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We generalize the notion of moment-angle manifold over a simple convex polytope to an arbitrary nice manifold with corners. For a nice manifold with corners Q, we first compute the stable decomposition of the moment-angle manifold $\mathcal{Z}_Q$ via a construction called rim-cubicalization of Q. From this, we derive a formula to compute the integral cohomology group of $\mathcal{Z}_Q$ via the strata of Q. This generalizes the Hochster's formula for the moment-angle manifold over a simple convex polytope. Moreover, we obtain a description of the integral cohomology ring of $\mathcal{Z}_Q$ using the idea of partial diagonal maps. In addition, we define the notion of polyhedral product of a sequence of based CW–complexes over Q and obtain similar results for these spaces as we do for $\mathcal{Z}_Q$. Using this general construction, we can compute the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to its canonical torus action from the Davis–Januszkiewicz space of Q. The result leads to the definition of a new notion called the topological face ring of Q, which generalizes the notion of face ring of a simple polytope. Moreover, the topological face ring of Q computes the equivariant cohomology of all locally standard torus actions with Q as the orbit space when the corresponding principal torus bundle over Q is trivial. Meanwhile, we obtain some parallel results for the real moment-angle manifold $\mathbb{R}\mathcal{Z}_Q$ over Q as well.

1 Introduction

The construction of a moment-angle manifold over a simple polytope is first introduced by Davis and Januszkiewicz in [17]. Suppose P is a simple (convex) polytope with *m facets* (codimension-one faces). A convex polytope in a Euclidean space is called *simple* if every codimension-k face is the intersection of exactly k facets of the polytope. The moment-angle manifold $\mathcal{Z}_P$ over P is a closed connected manifold with an effective action by the compact torus $T^m = (S^1)^m$ whose orbit space is P. It is shown in [17] that many important topological invariants of $\mathcal{Z}_P$ can be computed easily from the combinatorial structure of P. These manifolds play an important role in the research of toric topology. The reader is referred to Buchstaber and Panov [9; 10] for more discussions on the topological and geometrical aspects of moment-angle manifolds.

The notion of moment-angle manifold over a simple convex polytope has been generalized in many different ways. For example, Davis and Januszkiewicz [17] define a class of topological spaces now

called *moment-angle complexes* — named by Buchstaber and Panov in [8] — where the simple polytope is replaced by a simple polyhedral complex. Later, Lü and Panov [26] defined the notion of moment-angle complex of a *simplicial poset*. In addition, Ayzenberg and Buchstaber [1] defined the notion of moment-angle spaces over arbitrary convex polytopes (not necessarily simple). Note that in all these generalizations, the orbit spaces of the canonical torus actions are all contractible. Yet an even wider class of spaces called *generalized moment-angle complexes* or *polyhedral products* over simplicial complexes were introduced by Bahri, Bendersky, Cohen and Gitler in [3], which has become the major subject in the homotopy-theoretic study of toric topology.

In this paper, we generalize the construction of moment-angle manifolds by replacing the simple polytope P by a nice manifold with corners Q which is not necessarily contractible. Such a generalization has been considered by Poddar and Sarkar [29] for polytopes with simple holes.

A motive for the study of this generalized construction is to compute the equivariant cohomology ring of locally standard torus actions. Recall that an action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally modeled on the standard representation of T^n on $\mathbb{C}^n$. Then the orbit space $Q = M/T^n$ is a manifold with corners. Conversely, every manifold with a locally standard T^n–action and with Q as the orbit space is equivariantly homeomorphic to the quotient construction $Y/\sim$, where Y is a principal T^n–bundle over Q and $\sim$ is an equivalence relation determined by the characteristic function on Q; see Yoshida [35]. Generally speaking, it is difficult to compute the equivariant cohomology ring of M from the corresponding principal bundle Y and the characteristic function on Q. But we will see in Corollary 5.5 that when Y is the trivial T^n–bundle over Q, the equivariant cohomology ring of M can be computed from the strata of Q directly. Examples of such kind include many toric origami manifolds — see Ayzenberg, Masuda, Park and Zeng [2], Cannas da Silva, Guillemin and Pires [12] and Holm and Pires [22] — with coorientable folding hypersurface where the faces of the orbit spaces may be nonacyclic.

Recall that an *n–dimensional manifold with corners* Q is a Hausdorff space with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the transitional functions are homeomorphisms which preserve the codimension of each point. Here the *codimension $c(x)$* of a point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n_{\geq 0}$ is the number of x_i which are 0. So we have a well-defined map $c \colon Q \to \mathbb{Z}_{\geq 0}$, where $c(q)$ is the codimension of a point $q \in Q$. In particular, the interior Q° of Q consists of points of codimension 0, ie $Q^\circ = c^{-1}(0)$.

Suppose Q is an n–dimensional manifold with corners with $\partial Q \neq \varnothing$. An *open face* of Q of codimension k is a connected component of $c^{-1}(k)$. A (closed) *face* is the closure of an open face. A face of codimension one is called a *facet* of Q. Note that a face of codimension zero in Q is just a connected component of Q.

A manifold with corners Q is said to be *nice* if either its boundary ∂Q is empty or ∂Q is nonempty and any codimension-k face of Q is a component of the intersection of k different facets in Q.

Let Q be a nice n–manifold with corners. Let $\mathscr{F}(Q) = \{F_1, \ldots, F_m\}$ be the set of facets of Q. For any subset $J \subseteq [m] = \{1, \ldots, m\}$, let

$$F_J = \bigcup_{j \in J} F_j, \quad F_\varnothing = \varnothing, \quad F_{\cap J} = \bigcap_{j \in J} F_j, \quad F_{\cap \varnothing} = Q.$$

It is clear that, for all $J \subseteq J' \subseteq [m]$,

$$F_J \subseteq F_{J'}, \quad F_{\cap J'} \subseteq F_{\cap J}, \quad F_{\cap J} \subseteq F_J.$$

Let $\lambda \colon \mathscr{F}(Q) \to \mathbb{Z}^m$ be a map such that $\{\lambda(F_1), \ldots, \lambda(F_m)\}$ is a unimodular basis of $\mathbb{Z}^m \subset \mathbb{R}^m$. Since $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$, we can identify the m–torus $(S^1)^m = \mathbb{R}^m/\mathbb{Z}^m$. The *moment-angle manifold over Q* is defined by

$$(1) \qquad\qquad \mathscr{Z}_Q = Q \times (S^1)^m / \sim,$$

where $(x, g) \sim (x', g')$ if and only if $x = x'$ and $g^{-1}g' \in \mathbb{T}_x^\lambda$ where $\mathbb{T}_x^\lambda$ is the subtorus of $(S^1)^m$ determined by the linear subspace of $\mathbb{R}^m$ spanned by the set $\{\lambda(F_j) \mid x \in F_j\}$. There is a canonical action of $(S^1)^m$ on $\mathscr{Z}_Q$ defined by

$$(2) \qquad\qquad g' \cdot [(x, g)] = [(x, g'g)] \quad \text{for } x \in Q \text{ and } g, g' \in (S^1)^m.$$

Since the manifold with corners Q is nice and λ is unimodular, it is easy to see from the above definition that $\mathscr{Z}_Q$ is a manifold.

Convention In the rest of this paper, we assume that any nice manifold with corners Q can be equipped with a CW–complex structure such that every face of Q is a subcomplex. In addition, we assume that Q has only finitely many faces. Note that a compact smooth nice manifold with corners always satisfies these two conditions since it is triangulable; see Johnson [25]. But in general we do not require Q to be compact or smooth. We do not assume Q to be connected either.

Similarly to the stable decomposition of (generalized) moment-angle complexes obtained in [3], we have the following stable decomposition of $\mathscr{Z}_Q$.

Theorem 1.1 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. There is a homotopy equivalence*

$$(3) \qquad\qquad \mathbf{\Sigma}(\mathscr{Z}_Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}^{|J|+1}(Q/F_J),$$

where $\bigvee$ denotes the wedge sum and $\mathbf{\Sigma}$ denotes the reduced suspension.

Here we will not explicitly write down the basepoints for our spaces unless it is necessary to do so.

Corollary 1.2 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. The integral (reduced) cohomology group of $\mathscr{Z}_Q$ is given by*

$$(4) \quad H^p(\mathscr{Z}_Q) \cong \bigoplus_{J \subseteq [m]} H^{p-|J|}(Q, F_J), \quad \tilde{H}^p(\mathscr{Z}_Q) \cong \bigoplus_{J \subseteq [m]} \tilde{H}^{p-|J|}(Q/F_J) \quad \text{for all } p \in \mathbb{Z}.$$

Note that when $J = \varnothing$, $H^*(Q, F_\varnothing) = H^*(Q, \varnothing) = H^*(Q) \cong \widetilde{H}^*(Q) \oplus \mathbb{Z}$.

The term "cohomology" of a space X, denoted by $H^*(X)$, in this paper always means singular cohomology with integral coefficients if not specified otherwise.

When Q is acyclic (ie $\widetilde{H}^*(Q) = 0$), we have $H^p(Q, F_J) \cong \widetilde{H}^{p-1}(F_J)$ by a cohomology long exact sequence for the pair (Q, F_J). So, in this case,

$$H^p(\mathscr{Z}_Q) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^{p-|J|-1}(F_J) \quad \text{for all } p \in \mathbb{Z}.$$

This recovers Hochster's formula for the moment-angle manifold over a simple polytope in [10, Theorem 3.2.9]; see also [10, Proposition 3.2.11].

Remark 1.3 There is an analogue of $\mathscr{Z}_Q$ by replacing the group $(S^1)^m$ by $(\mathbb{Z}_2)^m$. The counterpart in the $(\mathbb{Z}_2)^m$ construction, denoted by $\mathbb{R}\mathscr{Z}_Q$, is a special case of the basic construction of Davis [16, Chapter 5] for a mirror space along with a Coxeter system. A formula parallel to Corollary 1.2 for computing the integral cohomology group of $\mathbb{R}\mathscr{Z}_Q$ is contained in Davis [15, Theorem A]; see also [16, Chapter 8]. We call $\mathbb{R}\mathscr{Z}_Q$ the *real moment-angle manifold over* Q.

Given a nice manifold with corners Q with facets $F_1, \ldots, F_m$, define

(5) $$\mathscr{R}_Q^* := \bigoplus_{J \subseteq [m]} H^*(Q, F_J).$$

There is a graded ring structure $\mathbb{U}$ on $\mathscr{R}_Q^*$ defined as follows:

- If $J \cap J' \neq \varnothing$, then $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\mathbb{U}} H^*(Q, F_{J \cup J'})$ is trivial.
- If $J \cap J' = \varnothing$, then $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\mathbb{U}} H^*(Q, F_{J \cup J'})$ is the relative cup product $\cup$; see Hatcher [20, page 209].

We can prove the following theorem via the above stable decomposition of $\mathscr{Z}_Q$.

Theorem 1.4 *Let Q be a nice manifold with corners with m facets $F_1, \ldots, F_m$. There exists a ring isomorphism (up to a sign) from $(\mathscr{R}_Q^*, \mathbb{U})$ to the integral cohomology ring of $\mathscr{Z}_Q$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_J)$ up by $|J|$ for every $J \subseteq [m]$.*

It is indicated in [10, Exercise 3.2.14] that Theorem 1.4 holds for any simple polytope. Moreover, we can generalize Theorem 1.4 to describe the cohomology ring of the polyhedral product of any $(\mathbb{D}, \mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j)\}_{j=1}^m$ over Q (see Theorem 4.8). In particular, we have the following result for $\mathbb{R}\mathscr{Z}_Q$.

Theorem 1.5 (Corollary 4.10) *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Then*

$$\Sigma(\mathbb{R}\mathscr{Z}_Q) \simeq \bigvee_{J \subseteq [m]} \Sigma(Q/F_J), \quad H^p(\mathbb{R}\mathscr{Z}_Q) \cong \bigoplus_{J \subseteq [m]} H^p(Q, F_J), \quad \text{for all } p \in \mathbb{Z}.$$

Moreover, *the integral cohomology ring of $\mathbb{R}\mathcal{Z}_Q$ is isomorphic as a graded ring to the ring $(\mathcal{R}_Q^*, \cup)$ where $\cup$ is the relative cup product*

$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\cup} H^*(Q, F_{J \cup J'}) \quad \text{for all } J, J' \subseteq [m].$$

We can describe the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to the canonical action of $(S^1)^m$ as follows.

Let $\boldsymbol{k}$ denote a commutative ring with a unit. For any $J \subseteq [m]$, let $R_{\boldsymbol{k}}^J$ be the subring of the polynomial ring $\boldsymbol{k}[x_1, \ldots, x_m]$ defined by

$$(6) \qquad R_{\boldsymbol{k}}^J := \begin{cases} \operatorname{span}_{\boldsymbol{k}} \{x_{j_1}^{n_1} \cdots x_{j_s}^{n_s} \mid n_1 > 0, \ldots, n_s > 0\} & \text{if } J = \{j_1, \ldots, j_s\} \neq \varnothing, \\ \boldsymbol{k} & \text{if } J = \varnothing. \end{cases}$$

We can multiply $f(x) \in R_{\boldsymbol{k}}^J$ and $f'(x) \in R_{\boldsymbol{k}}^{J'}$ in $\boldsymbol{k}[x_1, \ldots, x_m]$ and obtain an element $f(x)f'(x) \in R_{\boldsymbol{k}}^{J \cup J'}$.

Definition 1.6 (topological face ring) Let Q be a nice manifold with corners with m facets $F_1, \ldots, F_m$. For any coefficient ring $\boldsymbol{k}$, the *topological face ring of Q* over $\boldsymbol{k}$ is defined to be

$$(7) \qquad \boldsymbol{k}\langle Q \rangle := \bigoplus_{J \subseteq [m]} H^*(F_{\cap J}; \boldsymbol{k}) \otimes R_{\boldsymbol{k}}^J.$$

Here if $F_{\cap J} = \varnothing$, we use the convention $H^*(\varnothing; \boldsymbol{k}) = \{0\}$.

For any $J, J' \subseteq [m]$, the product $\star$ on $\boldsymbol{k}\langle Q \rangle$,

$$(H^*(F_{\cap J}; \boldsymbol{k}) \otimes R_{\boldsymbol{k}}^J) \otimes (H^*(F_{\cap J'}; \boldsymbol{k}) \otimes R_{\boldsymbol{k}}^{J'}) \xrightarrow{\star} (H^*(F_{\cap (J \cup J')}; \boldsymbol{k}) \otimes R_{\boldsymbol{k}}^{J \cup J'})$$

is defined, for $\phi \in H^*(F_{\cap J}; \boldsymbol{k})$, $f(x) \in R_{\boldsymbol{k}}^J$, $\phi' \in H^*(F_{\cap J'}; \boldsymbol{k})$ and $f'(x) \in R_{\boldsymbol{k}}^{J'}$, by

$$(8) \qquad (\phi \otimes f(x)) \star (\phi' \otimes f'(x)) := (\kappa_{J \cup J', J}^*(\phi) \cup \kappa_{J \cup J', J'}^*(\phi')) \otimes f(x)f'(x),$$

where $\kappa_{I', I} : F_{\cap I'} \to F_{\cap I}$ is the inclusion map for any $I \subseteq I' \subseteq [m]$ and $\kappa_{I', I}^* : H^*(F_{\cap I}; \boldsymbol{k}) \to H^*(F_{\cap I'}; \boldsymbol{k})$ is the induced homomorphism on cohomology.

In addition, we can consider $\boldsymbol{k}\langle Q \rangle$ as a graded ring if we choose a degree for every indeterminate x_j in $\boldsymbol{k}[x_1, \ldots, x_m]$ and define

$$\deg(\phi \otimes (x_{j_1}^{n_1} \cdots x_{j_s}^{n_s})) = \deg(\phi) + n_1 \deg(x_{j_1}) + \cdots + n_s \deg(x_{j_s}).$$

Theorem 1.7 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Then the equivariant cohomology ring of $\mathcal{Z}_Q$ (or $\mathbb{R}\mathcal{Z}_Q$) with $\mathbb{Z}$–coefficients (or $\mathbb{Z}_2$–coefficients) with respect to the canonical $(S^1)^m$–action (or $(\mathbb{Z}_2)^m$–action) is isomorphic as a graded ring to the topological face ring $\mathbb{Z}\langle Q \rangle$ (or $\mathbb{Z}_2\langle Q \rangle$) of Q by choosing $\deg(x_j) = 2$ (or $\deg(x_j) = 1$) for all $1 \leq j \leq m$.*

Moreover, the natural $H^*(BT^m)$–module structure on the integral equivariant cohomology ring $H_{T^m}^*(\mathcal{Z}_Q)$ is described in (52) where $T^m = (S^1)^m$.

Remark 1.8 A calculation of the equivariant cohomology group of $\mathscr{L}_Q$ with $\mathbb{Z}$–coefficients was announced earlier by T Januszkiewicz in a talk in 2020 [24]. The formula given in Januszkiewicz's talk is equivalent to our $\mathbb{Z}\langle Q \rangle$. But the ring structure of the equivariant cohomology of $\mathscr{L}_Q$ was not described in [24].

For a nice manifold with corners Q, there are two other notions which reflect the stratification of Q. One is the *face poset* of Q which is the set of all faces of Q ordered by inclusion, denoted by $\mathscr{S}_Q$ (note that each connected component of Q is also a face). The other one is the *nerve simplicial complex* of the covering of ∂Q by its facets, denoted by K_Q. The *face ring* (or *Stanley–Reisner ring*) of a simplicial complex is an important tool to study combinatorial objects in algebraic combinatorics and combinatorial commutative algebra; see [28; 30].

When Q is a simple polytope, all faces of Q, including Q itself, and all their intersections are acyclic. Then it is easy to see that the topological face ring of Q is isomorphic to the face ring of K_Q (see Example 5.2). But in general, the topological face ring of Q encodes more topological information of Q than the face ring of K_Q.

