)$ and $\check{\mathrm{C}}(\mathbb{S}^1; r)$ are known to obtain the homotopy types of all odd-dimensional spheres as r increases, the homotopy types of $\mathrm{VR}_\infty(\mathbb{S}^1; r)$ and $\check{\mathrm{C}}_\infty(\mathbb{S}^1; r)$ are still not proven.

Question 5.13 Is the simplicial complex $\mathrm{VR}(X; r)$ always homotopy equivalent to $\mathrm{VR}_\infty^{\mathrm{fin}}(X; r)$ or $\mathrm{VR}_\infty(X; r)$? Compare with [2, Remark 3.3].

Question 5.14 Theorem 5.2 of [52] states that, for X compact, bars in the persistent homology of the simplicial complex filtration $\mathrm{VR}(X, \cdot)$ are of the form $(a, b]$ or (a, ∞). Is the same true for $\mathrm{VR}_\infty(X; \cdot)$? Note that we are using the $<$ convention, ie a simplex is included in $\mathrm{VR}(X; r)$ when its diameter is strictly less than r, and a measure is included in $\mathrm{VR}_\infty(X; r)$ when the diameter of its support is strictly less than r.

5.4 The proof of stability

As mentioned above, the construction of interleaving maps between filtrations of metric thickenings is more intricate than in the case of simplicial complexes. A main idea behind the proof of our stability result, Theorem A, is to approximate the metric space X by a finite subspace (a net) U. The advantage is that it is easier to construct a continuous map with domain $\mathcal{P}_U$ than one with domain $\mathcal{P}_X$. The crucial next step is the construction of a continuous map from $\mathcal{P}_X$ to $\mathcal{P}_U$ by using a partition of unity subordinate to an open covering by δ–balls. This map distorts distances by a controlled amount, allowing us to approximate measures of $\mathcal{P}_X$ in $\mathcal{P}_U$.

We begin with some necessary lemmas.

Lemma 5.15 *Let U be a finite δ–net of a bounded metric space X, and let $\{\zeta_u^U\}_{u \in U}$ be a continuous partition of unity subordinate to the open covering $\bigcup_{u \in U} B(u; \delta)$ of X.*

We have the continuous map $\Phi_U \colon \mathcal{P}_X \to \mathcal{P}_U$ defined by

$$\alpha \mapsto \sum_{u \in U} \int_X \zeta_u^U(x)\, \alpha(dx)\, \delta_u.$$

For any $q \in [1, \infty]$, we have $d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) < \delta$.

Proof The map Φ_U is well-defined, as

$$\sum_{u \in U} \int_X \zeta_u^U(x)\, \alpha(dx) = \int_X \sum_{u \in U} \zeta_u^U(x)\, \alpha(dx) = \int_X \alpha(dx) = 1.$$

For the continuity, since the weak topology on $\mathcal{P}_X$ and $\mathcal{P}_U$ can be metrized, it suffices to show, for a weakly convergent sequence $\alpha_n \in \mathcal{P}_X$, that the image $\Phi_U(\alpha_n)$ is also weakly convergent. As U is a finite metric space, it suffices to show that, for any fixed $u_0 \in U$, the sequence of real numbers $(\Phi_U(\alpha_n)(\{u_0\}))_n$ is itself convergent. Note that

$$\Phi_U(\alpha_n)(\{u_0\}) = \int_X \zeta_{u_0}^U(x)\, \alpha_n(dx).$$

Since $\zeta_{u_0}^U$ is a bounded continuous function on X, we obtain the desired convergence through the weak convergence of α_n.

Lastly, we must show that $d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) < \delta$. For any $u \in U$, we use the notation $w_u^\alpha := \int_X \zeta_u^U(x)\, \alpha(dx)$, so that $\Phi_U(\alpha) = \sum_{u \in U} w_u^\alpha \delta_u$ and

$$\sum_{u \in U} w_u^\alpha = \sum_{u \in U} \int_X \zeta_u^U(x)\, \alpha(dx) = \int_X \sum_u \zeta_u^U(x)\, \alpha(dx) = \int_X \alpha(dx) = 1.$$

Let $\zeta_u^U \alpha$ denote the measure such that $\zeta_u^U \alpha(B) = \int_B \zeta_u^U(x)\, \alpha(dx)$ for any measurable set $B \subseteq X$. We have

$$\alpha = \sum_{u \in U} \zeta_u^U \alpha = \sum_{\substack{u \in U, \\ w_u^\alpha \neq 0}} w_u^\alpha \Big(\frac{1}{w_u^\alpha} \zeta_u^U \alpha \Big).$$

From this we get

$$d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) = d_{\mathrm{W},q}^X \Big(\sum_{\substack{u \in U, \\ w_u^\alpha \neq 0}} w_u^\alpha \Big(\frac{1}{w_u^\alpha} \zeta_u^U \alpha \Big), \sum_{\substack{u \in U, \\ w_u^\alpha \neq 0}} w_u^\alpha \delta_u \Big) \leq \Big(\sum_{\substack{u \in U, \\ w_u^\alpha \neq 0}} w_u^\alpha \Big(d_{\mathrm{W},q}^X \Big(\frac{1}{w_u^\alpha} \zeta_u^U \alpha, \delta_u \Big) \Big)^q \Big)^{\frac{1}{q}} < \delta.$$

The first inequality is by [Lemma 4.2](#). The last one comes from the fact that each $(1/w_u^\alpha)\zeta_u^U \alpha$ is a probability measure supported in $B(u; \delta)$, and thus $d_{\mathrm{W},q}^X((1/w_u^\alpha)\zeta_u^U \alpha, \delta_u) < \delta$. $\square$

For any two totally bounded metric spaces X and Y, through partitions of unity we can build continuous maps between $\mathcal{P}_X$ and $\mathcal{P}_Y$, as follows. Let X and Y be two totally bounded metric spaces, let $\eta > 2d_{\mathrm{GH}}(X, Y)$ and let $\delta > 0$. We fix finite δ–nets $U \subseteq X$ of X and $V \subseteq Y$ of Y. By the triangle inequality, $d_{\mathrm{GH}}(U, V) < \frac{1}{2}\eta + 2\delta$. So there exist maps $\varphi \colon U \to V$ and $\psi \colon V \to U$ with

$$\max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi, \psi)) \leq \eta + 4\delta.$$

We then define the maps $\hat{\Phi} \colon \mathcal{P}_X \to \mathcal{P}_Y$ and $\hat{\Psi} \colon \mathcal{P}_Y \to \mathcal{P}_X$ to be

$$\hat{\Phi} := \iota_V \circ \varphi_\sharp \circ \Phi_U \quad \text{and} \quad \hat{\Psi} := \iota_U \circ \psi_\sharp \circ \Phi_V,$$

where $\iota_U \colon \mathcal{P}_U \hookrightarrow \mathcal{P}_X$ and $\iota_V \colon \mathcal{P}_V \hookrightarrow \mathcal{P}_Y$ are inclusions and Φ_U and Φ_V are the maps defined in Lemma 5.15:

$$
\begin{array}{ccc}
\mathcal{P}_X & \xdashrightarrow{\hat{\Phi}} & \mathcal{P}_Y \\
\Phi_U \downarrow & & \uparrow \iota_V \\
\mathcal{P}_U & \xrightarrow{\varphi_\sharp} & \mathcal{P}_V
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}_Y & \xdashrightarrow{\hat{\Psi}} & \mathcal{P}_X \\
\Phi_V \downarrow & & \uparrow \iota_U \\
\mathcal{P}_V & \xrightarrow{\psi_\sharp} & \mathcal{P}_U
\end{array}
$$

Then both $\hat{\Phi}$ and $\hat{\Psi}$ are continuous maps by Lemma 5.15 and by the continuity of pushforwards of continuous maps.

The use of the finite δ–nets U and V is important, because they can be compared by continuous maps φ and ψ, which yield continuous maps $\varphi_\sharp$ and $\psi_\sharp$. The following lemma shows that $\hat{\Phi}$ and $\hat{\Psi}$ are homotopy equivalences.

Lemma 5.16 *With the above notation, we have homotopy equivalences $\hat{\Psi} \circ \hat{\Phi} \simeq \mathrm{id}_{\mathcal{P}_X}$ and $\hat{\Phi} \circ \hat{\Psi} \simeq \mathrm{id}_{\mathcal{P}_Y}$ via the linear families $H_t^X \colon \mathcal{P}_X \times [0,1] \to \mathcal{P}_X$ and $H_t^Y \colon \mathcal{P}_Y \times [0,1] \to \mathcal{P}_Y$ given by*

$$H_t^X(\alpha) := (1-t)\hat{\Psi} \circ \hat{\Phi}(\alpha) + t\alpha \quad \text{and} \quad H_t^Y(\beta) := (1-t)\hat{\Phi} \circ \hat{\Psi}(\beta) + t\beta.$$

Moreover, $d_{\mathrm{W},q}^X(H_t^X(\alpha), \alpha) < \eta + 6\delta$ and $d_{\mathrm{W},q}^Y(H_t^Y(\beta), \beta) < \eta + 6\delta$ for any $q \in [1, \infty]$ and any $t \in [0,1]$.

Proof The homotopies H_t^X and H_t^Y are continuous by Proposition 2.4 since $\hat{\Phi}$ and $\hat{\Psi}$ are continuous. We will only present the estimate $d_{\mathrm{W},q}^X(H_t^X(\alpha), \alpha) < \eta + 6\delta$, as the other inequality can be proved in a similar way. We first calculate the expression for $\hat{\Psi} \circ \hat{\Phi}(\alpha)$, obtaining

$$
\begin{aligned}
\hat{\Psi} \circ \hat{\Phi}(\alpha) &= \psi_\sharp(\Phi_V(\varphi_\sharp(\Phi_U(\alpha)))) = \psi_\sharp\Big(\Phi_V\Big(\varphi_\sharp\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_u\Big)\Big)\Big) \\
&= \psi_\sharp\Big(\Phi_V\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_{\varphi(u)}\Big)\Big) \\
&= \psi_\sharp\Big(\sum_{v \in V} \int_Y \zeta_v^V(y)\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_{\varphi(u)}\Big)(dy)\,\delta_v\Big) \\
&= \psi_\sharp\Big(\sum_{v \in V}\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\int_Y \zeta_v^V(y)\,\delta_{\varphi(u)}(dy)\Big)\delta_v\Big) \\
&= \psi_\sharp\Big(\sum_{v \in V}\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\zeta_v^V(\varphi(u))\Big)\delta_v\Big) \\
&= \sum_{v \in V}\Big(\sum_{u \in U} \int_X \zeta_u^U(x)\,\alpha(dx)\zeta_v^V(\varphi(u))\Big)\delta_{\psi(v)} = \sum_{v \in V}\Big(\sum_{u \in U} w_u^\alpha \zeta_v^V(\varphi(u))\Big)\delta_{\psi(v)},
\end{aligned}
$$

where we again use the notation $w_u^\alpha := \int_X \zeta_u^U(x)\,\alpha(dx)$. We have

$$H_t^X(\alpha) = (1-t)\sum_{v \in V}\sum_{u \in U}\big(w_u^\alpha \zeta_v^V(\varphi(u))\big)\delta_{\psi(v)} + t\alpha.$$

Now we show that $H_t^X(\alpha)$ is a convex combination of probability measures in $\mathcal{P}_X$ and hence itself lies in $\mathcal{P}_X$. That is, the sum of coefficients in front of all $\delta_{\psi(v)}$ and α in the above formula for H_t^X is 1:

$$(1-t)\sum_{v\in V}\sum_{u\in U}\left(w_u^\alpha \zeta_v^V(\varphi(u))\right)+t=(1-t)\sum_{u\in U}w_u^\alpha+t=1.$$

Next, we will bound the q–Wasserstein distance $d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)$. For this, we first rewrite α as

$$\alpha=(1-t)\sum_{u\in U}\zeta_u^U\alpha+t\alpha=(1-t)\sum_{v\in V}\sum_{u\in U}\zeta_v^V(\varphi(u))(\zeta_u^U\alpha)+t\alpha$$

$$=(1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\w_u^\alpha\neq0}}\left(w_u^\alpha\zeta_v^V(\varphi(u))\right)\left(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\right)+t\alpha.$$

Based on these observations, we then apply the inequality from [Lemma 4.2](#) to obtain

$$d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)$$

$$=d_{\mathrm{W},q}^X\left((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\w_u^\alpha\neq0}}\left(w_u^\alpha\zeta_v^V(\varphi(u))\right)\delta_{\psi(v)}+t\alpha,\ (1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\w_u^\alpha\neq0}}\left(w_u^\alpha\zeta_v^V(\varphi(u))\right)\left(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\right)+t\alpha\right)$$

$$\leq\left((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\w_u^\alpha\neq0}}\left(w_u^\alpha\zeta_v^V(\varphi(u))\right)\left(d_{\mathrm{W},q}^X\left(\delta_{\psi(v)},\frac{1}{w_u^\alpha}\zeta_u^U\alpha\right)\right)^p\right)^{\frac{1}{p}}$$

$$\leq\left((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\w_u^\alpha\neq0}}\left(w_u^\alpha\zeta_v^V(\varphi(u))\right)\left(d_{\mathrm{W},q}^X(\delta_{\psi(v)},\delta_u)+d_{\mathrm{W},q}^X\left(\delta_u,\frac{1}{w_u^\alpha}\zeta_u^U\alpha\right)\right)^p\right)^{\frac{1}{p}}.$$

For each nonzero term in the summand of the last expression we have $d_Y(v,\varphi(u))<\delta$, which implies $d_X(\psi(v),u)<5\delta+\eta$ via the codistortion assumption. Therefore $d_{\mathrm{W},q}^X(\delta_{\psi(v)},\delta_u)<5\delta+\eta$ as well. Moreover, since $(1/w_u^\alpha)\zeta_u^U\alpha$ is a probability measure supported in the δ–ball centered at the point u, we have $d_{\mathrm{W},q}^X(\delta_u,(1/w_u^\alpha)\zeta_u^U\alpha)<\delta$. Therefore $d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)<6\delta+\eta$. $\qquad\square$

Remark 5.17 The above lemma still holds if we replace $\mathcal{P}_X$ by $\mathcal{P}_X^{\mathrm{fin}}$ and $\mathcal{P}_Y$ by $\mathcal{P}_Y^{\mathrm{fin}}$, as all related maps naturally restrict to the set of finitely supported measures, $\mathcal{P}_X^{\mathrm{fin}}$.

