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Chow–Witt rings of Grassmannians

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We complement the previous computation of the Chow–Witt rings of classifying spaces of special linear groups by an analogous computation for the general linear groups. This case involves discussion of nontrivial dualities. The computation proceeds along the lines of the classical computation of the integral cohomology of $BO(n)$ with local coefficients, as done by Čadek. The computations of Chow–Witt rings of classifying spaces of GL_n are then used to compute the Chow–Witt rings of the finite Grassmannians. As before, the formulas are close parallels of the formulas describing integral cohomology rings of real Grassmannians.

14C15, 14F43; 14C17, 14M15

1 Introduction

The computation of the Chow ring of Grassmannians is fundamental in algebraic geometry. The computation for finite Grassmannians provides the basis of Schubert calculus and its applications in enumerative geometry, while the computation for the infinite Grassmannians (ie the classifying space $B\mathrm{GL}_n$ of vector bundles) describes the characteristic classes of vector bundles in terms of Chern classes. Given any cohomology theory, one can ask for similar computations and how these computations provide information on characteristic classes of vector bundles (in the case of infinite Grassmannians) or provide variations of Schubert calculus with relevance for combinatorics and representation theory (in the case of finite Grassmannians). Indeed, many such investigations have been done in recent years for algebraic versions of complex-oriented cohomology theories. On the other hand, several cohomology theories have recently been considered which detect aspects related to real algebraic geometry and the theory of quadratic forms; for example the Chow–Witt rings $\widetilde{\mathrm{CH}}^\bullet(X)$ introduced by Barge and Morel in [7] and studied in depth by Fasel — see eg [12; 13]. Other strongly related examples are given by the cohomology $H^\bullet(X, \mathbf{W})$ of the sheaf of Witt rings $\boldsymbol{W}$ on the small (Zariski or Nisnevich) site of a smooth scheme X — appearing, for example, in the study of the Gersten–Witt spectral sequence converging to Witt groups of X by Balmer and Walter [6] — and the cohomology $H^\bullet(X, \boldsymbol{I}^\bullet)$ of the sheaves $\boldsymbol{I}^n$ of powers of the fundamental ideal in the Witt ring (which by work of Jacobson [18] is strongly related to singular cohomology of the real realization). The goal of the present paper is to compute these cohomology theories for the (finite and infinite) Grassmannians. As mentioned above, the case of infinite Grassmannians serves to better understand the relevant characteristic classes of vector bundles which may be relevant for splitting questions — as in the original work of Barge and Morel [7] on Euler class obstructions for splitting, or more recently in the work of Asok and Fasel, eg [3; 4]. The case of finite Grassmannian allows to set up a variant of Schubert calculus combining features from complex and real Grassmannians, see [28], which could be useful for refined or $\mathbb{A}^1$–enumerative questions over general fields as considered recently by Kass and Wickelgren in [19], Levine in [20], and others.

The present paper is a sequel to work with Hornbostel [16], which computed the Chow–Witt rings of classifying spaces of the symplectic and special linear groups. We provide a similar computation of the total Chow–Witt ring of $B\mathrm{GL}_n$, essentially by a combination and extension of the techniques developed in [16] and Čadek's computation of cohomology of $BO(n)$ with twisted coefficients in [26]. Once the characteristic classes for vector bundles and their relations are known, we are also able to compute the Chow–Witt rings of the finite Grassmannians. It turns out that the formulas describing the $\boldsymbol{I}$–cohomology of Grassmannians are direct analogues of the classical formulas for integral cohomology of real Grassmannians, so the Chow–Witt rings capture information from the singular cohomology of both real and complex Grassmannians (via the $\boldsymbol{I}$–cohomology and Chow ring components, respectively).

1.1 Chow–Witt rings of infinite Grassmannians

We first formulate the results for the infinite Grassmannians $B\mathrm{GL}_n$. As these are not smooth schemes, there are two choices for talking about their cohomology — either using finite-dimensional approximations by smooth schemes (as in Totaro's approach to Chow groups of classifying spaces [24]) or in the framework of the motivic homotopy category of Morel and Voevodsky [22] (in which all the cohomology theories considered here happen to be representable). Both approaches yield equivalent results, but we adopt the former point of view for the present paper; see also the beginning of Section 3 for a slightly more detailed discussion.

Before going into a detailed description of the Chow–Witt ring of $B\mathrm{GL}_n$, we need to introduce some notation; see also the more detailed discussions around the relevant cohomology theories in Section 2. The ultimate goal is the computation of the Chow–Witt ring $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n)$ which is a graded algebra over the Grothendieck–Witt ring $\mathrm{GW}(F)$ of quadratic forms over the base field F. The Chow–Witt ring combines two pieces of information, via a cartesian square in point (1) of Theorem 1.1: one piece, described in point (2) of the theorem, comes from the Chow ring and is related to complex Grassmannians; the other piece, described in point (3), is the $\boldsymbol{I}$–cohomology ring which is related to the real Grassmannian. Finally, the two pieces are glued together by means of maps from both pieces to the mod 2 Chow ring $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ which are described in point (4) of the theorem.

The main new computations in this paper concern the $\boldsymbol{I}$–cohomology ring as a graded algebra over the Witt ring $W(F)$ of quadratic forms (obtained from $\mathrm{GW}(F)$ as a quotient by the ideal generated by the hyperbolic form). Both the Chow–Witt ring and the $\boldsymbol{I}$–cohomology ring involve possible twists by line bundles. These are related to orientability questions and are in some ways similar to the cohomology with local coefficients for real manifolds. There is consequently an additional grading by the mod 2 Picard group for the Chow–Witt and $\boldsymbol{I}$–cohomology ring, since tensor squares of line bundles do not change isomorphism classes of the cohomology groups. In the particular case of the Grassmannians (both finite and infinite), there are essentially only two line bundles, or dualities, to consider: the trivial duality and the nontrivial one given by the (dual of the) determinant $\det\gamma_n^\vee$ of the universal rank n bundle γ_n. Thus, the Chow–Witt and $\boldsymbol{I}$–cohomology rings of $B\mathrm{GL}_n$ are graded algebras with grading by $\mathbb{Z}\oplus\mathrm{Pic}(B\mathrm{GL}_n)/2\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. They are in fact graded-commutative (in the sense that the correction factor for switching cup product factors is determined by the cohomological degrees of the cohomology classes). For a detailed discussion of subtleties in the graded-commutativity of total $\boldsymbol{I}$–cohomology or Chow–Witt rings, see Remark 2.3.

As a last piece of notation, it turns out to be convenient for the description of the Chow–Witt rings to also upgrade the (integral and mod 2) Chow ring to a $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ algebra by defining a product on

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n)^{\oplus 2}=\{(\alpha,\mathscr{L})\mid \alpha\in\mathrm{CH}^\bullet(B\mathrm{GL}_n),\mathscr{L}\in\mathrm{Pic}(B\mathrm{GL}_n)/2\}$$

by $(\alpha,\mathscr{L})\cdot(\beta,\mathscr{L}'):=(\alpha\cup\beta,\mathscr{L}\otimes\mathscr{L}')$, ie by taking intersection products from the Chow ring combined with the group structure of $\mathrm{Pic}(B\mathrm{GL}_n)/2$. With this definition, the reduction map

$$\rho\colon H^q(B\mathrm{GL}_n,\boldsymbol{I}^q)\oplus H^q(B\mathrm{GL}_n,\boldsymbol{I}^q(\det\gamma_n^\vee))\to\mathrm{Ch}^q(B\mathrm{GL}_n)^{\oplus 2}$$

becomes a map of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebras. Since this map is compatible with the direct sum decomposition, we will also denote the summands by $\rho\colon H^q(B\mathrm{GL}_n,\boldsymbol{I}^q(\mathscr{L}))\to\mathrm{Ch}^q(B\mathrm{GL}_n)$.

With these preparations, the following result now describes the total Chow–Witt ring of $B\mathrm{GL}_n$; see Theorems 3.24 and 3.27 and Proposition 3.26.

Theorem 1.1 *Let F be a perfect field of characteristic $\neq 2$.*

(1) *The following square, induced from the pullback description of the Milnor–Witt K–theory sheaf, is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras:*

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\mathscr{O}\oplus\det\gamma_n^\vee) \arrow[r] \arrow[d] & \ker\partial_{\mathscr{O}}\oplus\ker\partial_{\det\gamma_n^\vee} \arrow[r, hook] \arrow[d, "\mathrm{mod}\ 2"] & \mathrm{CH}^\bullet(B\mathrm{GL}_n)^{\oplus 2} \\
H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n,\boldsymbol{I}^\bullet\oplus\boldsymbol{I}^\bullet(\det\gamma_n^\vee)) \arrow[r, "\rho"'] & \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2} &
\end{tikzcd}$$

In the upper-right corner of this diagram, we have the kernels of the (twisted) integral Bockstein maps

$$\partial_{\mathscr{L}}\colon\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\mathscr{L}}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1}(\mathscr{L})).$$

(2) *The kernels of the twisted integral Bockstein operations inside the Chow ring*

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}[c_1, \dots, c_n]$$

can be described as

$$\ker \partial_{\mathscr{O}} = \mathbb{Z}[c_i^2, c_1 c_{2i} + c_{2i+1}, c_1 c_n, (2)], \quad \ker \partial_{\det \gamma_n^\vee} = \langle c_{2i+1}, c_n, (2)\rangle_{\ker \partial_{\mathscr{O}}}.$$

The first is a subring of $\mathrm{CH}^\bullet(B\mathrm{GL}_n)$, *the second is a sub–*$\ker \partial_{\mathscr{O}}$*–module, and* (2) *denotes the ideal generated by* 2 *in* $\mathrm{CH}^\bullet(B\mathrm{GL}_n)$.

(3) *The cohomology ring* $H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$ *is generated by the following characteristic classes: the even Pontryagin classes* p_{2i} *in degree* $(4i, 0)$, *the Euler class* e_n *in degree* $(n, 1)$ *and the* (*twisted*) *Bocksteins of products of Stiefel–Whitney classes. The latter classes are defined as*

$$\beta_J = \beta_{\mathscr{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{and} \quad \tau_J = \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}),$$

with the index set J *running through the* (*possibly empty*) *sets* $\{j_1, \dots, j_l\}$ *of positive natural numbers such that* $0 < j_1 < \dots < j_l \le \left[\frac{1}{2}(n-1)\right]$; *in the special case of* $J = \varnothing$, *the corresponding classes are* $\beta_\varnothing = \beta_{\mathscr{O}}(1)$ *and* $\tau_\varnothing = \beta_{\det \gamma_n^\vee}(1)$. *For an index set* $J = \{j_1, \dots, j_l\}$, *the degree of* β_J *is* $\left(1 + 2\sum_{i=1}^l j_i, 0\right)$ *and the degree of* τ_J *is* $\left(1 + 2\sum_{i=1}^l j_i, 1\right)$. *The* $\boldsymbol{I}$*–cohomology ring can then be explicitly identified as the* $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$*–graded-commutative* $W(F)$*–algebra*

$$W(F)[p_2, p_4, \dots, p_{[(n-1)/2]}, e_n, \{\beta_J\}_J, \{\tau_J\}_J, \tau_\varnothing]$$

modulo the relations

(a) $I(F)\beta_J = I(F)\tau_J = I(F)\tau_\varnothing = 0$;

(b) *if* $n = 2k+1$ *is odd and* $k \ge 1$, *we have* $e_{2k+1} = \tau_{\{k\}}$, *and for* $n = 1$ *we have* $e_1 = \tau_\varnothing$;

(c) *for two index sets* J *and* J', *where* J' *can be empty*,

$$\begin{aligned}
\beta_J \cdot \beta_{J'} &= \sum_{k \in J} \beta_{\{k\}} \cdot p_{(J \setminus \{k\}) \cap J'} \cdot \beta_{\Delta(J \setminus \{k\}, J')}, \\
\beta_J \cdot \tau_{J'} &= \sum_{k \in J} \beta_{\{k\}} \cdot p_{(J \setminus \{k\}) \cap J'} \cdot \tau_{\Delta(J \setminus \{k\}, J')}, \\
\tau_J \cdot \beta_{J'} &= \beta_J \cdot \tau_{J'} + \tau_\varnothing \cdot p_{J \cap J'} \cdot \beta_{\Delta(J, J')}, \\
\tau_J \cdot \tau_{J'} &= \beta_J \cdot \beta_{J'} + \tau_\varnothing \cdot p_{J \cap J'} \cdot \tau_{\Delta(J, J')},
\end{aligned}$$

where we set $p_A = \prod_{i=1}^l p_{a_i}$ *for an index set* $A = \{a_1, \dots, a_l\}$.

(4) *The reduction morphism* ρ *is given by*

$$p_{2i} \mapsto \bar{c}_{2i}^2, \quad e_n \mapsto \bar{c}_n, \quad \beta_{\mathscr{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \mapsto \mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}).$$

Under the homomorphism $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathscr{O}) \to \mathrm{CH}^\bullet(B\mathrm{GL}_n)$, *the Chow–Witt-theoretic Pontryagin class is mapped as*

$$p_i \mapsto (-1)^i c_i^2 + 2 \sum_{j=\max\{0, 2i-n\}}^{i-1} (-1)^j c_j c_{2i-j}.$$

It can be shown that formulas similar to the above description of $\boldsymbol{I}^\bullet$–cohomology are true for real-étale cohomology, but as algebra over $H^0_{\text{rét}}(F,\mathbb{Z}) \cong \operatorname{colim}_n I^n(F)$. For $F=\mathbb{R}$, the real cycle class map induces an isomorphism

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee)) \xrightarrow{\cong} H^\bullet(BO(n), \mathbb{Z}\oplus\mathbb{Z}^{\mathrm{t}}),$$

where the target was computed in [26], and it sends algebraic characteristic classes to their topological counterparts. For this result and a discussion of the required compatibilities, eg between localization sequences and the real cycle class maps, see Hornbostel, Xie, Zibrowius and the author [17].

1.2 Chow–Witt rings of finite Grassmannians

The second point of the paper is to provide a computation of the Chow–Witt rings of the finite Grassmannians $\mathrm{Gr}(k,n)$. The full description is even longer than the description of the Chow–Witt ring of $B\mathrm{GL}_n$ above, so we will only give pointers to the main results in the text. First, the Chow–Witt ring is again given in terms of a cartesian square combining the kernels of integral Bockstein maps with $\boldsymbol{I}^\bullet$–cohomology; see Theorem 5.10. The $\boldsymbol{I}^\bullet$–cohomology of $\mathrm{Gr}(k,n)$ can be described as follows: the characteristic classes of the tautological rank k subbundle $\mathscr{S}_k$ and the tautological rank $n-k$ quotient bundle $\mathscr{Q}_{n-k}$ generate the $\boldsymbol{I}^\bullet$–cohomology, except in the case where $k(n-k)$ is odd, in which we have a new class R in degree $n-1$. They naturally satisfy the relations in the $\boldsymbol{I}^\bullet$–cohomology of $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$, and they also satisfy the relations which are consequences of the Whitney sum formula for the extension

$$0 \to \mathscr{S}_k \to \mathscr{O}^{\oplus n} \to \mathscr{Q}_{n-k} \to 0.$$

There are a few further relations involving the potential class R. All these statements are established in Theorem 5.7. The reduction morphisms

$$H^\bullet(\mathrm{Gr}(k,n), \boldsymbol{I}^\bullet(\mathscr{L})) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$$

are described in Proposition 5.8. Except for the new fact that $R \mapsto \bar{c}_{k-1}\bar{c}^{\perp}_{n-k}$, the description of the reduction morphisms follows directly from the ones for $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$. This also provides a description of the kernel of the integral Bockstein maps; cf Theorem 5.10. Again, similar formulas would be true in real-étale cohomology, and for $F=\mathbb{R}$ the above description recovers exactly the integral cohomology of the real Grassmannians $\mathrm{Gr}_k(\mathbb{R}^n)$ (with local coefficients); see [17].