There is another way to think of the topological face ring $\boldsymbol{k}\langle Q \rangle$. Let

$$\mathscr{R}^*_{\cap Q, \boldsymbol{k}} := \bigoplus_{J \subseteq [m]} H^*(F_{\cap J}; \boldsymbol{k}),$$

where product $*$ on $\mathscr{R}^*_{\cap Q, \boldsymbol{k}}$ is defined, for any $\phi \in H^*(F_{\cap J}; \boldsymbol{k})$ and $\phi' \in H^*(F_{\cap J'}; \boldsymbol{k})$, by

$$\phi * \phi' := \kappa^*_{J \cup J', J}(\phi) \cup \kappa^*_{J \cup J', J'}(\phi') \in H^*(F_{\cap (J \cup J')}; \boldsymbol{k}).$$

Moreover,

$$\boldsymbol{k}[x_1, \ldots, x_m] = \bigoplus_{J \subseteq [m]} R^J_{\boldsymbol{k}},$$

so we can think of both $\mathscr{R}^*_{\cap Q, \boldsymbol{k}}$ and $\boldsymbol{k}[x_1, \ldots, x_m]$ as $2^{[m]}$–graded rings where $2^{[m]} = \{J \subseteq [m]\}$ is the power set of $[m]$. Then the topological face ring $\boldsymbol{k}\langle Q \rangle$ can be thought of as the Segre product of $\mathscr{R}^*_{\cap Q, \boldsymbol{k}}$ and $\boldsymbol{k}[x_1, \ldots, x_m]$ with respect to their $2^{[m]}$–gradings. By definition, the *Segre product* of two rings R and S graded by a common semigroup $\mathscr{A}$ — using the notation of Hoa [21] — is

$$R \underline{\otimes} S = \bigoplus_{\boldsymbol{a} \in \mathscr{A}} R_{\boldsymbol{a}} \otimes S_{\boldsymbol{a}}.$$

So $R \underline{\otimes} S$ is a subring of the tensor product of R and S (as graded rings). The Segre product of two graded rings (or modules) is studied in algebraic geometry and commutative algebra; see Chow [13] and Fröberg and Hoa [19; 21], for example.

Here we can think of $2^{[m]}$ as a semigroup where the product of two subsets of $[m]$ is just their union. Then by this notation, we can write

$$\boldsymbol{k}\langle Q \rangle = \mathscr{R}^*_{\cap Q, \boldsymbol{k}} \underline{\otimes} \boldsymbol{k}[x_1, \ldots, x_m].$$

From this form, we see that $\boldsymbol{k}\langle Q \rangle$ is essentially determined by $\mathscr{R}^*_{\cap Q, \boldsymbol{k}}$.

The paper is organized as follows. In Section 2, we first construct an embedding of Q into $Q \times [0, 1]^m$ which is analogous to the embedding of a simple polytope into a cube. This induces an embedding of $\mathscr{Z}_Q$ into $Q \times (D^2)^m$, from which we can do the stable decomposition of $\mathscr{Z}_Q$ and give a proof of Theorem 1.1. Our argument proceeds along the same line as the argument given in [3, Section 6] but with some extra ingredients. In fact, we will not do the stable decomposition of $\mathscr{Z}_Q$ directly, but the stable decomposition of the disjoint union of $\mathscr{Z}_Q$ with a point. In Section 3, we obtain a description of the product structure of the cohomology of $\mathscr{Z}_Q$ using the stable decomposition of $\mathscr{Z}_Q$ and the partial diagonal map introduced in [4]. From this we give a proof of Theorem 1.4. In Section 4, we define the notion of polyhedral product of a sequence of based CW–complexes over a nice manifold with corners Q and obtain some results parallel to $\mathscr{Z}_Q$ for these spaces. In particular, we obtain a description of the integral cohomology ring of real moment-angle manifold $\mathbb{R}\mathscr{Z}_Q$ (see Corollary 4.10). In Section 5, we compute the equivariant cohomology ring of $\mathscr{Z}_Q$ and prove Theorem 1.7. In Section 6, we discuss more generalizations of the construction of $\mathscr{Z}_Q$ and extend our main theorems to some wider settings.

2 Stable decomposition of $\mathscr{Z}_Q$

Let Q be a nice manifold with corners with m facets. To obtain the stable decomposition of $\mathscr{Z}_Q$, we first construct a special embedding of Q into $Q \times [0, 1]^m$, called the rim-cubicalization of Q. This construction can be thought of as a generalization of the embedding of a simple polytope with m facets into $[0, 1]^m$ defined in [9, Chapter 4].

2.1 Rim-cubicalization of Q in $Q \times [0, 1]^m$

Let $F_1, \ldots, F_m$ be all the facets of Q. For a face f of Q, let I_f be the subset of $[m]$ called the *strata index* of f,
$$I_f = \{i \in [m] \mid f \subseteq F_i\} \subseteq [m].$$
Then we define a subset $\hat{f}$ of $Q \times [0, 1]^m$ associated to f as follows. We write
$$[0, 1]^m = \prod_{j \in [m]} [0, 1]_{(j)}$$
and define
$$\hat{f} = f \times \prod_{j \in I_f} [0, 1]_{(j)} \times \prod_{j \in [m] \setminus I_f} 1_{(j)}.$$
In particular,
$$\widehat{F_i} = F_i \times [0, 1]_{(i)} \times \prod_{j \in [m] \setminus \{i\}} 1_{(j)}, \quad 1 \leq i \leq m.$$
Let $\mathscr{S}_Q$ be the face poset of Q and define
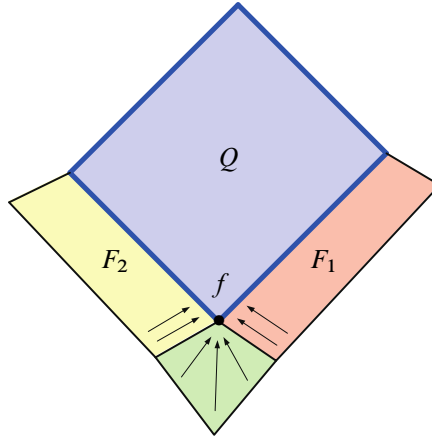$$(9) \qquad \hat{Q} = \bigcup_{f \in \mathscr{S}_Q} \hat{f} \subseteq Q \times [0, 1]^m.$$

Figure 1: Rim-cubicalization of Q in $Q \times [0, 1]^m$.

It is easy to see that $\widehat{Q}$ is a nice manifold with corners whose facets are $\widehat{F}_1, \ldots, \widehat{F}_m$. If we identify Q with the subspace $Q \times \prod_{j \in [m]} 1_{(j)} \subseteq \widehat{Q}$, then we can think of $\widehat{Q}$ as inductively gluing the product of all codimension-k strata of Q with a k–cube to ∂Q (see Figure 1),

$$Q \xleftarrow{\text{glue}} F_j \times [0, 1] \xleftarrow{\text{glue}} \cdots \xleftarrow{\text{glue}} f \times [0, 1]^{|I_f|} \xleftarrow{\text{glue}} \cdots .$$

Due to this viewpoint, we call $\widehat{Q}$ the *rim-cubicalization* of Q in $Q \times [0, 1]^m$.

Lemma 2.1 $\widehat{Q}$ *is homeomorphic to* Q *as a manifold with corners.*

Proof For any face f of Q and $0 \le t \le 1$, let

$$\hat{f}(t) = f \times \prod_{j \in I_f} [t, 1]_{(j)} \times \prod_{j \in [m] \setminus I_f} 1_{(j)}, \quad \widehat{Q}(t) = \bigcup_{f \in \mathscr{S}_Q} \hat{f}(t) \subseteq Q \times [t, 1]^m.$$

Then $\widehat{Q}(t)$ determines an isotopy (see Figure 1) from $\widehat{Q}(0) = \widehat{Q}$ to

$$\widehat{Q}(1) = \bigcup_{f \in \mathscr{S}_Q} \left(f \times \prod_{j \in [m]} 1_{(j)} \right) = Q \times \prod_{j \in [m]} 1_{(j)} \cong Q.$$

Around a codimension-k stratum of Q, the isotopy $\widehat{Q}(t)$ is locally equivalent to the standard isotopy from $C_k^n(-1)$ to $C_k^n(0)$ defined in Example 2.2.

Clearly, the isotopy $\widehat{Q}(t)$ sends each face $\hat{f}$ of $\widehat{Q}$ to $f \times \prod_{j \in [m]} 1_{(j)}$. So, under the identification of $Q \times \prod_{j \in [m]} 1_{(j)}$ with Q, $\widehat{Q}$ is homeomorphic to Q as a manifold with corners. $\square$

Example 2.2 Let $C_k^n(0)$ and $C_k^n(-1)$ be two subspaces of $\mathbb{R}^n$ defined by

$$C_k^n(0) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_1, \ldots, x_k < 1, -1 < x_{k+1}, \ldots, x_n < 1\},$$
$$C_k^n(-1) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid -1 \le x_1, \ldots, x_k < 1, -1 < x_{k+1}, \ldots, x_n < 1\}.$$
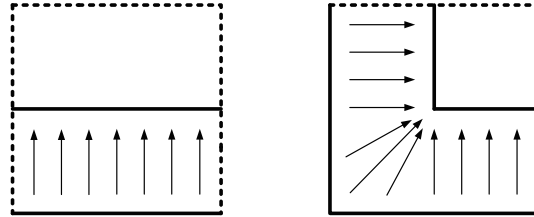
Figure 2: Isotopy from $C_k^n(-1)$ to $C_k^n(0)$.

There is a strong deformation retraction from $C_k^n(-1)$ to $C_k^n(0)$ defined by

$$H(x_1, \ldots, x_n, t) = (\delta_{x_1}(t) \cdot x_1, \ldots, \delta_{x_k}(t) \cdot x_k, x_{k+1}, \ldots, x_n)$$

for any $(x_1, \ldots, x_n) \in C_k^n(-1)$ and $t \in [0, 1]$, where

$$\delta_x(t) = \begin{cases} 1 - t & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

It is easy to see that for any $t \in [0, 1]$, the image of $H(\cdot, t)$ is

$$C_k^n(t - 1) = \{(x_1, \ldots, x_n) \mid t - 1 \leq x_1, \ldots, x_k < 1, -1 < x_{k+1}, \ldots, x_n < 1\}.$$

So H actually defines an isotopy from $C_k^n(-1)$ to $C_k^n(0)$ (see Figure 2).

2.2 Embedding $\mathfrak{Z}_Q$ into $Q \times (D^2)^m$

Using the above rim-cubicalization of Q in $Q \times [0, 1]^m$, we can embed the manifold $\mathfrak{Z}_Q$ into $Q \times (D^2)^m$, where $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$ is the unit disk.

In the following, we consider $[0, 1]$ as a subset of D^2 and the cube $[0, 1]^m$ as a subset of $(D^2)^m \subset \mathbb{C}^m$. For any $j \in [m]$, let $S_{(j)}^1$ and $D_{(j)}^2$ denote the corresponding spaces indexed by j.

There is a canonical action of $(S^1)^m$ on $Q \times (D^2)^m$ defined by

$$(g_1, \ldots, g_m) \cdot (x, z_1, \ldots, z_m) = (x, g_1 z_1, \ldots, g_m z_m),$$

where $x \in Q$, $g_j \in S_{(j)}^1$ and $z_j \in D_{(j)}^2$ for $1 \leq j \leq m$. The orbit space of this action can be identified with $Q \times [0, 1]^m$. We denote the quotient map by

$$p \colon Q \times (D^2)^m \to Q \times [0, 1]^m.$$

For any face f of Q, we define

$$(10) \qquad (D^2, S^1)^f := p^{-1}(\hat{f}) = f \times \prod_{j \in I_f} D_{(j)}^2 \times \prod_{j \in [m] \setminus I_f} S_{(j)}^1 \subseteq Q \times (D^2)^m,$$

$$(11) \qquad (D^2, S^1)^Q := \bigcup_{f \in \mathscr{S}_Q} (D^2, S^1)^f = \bigcup_{f \in \mathscr{S}_Q} \left(f \times \prod_{j \in I_f} D_{(j)}^2 \times \prod_{j \in [m] \setminus I_f} S_{(j)}^1 \right).$$

There is a canonical action of $(S^1)^m$ on $(D^2, S^1)^Q$ induced by the canonical action of $(S^1)^m$ on $Q \times (D^2)^m$.

Lemma 2.3 $(D^2, S^1)^Q$ *is equivariantly homeomorphic to* $\mathfrak{L}_Q$.

Proof By Lemma 2.1, it is equivalent to show that $(D^2, S^1)^Q$ is equivariantly homeomorphic to $\mathfrak{L}_{\hat{Q}}$. We consider $[0, 1)^m$ as a nice manifold with corners whose facets are $F_1^\square, \dots, F_m^\square$, where

$$F_i^\square = 0_{(i)} \times \prod_{j \in [m] \setminus \{i\}} [0, 1)_{(j)}, \quad 1 \le i \le m.$$

It is clear that $\mathfrak{L}_{[0,1)^m} = [0, 1)^m \times (S^1)^m / \sim$ is homeomorphic to $(D^2 \backslash S^1)^m$. The quotient map $[0, 1)^m \times (S^1)^m \to \mathfrak{L}_{[0,1)^m} = (D^2 \backslash S^1)^m$ extends to a map $\pi : [0, 1]^m \times (S^1)^m \to (D^2)^m$ which can be written explicitly as

(12) $\pi : [0, 1]^m \times (S^1)^m \to (D^2)^m, \quad ((t_1, \dots, t_m), (g_1, \dots g_m)) \mapsto (g_1 t_1, \dots, g_m t_m).$

Define

$$\pi_Q = \mathrm{id}_Q \times \pi : Q \times [0, 1]^m \times (S^1)^m \to Q \times (D^2)^m.$$

Notice that the facets of $\hat{Q}$ are the intersections of $\hat{Q}$ with $Q \times F_1^\square, \dots, Q \times F_m^\square$,

$$\hat{F}_i = \hat{Q} \cap (Q \times F_i^\square), \quad 1 \le i \le m.$$

We can easily check that the restriction of π_Q to $\hat{Q} \times (S^1)^m$ gives exactly $\mathfrak{L}_{\hat{Q}}$, ie

$$\mathfrak{L}_{\hat{Q}} = \pi_Q(\hat{Q} \times (S^1)^m).$$

Moreover, for any face f of Q,

$$\pi_Q(\hat{f} \times (S^1)^m) = \pi_Q \left(f \times \prod_{j \in I_f} ([0, 1]_{(j)} \times S^1_{(j)}) \times \prod_{j \in [m] \setminus I_f} S^1_{(j)} \right)$$

$$= f \times \prod_{j \in I_f} D^2_{(j)} \times \prod_{j \in [m] \setminus I_f} S^1_{(j)} = (D^2, S^1)^f.$$

So we have a homeomorphism

$$\mathfrak{L}_{\hat{Q}} \cong \pi_Q(\hat{Q} \times (S^1)^m) = \bigcup_{f \in \mathscr{S}_Q} \pi_Q(\hat{f} \times (S^1)^m) = \bigcup_{f \in \mathscr{S}_Q} (D^2, S^1)^f = (D^2, S^1)^Q.$$

Clearly, the above homeomorphism is equivariant with respect to the canonical actions of $(S^1)^m$ on $\mathfrak{L}_{\hat{Q}}$ and $(D^2, S^1)^Q$. $\square$

2.3 Viewing $\mathfrak{L}_Q$ as a colimit of CW–complexes

By Lemma 2.3, studying the stable decomposition of $\mathfrak{L}_Q$ is equivalent to studying that for $(D^2, S^1)^Q$. To do the stable decomposition as in [3], we want to first think of $(D^2, S^1)^Q$ as the colimit of a diagram of CW–complexes over a finite *poset* (partially ordered set). The following are some basic definitions; see [38].

- Let CW be the category of CW–complexes and continuous maps.

- Let CW_* be the category of based CW–complexes and based continuous maps.

- A *diagram* $\mathscr{D}$ of CW–complexes or based CW–complexes over a finite poset $\mathscr{P}$ is a functor

$$\mathscr{D} : \mathscr{P} \to CW \text{ or } CW_*$$

such that for every $p \leq p'$ in $\mathscr{P}$, there is a map $d_{pp'} : \mathscr{D}(p') \to \mathscr{D}(p)$ with

$$d_{pp} = \mathrm{id}_{\mathscr{D}(p)}, \quad d_{pp'} d_{p'p''} = d_{pp''} \quad \text{for all } p \leq p' \leq p''.$$

- The *colimit* of $\mathscr{D}$ is the space

$$\mathrm{colim}(\mathscr{D}) := \left(\coprod_{p \in \mathscr{P}} \mathscr{D}(p) \right) \Big/ \sim$$

where $\sim$ denotes the equivalence relation generated by requiring that for each $x \in \mathscr{D}(p')$, $x \sim d_{pp'}(x)$ for every $p < p'$.

To think of $(D^2, S^1)^Q$ as a colimit of CW–complexes, we need to introduce a finer decomposition of $(D^2, S^1)^Q$ as follows. By the notations in Section 2.1, for any face f of Q and any subset $L \subseteq I_f \subseteq [m]$, let

$$(13) \qquad\qquad (D^2, S^1)^{(f,L)} := f \times \prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)}.$$

Clearly, $(D^2, S^1)^{(f,L)} \subseteq (D^2, S^1)^{(f,L')}$ if and only if $L \supseteq L'$. So we have

$$(D^2, S^1)^f = (D^2, S^1)^{(f,\varnothing)} = \bigcup_{L \subseteq I_f} (D^2, S^1)^{(f,L)},$$

$$(14) \qquad (D^2, S^1)^Q = \bigcup_{f \in \mathscr{S}_Q} (D^2, S^1)^f = \bigcup_{f \in \mathscr{S}_Q} \bigcup_{L \subseteq I_f} (D^2, S^1)^{(f,L)}.$$

Corresponding to this decomposition, we define a poset associated to Q by

$$(15) \qquad\qquad \mathscr{P}_Q = \{ (f, L) \mid f \in \mathscr{S}_Q, L \subseteq I_f \subseteq [m] \},$$

where $(f, L) \leq (f', L')$ if and only if $f \supseteq f'$ and $I_f \setminus L \supseteq I_{f'} \setminus L'$. It follows from the definition (13) that

$$(f, L) \leq (f', L') \iff (D^2, S^1)^{(f',L')} \subseteq (D^2, S^1)^{(f,L)}.$$

Note that $\mathscr{P}_Q$ is a finite poset since by our convention Q only has finitely many faces.

Definition 2.4 Let $\boldsymbol{D} : \mathscr{P}_Q \to CW$ be a diagram of CW–complexes where

$$\boldsymbol{D}((f, L)) = (D^2, S^1)^{(f,L)} \quad \text{for all } (f, L) \in \mathscr{P}_Q.$$

For any $(f, L) \leq (f', L') \in \mathscr{P}_Q$, $d_{(f,L),(f',L')} : \boldsymbol{D}((f', L')) \to \boldsymbol{D}((f, L))$ is the natural inclusion.

Clearly, $(D^2, S^1)^Q$ is the colimit of the diagram $\boldsymbol{D}$. So we have

$$(16) \qquad\qquad \mathscr{Z}_Q \cong (D^2, S^1)^Q = \mathrm{colim}(\boldsymbol{D}) = \bigcup_{(f,L) \in \mathscr{P}_Q} (D^2, S^1)^{(f,L)}.$$

Remark 2.5 Here we do not write $(D^2, S^1)^Q$ as the colimit of a diagram of based CW–complexes. This is because in general it is not possible to choose a basepoint in each $(D^2, S^1)^{(f,L)}$ to adapt to the colimit construction of a diagram in CW_*.