We are now ready to prove our main result, [Theorem A](#).

Proof of [Theorem A] For $\mathfrak{i}$ an L–controlled invariant, for totally bounded metric spaces X and Y, and for any integer $k\geq0$, we must show

$$d_{\mathrm{I}}(H_k\circ[\mathcal{P}_X,\mathfrak{i}^X;\cdot],H_k\circ[\mathcal{P}_Y,\mathfrak{i}^Y;\cdot])\leq d_{\mathrm{HT}}((\mathcal{P}_X,\mathfrak{i}^X),(\mathcal{P}_Y,\mathfrak{i}^Y))\leq 2L\,d_{\mathrm{GH}}(X,Y),$$

$$d_{\mathrm{I}}(H_k\circ[\mathcal{P}_X^{\mathrm{fin}},\mathfrak{i}^X;\cdot],H_k\circ[\mathcal{P}_Y^{\mathrm{fin}},\mathfrak{i}^Y;\cdot])\leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}},\mathfrak{i}^X),(\mathcal{P}_Y^{\mathrm{fin}},\mathfrak{i}^Y))\leq 2L\,d_{\mathrm{GH}}(X,Y).$$

We prove only the first line above, as the finitely supported case in the second line has a nearly identical proof. The inequality involving d_{I} and d_{HT} follows from [Lemma 2.9](#). Hence it suffices to prove the inequality $d_{\mathrm{HT}}((\mathcal{P}_X,\mathfrak{i}^X),(\mathcal{P}_Y,\mathfrak{i}^Y))\leq 2L\,d_{\mathrm{GH}}(X,Y)$ involving d_{HT} and d_{GH}.

Following the construction in Lemma 5.16, let $\eta > 2\,d_{\mathrm{GH}}(X, Y)$ and $\delta > 0$. We fix finite δ–nets $U \subset X$ of X and $V \subset Y$ of Y. By the triangle inequality, $d_{\mathrm{GH}}(U, V) < \frac{1}{2}\eta + 2\delta$, and so there exist maps $\varphi \colon U \to V$ and $\psi \colon V \to U$ with

$$\max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi, \psi)) \le \eta + 4\delta.$$

For any $\delta > 0$, we will show that $(\mathcal{P}_X, \mathfrak{i}^X)$ and $(\mathcal{P}_Y, \mathfrak{i}^Y)$ are $(\eta + 6\delta)L$–homotopy equivalent. By the definition of a δ–homotopy (Definition 2.6), it suffices to show that

- $\hat{\Phi}$ is an $(\eta + 6\delta)L$–map from $(\mathcal{P}_X, \mathfrak{i}^X)$ to $(\mathcal{P}_Y, \mathfrak{i}^Y)$,
- $\hat{\Psi}$ is an $(\eta + 6\delta)L$–map from $(\mathcal{P}_Y, \mathfrak{i}^Y)$ to $(\mathcal{P}_X, \mathfrak{i}^X)$,
- $\hat{\Psi} \circ \hat{\Phi} \colon \mathcal{P}_X \to \mathcal{P}_X$ is $(2\eta + 12\delta)L$–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathfrak{i}^X, \mathfrak{i}^X)$,
- $\hat{\Phi} \circ \hat{\Psi} \colon \mathcal{P}_X \to \mathcal{P}_X$ is $(2\eta + 12\delta)L$–homotopic to $\mathrm{id}_{\mathcal{P}_Y}$ with respect to $(\mathfrak{i}^Y, \mathfrak{i}^Y)$.

We will only present the proof for the first and third items; the other two can be proved similarly. We have

$$\mathfrak{i}^Y(\hat{\Phi}(\alpha)) = \mathfrak{i}^Y(\varphi_\sharp \circ \Phi_U(\alpha)) \le \mathfrak{i}^X(\Phi_U(\alpha)) + (\eta + 4\delta)L \le \mathfrak{i}^X(\alpha) + (\eta + 6\delta)L,$$

where the first inequality is from the stability of the invariant $\mathfrak{i}$ under pushforward and from the bound on the distortion of φ, and where the second inequality is from the stability of the invariant with respect to Wasserstein distance and from the bound $d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) < \delta$ in Lemma 5.15. This proves the first item.

For the third item, we use the homotopy H_t^X in Lemma 5.16. Then it suffices to show, for any fixed $t \in [0, 1]$, that the map H_t^X is a $(2\eta + 12\delta)L$–map from $(\mathcal{P}_X, \mathfrak{i}^X)$ to $(\mathcal{P}_X, \mathfrak{i}^X)$, that is,

$$\mathfrak{i}^X(H_t^X(\alpha)) \le \mathfrak{i}^X(\alpha) + (2\eta + 12\delta)L.$$

This comes from the inequality $d_{\mathrm{W},\infty}^X(H_t^X(\alpha), \alpha) < \eta + 6\delta$ from Lemma 5.16, and from the stability assumption of $\mathfrak{i}$ with respect to Wasserstein distance.

Now, since $(\mathcal{P}_X, \mathfrak{i}^X)$ and $(\mathcal{P}_Y, \mathfrak{i}^Y)$ are $(\eta + 6\delta)L$–homotopy equivalent for any $\eta > 2\,d_{\mathrm{GH}}(X, Y)$ and $\delta > 0$, it follows from the definition of the d_{HT} distance (Definition 2.7) that $d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_Y, \mathfrak{i}^Y)) \le 2L\,d_{\mathrm{GH}}(X, Y)$. This completes the proof. $\qquad\square$

6 A Hausmann type theorem for 2–Vietoris–Rips and 2–Čech thickenings of Euclidean submanifolds

Hausmann's theorem [42] states that if X is a Riemannian manifold and if the scale $r > 0$ is sufficiently small (depending on the curvature X), then the Vietoris–Rips simplicial complex $\mathrm{VR}(X; r)$ is homotopy equivalent to X. A short proof using an advanced version of the nerve lemma is given in [73]. Versions of Hausmann's theorem have been proven for ∞–metric thickenings in [2], where X is equipped with the Riemannian metric, and in [6], where X is equipped with a Euclidean metric. In this section we prove a Hausmann type theorem for the 2–Vietoris–Rips and 2–Čech metric thickenings. Our result is closest to that in [6] (now with $p = 2$ instead of $p = \infty$): we work with any Euclidean subset X of positive reach. This includes (for example) any embedded C^k submanifold of $\mathbb{R}^n$ for $k \ge 2$, with or without boundary [70].

We begin with a definition and some lemmas that will be needed in the proof.

Let α be a measure in $\mathcal{P}_1(\mathbb{R}^n)$, the collection of all Radon measures with finite first moment. For any coordinate function x_i, where $1 \le i \le n$, we have

$$\int_{\mathbb{R}^n} |x_i| \, \alpha(dx) \le \int_{\mathbb{R}^n} \|x\| \, \alpha(dx) < \infty.$$

Therefore the *Euclidean mean* map $m \colon \mathcal{P}_1(\mathbb{R}^n) \to \mathbb{R}^n$ given by the following vector-valued integral is well-defined:

$$m(\alpha) := \int_{\mathbb{R}^n} x \, \alpha(dx).$$

Lemma 6.1 *Let X be a metric space and let $f \colon X \to \mathbb{R}^n$ be a bounded continuous function. Then the induced map $m \circ f_\sharp \colon \mathcal{P}_X \to \mathbb{R}^n$ is continuous.*

Proof For a sequence α_n that weakly converges to α, we have the vector-valued integral

$$m \circ f_\sharp(\alpha_n) = \int_{\mathbb{R}^n} x f_\sharp(\alpha_n)(dx) = \int_X f(x) \, \alpha_n(dx).$$

As f is bounded and continuous, the above limit converges to $m \circ f_\sharp(\alpha)$ as α_n converges to α. Therefore the map $m \circ f_\sharp$ is continuous. □

Lemma 6.2 *Let α be a probability measure in $\mathcal{P}_1(\mathbb{R}^n)$. Then, for $p \in [1, \infty]$, there is some $z \in \operatorname{supp}(\alpha)$ with*

$$\|m(\alpha) - z\| \le \operatorname{diam}_p(\alpha).$$

Proof As $\operatorname{diam}_1(\alpha) \le \operatorname{diam}_p(\alpha)$ for all $p \in [1, \infty]$, it suffices to show $\|m(\alpha) - z\| \le \operatorname{diam}_1(\alpha)$. We consider the formula

$$\int_{\mathbb{R}^n} \|m(\alpha) - z\| \, \alpha(dz) = \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} (x - z) \, \alpha(dx) \right\| \alpha(dz) \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|(x - z)\| \, \alpha(dx) \, \alpha(dz) = \operatorname{diam}_1(\alpha).$$

So there must be some $z \in \operatorname{supp}(\alpha)$ with $\|m(\alpha) - z\| \le \operatorname{diam}_1(\alpha)$. □

Lemma 6.3 *For any probability measure $\alpha \in \mathcal{P}_1(\mathbb{R}^n)$ and for any $x \in \mathbb{R}^n$, we can write the associated squared 2–Fréchet function $F_{\alpha,2}^2(x)$ as*

$$F_{\alpha,2}^2(x) = \|x - m(\alpha)\|^2 + F_{\alpha,2}^2(m(\alpha)).$$

Proof By the linearity of the inner product,

$$
\begin{aligned}
F_{\alpha,2}^2(x) &:= \int_{\mathbb{R}^n} \|x - y\|^2 \, \alpha(dy) = \int_{\mathbb{R}^n} (\|x\|^2 - 2\langle x, y \rangle + \|y\|^2) \, \alpha(dy) \\
&= \int_{\mathbb{R}^n} (\|x\|^2 - 2\langle x, y \rangle + \|m(\alpha)\|^2 - \|m(\alpha)\|^2 + \|y\|^2) \, \alpha(dy) \\
&= \|x\|^2 - 2\langle x, m(\alpha) \rangle + \|m(\alpha)\|^2 + \int_{\mathbb{R}^n} (-\|m(\alpha)\|^2 + \|y\|^2) \, \alpha(dy) \\
&= \|x - m(\alpha)\|^2 + \int_{\mathbb{R}^n} (\|m(\alpha)\|^2 - 2\langle m(\alpha), y \rangle + \|y\|^2) \, \alpha(dy) \\
&= \|x - m(\alpha)\|^2 + \int_{\mathbb{R}^n} \|m(\alpha) - y\| \, \alpha(dy) = \|x - m(\alpha)\|^2 + F_{\alpha,2}^2(m(\alpha)).
\end{aligned}
$$ □

Recall the *medial axis* of X is defined as the closure

$$\mathrm{med}(X) = \overline{\{y \in \mathbb{R}^n \mid \text{there exist } x_1 \neq x_2 \in X \text{ with } \|y - x_1\| = \|y - x_2\| = \inf_{x \in X} \|y - x\|\}}.$$

The *reach* τ of X is the closest distance $\tau := \inf_{x \in X, y \in \mathrm{med}(X)} \|x - y\|$ between points in X and $\mathrm{med}(X)$. Let $U_\tau(X)$ be the τ–neighborhood of X in $\mathbb{R}^n$, that is, the set of points $y \in \mathbb{R}^n$ such that there is some $x \in X$ with $\|y - x\| < \tau$. The definition of reach implies that, for any point y in the open neighborhood $U_\tau(X)$, there is a unique closest point $x \in X$. The associated nearest projection map $\pi : U_\tau(X) \to X$ is continuous; see [36, Theorem 4.8(8)] or [6, Lemma 3.7].

Lemma 6.4 *Let X be a bounded subset of $\mathbb{R}^n$ with reach $\tau(X) > 0$. Let $\alpha \in \mathcal{P}_X$ have its Euclidean mean $m(\alpha)$ in the neighborhood $U_\tau(X)$. Then along the linear interpolation family*

$$\alpha_t := (1 - t)\alpha + t\delta_{\phi(\alpha)},$$

where ϕ is the composition $\pi \circ m : \mathcal{P}_X \to X$, both diam_2 and rad_2 obtain their maximum at $t = 0$.