1.3 Decomposition of $\boldsymbol{I}$–cohomology

The present paper is a significantly revised version of its predecessor. While the previous version established the results mostly following the proof strategy of [16] fairly closely, the revised proofs follow a different strategy. The key new insight arises from a decomposition of $\boldsymbol{I}$–cohomology, described in Section 2.4,

$$0 \to \operatorname{Im}\beta_{\mathscr{L}}(X) \to H^q(X, \boldsymbol{I}^q(\mathscr{L})) \to H^q(X, \boldsymbol{W}(\mathscr{L})) \to 0.$$

This decomposition arises from the twisted Bär sequence, an algebraic analogue of the long exact Bockstein sequence in topology, which is discussed in more detail in Section 2.1. The cohomology with coefficients in the sheaf $\boldsymbol{W}$ of Witt rings is a theory in which η is invertible, and much more amenable to long exact sequence calculations than $\boldsymbol{I}$–cohomology; see work of Ananyevskiy [1]. Moreover, if $\boldsymbol{W}$–cohomology is free, the above sequence splits. In that case, the reduction morphism $\rho\colon H^q(X, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^q(X)$ is injective on the image of $\beta_{\mathscr{L}}$; hence the torsion classes in $\operatorname{Im}\beta_{\mathscr{L}}$ can be computed from the knowledge of the Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ on the mod 2 Chow ring. This way, the computation of $\boldsymbol{I}$–cohomology splits into two significantly easier parts, the computation of $\boldsymbol{W}$–cohomology which can be done by the same methods as calculations of rational cohomology of real Grassmannians — see Milnor and Stasheff [21] and Sadykov [23] — and the computation of $\operatorname{Im}\beta$ which only requires knowledge of the mod 2 Chow theory. The freeness of $\boldsymbol{W}$–cohomology, which implies the above decomposition of $\boldsymbol{I}$–cohomology, can therefore be seen as the algebraic analogue of the statement that "all torsion in the cohomology of real Grassmannians is 2–torsion". This gets rid of problems as in Remark 2.2 or Remark 7.2 of [16]. Moreover, the proof of freeness of $\boldsymbol{W}$–cohomology, and therefore the torsion statement, is significantly easier than in topology (where it is not clear that integral cohomology modulo the image of the integral Bockstein maps is even a cohomology theory). The $\operatorname{Im}\beta$–$\boldsymbol{W}$–decomposition of $\boldsymbol{I}$–cohomology will be a useful tool for a number of upcoming computations (where the real topological counterparts have only 2–torsion), such as classifying spaces of orthogonal groups and flag varieties.

The shorter, alternative way to describe the structure of the $\boldsymbol{I}$–cohomology of $B\mathrm{GL}_n$ or $\mathrm{Gr}(k,n)$ (at least additively) is then the following. The $\boldsymbol{I}$–cohomology splits as a direct sum of the image of $\beta_{\mathscr{L}}$, which is a 2–torsion group with the same structure as in the integral cohomology of the real Grassmannians, and the $\boldsymbol{W}$–cohomology, which is a free $W(F)$–algebra having the same presentation as the rational cohomology of the real Grassmannians. The multiplication on the torsion part can be described completely by reduction to mod 2 Chow theory where we have the classical formulas from Schubert calculus. Conceptual descriptions for the multiplication can be found in Casian and Stanton [10] (with an interesting link to representation theory of real Lie groups) and Casian and Kodama [9] (explicitly in terms of signed Young diagrams); see also the discussion of checkerboard fillings for Young diagrams to compute $\mathrm{Sq}^2_{\mathscr{L}}$ in [28]. The description of Chow–Witt rings of finite Grassmannians is used in a sequel [28] to develop an oriented Schubert calculus which allows us to establish arithmetic refinements of classical Schubert calculus.

Structure of the paper We provide a recollection on relevant statements from Chow–Witt theory, in particular the twisted Steenrod squares, in Section 2. The relevant characteristic classes for vector bundles are recalled in Section 3, where we also formulate the main structural results on the Chow–Witt ring of $B\mathrm{GL}_n$. The inductive computation of the $\boldsymbol{I}^\bullet$–cohomology is done in Section 4. The results on Chow–Witt rings of finite Grassmannians are formulated in Section 5 and the proofs are given in Section 6.

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Jens Hornbostel, Thomas Hudson, Lorenzo Mantovani, Toan Manh Nguyen and Konrad Voelkel clarified that the use of the $\mathrm{Im}\,\beta$–$\boldsymbol{W}$–decomposition of $\boldsymbol{I}$–cohomology could improve structural statements and streamline proofs. I am grateful to Marc Levine and Ákos Matszangosz for various corrections and to an anonymous referee at AGT for many and detailed comments that helped eliminate a number of mistakes and greatly improve the presentation of the paper.

In some of the results describing kernels of Steenrod squares Sq^2 and Bockstein maps β, the obvious and necessary generators in the image of the Steenrod square have been omitted: in Theorem 1.1(2), Corollary 3.13, and Proposition 3.26, Steenrod squares of products of even Stiefel–Whitney classes $\mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ have to be added to the list of generators. Formulations are correct for Corollary 3.14 and Theorem 5.10. This doesn't affect either proofs or main results. I would like to thank Jan Hennig and Marc Levine for pointing this out.

2 Recollection on Chow–Witt rings

Throughout this article, we consider a perfect base field F of characteristic $\mathrm{char}(F)\neq 2$. All the relevant cohomology groups will be Nisnevich cohomology groups, ie Ext-groups between Nisnevich sheaves on the small site of a smooth scheme. Note, however, that for all the sheaves we consider, Nisnevich and Zariski cohomology are isomorphic. The relevant sheaves will be denoted by boldface letters, such as the sheaves $\boldsymbol{K}^{\mathrm{M}}_n$ and $\boldsymbol{K}^{\mathrm{MW}}_n$ of Milnor and Milnor–Witt K–groups, respectively, the sheaf $\boldsymbol{W}$ of Witt rings and the sheaves $\boldsymbol{I}^n$ of powers of the fundamental ideal in $\boldsymbol{W}$. Sections of these sheaves are mostly taken over fields and are denoted by the more usual letters K^{M}_n, K^{MW}_n, W and I^n, respectively.

Since this is a sequel to [16], most of the general facts concerning Chow–Witt rings relevant for the computation in the present paper can already be found in the discussion of [16, Section 2] (or, of course, in the original literature; see [loc. cit.] for references). The same applies to the general discussion of Chow–Witt rings of classifying spaces; all the statements relevant for the present paper can be found in [16, Section 3]. We freely use the definitions, facts and notation from [16].

2.1 Twisted coefficients and cohomology operations

What has to be discussed in slightly more detail is the use of twisted coefficients in Chow–Witt groups and $\boldsymbol{I}^\bullet$–cohomology which were only mentioned in passing in [16]. If $\mathscr{L}$ is a line bundle on a smooth scheme X, then there are twisted sheaves $\boldsymbol{I}^n(\mathscr{L})$ and $\boldsymbol{K}^{\mathrm{MW}}_n(\mathscr{L})$. For the construction as well as a description of Gersten-type complexes computing $H^n_{\mathrm{Nis}}(X,\boldsymbol{I}^n(\mathscr{L}))$ and $H^n_{\mathrm{Nis}}(X,\boldsymbol{K}^{\mathrm{MW}}_n(\mathscr{L}))\cong\widetilde{\mathrm{CH}}^n(X,\mathscr{L})$, see [13, Section 10] and [4, Section 2]. In particular, Theorem 2.3.4 of [4] provides an identification of the definition of twisted Chow–Witt groups in [13] with the Nisnevich cohomology of the twisted Milnor–Witt K–theory sheaves. If $\mathscr{L}$ and $\mathscr{N}$ are two line bundles on X, then there are canonical isomorphisms

$$\widetilde{\mathrm{CH}}^\bullet(X,\mathscr{N})\cong\widetilde{\mathrm{CH}}^\bullet(X,\mathscr{L}^2\otimes\mathscr{N}).$$

The twisted versions of Chow–Witt groups and $\boldsymbol{I}^n$–cohomology have functorial pullbacks, pushforwards and a localization sequence (where the cohomology of the closed subscheme appears with twist by the normal bundle of the inclusion). Formulations and references to the relevant literature can all be found in [16, Section 2.1].

We also need to discuss twisted analogues of the facts on cohomology operations discussed in [16, Section 2.3]. If X is a smooth scheme and $\mathscr{L}$ is a line bundle on X, we can twist the exact sequence of fundamental ideals by $\mathscr{L}$ to get an exact sequence of strictly $\mathbb{A}^1$–invariant Nisnevich sheaves of abelian groups

$$0 \to \boldsymbol{I}^{n+1}(\mathscr{L}) \to \boldsymbol{I}^n(\mathscr{L}) \to \boldsymbol{K}_n^{\mathrm{M}}/2 \to 0.$$

This is analogous to the topological exact sequence $0 \to \mathbb{Z}^{\mathrm{t}} \to \mathbb{Z}^{\mathrm{t}} \to \mathbb{Z}/2\mathbb{Z} \to 0$ for a local system $\mathbb{Z}^{\mathrm{t}}$ with fiber $\mathbb{Z}$. Associated to the previous exact sequence of Nisnevich sheaves, we get a twisted analogue of the Bär sequence used in [16],

$$\cdots \to H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \xrightarrow{\rho} \mathrm{Ch}^n(X) \xrightarrow{\beta_{\mathscr{L}}} H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \to \cdots.$$

This is an analogue of the long exact Bockstein sequence in topology. The maps in this sequence are

(1) the connecting map $\beta_{\mathscr{L}}$, a Chow–Witt analogue of the Bockstein operation twisted by a local system in classical algebraic topology,

(2) the map η induced by the inclusion $\boldsymbol{I}^{n+1} \hookrightarrow \boldsymbol{I}^n$ and

(3) the reduction map ρ induced by the quotient map $\boldsymbol{I}^n \to \boldsymbol{K}_n^{\mathrm{M}}/2$.

A discussion of the twisted Bockstein maps in the topological context of cohomology of $BO(n)$ can be found in [26].

Remark 2.1 A funny side remark on some of the differences between $\mathrm{Ch}^n(X)$ and mod 2 singular cohomology: By the universal coefficient formula, mod 2 singular cohomology H^n in general contains mod 2 reductions of integral classes in H^n *as well as* classes related to 2–torsion classes in H^{n-1}. This is not true for $\mathrm{Ch}^n(X)$, viewed as mod 2 reduction of $\mathrm{CH}^n(X)$ — by definition all classes in $\mathrm{Ch}^n(X)$ are simply mod 2 reductions of $\mathrm{CH}^n(X)$. However, the Bär sequence encodes a behavior of $\mathrm{Ch}^n(X)$ exactly analogous to mod 2 singular cohomology: there are some classes which lift to integral cohomology $H^n(X, \boldsymbol{I}^n)$, and some classes which don't (because they have nontrivial images under the Bockstein operation).

For a line bundle $\mathscr{L}$ on a smooth scheme X, the twisted Bockstein map

$$\beta_{\mathscr{L}}\colon \mathrm{Ch}^n(X) \to H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L}))$$

can be used to define twisted versions of integral Stiefel–Whitney classes analogous to those defined in [14]. The composition with the reduction morphism $\rho\colon H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \to \mathrm{Ch}^{n+1}(X)$ has been identified in [2, Theorem 3.4.1]. This is a twisted version of Totaro's identification [25, Theorem 1.1] and a Chow–Witt version of [26, Lemma 2].

Proposition 2.2 *Let X be a smooth scheme and $\mathscr{L}$ be a line bundle over X. Denote by*

$$\beta_{\mathscr{L}}\colon \mathrm{Ch}^i(X) \to H^{i+1}(X, \boldsymbol{I}^{i+1}(\mathscr{L}))$$

the twisted Bockstein map. Then for all $x \in \mathrm{Ch}^i(X)$,

$$\rho \circ \beta_{\mathscr{L}}(x) = \bar{c}_1(\mathscr{L}) \cdot x + \mathrm{Sq}^2(x) =: \mathrm{Sq}^2_{\mathscr{L}}(x).$$

2.2 Oriented intersection product and total Chow–Witt ring

The oriented intersection product for the Chow–Witt ring has the form

$$\widetilde{\mathrm{CH}}^i(X, \mathscr{L}_1) \times \widetilde{\mathrm{CH}}^j(X, \mathscr{L}_2) \to \widetilde{\mathrm{CH}}^{i+j}(X, \mathscr{L}_1 \otimes \mathscr{L}_2);$$

there is a similar product on twisted $\boldsymbol{I}^\bullet$–cohomology rings. With these products, the Chow–Witt ring is a $\langle -1 \rangle$–graded commutative algebra over the Grothendieck–Witt ring $\mathrm{GW}(F)$, and $\bigoplus_n H^n(X, \boldsymbol{I}^n)$ is a (-1)–graded commutative algebra over the Witt ring $W(F)$; see eg [16, Section 2.2].

The total Chow–Witt ring of a smooth scheme X is defined by

$$\bigoplus_{\mathscr{L} \in \mathrm{Pic}(X)/2} \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L});$$

see eg [12, Definition 6.10]. Strictly speaking, a total Chow–Witt ring doesn't exist because identifications $\widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L}) \cong \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{N})$ for isomorphic line bundles $\mathscr{L}$ and $\mathscr{N}$ depend on the choice of isomorphism between the line bundles. However, the technical inaccuracy of neglecting such choices of isomorphisms between different representatives of isomorphism classes of line bundles can be fixed by the methods in [5]. The same goes for the total $\boldsymbol{I}$–cohomology ring

$$\bigoplus_{\mathscr{L} \in \mathrm{Pic}(X)/2, q \in \mathbb{N}} H^q(X, \boldsymbol{I}^q(\mathscr{L})).$$

Note that $\mathrm{Pic}(B\mathrm{GL}_n) \cong \mathbb{Z}$ and $\mathrm{Pic}(\mathrm{Gr}(k,n)) \cong \mathbb{Z}$; in particular, there are only two nontrivial dualities to consider for the total Chow–Witt rings of $B\mathrm{GL}_n$ and $\mathrm{Gr}(k,n)$. For $B\mathrm{GL}_n$, the nontrivial element of $\mathrm{Pic}(B\mathrm{GL}_n)/2$ is given by $\det \gamma_n^\vee$, the dual of the determinant of the universal rank n bundle. Note that this corresponds precisely to the well-known topological fact that there are exactly two isomorphism classes of local systems on $BO(n)$, the trivial one and the one for the sign representation of $\pi_1(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ on the coefficient ring $\mathbb{Z}$.