2.4 Stable decomposition of $\mathscr{Z}_Q$

First of all, let us recall a well-known theorem — see [23; 34] — which allows us to decompose the Cartesian product of a collection of based CW–complexes into a wedge of spaces after doing a suspension.

Let (X_i, x_i) for $1 \leq i \leq m$ be based CW–complexes. For $I = \{i_1, \ldots, i_k\} \subseteq [m]$ with $1 \leq i_1 < \cdots < i_k \leq m$, define

$$\widehat{X}^I = X_{i_1} \wedge \cdots \wedge X_{i_k},$$

which is the quotient space of $X^I = X_{i_1} \times \cdots \times X_{i_k}$ by the subspace given by

$$FW(X^I) = \{(y_{i_1}, \ldots, y_{i_k}) \in X^I \mid y_{i_j} \text{ is the basepoint } x_{i_j} \in X_{i_j} \text{ for at least one } i_j\}.$$

Theorem 2.6 *Let (X_i, x_i) for $1 \leq i \leq m$ be based connected CW–complexes. There is a based, natural homotopy equivalence*

$$h \colon \Sigma(X_1 \times \cdots \times X_m) \to \Sigma\left(\bigvee_{\varnothing \neq I \subseteq [m]} \widehat{X}^I\right),$$

where I runs over all the nonempty subsets of $[m]$. Furthermore, the map h commutes with colimits.

In our proof later, we need a slightly generalized version of Theorem 2.6. Before that, let us first prove three simple lemmas.

Lemma 2.7 *If (X, x_0) and (Y, y_0) are based CW–complexes with X contractible, then $X \wedge Y$ is also contractible.*

Proof The deformation retraction from X to x_0 naturally induces a deformation retraction from

$$X \wedge Y = X \times Y / (\{x_0\} \times Y) \cup (X \times \{y_0\})$$

to its canonical basepoint $[(x_0, y_0)] = [(\{x_0\} \times Y) \cup (X \times \{y_0\})]$.   □

Lemma 2.8 *Suppose a CW–complex X has N connected components $X_1, \ldots, X_N$. Then there is a homotopy equivalence*

$$\Sigma(X) \simeq \Sigma(X_1) \vee \cdots \vee \Sigma(X_N) \vee \bigvee_{N-1} S^1.$$

where $\bigvee_{N-1} S^1$ is the wedge sum of $N-1$ copies of S^1.

Proof This follows easily from the definition of reduced suspension.   □

Lemma 2.9 Let $X_1 = X_1' \cup \{x_1\}$ where X_1' is a connected based CW–complex and $x_1 \notin X_1'$ is the basepoint of X_1.

(a) *For any connected based CW–complex X_2, there is a homotopy equivalence*

$$\Sigma(X_1 \wedge X_2) \simeq \Sigma(X_2) \vee \Sigma(X_1' \wedge X_2).$$

(b) *If $X_2 = X_2' \cup \{x_2\}$, where X_2' is a connected based CW–complex and $x_2 \notin X_2'$ is the basepoint of X_2, then $X_1 \wedge X_2$ is the disjoint union of $X_1' \wedge X_2'$ and a point represented by $\{x_1\} \times \{x_2\}$.*

Proof (a) By the definition of smash product, we have a homeomorphism

$$X_1 \wedge X_2 = (X_1' \cup \{x_1\}) \times X_2/(\{x_1\} \times X_2 \cup X_1' \times \{x_2\}) \cong X_1' \times X_2/X_1' \times \{x_2\}.$$

Then we have

$$\begin{aligned}
\Sigma(X_1 \wedge X_2) &= \Sigma(X_1' \times X_2/X_1' \times \{x_2\}) \\
&\simeq \Sigma(X_1' \times X_2)/\Sigma(X_1' \times \{x_2\}) \\
&\simeq \big(\Sigma(X_1') \vee \Sigma(X_2) \vee \Sigma(X_1' \wedge X_2)\big)/\Sigma(X_1') \quad \text{(by Theorem 2.6)} \\
&\simeq \Sigma(X_2) \vee \Sigma(X_1' \wedge X_2).
\end{aligned}$$

(b) This follows directly from the definition of smash product. $\square$

We can generalize Theorem 2.6 to the following form.

Theorem 2.10 *Let (X_i, x_i) for $1 \le i \le m$, be based CW–complexes. Assume that for some $1 \le n \le m$,*

- $X_i = Y_i \cup \{x_i\}$ *for $1 \le i \le n$, where Y_i is a connected CW–complex and $x_i \notin Y_i$.*
- X_i, *for $n+1 \le i \le m$, is a connected CW–complex.*

There is a based, natural homotopy equivalence which commutes with colimits,

$$h \colon \Sigma(X_1 \times \cdots \times X_m) \to \Sigma\left(\bigvee_{\varnothing \ne I \subseteq [m]} \widehat{X}^I\right).$$

Proof For brevity, let $[n_1, n_2] = \{n_1, \dots, n_2\}$ for any integer $n_1 \le n_2$. Let

$$x^I = \{x_{i_1}\} \times \cdots \times \{x_{i_k}\}, \quad Y^I = Y_{i_1} \times \cdots \times Y_{i_k}, \quad I = \{i_1, \dots, i_k\}, \quad i_1 < \cdots < i_k.$$

There are 2^n connected components in $X^{[m]} = X_1 \times \cdots \times X_m$, which are

$$\{x^{[n] \setminus I} \times Y^I \times X^{[n+1, m]} \mid I \subseteq [1, n]\}.$$

We choose a basepoint for each Y_i with $1 \le i \le n$. So, by Lemma 2.8,

$$\Sigma(X_1 \times \cdots \times X_m) \simeq \bigvee_{I \subseteq [1, n]} \Sigma(Y^I \times X^{[n+1, m]}) \vee \bigvee_{2^n - 1} S^1.$$

Since $Y_1, \ldots, Y_n, X_{n+1}, \ldots, X_m$ are all connected based CW–complexes, we can apply Theorem 2.6 to each $Y^I \times X^{[n+1,m]}$ and obtain

$$(17) \qquad \boldsymbol{\Sigma}(X_1 \times \cdots \times X_m) \simeq \bigvee_{I \subseteq [1,n]} \bigvee_{\substack{L \cup J \neq \varnothing, L \subseteq I \\ J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J) \vee \bigvee_{2^n-1} S^1.$$

On the other hand, for any $I = \{i_1, \ldots, i_k\} \subseteq [1,n]$ and $J \subseteq [n+1,m]$,

$$\widehat{X}^I \wedge \widehat{X}^J = X_{i_1} \wedge \cdots \wedge X_{i_k} \wedge \widehat{X}^J = (Y_{i_1} \cup \{x_{i_1}\}) \wedge \cdots \wedge (Y_{i_k} \cup \{x_{i_k}\}) \wedge \widehat{X}^J.$$

If $J \neq \varnothing$, $\widehat{X}^I \wedge \widehat{X}^J$ is a connected CW–complex. Then by iteratively using Lemma 2.9(a), we obtain

$$\boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J) \simeq \bigvee_{L \subseteq I} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J).$$

Alternatively, if $J = \varnothing$ and $I \neq \varnothing$, by iteratively using Lemma 2.9(b), we can deduce that $\widehat{X}^I$ is the disjoint union of $\widehat{Y}^I$ and a point represented by x^I. So, by Lemma 2.8, $\boldsymbol{\Sigma}(\widehat{X}^I) \simeq \boldsymbol{\Sigma}(\widehat{Y}^I) \vee S^1$.

So we have

$$\bigvee_{\substack{H \subseteq [m] \\ H \neq \varnothing}} \boldsymbol{\Sigma}(\widehat{X}^H) = \bigvee_{\substack{I \cup J \neq \varnothing, I \subseteq [1,n] \\ J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J)$$

$$= \left(\bigvee_{\substack{I \subseteq [1,n] \\ \varnothing \neq J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J) \right) \vee \left(\bigvee_{\varnothing \neq I \subseteq [1,n]} \boldsymbol{\Sigma}(\widehat{X}^I) \right)$$

$$\simeq \left(\bigvee_{\substack{I \subseteq [1,n] \\ \varnothing \neq J \subseteq [n+1,m]}} \bigvee_{L \subseteq I} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J) \right) \vee \left(\bigvee_{\varnothing \neq I \subseteq [1,n]} (\boldsymbol{\Sigma}(\widehat{Y}^I) \vee S^1) \right).$$

By comparing the above expression with (17), we prove the theorem. $\qquad\square$

Remark 2.11 By Theorem 2.10, it is not hard to see that all the main theorems in [3] also hold for based CW–complex pairs $\{(X_i, A_i, a_i)\}_{i=1}^m$ where each of X_i and A_i is either connected or is a disjoint union of a connected CW–complex with its basepoint. In particular, [3, Corollary 2.24] also holds for (D^1, S^0).

Remark 2.12 It is possible to extend Theorem 2.10 further to deal with spaces which are a disjoint union of a connected CW–complex with finitely many points. But since Theorem 2.10 is already enough for our discussion in this paper, we leave the more generalized statement to the reader.

Definition 2.13 For any based CW–complexes (X, x_0) and (Y, y_0), let

$$X \rtimes Y := X \times Y / x_0 \times Y, \quad X \ltimes Y := X \times Y / X \times y_0.$$

If each of X and Y is either connected or is a disjoint union of a connected CW–complex with its basepoint, there is a homotopy equivalence by Theorem 2.10,

$$(18) \qquad \mathbf{\Sigma}(X \rtimes Y) \simeq \mathbf{\Sigma}(X \times Y)/\mathbf{\Sigma}(x_0 \times Y) \simeq \mathbf{\Sigma}(X) \vee \mathbf{\Sigma}(X \wedge Y),$$

$$(19) \qquad \mathbf{\Sigma}(X \ltimes Y) \simeq \mathbf{\Sigma}(X \times Y)/\mathbf{\Sigma}(X \times y_0) \simeq \mathbf{\Sigma}(Y) \vee \mathbf{\Sigma}(X \wedge Y).$$

We can further generalize Theorem 2.6 to the following form. We will use the following convention in the rest of the paper:

Convention For any based space Y, define $Y \wedge \hat{X}^I := Y$ when $I = \varnothing$.

Theorem 2.14 *Let (X_i, x_i) for $1 \leq i \leq m$ and (B, b_0) be a collection of based CW–complexes where each of X_i and B is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a based, natural homotopy equivalence which commutes with colimits,*

$$h \colon \mathbf{\Sigma}(B \rtimes (X_1 \times \cdots \times X_m)) \to \mathbf{\Sigma}\left(\bigvee_{I \subseteq [m]} B \wedge \hat{X}^I \right).$$

Proof By definition,

$$\mathbf{\Sigma}(B \wedge (X_1 \times \cdots \times X_m)) = \mathbf{\Sigma}(B \times (X_1 \times \cdots \times X_m)/B \vee (X_1 \times \cdots \times X_m))$$

$$\simeq \mathbf{\Sigma}(B \times X_1 \times \cdots \times X_m)/\mathbf{\Sigma}(B) \vee \mathbf{\Sigma}(X_1 \times \cdots \times X_m)$$

$$\simeq \bigvee_{\varnothing \neq I \subseteq [m]} \mathbf{\Sigma}(B \wedge \hat{X}^I) \quad \text{(by Theorem 2.10).}$$

Then, by (18),

$$\mathbf{\Sigma}(B \rtimes (X_1 \times \cdots \times X_m)) = \mathbf{\Sigma}(B) \vee \mathbf{\Sigma}(B \wedge (X_1 \times \cdots \times X_m))$$

$$\simeq \mathbf{\Sigma}(B) \vee \bigvee_{\varnothing \neq I \subseteq [m]} \mathbf{\Sigma}(B \wedge \hat{X}^I)$$

$$\simeq \mathbf{\Sigma}\left(\bigvee_{I \subseteq [m]} B \wedge \hat{X}^I \right). \qquad \square$$

To apply the above stable decomposition lemmas to $(D^2, S^1)^Q$, we need to choose a basepoint for each $(D^2, S^1)^{(f,L)}$ in the first place. But by Remark 2.5, there is no good way to choose a basepoint inside each $(D^2, S^1)^{(f,L)}$ to adapt to the colimit construction of $(D^2, S^1)^Q$. So, in the following, we add an auxiliary point to all $(D^2, S^1)^{(f,L)}$ as their common basepoint:

- Let $1_{(j)}$ be the basepoint of $S^1_{(j)}$ and $D^2_{(j)}$ for every $j \in [m]$.
- Let $Q_+ = Q \cup q_0$ where $q_0 \notin Q$ is the basepoint of Q_+.
- For any face f of Q, let $f_+ = f \cup q_0$ with basepoint q_0.
- For any $(f, L) \in \mathscr{P}_Q$, define $(D^2, S^1)_+^{(f,L)} := (D^2, S^1)^{(f,L)} \cup \hat{q}_0$, where $\hat{q}_0 = q_0 \times \prod_{j \in [m]} 1_{(j)}$ is the basepoint.
- Let $(D^2, S^1)_+^Q = (D^2, S^1)^Q \cup \hat{q}_0$ with basepoint $\hat{q}_0$.

Let $\boldsymbol{D}_+\colon \mathscr{P}_Q \to \mathrm{CW}_*$ be a diagram of based CW–complexes, where

$$\boldsymbol{D}_+((f,L)) = (D^2, S^1)_+^{(f,L)} \quad \text{for all } (f,L) \in \mathscr{P}_Q,$$

and $(d_+)_{(f,L),(f',L')}\colon \boldsymbol{D}_+((f',L')) \to \boldsymbol{D}_+((f,L))$ is the natural inclusion for any $(f,L) \le (f',L') \in \mathscr{P}_Q$. Then it is clear that

$$(20) \qquad (D^2,S^1)_+^Q = (D^2,S^1)^Q \cup \hat{q}_0 = \mathrm{colim}(\boldsymbol{D}) \cup \hat{q}_0 = \mathrm{colim}(\boldsymbol{D}_+).$$

Next, we analyze the reduced suspension $\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+))$ from the colimit point of view. Since all the $(D^2,S^1)_+^{(f,L)}$ share the same basepoint $\hat{q}_0$,

$$\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+)) = \boldsymbol{\Sigma}\left(\bigcup_{(f,L) \in \mathscr{P}_Q} (D^2,S^1)_+^{(f,L)} \right) = \bigcup_{(f,L) \in \mathscr{P}_Q} \boldsymbol{\Sigma}((D^2,S^1)_+^{(f,L)}).$$

Lemma 2.15 *For any $(f,L) \in \mathscr{P}_Q$, there is a natural homeomorphism which commutes with taking the colimit,*

$$(D^2,S^1)_+^{(f,L)} \cong f_+ \rtimes \left(\prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)} \right).$$

Proof By our definitions,

$$f_+ \rtimes \prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)}$$

$$= (f \cup q_0) \times \prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)} \Big/ q_0 \times \prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)}$$

$$\cong \left(f \times \prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)} \right) \cup \hat{q}_0 = (D^2,S^1)_+^{(f,L)}.$$

The above homeomorphism "$\cong$" is induced by the global homeomorphism

$$Q_+ \rtimes \prod_{j \in [m]} D^2_{(j)} \to \left(Q \times \prod_{j \in [m]} D^2_{(j)} \right) \cup \hat{q}_0$$

which identifies $q_0 \times \prod_{j \in [m]} D^2_{(j)} \big/ q_0 \times \prod_{j \in [m]} D^2_{(j)}$ with $\hat{q}_0$. $\qquad \square$

Since we assume that each face f of Q is a CW–complex in our convention, we can deduce from Theorem 2.14 and Lemma 2.15 that

$$(21) \qquad \boldsymbol{\Sigma}((D^2,S^1)_+^{(f,L)}) \cong \boldsymbol{\Sigma}\left(f_+ \rtimes \left(\prod_{j \in I_f \setminus L} D^2_{(j)} \times \prod_{j \in [m] \setminus (I_f \setminus L)} S^1_{(j)} \right) \right)$$

$$\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}\left(f_+ \wedge \bigwedge_{j \in J \cap (I_f \setminus L)} D^2_{(j)} \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} S^1_{(j)} \right).$$

According to (21), we define a family of diagrams of based CW–complexes

$$\hat{\boldsymbol{D}}_+^J : \mathscr{P}_Q \to \mathrm{CW}_*, \quad J \subseteq [m],$$

(22) $$\hat{\boldsymbol{D}}_+^J((f, L)) := f_+ \wedge \bigwedge_{j \in J \cap (I_f \setminus L)} D_{(j)}^2 \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} S_{(j)}^1 \subseteq Q_+ \wedge \bigwedge_{j \in J} D_{(j)}^2,$$

where $(\hat{d}_+^J)_{(f,L),(f',L')} : \hat{\boldsymbol{D}}_+^J((f', L')) \to \hat{\boldsymbol{D}}_+^J((f, L))$ is the natural inclusion for $(f, L) \leq (f', L') \in \mathscr{P}_Q$. The basepoint of $\hat{\boldsymbol{D}}_+^J((f, L))$ is

$$[\hat{q}_0^J] := \left[q_0 \times \prod_{j \in J} 1_{(j)} \right].$$

Since here the reduced suspension commutes with colimits up to homotopy equivalence [3, Theorem 4.3], we obtain a homotopy equivalence

(23) $$\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+)) \simeq \mathrm{colim}(\boldsymbol{\Sigma}(\boldsymbol{D}_+)) \simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\mathrm{colim}(\hat{\boldsymbol{D}}_+^J)).$$

The following theorem from [3] will be useful in our proof of Theorem 1.1. It is a modification of the "homotopy lemma" given in [31; 33; 38].