Proof According to Lemma 6.3, for any $x \in X$,

$$F_{\alpha,2}^2(x) = \|x - m(\alpha)\|^2 + F_{\alpha,2}^2(m(\alpha)).$$

As $m(\alpha)$ is inside $U_\tau(X)$, the first term $\|x - m(\alpha)\|$ is minimized over $x \in X$ at $x = \phi(\alpha)$, and hence so is $F_{\alpha,2}(x)$. Therefore $\mathrm{rad}_2(\alpha) = \inf_{x \in X} F_{\alpha,2}(x) = F_{\alpha,2}(\phi(\alpha))$. So for $\mathrm{rad}_2(\alpha_t)$,

$$\begin{aligned}
(\mathrm{rad}_2(\alpha_t))^2 \leq F_{\alpha_t,2}^2(\phi(\alpha)) &= \int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \, \alpha_t(dx) \\
&= (1-t) \int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \, \alpha(dx) + t \int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \delta_{\phi(\alpha)}(dx) \\
&= (1-t) \int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \, \alpha(dx) = (1-t) F_{\alpha,2}^2(\phi(\alpha)) \leq (\mathrm{rad}_2(\alpha))^2.
\end{aligned}$$

We now consider diam_2. Recall that $F_{\alpha,2}(\phi(\alpha)) \leq F_{\alpha,2}(x)$ for all $x \in X$. From this, we have $(\mathrm{diam}_2(\alpha))^2 = \int_X F_{\alpha,2}^2(x) \, \alpha(dx) \geq F_{\alpha,2}^2(\phi(\alpha))$. Therefore

$$\begin{aligned}
(\mathrm{diam}_2(\alpha_t))^2 &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, \alpha_t(dx) \, \alpha_t(dy) \\
&= (1-t)^2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, \alpha(dx) \, \alpha(dy) + 2t(1-t) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, \alpha(dx) \, \delta_{\phi(\alpha)}(dy) \\
&\qquad\qquad + t^2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, \delta_{\phi(\alpha)}(dx) \, \delta_{\phi(\alpha)}(dy) \\
&= (1-t)^2 (\mathrm{diam}_2(\alpha))^2 + 2t(1-t) F_{\alpha,2}^2(\phi(\alpha)) \leq (1-t)^2 (\mathrm{diam}_2(\alpha))^2 + 2t(1-t)(\mathrm{diam}_2(\alpha))^2 \\
&= (1-t^2)(\mathrm{diam}_2(\alpha))^2 \leq (\mathrm{diam}_2(\alpha))^2. \qquad\qquad \square
\end{aligned}$$

Theorem C *Let X be a bounded subset of $\mathbb{R}^n$ with positive reach τ. Then for all $0 < r \leq \tau$, the isometric embeddings $X \hookrightarrow \mathrm{VR}_2(X; r)$, $X \hookrightarrow \mathrm{VR}_2^{\mathrm{fin}}(X; r)$, $X \hookrightarrow \check{\mathrm{C}}_2(X; r)$ and $X \hookrightarrow \check{\mathrm{C}}_2^{\mathrm{fin}}(X; r)$ are homotopy equivalences.*

Proof   We begin with the 2–Vietoris–Rips metric thickening. Let α be a measure in $\mathrm{VR}_2(X;r)$. Lemma 6.2 shows that there exists some $x \in X$ with $\|m(\alpha) - x\| \leq \mathrm{diam}_2(\alpha) < r \leq \tau$. Hence $m(\alpha)$ is in the neighborhood $U_\tau(X)$, and $\phi := \pi \circ m$ is well defined on $\mathrm{VR}_2(X;r)$. As both π and m are continuous, so is their composition ϕ. Let i be the isometric embedding of X into $\mathrm{VR}_2(X;r)$. Clearly we have $\phi \circ i = \mathrm{id}_X$. Then it suffices to show the homotopy equivalence $i \circ \phi \simeq \mathrm{id}_{\mathrm{VR}_2(X;r)}$.

For this, we use the linear family $(\alpha_t)_{t \in [0,1]}$, where $\alpha_t := (1-t)\alpha + t\delta_{\phi(\alpha)}$. According to Lemma 6.4, we know this family is inside $\mathrm{VR}_2(X;r)$, and therefore α_t is well defined. By Proposition 2.4, this is a continuous family as the map ϕ is continuous. Therefore the linear family indeed provides a homotopy $i \circ \phi \simeq \mathrm{id}_{\mathrm{VR}_2(X;r)}$.

If α is a finitely supported measure, then the above construction resides inside finitely supported measures as well and therefore we get the result for $\mathrm{VR}_2^{\mathrm{fin}}(X;r)$.

For the Čech case, we again map $\check{\mathrm{C}}_2(X;r)$ onto X using the map $\alpha \mapsto \phi(\alpha) := \pi(m(\alpha))$. To see this is well defined, note that as $\alpha \in \check{\mathrm{C}}_2(X;r)$, there is some $z \in X$ with $F_\alpha(z) = \|z - m(\alpha)\|^2 + F_\alpha(m(\alpha)) < r^2$. This implies $\|z - m(\alpha)\| < r \leq \tau$ and so $m(\alpha)$ lies inside $U_\tau(X)$. We then get the result by the same homotopy construction as before. The finitely supported Čech case holds similarly.   □

7   The 2–Vietoris–Rips and 2–Čech thickenings of spheres with Euclidean metric

Let $\mathbb{S}^n$ be the n–dimensional sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, and let $(\mathbb{S}^n, \ell_2)$ denote the unit sphere equipped with the Euclidean metric. In this section we determine the successive homotopy types of the 2–Vietoris–Rips and 2–Čech metric thickening filtrations $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); \cdot)$ and $\check{\mathrm{C}}_2((\mathbb{S}^n, \ell_2); \cdot)$. We begin with a lemma.

Lemma 7.1   *For any measure $\alpha \in \mathcal{P}_{(\mathbb{S}^n, \ell_2)}$, we have*

$$\mathrm{diam}_2(\alpha) = (2 - 2\|m(\alpha)\|^2)^{1/2} \leq \sqrt{2} \quad \text{and} \quad \mathrm{rad}_2(\alpha) = (2 - 2\|m(\alpha)\|)^{1/2} \leq \sqrt{2}.$$

Note that since $\|m(\alpha)\| \leq 1$, we have that indeed $\mathrm{rad}_2(\alpha) \leq \mathrm{diam}_2(\alpha)$, and by the same token $\mathrm{diam}_2(\alpha) \leq 2\,\mathrm{rad}_2(\alpha)$, in agreement with Proposition 3.1.

Proof of Lemma 7.1   We can calculate $\mathrm{diam}_2(\alpha)$ as follows:

$$\mathrm{diam}_2(\alpha) = \left(\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \|x - y\|^2 \, \alpha(dx)\,\alpha(dy) \right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} 2 - 2\langle x, y \rangle \, \alpha(dx)\,\alpha(dy) \right)^{\frac{1}{2}}$$

$$= \left(2 - 2\Big\langle \int_{\mathbb{S}^n} x\,\alpha(dx), \int_{\mathbb{S}^n} y\,\alpha(dy) \Big\rangle \right)^{\frac{1}{2}} = (2 - 2\|m(\alpha)\|^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

The calculation for $\mathrm{rad}_2(\alpha)$ gives

$$
\mathrm{rad}_2(\alpha) = \inf_{x \in \mathbb{S}^n} \left(\int_{\mathbb{S}^n} \|x - y\|^2 \, \alpha(dy) \right)^{\frac{1}{2}} = \inf_{x \in \mathbb{S}^n} \left(2 - 2 \int_{\mathbb{S}^n} \langle x, y \rangle \, \alpha(dy) \right)^{\frac{1}{2}}
$$

$$
= \inf_{x \in \mathbb{S}^n} (2 - 2\langle x, m(\alpha) \rangle)^{1/2} = \left(2 - 2 \left\langle \frac{m(\alpha)}{\|m(\alpha)\|}, m(\alpha) \right\rangle \right)^{\frac{1}{2}} = (2 - 2\|m(\alpha)\|)^{1/2} \le \sqrt{2}. \quad \square
$$

We now see that, for spheres with euclidean metric, both the 2–Vietoris–Rips and the 2–Čech metric thickenings attain *only two* homotopy types:

Theorem D *The isometric embeddings* $\mathbb{S}^n \hookrightarrow \mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ *and* $\mathbb{S}^n \hookrightarrow \check{\mathrm{C}}_2((\mathbb{S}^n, \ell_2); r)$ *are homotopy equivalences for any* $r \le \sqrt{2}$. *When* $r > \sqrt{2}$, *the spaces* $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ *and* $\check{\mathrm{C}}_2((\mathbb{S}^n, \ell_2); r)$ *are contractible. By restricting to finitely supported measures, we get the same result for* $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n, \ell_2); r)$ *and* $\check{\mathrm{C}}_2^{\mathrm{fin}}((\mathbb{S}^n, \ell_2); r)$.

Remark 7.2 A similar phenomenon is found for the Vietoris–Rips filtration of $(\mathbb{S}^n, \ell_\infty)$, where $\mathrm{VR}((\mathbb{S}^n, \ell_\infty); r)$ has the homotopy type of $\mathbb{S}^n$ when $r \le 2/\sqrt{n+1}$ and becomes contractible when $r > 2/\sqrt{n+1}$; see [52, Section 7.3]. In contrast, the homotopy types of the Vietoris–Rips and Čech filtrations of geodesic circles include all possible odd-dimensional spheres [1]. The first new homotopy type of the ∞–Vietoris–Rips metric thickening of the n–sphere (with either the geodesic or the Euclidean metric) is known [2, Theorem 5.4], but so far only for a single (nonpersistent) scale parameter, and only when using the convention $\mathrm{diam}_\infty(\alpha) \le r$ instead of $\mathrm{diam}_\infty(\alpha) < r$.

We note that Theorem C only implies that $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r) \simeq \mathbb{S}^n$ for $r \le \tau(\mathbb{S}^n) = 1$, whereas Theorem D extends this range to $r \le \sqrt{2}$.

Proof of Theorem D From Lemma 7.1 we know that $\sqrt{2}$ is the maximal possible 2–diameter on $\mathcal{P}_{(\mathbb{S}^n, \ell_2)}$. Therefore, when $r > \sqrt{2}$, the spaces $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r) = \mathcal{P}_{(\mathbb{S}^n, \ell_2)}$ and $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n, \ell_2); r) = \mathcal{P}_{(\mathbb{S}^n, \ell_2)}^{\mathrm{fin}}$ are both convex and hence contractible.

When $0 < r \le \sqrt{2}$, by Lemma 7.1 we get that $m(\alpha)$ is not the origin for any $\alpha \in \mathcal{P}_{(\mathbb{S}^n, \ell_2)}$ (since otherwise we would have $\mathrm{diam}_2(\alpha) = \sqrt{2}$, which is too large). Therefore $m(\alpha)$ is inside the tubular neighborhood $U_1((\mathbb{S}^n, \ell_2))$. Let i be the inclusion map $(\mathbb{S}^n, \ell_2) \hookrightarrow \mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ mapping points to delta measures, and let ϕ be the composition $\pi \circ m \colon \mathrm{VR}_2((\mathbb{S}^n, \ell_2); r) \to (\mathbb{S}^n, \ell_2)$, which is well defined as $m(\alpha)$ is not the origin. Then we naturally have $\phi \circ i = \mathrm{id}_{(\mathbb{S}^n, \ell_2)}$. By applying Lemma 6.4, the linear family $\alpha_t = (1-t)\alpha + t \, \delta_{\phi(\alpha)}$ lies inside $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ and therefore provides the desired homotopy between $i \circ \phi$ and $\mathrm{id}_{\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)}$. By restricting to the set of finitely supported measures, we get the result for $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n, \ell_2); r)$.

The proof is identical for the Čech case. $\quad \square$

Remark 7.3 The above calculation gives, for any two positive integers $n \neq m$, that

$$d_{\mathrm{B}}(\mathrm{dgm}_{n,2}^{\mathrm{VR}}(\mathbb{S}^n, \ell_2), \mathrm{dgm}_{n,2}^{\mathrm{VR}}(\mathbb{S}^m, \ell_2)) = \tfrac{\sqrt{2}}{2} \quad \text{and} \quad d_{\mathrm{B}}(\mathrm{dgm}_{n,2}^{\check{\mathrm{C}}}(\mathbb{S}^n, \ell_2), \mathrm{dgm}_{n,2}^{\check{\mathrm{C}}}(\mathbb{S}^m, \ell_2)) = \tfrac{\sqrt{2}}{2}.$$

The stability theorem, Theorem B, gives

$$(5) \hspace{4cm} d_{\mathrm{GH}}((\mathbb{S}^n, \ell_2), (\mathbb{S}^m, \ell_2)) \geq \tfrac{\sqrt{2}}{4}.$$

Remark 7.4 In a more detailed analysis, Proposition 9.13 of [53] provides the lower bound $\frac{1}{2}$ for $d_{\mathrm{GH}}((\mathbb{S}^n, \ell_2), (\mathbb{S}^m, \ell_2))$ when $n \neq m$, which is larger than the lower bound $\frac{\sqrt{2}}{4}$ given in (5). As for the geodesic distance case, one can use [53, Corollary 9.8(1)] to obtain $d_{\mathrm{GH}}(\mathbb{S}^n, \mathbb{S}^m) \geq \arcsin\left(\frac{\sqrt{2}}{4}\right)$ from (5), where $d_{\mathrm{GH}}(\mathbb{S}^n, \mathbb{S}^m)$ is the Gromov–Hausdorff distance between spheres endowed with their geodesic distances for any $0 < m < n$. A better lower bound is found in [53]:

$$d_{\mathrm{GH}}(\mathbb{S}^n, \mathbb{S}^m) \geq \tfrac{1}{2} \arccos\left(\frac{-1}{m+1}\right) \geq \tfrac{\pi}{4},$$

which arises from considerations other than stability. The lower bound given by the quantity in the middle is shown to coincide with the exact Gromov–Hausdorff distance between $\mathbb{S}^1$ and $\mathbb{S}^2$, between $\mathbb{S}^1$ and $\mathbb{S}^3$, and also between $\mathbb{S}^2$ and $\mathbb{S}^3$.