Remark 2.3 There is a serious subtlety concerning the graded commutativity of the total $\boldsymbol{I}$–cohomology ring which we want to discuss at this point. For the correct formulation of graded commutativity of twisted Chow–Witt/$\boldsymbol{I}$–cohomology/$\boldsymbol{W}$–cohomology groups, one has to use graded line bundles, as explained, for example, in [15]. Specializing to the $\boldsymbol{I}$–cohomology situation, the correction factor for graded commutativity of the cup product

$$H^i(X, \boldsymbol{I}^i(\mathscr{L}_1, a_1)) \times H^j(X, \boldsymbol{I}^j(\mathscr{L}_2, a_2)) \to H^{i+j}(X, \boldsymbol{I}^{i+j}(\mathscr{L}_1 \otimes \mathscr{L}_2, a_1 + a_2))$$

would be $(-1)^{(a_1+i)(a_2+j)}$; see [15, Section 3.4].[1] In particular, the degrees of the graded line bundles play an important role. However, for the specific situation of the present paper, this problem is rather invisible: for classifying spaces $B\mathrm{GL}_n$ and Grassmannians $\mathrm{Gr}(k,n)$, the correct graded line bundle for the nontrivial twists would be $(\det\mathscr{V}, \mathrm{rk}\,\mathscr{V})$ for $\mathscr{V}=\gamma_n$ the universal rank n bundle on $B\mathrm{GL}_n$ or $\mathscr{V}=\mathscr{S}_k, \mathscr{Q}_{n-k}$ the tautological subbundle and quotient bundle on the Grassmannian $\mathrm{Gr}(k,n)$. Since the classes in twisted $\boldsymbol{I}$–cohomology of these spaces only appear as multiples of Euler classes of *even rank bundles*, the contributions of the degrees of the twist bundles play no role in the correction term $(-1)^{(a_1+i)(a_2+j)}$. In particular, graded commutativity for the total $\boldsymbol{I}$–cohomology ring in the present paper always means that the correction factor is $(-1)^{ij}$, ie it only depends on the cohomological degrees of the cohomology classes involved. In fact, a posteriori, the $\boldsymbol{I}$–cohomology ring turns out to be commutative after all: all the nontorsion classes in the $\boldsymbol{I}$–cohomology (Pontryagin classes and Euler classes) have even degrees, and all torsion classes are in fact 2–torsion, so signs don't matter.

2.3 The fundamental square

After having discussed all the relevant preliminaries, there are now twisted analogues of the key diagram from [16], for any line bundle $\mathscr{L}$ on X:

$$\begin{tikzcd}
 & \mathrm{CH}^n(X) \arrow[r, equal] \arrow[d] & \mathrm{CH}^n(X) \arrow[d, "2"] & \\
H^n(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \arrow[r] \arrow[d, equal] & \widetilde{\mathrm{CH}}^n(X,\mathscr{L}) \arrow[r] \arrow[d] & \mathrm{CH}^n(X) \arrow[r, "\partial_{\mathscr{L}}"] \arrow[d, "\mathrm{mod}\ 2"] & H^{n+1}(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \arrow[d, equal] \\
H^n(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \arrow[r, "\eta"'] & H^n(X,\boldsymbol{I}^{n}(\mathscr{L})) \arrow[r, "\rho"'] \arrow[d] & \mathrm{Ch}^n(X) \arrow[r, "\beta_{\mathscr{L}}"] \arrow[d] \arrow[dr, "\mathrm{Sq}^2_{\mathscr{L}}"'] & H^{n+1}(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \arrow[d, "\rho"] \\
 & 0 \arrow[r] & 0 & \mathrm{Ch}^{n+1}(X)
\end{tikzcd}$$

As already mentioned in [16], there is a twisted analogue of [16, Proposition 2.11], which states that for F a perfect field of characteristic unequal to 2 and a smooth scheme X over F, the canonical map

$$c\colon \widetilde{\mathrm{CH}}^\bullet(X,\mathscr{L}) \to H^\bullet(X,\boldsymbol{I}^\bullet(\mathscr{L})) \times_{\mathrm{Ch}^\bullet(X)} \ker\partial_{\mathscr{L}}$$

induced from the above key square is always surjective, and is injective if $\mathrm{CH}^\bullet(X)$ has no nontrivial 2–torsion. This way we can determine the additive structure of twisted Chow–Witt groups; if we consider the total Chow–Witt ring (ie the direct sum of twisted Chow–Witt groups over $\mathrm{Pic}(X)/2$), the fiber square also describes the oriented intersection product. The result applies, in particular, to $B\mathrm{GL}_n$ and the Grassmannians $\mathrm{Gr}(k,n)$ (or more generally flag varieties G/P for reductive groups) because these are

[1]The formulation in [15] is for Chow–Witt rings and consequently the correction factor there is $\langle(-1)^{(a_1+i)(a_2+j)}\rangle$. For $\boldsymbol{I}$–cohomology, this reduces to $(-1)^{(a_1+i)(a_2+j)}$; see also the discussion of graded commutativity of the various cohomology theories in [16, Definition 2.4 and Proposition 2.5].

known to have 2–torsion-free Chow groups. This implies that we only need to determine the individual terms of the fiber product to get a description of the Chow–Witt ring.

2.4 Decomposing $\boldsymbol{I}$–cohomology into $\boldsymbol{W}$–cohomology and the image of β

One of the features which is new and has not been used in either [16] or the previous version of the present paper is the use of $\boldsymbol{W}$–cohomology. For a smooth F–scheme X, we can consider the restriction of the Nisnevich sheaf $\boldsymbol{W}$ of Witt groups to the small Nisnevich site of X, and then take its Nisnevich cohomology $H^\bullet(X, \boldsymbol{W})$. This cohomology theory has been considered before, as it appears in the context of the Gersten–Witt spectral sequence converging to the Witt groups of a smooth scheme X; see [6]. Some of its properties discussed below make it also very suitable as a stepping stone in computations of $\boldsymbol{I}$–cohomology and Chow–Witt rings.

As before, if $\mathscr{L}$ is a line bundle on X, we can consider the twisted $\boldsymbol{W}$–cohomology groups $H^\bullet(X, \boldsymbol{W}(\mathscr{L}))$. The product structure on the Witt rings induces an intersection product

$$H^i(X, \boldsymbol{W}(\mathscr{L}_1)) \times H^j(X, \boldsymbol{W}(\mathscr{L}_2)) \to H^{i+j}(X, \boldsymbol{W}(\mathscr{L}_1 \otimes \mathscr{L}_2)),$$

and we can consider the total $\boldsymbol{W}$–cohomology ring $\bigoplus_{q,\mathscr{L}\in \mathrm{Pic}(X)/2} H^q(X, \boldsymbol{W}(\mathscr{L}))$ (again using [5] to make sense of this). Similar to the $\boldsymbol{I}$–cohomology ring, the total $\boldsymbol{W}$–cohomology ring is a (-1)–graded commutative algebra over the Witt ring $W(F)$; see Remark 2.3.

There is a morphism $(\boldsymbol{I}^n)_{n\in\mathbb{Z}} \to (\boldsymbol{W})_{n\in\mathbb{Z}}$ which in degree n is given by the natural inclusion $\boldsymbol{I}^n \hookrightarrow \boldsymbol{W}$, with the usual convention of $\boldsymbol{I}^n = \boldsymbol{W}$ for $n \le 0$. This morphism induces a $W(F)$–algebra homomorphism

$$\bigoplus_{q,\mathscr{L}} H^q(X, \boldsymbol{I}^q(\mathscr{L})) \to \bigoplus_{q,\mathscr{L}} H^q(X, \boldsymbol{W}(\mathscr{L}))$$

from the total $\boldsymbol{I}$–cohomology ring to the total $\boldsymbol{W}$–cohomology ring.

The relation with $\boldsymbol{I}$–cohomology can be made more precise. The pieces

$$H^{i-1}(X, \boldsymbol{K}^{\mathrm{M}}_{n-1}/2) \to H^i(X, \boldsymbol{I}^n(\mathscr{L})) \to H^i(X, \boldsymbol{I}^{n-1}(\mathscr{L})) \to H^i(X, \boldsymbol{K}^{\mathrm{M}}_{n-1}/2)$$

of the Bär sequence provide isomorphisms $H^i(X, \boldsymbol{I}^n(\mathscr{L})) \to H^i(X, \boldsymbol{I}^{n-1}(\mathscr{L}))$ for $i > n$, because the outer terms vanish. This can be seen from the Gersten resolution for mod 2 Milnor K–theory together with the fact that $(\boldsymbol{K}^{\mathrm{M}}_{n-1}/2)_{-c} = 0$ for $c > n-1$. Moreover, from the Gersten resolution for $\boldsymbol{I}^n$, we also see that the natural morphisms $H^i(X, \boldsymbol{I}^n(\mathscr{L})) \to H^i(X, \boldsymbol{W}(\mathscr{L}))$ are isomorphisms for $i > n$. Now with this reinterpretation, we can consider the piece of the Bär sequence for the boundary case $i = n$,

$$\mathrm{Ch}^{n-1}(X) \cong H^{n-1}(X, \boldsymbol{K}^{\mathrm{M}}_{n-1}/2) \xrightarrow{\beta_{\mathscr{L}}} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to H^n(X, \boldsymbol{W}(\mathscr{L})) \to 0.$$

In particular, $\boldsymbol{I}$–cohomology is a combination of $\boldsymbol{W}$–cohomology with the image of the Bockstein morphism β. We get a stronger splitting result if the $\boldsymbol{W}$–cohomology is free:

Lemma 2.4 *Let X be a smooth scheme over a field F of characteristic $\neq 2$, and let $\mathscr{L}$ be a line bundle on X. If $H^n(X, \boldsymbol{W}(\mathscr{L}))$ is free as a $W(F)$–module, then we have a splitting*

$$H^n(X, \boldsymbol{I}^n(\mathscr{L})) \cong \operatorname{Im} \beta_{\mathscr{L}} \oplus H^n(X, \boldsymbol{W}(\mathscr{L})).$$

In this case, the reduction morphism $\rho\colon H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to \mathrm{Ch}^n(X)$ is injective on the image of $\beta_{\mathscr{L}}$.

Proof The Bär sequence is a long exact sequence of $W(F)$–modules. The first claim follows from the piece

$$\mathrm{Ch}^{n-1}(X) \xrightarrow{\beta_{\mathscr{L}}} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to H^n(X, \boldsymbol{W}(\mathscr{L})) \to 0.$$

If the last group is free as $W(F)$–module, then the sequence splits as claimed.

For the second claim, we first note that the existence of a splitting implies

$$\big(I(F) \cdot H^n(X, \boldsymbol{I}^n(\mathscr{L}))\big) \cap \operatorname{Im} \beta_{\mathscr{L}} = 0.$$

This follows since the splitting map $H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to \operatorname{Im} \beta_{\mathscr{L}}$ is a $W(F)$–module map, and the $W(F)$–module structure on $\mathrm{Ch}^{n-1}(X)$ and hence $\operatorname{Im} \beta_{\mathscr{L}}$ is a direct sum of copies of $W(F)/I(F)$. Thus the splitting map necessarily sends every element in $I(F) \cdot H^n(X, \boldsymbol{I}^n(\mathscr{L}))$ to zero. But the splitting map is the identity on $\operatorname{Im} \beta_{\mathscr{L}}$, which implies the claim concerning the intersection.

Now to prove the second claim of the lemma, we want to use the exact piece of the Bär sequence

$$H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \xrightarrow{\rho} \mathrm{Ch}^n(X).$$

If we can show $\operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta = 0$, then the injectivity claim of the lemma follows. Suppose we have a nonzero element $\alpha \in \operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta$. Then we can factor the inclusion of the corresponding $W(F)/I(F)$–summand as

$$W(F)/I(F) \to H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})).$$

This map is now the inclusion of a direct summand (as $W(F)$–modules), but it is also multiplication by η from the factorization. Therefore, it is the zero map, contradicting the assumption $0 \neq \alpha \in \operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta$, which proves injectivity. □

Corollary 2.5 *Let X be a smooth scheme over a field F of characteristic $\neq 2$, and let $\mathscr{L}$ be a line bundle on X. If the total $\boldsymbol{W}$–cohomology ring is free as a $W(F)$–module, then the image of the maps $\beta_{\mathscr{L}}$ for $\mathscr{L} \in \mathrm{Pic}(X)/2$ coincides exactly with the $W(F)$–torsion in $\boldsymbol{I}^\bullet$–cohomology. In particular, the image of the maps $\beta_{\mathscr{L}}$ for $\mathscr{L} \in \mathrm{Pic}(X)/2$ is an ideal in the total $\boldsymbol{I}^\bullet$–cohomology ring.*

Remark 2.6 The freeness of $\boldsymbol{W}$–cohomology in this lemma will play an important role in our computations. It is an algebraic replacement of the classical statement that "all torsion in the cohomology of the Grassmannians is 2–torsion", as formulated in eg [8, Lemma 2.2]. Using the splitting in Lemma 2.4 is a different strategy than the cumbersome proofs in [16] which were needed to establish that ρ is injective on the image of β; see Remark 7.2 and the discussion before Proposition 8.6 in [16].

There are two reasons why the decomposition of $\boldsymbol{I}$–cohomology as a direct sum of $\boldsymbol{W}$–cohomology and the image of β is so effective as a computational tool. On the one hand, the image of β is basically known in the relevant cases — all it requires is knowledge of the Chow ring together with the action of Sq^2. On the other hand, computations in $\boldsymbol{W}$–cohomology are simpler than for $\boldsymbol{I}$–cohomology because the localization sequence takes the following simplified form: Assume X is a smooth scheme, $Z \subseteq X$ a smooth closed subscheme of pure codimension c with open complement $U = X \setminus Z$, and $\mathscr{L}$ is a line bundle on X. Denote the inclusions by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$, and denote by $\mathscr{N}$ the determinant of the normal bundle for Z in X. Then we have a localization sequence for $\boldsymbol{W}$–cohomology,

$$\cdots \to H^i(U, \boldsymbol{W}(\mathscr{L})) \xrightarrow{\partial} H^{i-c+1}(Z, \boldsymbol{W}(\mathscr{L} \otimes \mathscr{N}_Z)) \xrightarrow{i_*} H^{i+1}(X, \boldsymbol{W}(\mathscr{L})) \xrightarrow{j^*} H^{i+1}(U, \boldsymbol{W}(\mathscr{L})) \to \cdots .$$

This has the distinct advantage that there are no index shifts in the coefficients (such as what happens for $\boldsymbol{I}$–cohomology) and we really get an honest long exact sequence (as opposed to only a piece of a long exact sequence containing the "geometric bidegrees"). This way, computations of $\boldsymbol{W}$–cohomology can follow their classical topology counterparts much more closely than is possible for $\boldsymbol{I}$–cohomology.

Remark 2.7 One explanation of the simplified form of the localization sequence is that the $\boldsymbol{W}$–cohomology ring $\bigoplus_n H^\bullet(X, \boldsymbol{W})$ considered above is part of the η–inverted Witt group theory considered, for example, in [1]. Essentially, it is the quotient of the η–inverted Witt ring of X modulo $\eta - 1$. Some of the formulas for $\boldsymbol{W}$–cohomology of Grassmannians we develop in this paper already appear in [loc. cit.]. On the other hand, some of the computations for $\boldsymbol{W}$–cohomology in Section 6 could surely be done more generally for other cohomology theories in which η is invertible.

3 Characteristic classes for vector bundles

The next two sections will provide a computation of the Chow–Witt ring of $B\mathrm{GL}_n$. The global structure of the argument is similar to the computation of integral cohomology with local coefficients of $BO(n)$; see [26]. Some of the relevant adaptations to the Chow–Witt setting have already been made in [16]. Additionally, the decomposition of $\boldsymbol{I}$–cohomology into the image of β and $\boldsymbol{W}$–cohomology will significantly simplify the approach of [16], rendering the arguments even closer to their topological counterparts.