Theorem 2.16 [3, Corollary 4.5] *Let D and E be two diagrams over a finite poset $\mathscr{P}$ with values in CW_* for which the maps $\mathrm{colim}_{q>p} D(q) \hookrightarrow D(p)$ and $\mathrm{colim}_{q>p} E(q) \hookrightarrow E(p)$ are all closed cofibrations. If f is a map of diagrams over $\mathscr{P}$ such that for every $p \in \mathscr{P}$, $f_p : D(p) \to E(p)$ is a homotopy equivalence, then f induces a homotopy equivalence $f : \mathrm{colim}(D(P)) \to \mathrm{colim}(E(P))$.*

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1   By (20) and (23), we obtain a homotopy equivalence

(24) $$\boldsymbol{\Sigma}((D^2, S^1)_+^Q) \simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\mathrm{colim}(\hat{\boldsymbol{D}}_+^J)).$$

Notice that when $J \cap (I_f \setminus L) \neq \varnothing$, $\hat{\boldsymbol{D}}_+^J((f, L))$ is contractible by Lemma 2.7. So, for any $J \subseteq [m]$, we define another diagram of based CW–complexes

$$\hat{\boldsymbol{E}}_+^J : \mathscr{P}_Q \to \mathrm{CW}_*,$$

(25) $$\hat{\boldsymbol{E}}_+^J((f, L)) := \begin{cases} \hat{\boldsymbol{D}}_+^J((f, L)) = f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1 & \text{if } J \cap (I_f \setminus L) = \varnothing, \\ [\hat{q}_0^J] & \text{if } J \cap (I_f \setminus L) \neq \varnothing. \end{cases}$$

For $(f, L) \leq (f', L') \in \mathscr{P}_Q$, $(\hat{e}_+^J)_{(f,L),(f',L')} : \hat{\boldsymbol{E}}_+^J((f', L')) \to \hat{\boldsymbol{E}}_+^J((f, L))$ is either the natural inclusion or the constant map $c_{[\hat{q}_0^J]}$ (mapping all points to $[\hat{q}_0^J]$). The basepoint of $\hat{\boldsymbol{E}}_+^J((f, L))$ is $[\hat{q}_0^J]$.

Moreover, let $\Phi^J : \hat{\boldsymbol{D}}_+^J \to \hat{\boldsymbol{E}}_+^J$ be a map of diagrams over $\mathscr{P}_Q$ defined by

$$\Phi_{(f,L)}^J : \hat{\boldsymbol{D}}_+^J((f, L)) \to \hat{\boldsymbol{E}}_+^J((f, L)), \qquad \Phi_{(f,L)}^J = \begin{cases} \mathrm{id}_{\hat{\boldsymbol{D}}_+^J((f,L))} & \text{if } J \cap (I_f \setminus L) = \varnothing, \\ c_{[\hat{q}_0^J]} & \text{if } J \cap (I_f \setminus L) \neq \varnothing. \end{cases}$$

Then by Theorem 2.16, there exists a homotopy equivalence

$$\mathrm{colim}(\hat{\boldsymbol{D}}_+^J) \simeq \mathrm{colim}(\hat{\boldsymbol{E}}_+^J).$$

Note that we always have $\hat{\boldsymbol{D}}_+^J((f, I_f)) \subseteq \hat{\boldsymbol{D}}_+^J((f, L))$ for any $L \subsetneq I_f \neq \varnothing$. So we can ignore the terms $\{\hat{\boldsymbol{D}}_+^J((f, I_f)) \mid I_f \neq \varnothing, f \in \mathscr{S}_Q\}$ when computing $\mathrm{colim}(\hat{\boldsymbol{D}}_+^J)$. If $I_f = \varnothing$, then $f \subseteq Q^\circ$ and $\hat{\boldsymbol{D}}_+^J((f, L)) = \hat{\boldsymbol{D}}_+^J((f, \varnothing)) = f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1$.

To understand $\mathrm{colim}(\hat{\boldsymbol{E}}_+^J)$, we need to figure out in (25) what are those faces f of Q with some $L \subsetneq I_f$ such that $J \cap (I_f \backslash L) \neq \varnothing$.

- There exists $L \subsetneq I_f$ with $J \cap (I_f \backslash L) \neq \varnothing$ if and only if $J \cap I_f \neq \varnothing$, which is equivalent to $f \subseteq F_J$. Conversely,

$$F_J = \bigcup_{f \in \mathscr{S}_Q} f \cap F_J = \bigcup_{f \in \mathscr{S}_Q} \bigcup_{j \in J} f \cap F_j = \bigcup_{f \in \mathscr{S}_Q} \bigcup_{j \in J \cap I_f} f \cap F_j \subseteq \bigcup_{f \in \mathscr{S}_Q} \bigcup_{J \cap I_f \neq \varnothing} f.$$

  This implies

(26)
$$\bigcup_{\substack{f \in \mathscr{S}_Q}} \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \backslash L) \neq \varnothing}} f = F_J.$$

- There exists $L \subsetneq I_f$ with $J \cap (I_f \backslash L) = \varnothing$ if and only if $f \subseteq F_{[m] \backslash J}$. So

(27)
$$\bigcup_{\substack{f \in \mathscr{S}_Q}} \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \backslash L) = \varnothing}} f = F_{[m] \backslash J}.$$

The above discussion implies

(28)
$$\bigcup_{\substack{f \in \mathscr{S}_Q}} \bigcup_{\substack{\exists L, L' \subsetneq I_f \\ J \cap (I_f \backslash L) = \varnothing \\ J \cap (I_f \backslash L') \neq \varnothing}} f = F_{[m] \backslash J} \cap F_J.$$

By the definition of $\hat{\boldsymbol{E}}_+^J$, if we have a face f of Q and two subsets $L, L' \subsetneq I_f$ such that $J \cap (I_f \backslash L) = \varnothing$ while $J \cap (I_f \backslash L') \neq \varnothing$, then $J \cap (I_f \backslash (L \cup L')) = \varnothing$ and $J \cap (I_f \backslash (L \cap L')) \neq \varnothing$. So, in this case,

- $(\hat{e}_+^J)_{(f,L'),(f,L \cup L')} \colon \hat{\boldsymbol{E}}_+^J((f, L \cup L')) \to \hat{\boldsymbol{E}}_+^J((f, L'))$ is the constant map $c_{[\hat{q}_0^J]}$,

$$f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1 \mapsto [\hat{q}_0^J];$$

- $(\hat{e}_+^J)_{(f,L),(f,L \cup L')} \colon \hat{\boldsymbol{E}}_+^J((f, L \cup L')) \to \hat{\boldsymbol{E}}_+^J((f, L))$ is identity map,

$$f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1 \mapsto f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1.$$

Then in $\mathrm{colim}(\hat{\boldsymbol{E}}_+^J)$, the image of any of such $f_+ \wedge \bigwedge_{j \in J} S_{(j)}^1$ is equivalent to the point $[\hat{q}_0^J]$. So we can deduce that:

- For $J \neq \varnothing$,

$$\mathrm{colim}(\hat{E}_+^J) \cong \left(Q^\circ \cup \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \setminus L) = \varnothing}} f_+ \right) \wedge \bigwedge_{j \in J} S_{(j)}^1 \Big/ \left(\bigcup_{\substack{\exists L, L' \subsetneq I_f \\ J \cap (I_f \setminus L) = \varnothing \\ J \cap (I_f \setminus L') \neq \varnothing}} f_+ \right) \wedge \bigwedge_{j \in J} S_{(j)}^1$$

$$\cong \left(((Q^\circ \cup F_{[m] \setminus J}) \cup q_0) \big/ ((F_{[m] \setminus J} \cap F_J) \cup q_0) \right) \wedge \bigwedge_{j \in J} S_{(j)}^1 \quad \text{(by (27) and (28))}$$

$$\cong (Q / F_J) \wedge \bigwedge_{j \in J} S_{(j)}^1 \cong \mathbf{\Sigma}^{|J|}(Q / F_J).$$

- For $J = \varnothing$, $\mathrm{colim}(\hat{E}_+^\varnothing) \cong Q^\circ \cup F_{[m]} \cup q_0 = Q \cup q_0 = Q_+$.

Combining all the above arguments, we obtain homotopy equivalences

$$\mathbf{\Sigma}((D^2, S^1)_+^Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}(\mathrm{colim}(\hat{D}_+^J))$$

$$\simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}(\mathrm{colim}(\hat{E}_+^J))$$

$$\simeq \mathbf{\Sigma}(Q_+) \vee \bigvee_{\varnothing \neq J \subseteq [m]} \mathbf{\Sigma}^{|J|+1}(Q / F_J)$$

$$\simeq S^1 \vee \mathbf{\Sigma}(Q) \vee \bigvee_{\varnothing \neq J \subseteq [m]} \mathbf{\Sigma}^{|J|+1}(Q / F_J)$$

$$\simeq S^1 \vee \bigvee_{J \subseteq [m]} \mathbf{\Sigma}^{|J|+1}(Q / F_J).$$

On the other hand,

$$\mathbf{\Sigma}((D^2, S^1)_+^Q) = \mathbf{\Sigma}((D^2, S^1)^Q \cup \hat{q}_0) \simeq S^1 \vee \mathbf{\Sigma}((D^2, S^1)^Q) \cong S^1 \vee \mathbf{\Sigma}(\mathscr{Z}_Q). \qquad \square$$

3 Cohomology ring structure of $\mathscr{Z}_Q$

The cohomology ring of the moment-angle complex over a simplicial complex K was computed by Franz [18] and Baskakov, Buchstaber and Panov [7]. The cohomology rings of a much wider class of spaces called *generalized moment-angle complexes* or *polyhedral products* were computed by Bahri, Bendersky, Cohen and Gitler [4] via partial diagonal maps and by Bahri, Bendersky, Cohen and Gitler [5] by a spectral sequence under certain freeness conditions (coefficients in a field for example). The study in this direction is further extended in [6]. A computation using different methods was carried out by Wang and Zheng [32] and Zheng [37].

It was shown by Bahri, Bendersky, Cohen and Gitler [4] that the product structure on the cohomology of a polyhedral product over a simplicial complex can be formulated in terms of the stable decomposition and partial diagonal maps of the polyhedral product. For a nice manifold with corners Q, since we also

have the stable decomposition of $\mathscr{L}_Q$, we should be able to describe the cohomology ring of $\mathscr{L}_Q$ in a similar way.

Let us first recall the definition of partial diagonal in product spaces from [4]. Let $X_1, \ldots, X_m$ be a collection of based CW–complexes. Using the notations in Section 2.4, for any $I \subseteq [m]$, there are natural projections $X^{[m]} \to \widehat{X}^I$ obtained as the composition

$$\widehat{\Pi}_I : X^{[m]} \xrightarrow{\Pi_I} X^I \xrightarrow{\rho_I} \widehat{X}^I,$$

where $\Pi_I : X^{[m]} \to X^I$ is the natural projection and ρ_I is the quotient map in the definition of the smash product $\widehat{X}^I$. In addition, let

$$W_I^{J,J'} := \bigwedge_{|J|+|J'|} W_i, \quad J \cup J' = I,$$

where

$$W_i = \begin{cases} X_i & \text{if } i \in I \backslash (J \cap J'), \\ X_i \wedge X_i & \text{if } i \in J \cap J'. \end{cases}$$

Note that if $J \cup J' = I$ and $J \cap J' = \varnothing$, $W_I^{J,J'} = \widehat{X}^I$.

Define $\psi_I^{J,J'} : \widehat{X}^I \to W_I^{J,J'}$ as $\psi_I^{J,J'} = \bigwedge_{i \in I} \psi_i$, where $\psi_i : X_i \to W_i$ is defined by

$$\psi_i = \begin{cases} \text{id} & \text{if } i \in I \backslash (J \cap J'), \\ \Delta_i : X_i \to X_i \wedge X_i & \text{if } i \in J \cap J', \end{cases}$$

where $\Delta_i : X_i \to X_i \times X_i \to X_i \wedge X_i$ is the *reduced diagonal* of X_i.

Note that the smash products $W_I^{J,J'}$ and $\widehat{X}^J \wedge \widehat{X}^{J'}$ have the same factors, but in a different order arising from the natural shuffles. Let

(29) $$\Theta_I^{J,J'} : W_I^{J,J'} \to \widehat{X}^J \wedge \widehat{X}^{J'}, \quad J \cup J' = I,$$

be the natural homeomorphism given by a shuffle. Define the *partial diagonal*

(30) $$\widehat{\Delta}_I^{J,J'} : \widehat{X}^I \xrightarrow{\psi_I^{J,J'}} W_I^{J,J'} \xrightarrow{\Theta_I^{J,J'}} \widehat{X}^J \wedge \widehat{X}^{J'}$$

be the composition of $\Theta_I^{J,J'}$ and $\psi_I^{J,J'}$. There is a commutative diagram

$$\begin{array}{ccc} X^{[m]} & \xrightarrow{\Delta_{[m]}^X} & X^{[m]} \wedge X^{[m]} \\ {\scriptstyle \widehat{\Pi}_I} \downarrow & & \downarrow {\scriptstyle \widehat{\Pi}_J \wedge \widehat{\Pi}_{J'}} \\ \widehat{X}^I & \xrightarrow{\widehat{\Delta}_I^{J,J'}} & \widehat{X}^J \wedge \widehat{X}^{J'} \end{array}$$

where $\Delta_{[m]}^X$ is the reduced diagonal map of $X^{[m]}$.

Let k denote a commutative ring with a unit. For any $J \subseteq [m]$, there is a homomorphism of rings given by the *reduced cross product* $\times$ (see [20, page 223]),

$$\bigotimes_{j \in J} \widetilde{H}^*(X_j; k) \xrightarrow{\times} \widetilde{H}^*(\widehat{X}^J; k).$$

In particular, this ring homomorphism becomes a ring isomorphism if all (possibly except one) $\tilde{H}^*(X_j; \boldsymbol{k})$ are free $\boldsymbol{k}$–modules; see [20, Theorem 3.21].

Lemma 3.1 *For any $\phi_j \in \tilde{H}^*(X_j; \boldsymbol{k})$ with $j \in J$ and any $\phi'_j \in \tilde{H}^*(X_j; \boldsymbol{k})$ with $j \in J'$,*

$$(\hat{\Delta}_I^{J,J'})^* : H^*(\hat{X}^J \wedge \hat{X}^{J'}; \boldsymbol{k}) \to H^*(\hat{X}^I; \boldsymbol{k}), \quad I = J \cup J',$$

$$(\hat{\Delta}_I^{J,J'})^*\left(\left(\underset{j \in J}{\times} \phi_j\right) \times \left(\underset{j \in J'}{\times} \phi'_j\right)\right) = \left(\underset{j \in J \setminus J'}{\times} \phi_j\right) \times \left(\underset{j \in J' \setminus J}{\times} \phi'_j\right) \times \left(\underset{j \in J \cap J'}{\times} \Delta_j^*(\phi_j \times \phi'_j)\right).$$

Proof The above formula follows easily from the definition of $\hat{\Delta}_I^{J,J'}$. Note that the shuffle $\Theta_I^{J,J'}$ (see (29)) sorts all the cohomology classes $\{\phi_j\}_{j \in J}$ and $\{\phi'_j\}_{j \in J'}$ in order without introducing any $\pm$ sign. This is because for any space X and Y,

$$T : X \wedge Y \to Y \wedge X, \quad [(x, y)] \mapsto [(y, x)],$$

induces a group isomorphism $T^* : H^*(Y \wedge X; \boldsymbol{k}) \to H^*(X \wedge Y; \boldsymbol{k})$ such that

$$T^*(\phi_Y \times \phi_X) = \phi_X \times \phi_Y, \quad \phi_X \in H^*(X; \boldsymbol{k}), \phi_Y \in H^*(Y; \boldsymbol{k}).$$

So, when $\Theta_I^{J,J'}$ transposes the space factors, the cohomology classes in the reduced cross product are transposed accordingly. $\quad\square$

The following lemma will be useful for our proof of Theorem 1.4 later.

Lemma 3.2 *Let X be a CW–complex and A and B be two subcomplexes of X. The relative cup product $H^*(X, A) \otimes H^*(X, B) \xrightarrow{\cup} H^*(X, A \cup B)$ induces a product*

$$\tilde{H}^*(X/A) \otimes \tilde{H}^*(X/B) \xrightarrow{\tilde{\cup}} \tilde{H}^*(X/(A \cup B)),$$

which can be factored as

$$\phi \,\tilde{\cup}\, \phi' = \Delta_X^*(\phi \times \phi'), \quad \phi \in \tilde{H}^*(X/A), \phi' \in \tilde{H}^*(X/B)$$

where $\Delta_X : X \to X \times X$ is the diagonal map and $\phi \times \phi'$ is the reduced cross product of ϕ and ϕ'.

Proof This can be verified directly from the following diagram when A and B are nonempty:

$$
\begin{array}{ccccc}
H^*(X,A) \otimes H^*(X,B) & \xrightarrow[\text{relative cross product}]{\times} & H^*(X \times X, (A \times X) \cup (X \times B)) & \xrightarrow{\Delta_X^*} & H^*(X, A \cup B) \\
\cong \uparrow & & \cong \uparrow & & \cong \uparrow \\
H^*(X/A, A/A) \otimes H^*(X/B, B/B) & \xrightarrow{\times} & H^*\big(X/A \times X/B, (A/A \times X/B) \cup (X/A \times B/B)\big) & & H^*\big(X/(A \cup B), A \cup B/A \cup B\big) \\
\cong & & \cong & & \cong \\
\tilde{H}^*(X/A) \otimes \tilde{H}^*(X/B) & & \tilde{H}^*(X/A \wedge X/B) & & \tilde{H}^*(X/(A \cup B))
\end{array}
$$

where the lower $\xrightarrow{\times}$ is the reduced cross product on $\tilde{H}^*(X/A) \otimes \tilde{H}^*(X/B)$.

If A or B is empty, we should replace $\tilde{H}^*(X/A)$ or $\tilde{H}^*(X/B)$ by $H^*(X)$ in the above diagram. Moreover, since $H^*(X) \cong \tilde{H}^*(X) \oplus \mathbb{Z}$, the $\tilde{\cup}$ on $\tilde{H}^*(X)$ is just the restriction of $\cup$ from $H^*(X)$. $\quad\square$

Another useful fact is when X_i is the suspension of some space, the reduced diagonal $\Delta_i : X_i \to X_i \wedge X_i$ is nullhomotopic; see [4]. So we have the following lemma.

Lemma 3.3 *If for some $j \in J \cap J'$, X_j is a suspension space, then the partial diagonal*

$$\hat{\Delta}_I^{J,J'} : \hat{X}^I \to \hat{X}^J \wedge \hat{X}^{J'}$$

is nullhomotopic, where $I = J \cup J'$.

Now we are ready to give a proof of Theorem 1.4. Our argument is parallel to the argument used in the proof of [4, Theorem 1.4].