8 Bounding barcode length via spread

The *spread* of a metric space is defined by Katz [45], and used in [52, Section 9] to bound the length of intervals in Vietoris–Rips simplicial complex persistence diagrams. In this section we identify a measure theoretic version of the notion of spread, and we use it to bound the length of intervals in p–Vietoris–Rips and p–Čech metric thickening persistence diagrams.

Definition 8.1 The p–spread of a bounded metric space X is defined as

$$\mathrm{spread}_p(X) := \inf_{\alpha_* \in \mathcal{P}_X^{\mathrm{fin}}} \sup_{\alpha \in \mathcal{P}_X} d_{\mathrm{W},p}^X(\alpha_*, \alpha).$$

Note that, for any $1 \leq p \leq q \leq \infty$,

$$\mathrm{spread}_1(X) \leq \mathrm{spread}_p(X) \leq \mathrm{spread}_q(X) \leq \mathrm{spread}_\infty(X) \leq \mathrm{rad}(X).$$

Remark 8.2 When $p \in [1, \infty)$, the space $\mathcal{P}_X^{\mathrm{fin}}$ is dense in $\mathcal{P}_{p,X}$, the set of Radon measures with finite moment of order p; see [15, Corollary 3.3.5]. Therefore, for a bounded metric space, the p–spread is just the radius of the metric space $(\mathcal{P}_X, d_{\mathrm{W},p}^X)$.

Proposition 8.3 *Let X be a bounded metric space and let $\mathfrak{i}$ be an L–controlled invariant. For any $r > 0$ and any $\xi > \mathrm{spread}_\infty(X)$, the space $[\mathcal{P}_X, \mathfrak{i}^X; r]$ can be contracted inside of $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$. In particular, any homology class of $[\mathcal{P}_X, \mathfrak{i}^X; r]$ will vanish in $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$.*

Proof As $\xi > \mathrm{spread}_\infty(X)$, there is some $\alpha_\xi \in \mathcal{P}_X^{\mathrm{fin}}$ such that, for all $\alpha \in \mathcal{P}_X$, we have $d_{W,\infty}^X(\alpha_\xi, \alpha) < \xi$. Consider the linear homotopy

$$h_t : [0,1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$$

defined by

$$(t, \alpha) \mapsto (1-t)\alpha + t\alpha_\xi.$$

This gives a homotopy between the inclusion from $[\mathcal{P}_X, \mathfrak{i}^X; r]$ to $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$ and a constant map, and therefore implies our conclusion so long as the homotopy is well defined. It then suffices to show $\mathfrak{i}^X(h_t(\alpha)) < r + 2L\xi$. This comes from the stability condition of the invariant with respect to Wasserstein distances and from [Lemma 4.2](#):

$$\mathfrak{i}^X(h_t(\alpha)) \leq \mathfrak{i}^X(\alpha) + 2L\, d_{W,\infty}^X(h_t(\alpha), \alpha) \leq \mathfrak{i}^X(\alpha) + 2L\, d_{W,\infty}^X(\alpha_\xi, \alpha) < r + 2L\xi. \qquad \square$$

Remark 8.4 It is not difficult to see that for the $\mathfrak{i}_{q,p}$ invariants we can improve the bound $\xi > \mathrm{spread}_\infty(X)$ to $\xi > \max\{\mathrm{spread}_p(X), \mathrm{spread}_q(X)\}$.

We also have the following stronger contractibility conclusion for $\mathrm{VR}_p(X; \cdot)$ and $\check{\mathrm{C}}_p(X; \cdot)$:

Theorem E *Let X be a bounded metric space. For any $p \in [1, \infty]$ and any $r > \mathrm{spread}_p(X)$, both $\mathrm{VR}_p(X; r)$ and $\check{\mathrm{C}}_p(X; r)$ are contractible.*

Proof As $r > \mathrm{spread}_p(X)$, there is some $\alpha_r \in \mathcal{P}_X^{\mathrm{fin}}$ such that, for all $\alpha \in \mathcal{P}_X$, we have $d_{W,p}^X(\alpha_r, \alpha) < r$. In particular, this implies $\mathrm{diam}_p(\alpha_r) = (\int_X F_{\alpha_r,p}^p(x)\, \alpha_r(dx))^{1/p} < r$ and $\mathrm{rad}_p(\alpha_r) = \inf_{x \in X} F_{\alpha_r,p}(x) < r$. Now let $\mathfrak{i}$ be either diam_p or rad_p. Consider the linear homotopy

$$h_t : [0,1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r]$$

defined by

$$(t, \alpha) \mapsto (1-t)\alpha + t\alpha_r.$$

It then suffices to show, for all $t \in [0,1]$, that $\mathrm{diam}_p(h_t(\alpha)) < r$ and $\mathrm{rad}_p(h_t(\alpha)) < r$, so that this linear homotopy from the identity map on $[\mathcal{P}_X, \mathfrak{i}^X; r]$ to the constant map to α_r is well-defined.

In the diam_p case, we have

$$\mathrm{diam}_p^p(h_t(\alpha)) = (1-t)^2 \iint_{X \times X} d_X^p(x, x')\, \alpha(dx)\, \alpha(dx') + 2t(1-t) \iint_{X \times X} d_X^p(x, x')\, \alpha_r(dx)\, \alpha(dx')$$
$$+ t^2 \iint_{X \times X} d_X^p(x, x')\, \alpha_r(dx)\, \alpha_r(dx')$$

$$= (1-t)^2 \mathrm{diam}_p^p(\alpha) + 2t(1-t) \int_X (d_{W,p}^X(\alpha_r, \delta_{x'}))^p\, \alpha(dx') + t^2 \int_X (d_{W,p}^X(\alpha_r, \delta_{x'}))^p\, \alpha_r(dx')$$

$$< (1-t)^2 r^p + 2t(1-t) r^p + t^2 r^p = r^p.$$

In the rad_p case, as $\mathrm{rad}_p(\alpha) < r$, there exists a point $y \in X$ such that $d_{W,p}^X(\delta_y, \alpha) < r$. Thus

$$\mathrm{rad}_p^p(h_t(\alpha)) = \left(\inf_{x \in X} d_{W,p}^X(\delta_x, h_t(\alpha)) \right)^p \leq \left(d_{W,p}^X(\delta_y, h_t(\alpha)) \right)^p$$

$$= (1-t)(d_{W,p}^X(\delta_y, \alpha))^p + t(d_{W,p}^X(\delta_y, \alpha_r))^p < (1-t)r^p + t r^p = r^p. \qquad \square$$

The upper bound for the lifetime of the persistent homology features in $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ for L–controlled invariants is also related to the metric spread defined in [45].

Definition 8.5 The *metric spread* of a metric space X is defined to be

$$\text{spread}(X) := \inf_{U \subset X, |U| < \infty} \max(d_{\mathrm{H}}(U, X), \text{diam}(U)),$$

where d_{H} is the Hausdorff distance.

Let $\mathfrak{i}$ be an L–controlled invariant and let $\max(\mathfrak{i}^X)$ be the maximum of the function $\mathfrak{i}^X$ on $\mathcal{P}_X$, for X a bounded metric space. For any $r > \max(\mathfrak{i}^X)$ we have $[\mathcal{P}_X, \mathfrak{i}^X; r] = \mathcal{P}_X$, and therefore $[\mathcal{P}_X, \mathfrak{i}^X; r]$ is contractible. Inspired by the definition of the spread, we have the following result:

Lemma 8.6 Let X be a metric space, let $\mathfrak{i}$ be an L–controlled invariant and let U be a finite subset of X. Then, for any $\xi > \max(d_{\mathrm{H}}(U, X), (\max(\mathfrak{i}^U) - r)/(2L))$, the space $[\mathcal{P}_X, \mathfrak{i}^X; r]$ is contractible in $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$. In particular, any homology class of $[\mathcal{P}_X, \mathfrak{i}^X; r]$ will vanish in $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$.

Proof As $\xi > d_{\mathrm{H}}(U, X)$, the balls $\{B(u; \xi)\}_{u \in U}$ form a open covering of X. We choose a partition of unity subordinate to the covering and build the map $\Phi_U : [\mathcal{P}_X, \mathfrak{i}^X; r] \to \mathcal{P}_U$ as in Lemma 5.15. Since $r + 2L\xi > \max(\mathfrak{i}^U)$, we know $\mathcal{P}_U = [\mathcal{P}_U, \mathfrak{i}^U; r + 2L\xi]$. Since the inclusion map $\iota_U : U \to X$ is of zero distortion, $\mathfrak{i}^X((\iota_U)_\sharp(\beta)) \leq \mathfrak{i}^U(\beta) \leq \max(\mathfrak{i}^U)$ for all $\beta \in \mathcal{P}_U$. This implies that the contractible set $\mathcal{P}_U$ is inside $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$, and so we have the following diagram:

$$[\mathcal{P}_X, \mathfrak{i}^X; r] \xrightarrow{\nu^X_{r, r+2L\xi}} [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$$

$$\Phi_U \searrow \qquad \uparrow \iota_U$$

$$\mathcal{P}_U$$

As the image of $\iota_U \circ \Phi_U$ maps $[\mathcal{P}_X, \mathfrak{i}^X; r]$ into a contractible subset $\mathcal{P}_U \subset [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$, it is homotopy equivalent to a constant map. To obtain this conclusion, it thus suffices to show that $\iota_U \circ \Phi_U$ is homotopy equivalent to the inclusion given by the structure map $\nu^X_{r, r+2L\xi}$. Consider the linear homotopy

$$h_t : [0, 1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$$

defined by

$$(t, \alpha) \mapsto (1 - t)\alpha + t\,\Phi_U(\alpha).$$

From the stability property of $\mathfrak{i}$, Lemma 4.2 and the estimate in Lemma 5.15, we have

$$\mathfrak{i}^X(h_t(\alpha)) \leq \mathfrak{i}^X(\alpha) + 2L\, d^X_{\mathrm{W},\infty}(\alpha, h_t(\alpha)) \leq \mathfrak{i}^X(\alpha) + 2L\, d^X_{\mathrm{W},\infty}(\alpha, \Phi_U(\alpha)) \leq \mathfrak{i}^X(\alpha) + 2L\xi < r + 2L\xi.$$

This shows that the homotopy h_t from $\iota_U \circ \Phi_U$ to $\nu^X_{r, r+2L\xi}$ is well defined. $\qquad\square$

Remark 8.7 For the 1–controlled $\mathfrak{i}_{q,p}$ invariant on a bounded metric space X, the maximum of $\mathfrak{i}^X_{q,p}$ is bounded by $\text{diam}(X)$. Therefore, for any finite subset U,

$$\max(d_{\mathrm{H}}(U, X), \text{diam}(U)) \geq \max\big(d_{\mathrm{H}}(U, X), \tfrac{1}{2}(\max(\mathfrak{i}^U_{q,p}) - r)\big).$$

Hence, the previous lemma implies that the lifetime of features in $[\mathcal{P}_X, i^X_{q,p}; \cdot]$ is bounded by $2\operatorname{spread}(X)$. In the diam_p case the factor 2 can be removed, as we show next, and therefore matches [52, Proposition 9.6] for Vietoris–Rips simplicial complexes.