In this section, we begin by setting up the localization sequence and defining the relevant characteristic classes for vector bundles. We formulate the main structure results concerning the Chow–Witt and $\boldsymbol{I}^\bullet$–cohomology ring of $B\mathrm{GL}_n$ and establish the basic relations between the characteristic classes. The inductive proof of the structure theorem will be done in the next section.

Before embarking on the computation of cohomology of $B\mathrm{GL}_n$, we need to briefly discuss the issues arising from the classifying spaces not being smooth (in particular finite-dimensional) schemes; see also similar discussions in [16]. The cohomology theories discussed in Section 2 are usually applied to smooth schemes, and some techniques like Gersten-style complexes only work for smooth schemes.

There are then two approaches to extend the definition and computational tools like localization sequences to classifying spaces:

(i) One possibility is to use finite-dimensional approximations to the classifying space, built from representations of the group in question, as in Totaro's definition of Chow groups of classifying spaces [24]. In this approach, only finite-dimensional schemes are considered. Any particular such finite-dimensional approximation of the classifying space only captures the cohomology in a limited range of degrees. On the other hand, stabilization results imply that for any degree, one can always find a suitably high-dimensional approximation which correctly computes cohomology in this degree. This hinges on the fact that the cohomology theories we consider are based on cycles which implies that the degree q cohomology reflects the structure of codimension q subvarieties in a smooth scheme (as opposed to what would happen for algebraic or hermitian K–theory, for example).

(ii) The other possibility is to extend the cohomology theories discussed in Section 2 to all spaces in the Morel–Voevodsky motivic homotopy category [22]. All these cohomology theories are representable in motivic homotopy, because they satisfy Nisnevich descent and homotopy invariance. The classifying spaces can be constructed as spaces in the motivic homotopy category. This provides a definition of cohomology of classifying spaces which correctly computes all degrees at the same time.

For the present paper, we will work with the first viewpoint, using finite-dimensional approximations, as discussed below. In particular, all cohomology groups will in fact be cohomology groups of smooth schemes. Referring to $H^\bullet(B\mathrm{GL}_n)$ means that whenever we are interested in a particular cohomological degree q, we are actually considering a suitably high-dimensional smooth scheme X and compute $H^q(X)$. All the discussions (in particular ones using localization sequences or intersection products) will always only involve a finite number of degrees, so that this is indeed possible.

3.1 Setup of localization sequence

We begin by setting up the localization sequence for the inductive computation of the cohomology of $B\mathrm{GL}_n$, following the procedure for SL_n in [16, Section 5.1].

Let V be a finite-dimensional representation of GL_n such that outside a closed GL_n–stable subset Y of codimension s, the action of GL_n is free and the quotient $X(V) := (V \setminus Y)/\mathrm{GL}_n$ is a GL_n–torsor (ie a GL_n–principal bundle). For any s, there is a GL_n–representation satisfying this requirement; see the discussion [24, Section 1, Remark 1.4]. Then the Chow–Witt group $\widetilde{\mathrm{CH}}^q(X(V), \mathscr{L})$ is up to isomorphism independent of the choice of representation V for $q \le s-2$, ie computes $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathscr{L})$ in degrees $\le s-2$. Moreover, a finite-dimensional model for the universal GL_n–torsor is given by the projection $p\colon V \setminus Y \to X(V)$. The tautological GL_n–representation on $\mathbb{A}^n$ gives rise to a vector bundle $\gamma_V\colon E_n(V) \to X(V)$ associated to the GL_n–torsor $p\colon V \setminus Y \to X(V)$.

Denote by $S_n(V)$ the complement of the zero-section of $\gamma_V\colon E_n(V) \to X(V)$. As in the case of SL_n, the complement $S_n(V)$ can be identified as an approximation of the classifying space $B\mathrm{GL}_{n-1}$. Moreover, the quotient map $q\colon (V\setminus Y)/\mathrm{GL}_{n-1}\to X(V)$ induces a morphism

$$\widetilde{\mathrm{CH}}^\bullet(X(V),\mathscr{L})\xrightarrow{\gamma_V^*}\widetilde{\mathrm{CH}}^\bullet(S_n(V),\gamma_V^*(\mathscr{L}))\cong\widetilde{\mathrm{CH}}^\bullet((V\setminus Y)/\mathrm{GL}_{n-1},q^*(\mathscr{L}))$$

which models the stabilization map $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_{n-1},\iota^*\mathscr{L})$ for the standard inclusion $\iota\colon B\mathrm{GL}_{n-1}\to B\mathrm{GL}_n$. Consequently, we get the following localization sequence:

Proposition 3.1 *There is a long exact sequence of Chow–Witt groups of classifying spaces*

$$\cdots\to\widetilde{\mathrm{CH}}^{q-n}(B\mathrm{GL}_n,\mathscr{L}\otimes\det\gamma_n)\to\widetilde{\mathrm{CH}}^q(B\mathrm{GL}_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^q(B\mathrm{GL}_{n-1},\iota^*(\mathscr{L}))$$
$$\to H^{q+1-n}(B\mathrm{GL}_n,\boldsymbol{K}^{\mathrm{MW}}_{q-n}(\mathscr{L}\otimes\det\gamma_n))\to H^{q+1}(B\mathrm{GL}_n,\boldsymbol{K}^{\mathrm{MW}}_q(\mathscr{L}))\to\cdots.$$

The first map is the composition of the dévissage isomorphism with the forgetting of support — alternatively "multiplication with the Euler class of the universal bundle γ_n". The second map is the restriction along the stabilization inclusion $\iota\colon \mathrm{GL}_{n-1}\to\mathrm{GL}_n$.

There are similar exact sequences for the other coefficients, $\boldsymbol{I}^\bullet(\mathscr{L})$, $\boldsymbol{K}^{\mathrm{M}}_\bullet$ and $\boldsymbol{W}(\mathscr{L})$, and the change-of-coefficients maps induce commutative ladders of exact sequences. Notably, the localization sequence for $\boldsymbol{W}$–cohomology is

$$\cdots\to H^{q-n}(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}\otimes\det\gamma_n))\xrightarrow{e_n}H^q(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}))$$
$$\xrightarrow{\iota^*}H^q(B\mathrm{GL}_{n-1},\boldsymbol{W}(\iota^*\mathscr{L}))\xrightarrow{\partial}H^{q-n+1}(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}\otimes\det\gamma_n))\to\cdots.$$

The proof is the same line of argument as for the case SL_n in [16, Proposition 5.1].

Remark 3.2 Note also that for $\mathscr{L}=\det\gamma_n^\vee$, with $\gamma_n^\vee$ the dual of the universal rank n bundle on $B\mathrm{GL}_n$, we have $\iota^*\mathscr{L}\cong\det\gamma_{n-1}^\vee$. Multiplication with the Euler class changes the dualities.

3.2 Euler class

Recall from [16, Definition 5.9] how the Chow–Witt-theoretic Euler class of [4] gives rise to an Euler class in $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\det\gamma_n^\vee)$. For a smooth scheme X, the Chow–Witt-theoretic Euler class of a vector bundle $p\colon\mathscr{E}\to X$ of rank n is defined via the formula

$$e_n(p\colon\mathscr{E}\to X):=(p^*)^{-1}s_{0*}(1)\in\widetilde{\mathrm{CH}}^n(X,\det(p)^\vee),$$

where $s_0\colon X\to\mathscr{E}$ is the zero section. Using smooth finite-dimensional approximations to the classifying space $B\mathrm{GL}_n$ provides a well-defined Euler class

$$e_n\in\widetilde{\mathrm{CH}}^n(B\mathrm{GL}_n,\det(\gamma_n)^\vee).$$

In the localization sequence of Proposition 3.1, the Euler class corresponds under the dévissage isomorphism to the Thom class for the universal rank n vector bundle γ_n on $B\mathrm{GL}_n$. This justifies calling the composition

$$\widetilde{\mathrm{CH}}^{q-n}(B\mathrm{GL}_n,\mathscr{L}\otimes\det\gamma_n)\cong\widetilde{\mathrm{CH}}^q_{B\mathrm{GL}_n}(E_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^q(E_n,\mathscr{L})\cong\widetilde{\mathrm{CH}}^q(B\mathrm{GL}_n,\mathscr{L})$$

"multiplication with the Euler class". There are corresponding notions of Euler classes in $\boldsymbol{I}^\bullet$–cohomology, $\boldsymbol{W}$–cohomology, as well as Chow theory; these are compatible with the change of coefficients. The Euler classes are compatible with pullbacks of morphisms between smooth schemes; see [4, Proposition 3.1.1].

3.3 Chern classes

A direct consequence of the above localization sequence for Chow theory is the computation of the Chow ring (with integral and mod 2 coefficients) of the classifying space $B\mathrm{GL}_n$. The formulas are the standard ones found in any intersection theory handbook; see also [16, Proposition 5.2]. As in [loc. cit.], the Chern classes are uniquely determined by their compatibility with stabilization and the identification of the top Chern class with the Euler class of the universal bundle.

Proposition 3.3 *There are unique classes $c_i(\mathrm{GL}_n) \in \mathrm{CH}^i(B\mathrm{GL}_n)$ for $1 \le i \le n$, such that the natural stabilization morphism $\iota\colon B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$ satisfies $\iota^* c_i(\mathrm{GL}_n) = c_i(\mathrm{GL}_{n-1})$ for $i < n$ and $c_n(\mathrm{GL}_n) = e_n(\mathrm{GL}_n)$. In particular, the Chow–Witt-theoretic Euler class reduces to the top Chern class in the Chow theory. There is a natural isomorphism*

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n].$$

The restriction along the Whitney sum $B\mathrm{GL}_m \times B\mathrm{GL}_{n-m} \to B\mathrm{GL}_n$ maps the Chern classes as

$$c_i \mapsto \sum_{j=i+m-n}^{m} c_j \boxtimes c_{i-j}.$$

Remark 3.4 From the above computations of the Chow ring of $B\mathrm{GL}_n$ we also see the standard fact that $\mathrm{Pic}(B\mathrm{GL}_n) \cong \mathbb{Z}$. Note that for any smooth scheme X and any two line bundles $\mathscr{L}$ and $\mathscr{N}$ over X such that the class of $\mathscr{L}$ in $\mathrm{Pic}(X)$ is divisible by 2,

$$\widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L} \otimes \mathscr{N}) \cong \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{N}).$$

In particular, there are only two relevant dualities to consider for $B\mathrm{GL}_n$: the trivial duality corresponding to the trivial line bundle on $B\mathrm{GL}_n$, and the nontrivial duality corresponding to the determinant of the universal bundle. This closely resembles the classical situation where $\pi_1(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ and so there are only two isomorphism classes of local systems on $BO(n)$.

3.4 Pontryagin classes

Recall from [16, Definition 5.6] that the Pontryagin classes of vector bundles are defined as the images of $p_i \in \widetilde{\mathrm{CH}}^\bullet(B\mathrm{Sp}_{2n})$ of [16, Theorem 4.10] under the homomorphism $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{Sp}_{2n}) \to \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n)$, which is induced from the symplectification morphism (ie the standard hyperbolic functor) $B\mathrm{GL}_n \to B\mathrm{Sp}_{2n}$. Note that this means that the Pontryagin classes of vector bundles are elements in the Chow–Witt ring with trivial duality (because they are induced from the symplectic group). As for the special linear groups — see [16, Proposition 5.8] — the Pontryagin classes are compatible with stabilization in the sense that

$$\iota^*(p_i(\mathrm{GL}_n)) = p_i(\mathrm{GL}_{n-1}),$$

where $i < n$ and $\iota^*\colon \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n) \to \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_{n-1})$ is induced from the natural stabilization map $\mathrm{GL}_{n-1} \to \mathrm{GL}_n$. There are corresponding definitions of Pontryagin classes for $\boldsymbol{I}^\bullet$–cohomology and $\boldsymbol{W}$–cohomology, compatible with the natural change-of-coefficient maps

$$\widetilde{\mathrm{CH}}^q(X) \to H^q(X, \boldsymbol{I}^q) \to H^q(X, \boldsymbol{W}).$$

Remark 3.5 (NB concerning odd Pontryagin classes) The above definition produces Pontryagin classes $p_1, \dots, p_n$ for GL_n; in particular, we get *odd* Pontryagin classes p_{2i+1}. These classes turn out to be torsion and are not included explicitly in the presentation of $\boldsymbol{I}$–cohomology — see, for example, in Theorem 3.24 — because they can be expressed as Bockstein classes; see Theorem 3.27. Note that the indexing convention here differs from the one employed in topology, where only the even Chern classes are used in the definition of Pontryagin classes of bundles — the Pontryagin classes in topology correspond to the *even* Pontryagin classes in the present paper. The reason for this choice of indexing here — and also in [16] — is the easier formulation of the Whitney sum formula for Pontryagin classes; see Proposition 3.28 and Remark 3.29.

3.5 Stiefel–Whitney classes and their (twisted) Bocksteins

The localization sequence of Proposition 3.1 immediately implies a theory of Stiefel–Whitney classes which are determined by the compatibility with stabilization and the identification of the top Stiefel–Whitney class with the Euler class of the respective universal bundle; see [16, Proposition 5.4].

Proposition 3.6 *There are unique classes $\bar{c}_i(\mathrm{GL}_n) \in \mathrm{Ch}^i(B\mathrm{GL}_n)$ for $1 \le i \le n$, such that the natural stabilization morphism $\iota\colon B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$ satisfies $\iota^*\bar{c}_i(\mathrm{GL}_n) = \bar{c}_i(\mathrm{GL}_{n-1})$ for $i < n$ and $\bar{c}_n(\mathrm{GL}_n) = e_n(\mathrm{GL}_n)$. These agree with the Stiefel–Whitney classes in* [14, Definition 4.2]. *There is a natural isomorphism*

$$\mathrm{Ch}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \dots, \bar{c}_n].$$

Again, this is a very classical formula. We include it just for the following discussion of the (twisted) Bockstein classes and the action of the respective (twisted) Steenrod squares on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

Recall from Section 2 that for a scheme X and a line bundle $\mathscr{L}$, we have a Bockstein map

$$\beta_{\mathscr{L}}\colon \mathrm{Ch}^n(X) \to H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})).$$

For the specific case of $B\mathrm{GL}_n$, there are two relevant line bundles to consider: $\mathcal{O}$ and $\det \gamma_n^\vee$; see Remark 3.4. This leads to two types of Bockstein classes for vector bundles:

Definition 3.7 For a (possibly empty) set $J = \{j_1, \dots, j_l\}$ of positive natural numbers

$$0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right],$$

there are classes

$$\beta_J := \beta_{\mathcal{O}}(\bar{c}_{2j_1}\bar{c}_{2j_2}\cdots\bar{c}_{2j_l}) \in H^{d+1}(B\mathrm{GL}_n, \boldsymbol{I}^{d+1}),$$
$$\tau_J := \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\bar{c}_{2j_2}\cdots\bar{c}_{2j_l}) \in H^{d+1}(B\mathrm{GL}_n, \boldsymbol{I}^{d+1}(\det\gamma_n^\vee)),$$

where $d = \sum_{a=1}^{l} 2j_a$.