Proof of Theorem 1.4 For brevity, we will use the following notation in the proof:

$$Q_+ \times (D^2)^{[m]} := Q_+ \times \prod_{j \in [m]} D^2_{(j)}, \quad Q_+ \wedge (\hat{D}^2)^J := Q_+ \wedge \bigwedge_{j \in J} D^2_{(j)}.$$

Considering the partial diagonals (30) for $Q_+ \times (D^2)^{[m]}$, we obtain a map

$$\hat{\Delta}_{J \cup J', Q_+}^{J,J'} : Q_+ \wedge (\hat{D}^2)^{J \cup J'} \to (Q_+ \wedge (\hat{D}^2)^J) \wedge (Q_+ \wedge (\hat{D}^2)^{J'})$$

for any $J, J' \subseteq [m]$ and a commutative diagram

(31)
$$
\begin{array}{ccc}
Q_+ \times (D^2)^{[m]} & \xrightarrow{\Delta_{[m]}^{Q_+, D^2}} & (Q_+ \times (D^2)^{[m]}) \wedge (Q_+ \times (D^2)^{[m]}) \\
\downarrow{\scriptstyle \hat{\Pi}_{J \cup J'}} & & \downarrow{\scriptstyle \hat{\Pi}_J \wedge \hat{\Pi}_{J'}} \\
Q_+ \wedge (\hat{D}^2)^{J \cup J'} & \xrightarrow{\hat{\Delta}_{J \cup J', Q_+}^{J,J'}} & (Q_+ \wedge (\hat{D}^2)^J) \wedge (Q_+ \wedge (\hat{D}^2)^{J'})
\end{array}
$$

where $\Delta_{[m]}^{Q_+, D^2}$ is the reduced diagonal map of $Q_+ \times (D^2)^{[m]}$. By restricting the above diagram to $\mathrm{colim}(\boldsymbol{D}_+)$, we obtain a commutative diagram for all $J, J' \subseteq [m]$,

(32)
$$
\begin{array}{ccc}
\mathrm{colim}(\boldsymbol{D}_+) & \xrightarrow{\Delta_{[m]}^{Q_+, D^2}} & \mathrm{colim}(\boldsymbol{D}_+) \wedge \mathrm{colim}(\boldsymbol{D}_+) \\
\downarrow{\scriptstyle \hat{\Pi}_{J \cup J'}} & & \downarrow{\scriptstyle \hat{\Pi}_J \wedge \hat{\Pi}_{J'}} \\
\mathrm{colim}(\hat{\boldsymbol{D}}_+^{J \cup J'}) & \xrightarrow{\hat{\Delta}_{J \cup J', Q_+}^{J,J'}} & \mathrm{colim}(\hat{\boldsymbol{D}}_+^J) \wedge \mathrm{colim}(\hat{\boldsymbol{D}}_+^{J'})
\end{array}
$$

Given cohomology classes $u \in \tilde{H}^*(\mathrm{colim}(\hat{\boldsymbol{D}}_+^J))$ and $v \in \tilde{H}^*(\mathrm{colim}(\hat{\boldsymbol{D}}_+^{J'}))$, let

(33)
$$u \circledast v = (\hat{\Delta}_{J \cup J', Q_+}^{J,J'})^*(u \times v) \in \tilde{H}^*(\mathrm{colim}(\hat{\boldsymbol{D}}_+^{J \cup J'})),$$

where $u \times v \in \tilde{H}^*(\mathrm{colim}(\hat{\boldsymbol{D}}_+^J) \wedge \mathrm{colim}(\hat{\boldsymbol{D}}_+^{J'}))$ is the reduced cross product of u and v. This defines a ring structure on $\bigoplus_{J \subseteq [m]} \tilde{H}^*(\mathrm{colim}(\hat{\boldsymbol{D}}_+^J))$.

The commutativity of diagram (32) implies

$$\hat{\Pi}_{J \cup J'}^*(u \circledast v) = \hat{\Pi}_J^*(u) \cup \hat{\Pi}_{J'}^*(v),$$

where $\cup$ is the cup product for $\mathrm{colim}(\boldsymbol{D}_+)$.

By (23), the direct sum of $\widehat{\Pi}_J^*$ induces an additive isomorphism

$$(34) \qquad \bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^* : \bigoplus_{J \subseteq [m]} \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}_+^J)) \to \widetilde{H}^*(\mathrm{colim}(\boldsymbol{D}_+)) = \widetilde{H}^*((D^2, S^1)_+^Q).$$

Then since $\widehat{\Pi}_J^* : \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}_+^J)) \to \widetilde{H}^*(\mathrm{colim}(\boldsymbol{D}_+))$ is a ring homomorphism for every $J \subseteq [m]$, we can assert that $\bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^*$ is a ring isomorphism. Then by the proof of Theorem 1.1, this induces a ring isomorphism

$$(35) \qquad \bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^* : \left(\bigoplus_{J \subseteq [m]} \widetilde{H}^*(Q/F_J \wedge \bigwedge_{j \in J} S_{(j)}^1), \circledast \right) \to \widetilde{H}^*((D^2, S^1)^Q).$$

Finally, let us define a ring isomorphism from $(\mathcal{R}_Q^*, \mathbb{U})$ to the cohomology ring $H^*((D^2, S^1)^Q)$ via $\bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^*$.

For any $1 \leq j \leq m$, let $\iota_{(j)}^1$ denote a generator of $\widetilde{H}^1(S_{(j)}^1)$. Then for any subset $J = \{j_1, \ldots, j_s\} \subseteq [m]$ with $j_1 < \cdots < j_s$, we have a generator

$$\iota^J = \iota_{(j_1)}^1 \times \cdots \times \iota_{(j_s)}^1 \in \widetilde{H}^{|J|}\left(\bigwedge_{j \in J} S_{(j)}^1 \right).$$

For each $J \subseteq [m]$, there is a canonical linear isomorphism (see [20, page 223]),

$$\widetilde{H}^*(Q/F_J) \xrightarrow{\cong} \widetilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J} S_{(j)}^1 \right) \cong \widetilde{H}^*(\Sigma^{|J|}(Q/F_J)), \quad \phi \mapsto \phi \times \iota^J.$$

Let

$$\widetilde{\mathcal{R}}_Q^* := \bigoplus_{J \subseteq [m]} \widetilde{H}^*(Q/F_J).$$

Then $\mathcal{R}_Q^* = \widetilde{\mathcal{R}}_Q^* \oplus \mathbb{Z}$. By Lemma 3.2, there is natural ring structure on $\widetilde{\mathcal{R}}_Q^*$, denoted by $\widetilde{\mathbb{U}}$, that is induced from the product $\mathbb{U}$ on $\mathcal{R}_Q^*$ (see (5)). We have a commutative diagram

$$(36) \qquad \begin{array}{ccc} H^*(Q, F_J) \otimes H^*(Q, F_{J'}) & \xrightarrow{\mathbb{U}} & H^*(Q, F_{J \cup J'}) \\ \downarrow & & \downarrow \\ \widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) & \xrightarrow{\widetilde{\mathbb{U}}} & \widetilde{H}^*(Q/F_{J \cup J'}) \end{array}$$

such that for any $J, J' \subseteq [m]$ with $J \cap J' \neq \varnothing$, $\widetilde{\mathbb{U}}$ is trivial; and for any $J, J' \subseteq [m]$ with $J \cap J = \varnothing$, $\widetilde{\mathbb{U}} = \widetilde{\cup}$ is induced from the relative cup product $\cup$ on $H^*(Q, F_J) \otimes H^*(Q, F_{J'})$ (see Lemma 3.2).

It is clear that $(\mathcal{R}_Q^*, \mathbb{U})$ and $(\widetilde{\mathcal{R}}_Q^*, \widetilde{\mathbb{U}})$ determine each other.

- When $J \cap J' \neq \varnothing$, since $(D^2, S^1) \cong (\Sigma D^1, \Sigma S^0)$ is a pair of suspension spaces, Lemma 3.3 implies that

$$\widehat{\Delta}_{J \cup J', Q_+}^{J, J'} : \mathrm{colim}(\widehat{\boldsymbol{D}}_+^{J \cup J'}) \to \mathrm{colim}(\widehat{\boldsymbol{D}}_+^J) \wedge \mathrm{colim}(\widehat{\boldsymbol{D}}_+^{J'})$$

  is nullhomotopic. So, by (33), $\circledast$ is trivial in this case which corresponds to the definition of $\widetilde{\mathbb{U}}$ on $\widetilde{\mathcal{R}}_Q^*$.

- When $J \cap J' = \varnothing$, suppose in (35), we have elements

$$u = \phi \times \iota^J \in \widetilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J} S^1_{(j)}\right) = \widetilde{H}^*(\mathbf{\Sigma}^{|J|}(Q/F_J)),$$

$$v = \phi' \times \iota^{J'} \in \widetilde{H}^*\left(Q/F_{J'} \wedge \bigwedge_{j \in J'} S^1_{(j)}\right) = \widetilde{H}^*(\mathbf{\Sigma}^{|J'|}(Q/F_{J'})).$$

Then Lemmas 3.1 and 3.2 imply that

$$u \circledast v = (\widehat{\Delta}^{J,J'}_{J \cup J', Q_+})^*((\phi \times \iota^J) \times (\phi' \times \iota^{J'})) = (-1)^{|J||\phi'|}(\phi \,\widetilde{\cup}\, \phi') \times \iota^{J \cup J'}.$$

So we have a commutative diagram

(37)
$$
\begin{array}{ccc}
\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) & \xrightarrow{\;\;\widetilde{\mathbb{U}}\;\;} & \widetilde{H}^*(Q/F_{J \cup J'}) \\
{\scriptstyle \times \iota^J \otimes \times \iota^{J'}} \downarrow & & \downarrow {\scriptstyle \times \iota^{J \cup J'}} \\
\widetilde{H}^*(\mathbf{\Sigma}^{|J|}(Q/F_J)) \otimes \widetilde{H}^*(\mathbf{\Sigma}^{|J'|}(Q/F_{J'})) & \xrightarrow{\;\;\circledast\;\;} & \widetilde{H}^*(\mathbf{\Sigma}^{|J \cup J'|}(Q/F_{J \cup J'}))
\end{array}
$$

which implies that the product $\widetilde{\mathbb{U}}$ on $\widetilde{\mathscr{R}}^*_Q$ corresponds to the product $\circledast$ in (35) in this case.

Combining the above arguments, we obtain isomorphisms of rings,

$$(\widetilde{\mathscr{R}}^*_Q, \widetilde{\mathbb{U}}) \xrightarrow{\;\cong\;} \left(\bigoplus_{J \subseteq [m]} \widetilde{H}^*(Q/F_J \wedge \bigwedge_{j \in J} S^1_{(j)}), \circledast\right) \xrightarrow{\;\oplus_{J \subseteq [m]} \widehat{\Pi}^*_J\;} \widetilde{H}^*((D^2, S^1)^Q) = \widetilde{H}^*(\mathscr{Z}_Q).$$

It follows that there is a ring isomorphism (up to a sign) from $(\mathscr{R}^*_Q, \mathbb{U})$ to $H^*(\mathscr{Z}_Q)$.

Note that the above ring isomorphism is not degree-preserving. But by the diagram in (37), we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_J)$ up by $|J|$ for every $J \subseteq [m]$. $\qquad \square$

4   Polyhedral product over a nice manifold with corners

Let Q be a nice manifold with corners whose facets are $F_1, \ldots, F_m$. Let $(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m$, where X_j and A_j are CW–complexes with a basepoint $a_j \in A_j \subseteq X_j$.

For any face f of Q, define

$$(\mathbb{X}, \mathbb{A})^f := f \times \prod_{j \in I_f} X_j \times \prod_{j \in [m] \setminus I_f} A_j, \quad (\mathbb{X}, \mathbb{A})^Q := \bigcup_{f \in \mathscr{S}_Q} (\mathbb{X}, \mathbb{A})^f \subseteq Q \times \prod_{j \in [m]} X_j.$$

If $(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j) = (X, A, a_0)\}_{j=1}^m$, we also denote $(\mathbb{X}, \mathbb{A})^Q$ by $(X, A)^Q$.

We call $(\mathbb{X}, \mathbb{A})^Q$ the *polyhedral product* of $(\mathbb{X}, \mathbb{A})$ over Q. Note that in general, the homeomorphism type of $(\mathbb{X}, \mathbb{A})^Q$ depends on the ordering of the facets of Q and the ordering of the X_j. We consider $(\mathbb{X}, \mathbb{A})^Q$ as an analogue of polyhedral products over a simplicial complex; see [8].

In the rest of this section, we assume that each of X_j and A_j in $(\mathbb{X}, \mathbb{A})$ is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then we can study the stable decomposition and cohomology ring of $(\mathbb{X}, \mathbb{A})^Q$ in the same way as we do for $\mathscr{Z}_Q$.

- Let $Q_+ = Q \cup q_0$ where $q_0 \notin Q$ is the basepoint of Q_+.

- For any face f of Q, let $f_+ = f \cup q_0$ with basepoint q_0.

- For any $(f, L) \in \mathscr{P}_Q$, define

$$\tag{38} (\mathbb{X}, \mathbb{A})^{(f,L)} := f \times \prod_{j \in I_f \setminus L} X_j \times \prod_{j \in [m] \setminus (I_f \setminus L)} A_j, \quad (\mathbb{X}, \mathbb{A})^{(f,L)}_+ := (\mathbb{X}, \mathbb{A})^{(f,L)} \cup \hat{q}_0^{(\mathbb{X}, \mathbb{A})},$$

  where $\hat{q}_0^{(\mathbb{X}, \mathbb{A})}$ is the basepoint defined by $\hat{q}_0^{(\mathbb{X}, \mathbb{A})} := q_0 \times \prod_{j \in [m]} a_j$.

- Let $(\mathbb{X}, \mathbb{A})^Q_+ = (\mathbb{X}, \mathbb{A})^Q \cup \hat{q}_0^{(\mathbb{X}, \mathbb{A})}$ with basepoint $\hat{q}_0^{(\mathbb{X}, \mathbb{A})}$.

Let $\boldsymbol{D}_{(\mathbb{X}, \mathbb{A})+} : \mathscr{P}_Q \to \mathrm{CW}_*$ be the diagram of based CW–complexes, where

$$\boldsymbol{D}_{(\mathbb{X}, \mathbb{A})+}((f, L)) := (\mathbb{X}, \mathbb{A})^{(f,L)}_+ \quad \text{for all } (f, L) \in \mathscr{P}_Q,$$

and let $(d_{(\mathbb{X}, \mathbb{A})+})_{(f,L),(f',L')} : \boldsymbol{D}_{(\mathbb{X}, \mathbb{A})+}((f', L')) \to \boldsymbol{D}_{(\mathbb{X}, \mathbb{A})+}((f, L))$ be the natural inclusion for any $(f, L) \leq (f', L') \in \mathscr{P}_Q$. Then

$$\tag{39} (\mathbb{X}, \mathbb{A})^Q_+ = \mathrm{colim}(\boldsymbol{D}_{(\mathbb{X}, \mathbb{A})+}) = \bigcup_{(f,L) \in \mathscr{P}_Q} (\mathbb{X}, \mathbb{A})^{(f,L)}_+.$$

By Theorem 2.10, we can prove the following lemma parallel to Lemma 2.15.

Lemma 4.1 *For any $(f, L) \in \mathscr{P}_Q$, there is a natural homeomorphism which commutes with taking the colimit*

$$(\mathbb{X}, \mathbb{A})^{(f,L)}_+ \cong f_+ \ltimes \left(\prod_{j \in I_f \setminus L} X_j \times \prod_{j \in [m] \setminus (I_f \setminus L)} A_j \right).$$

So, by Theorem 2.14,

$$\tag{40} \boldsymbol{\Sigma}((\mathbb{X}, \mathbb{A})^{(f,L)}_+) \cong \boldsymbol{\Sigma}\left(f_+ \ltimes \left(\prod_{j \in I_f \setminus L} X_j \times \prod_{j \in [m] \setminus (I_f \setminus L)} A_j \right) \right)$$

$$\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}\left(f_+ \wedge \bigwedge_{j \in J \cap (I_f \setminus L)} X_j \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} A_j \right).$$

Accordingly, we define a family of diagrams of based CW–complexes,

$$\hat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+} : \mathscr{P}_Q \to \mathrm{CW}_*, \quad J \subseteq [m],$$

$$\hat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)) := f_+ \wedge \bigwedge_{j \in J \cap (I_f \setminus L)} X_j \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} A_j \quad \text{for all } (f, L) \in \mathscr{P}_Q.$$

Define $(\hat{d}^J_{(\mathbb{X},\mathbb{A})+})_{(f,L),(f',L')} \colon \hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f',L')) \to \hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ to be the natural inclusion for any $(f,L) \le (f',L') \in \mathscr{P}_Q$. The basepoint of $\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ is $\left[q_0 \times \prod_{j\in J} a_j\right]$. So we have the following theorem by [3, Theorem 4.3].

Theorem 4.2 *Let $(\mathbb{X},\mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m$ where each X_j and A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there are homotopy equivalences*

$$S^1 \vee \boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q) \simeq \boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q_+) = \boldsymbol{\Sigma}(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})).$$

This implies

$$H^*((\mathbb{X},\mathbb{A})^Q) \cong \widetilde{H}^*(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})).$$

Moreover, using the partial diagonal map for $Q_+ \times \prod_{j\in[m]} X_j$ as in the proof of Theorem 1.4, we have a diagram parallel to diagram (32) for any $J, J' \subseteq [m]$,

$$(41) \quad \begin{array}{ccc} \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) & \xrightarrow{\Delta^{Q_+,\mathbb{X}}_{[m]}} & \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) \wedge \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) \\ {\scriptstyle \hat{\Pi}_{J\cup J'}}\downarrow & & \downarrow{\scriptstyle \hat{\Pi}_J \wedge \hat{\Pi}_{J'}} \\ \operatorname{colim}(\hat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{X},\mathbb{A})+}) & \xrightarrow{\hat{\Delta}^{J,J'}_{J\cup J',Q_+}} & \operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}) \wedge \operatorname{colim}(\hat{\boldsymbol{D}}^{J'}_{(\mathbb{X},\mathbb{A})+}) \end{array}$$

Similarly, we can obtain the following theorem parallel to Theorem 1.4.

Theorem 4.3 *Let $(\mathbb{X},\mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m$ where each X_j and A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a ring isomorphism*

$$\bigoplus_{J\subseteq[m]} \hat{\Pi}^*_J \colon \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})) \to \widetilde{H}^*(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \cong H^*((\mathbb{X},\mathbb{A})^Q),$$

where the product $\circledast$ on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}))$ is defined by

$$(42) \quad \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})) \otimes \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^{J'}_{(\mathbb{X},\mathbb{A})+})) \xrightarrow{\circledast} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{X},\mathbb{A})+})),$$

$$u \circledast v := (\hat{\Delta}^{J,J'}_{J\cup J',Q_+})^*(u \times v).$$

In the following two subsections, we will study the stable decomposition and cohomology ring of $(\mathbb{X},\mathbb{A})^Q$ under some special conditions on $(\mathbb{X},\mathbb{A})$.

4.1 The case of $(\mathbb{X},\mathbb{A})^Q$ with each X_j contractible

Observe that in the proof of Theorem 1.1, the only properties of (D^2, S^1) that we actually use are that

(i) D^2 is contractible;

(ii) $X \wedge S^1$ is homeomorphic to $\boldsymbol{\Sigma}(X)$ for any based CW–complex X.