Proposition 8.8 *For any $r > 0$, $p \in [1, \infty]$ and $\xi > \operatorname{spread}(X)$, the space $\operatorname{VR}_p(X; r)$ is contractible in $\operatorname{VR}_p(X; r + \xi)$. In particular, any homology class on $\operatorname{VR}_p(X; r)$ will vanish in $\operatorname{VR}_p(X; r + \xi)$.*

Proof As $\xi > \operatorname{spread}(X)$, there exists some $U \subset X$ such that $\operatorname{diam}(U) < \xi$ and $d_{\mathrm{H}}(U, X) < \xi$. As the maximum of diam^U_p on $\mathcal{P}_U$ is bounded by $\operatorname{diam}(U)$, the contractible subset $\mathcal{P}_U$ lies inside $\operatorname{VR}_p(X; r + \xi)$, and therefore $\iota_U \circ \Phi_U$ is homotopy equivalent to a constant map. Let h_t be the linear homotopy used in Lemma 8.6. What is left to get a homotopy equivalence between $\iota_U \circ \Phi_U$ and $v^X_{r,r+\xi}$ is to show $\operatorname{diam}_p(h_t(\alpha)) < r + \xi$. This comes from the following calculation:

$$\operatorname{diam}^p_p(h_t(\alpha)) = (1-t)^2 \iint_{X \times X} d^p(x, x')\, \alpha(dx)\, \alpha(dx') + 2t(1-t) \iint_{X \times X} d^p(x, x')\, \alpha(dx)\, \Phi_U(\alpha)(dx')$$
$$+ t^2 \iint_{X \times X} d^p(x, x')\, \Phi_U(\alpha)(dx)\, \Phi_U(\alpha)(dx')$$
$$= (1-t)^2 \operatorname{diam}^p_p(\alpha) + 2t(1-t) \int_X (d^X_{\mathrm{W},p}(\alpha, \delta_{x'}))^p\, \Phi_U(\alpha)(dx') + t^2 \operatorname{diam}^p_p(\Phi_U(\alpha))$$
$$\leq (1-t)^2 \operatorname{diam}^p_p(\alpha) + 2t(1-t) \int_X \big(d^X_{\mathrm{W},p}(\alpha, \Phi_U(\alpha)) + d^X_{\mathrm{W},p}(\Phi_U(\alpha), \delta_{x'})\big)^p\, \Phi_U(\alpha)(dx')$$
$$+ t^2 \operatorname{diam}^p_p(\Phi_U(\alpha))$$
$$\leq (1-t)^2 \operatorname{diam}^p_p(\alpha) + 2t(1-t)\big(\xi + \operatorname{diam}_p(\Phi_U(\alpha))\big)^p + t^2 \operatorname{diam}^p_p(\Phi_U(\alpha))$$
$$< (1-t)^2 r^p + 2t(1-t)(r+\xi)^p + t^2 \xi^p < (r+\xi)^p. \qquad \square$$

The following result shows the lifetime of features of $\check{C}_p(X; \cdot)$ is also bounded by the metric spread of X, $\operatorname{spread}(X)$.

Proposition 8.9 *For any $r > 0$, $p \in [1, \infty]$ and $\xi > \operatorname{spread}(X)$, the space $\check{C}_p(X; r)$ is contractible in $\check{C}_p(X; r + \xi)$. In particular, any homology class on $\check{C}_p(X; r)$ will vanish in $\check{C}_p(X; r + \xi)$.*

Proof As $\xi > \operatorname{spread}(X)$, there exists some $U \subset X$ such that $\operatorname{diam}(U) < \xi$ and $d_{\mathrm{H}}(U, X) < \xi$. The maximum of rad^U_p on $\mathcal{P}_U$ is bounded by $\operatorname{diam}(U)$, and the contractible subset $\mathcal{P}_U$ lies inside $\check{C}_p(X; r + \xi)$. This shows $\iota_U \circ \Phi_U$ is homotopy equivalent to a constant map. Let $h_t = (1-t)\alpha + t\,\Phi_U(\alpha)$ be the linear homotopy used in Lemma 8.6. What is left is to show is $\operatorname{rad}_p(h_t(\alpha)) < r + \xi$. By Lemmas 4.2 and 4.9,

$$\operatorname{rad}_p(h_t(\alpha)) \leq \operatorname{rad}_p(\alpha) + d^X_{\mathrm{W},p}(\alpha, h_t(\alpha)) \leq r + t^{1/p} d^X_{\mathrm{W},p}(\alpha, \Phi_U(\alpha)) \leq r + \xi. \qquad \square$$

9 Conclusion

Filtrations, ie increasing sequences of spaces, play a foundational role in applied and computational topology, as they are the input to persistent homology. To produce a filtration from a metric space X, one

often considers a Vietoris–Rips or Čech simplicial complex, with X as its vertex set, as the scale parameter increases. Since a point in the geometric realization of a simplicial complex is a convex combination of the vertices of the simplex $[x_0, x_1, \ldots, x_k]$ in which it lies, each such point can alternatively be identified with a probability measure: a convex combination of Dirac delta masses δ_{x_0}, δ_{x_1}, $\ldots$, δ_{x_k}. We can therefore reinterpret the Vietoris–Rips and Čech simplicial complex filtrations instead as filtrations in the space of probability measures, which are referred to as the Vietoris–Rips and Čech *metric thickenings*. In [2] it is argued that the metric thickenings have nicer properties for some purposes. For example, the inclusion from metric space X into the metric thickening is always an isometry onto its image, whereas an inclusion from metric space X into the vertex set of a simplicial complex is not even continuous unless X is discrete. We prove that these two perspectives are compatible: the ∞–Vietoris–Rips (resp. Čech) metric thickening filtration has the same persistent homology as the Vietoris–Rips (resp. Čech) simplicial complex filtration when X is totally bounded. Therefore, when analyzing these filtrations, one can choose to apply either simplicial techniques (simplicial homology, simplicial collapses, discrete Morse theory) or measure-theoretic techniques (optimal transport, Karcher or Fréchet means), whichever is more convenient for the task at hand.

The measure-theoretic perspective motivates new filtrations to build on top of a metric space X. Though the Vietoris–Rips simplicial complex filtration is closely related (at interleaving distance zero) to the metric thickening filtration obtained by looking at sublevel sets in the space of probability measures of the ∞–diameter functional, one can instead consider sublevel sets of the p–diameter functional for any $1 \le p \le \infty$. The same is true upon replacing Vietoris–Rips with Čech and replacing p–diameter with p–radius. These relaxed p–Vietoris–Rips and p–Čech metric thickenings enjoy the same stability results underlying the use of persistent homology: nearby metric spaces produce nearby persistence modules. The generalization to $p < \infty$ is a useful one: though determining the homotopy types of ∞–Vietoris–Rips thickenings of n–spheres is a hard open problem, we give a complete description of the homotopy types of 2–Vietoris–Rips thickenings of n–spheres for all n. We also prove a Hausmann-type theorem in the case $p = 2$, and ask if the $p < \infty$ metric thickenings may be amenable to study using tools from Morse theory.

More generally, one can consider sublevel sets of any L–controlled function on the space of probability measures on X. We prove stability in this much more general context. This allows one to consider metric thickenings that are tuned to a particular task, perhaps incorporating other geometric notions besides just proximity, such as curvature, centrality, eccentricity, etc. One can design an L–controlled functional to highlight specific features that may be useful for a particular data science task.

We hope these contributions inspire more work on metric thickenings and their relaxations. We end with some open questions.

(i) For X totally bounded, is the $p = \infty$ metric thickening $\mathrm{VR}_\infty(X; r)$ homotopy equivalent to the simplicial complex $\mathrm{VR}(X; r)$, and similarly is $\check{\mathrm{C}}_\infty(X; r)$ homotopy equivalent to $\check{\mathrm{C}}(X; r)$? Note, we are using the $<$ convention.

(ii) For X totally bounded, is $\mathrm{VR}_p(X;r)$ homotopy equivalent to $\mathrm{VR}_p^{\mathrm{fin}}(X;r)$, is $\check{\mathrm{C}}_p(X;r)$ homotopy equivalent to $\check{\mathrm{C}}_p^{\mathrm{fin}}(X;r)$ and, for $\mathfrak{i}$ a controlled invariant, is $[\mathcal{P}_X, \mathfrak{i}^X; r]$ homotopy equivalent to $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; r]$?

(iii) Is there an analogue to the Hausmann-type Theorem C which holds for $p \in (2, \infty)$? The case $p = \infty$ was tackled in [6]. In a similar spirit, it seems interesting to explore whether analogous theorems hold when the ambient space is a more general Hadamard space instead of $\mathbb{R}^d$.

(iv) Can one prove Latschev-type theorems [50] for p–metric thickenings?

(v) For $p \neq 2$, what are the homotopy types of p–Vietoris–Rips and p–Čech thickenings of spheres at all scales? Is the homotopy connectivity a nondecreasing function of the scale, and if so, how quickly does the homotopy connectivity increase?

(vi) What are the homotopy types of p–Vietoris–Rips and p–Čech metric thickenings of other manifolds, such as ellipses (see [3]), ellipsoids, tori and projective spaces [5; 45; 49]?

(vii) What versions of Morse theory [60] can be developed in order to analyze the homotopy types of p–metric thickenings of manifolds as the scale increases? See Section A.2 for some initial ideas in the case of p–Čech thickenings. For homogeneous spaces such as spheres, versions of Morse–Bott theory [16; 17; 18] may be needed.

(viii) For X finite, is $\mathrm{VR}_p(X;r)$ always homotopy equivalent to a subcomplex of the complete simplex on the vertex set X? See Section A.3 for a proof of the Čech case.

(ix) For X finite with $n + 1$ points, the space $\mathcal{P}_X$ is an n–simplex in $\mathbb{R}^{n+1}$ where coordinates are the weights of a measure at each point. In this case, diam_p is a quadratic polynomial on $\mathbb{R}^{n+1}$ and rad_p is the minimum of $n + 1$ linear equations. Therefore, both $\mathrm{VR}_p(X;r)$ and $\check{\mathrm{C}}_p(X;r)$ are semialgebraic sets in $\mathbb{R}^{n+1}$. Can one use linear programming along with the results of Section A.3 to calculate the homology groups of $\check{\mathrm{C}}(X;r)$, and the work on quadratic semialgebraic sets [9; 11; 24] to calculate the homology groups of $\mathrm{VR}_p(X;r)$? See also the recent paper [10], which provides a singly exponential complexity algorithm for computing the sublevel set persistent homology induced by a polynomial on a semialgebraic set up to some fixed homological dimension.

Appendix

The appendix contains results which are related to but not central to the main thread of the paper. In Section A.1 we explain how the q–Wasserstein distance metrizes the weak topology for $1 \leq q < \infty$. We describe connections to min-type Morse theories in Section A.2, and ask what can be gained from these connections. In Section A.3 we show that p–Čech thickenings of finite metric spaces are homotopy equivalent to simplicial complexes with one vertex for each point in the metric space. We derive the persistent homology diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of a family of discrete metric spaces in Section A.4, and we describe the 0–dimensional persistent homology of the p–Vietoris–Rips and p–Čech metric thickenings of an arbitrary metric space in Section A.5. We consider

crushings in Section A.6. In Section A.7, we show that the main properties we prove for the (intrinsic) p–Čech metric thickening also hold for the ambient p–Čech metric thickening.

A.1 Metrization of the weak topology

For $1 \leq q < \infty$ the q–Wasserstein distance metrizes the weak topology, as we explain here for completeness.

Definition A.1 Let X be a metric space. The Lévy–Prokhorov metric on $\mathcal{P}X$ is given by the formula

$$d_{\mathrm{LP}}(\alpha, \beta) := \inf\{\varepsilon > 0 \mid \alpha(E) \leq \beta(E^{\varepsilon}) + \varepsilon, \beta(E) \leq \alpha(E^{\varepsilon}) + \varepsilon \text{ for all } E \in \mathfrak{B}(X)\}.$$

Here $E^{\varepsilon} = \bigcup_{x \in E} B_{\varepsilon}(x)$ is the open ε–neighborhood of E in $\mathcal{P}X$, and $\mathfrak{B}(X)$ is the Borel σ–algebra.

We state two theorems that we will use:[3]

Theorem [15, Theorem 3.1.4] *The weak topology on the set $\mathcal{P}X$ is generated by the Lévy–Prokhorov metric.*[4]

Theorem [39, Theorem 2] *On a metric space X, for any α and β in $\mathcal{P}X$, one has*

$$(d_{\mathrm{LP}})^2 \leq d_{\mathrm{W},1}^X \leq (\mathrm{diam}(X) + 1)d_{\mathrm{LP}}.$$

Corollary A.2 *On a bounded metric space X, for any $q \in [1, \infty)$, the q–Wasserstein metric generates the weak topology on $\mathcal{P}X$.*

Proof From [39, Theorem 2; 15, Theorem 3.1.4], we know $d_{\mathrm{W},1}^X$ generates the weak topology. For other values of q, notice that, for any coupling μ between α and β,

$$\left(\int_X d_X^q(x, x') \, \mu(dx \times dx')\right)^{\frac{1}{q}} \leq (\mathrm{diam}(X))^{(q-1)/q}\left(\int_X d_X(x, x') \, \mu(dx \times dx')\right)^{\frac{1}{q}}.$$

This implies $d_{\mathrm{W},q}^X \leq (\mathrm{diam}(X))^{(q-1)/q}(d_{\mathrm{W},1}^X)^{1/q}$. Along with $d_{\mathrm{W},1}^X \leq d_{\mathrm{W},q}^X$, this implies that, on a bounded metric space, all q–Wasserstein metrics with q finite are equivalent and generate the weak topology on $\mathcal{P}X$.   □

A.2 Min-type Morse theory

The paper [8] by Baryshnikov, Bubenik, and Kahle studies a Morse theory for min-type functions; see also [21; 38; 54]. The following notation is from [8, Section 3.1]. Let X be a compact metric space (called the *parameter space*), let M be a compact smooth manifold perhaps with boundary, and let $f : X \times M \to \mathbb{R}$ be a continuous function. For each $x \in X$, define $f_x : M \to \mathbb{R}$ by $f_x(m) = f(x, m)$. Let $\nabla f_x : M \to \mathbb{R}$ be the gradient of f_x with respect to m. We furthermore assume that the function

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[3]The first statement is from [15]. We emphasize that we use $\mathcal{P}X$ to denote the set of all Radon probability measures on X whereas the author instead uses $\mathcal{P}_r(X)$.