Remark 3.8 We discuss the special cases with $J = \varnothing$. The Bockstein class $\beta(\varnothing)$ is trivial; see [16, Remark 5.12]. However, the class $\tau(\varnothing)$ is nontrivial; more precisely,

$$\rho(\tau(\varnothing)) = \mathrm{Sq}^2_{\det\gamma_n^\vee}(1) = \bar{c}_1.$$

As a matter of convention, whenever products $c_{2j_1}\cdots c_{2j_l}$ of Chern classes (or their mod 2 reductions, the Stiefel–Whitney classes) appear in the paper, the indices will be positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right]$.

Lemma 3.9 *For a (possibly empty) set $J = \{j_1, \dots, j_l\}$ of positive natural numbers*

$$0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right],$$

we have

$$I(F)\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = 0 \quad \text{and} \quad I(F)\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = 0$$

in $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet)$ and $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet(\det\gamma_n^\vee))$, respectively.

Proof As in [16, Lemma 7.3], this is formal from the $W(F)$–linearity of the maps in the exact Bär sequence. □

Proposition 3.10 *With the notation from Definition 3.7, if $n = 2k+1$,*

$$e_n = \beta_{\det\gamma_n^\vee}(\bar{c}_{n-1}) = \tau_{\{k\}}.$$

Proof This is proved in [14, Theorem 10.1], noting that our Stiefel–Whitney classes in Proposition 3.6 agree with those in [loc. cit.]; see also [16, Proposition 7.5]. □

Combining Lemma 3.9 and Proposition 3.10, we see that the Euler class $e_n \in H^n(B\mathrm{GL}_n, \boldsymbol{I}^n(\det\gamma_n^\vee))$ is $I(F)$–torsion if n is odd.

Remark 3.11 On $B\mathrm{GL}_n$, the Bockstein classes don't contain more information than the Stiefel–Whitney classes; it will follow from Proposition 4.5 combined with Lemma 2.4 that the reduction morphism

$$\rho\colon H^m(B\mathrm{GL}_n, \boldsymbol{I}^m(\mathscr{L})) \to \mathrm{Ch}^m(B\mathrm{GL}_n)$$

is injective on the image of $\beta_{\mathscr{L}}$. However, for a smooth scheme X, it is possible that the Bockstein class is nontrivial while its reduction in the mod 2 Chow ring is trivial. Topologically, this happens if the integral Stiefel–Whitney class is divisible by 2; divisibility results for the integral Stiefel–Whitney classes arise, for example, in Massey's discussion of the obstruction theory for existence of almost complex structures.

3.6 The Wu formula for the Chow ring

We briefly discuss the action of the Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. Essentially, this is described by the Wu formula. It's well known, and there are several ways to prove it, for instance deducing it from the Wu formula in singular cohomology via the cycle class map. We give a sketch of argument relying mostly on Fasel's computations with integral Stiefel–Whitney classes in [14].

Proposition 3.12 *The untwisted Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *is given by*

$$\mathrm{Sq}^2_{\mathbb{O}}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n), \quad \bar{c}_j \mapsto \bar{c}_1\bar{c}_j + (j-1)\bar{c}_{j+1}.$$

The twisted Steenrod square $\mathrm{Sq}^2_{\det\gamma_n}$ *is given by*

$$\mathrm{Sq}^2_{\det\gamma_n}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n), \quad \bar{c}_j \mapsto (j-1)\bar{c}_{j+1}.$$

The (twisted) Steenrod squares $\mathrm{Sq}^2_{\mathbb{O}}$ *and* $\mathrm{Sq}^2_{\det\gamma_n}$ *of other elements are determined by the above formulas, the derivation property of the Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *and the relation*

$$\mathrm{Sq}^2_{\det\gamma_n^\vee}(x) = \bar{c}_1\cdot x + \mathrm{Sq}^2_{\mathbb{O}}(x).$$

Proof The first and second statement are equivalent by Proposition 2.2 and noting that $\bar{c}_1(\det\gamma_n) = \bar{c}_1$. So it suffices to prove the claims concerning $\mathrm{Sq}^2_{\det\gamma_n}$.

The second statement about the twisted Steenrod square in case of even Stiefel–Whitney classes is proved in [14, Proposition 10.3, Remark 10.5]. For odd Stiefel–Whitney classes, the vanishing of $\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2n+1})$ is a consequence of the following computation, applied to $x = \bar{c}_{2n}$ and using $\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2n}) = \bar{c}_{2n+1}$:

$$\begin{aligned}\mathrm{Sq}^2_{\det\gamma_n}\circ\mathrm{Sq}^2_{\det\gamma_n}(x) &= \bar{c}_1\cdot\mathrm{Sq}^2_{\det\gamma_n}(x) + \mathrm{Sq}^2_{\mathbb{O}}\circ\mathrm{Sq}^2_{\det\gamma_n}(x)\\ &= \bar{c}_1^2\cdot x + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + \mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_1\cdot x) + \mathrm{Sq}^2_{\mathbb{O}}\circ\mathrm{Sq}^2_{\mathbb{O}}(x)\\ &= \bar{c}_1^2\cdot x + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + x\cdot\mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_1)\\ &= 0.\end{aligned}$$

□

Corollary 3.13 *The kernel of the untwisted Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *is given by the subring*

$$\mathbb{Z}/2\mathbb{Z}[\bar{c}_i^2, \bar{c}_1\bar{c}_{2i} + \bar{c}_{2i+1}, \bar{c}_1\bar{c}_n] \subseteq \mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_n] = \mathrm{Ch}^\bullet(B\mathrm{GL}_n).$$

The kernel of the twisted Steenrod square $\mathrm{Sq}^2_{\det\gamma_n}$ *is given by the submodule (over the kernel of* $\mathrm{Sq}^2_{\mathbb{O}}$*)*

$$\langle\bar{c}_{2i+1}, \bar{c}_n\rangle_{\ker\mathrm{Sq}^2_{\mathbb{O}}} \subseteq \mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_n] = \mathrm{Ch}^\bullet(B\mathrm{GL}_n).$$

Proof The claims follow from the Wu formula in Proposition 3.12. The odd Chern classes are in the kernel of the twisted Steenrod square, and $\bar{c}_n$ is the image of the Euler class. Even though the twisted Steenrod square for Chern classes is given essentially by the same formula as the Steenrod square in $\mathrm{Ch}^\bullet(B\mathrm{SL}_n)$, the formula differs from [16] since $\mathrm{Sq}^2_{\det\gamma_n^\vee}(\bar{c}_{2i}^2) = \bar{c}_1\bar{c}_{2i}^2$. For the untwisted Steenrod square $\mathrm{Sq}^2_{\mathbb{O}}$, the even classes $\bar{c}_{2i}$ map to $\bar{c}_1\bar{c}_{2i} + \bar{c}_{2i+1}$. Hence the latter classes are in the kernel of the Steenrod square; similarly for $\bar{c}_1\bar{c}_n$. The description of the kernels follow from that; see also [26, page 285]. □

Corollary 3.14 *Consider the mod 2 Chow ring $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. The images of the Steenrod square maps $\mathrm{Sq}^2_{\mathscr{L}}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n)\to \mathrm{Ch}^{\bullet+1}(B\mathrm{GL}_n)$ for $\mathscr{L}=\mathscr{O},\det\gamma_n^\vee$ are contained in the subring generated by the classes $\bar c_1=\mathrm{Sq}^2_{\det\gamma_n}(1)$, $\bar c_{2i}^2$, and $\bar c_n$ as well as $\mathrm{Sq}^2_{\mathscr{O}}(\bar c_{2j_1}\cdots\bar c_{2j_l})$ and $\mathrm{Sq}^2_{\det\gamma_n}(\bar c_{2j_1}\cdots\bar c_{2j_l})$ for (possibly empty) sequences of positive natural numbers $0<j_1<j_2<\cdots<j_l\le\left[\frac{1}{2}(n-1)\right]$.*

Proof The Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ are linear. To determine generators of the image, it thus suffices to consider Steenrod squares of monomials in the Chern classes.

Since $\mathrm{Sq}^2_{\mathscr{O}}$ is a derivation, $\mathrm{Sq}^2_{\mathscr{O}}(x^2)=2x\mathrm{Sq}^2_{\mathscr{O}}(x)=0$ and $\mathrm{Sq}^2_{\mathscr{O}}(x^2y)=x^2\mathrm{Sq}^2_{\mathscr{O}}(y)$. In particular, we can always pull out squares. For even Stiefel–Whitney classes, these squares are explicitly included as generators in the statement. For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\mathscr{O}}(\bar c_{2i}\bar c_{2i+1})=\bar c_{2i}\mathrm{Sq}^2_{\mathscr{O}}(\bar c_{2i+1})+\bar c_{2i+1}\mathrm{Sq}^2_{\mathscr{O}}(\bar c_{2i})=2\bar c_1\bar c_{2i}\bar c_{2i+1}+\bar c_{2i+1}^2.$$

It thus suffices to show that the Steenrod squares of all products $\bar c_{j_1}\cdots\bar c_{j_m}$ with no repeating factors are contained in the subring as claimed.

For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\mathscr{O}}(\bar c_{2i+1}x)=\bar c_{2i+1}\mathrm{Sq}^2(x)+\bar c_1\bar c_{2i+1}x=\bar c_{2i+1}\mathrm{Sq}^2_{\det\gamma_n}(x).$$

Since $\bar c_{2i+1}=\mathrm{Sq}^2_{\det\gamma_n}(\bar c_{2i})$ with the special case $\bar c_1=\mathrm{Sq}^2_{\det\gamma_n}(1)$, the odd Stiefel–Whitney classes are also among the generators of the subring listed in the claim. Therefore, we can also pull out all the odd Stiefel–Whitney classes from the products $\bar c_{j_1}\cdots\bar c_{j_m}$. A similar calculation shows that we can also pull out $\bar c_n$, which is also included explicitly among the generators. We have thus established the claim for $\mathrm{Sq}^2_{\mathscr{O}}$.

To show the claim for $\mathrm{Sq}^2_{\det\gamma_n}$, we first have $\mathrm{Sq}^2_{\det\gamma_n}(x^2)=\bar c_1x^2$ and

$$\mathrm{Sq}^2_{\det\gamma_n}(x^2y)=\bar c_1x^2y+x^2\mathrm{Sq}^2_{\mathscr{O}}(y)=x^2\mathrm{Sq}^2_{\det\gamma_n}(y).$$

This tells us again that we can always pull out squares. For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\det\gamma_n}(\bar c_{2i+1}x)=\bar c_1\bar c_{2i+1}x+\bar c_{2i+1}\mathrm{Sq}^2_{\det\gamma_n}(x)=\bar c_{2i+1}\mathrm{Sq}^2_{\mathscr{O}}(x).$$

Therefore,we can also pull out odd Stiefel–Whitney classes (and by a similar computation also $\bar c_n$). □

3.7 The candidate presentation

We define an appropriate graded ring $\mathscr{R}_n/\mathscr{I}_n$ which we will prove to be isomorphic to

$$H^\bullet(B\mathrm{GL}_n,\boldsymbol{I}^\bullet\oplus\boldsymbol{I}^\bullet(\det\gamma_n^\vee)).$$

The ring will be graded by $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, where the degrees $(n,0)$ are those with $\boldsymbol{I}^\bullet$–coefficients, and the degrees $(n,1)$ are those with $\boldsymbol{I}^\bullet(\det\gamma_n^\vee)$–coefficients. The ring will in fact be graded-commutative; see the discussion in Remark 2.3. Following [26], we use the notation $\Delta(J,J')=(J\cup J')\setminus(J\cap J')$ for the symmetric difference of two subsets J and J' of a given set.

Definition 3.15 Let F be a field of characteristic $\neq 2$ and denote by $W(F)$ the Witt ring of quadratic forms over F with its fundamental ideal $I(F) \subseteq W(F)$ of even-dimensional forms. For a natural number $n \geq 1$, we define the $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$*–graded-commutative* $W(F)$–algebra

$$\mathcal{R}_n = W(F)[P_1, \dots, P_{[(n-1)/2]}, X_n, B_J, T_J, T_\varnothing].$$

The classes P_i have degree $(4i, 0)$ and the class X_n has degree $(n, 1)$. For the classes B_J and T_J, the index set J runs through the (possibly empty) sets $\{j_1, \dots, j_l\}$ of positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-1)\right]$, and the degrees of B_J and T_J are $(d, 0)$ and $(d, 1)$, respectively, with $d = 1 + 2\sum_{a=1}^{l} j_a$. By convention $B_\varnothing = 0$.

Let $\mathcal{I}_n \subset \mathcal{R}_n$ be the ideal generated by the following relations:

(1) $I(F)B_J = I(F)T_J = I(F)T_\varnothing = 0$.

(2) If $n = 2k+1$ is odd and $k \geq 1$, then $X_{2k+1} = T_{\{k\}}$; for $n = 1$ we have $X_1 = T_\varnothing$.

(3) For any two index sets J and J', where J' can be empty,

$$B_J \cdot B_{J'} = \sum_{k \in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot B_{\Delta(J\setminus\{k\}, J')}, \tag{3-1}$$

$$B_J \cdot T_{J'} = \sum_{k \in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot T_{\Delta(J\setminus\{k\}, J')}, \tag{3-2}$$

$$T_J \cdot B_{J'} = B_J \cdot T_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot B_{\Delta(J, J')}, \tag{3-3}$$

$$T_J \cdot T_{J'} = B_J \cdot B_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot T_{\Delta(J, J')}. \tag{3-4}$$

Here we set $P_A = \prod_{i=1}^{l} P_{a_i}$ for an index set $A = \{a_1, \dots, a_l\}$, with the usual convention that $P_\varnothing = 1$ (in the degree $(0,0)$ component of $\mathcal{R}_n$).

Example 3.16 We briefly discuss the edge case $n = 1$. In this case, no classes P_i appear, and there are no classes B_J or T_J with J nonempty. The only relevant generators are X_1 and $T_\varnothing$. From relation (2), we get $X_1 = T_\varnothing$. Relation (1) implies that this class is $I(F)$–torsion. From the relations in (3), only (3-4) would be applicable, but that trivializes to $T_\varnothing^2 = T_\varnothing^2$. The resulting ring has $W(F)$ in degree 0, generated by 1, and has a $W(F)/I(F) \cong \mathbb{Z}/2\mathbb{Z}$–summand generated by $T_\varnothing^i$ in degree $(i, i \bmod 2)$ for each $i \geq 1$.

Remark 3.17 We will show in Theorem 3.24 that the ring $\mathcal{R}_n/\mathcal{I}_n$ is isomorphic to the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_n$. The classes P_i correspond to Pontryagin classes, and the class X_n to the Euler class. The classes B_J and T_J for a (possibly empty) index set $J = \{j_1, \dots, j_l\}$ of positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-1)\right]$ correspond to nontwisted and twisted Bockstein classes, respectively,

$$B_J \mapsto \beta_{\mathbb{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}), \quad T_J \mapsto \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}).$$

The special class $T_\varnothing$ corresponds to $\beta_{\det \gamma_n^\vee}(1)$ whose reduction in $\mathrm{Ch}^1(B\mathrm{GL}_n)$ is the first Stiefel–Whitney class $\bar{c}_1 = \mathrm{Sq}^2_{\det \gamma_n^\vee}(1)$.