So, if we assume that every X_j in $(\mathbb{X}, \mathbb{A})$ is contractible, we can obtain the following theorem parallel to Theorem 1.1.

Theorem 4.4 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Let*

$$(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m,$$

where each X_j is contractible and each A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a homotopy equivalence

$$\mathbf{\Sigma}((\mathbb{X}, \mathbb{A})^Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}\left(Q/F_J \wedge \bigwedge_{j \in J} A_j\right).$$

So the reduced cohomology group

$$\widetilde{H}^*((\mathbb{X}, \mathbb{A})^Q) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J} A_j\right).$$

Proof We can easily extend the argument in the proof of Theorem 1.1 to show

$$\mathrm{colim}(\widehat{\boldsymbol{D}}_{(\mathbb{X}, \mathbb{A})+}^J) \simeq \begin{cases} Q/F_J \wedge \bigwedge_{j \in J} A_j & \text{if } J \neq \varnothing, \\ Q_+ & \text{if } J = \varnothing. \end{cases}$$

Then the statements of the theorem follow from Theorem 4.2 and the fact that $\mathbf{\Sigma}(Q_+) \simeq S^1 \vee \mathbf{\Sigma}(Q)$. $\square$

Moreover, we have the following theorem which is parallel to [4, Theorem 1.4].

Theorem 4.5 *Under the condition in Theorem 4.4, there is a ring isomorphism*

$$\left(\bigoplus_{J \subseteq [m]} \widetilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J} A_j\right), \circledast\right) \to \widetilde{H}^*((\mathbb{X}, \mathbb{A})^Q) \quad \text{induced by} \quad \bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^*.$$

Remark 4.6 If any combination of Q/F_J and A_j's satisfies the *strong smash form of the Künneth formula* as defined in [3, page 1647] over a coefficient ring $\boldsymbol{k}$, ie the natural map

$$\widetilde{H}^*(Q/F_J; \boldsymbol{k}) \otimes \bigotimes_{j \in I} \widetilde{H}^*(A_j; \boldsymbol{k}) \to \widetilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in I} A_j; \boldsymbol{k}\right)$$

is an isomorphism for any $I, J \subseteq [m]$, then we can write the cohomology ring structure of $(\mathbb{X}, \mathbb{A})^Q$ with $\boldsymbol{k}$–coefficients more explicitly via the formula in Lemma 3.1.

In the following, we demonstrate the product $\circledast$ for $(\mathbb{D}, \mathbb{S})^Q$ where

$$(\mathbb{D}, \mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j)\}_{j=1}^m.$$

Here D^{n+1} is the unit ball in $\mathbb{R}^{n+1}$ and $S^n = \partial D^{n+1}$.

In particular, if $(\mathbb{D}, \mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j) = (D^{n+1}, S^n, a_0)\}_{j=1}^m$, we also write

$$(\mathbb{D}, \mathbb{S})^Q = (D^{n+1}, S^n)^Q.$$

Example 4.7 $\mathscr{Z}_Q \cong (D^2, S^1)^Q$ and $\mathbb{R}\mathscr{Z}_Q \cong (D^1, S^0)^Q$ (see Remark 1.3).

We define a graded ring structure $\mathbb{U}^{(\mathbb{D},\mathbb{S})}$ on $\mathscr{R}_Q^*$ according to $(\mathbb{D}, \mathbb{S})$ as follows.

- If $J \cap J' = \varnothing$ or $J \cap J' \neq \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$, then

$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\mathbb{U}^{(\mathbb{D},\mathbb{S})}} H^*(Q, F_{J \cup J'})$$

  is the relative cup product $\cup$.

- If $J \cap J' \neq \varnothing$ and there exists $n_j \geq 1$ for some $j \in J \cap J'$, then

$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\mathbb{U}^{(\mathbb{D},\mathbb{S})}} H^*(Q, F_{J \cup J'})$$

  is trivial.

By Lemma 3.2, the product $\mathbb{U}^{(\mathbb{D},\mathbb{S})}$ on $\mathscr{R}_Q^*$ induces a product $\widetilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}$ on $\widetilde{\mathscr{R}}_Q^*$.

- If $J \cap J' = \varnothing$ or $J \cap J' \neq \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$, then

$$\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) \xrightarrow{\widetilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}} \widetilde{H}^*(Q/F_{J \cup J'})$$

  is the product $\widetilde{\cup}$ induced by the relative cup product $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\cup} H^*(Q, F_{J \cup J'})$.

- If $J \cap J' \neq \varnothing$ and there exists $n_j \geq 1$ for some $j \in J \cap J'$, then

$$\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) \xrightarrow{\widetilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}} \widetilde{H}^*(Q/F_{J \cup J'})$$

  is trivial.

We have the following theorem which generalizes Theorems 1.1 and 1.4.

Theorem 4.8 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Then for any*

$$(\mathbb{D}, \mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j)\}_{j=1}^m,$$

(a) *there is a homotopy equivalence*

$$\mathbf{\Sigma}((\mathbb{D}, \mathbb{S})^Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}\left(Q/F_J \wedge \bigwedge_{j \in J} S^{n_j}\right) \cong \bigvee_{J \subseteq [m]} \mathbf{\Sigma}^{1 + \sum_{j \in J} n_j}(Q/F_J),$$

  which implies

$$H^p((\mathbb{D}, \mathbb{S})^Q) \cong \bigoplus_{J \subseteq [m]} H^{p - \sum_{j \in J} n_j}(Q, F_J) \quad \text{for all } p \in \mathbb{Z};$$

(b) *there is a ring isomorphism (up to a sign) from $(\mathscr{R}_Q^*, \mathbb{U}^{(\mathbb{D},\mathbb{S})})$ to the integral cohomology ring of $(\mathbb{D}, \mathbb{S})^Q$; moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of the elements in $H^*(Q, F_J)$ for every $J \subseteq [m]$.*

Proof For brevity, we use the notation

$$N_J = \sum_{j \in J} n_j, \quad J \subseteq [m].$$

Statement (a) follows from Theorem 4.4 and the simple fact that

$$Q/F_J \wedge \bigwedge_{j \in J} S^{n_j} \cong Q/F_J \wedge S^{N_J} \cong \Sigma^{N_J}(Q/F_J).$$

For statement (b), note that, by Theorem 4.5, we have a ring isomorphism

(43)
$$\bigoplus_{J \subseteq [m]} \hat{\Pi}_J^* : \left(\bigoplus_{J \subseteq [m]} \tilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J} S^{n_j} \right), \circledast \right) \to \tilde{H}^*((\mathbb{D}, \mathbb{S})^Q).$$

For any $1 \le j \le m$, let ι^{n_j} denote a generator of $\tilde{H}^{n_j}(S^{n_j})$. Let

$$\iota_{(\mathbb{D},\mathbb{S})}^J = \bigtimes_{j \in J} \iota^{n_j} \in \tilde{H}^{N_J}\left(\bigwedge_{j \in J} S^{n_j} \right)$$

be a generator.

(i) Assume $J \cap J' \ne \varnothing$ and there exists $n_j \ge 1$ for some $j \in J \cap J'$. Then since S^{n_j} is a suspension space, the map $\hat{\Delta}_{J \cup J', Q_+}^{J, J'}$ in (41) is nullhomotopic. This implies that the product $\circledast$ in (43) is trivial which corresponds to the definition of $\tilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}$ on $\tilde{\mathscr{R}}_Q^*$ in this case.

(ii) Assume $J \cap J' \ne \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$. Let

$$J_0 = \{j \in [m] \mid n_j = 0\} \subseteq [m].$$

So the condition on J and J' is equivalent to $J \cap J' \subseteq J_0$ which implies

(44)
$$(J \setminus J_0) \cap (J' \setminus J_0) = \varnothing.$$

Since $X \wedge S^0 \cong X$ for any based space X, we have for any $J \subseteq [m]$,

$$Q/F_J \wedge \bigwedge_{j \in J} S^{n_j} \cong Q/F_J \wedge \bigwedge_{j \in J \setminus J_0} S^{n_j} \cong \Sigma^{N_{J \setminus J_0}}(Q/F_J).$$

By Lemmas 3.1 and 3.2, we can derive an explicit formula for the product $\circledast$ in (43) as follows. For any elements

$$u = \phi \times \iota_{(\mathbb{D},\mathbb{S})}^{J \setminus J_0} \in \tilde{H}^*\left(Q/F_J \wedge \bigwedge_{j \in J \setminus J_0} S^{n_j} \right) = \tilde{H}^*(\Sigma^{N_{J \setminus J_0}}(Q/F_J)),$$

$$v = \phi' \times \iota_{(\mathbb{D},\mathbb{S})}^{J' \setminus J_0} \in \tilde{H}^*\left(Q/F_{J'} \wedge \bigwedge_{j \in J' \setminus J_0} S^{n_j} \right) = \tilde{H}^*(\Sigma^{N_{J' \setminus J_0}}(Q/F_{J'})),$$

we have

$$u \circledast v = (\hat{\Delta}_{(J \cup J') \setminus J_0, Q_+}^{J \setminus J_0, J' \setminus J_0})^* \big((\phi \times \iota_{(\mathbb{D},\mathbb{S})}^{J \setminus J_0}) \times (\phi' \times \iota_{(\mathbb{D},\mathbb{S})}^{J' \setminus J_0}) \big) = (-1)^{N_{J \setminus J_0} |\phi'|}(\phi \tilde{\cup} \phi') \times \iota_{(\mathbb{D},\mathbb{S})}^{(J \cup J') \setminus J_0}$$

by (44). So we have a commutative diagram parallel to diagram (37),

$$
\begin{array}{ccc}
\tilde{H}^*(Q/F_J) \otimes \tilde{H}^*(Q/F_{J'}) & \xrightarrow{\;\;\tilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}\;\;} & \tilde{H}^*(Q/F_{J\cup J'}) \\
{\scriptstyle \times\iota^{J\backslash J_0}_{(\mathbb{D},\mathbb{S})}\otimes\times\iota^{J'\backslash J_0}_{(\mathbb{D},\mathbb{S})}}\Big\downarrow & & \Big\downarrow{\scriptstyle \times\iota^{(J\cup J')\backslash J_0}_{(\mathbb{D},\mathbb{S})}} \\
\tilde{H}^*(\mathbf{\Sigma}^{N_{J\backslash J_0}}(Q/F_J)) \otimes \tilde{H}^*(\mathbf{\Sigma}^{N_{J'\backslash J_0}}(Q/F_{J'})) & \xrightarrow{\;\;\circledast\;\;} & \tilde{H}^*(\mathbf{\Sigma}^{N_{(J\cup J')\backslash J_0}}(Q/F_{J\cup J'}))
\end{array}
$$

This implies that the product $\tilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}$ on $\tilde{\mathscr{R}}^*_Q$ corresponds to the product $\circledast$ in (43) in this case.

(iii) When $J \cap J' = \varnothing$, the proof of the correspondence between the product $\tilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}$ on $\tilde{\mathscr{R}}^*_Q$ and the product $\circledast$ in (43) is the same as case (ii).

The above discussion implies that there is an isomorphism of rings

$$
(\tilde{\mathscr{R}}^*_Q, \tilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}) \to \left(\bigoplus_{J\subseteq[m]} \tilde{H}^*\Big(Q/F_J \wedge \bigwedge_{j\in J} S^{n_j}\Big), \circledast \right) \xrightarrow{\;\oplus_{J\subseteq[m]}\hat{\Pi}^*_J\;} \tilde{H}^*((\mathbb{D},\mathbb{S})^Q).
$$

This implies that $(\mathscr{R}^*_Q, \mathbb{U}^{(\mathbb{D},\mathbb{S})})$ is isomorphic (up to a sign) to the integral cohomology ring $H^*((\mathbb{D},\mathbb{S})^Q)$. Moreover, according to the above diagram, we can make the ring isomorphism between $(\mathscr{R}^*_Q, \mathbb{U}^{(\mathbb{D},\mathbb{S})})$ and $H^*((\mathbb{D},\mathbb{S})^Q)$ degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_J)$ up by $N_{J\backslash J_0}$ for every $J \subseteq [m]$. $\qquad\square$

Remark 4.9 S^0 is not a suspension of any space and the reduced diagonal map

$$
\Delta_{S^0} = \mathrm{id}_{S^0} \colon S^0 \to S^0 \wedge S^0 \cong S^0
$$

is not nullhomotopic. This is the essential reason why for a general $(\mathbb{D},\mathbb{S})$, the cohomology ring of $(\mathbb{D},\mathbb{S})^Q$ is more subtle than that of $\mathscr{Z}_Q$.

A very special case of Theorem 4.8 is $(D^1, S^0)^Q = \mathbb{R}\mathscr{Z}_Q$ where the product $\mathbb{U}^{(D^1,S^0)}$ on $\mathscr{R}^*_Q$ is exactly the relative cup product for all $J, J' \subseteq [m]$.

Corollary 4.10 *Let Q be a nice manifold with corners with facets $F_1,\dots,F_m$. Then*

$$
\mathbf{\Sigma}(\mathbb{R}\mathscr{Z}_Q) \simeq \bigvee_{J\subseteq[m]} \mathbf{\Sigma}(Q/F_J), \quad H^p(\mathbb{R}\mathscr{Z}_Q) \cong \bigoplus_{J\subseteq[m]} H^p(Q, F_J) \quad \text{for all } p \in \mathbb{Z}.
$$

Moreover, the integral cohomology ring of $\mathbb{R}\mathscr{Z}_Q$ is isomorphic as a graded ring to the ring $(\mathscr{R}^*_Q, \cup)$ where $\cup$ is the relative cup product

$$
H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\;\cup\;} H^*(Q, F_{J\cup J'}) \quad \text{for all } J, J' \subseteq [m].
$$

Note that in the case of $\mathbb{R}\mathscr{Z}_Q$, the sign factor of the isomorphism between $(\mathscr{R}^*_Q, \cup)$ and $H^*(\mathbb{R}\mathscr{Z}_Q)$ is trivial because the degree of $\iota^J_{(D^1,S^0)}$ is always zero.

Remark 4.11 When Q is a simple polytope, the ring structure of the integral cohomology of $\mathbb{R}\mathscr{Z}_Q$ was studied in [11] via a different method.

4.2 The case of $(\mathbb{X}, \mathbb{A})^Q$ with each A_j contractible

If in $(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m$, each A_j is contractible, we can derive the stable decomposition of $(\mathbb{X}, \mathbb{A})^Q$ from Theorem 4.2 as follows.

Theorem 4.12 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Let*

$$(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m,$$

where each A_j is contractible and each X_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a homotopy equivalence

$$S^1 \vee \mathbf{\Sigma}((\mathbb{X}, \mathbb{A})^Q) \simeq \mathbf{\Sigma}((\mathbb{X}, \mathbb{A})^Q_+) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}\left((F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \right).$$

So we have $\mathbf{\Sigma}((\mathbb{X}, \mathbb{A})^Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}\left(F_{\cap J} \ltimes \bigwedge_{j \in J} X_j \right)$ and

$$H^*((\mathbb{X}, \mathbb{A})^Q) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^*\left((F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \right).$$

Proof By Lemma 2.7 and our assumption on A_j, when $J \setminus (I_f \setminus L) \neq \varnothing$,

$$\widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)) = f_+ \wedge \bigwedge_{j \in J \cap (I_f \setminus L)} X_j \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} A_j$$

is contractible. So, for any $J \subseteq [m]$, we define a diagram of based CW–complexes

$$\widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+} : \mathscr{P}_Q \to \mathrm{CW}_*$$

(45) $\qquad \widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)) := \begin{cases} \widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)) = f_+ \wedge \bigwedge_{j \in J} X_j & \text{if } J \setminus (I_f \setminus L) = \varnothing, \\ [\hat{q}_0^J] & \text{if } J \setminus (I_f \setminus L) \neq \varnothing, \end{cases}$

where $(\hat{g}^J_{(\mathbb{X}, \mathbb{A})+})_{(f,L),(f',L')} : \widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}((f', L')) \to \widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L))$ is either the natural inclusion or the constant map $c_{[\hat{q}_0^J]}$ for any $(f, L) \leq (f', L') \in \mathscr{P}_Q$. The basepoint of $\widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L))$ is $[\hat{q}_0^J]$.

Let $\Psi^J_{(\mathbb{X}, \mathbb{A})+} : \widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+} \to \widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}$ be a map of diagrams over $\mathscr{P}_Q$ defined by

$$(\Psi^J_{(\mathbb{X}, \mathbb{A})+})_{(f,L)} : \widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)) \to \widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}((f, L)),$$

$$(\Psi^J_{(\mathbb{X}, \mathbb{A})+})_{(f,L)} = \begin{cases} \mathrm{id}_{\widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}((f,L))} & \text{if } J \setminus (I_f \setminus L) = \varnothing, \\ c_{[\hat{q}_0^J]} & \text{if } J \setminus (I_f \setminus L) \neq \varnothing. \end{cases}$$

Then by Theorem 2.16, there exists a homotopy equivalence

$$\mathrm{colim}(\widehat{\boldsymbol{D}}^J_{(\mathbb{X}, \mathbb{A})+}) \simeq \mathrm{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+}), \quad J \subseteq [m].$$

To understand $\mathrm{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X}, \mathbb{A})+})$, we need to figure out in (45) what are those faces f of Q with some $L \subseteq I_f$ such that $J \setminus (I_f \setminus L) = \varnothing$.

- There exists $L \subseteq I_f$ such that $J \setminus (I_f \setminus L) = \varnothing$ if and only if $J \subseteq I_f$. So

$$(46) \qquad \bigcup_{\substack{f \in \mathscr{S}_Q \\ \exists L \subseteq I_f \\ J \setminus (I_f \setminus L) = \varnothing}} f = \bigcup_{f \in \mathscr{S}_Q} \bigcup_{J \subseteq I_f} f = F_{\cap J}.$$

- There exists $L \subseteq I_f$ such that $J \setminus (I_f \setminus L) \neq \varnothing$ if and only if $J \neq \varnothing$.