[4]Theorem 3.1.4 in [15] proves a stronger result, namely that the claim is true within the set of τ–additive probability measures. The restricted version we use follows from the fact that every Radon measure is τ–additive; see [14, Proposition 7.2.2(i)].

$X \times M \to \mathbb{R}$ defined by $(x, m) \mapsto \nabla f_x(m)$ is continuous. There is then an analogue of Morse theory, called "min-type Morse theory", for the min-type function $\tau \colon M \to \mathbb{R}$ defined by $\tau(m) = \min_{x \in X} f_x(m)$. Whereas [8] uses min-type Morse theory to study configurations of hard spheres, we instead propose the use of min-type Morse theory to study p–Čech metric thickenings, as follows. Let X be a *finite* metric space with $n + 1$ points. Define $M = \Delta_n$ to be the n–simplex on $n + 1$ vertices; a point $m \in \Delta_n$ is given in barycentric coordinates as $m = (m_0, \ldots, m_n)$ with $m_i \geq 0$ and $\sum_i m_i = 1$. For $x \in X$, let $f_x \colon \Delta_n \to \mathbb{R}$ be defined by $f_x(m) = \sum_i m_i d_X^p(x, x_i)$; note that this is equal to the pth power of the p–Fréchet function, namely to $F_{\alpha,p}^p(x) = \sum_i m_i d_X^p(x, x_i)$, when α is the measure $\alpha = \sum_i m_i \delta_{x_i}$. So $f \colon X \times \Delta_n \to \mathbb{R}$ is defined by $f(x, m) = f_x(m)$. Note that each gradient $\nabla f_x \colon \Delta_n \to \mathbb{R}$ is linear, and hence continuous, and so the joint function $X \times \Delta_n \to \mathbb{R}$ given by $(x, m) \mapsto \nabla f_x(m)$ is continuous since X is discrete. The function $\tau \colon \Delta_n \to \mathbb{R}$ is then defined by $\tau(m) = \min_{x \in X} f_x(m)$; note that this is equal to $\text{rad}_p^p(\alpha) = \inf_{x \in X} F_{\alpha,p}^p(x)$ for $\alpha = \sum_i m_i \delta_{x_i}$. By Lemma 4.11 and its proof, we have not only a homeomorphism $\mathcal{P}_X \cong \Delta_n$, but also a homeomorphism

$$\check{C}_p(X; r) = \text{rad}_p^{-1}((-\infty, r)) \cong \tau^{-1}((-\infty, r^p)),$$

meaning that the p–Čech metric thickenings are homeomorphic to the sublevel sets of the min-type function τ. In this setting, we have the additional convenience that each function f_x is affine.

Question A.3 Can the machinery from [8] be used to prove new results about p–Čech metric thickenings, such as homotopy types? A first step in this direction might be to use their balanced criterion (which derives from Farkas' lemma) to help identify which points are topological regular points or critical points of τ.

Question A.4 We have restricted to X finite (with $n + 1$ points) so that $M = \Delta_n$ will be a manifold with boundary. Can one build up towards letting X be a manifold, such as a circle or n–sphere?

A.3 Finite p–Čech metric thickenings are homotopy equivalent to simplicial complexes

The p–Vietoris–Rips and p–Čech metric thickenings we consider are based on the corresponding simplicial complexes and are closely related to them. The most direct relationship is given by Lemma 4.12 in the case of finite metric spaces and $p = \infty$. Further similarities in the case of totally bounded metric spaces and $p = \infty$ are observed in the persistence diagrams, as shown in Corollary 5.10. For $p < \infty$, the metric thickenings are not as directly related to the corresponding simplicial complexes. However, in Section A.3, we establish that all p–Čech metric thickenings on finite metric spaces are homotopy equivalent to simplicial complexes (although generally not the corresponding Čech simplicial complexes).

Theorem F *Let (X, d_X) be a finite metric space with $n + 1$ points. For any $p \in [1, \infty]$ and any $r > 0$, $\check{C}_p(X; r)$ is homotopy equivalent to a simplicial complex on $n + 1$ vertices, consisting of the simplices contained in the homeomorphic image of $\check{C}_p(X; r)$ in the standard n–simplex.*

Proof The result holds for the case $p = \infty$ by Lemma 4.12, so we will suppose $p \in [1, \infty)$. We begin with some background notation and observations. Let σ be a simplex in a Euclidean space, let $z_0 \in \sigma$ and

let σ_{z_0} be the union of the closed faces of σ not containing z_0 (we always have $\sigma_{z_0} \subseteq \partial\sigma$, and $\sigma_{z_0} = \partial\sigma$ if z_0 is in the interior of σ). Then there is a continuous function $P: \sigma \setminus \{z_0\} \to \sigma_{z_0}$ defined by projecting radially from z_0. Furthermore, if $C \subseteq \sigma$ is convex and contains z_0, then for any $z \in \sigma \setminus C$ the line segment connecting z and $P(z)$ is contained in $\sigma \setminus C$, since this lies in the line segment connecting z_0 and $P(z)$. Therefore a linear homotopy shows that $P|_{\sigma \setminus C}: \sigma \setminus C \to \sigma_{z_0} \setminus C$ is a strong deformation retraction. We will apply such retractions successively to a simplex and its faces as described below.

Let $X = \{x_0, \ldots, x_n\}$. If $\alpha = \sum_i a_i \delta_{x_i}$, then using the notation for the Fréchet function from Section 2, $F_{\alpha,p}^p(x_j) = \sum_i a_i d_X^p(x_i, x_j)$. Thus $\alpha \in \check{C}_p(X; r)$ if and only if $\sum_i a_i d_X^p(x_i, x_j) < r^p$ for some j. Let $\Delta = \{(y_0, \ldots, y_n) \in \mathbb{R}^{n+1} \mid \sum_i y_i = 1 \text{ with } y_i \geq 0 \text{ for all } i\}$ be the standard n–simplex in $\mathbb{R}^{n+1}$. By Lemma 4.11, $\check{C}_p(X; r)$ is homeomorphic to

$$Y = \left\{ (y_0, \ldots, y_n) \in \Delta \ \middle| \ \sum_i y_i d_X^p(x_i, x_j) < r^p \text{ for some } j \right\}.$$

Equivalently, Y is a sublevel set of the function $\rho: \Delta \to \mathbb{R}$ given by

$$\rho(y_0, \ldots, y_n) = \min_j \left\{ \sum_i y_i d_X^p(x_i, x_j) \right\}.$$

We note that Y contains the vertices of Δ by the assumption that $r > 0$. If $Y = \Delta$, then $\check{C}_p(X; r) \cong \Delta$, and Δ is homeomorphic to a simplex with vertex set X, as required. If not, then $\Delta \setminus Y$ is convex as it is the intersection of half-spaces and Δ, so we may project radially from any point $z_0 \in \Delta \setminus Y$, as above. This shows $Y \simeq Y \cap \Delta_{z_0}$, where Δ_{z_0} is the union of the closed faces of Δ not containing z_0.

Since $Y \cap \Delta_{z_0}$ is contained in the boundary of Δ, we will next verify that we can define retractions within the $(n-1)$–dimensional faces of Δ contained in Δ_{z_0} that contain a point not in Y. More generally, we will repeat, for successively lower dimensional faces, inductively obtaining a sequence of strong deformation retractions $Y_n \to Y_{n-1} \to \cdots \to Y_0$, where $Y_n = Y$. Each Y_k will consist of all closed faces of Δ contained in Y, along with some subset of the remaining k–dimensional closed faces intersected with Y. Thus Y_0 will be a simplicial complex consisting of the closed faces of Δ contained in Y and will be homotopy equivalent to $\check{C}_p(X; r)$, as required.

We will use induction, so suppose Y_k meets the description above. Let $\sigma_1, \ldots, \sigma_m$ be those remaining k–dimensional faces of Δ whose intersections with Y are contained in Y_k and that contain a point in their
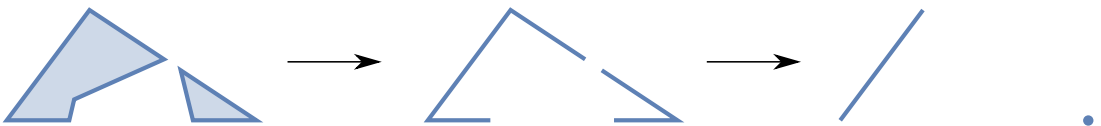
Figure 2: The sequence of deformation retractions used in the proof of Theorem F collapses a subset of a simplex to a simplicial complex on its vertices. In this example $n = 2$, and the sequence of deformation retractions is $Y_2 \to Y_1 \to Y_0$.

interior that is not in Y_k. Let $\tau_1, \ldots, \tau_{m'}$ be the remaining k–dimensional faces whose intersections with Y are contained in Y_k and that have only some boundary point not in Y_k. On each σ_l, we may choose an interior point not in Y_k and project radially from this point as above. The homotopy is constant on the boundary, so we may define a homotopy on Y_k that simultaneously retracts each $\sigma_l \cap Y_k$ to $\partial \sigma_l \cap Y_k$ and is constant on all points not in any σ_l. Thus Y_k deformation retracts onto a subset Y_k' consisting of faces of Δ that are contained in Y, along with $\tau_l \cap Y_k$ for all l.

Next, choose any point $y \in \partial \tau_1 \setminus Y_k'$. Note that projecting radially from y as above affects any simplex in $\partial \tau_1$ that contains y, so we must extend this to a homotopy H on Y_k' in a way that is consistent on the simplices containing y. For any τ_l containing y, let H be defined on $\tau_l \cap Y_k'$ by the radial projection from y, as above. For any two τ_l and $\tau_{l'}$ containing y, the two definitions of H on $\tau_l \cap \tau_{l'} \cap Y$ are consistent, as they are both linear homotopies. We also let H be constant on any point in a face of Δ not containing y. This includes all faces of Δ contained in Y and all $\tau_l \cap Y$ such that $y \notin \tau_l$. The definitions are consistent on their overlap, since if τ_l contains y, then radial projection from y is constant on the faces of τ_l that do not contain y. Therefore H is a well-defined homotopy, which shows that Y_k' deformation retracts onto the subset that excludes the interiors of all τ_l containing y. We can repeat this for the remaining set of τ_l until all have been retracted. Composing these deformation retractions, we have shown that Y_k deformation retracts onto a subset Y_{k-1} consisting of faces of Δ contained in Y along with the intersections of Y with a subset of the remaining $(k-1)$–dimensional faces of Δ (subsets of the boundaries of those k–dimensional simplices that were collapsed). This completes the inductive step, so we obtain the sequence $Y_n \to Y_{n-1} \to \cdots \to Y_0$ of strong deformation retractions, as required. $\square$

Corollary A.5 *Let (X, d_X) be a finite metric space, let $p \in [1, \infty]$ and, for any $r > 0$, let $\mathrm{S}(X; r)$ be the simplicial complex from Theorem F that is homotopy equivalent to $\check{\mathrm{C}}_p(X; r)$. These simplicial complexes form a filtration $\mathrm{S}(X; \cdot)$, and for any integer $k \geq 0$, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \mathrm{S}(X; \cdot)$ are isomorphic persistence modules.*

Proof The $p = \infty$ case again holds by Lemma 4.12, so let $p \in [1, \infty)$. We will identify $\check{\mathrm{C}}_p(X; r)$ with its homeomorphic image in the standard simplex. By Theorem F, $\mathrm{S}(X; r)$ consists of the simplices contained in $\check{\mathrm{C}}_p(X; r)$. So if $r_1 \leq r_2$, the fact that $\check{\mathrm{C}}_p(X; r_1) \subseteq \check{\mathrm{C}}_p(X; r_2)$ implies $\mathrm{S}(X; r_1) \subseteq \mathrm{S}(X; r_2)$, and thus $\mathrm{S}(X; \cdot)$ is a filtration of simplicial complexes.

Since in the proof of the theorem we constructed a deformation retraction $\check{\mathrm{C}}_p(X; r) \to \mathrm{S}(X; r)$, the inclusion $\mathrm{S}(X; r) \hookrightarrow \check{\mathrm{C}}_p(X; r)$ is a homotopy equivalence for each r. Therefore the induced maps on homology give isomorphisms $H_k(\mathrm{S}(X; r)) \cong H_k(\check{\mathrm{C}}_p(X; r))$ for each r. Furthermore, the following diagram commutes for all $r_1 < r_2$, since all maps are inclusions:

$$
\begin{array}{ccc}
\check{\mathrm{C}}_p(X; r_1) & \longhookrightarrow & \check{\mathrm{C}}_p(X; r_2) \\
\uparrow & & \uparrow \\
\mathrm{S}(X; r_1) & \longhookrightarrow & \mathrm{S}(X; r_2)
\end{array}
$$

This shows that the induced maps on homology commute, giving a morphism of persistence modules. Since each vertical map is an isomorphism, this is an isomorphism of persistence modules.   □

These results show that p–Čech persistent homology can be computed, at least in principle. The complex $S(X; r)$ in Corollary A.5 consists of all simplices contained in the r^p–sublevel set of the function ρ in the proof of Theorem F. So finding the filtration $S(X; \cdot)$ would require finding the maximum of ρ on each face. Calculating persistent homology would involve finding the maximum of ρ on each face of the necessary dimensions.