Remark 3.18 There are slight differences in the indexing sets between the formulas in [8] and [26]. For the Pontryagin classes, this difference is due to fact that the Euler class squares to the top Pontryagin class. So in Čadek's presentation, there is no need to introduce the top Pontryagin class; on the other hand, Brown only computes cohomology with trivial coefficients and he has to introduce the top Pontryagin class separately. The same thing is true for the Bockstein classes:

$$\beta_{\mathbb{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})e_{n-1} \quad \text{if } j_l = \tfrac{1}{2}(n-1),$$

and this relation cannot be expressed in cohomology with trivial coefficients. Moreover, the reason why Brown's additional $\bar{c}_1$–factors in the Bockstein classes can be omitted in Čadek's presentation is given by the formula

$$\beta_{\mathbb{O}}(\bar{c}_1\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})\beta_{\det\gamma_n^\vee}(1).$$

Definition 3.19 Let $n \geq 2$ be a natural number. Define the $W(F)$–algebra homomorphism $\Phi_n\colon \mathcal{R}_n \to \mathcal{R}_{n-1}$ by

(1) the element P_i maps to P_i if $i < \frac{1}{2}(n-1)$ and maps to X_{n-1}^2 if $i = \frac{1}{2}(n-1)$,

(2) the element X_n maps to 0,

(3) for any index set $J = \{j_1, \dots, j_l\}$,

$$B_J \mapsto \begin{cases} B_J & \text{if } j_l < \frac{1}{2}(n-1), \\ T_{J'} \cdot X_{n-1} & \text{if } j_l = \frac{1}{2}(n-1),\ J = J' \sqcup \{j_l\}, \end{cases}$$
$$T_J \mapsto \begin{cases} T_J & \text{if } j_l < \frac{1}{2}(n-1), \\ B_{J'} \cdot X_{n-1} & \text{if } j_l = \frac{1}{2}(n-1),\ J = J' \sqcup \{j_l\}. \end{cases}$$

Remark 3.20 The above formulas model the restriction of classes from $B\mathrm{GL}_n$ to $B\mathrm{GL}_{n-1}$. On the level of mod 2 Chow rings,

$$\mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})\bar{c}_{2j_l} + \bar{c}_1\bar{c}_{2j_1}\cdots\bar{c}_{2j_l} = \mathrm{Sq}^2_{\mathcal{L}\otimes\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})e_{n-1},$$

using Proposition 2.2. Note that the formulas for restriction on the bottom of page 283 in [26] contain some typos, the classes having the wrong degrees.

Proposition 3.21 *With the notation from Definitions 3.15 and 3.19,*

$$\Phi_n(\mathcal{I}_n) \subseteq \mathcal{I}_{n-1}.$$

In particular, the map Φ_n descends to a well-defined ring homomorphism

$$\overline{\Phi}_n\colon \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1}.$$

Proof We first deal with the relations of type (1). Recall that the map Φ_n is by definition $W(F)$–linear; in particular, it will send $I(F)$ to $I(F)$. Since Φ_n sends B_J to either B_J or $T_{J'} \cdot X_{n-1}$ (and similarly T_J to either T_J or $B_{J'} \cdot X_{n-1}$, with the special case $B_\varnothing = 0$) it is clear the relations of type (1) are preserved.

The relations of type (2) are also preserved since both X_{2k+1} and T_k are mapped to 0 by Φ_n.

It remains to deal with relations of type (3). These relations are trivially preserved if neither J nor J' contains the highest possible index $j_l = \frac{1}{2}(n-1)$. In this case, all the relevant B_J, T_J and P_J will exist both in $\mathcal{R}_n$ and $\mathcal{R}_{n-1}$, and the corresponding relation in $\mathcal{R}_n$ is just mapped to the same relation in $\mathcal{R}_{n-1}$. For the rest of the proof, we will use the numbering (3-1)–(3-4) specified in Definition 3.15.

For relations of type (3-1), assume that $j_l \in J'$ and $j_l \notin J$. On the left-hand side, $B_{J'}$ restricts to $T_{J'\setminus\{j_l\}} \cdot X_{n-1}$ and on the right-hand side, $B_{\Delta(J\setminus\{k\},J')}$ restricts to $B_{\Delta(J\setminus\{k\},J'\setminus\{j_l\})} \cdot X_{n-1}$. The result is the product of a relation of type (3-2) with X_{n-1}. Conversely, if $j_l \in J$ and $j_l \notin J'$, then the left-hand side restricts to $T_{J\setminus\{j_l\}} \cdot X_{n-1} \cdot B_{J'}$. The right-hand side restricts to

$$\sum_{k \in J\setminus\{j_l\}} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot T_{\Delta(J\setminus\{k,j_l\},J')} \cdot X_{n-1} + T_{\varnothing} \cdot X_{n-1} \cdot P_{(J\setminus\{j_l\})\cap J'} \cdot B_{\Delta(J\setminus\{j_l\},J')}.$$

But this is the product of a relation of type (3-3) and X_{n-1}. Finally, when $j_l \in J \cap J'$, the left-hand side restricts to $T_{J\setminus\{j_l\}} \cdot T_{J'\setminus\{j_l\}} \cdot X_{n-1}^2$. The right-hand side restricts to

$$\sum_{k \in J\setminus\{j_l\}} B_{\{k\}} \cdot P_{(J\setminus\{k,j_l\})\cap J'} \cdot X_{n-1}^2 \cdot B_{\Delta(J\setminus\{k\},J')} + T_{\varnothing} \cdot P_{(J\setminus\{j_l\})\cap J'} \cdot T_{\Delta(J,J')} \cdot X_{n-1}^2.$$

This is a product of a relation of type (3-4) with X_{n-1}^2. The argument for restriction of relations of type (3-2) is completely analogous.

For the restriction of relations of type (3-4), if $j_l \in J$ and $j_l \notin J'$, the left-hand side restricts to $B_J \cdot T_{J'} \cdot X_{n-1}$. The right-hand side restricts to $T_J \cdot B_{J'} \cdot X_{n-1} + T_{\varnothing} \cdot P_{J\cap J'} \cdot B_{\Delta(J\setminus\{j_l\},J')} \cdot X_{n-1}$. This is the product of a relation of type (3-3) with X_{n-1}, noting that all terms here are 2–torsion. All the other cases are done similarly, and the argument for relations (3-3) is again analogous.

Since $\Phi_n(\mathcal{I}_n) \subset \mathcal{I}_{n-1}$, it follows that the restriction map descends to a $W(F)$–algebra map

$$\overline{\Phi}_n : \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1},$$

as claimed. □

Lemma 3.22 *If n is even, then we have an isomorphism*

$$\mathcal{R}_n/\mathcal{I}_n \cong \mathcal{R}_{n-1}/\mathcal{I}_{n-1}[X_n].$$

In particular, the restriction map $\overline{\Phi}_n : \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ is surjective.

Proof The index sets for the elements P_i are the same for n and $n-1$. In particular, $i \neq \frac{1}{2}(n-1)$ which means that the P_i in $\mathcal{R}_n$ are just mapped to the P_i in $\mathcal{R}_{n-1}$. The same is true for the index sets for B_J and T_J. Moreover, in $\mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ we have $X_{n-1} = T_{\{(n-2)/2\}}$. This proves the surjectivity of Φ_n. The claim about the polynomial ring follows since X_n doesn't appear in any relation in $\mathcal{R}_n$. □

Lemma 3.23 *If n is odd, then there is an exact sequence of graded $W(F)$–algebras*

$$\mathcal{R}_n/\mathcal{I}_n \xrightarrow{\overline{\Phi}_n} \mathcal{R}_{n-1}/\mathcal{I}_{n-1} \to W(F)[X_{n-1}]/(X_{n-1}^2) \to 0.$$

Proof The elements $P_i \in \mathcal{R}_n$ with $i < \frac{1}{2}(n-1)$ are mapped under Φ_n to the elements with the same name in $\mathcal{R}_{n-1}$. The same holds for the elements B_J and T_J where the index set J doesn't contain $\frac{1}{2}(n-1)$. In particular, the subalgebra of $\mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ generated by all P_i, B_J and T_J is in the image. The only elements in $\mathcal{R}_n$ we have not yet considered so far are the new $P_{(n-1)/2}$ and the elements B_J and T_J where J contains $\frac{1}{2}(n-1)$. The element X_{n-1}^2 is in the image of $P_{(n-1)/2}$, the elements $B'_J X_{n-1}$ are in the image of T_J and the elements $T'_J X_{n-1}$ are in the image of B_J. However, the element X_{n-1} itself is not in the image since we noted in Lemma 3.22 that it is a polynomial variable in $\mathcal{R}_{n-1}$. Consequently, defining the morphism $\mathcal{R}_{n-1}/\mathcal{I}_{n-1} \to W(F)[X_{n-1}]/(X_{n-1}^2)$ by sending X_{n-1} to itself and all the other generators to 0 yields the desired exact sequence. $\square$

3.8 Statement of results

Now we are ready to state the main theorem describing the $\boldsymbol{I}^\bullet$–cohomology and Chow–Witt ring of $B\mathrm{GL}_n$. For the $\boldsymbol{I}^\bullet$–cohomology, the result is very close to Čadek's computation of the integral cohomology of $BO(n)$ with twisted coefficients; see [26].

Theorem 3.24 *Let $n \geq 1$ be a natural number.*

(1) *The ring homomorphism*

$$\begin{aligned}\theta_n : \mathcal{R}_n &\to \bigoplus_q H^q(B\mathrm{GL}_n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_n^\vee)),\\ P_i &\mapsto p_{2i},\\ X_n &\mapsto e_n,\\ B_J &\mapsto \beta_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{for } J = \{j_1, \dots, j_l\},\\ T_J &\mapsto \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{for } J = \{j_1, \dots, j_l\},\\ T_\varnothing &\mapsto \beta_{\det \gamma_n^\vee}(1),\end{aligned}$$

induces a ring isomorphism $\bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \xrightarrow{\cong} H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$.

(2) *For any line bundle $\mathcal{L}$ on $B\mathrm{GL}_n$, the reduction morphism*

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet(\mathcal{L})) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$$

induced from the projection $\boldsymbol{I}^n(\mathcal{L}) \to \boldsymbol{K}_n^{\mathrm{M}}/2$ is explicitly given by mapping

$$p_{2i} \mapsto \bar{c}_{2i}^2, \quad \beta_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \mapsto \mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}), \quad e_n \mapsto \bar{c}_n.$$

(3) *Any class x in the ideal of $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$ generated by β_J and τ_J is trivial if and only if its reduction $\rho(x) \in \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ is trivial.*

Remark 3.25 This theorem is one of the key components of Theorem 1.1. More concretely, the theorem together with Definition 3.15 provides a generators-and-relations description of the total $\boldsymbol{I}$–cohomology ring

$$\bigoplus_q H^q(B\mathrm{GL}_n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_n^\vee))$$

as a $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebra over the Witt ring $W(F)$. The generators are the Pontryagin classes $p_1,\dots,p_{[(n-1)/2]}$, the Euler class e_n and the (nontwisted and twisted) Bockstein classes

$$\beta_J=\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})\quad\text{and}\quad \tau_J=\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$$

for an index set $J=\{j_1,\dots,j_l\}$ of natural numbers $0<j_1<j_2<\cdots<j_l\le\left[\frac{1}{2}(n-1)\right]$, plus the additional $\beta_{\det\gamma_n^\vee}(1)$. Spelling out Definition 3.15, the relations between these classes are as follows:

(1) The Bockstein classes $\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$, $\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ and $\beta_{\det\gamma_n^\vee}(1)$ are $I(F)$–torsion.

(2) For $n=2k+1$, the Euler class is a twisted Bockstein class: $e_n=\beta_{\det\gamma_n^\vee}(\bar{c}_{2k})$.

(3) For any two index sets J and J', where J' can be empty, multiplication of Bockstein classes is explicitly given by

$$\beta_J\cdot\beta_{J'}=\sum_{k\in J}\beta_{\{k\}}\cdot p_{(J\setminus\{k\})\cap J'}\cdot\beta_{\Delta(J\setminus\{k\},J')},$$
$$\beta_J\cdot\tau_{J'}=\sum_{k\in J}\beta_{\{k\}}\cdot p_{(J\setminus\{k\})\cap J'}\cdot\tau_{\Delta(J\setminus\{k\},J')},$$
$$\tau_J\cdot\beta_{J'}=\beta_J\cdot\tau_{J'}+\tau_\varnothing\cdot p_{J\cap J'}\cdot\beta_{\Delta(J,J')},$$
$$\tau_J\cdot\tau_{J'}=\beta_J\cdot\beta_{J'}+\tau_\varnothing\cdot p_{J\cap J'}\cdot\tau_{\Delta(J,J')}.$$

In the above, $p_A=\prod_{i=1}^{l}p_{a_i}$ for an index set $A=\{a_1,\dots,a_l\}$ with the special case $p_\varnothing=1$.

Compared to the known integral singular cohomology ring of $BO(n)$, the Bockstein classes generate what is the 2–torsion part of the integral singular cohomology. This part of the $\boldsymbol{I}$–cohomology ring is always the same, independent of the base field F. The part generated by the Pontryagin classes (plus the Euler class for even n) corresponds to the torsion-free part of integral singular cohomology. In $\boldsymbol{I}$–cohomology, it is a free $W(F)$–module. It depends on the base field via $W(F)$, but its rank as a $W(F)$–module is again independent of the base field.

The proof will be given in Section 4. For now we draw some consequences concerning the structure of the Chow–Witt ring of $B\mathrm{GL}_n$.

Proposition 3.26 (1) *The kernel of the composition*

$$\partial_{\mathcal{O}}:\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\mathcal{O}}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1})$$

is the subring

$$\ker\partial_{\mathcal{O}}=\mathbb{Z}\left[\{c_i^2\}_{1\le i\le n},\{c_1c_{2k}+c_{2k+1}\}_{1\le k\le[(n-1)/2]},c_1c_n,(2)\right]\subseteq\mathbb{Z}[c_1,\dots,c_n]\cong\mathrm{CH}^\bullet(B\mathrm{GL}_n).$$

(2) *The kernel of the composition*

$$\partial_{\det\gamma_n^\vee}:\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\det\gamma_n^\vee}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1}(\det\gamma_n^\vee))$$

is the $\ker\partial_{\mathcal{O}}$*–submodule of* $\mathbb{Z}[c_1,\dots,c_n]\cong\mathrm{CH}^\bullet(B\mathrm{GL}_n)$,

$$\ker\partial_{\det\gamma_n^\vee}=\langle\{c_{2k+1}\}_{1\le k\le[(n-1)/2]},c_n,(2)\rangle_{\ker\partial_{\mathcal{O}}}\subseteq\mathrm{CH}^\bullet(B\mathrm{GL}_n).$$

Proof By (3) of Theorem 3.24 and Proposition 2.2, the kernel of $\beta_{\mathscr{L}}$ equals the kernel of $\mathrm{Sq}^2_{\mathscr{L}}$ and the latter is determined by the Wu formula; see Corollary 3.13. Then statements (1) and (2) for the Chow ring follow directly from the corresponding statement for the mod 2 Chow ring in Corollary 3.13, adding as additional generators the elements of the ideal (2). □

The following theorem now establishes the first item of Theorem 1.1. The structure of the $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebra on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}$ is discussed before the statement of Theorem 1.1.