Then for any $\varnothing \neq J \subseteq [m]$,

$$\mathrm{colim}(\hat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}) \cong \bigcup_{\substack{\exists L \subseteq I_f \\ J \setminus (I_f \setminus L) = \varnothing}} \left(f_+ \wedge \bigwedge_{j \in J} X_j \right) \Big/ \bigcup_{\substack{\exists L \subseteq I_f \\ J \setminus (I_f \setminus L) \neq \varnothing}} \left(f_+ \wedge \bigwedge_{j \in I_f \setminus L} X_j \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} A_j \right)$$

$$(47) \qquad \simeq \left(\left(\bigcup_{J \subseteq I_f} f_+ \right) \wedge \bigwedge_{j \in J} X_j \right) \Big/ \bigcup_{\substack{f \in \mathscr{S}_Q, L \subseteq I_f \\ J \setminus (I_f \setminus L) \neq \varnothing}} \left(f_+ \wedge \bigwedge_{j \in I_f \setminus L} X_j \wedge \bigwedge_{j \in J \setminus (I_f \setminus L)} a_j \right)$$

$$(48) \qquad \cong \left(\bigcup_{J \subseteq I_f} f_+ \right) \wedge \bigwedge_{j \in J} X_j \Big/ [\hat{q}_0^J] = (F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \quad \text{(by (46))},$$

where the "$\simeq$" in (47) is because each A_j is contractible, so A_j deformation retracts to its basepoint a_j, and the "$\cong$" in (48) is because $f_+ \times \prod_{j \in I_f \setminus L} X_j \times \prod_{j \in J \setminus (I_f \setminus L)} a_j$ is equivalent to the basepoint $[\hat{q}_0^J]$ in $f_+ \wedge \bigwedge_{j \in J} X_j$ since a_j is the basepoint of X_j.

When $J = \varnothing$,

$$\mathrm{colim}(\hat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}) = \bigcup_{f \in \mathscr{S}_Q} f_+ = Q \cup q_0 = F_{\cap \varnothing} \cup q_0.$$

So, by Theorem 4.2, we have homotopy equivalences

$$\boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})_+^Q) = \boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\mathrm{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}))$$

$$\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\mathrm{colim}(\hat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+})) \simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}\left((F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \right).$$

By Definition 2.13,

$$(F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \cong \begin{cases} F_{\cap J} \ltimes \bigwedge_{j \in J} X_j & \text{if } J \neq \varnothing, \\ Q \cup q_0 & \text{if } J = \varnothing. \end{cases}$$

Then since $\boldsymbol{\Sigma}(Q \cup q_0) \simeq S^1 \vee \boldsymbol{\Sigma}(Q)$, the theorem is proved. $\qquad\square$

The cohomology ring structure of $(\mathbb{X},\mathbb{A})^Q$ can be computed by Theorem 4.3. In particular, if any combination of $F_{\cap J}$ and X_j's satisfies the strong smash form of the Künneth formula over a coefficient ring $\boldsymbol{k}$, we can give an explicit description of the cohomology ring of $(\mathbb{X},\mathbb{A})^Q$ with $\boldsymbol{k}$–coefficients. Indeed,

by Theorems 4.3 and 4.12 we obtain an isomorphism of rings

$$\text{(49)} \qquad \bigoplus_{J \subseteq [m]} \hat{\Pi}_J^* : \bigoplus_{J \subseteq [m]} \left(H^*(F_{\cap J}; \boldsymbol{k}) \otimes \bigotimes_{j \in J} \tilde{H}^*(X_j; \boldsymbol{k}) \right) \to H^*((\mathbb{X}, \mathbb{A})^Q; \boldsymbol{k}),$$

where the product $\circledast$ on the left-hand side is defined by (42) via the partial diagonal maps. We will do some computation of this kind in the next section to describe the equivariant cohomology ring of the moment-angle manifold $\mathcal{Z}_Q$.

5 Equivariant cohomology ring of $\mathcal{Z}_Q$ and $\mathbb{R}\mathcal{Z}_Q$

Let Q be a nice manifold with corners whose facets are $F_1, \ldots, F_m$. Since there is a canonical action of $(S^1)^m$ on $\mathcal{Z}_Q$ (see (2)), it is a natural problem to compute the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to this action.

For a simple polytope P, it is shown in Davis and Januszkiewicz [17] that the equivariant cohomology of $\mathcal{Z}_P$ with integral coefficients is isomorphic to the face ring (or Stanley–Reisner ring) $\mathbb{Z}[P]$ of P defined by

$$\mathbb{Z}[P] = \mathbb{Z}[x_1, \ldots, x_m]/\mathcal{I}_P,$$

where $\mathcal{I}_P$ is the ideal generated by all square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ such that $F_{i_1} \cap \cdots \cap F_{i_s} = \varnothing$ in P. A liner basis of $\mathbb{Z}[P]$ is given by

$$\text{(50)} \qquad \{1\} \cup \{x_{i_1}^{n_1} \cdots x_{i_s}^{n_s} \mid F_{i_1} \cap \cdots \cap F_{i_s} \neq \varnothing, \, n_1 > 0, \ldots, n_s > 0\}.$$

We can also think of $\mathbb{Z}[P]$ as the face ring of ∂P^* where P^* is the dual simplicial polytope of P; see [9, Chapter 3].

For brevity, let $T^m = (S^1)^m$. By definition, the *equivariant cohomology* of $\mathcal{Z}_Q$, denoted by $H_{T^m}^*(\mathcal{Z}_Q)$, is the cohomology of the *Borel construction*

$$ET^m \times_{T^m} \mathcal{Z}_Q = ET^m \times \mathcal{Z}_Q/\sim,$$

where $(e, x) \sim (eg, g^{-1}x)$ for any $e \in ET^m$, $x \in \mathcal{Z}_Q$ and $g \in T^m$. Here we let

$$ET^m = (ES^1)^m = (S^\infty)^m.$$

Associated to the Borel construction, there is a canonical fiber bundle

$$\text{(51)} \qquad \mathcal{Z}_Q \to ET^m \times_{T^m} \mathcal{Z}_Q \to BT^m,$$

where $BT^m = (BS^1)^m = (S^\infty/S^1)^m = (\mathbb{C}P^\infty)^m$ is the *classifying space* of T^m.

By Lemma 2.3, $\mathcal{Z}_Q$ is equivariantly homeomorphic to $(D^2, S^1)^Q$. So computing the equivariant cohomology of $\mathcal{Z}_Q$ is equivalent to computing that for $(D^2, S^1)^Q$.

By the colimit construction of $(D^2, S^1)^Q$ in (16) and our notation for polyhedral products (38), the Borel construction

$$ET^m \times_{T^m} (D^2, S^1)^Q = \bigcup_{(f,L)\in\mathscr{P}_Q} ET^m \times_{T^m} (D^2, S^1)^{(f,L)}$$

$$= \bigcup_{(f,L)\in\mathscr{P}_Q} (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^{(f,L)}$$

$$= (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^Q.$$

Then, by the homotopy equivalence of the pairs

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1) \to (\mathbb{C}P^\infty, *),$$

we can derive from Theorem 2.16 that there is a homotopy equivalence

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^Q \simeq (\mathbb{C}P^\infty, *)^Q.$$

We call $(\mathbb{C}P^\infty, *)^Q$ the *Davis–Januszkiewicz space* of Q, denoted by $\mathscr{DJ}(Q)$. So the equivariant cohomology ring of $\mathscr{Z}_Q$ is isomorphic to the ordinary cohomology ring of $\mathscr{DJ}(Q)$.

Similarly, we can prove that the Borel construction of $\mathbb{R}\mathscr{Z}_Q$ with respect to the canonical $(\mathbb{Z}_2)^m$–action is $(\mathbb{R}P^\infty, *)^Q$.

Proof of Theorem 1.7 By the proof of Theorem 4.12 and the fact that $H^*(\mathbb{C}P^\infty)$ is torsion free, we can deduce from (48) that

$$\widetilde{H}^*(\widehat{\boldsymbol{D}}^J_{(\mathbb{C}P^\infty,*)+}) \cong \widetilde{H}^*(\widehat{\boldsymbol{G}}^J_{(\mathbb{C}P^\infty,*)+}) \cong H^*(F_{\cap J}) \otimes \bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}) \quad \text{for all } J \subseteq [m],$$

where $(\mathbb{C}P^\infty)^m = \prod_{j\in[m]} \mathbb{C}P^\infty_{(j)}$. Then we obtain a ring isomorphism from (49),

$$\bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J : \bigoplus_{J\subseteq[m]} \left(H^*(F_{\cap J}) \otimes \bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}) \right) \to H^*((\mathbb{C}P^\infty, *)^Q) \cong H^*_{T^m}(\mathscr{Z}_Q),$$

where the product $\circledast$ on the left-hand side is defined by (42) via the partial diagonal maps

$$\mathrm{colim}(\widehat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{C}P^\infty,*)+}) \xrightarrow{\widehat{\Delta}^{J,J'}_{J\cup J',Q_+}} \mathrm{colim}(\widehat{\boldsymbol{D}}^J_{(\mathbb{C}P^\infty,*)+}) \wedge \mathrm{colim}(\widehat{\boldsymbol{D}}^{J'}_{(\mathbb{C}P^\infty,*)+}).$$

Example 5.1 If $Q = [0, 1)$, the moment-angle manifold $\mathscr{Z}_{[0,1)} = D^2 \backslash S^1$, whose Borel construction is homotopy equivalent to $\mathbb{C}P^\infty$. Then

$$H^*_{S^1}(\mathscr{Z}_{[0,1)}) \cong H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \ \deg(x) = 2.$$

The above ring isomorphism implies that the homomorphism $\Delta^*_{\mathbb{C}P^\infty}$ induced by the reduced diagonal map $\Delta_{\mathbb{C}P^\infty} : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty$ on the integral cohomology is given by

$$\Delta^*_{\mathbb{C}P^\infty} : \widetilde{H}^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \cong \widetilde{H}^*(\mathbb{C}P^\infty) \otimes \widetilde{H}^*(\mathbb{C}P^\infty) \to \widetilde{H}^*(\mathbb{C}P^\infty), \quad \theta \otimes \theta' \to \theta \cup \theta'.$$

Then, by Lemma 3.1 and the above example, for any elements

$$u = \phi \otimes \bigotimes_{j \in J} \theta_j, \qquad \phi \in H^*(F_{\cap J}), \; \theta_j \in \tilde{H}^*(\mathbb{C}P^\infty_{(j)}),$$

$$v = \phi' \otimes \bigotimes_{j \in J'} \theta'_j, \quad \phi' \in H^*(F_{\cap J'}), \; \theta'_j \in \tilde{H}^*(\mathbb{C}P^\infty_{(j)}),$$

we have

$$u \circledast v = (\kappa^*_{J \cup J', J}(\phi) \cup \kappa^*_{J \cup J', J'}(\phi')) \otimes \bigotimes_{J \setminus J'} \theta_j \otimes \bigotimes_{J' \setminus J} \theta'_j \otimes \bigotimes_{j \in J \cap J'} (\theta_j \cup \theta'_j),$$

where $\kappa_{I', I} \colon F_{\cap I'} \to F_{\cap I}$ is the inclusion map for any subsets $I \subseteq I' \subseteq [m]$.

Finally, since there is a graded ring isomorphism

$$H^*((\mathbb{C}P^\infty)^m) \cong \mathbb{Z}[x_1, \dots, x_m], \; \deg(x_1) = \dots = \deg(x_m) = 2,$$

it is easy to check that $\bigoplus_{J \subseteq [m]} \left(H^*(F_{\cap J}) \otimes \bigotimes_{j \in J} \tilde{H}^*(\mathbb{C}P^\infty_{(j)}) \right)$ with the product $\circledast$ is isomorphic to the topological face ring $\mathbb{Z}\langle Q \rangle = \bigoplus_{J \subseteq [m]} H^*(F_{\cap J}) \otimes R_\mathbb{Z}^J$, where $\bigotimes_{j \in J} \tilde{H}^*(\mathbb{C}P^\infty_{(j)})$ corresponds to $R_\mathbb{Z}^J$; see (6).

By replacing (D^2, S^1) with (D^1, S^0), $(S^1)^m$ with $(\mathbb{Z}_2)^m$ and $\mathbb{C}P^\infty$ with $\mathbb{R}P^\infty$ in the above argument, and by the fact $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$, $\deg(x) = 1$, we obtain the parallel result for $\mathbb{R}\mathcal{Z}_Q$. $\qquad \square$

From the canonical fiber bundle associated to the Borel construction in (51), we have a natural $H^*(BT^m)$–module structure on $H^*_{T^m}(\mathcal{Z}_Q)$. By the identification

$$H^*(BT^m) = \mathbb{Z}[x_1, \dots, x_m],$$

we can write the $H^*(BT^m)$–module structure on $H^*_{T^m}(\mathcal{Z}_Q)$ as, for each $1 \leq i \leq m$,

$$(52) \qquad x_i \cdot (\phi \otimes f(x)) = (1 \otimes x_i) \star (\phi \otimes f(x)) = \kappa^*_{J \cup \{i\}, J}(\phi) \otimes x_i f(x) \quad \text{(by (8))},$$

where $\phi \in H^*(F_{\cap J})$ and $f(x) \in R_\mathbb{Z}^J$, $J \subseteq [m]$.

Example 5.2 Let P be a simple polytope with facets $F_1, \dots, F_m$. For a subset $J \subseteq [m]$, $F_{\cap J}$ is either empty or a face of P and hence acyclic. So we can write the topological face ring of P as

$$\mathbb{Z}\langle P \rangle \cong \left(\bigoplus_{\substack{F_{\cap J} \neq \varnothing \\ J \subseteq [m]}} R_\mathbb{Z}^J, \star \right),$$

where for any $f(x) \in R_\mathbb{Z}^J$ and $f'(x) \in R_\mathbb{Z}^{J'}$ with $F_{\cap J} \neq \varnothing$ and $F_{\cap J'} \neq \varnothing$,

$$f(x) \star f'(x) = \begin{cases} f(x) f'(x) & \text{if } F_{\cap (J \cup J')} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

According to the linear basis of the face ring $\mathbb{Z}[P]$ in (50), we can easily check that $\mathbb{Z}\langle P \rangle$ is isomorphic to $\mathbb{Z}[P]$.

Theorem 5.3 Let Q be a nice manifold with corners with m facets. If a subtorus $H \subseteq T^m = (S^1)^m$ acts freely on $\mathscr{Z}_Q$ through the canonical action, the equivariant cohomology ring with $\mathbb{Z}$–coefficients of the quotient space $\mathscr{Z}_Q/H$ with respect to the induced action of T^m/H is isomorphic to the topological face ring $\mathbb{Z}\langle Q \rangle$ of Q.

Proof Suppose $T^m/H \cong T^k$. Since H acts freely on $\mathscr{Z}_Q$, the Borel constructions of $\mathscr{Z}_Q/H$ and $\mathscr{Z}_Q$ are homotopy equivalent by

$$(53) \qquad ET^m \times_{T^m} \mathscr{Z}_Q \cong EH \times (E(T^m/H) \times_{T^m/H} (\mathscr{Z}_Q/H)) \simeq ET^k \times_{T^k} \mathscr{Z}_Q/H.$$

So the equivariant cohomology ring of $\mathscr{Z}_Q/H$ is isomorphic to the equivariant cohomology ring of $\mathscr{Z}_Q$. Then the theorem follows from Theorem 1.7.   □

In Theorem 5.3, the group homomorphism $T^m \to T^m/H \cong T^k$ induces a commutative diagram

$$\begin{array}{ccc} ET^m & \longrightarrow & E(T^m/H) \\ \downarrow & & \downarrow \\ BT^m & \longrightarrow & B(T^m/H) \end{array}$$

which, along with the maps in (53), induces the diagram

$$(54) \qquad \begin{array}{ccc} ET^m \times_{T^m} \mathscr{Z}_Q & \overset{\simeq}{\longrightarrow} & E(T^m/H) \times_{T^m/H} (\mathscr{Z}_Q/H) \\ \downarrow & & \downarrow \\ BT^m & \longrightarrow & B(T^m/H) \end{array}$$

We can describe the natural $H^*(B(T^m/H))$–module structure of the integral equivariant cohomology ring of $\mathscr{Z}_Q/H$ as follows. The inclusion $H \hookrightarrow T^m$ induces a monomorphism $\varphi_H \colon \mathbb{Z}^{m-k} \to \mathbb{Z}^m$ whose image is a direct summand in $\mathbb{Z}^m$. This determines an integer $m \times (m-k)$ matrix $S = (s_{ij})$ if we choose a basis for each of $\mathbb{Z}^{m-k}$ and $\mathbb{Z}^m$. Then since the image of φ_H is a direct summand in $\mathbb{Z}^m$, there is an integer $k \times m$ matrix $R = (r_{ij})$ of rank k such that $R \cdot S = 0$ which defines the homomorphism $T^m \to T^m/H$.

If we write $H^*(B(T^m/H)) = H^*(BT^k) = \mathbb{Z}[y_1, \ldots, y_k]$, it follows from the diagram (54) that the natural $H^*(B(T^m/H))$–module structure of the integral equivariant cohomology ring of $\mathscr{Z}_Q/H$ is determined by the formula in (52) along with the map $H^*(B(T^m/H)) \to H^*(BT^m)$ given by

$$\mathbb{Z}[y_1, \ldots, y_k] \to \mathbb{Z}[x_1, \ldots, x_m], \quad y_i \mapsto r_{i1}x_1 + \cdots + r_{im}x_m.$$

The above formula is parallel to the formula given in [9, Theorem 7.37] (where Q is a simple polytope).

Remark 5.4 If a subtorus $H \subseteq T^m$ of dimension $m - \dim(Q)$ acts freely on $\mathscr{Z}_Q$ through the canonical action, the quotient space $\mathscr{Z}_Q/H$ with the induced action of $T^m/H \cong T^{\dim(Q)}$ can be considered as a generalization of *quasitoric manifold* over a simple polytope defined by Davis and Januszkiewicz [17].

The following is an application of Theorem 5.3 to locally standard torus actions on closed manifolds. Recall that an action of T^n on a closed $2n$–manifold M^{2n} is called *locally standard* — see [17, Section 1] — if every point in M^{2n} has a T^n–invariant neighborhood that is weakly equivariantly diffeomorphic an open subset of $\mathbb{C}^n$ invariant under the *standard T^n–action*,

$$(g_1, \dots, g_n) \cdot (z_1, \dots, z_n) = (g_1 z_1, \dots, g_n z_n), \quad g_i \in S^1, z_i \in \mathbb{C}, 1 \leq i \leq n.$$

Corollary 5.5 *Let M^{2n} be a closed smooth $2n$–manifold with a smooth locally standard T^n–action and the free part of the action is a trivial T^n–bundle. Then the integral equivariant cohomology ring $H^*_{T^n}(M^{2n})$ of M^{2n} is isomorphic to the topological face ring $\mathbb{Z}\langle M^{2n}/T^n \rangle$.*

Proof The orbit space $Q = M^{2n}/T^n$ is a smooth nice manifold with corners since the T^n–action is locally standard and smooth. Then Q is triangulable — by [25] — and hence all our theorems can be applied to Q. In addition, using the characteristic function argument in Davis and Januszkiewicz [17] — see also [27, Section 4.2] or [35] — we can prove that M^{2n} is a free quotient space of $\mathcal{Z}_Q$ by a canonical action of some torus. Then this corollary follows from Theorem 5.3. □

Remark 5.6 The equivariant cohomology ring $H^*_{T^n}(M^{2n})$ in the above corollary was also computed by Ayzenberg, Masuda, Park and Zeng [2, Proposition 5.2] under an extra assumption that all the proper faces of M^{2n}/T^n are acyclic. We leave it as an exercise for the reader to check that the formula for $H^*_{T^n}(M^{2n})$ given in [2] is isomorphic to $\mathbb{Z}\langle M^{2n}/T^n \rangle$.