A.4   Persistence diagrams of $\mathrm{VR}_p(Z_{n+1}; \cdot)$ and $\check{C}_p(Z_{n+1}; \cdot)$

In this section, we will calculate the persistence diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of the metric space Z_{n+1}, consisting of $n + 1$ points with all interpoint distances equal to 1. Let Δ_n be the n–dimensional simplex on $n + 1$ points, and let $\Delta_n^{(k)}$ denote its k–skeleton.

Proposition A.6   *Let Z_{n+1} be the metric space consisting of $n + 1$ points with all interpoint distances equal to 1. For $(k/(k + 1))^{1/p} < r \leq ((k + 1)/(k + 2))^{1/p}$ with $0 \leq k \leq n - 1$, we have*

$$\mathrm{VR}_p(Z_{n+1}; r) \simeq \check{C}_p(Z_{n+1}; r) \simeq \Delta_n^{(k)},$$

and when $r > (n/(n + 1))^{1/p}$, both $\mathrm{VR}_p(Z_{n+1}; r)$ and $\check{C}_p(Z_{n+1}; r)$ become the n–simplex $\mathcal{P}_{Z_{n+1}}$, which is contractible.

Proof   We will use the following observation, for which we omit the proof. Let α be a measure in $\mathcal{P}_{Z_{m+1}}$, where $m + 1$ is any positive integer. Then both diam_p and rad_p will obtain the maximum $(m/(m + 1))^{1/p}$ only at the uniform measure supported on the $m + 1$ points of the metric space Z_{m+1}, ie the barycenter of the m–simplex $\mathcal{P}_{Z_{m+1}}$.

Let k be an integer where $0 \leq k \leq n - 1$. For any r in $((k/(k + 1))^{1/p}, ((k + 1)/(k + 2))^{1/p}]$, we will show both $\mathrm{VR}_p(Z_{n+1}; r)$ and $\check{C}_p(Z_{n+1}; r)$ can be deformation retracted to the k–skeleton of the n–simplex $\mathcal{P}_{Z_{n+1}}$, denoted by $\Delta_n^{(k)}$. Since $(k/(k + 1))^{1/p} < r \leq ((k + 1)/(k + 2))^{1/p}$, we know both $\mathrm{VR}_p(Z_{n+1}; r)$ and $\check{C}_p(Z_{n+1}; r)$ contain the k–skeleton of Δ_n, but not any higher-dimensional skeleta. We can then use the radial projection as in Corollary A.5 to work out the retraction. From the above observation, the barycenter of any simplex in Δ_n with dimension higher than k is not in $\mathrm{VR}_p(Z_{n+1}; r)$ and $\check{C}_p(Z_{n+1}; r)$. We can use these barycenters as the basepoint for radial projection. We start from Δ_n: if the radial projection based at its barycenter can be restricted to $\mathrm{VR}_p(Z_{n+1}; r)$ or $\check{C}_p(Z_{n+1}; r)$, then it will retract them onto their intersections with the $(n-1)$–skeleton $(\Delta_n)^{(n-1)}$. From the proof of Corollary A.5, we know the restriction is well defined for $\check{C}_p(Z_{n+1}; r)$. Here, it is also well defined for $\mathrm{VR}_p(Z_{n+1}; r)$, because our basepoint for the radial projection is an interior maximum of the quadratic function diam_p^p, and diam_p^p will be concave along any line that passes through the basepoint. This in turn shows that

the line segment connecting two points outside of $\text{VR}_p(Z_{n+1}; r)$ is also outside of $\text{VR}_p(Z_{n+1}; r)$. We can continue the retraction process inductively on $\text{VR}_p(Z_{n+1}; r) \cap \Delta_n^{(l)}$ and $\check{C}_p(Z_{n+1}; r) \cap \Delta_n^{(l)}$, where $k < l \leq n-1$, with radial projections based at barycenters of l–simplices of Δ_n. This results in a retraction onto $\text{VR}_p(Z_{n+1}; r) \cap \Delta_n^{(l-1)}$ and $\check{C}_p(Z_{n+1}; r) \cap \Delta_n^{(l-1)}$, respectively. Eventually, we will deformation retract onto $\text{VR}_p(Z_{n+1}; r) \cap \Delta_n^{(k)}$ and $\check{C}_p(Z_{n+1}; r) \cap \Delta_n^{(k)}$, which are both equal to $\Delta_n^{(k)}$. $\qquad\square$

Corollary A.7 *Let n be a positive integer and let Z_{n+1} be the metric space consisting of $n + 1$ points with interpoint distance equal to 1. The persistence diagrams of $\text{VR}_p(Z_{n+1}; \cdot)$ and $\check{C}_p(Z_{n+1}; \cdot)$ are the same and are of the form*

$$\text{dgm}_{k,p}^{\text{VR}}(Z_{n+1}) = \text{dgm}_{k,p}^{\check{C}}(Z_{n+1}) = \begin{cases} \left(0, \left(\tfrac{1}{2}\right)^{1/p}\right)^{\otimes n} \oplus (0, \infty) & \text{if } k = 0, \\ \left(\left(k/(k+1)\right)^{1/p}, \left((k+1)/(k+2)\right)^{1/p}\right)^{\otimes \binom{n}{k+1}} & \text{if } 0 < k \leq n-1, \\ \varnothing & \text{if } k > n-1. \end{cases}$$

The superscripts denote the multiplicity of a point in the persistence diagram.

Proof For any integer k with $0 < k < n-1$, we know the homology of the k–skeleton of an n–simplex Δ_n is given by

$$H_l(\Delta_n^{(k)}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } l = 0, \\ \mathbb{Z}^{\binom{n}{k+1}} & \text{if } l = k, \\ \varnothing & \text{otherwise.} \end{cases}$$

We get the result by combining this with the previous result on the homotopy types of $\text{VR}_p(Z_{n+1}; \cdot)$ and $\check{C}_p(Z_{n+1}; \cdot)$. $\qquad\square$

A.5 Zero-dimensional persistent homology of $\text{VR}_p(X; \cdot)$ and $\check{C}_p(X; \cdot)$

For a finite metric space X, we will show that the 0–dimensional persistent homology of $\text{VR}_p(X; \cdot)$ and $\check{C}_p(X; \cdot)$ are the same, and that they both recover the single-linkage clustering up to a constant factor related to p.

Lemma A.8 *For X a finite metric space, the birth time for all intervals in the 0–dimensional barcodes of $\text{VR}_p(X; r)$ and $\check{C}_p(X; r)$ is zero.*

Proof Since all delta measures δ_x have diam_p and rad_p equal to zero, it suffices to show, for any measure α in $\text{VR}_p(X; r)$ (or $\check{C}_p(X; r)$), that there is a path in $\text{VR}_p(X; r)$ (or $\check{C}_p(X; r)$) that connects α with some delta measure.

For α a measure in $\text{VR}_p(X; r)$,

$$\int_X (d_{\text{W},p}^X(\alpha, \delta_x))^p \, \alpha(dx) = (\text{diam}_p(\alpha))^p < r^p.$$

So there is some $x_0 \in \mathrm{supp}(\alpha)$ such that $d^X_{W,p}(\alpha, \delta_{x_0}) < r^p$. We then pick the path $\alpha_t = (1-t)\alpha + t\delta_{x_0}$ for $t \in [0,1]$. Now consider

$$\mathrm{diam}_p(\alpha_t) = \left(\int_X (d^X_{W,p}(\alpha_t, \delta_x))^p \, \alpha_t(dx) \right)^{\frac{1}{p}} = \left(\int_X ((1-t)d^X_{W,p}(\alpha, \delta_x) + t d^X_{W,p}(\delta_{x_0}, \delta_x))^p \, \alpha_t(dx) \right)^{\frac{1}{p}}$$

$$\leq (1-t)\left(\int_X (d^X_{W,p}(\alpha, \delta_x))^p \, \alpha_t(dx) \right)^{\frac{1}{p}} + t\left(\int_X (d^X_{W,p}(\delta_{x_0}, \delta_x))^p \, \alpha_t(dx) \right)^{\frac{1}{p}}$$

$$\leq (1-t)\left((1-t)\mathrm{diam}_p^p(\alpha) + t(d^X_{W,p}(\alpha, \delta_{x_0}))^p \right)^{1/p} + t\left((1-t)(d^X_{W,p}(\alpha, \delta_{x_0}))^p \right)^{1/p}$$

$$< (1-t)r + t(1-t)^{1/p}r < r.$$

This shows $\alpha_t \in \mathrm{VR}_p(X; r)$, and the path is continuous by Proposition 2.4.

For α a measure in $\check{\mathrm{C}}_p(X; r)$, there is some x_0 with $d^X_{W,p}(\alpha, \delta_{x_0}) < r$. Then $d^X_{W,p}((1-t)\alpha + t\delta_{x_0}, \delta_{x_0}) = (1-t)d^X_{W,p}(\alpha, \delta_{x_0}) < r$. This shows α_t is a continuous path in $\check{\mathrm{C}}_p(X; r)$. □

Proposition A.9 Let (X, d_X) be a finite metric space. Then the 0–dimensional persistence modules of $\mathrm{VR}_p(X; \cdot)$ and $\check{\mathrm{C}}_p(X; \cdot)$ are both equal to the 0–dimensional persistence module of the Vietoris–Rips simplicial complex filtration of the rescaled metric space $(X, (\frac{1}{2})^p d_X)$.

Proof From Lemma A.8, we know that all the bars for the 0–dimensional persistence module are born at 0. Let x be a point in (X, d), and let x' be a closest point to x in the finite metric space X. Then it suffices to show that δ_x and $\delta_{x'}$ will only be in the same connected component of $\mathrm{VR}_p((X, d); r)$ or $\check{\mathrm{C}}_p((X, d); r)$ for any $r > (\frac{1}{2})^p d_X(x, x')$.

Since the path $\gamma_t = (1-t)\delta_x + t\delta_{x'}$ has maximal diam_p and rad_p given by $(\frac{1}{2})^p d_X(x, x')$, we know δ_x and $\delta_{x'}$ will be in the same connected component of $\mathrm{VR}_p((X, d); r)$ or $\check{\mathrm{C}}_p((X, d); r)$ for any $r > (\frac{1}{2})^p d_X(x, x')$.

On the other hand, let (Y, d_Y) be the metric space $(\{x, x'\}, d_X|_{\{x, x'\}})$. Then the map $g_{x,x'} : (X, d) \to (Y, d_Y)$ sending x to x and all other points to x' is a 1–Lipschitz map. Also, $g_{x,x'}$ induces a continuous map $G_{x,x'} : \mathrm{VR}_p(X; r) \to \mathrm{VR}_p(Y; r)$ for any $r > 0$ via pushforward. Note that, for any $r \leq (\frac{1}{2})^p d_X(x, x')$, the images of δ_x and $\delta_{x'}$ are not in the same connected component of $\mathrm{VR}_p(Y; r)$. By the continuity of $G_{x,x'}$, the delta masses δ_x and $\delta_{x'}$ cannot be in the same connected component of $\mathrm{VR}_p(X; r)$ either. A similar argument works for $\check{\mathrm{C}}_p(X; r)$. □

A.6 Crushings and the homotopy type distance

A crushing is a particular type of deformation retraction that doesn't increase distances. In this section we consider the effects of a crushing applied to the metric space underlying a metric thickening.

Let X be a metric space and $A \subseteq X$ a subspace. Following [42], a *crushing* from X to A is defined as a distance-nonincreasing strong deformation retraction from X to A, that is, a continuous map

$H : X \times [0,1] \to X$ satisfying

(i) $H(x,1) = x$, $H(x,0) \in A$ and $H(a,t) = a$ if $a \in A$,

(ii) $d_X(H(x,t'), H(y,t')) \le d_X(H(x,t), H(y,t))$ whenever $t' \le t$.

When this happens, we say that X *can be crushed onto* A.