Theorem 3.27 *There is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras*

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathcal{O}\oplus\det\gamma_n^\vee) \arrow[r] \arrow[d] & \ker\partial_{\mathcal{O}}\oplus\ker\partial_{\det\gamma_n^\vee} \arrow[d, "\mathrm{mod}\ 2"] \\
H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet\oplus\boldsymbol{I}^\bullet(\det\gamma_n^\vee)) \arrow[r, "\rho"'] & \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}
\end{tikzcd}$$

The right vertical morphism is the natural reduction mod 2 restricted to the kernels of the two boundary maps, and the lower horizontal morphism is the reduction morphism described in Theorem 3.24. The Chow–Witt-theoretic Euler class satisfies $e_n^{\widetilde{\mathrm{CH}}} = (e_n^{\boldsymbol{I}}, c_n)$ with $c_n \in \ker\partial_{\det\gamma_n^\vee}$.[2] *For the Chow–Witt-theoretic Pontryagin classes,*

$$p_i^{\widetilde{\mathrm{CH}}} = \Bigl(p_i^{\boldsymbol{I}}, (-1)^i c_i^2 + 2\sum_{j=\max\{0,2i-n\}}^{i-1} (-1)^j c_j c_{2i-j}\Bigr),$$

where the odd Pontryagin classes[3] *in $\boldsymbol{I}$–cohomology are $I(F)$–torsion and satisfy*

$$p_{2i+1} = (\beta_{\mathcal{O}}(\bar{c}_{2i}))^2 + p_{2i}\beta_{\mathcal{O}}(\bar{c}_1) = \beta_{\mathcal{O}}(\bar{c}_{2i}\bar{c}_{2i+1}).$$

The top Pontryagin class $p_n \in \widetilde{\mathrm{CH}}^{2n}(B\mathrm{Sp}_{2n})$ maps to $e_n^2 \in \widetilde{\mathrm{CH}}^{2n}(B\mathrm{GL}_n, \mathcal{O})$.

Proof The statement about the cartesian square follows directly from [16, Proposition 2.11]. The claims about the reduction from the Chow–Witt ring to $\boldsymbol{I}$–cohomology follows from the definition of the characteristic classes. The statement about e_n and c_n follows from Proposition 3.3. The statement about the p_i has been proved in [16, Theorem 6.10]. For the description of odd Pontryagin classes in terms of Bockstein classes, we first note that the injectivity of restriction to $B\mathrm{SL}_n$ as discussed at the end of the proof of Proposition 4.5 combined with [16, Theorem 6.10 or Proposition 8.16] implies that odd Pontryagin classes are $I(F)$–torsion. The alternative description follows using Theorem 3.24(3) by showing the equality after reduction in $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$, which is Proposition 4.10; see also [16, Remark 8.17]. The statement about the top Pontryagin class is proved in Proposition 4.5, or [16, Proposition 7.9]. □

[2] Here and in the formula for Pontryagin classes, upper indices have been added to clarify the relevant cohomology theory: Chow–Witt on the left-hand side and $\boldsymbol{I}$–cohomology on the right-hand side.

[3] For a remark on odd Pontryagin classes, see Remark 3.5.

Proposition 3.28 *The restriction along the Whitney sum map $B(\mathrm{GL}_n \times \mathrm{GL}_m) \to B\mathrm{GL}_{n+m}$ maps the Pontryagin classes as*

$$p_i \mapsto \sum_{j=\max\{0,i-m\}}^{\min\{i,n\}} p_j \otimes p_{i-j},$$

where the sum is over the indices j such that p_j and p_{i-j} are Pontryagin classes for GL_n and GL_m, respectively.

Proof The Whitney sum formula follows directly from the Whitney sum formula for the Pontryagin classes of symplectic bundles and the compatibility of Whitney sum and symplectification; see [16]. □

Remark 3.29 The Whitney sum formula above is exactly the classical one from [8]. It is easier to state simply by our conventions — see [16, Remark 5.7] — concerning indexing of the Pontryagin classes.

Example 3.30 To clarify the relation between the cohomology of $B\mathrm{GL}_n$ and $B\mathrm{SL}_n$ — see [16, Example 6.12] — we conclude this section with a detailed description of the cartesian square for $B\mathrm{GL}_3$ with both dualities.

For the trivial duality, we have the cartesian square

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_3, \mathcal{O}) \arrow[r] \arrow[d] & \mathbb{Z}[c_i^2, c_1c_2 + c_3, c_1c_3, (2)] \arrow[d] \\
W(F)[p_2, \beta_{\mathcal{O}}(\bar{c}_1), \beta_{\mathcal{O}}(\bar{c}_2), \beta_{\mathcal{O}}(\bar{c}_1\bar{c}_2)]/\mathcal{I}_{3,\mathcal{O}} \arrow[r, "\rho"'] & \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \bar{c}_2, \bar{c}_3]
\end{tikzcd}$$

In the upper-right corner, we have the subring of the Chow ring (in particular containing $1 \in \mathrm{CH}^0(B\mathrm{GL}_3)$) generated by everything 2–divisible, squares of Chern classes and the classes $c_1c_2 + c_3$ and c_1c_3.

The structure of the $W(F)$–algebra in the lower left can be made more explicit using the $\mathrm{Im}\,\beta$–$\boldsymbol{W}$–decomposition: the $\boldsymbol{W}$–cohomology is

$$H^\bullet(B\mathrm{GL}_3, \boldsymbol{W}) \cong W(F)[p_2].$$

This is a part that depends on the underlying field via $W(F)$, but the presentation as $W(F)$–algebra is independent of the field — it is always a polynomial $W(F)$–algebra in p_2.

The other summand of the $\boldsymbol{I}$–cohomology in the lower-left corner is the image of $\beta_{\mathcal{O}}$, this part is independent of the base, it is the same as the 2–torsion in the integral singular cohomology of $BO(3)$. As generators, we have $\beta_{\mathcal{O}}(\bar{c}_2)$, the odd Pontryagin class $p_1 = \beta_{\mathcal{O}}(\bar{c}_1) = \beta_{\det\gamma_n^\vee}(1)^2$ and the class

$$\beta_{\mathcal{O}}(\bar{c}_3) = \beta_{\mathcal{O}}(\bar{c}_1\bar{c}_2) = \beta_{\det\gamma_n^\vee}(1)\beta_{\det\gamma_n^\vee}(\bar{c}_2).$$

Note that the class $\beta_{\mathcal{O}}(\bar{c}_2)$ is the generator β_J for $J = \{1\}$ and is actually the only generator of this form in this case. All the other classes written above are products of twisted Bockstein classes, as indicated.[4]

[4]See also the discussion of the relation between the presentations of Brown and Čadek in Remark 3.18.

Using that ρ is injective on the image of β, one can use an argument as in the proof of Corollary 3.14 to show that any torsion class is contained in the subring generated by these classes together with p_2. For example, the third Pontryagin class can be expressed as

$$p_3 = \beta_{\mathbb{O}}(\bar{c}_2\bar{c}_3) = \beta_{\mathbb{O}}(\bar{c}_2)^2 + p_2\beta_{\mathbb{O}}(\bar{c}_1).$$

The reductions of the Pontryagin classes are the squares of Chern classes, and the reductions of the other two classes are

$$\rho(\beta_{\mathbb{O}}(\bar{c}_2)) = \bar{c}_1\bar{c}_2 + \bar{c}_3, \quad \rho(\beta_{\mathbb{O}}(\bar{c}_3)) = \bar{c}_1\bar{c}_3.$$

In particular, we recover exactly the generators of $\ker \beta_{\mathbb{O}} \subseteq \mathrm{Ch}^\bullet(B\mathrm{GL}_3)$. As an example, the class $(\beta_{\mathbb{O}}(\bar{c}_2), c_1c_2 + c_3)$ is then a class in the Chow–Witt ring, because both classes have the same mod 2 reductions.

For the nontrivial duality, we have the square

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_3, \det\gamma_n^\vee) \arrow[r] \arrow[d] & \langle c_1, c_3, (2)\rangle_{\ker\partial_{\mathbb{O}}} \arrow[d] \\
\langle \beta_{\det\gamma_n^\vee}(1), \beta_{\det\gamma_n^\vee}(\bar{c}_2)\rangle \arrow[r] & \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \bar{c}_2, \bar{c}_3]
\end{tikzcd}$$

Here the upper-right corner is the sub–$\ker_{\mathbb{O}}$–module of $\mathrm{CH}^\bullet(B\mathrm{GL}_3)$ with the indicated generators. For the lower-left corner, the twisted $\boldsymbol{W}$–cohomology is trivial, since the $\boldsymbol{W}$–cohomological Euler class for odd-rank vector bundles is trivial. So the $\boldsymbol{I}$–cohomology is $I(F)$–torsion. As a module over $\bigoplus_q H^q(B\mathrm{GL}_3, \boldsymbol{I}^q)$ — the lower-left corner of the upper diagram for the nontwisted case — it is generated by $\beta_{\det\gamma_n^\vee}(1)$ and the Euler class $e_3 = \beta_{\det\gamma_n^\vee}(\bar{c}_2)$, which in the notation of Remark 3.25 are $\tau_\varnothing$ and $\tau_{\{1\}}$, respectively. The remaining torsion relations, in particular describing further multiplication rules, are not completely spelled out for typesetting reasons.

4 The Chow–Witt ring of $B\mathrm{GL}_n$: proofs

The main goal of this section is to prove Theorem 3.24 which is a Chow–Witt analogue of Čadek's description of integral cohomology of $BO(n)$ with local coefficients. The arguments are based on the decomposition into $\boldsymbol{W}$–cohomology and the image of β.

4.1 Projective spaces

As a first step we need to recall the computations of the $\boldsymbol{I}^\bullet$–cohomology and Chow–Witt rings of projective spaces $\mathbb{P}^n$ from [14]. Since $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, there are only two possible dualities to consider, given by the line bundles $\mathbb{O}_{\mathbb{P}^n}$ and $\mathbb{O}_{\mathbb{P}^n}(1)$.

It is a most classical computation that $\mathrm{Ch}^\bullet(\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1]/(\bar{c}_1^{n+1})$. The Steenrod squares are given by $\mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_1) = \bar{c}_1^2$ and $\mathrm{Sq}^2_{\mathbb{O}(1)}(\bar{c}_1) = 0$. In particular, $\ker \mathrm{Sq}^2_{\mathbb{O}} = \mathbb{Z}/2\mathbb{Z}[\bar{c}_1^2]$, and the kernel of $\mathrm{Sq}^2_{\mathbb{O}(1)}$ is the submodule of $\mathrm{Ch}^\bullet(\mathbb{P}^n)$ generated by odd powers of $\bar{c}_1$.

The following is a direct reformulation of the computations in [14, Section 11].

Proposition 4.1 (1) *If n is odd, then*

$$\bigoplus_q H^q(\mathbb{P}^n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1, R]/(I(F)\cdot e_1, e_1^{n+1}, e_1 R, R^2).$$

Moreover, $e_1 = \beta_{\mathcal{O}(1)}(1)$ and $R \in H^n(\mathbb{P}^n, \boldsymbol{I}^n)$ is the fundamental class of $\mathbb{P}^n$ (which is orientable in this case). The image of R under the reduction morphism ρ is $\bar{c}_n \in \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

(2) *If n is even, then*

$$\bigoplus_q H^q(\mathbb{P}^n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1, e_n^\perp]/(I(F)\cdot e_1, e_1^{n+1}, e_1 e_n^\perp, (e_n^\perp)^2).$$

Again, $e_1 = \beta_{\mathcal{O}(1)}(1)$, and the class $e_n^\perp \in H^n(\mathbb{P}^n, \boldsymbol{I}^n(\det\gamma_1^\vee))$ is the Euler class of the rank n hyperplane bundle on $\mathbb{P}^n \cong (\mathbb{P}^n)^\vee$.

Proof Note that [14] only establishes the additive structure statements, not quite the full presentation of the ring structure as formulated. Nevertheless, the statements about the ring structure follow from this: since we already know some characteristic classes of vector bundles, we obtain a ring homomorphism from our claimed presentation to the cohomology ring of $\mathbb{P}^n$. Additively, we also know that the Euler class reduces to $\bar{c}_1$; in particular the nontriviality of the powers of the Euler class is then immediate and this already deals with all the torsion classes. The statement for the nontorsion classes R and $e_n^\perp$ follows directly, since these cannot have nontrivial intersections with anything else for dimension reasons. □

Remark 4.2 The classical presentations of the integral cohomology of real projective spaces are recovered exactly for $F = \mathbb{R}$. The algebraic Euler class maps to the topological Euler class under real realization, so the real realization morphism also induces an isomorphism from $\boldsymbol{I}^\bullet$–cohomology to the integral cohomology of real projective space; see [17].

The following is the Chow–Witt version of [26, Lemma 1]. This is a consequence of the above restatement of the computations in [14, Section 11], noting that $B\mathrm{GL}_1 \cong \mathbb{P}^\infty$.

Proposition 4.3 *The Euler class $e_1 \in H^1(\mathbb{P}^\infty, \boldsymbol{I}^1(\det\gamma_1^\vee))$ is nontrivial. Moreover, $e_1 = \beta_{\det\gamma_1^\vee}(1)$. There is an isomorphism*

$$\bigoplus_q H^q(\mathbb{P}^\infty, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1]/(I(F)\cdot e_1).$$

The reduction morphism $H^1(\mathbb{P}^\infty, \boldsymbol{I}^1(\det\gamma_1^\vee)) \to \mathrm{Ch}^1(\mathbb{P}^\infty)$ maps e_1 to $\mathrm{Sq}^2_{\det\gamma_1^\vee}(1) = \bar{c}_1$. In particular, Theorem 3.24 is true for $n = 1$.

Remark 4.4 Alternatively, we can formulate the description of the $\boldsymbol{I}$–cohomology of projective space in terms of the decomposition into $\boldsymbol{W}$–cohomology and the image of β. The $\boldsymbol{W}$–cohomology of $\mathbb{P}^n$

is an exterior $W(F)$–algebra on one generator, which is $e_n^{\perp} \in H^n\big(\mathbb{P}^n, \boldsymbol{W}(\mathcal{O}(1))\big)$ for n even and $R = [\mathrm{pt}] \in H^n(\mathbb{P}^n, \boldsymbol{W})$ for n odd. The image of $\beta_{\mathcal{L}}$ is identified with the image of $\mathrm{Sq}^2_{\mathcal{L}}$ and consists of the appropriate powers of e_1. Multiplication with torsion classes can be computed after reduction in $\mathrm{Ch}^\bullet(\mathbb{P}^n)$.

4.2 Computation of $\boldsymbol{W}$–cohomology

The next step is the computation of the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$.

Proposition 4.5 *The $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is given by*

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{W} \oplus \boldsymbol{W}(\det \gamma_n^\vee)) \cong \begin{cases} W(F)[p_2, p_4, \dots, p_{n-2}, e_n] & \text{if } n \equiv 0 \bmod 2, \\ W(F)[p_2, p_4, \dots, p_{n-1}] & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

The morphisms $H^\bullet(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L})) \to H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\mathcal{L}))$, induced by the stabilization morphism $B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$, are compatible with Pontryagin classes. The restriction along $B\mathrm{GL}_{2n+1} \to B\mathrm{GL}_{2n}$ maps p_{2n} to e_{2n}^2.

Proof We note that the compatibility of the Pontryagin classes with stabilization follows from their definition; see [16, Proposition 5.8].

The result is proved by induction. The base case for the induction is given by $B\mathrm{GL}_1 \cong \mathbb{P}^\infty$. In this case, the claim is that

$$H^q(\mathbb{P}^\infty, \boldsymbol{W}(\mathcal{L})) \cong \begin{cases} W(F) & \text{if } q = 0 \text{ and } \mathcal{L} = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

This follows from Fasel's computations; see Proposition 4.3.