6 Generalizations

Let Q be a nice manifold with corners with facets $\mathcal{F}(Q) = \{F_1, \dots, F_m\}$. Observe that neither in the construction of $\mathcal{Z}_Q$ nor in the proof of Theorems 1.1 and 1.4 do we really use the connectedness of each facet F_j. So we have the following generalization of $\mathcal{Z}_Q$.

Let $\mathcal{J} = \{J_1, \dots, J_k\}$ be a *partition* of $[m] = \{1, \dots, m\}$, ie the J_i are disjoint subsets of $[m]$ with $J_1 \cup \dots \cup J_k = [m]$. So $\partial Q = F_{J_1} \cup \dots \cup F_{J_k}$. Moreover, we require $\mathcal{J}$ to satisfy

(55) for any $1 \leq i \leq k$, if $j, j' \in J_i$, then $F_j \cap F_{j'} = \varnothing$.

From Q and the partition $\mathcal{J}$, we can construct the following manifold.

Let $\{e_1, \dots, e_k\}$ be a unimodular basis of $\mathbb{Z}^k$. Let $\mu \colon \mathcal{F}(Q) \to \mathbb{Z}^k$ be the map which sends all the facets in F_{J_i} to e_i for every $1 \leq i \leq k$. Define

$$\mathcal{Z}_{Q,\mathcal{J}} := Q \times (S^1)^k / \sim,$$

where $(x, g) \sim (x', g')$ if and only if $x = x'$ and $g^{-1}g' \in \mathbb{T}_x^\mu$ where $\mathbb{T}_x^\mu$ is the subtorus of $(S^1)^k = \mathbb{R}^k / \mathbb{Z}^k$ determined by the linear subspace of $\mathbb{R}^k$ spanned by the set $\{\mu(F_j) \mid x \in F_j\}$. There is a canonical action of $(S^1)^k$ on $\mathcal{Z}_{Q,\mathcal{J}}$ defined by

$$(56) \qquad\qquad g' \cdot [(x, g)] = [(x, g'g)], \quad x \in Q, \ g, g' \in (S^1)^k.$$

If $\mathcal{J}_0 = \{\{1\}, \dots, \{m\}\}$ is the trivial partition of $[m]$, then $\mathcal{Z}_{Q,\mathcal{J}_0} = \mathcal{Z}_Q$.

Note that here $\{F_{J_i}\}$ play the role of facets $\{F_j\}$ in the definition of $\mathcal{Z}_Q$, but F_{J_i} may not be connected. Using the term defined in Davis [14], the decomposition of ∂Q into $\{F_{J_i}\}$ is called a *panel structure on Q* and each F_{J_i} is called a *panel*.

Remark 6.1 For a general partition $\mathcal{J}$ of $[m]$, it is possible that $F_j \cap F_{j'} \neq \varnothing$ for some $j, j' \in J_i$. Although the definition of $\mathcal{Z}_{Q,\mathcal{J}}$ still makes sense in the general setting, the orbit space of the $(S^1)^k$–action on $\mathcal{Z}_{Q,\mathcal{J}}$ may not be Q (as a manifold with corners). It would be Q with some corners smoothed. But for a general partition of $[m]$, one can always reduce to the case where the condition (55) is satisfied by smoothing the corners of the orbit space.

For any subset $\omega \subseteq [k] = \{1, \dots, k\}$, let

$$F_\omega = \bigcup_{i \in \omega} F_{J_i}, \quad F_\varnothing = \varnothing, \quad F_{\cap \omega} = \bigcap_{i \in \omega} F_{J_i}, \quad F_{\cap \varnothing} = Q.$$

Theorem 6.2 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. For any partition $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[m] = \{1, \dots, m\}$,*

$$\mathbf{\Sigma}(\mathcal{Z}_{Q,\mathcal{J}}) \simeq \bigvee_{\omega \subseteq [k]} \mathbf{\Sigma}^{|\omega|+1}(Q/F_\omega), \quad H^p(\mathcal{Z}_{Q,\mathcal{J}}) \cong \bigoplus_{\omega \subseteq [k]} H^{p-|\omega|}(Q, F_\omega) \quad \text{for all } p \in \mathbb{Z}.$$

Proof We can generalize the rim-cubicalization of Q in Section 2.1 as follows. For any face f of Q, let

$$I_f^{\mathcal{J}} = \{i \in [k] \mid f \subseteq F_{J_i}\} \subseteq [k].$$

Then define

$$\hat{f}^{\mathcal{J}} = f \times \prod_{i \in I_f^{\mathcal{J}}} [0, 1]_{(i)} \times \prod_{i \in [k] \setminus I_f^{\mathcal{J}}} 1_{(i)}, \quad \hat{Q}^{\mathcal{J}} = \bigcup_{f \in \mathscr{S}_Q} \hat{f}^{\mathcal{J}} \subseteq Q \times [0, 1]^k.$$

By the same argument as in the proof of Lemma 2.1, we can show that $\hat{Q}^{\mathcal{J}}$ with faces $\hat{f}^{\mathcal{J}}$ is homeomorphic to Q as a manifold with corners. The partition $\mathcal{J}$ of the facets of Q naturally induces a partition of the corresponding facets of $\hat{Q}^{\mathcal{J}}$, also denoted by $\mathcal{J}$. So we have $\mathcal{Z}_{\hat{Q}^{\mathcal{J}},\mathcal{J}} \cong \mathcal{Z}_{Q,\mathcal{J}}$.

For any face f of Q, let

$$(D^2, S^1)_{\mathcal{J}}^f := f \times \prod_{i \in I_f^{\mathcal{J}}} D_{(i)}^2 \times \prod_{i \in [k] \setminus I_f^{\mathcal{J}}} S_{(i)}^1, \quad (D^2, S^1)_{\mathcal{J}}^Q := \bigcup_{f \in \mathscr{S}_Q} (D^2, S^1)_{\mathcal{J}}^f \subseteq Q \times (D^2)^k.$$

There is a canonical $(S^1)^k$–action on $(D^2, S^1)_{\mathcal{J}}^Q$ induced from the canonical $(S^1)^k$–action on $Q \times (D^2)^k$. And parallel to Lemma 2.3, we can prove that there is an equivariant homeomorphism from $(D^2, S^1)_{\mathcal{J}}^Q$ to $\mathcal{Z}_{\widehat{Q}^{\mathcal{J}}, \mathcal{J}} \cong \mathcal{Z}_{Q, \mathcal{J}}$.

For any subset $L \subseteq I_f^{\mathcal{J}}$, let

$$(D^2, S^1)_{\mathcal{J}}^{(f,L)} := f \times \prod_{i \in I_f^{\mathcal{J}} \setminus L} D_{(i)}^2 \times \prod_{i \in [k] \setminus (I_f^{\mathcal{J}} \setminus L)} S_{(i)}^1.$$

We can translate the proof of Theorem 1.1 to obtain the desired stable decomposition of $\mathcal{Z}_{Q, \mathcal{J}} \cong (D^2, S^1)_{\mathcal{J}}^Q$ by the following correspondence of symbols:

$$\text{Theorem 1.1} \longrightarrow \text{Theorem 6.2}$$
$$J \subseteq [m] \longrightarrow \omega \subseteq [k]$$
$$F_J \longrightarrow F_\omega$$
$$I_f \subseteq [m] \longrightarrow I_f^{\mathcal{J}} \subseteq [k]$$
$$D_{(j)}^2, S_{(j)}^1, j \in [m] \longrightarrow D_{(i)}^2, S_{(i)}^1, i \in [k]$$
$$(D^2, S^1)^{(f,L)} \longrightarrow (D^2, S^1)_{\mathcal{J}}^{(f,L)} \qquad \square$$

Remark 6.3 Theorem 6.2 is an analogue of [36, Theorem 1.3].

To describe the cohomology ring of $\mathcal{Z}_{Q, \mathcal{J}}$, let

$$(57) \qquad \mathcal{R}_{Q, \mathcal{J}}^* := \bigoplus_{\omega \subseteq [k]} H^*(Q, F_\omega).$$

There is a graded ring structure $\mathbb{U}_{\mathcal{J}}$ on $\mathcal{R}_{Q, \mathcal{J}}^*$ defined as follows:

- If $\omega \cap \omega' \neq \varnothing$, then $H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\mathbb{U}_{\mathcal{J}}} H^*(Q, F_{\omega \cup \omega'})$ is trivial.

- If $\omega \cap \omega' = \varnothing$, then $H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\mathbb{U}_{\mathcal{J}}} H^*(Q, F_{\omega \cup \omega'})$ is the relative cup product $\cup$.

To describe the equivariant cohomology ring of $\mathcal{Z}_{Q, \mathcal{J}}$, let

$$\boldsymbol{k}^{\mathcal{J}}\langle Q \rangle := \bigoplus_{\omega \subseteq [k]} H^*(F_{\cap \omega}; \boldsymbol{k}) \otimes R_{\boldsymbol{k}}^\omega,$$

where the product on $\boldsymbol{k}^{\mathcal{J}}\langle Q \rangle$ is defined in the same way as $\boldsymbol{k}\langle Q \rangle$ in Definition 1.6.

The following theorem generalizes Theorems 1.4 and 1.7. The proof is omitted since it is completely parallel to the proof of these two theorems.

Theorem 6.4 *Let Q be a nice manifold with corners with m facets $F_1, \ldots, F_m$ and let $\mathcal{J} = \{J_1, \ldots, J_k\}$ be a partition of $[m]$.*

- *There is a ring isomorphism (up to a sign) from $(\mathscr{R}^*_{Q,\mathscr{J}}, \mathbb{U}_{\mathscr{J}})$ to the integral cohomology ring of $\mathfrak{X}_{Q,\mathscr{J}}$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_\omega)$ up by $|\omega|$ for every $\omega \subseteq [k]$.*

- *There is a graded ring isomorphism from the equivariant cohomology ring of $\mathfrak{X}_{Q,\mathscr{J}}$ with integral coefficients to $\mathbb{Z}^{\mathscr{J}}\langle Q \rangle$ by choosing $\deg(x_i) = 2$ for all $1 \le i \le k$.*

By combining the constructions in Theorems 4.4 and 6.2, we have the following definitions which provide the most general setting for our study.

Let $\mathscr{J} = \{J_1, \dots, J_k\}$ be a *partition* of $[m] = \{1, \dots, m\}$ and let

$$(\mathbb{X}, \mathbb{A}) = \{(X_i, A_i, a_i)\}_{i=1}^k,$$

where X_i and A_i are CW–complexes with a basepoint $a_i \in A_i \subseteq X_i$.

For any face f of Q, let

$$(\mathbb{X}, \mathbb{A})_{\mathscr{J}}^f := f \times \prod_{i \in I_f^{\mathscr{J}}} X_i \times \prod_{i \in [k] \setminus I_f^{\mathscr{J}}} A_i, \quad (\mathbb{X}, \mathbb{A})_{\mathscr{J}}^Q := \bigcup_{f \in \mathscr{S}_Q} (\mathbb{X}, \mathbb{A})_{\mathscr{J}}^f \subseteq Q \times \prod_{i \in [k]} X_i.$$

The following theorem generalizes Theorems 4.4 and 4.5.

Theorem 6.5 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Let*

$$(\mathbb{X}, \mathbb{A}) = \{(X_i, A_i, a_i)\}_{i=1}^k,$$

where each X_i is contractible and each A_i is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then for any partition $\mathscr{J} = \{J_1, \dots, J_k\}$ of $[m]$, there is a homotopy equivalence

$$\Sigma((\mathbb{X}, \mathbb{A})_{\mathscr{J}}^Q) \simeq \bigvee_{\omega \subseteq [k]} \Sigma\left(Q/F_\omega \wedge \bigwedge_{i \in \omega} A_i\right).$$

In addition, there is a ring isomorphism

$$\left(\bigoplus_{\omega \subseteq [k]} \tilde{H}^*\left(Q/F_\omega \wedge \bigwedge_{i \in \omega} A_i\right), \circledast\right) \to \tilde{H}^*((\mathbb{X}, \mathbb{A})_{\mathscr{J}}^Q),$$

where $\circledast$ is defined in the same way as in (42).

In particular, for $(\mathbb{D}, \mathbb{S}) = \{(D^{n_i+1}, S^{n_i}, a_i)\}_{i=1}^k$, we can describe the integral cohomology ring of $(\mathbb{D}, \mathbb{S})_{\mathscr{J}}^Q$ explicitly as follows. Define a graded ring structure $\mathbb{U}_{\mathscr{J}}^{(\mathbb{D}, \mathbb{S})}$ on $\mathscr{R}^*_{Q, \mathscr{J}}$ according to $(\mathbb{D}, \mathbb{S})$ by:

- If $\omega \cap \omega' = \varnothing$ or $\omega \cap \omega' \ne \varnothing$ but $n_i = 0$ for all $i \in \omega \cap \omega'$, then

$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\mathbb{U}_{\mathscr{J}}^{(\mathbb{D}, \mathbb{S})}} H^*(Q, F_{\omega \cup \omega'})$$

is the relative cup product.

- If $\omega \cap \omega' \neq \varnothing$ and there exists $n_i \geq 1$ for some $i \in \omega \cap \omega'$, then

$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\ \mathbb{U}_{\mathscr{J}}^{(\mathbb{D},\mathbb{S})}\ } H^*(Q, F_{\omega \cup \omega'})$$

is trivial.

Theorem 6.6 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. For any partition $\mathscr{J} = \{J_1, \ldots, J_k\}$ of $[m]$ and $(\mathbb{D}, \mathbb{S}) = \{(D^{n_i+1}, S^{n_i}, a_i)\}_{i=1}^k$, there is a homotopy equivalence*

$$\mathbf{\Sigma}((\mathbb{D}, \mathbb{S})_{\mathscr{J}}^Q) \simeq \bigvee_{\omega \subseteq [k]} \mathbf{\Sigma}^{1+\sum_{i\in\omega} n_i}(Q/F_\omega),$$

and there is a ring isomorphism (up to a sign) from $(\mathscr{R}_{Q,\mathscr{J}}^*, \mathbb{U}_{\mathscr{J}}^{(\mathbb{D},\mathbb{S})})$ to the integral cohomology ring of $(\mathbb{D}, \mathbb{S})_{\mathscr{J}}^Q$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of the elements in $H^*(Q, F_\omega)$ for every $\omega \subseteq [k]$.

When $(\mathbb{D}, \mathbb{S}) = \{(D^1, S^0, a_0)\}_{i=1}^k$, we denote $(\mathbb{D}, \mathbb{S})_{\mathscr{J}}^Q$ also by $\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}$, which is the real analogue of $\mathscr{Z}_{Q,\mathscr{J}}$. Then we have the following corollary which generalizes Corollary 4.10 and Theorem 1.7.

Corollary 6.7 *Let Q be a nice manifold with corners with facets $F_1, \ldots, F_m$. Then for any partition $\mathscr{J} = \{J_1, \ldots, J_k\}$ of $[m]$,*

$$\mathbf{\Sigma}(\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}) \simeq \bigvee_{\omega \subseteq [k]} \mathbf{\Sigma}(Q/F_\omega), \quad H^p(\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}) \cong \bigoplus_{\omega \subseteq [k]} H^p(Q, F_\omega) \quad \text{for all } p \in \mathbb{Z}.$$

Furthermore, the integral cohomology ring of $\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}$ is isomorphic as a graded ring to the ring $(\mathscr{R}_{Q,\mathscr{J}}^*, \cup)$, where $\cup$ is the relative cup product

$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\ \cup\ } H^*(Q, F_{\omega \cup \omega'}) \quad \text{for all } \omega, \omega' \subseteq [k],$$

and there is a graded ring isomorphism from the equivariant $\mathbb{Z}_2$–cohomology ring of $\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}$ to $\mathbb{Z}_2^{\mathscr{J}}\langle Q \rangle$ by choosing $\deg(x_i) = 1$ for all $1 \leq i \leq k$.

The proofs of Theorems 6.5 and 6.6 and Corollary 6.7 are almost the same as their counterparts in Sections 4 and 5, hence omitted.

For any partition $\mathscr{J} = \{J_1, \ldots, J_k\}$ of $[m]$,

- we can think of $\mathscr{Z}_{Q,\mathscr{J}}$ as the quotient space of $\mathscr{Z}_Q$ by the canonical action of an $(m-k)$–dimensional subtorus $\mathbb{T}^{\mathscr{J}}$ of $(S^1)^m$ determined by (see (1))

$$\{\lambda(F_j) - \lambda(F_{j'}) \mid j, j' \text{ belong to the same } J_i \text{ for some } 1 \leq i \leq k\} \subseteq \mathbb{Z}^m;$$

- similarly, we can think of $\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}$ as the quotient space of $\mathbb{R}\mathscr{Z}_Q$ by the canonical action of a subgroup of rank $m - k$ in $(\mathbb{Z}_2)^m$.

Note that the canonical action of $\mathbb{T}^{\mathscr{J}}$ on $\mathscr{Z}_Q$ may not be free. But when the action is free, the integral equivariant cohomology ring of $\mathscr{Z}_Q/\mathbb{T}^{\mathscr{J}} = \mathscr{Z}_{Q,\mathscr{J}}$ is isomorphic to $\mathbb{Z}\langle Q \rangle$ by Theorem 5.3. So, by Theorem 6.4, $\mathbb{Z}^{\mathscr{J}}\langle Q \rangle$ is isomorphic as a ring to $\mathbb{Z}\langle Q \rangle$ in this case. But this ring isomorphism is not obvious from the algebraic definition of $\mathbb{Z}^{\mathscr{J}}\langle Q \rangle$ and $\mathbb{Z}\langle Q \rangle$.

Remark 6.8 For any partition $\mathscr{J} = \{J_1, \ldots, J_k\}$ of $[m]$ with $k = \dim(Q)$, the $\mathscr{Z}_{Q,\mathscr{J}}$ and $\mathbb{R}\mathscr{Z}_{Q,\mathscr{J}}$ can be considered as a generalization of the *pull-back from the linear model* — see [17, Example 1.15] — in the study of quasitoric manifolds and small covers.

Acknowledgements

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