In [42, Proposition 2.2], Hausmann proves that, if X can be crushed onto A, then inclusion $\mathrm{VR}(A;r) \hookrightarrow \mathrm{VR}(X;r)$ of Vietoris–Rips simplicial complexes is a homotopy equivalence. A similar result is proven for Vietoris–Rips and Čech metric thickenings with $p = \infty$ in [2, Appendix B]. We study how more general metric thickenings behave with respect to crushings. We begin with some preliminaries:

Lemma A.10   *Let X be a complete separable metric space and let ϕ be an L–Lipschitz function that is absolutely bounded by a constant $C > 0$. Then, for any $\alpha, \beta \in \mathcal{P}_X$,*

$$\left| \int_X \phi(x)\,\alpha(dx) - \int_X \phi(x)\,\beta(dx) \right| \le (L + 2C) d_{\mathrm{LP}}(\alpha, \beta).$$

Proof   Let $\varepsilon = d_{\mathrm{LP}}(\alpha, \beta)$. Then by the expression of the Lévy–Prokhorov metric in [15, Theorem 3.1.5], there is a coupling between α and β by $\mu \in \mathcal{P}_{X \times X}$ such that

$$\mu(\{(x, x') \in X \times X \mid d_X(x, x') > \varepsilon\}) \le \varepsilon.$$

Let E be the set $\{(x, x') \in X \times X \mid d_X(x, x') > \varepsilon\}$. Then

$$
\begin{aligned}
\left| \int_X \phi(x)\,\alpha(dx) - \int_X \phi(x)\,\beta(dx) \right| &= \left| \iint_{X \times X} \phi(x) - \phi(x')\,\mu(dx \times dx') \right| \\
&\le \iint_E |\phi(x) - \phi(x')|\,\mu(dx \times dx') + \iint_{E^c} |\phi(x) - \phi(x')|\,\mu(dx \times dx') \\
&\le 2C\varepsilon + L \iint_{E^c} d_X(x, x')\,\mu(dx \times dx') = (L + 2C) d_{\mathrm{LP}}(\alpha, \beta). \qquad \square
\end{aligned}
$$

Proposition A.11   *Let X be a complete separable metric space such that there is a crushing from X onto a subset $A \subset X$. Then there is an induced deformation retraction from $\mathcal{P}_X$ onto $\mathcal{P}_A$.*

Proof   Let H be a crushing from X to A, so H is a continuous map from $X \times [0,1]$ to X. We use the notation $f_t(x)$ to denote the map $H(x,t) : X \to X$ for any fixed t in $[0,1]$. Then we can define a map $\tilde{H} : \mathcal{P}_X \times [0,1] \to \mathcal{P}_X$ via

$$\tilde{H}(\alpha, t) = (f_t)_\sharp(\alpha).$$

For continuity, let (α_n, t_n) be a sequence that converges to $(\alpha_\infty, t_\infty)$. Then, for any bounded continuous function $\gamma(x)$ on X, we have

$$\int_X \gamma(x)(f_{t_n})_\sharp(\alpha_n)(dx) = \int_X \gamma \circ f_{t_n}(x)\,\alpha_n(dx).$$

Without loss of generality, we may assume γ is 1–Lipschitz and absolutely bounded by $C > 0$. Therefore, every $\gamma \circ f_{t_n}(x)$ is 1–Lipschitz and bounded by C for any n. We use the notation $I_{i,j}$ to denote $\int_X \gamma \circ f_{t_i}(x)\,\alpha_j(dx)$. Then $|I_{i,j}|$ is uniformly bounded by C. Lemma A.10 then shows that, for any i,

$$|I_{i,j} - I_{i,\infty}| \le (1 + 2C) d_{\mathrm{LP}}(\alpha_j, \alpha_\infty).$$

The uniform bound on $|I_{i,\infty}|$ implies that $I_{i,\infty}$ converges to $I_{\infty,\infty}$. For any $\varepsilon > 0$, we can find an N such that, for any $n > N$, $|I_{n,\infty} - I_{\infty,\infty}| \leq \frac{1}{2}\varepsilon$ and

$$d_{\mathrm{LP}}(\alpha_n, \alpha_\infty) \leq \frac{\varepsilon}{2(1+2C)}.$$

Then

$$|I_{n,n} - I_{\infty,\infty}| \leq |I_{n,n} - I_{n,\infty}| + |I_{n,\infty} - I_{\infty,\infty}| \leq \varepsilon.$$

This shows that $I_{i,i}$ converges to $I_{\infty,\infty}$, and therefore $\tilde{H}$ is continuous. As H satisfies $H(x,1) = x$, $H(x,0) = f_0(x) \in A$ and $H(a,t) = f_t(a) = a$ if $a \in A$, we get $\tilde{H}(\alpha, 1) = (\mathrm{id}_{\mathcal{P}_X})_\sharp(\alpha) = \alpha$, $\tilde{H}(\alpha, 0) = (f_0)_\sharp(\alpha) \in \mathcal{P}_A$ for any $\alpha \in \mathcal{P}_X$, and $\tilde{H}(\beta, t) = \beta$ for $\beta \in \mathcal{P}_A$. Therefore $\tilde{H}$ is a indeed a strong deformation retraction from $\mathcal{P}_X$ to $\mathcal{P}_A$. □

In the spirit of [2, Lemma B.1], we can apply the above deformation retraction to the sublevel set filtrations of a set of invariants that includes $\mathfrak{i}_{q,p}$ and rad_p.

Theorem G *Let $\mathfrak{i}$ be an invariant such that, for any metric spaces X and Y and any 1–Lipschitz map $f : X \to Y$, the induced map on $\mathcal{P}_X$ does not increase the values of $\mathfrak{i}$. More precisely, for any $\alpha \in \mathcal{P}_X$, we require $\mathfrak{i}^Y(f_\sharp(\alpha)) \leq \mathfrak{i}^X(\alpha)$. Then, for any complete separable metric space X and any subset A such that X can crushed onto A, we have*

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_A, \mathfrak{i}^A)) = 0.$$

Proof Let H be the crushing from X onto A and let f be $H(x,0)$. Then as both f and the inclusion $\iota : A \to X$ are 1–Lipschitz maps, the above condition on the invariant $\mathfrak{i}$ implies

- $f_\sharp$ is a 0–map from $(X, \mathfrak{i}^X)$ to $(A, \mathfrak{i}^A)$, and
- $\iota_\sharp$ is a 0–map from $(A, \mathfrak{i}^A)$ to $(X, \mathfrak{i}^X)$.

Note that $f_\sharp \circ \iota_\sharp$ is the identify map on $\mathcal{P}_A$ and $\iota_\sharp \circ f_\sharp$ is 0–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathfrak{i}^X, \mathfrak{i}^X)$. Therefore $d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_A, \mathfrak{i}^A)) = 0$. □

Remark A.12 The crushing result could be leveraged to analyze the persistent homology of a space using the persistent homology of embedded submanifolds; see [74].

A.7 Ambient filtrations from Lipschitz invariants

We now show that the main properties for intrinsic p–Čech metric thickenings also hold for ambient p–Čech metric thickenings.

Let M be a metric space and let X be a subset of M. Then any function $\mathfrak{i}^M$ naturally restricts to $\mathcal{P}_X$ and induces a filtration. We have the following stability result, given that $\mathfrak{i}^M$ is C–Lipschitz with respect to $d_{\mathrm{W},\infty}^M$ for some $C > 0$.

Theorem H *Let M be a metric space and let $\mathfrak{i}^M$ be a C–Lipschitz function on $\mathcal{P}_X$ with respect to $d_{\mathrm{W},\infty}^M$. Then, for any two totally bounded subsets X and Y in M,*

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^M), (\mathcal{P}_Y, ^M)) \leq Cd_{\mathrm{H}}^M(X,Y) \quad \text{and} \quad d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^M), (\mathcal{P}_Y^{\mathrm{fin}}, ^M)) \leq Cd_{\mathrm{H}}^M(X,Y).$$

Here d_{H}^M is the Hausdorff distance in M.

Proof Overall, the proof follows a construction similar to that in Lemma 5.16 in the setting of Hausdorff distance. For $\eta > 2\,d_{\mathrm{H}}^M(X, Y)$ and $\delta > 0$, we fix finite δ–nets $U \subset X$ of X and $V \subset Y$ of Y. By the triangle inequality, $d_{\mathrm{H}}(U, V) < \frac{1}{2}\eta + 2\delta$. For any point $u \in U$ there is a point v in V with $d_M(u, v) < \frac{1}{2}\eta + 2\delta$. Through this construction, there are maps $\varphi: U \to V$ and $\psi: V \to U$ with

- $d_M(u, \varphi(u)) < \frac{1}{2}\eta + 2\delta$ for any $u \in U$,
- $d_M(v, \psi(v)) < \frac{1}{2}\eta + 2\delta$ for any $v \in V$,
- $\max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi, \psi)) \le \eta + 4\delta$.

We use the notations $\widehat{\Phi}$, $\widehat{\Psi}$, H_t^X and H_t^Y as in Lemma 5.16. Let α be a measure in $\mathcal{P}_X$ and β a measure in $\mathcal{P}_Y$. The last item implies that the following bound from Lemma 5.16 still holds here: we have $d_{\mathrm{W},\infty}^M(H_t^X(\alpha), \alpha) < \eta + 6\delta$ and $d_{\mathrm{W},\infty}^M(H_t^Y(\beta), \beta) < \eta + 6\delta$. Similar to the proof of Theorem A, it suffices to show that

- $\widehat{\Phi}: \mathcal{P}_X \to \mathcal{P}_Y$ is a $\left(\frac{1}{2}\eta + 3\delta\right)C$–map from $(\mathcal{P}_X, \mathfrak{i}^M)$ to $(\mathcal{P}_Y, \mathfrak{i}^M)$,
- $\widehat{\Psi}: \mathcal{P}_Y \to \mathcal{P}_X$ is a $\left(\frac{1}{2}\eta + 3\delta\right)C$–map from $(\mathcal{P}_Y, \mathfrak{i}^M)$ to $(\mathcal{P}_X, \mathfrak{i}^M)$,
- $\widehat{\Psi} \circ \widehat{\Phi}: \mathcal{P}_X \to \mathcal{P}_X$ is $(\eta + 6\delta)C$–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathfrak{i}^M, \mathfrak{i}^M)$,
- $\widehat{\Phi} \circ \widehat{\Psi}: \mathcal{P}_X \to \mathcal{P}_X$ is $(\eta + 6\delta)C$–homotopic to $\mathrm{id}_{\mathcal{P}_Y}$ with respect to $(\mathfrak{i}^M, \mathfrak{i}^M)$.

We will only show the first and the third items; the rest can be proved similarly. For the first item, using the fact that $\mathfrak{i}^M$ is C–Lipschitz with respect to $d_{\mathrm{W},\infty}^M$ and the estimate in Lemma 5.15, we get

$$
\begin{aligned}
\mathfrak{i}^M(\widehat{\Phi})(\alpha) = \mathfrak{i}^M\left(\varphi_\sharp(\Phi_U(\alpha))\right) &\le \mathfrak{i}^M(\Phi_U(\alpha)) + d_{\mathrm{W},\infty}^M\left(\varphi_\sharp(\Phi_U(\alpha)), \Phi_U(\alpha)\right)C \\
&\le \mathfrak{i}^M(\Phi_U(\alpha)) + \left(\tfrac{1}{2}\eta + 2\delta\right)C \\
&\le \mathfrak{i}^M(\alpha) + d_{\mathrm{W},\infty}^M(\alpha, \Phi_U(\alpha))C + \left(\tfrac{1}{2}\eta + 2\delta\right)C \\
&\le \mathfrak{i}^M(\alpha) + \left(\tfrac{1}{2}\eta + 3\delta\right)C.
\end{aligned}
$$

For the third item, by the inequality $d_{\mathrm{W},\infty}^M(H_t^X(\alpha), \alpha) < \eta + 6\delta$, we have

$$
\mathfrak{i}^M(H_t^X(\alpha)) \le \mathfrak{i}^M(\alpha) + (\eta + 6\delta)C. \qquad \square
$$

An interesting case is the ambient p–radius, which leads to p–ambient Čech filtrations. Let X be a bounded subset in a metric space M. For any $\alpha \in \mathcal{P}_X$, we define the *p–ambient radius* of α to be

$$
\mathrm{rad}_p^M(\alpha) := \inf_{m \in M} F_{\alpha, p}(m) = \inf_{m \in M} d_{\mathrm{W}, p}^X(\delta_m, \alpha).
$$

Definition A.13 (*p–ambient Čech filtration*) Let $X \subseteq M$ be metric spaces. For each $r > 0$ and $p \in [1, \infty]$, let the *p–ambient Čech metric thickening at scale r* be

$$
\check{\mathrm{C}}_p(X, M; r) := \{\alpha \in \mathcal{P}_X \mid \mathrm{rad}_p^M(\alpha) < r\}.
$$

Similarly, the *p–ambient Čech metric thickening at scale r with finite support* is defined as

$$
\check{\mathrm{C}}_p^{\mathrm{fin}}(X, M; r) := \{\alpha \in \mathcal{P}_X^{\mathrm{fin}} \mid \mathrm{rad}_p^M(\alpha) < r\}.
$$

Therefore, as a special instance of Theorem H:

Theorem I *Let X and Y be two totally bounded spaces sitting inside a metric space M. Then*

$$d_I(H_k \circ \check{C}_p(X, M; \cdot), H_k \circ \check{C}_p(Y, M; \cdot)) \leq d_{HT}((\mathcal{P}_X, \mathrm{rad}_p^M), (\mathcal{P}_Y, {}^M)) \leq d_H^M(X, Y),$$

$$d_I(H_k \circ \check{C}_p^{\mathrm{fin}}(X, M; \cdot), H_k \circ \check{C}_p^{\mathrm{fin}}(Y, M; \cdot)) \leq d_{HT}((\mathcal{P}_X^{\mathrm{fin}}, \mathrm{rad}_p^M), (\mathcal{P}_Y^{\mathrm{fin}}, {}^M)) \leq d_H^M(X, Y).$$

Proof According to Lemma 4.9, rad_p^M is 1–Lipschitz with respect to $d_{W,\infty}^M$. We apply Theorem H to get the inequality on the right. The inequality on the left follows from Lemma 2.9. □

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