For the inductive step, we use the localization sequence of Proposition 3.1,

$$\cdots \to H^{q-n}(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L} \otimes \det \gamma_n)) \xrightarrow{e_n} H^q(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L})) \\ \xrightarrow{\iota^*} H^q(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\iota^*\mathcal{L})) \xrightarrow{\partial} H^{q-n+1}(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L} \otimes \det \gamma_n)) \to \cdots.$$

If n is even, then by the induction hypothesis $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\mathcal{L}))$ is a polynomial $W(F)$–algebra generated by the Pontryagin classes $p_2, \dots, p_{n-2}$. Since the stabilization morphism ι^* is compatible with the Pontryagin classes, it is surjective, hence $\partial = 0$. Thus, e_n is injective. Induction on the cohomological degree proves the claim that e_n is a new polynomial generator; alternatively, we can use the splitting principle of [16, Proposition 7.8] to show independence of e_n from the Pontryagin classes.

If n is odd, we know that $e_n = 0$ in $\boldsymbol{W}$–cohomology, since by Proposition 3.10 it is in the image of β. Therefore, the boundary map

$$\partial\colon H^{n-1}(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\det \gamma_{n-1}^\vee)) \to H^0(B\mathrm{GL}_n, \boldsymbol{W})$$

is surjective. The target is a cyclic $W(F)$–module generated by 1, and by the inductive assumption the image is a cyclic $W(F)$–module generated by ∂e_{n-1}. In particular, $\partial e_{n-1} = 1$, up to a unit in $W(F)$. By the derivation property for ∂, the boundary map is trivial on $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W})$ and injective on $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\det \gamma_{n-1}^\vee))$. This implies that the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is a polynomial $W(F)$–algebra generated by the Pontryagin classes $p_2, \dots, p_{n-1}$.

Finally, to prove the claim concerning restriction of the top Pontryagin class, consider the morphism

$$o^*\colon H^\bullet(B\mathrm{GL}_{2n(+1)}, \boldsymbol{W}(\det \gamma_n^\vee)) \to H^\bullet(B\mathrm{SL}_{2n(+1)}, \boldsymbol{W})$$

given by pullback to the orientation cover. This maps the Pontryagin classes and Euler class to their respective counterparts for $B\mathrm{SL}_{2n(+1)}$. From the present computation of the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_{2n(+1)}$ and the computations in [16, Theorem 1.3] for $B\mathrm{Sl}_{2n(+1)}$ we conclude that o^* is injective. Moreover, $p_{2n} - e_{2n}^2$ is mapped to 0 by [16, Theorem 1.3] which proves the claim. □

Remark 4.6 For the case $B\mathrm{SL}_n$, the analogous formulas can be obtained from the general machinery for η–inverted cohomology theories in [1].

4.3 Relations in the mod 2 Chow ring

In this subsection we show that the ideal $\mathcal{I}_n$ of relations between characteristic classes is annihilated by the composition

$$\mathcal{R}_n \xrightarrow{\theta_n} H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee)) \xrightarrow{\rho} \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}.$$

Lemma 4.7 *Assume n is odd. With the above notation,*

$$\rho(e_n) = \rho \circ \beta_{\det \gamma_n^\vee}(\bar{c}_{n-1}) = \bar{c}_n.$$

Proof This follows from [14, Proposition 10.3, Remark 10.5], the identification $\mathrm{Sq}^2_{\det \gamma_n^\vee} = \rho \circ \beta_{\det \gamma_n^\vee}$ from Proposition 2.2, and the identification of Stiefel–Whitney classes with reductions of Chern classes in Proposition 3.6. □

Proposition 4.8 *For two index sets J and J', the elements*

$$\begin{gathered}
B_J \cdot B_{J'} - \sum_{k \in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot B_{\Delta(J\setminus\{k\}, J')}, \\
B_J \cdot T_{J'} - \sum_{k \in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot T_{\Delta(J\setminus\{k\}, J')}, \\
T_J \cdot B_{J'} - B_J \cdot T_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot B_{\Delta(J,J')}, \\
T_J \cdot T_{J'} - B_J \cdot B_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot T_{\Delta(J,J')}
\end{gathered}$$

have trivial images under the composition $\rho \circ \theta_n \colon \mathcal{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

Proof The first relation can be established as in [16, Proposition 7.13]. Note that $\rho\circ\theta_n$ maps the elements B_J and T_J to the elements $\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_k})$ and $\mathrm{Sq}^2_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_k})$, respectively; see Proposition 2.2. The proofs of the other relations can be done by the same manipulations as detailed in [26, Lemma 4]. □

Corollary 4.9 *The composition $\rho \circ \theta_n \colon \mathcal{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}$ factors through the quotient $\mathcal{R}_n/\mathcal{I}_n$.*

Proof This follows directly from Lemma 4.7 and Proposition 4.8. □

Proposition 4.10 *Let $2i+1 \leq n$ be an odd natural number.*[5] *Then*

$$\rho(p_{2i+1}) = (\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}))^2 + \rho(p_{2i})\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_1) = \mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}\bar{c}_{2i+1}).$$

Proof The claim follows from the computations

$$\begin{aligned}\rho(p_{2i+1})(\mathscr{E}_n) = \bar{c}_{4i+2}(\mathscr{E}_n \oplus \overline{\mathscr{E}}_n) = \bar{c}_{4i+2}(\mathscr{E}_n^{\oplus 2}) = \bar{c}_{2i+1}(\mathscr{E}_n)^2,\\ (\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}))^2 + \rho(p_{2i})\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_1) = (\bar{c}_{2i+1} + \bar{c}_1\bar{c}_{2i})^2 + \bar{c}_{2i}^2\bar{c}_1^2 = \bar{c}_{2i+1}^2,\\ \mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}\bar{c}_{2i+1}) = \bar{c}_{2i}\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i+1}) + \bar{c}_{2i+1}\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}) = \bar{c}_{2i+1}^2;\end{aligned}$$

see [8, page 288]. □

4.4 Proof of Theorem 3.24

We first note that Proposition 4.5, in combination with Lemma 2.4, a priori implies a splitting of $\boldsymbol{I}$–cohomology into $\boldsymbol{W}$–cohomology and the image of β, and this is the key tool in the proof. This already establishes part (3) of the theorem.

Part (2) of the theorem follows from Proposition 2.2 for the Bockstein classes and [16, Corollary 7.11] for the Pontryagin and Euler classes.

To prove part (1) of the theorem, consider the ring homomorphism

$$\theta_n : \mathscr{R}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L}))$$

defined in Theorem 3.24. The first step is to show that θ_n factors through the quotient $\mathscr{R}_n/\mathscr{I}_n$, ie that $\theta_n(\mathscr{I}_n) = 0$. We consider the relations generating $\mathscr{I}_n$ given in Definition 3.15. Relations of type (1) hold by Lemma 3.9, relations of type (2) by Proposition 3.10. Relations of type (3) are annihilated by the composition $\rho \circ \theta_n : \mathscr{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ by Proposition 4.8. By Proposition 4.5, the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is free, hence Lemma 2.4 implies that the reduction $\rho\colon H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^q(B\mathrm{GL}_n)$ is injective on the image of $\beta_{\mathscr{L}}$. Since all relations of type (3) are in the image of $\beta_{\mathscr{L}}$, those relations have trivial image under θ_n. Therefore, we get a well-defined ring homomorphism

$$\bar{\theta}_n : \mathscr{R}_n/\mathscr{I}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})).$$

We now prove that the ring homomorphism $\bar{\theta}_n$ is surjective. First, we note that $\bar{\theta}_n$ surjects onto $\operatorname{Im}\beta_{\mathscr{L}}$ if and only if the composition

$$\rho \circ \bar{\theta}_n : \mathscr{R}_n/\mathscr{I}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$$

surjects onto the image of $\mathrm{Sq}^2_{\mathscr{L}}\colon \mathrm{Ch}^{\bullet-1}(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. By Corollary 3.14, we know that the image of $\mathrm{Sq}^2_{\mathscr{L}}$ is contained in the subring generated by the classes $\mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$, $\mathrm{Sq}^2_{\det\gamma_n}(1)$, $\bar{c}_{2i}^2$ and $\bar{c}_n$. By part (2) of the theorem, all these classes are reductions of classes in the image of θ_n, proving

[5] Recall that the definition of Pontryagin classes includes a definition of odd Pontryagin classes; see Remark 3.5.

that $\bar{\theta}_n$ surjects onto the image of β. It then suffices to show that the composition

$$\mathcal{R}_n/\mathcal{I}_n \xrightarrow{\bar{\theta}_n} \bigoplus_{q,\mathcal{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathcal{L})) \to \bigoplus_{q,\mathcal{L}} H^q(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L}))$$

is surjective, where the second map is the projection onto $\boldsymbol{W}$–cohomology. But this follows from Proposition 4.5, finishing the surjectivity proof.

Finally, we prove that $\bar{\theta}_n$ is injective. First, we consider the $W(F)$–torsion-free part of $\mathcal{R}_n/\mathcal{I}_n$ which is generated, as a commutative graded $W(F)$–algebra, by the P_i, and X_{2n} if applicable. The restriction of $\bar{\theta}_n$ to that subalgebra is injective by Proposition 4.5. The injectivity on the torsion part, ie the ideal generated by the classes B_J, T_J for $J = \{j_1, \dots, j_l\}$ and $T_\varnothing$ can be checked after composition with ρ, by the decomposition of Lemma 2.4 (and Proposition 4.5) and the resulting fact that ρ is injective on the image of β. The direct translation (replacing w_i by $\bar{c}_i$ and Sq^1 by Sq^2) of the argument on page 285 of [26] takes care of that; see also [16, Proposition 8.15].

5 Chow–Witt rings of finite Grassmannians: statement of results

In the following two sections, we compute the Chow–Witt rings of the finite Grassmannians $\mathrm{Gr}(k,n)$. The results are stated in the present section, and the proofs are deferred to the next section.

5.1 Generators from characteristic classes

The first step is to get enough classes in $\widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n),\mathcal{L})$. We realize the Grassmannian $\mathrm{Gr}(k,n)$ over the field F as the variety of k–dimensional F–subspaces of $V = F^n$. Recall that we have an exact sequence of vector bundles on $\mathrm{Gr}(k,n)$,

$$0 \to \mathcal{S}_k \to \mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n} \to \mathcal{Q}_{n-k} \to 0.$$

Here, $\mathcal{S}_k$ is the *tautological subbundle*, mapping a point $[W]$ corresponding to a k–dimensional subspace $W \subset V$ to W, and $\mathcal{Q}_{n-k}$ is the *tautological quotient bundle*, mapping a point $[W]$ to the quotient space V/W.

There is a vector bundle torsor $f\colon \mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k}) \to \mathrm{Gr}(k,n)$ over the Grassmannian. This is an $\mathbb{A}^1$–weak equivalence, and the above exact sequence of vector bundles splits over $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$. Consequently, we obtain an $\mathbb{A}^1$–fiber sequence

$$\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k}) \to B\mathrm{GL}_k \times B\mathrm{GL}_{n-k} \xrightarrow{\oplus} B\mathrm{GL}_n,$$

where the second map is the Whitney sum map and the first map classifies the pair $(f^*\mathcal{S}_k, f^*\mathcal{Q}_{n-k})$. We can also consider the map $c\colon \mathrm{Gr}(k,n) \to B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}$ obtained by composing a homotopy inverse of f with the inclusion of the homotopy fiber, and this map classifies the pair $(\mathcal{S}_k, \mathcal{Q}_{n-k})$.

Note that there are two possible dualities on $B\mathrm{GL}_k$, corresponding to the line bundles $\mathcal{O}$ and $\det \gamma_k^\vee$; and similarly there are two possible dualities on $B\mathrm{GL}_{n-k}$ corresponding to $\mathcal{O}$ and $\det \gamma_{n-k}^\vee$. Consequently, there are four possible dualities on $B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}$, given by the four possible exterior products of the above line bundles. For any choice of line bundles $\mathscr{L}_k$ and $\mathscr{L}_{n-k}$ on $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$, respectively, the classifying map c above induces homomorphisms of Chow–Witt groups

$$\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}, \mathscr{L}_k \boxtimes \mathscr{L}_{n-k}) \to \widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})).$$

Note that the bundle $c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})$ is trivial (modulo squares of line bundles) if and only if $\mathscr{L}_k$ and $\mathscr{L}_{n-k}$ are either both trivial or both nontrivial. This follows from the fact that the assignment $(\mathscr{L}_k, \mathscr{L}_{n-k}) \mapsto c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})$ can be computed by pulling back both line bundles to the Grassmannian and then taking the tensor product; hence it induces the addition

$$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \cong \mathrm{Ch}^1(B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}) \to \mathrm{Ch}^1(\mathrm{Gr}(k,n)) \cong \mathbb{Z}/2\mathbb{Z}.$$

The induced homomorphisms assemble into a ring homomorphism of the total Chow–Witt rings (to the extent that this makes sense; see the remarks on [5] in Section 2).

This means that the characteristic classes of the tautological bundles $\mathscr{S}_k$ and $\mathscr{Q}_{n-k}$ provide classes in the Chow–Witt ring of $\mathrm{Gr}(k,n)$. For the definition of these classes and relations satisfied by them; see Section 3 and in particular Theorem 3.24, or the main result Theorem 1.1. The characteristic classes for the tautological subbundle $\mathscr{S}_k$ are

(1) the Pontryagin classes $p_1, p_2, \dots, p_{k-1}$,
(2) the Euler class e_k,
(3) the (twisted) Bockstein classes $\beta_{\mathscr{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ and $\beta_{\det \gamma_k^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$, and
(4) the Chern classes c_i.

Similarly, for the tautological quotient bundle $\mathscr{Q}_{n-k}$, we have the same characteristic classes (with different index sets); these will be denoted by an additional superscript $(-)^\perp$.[6] This provides a number of canonical elements in $\widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), \mathscr{L})$. It turns out that in the cases where $\dim \mathrm{Gr}(k,n) = k(n-k)$ is even, these classes generate the Chow–Witt ring; in the case where the dimension is odd, there is essentially one additional class arising as lift of an Euler class.

Remark 5.1 We follow the convention of [16, Remark 5.7], including all Pontryagin classes without added signs or reindexing. While the odd Pontryagin classes are $I(F)$–torsion and can be expressed in terms of Bockstein classes, this convention makes the Whitney sum formula for Pontryagin classes easier to state; see Proposition 3.28 and the subsequent remark.

5.2 Chow rings of Grassmannians

Before giving the statement concerning the structure of the Chow–Witt rings, we discuss the Chow rings of the Grassmannians. This result is very well-known and can be found in the relevant books on intersection theory, such as [11].

[6] The notation is suggestive that $\mathscr{Q}_{n-k}$ is the complement of $\mathscr{S}_k$ in $\mathscr{O}^{\oplus n}$, after pulling back to $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$.

Remark 5.2 The key relation in the Chow ring (with integral or mod 2 coefficients) is the Whitney sum formula. For this, in the statements below, we will use the notation $c = \sum_{i=0}^{k} c_i$ and $c^\perp = \sum_{i=0}^{n-k} c_i^\perp$ for the total Chern classes of the tautological subbundles and quotient bundles $\mathcal{S}_k$ and $\mathcal{Q}_{n-k}$, respectively. Similarly, we will use the notation $\bar{c} = \sum_{i=0}^{k} \bar{c}_i$ and $\bar{c}^\perp = \sum_{i=0}^{n-k} \bar{c}_i^\perp$ for the total Stiefel–Whitney classes of the tautological subbundles and quotient bundles, respectively. The Whitney sum formula for the extension

$$0 \to \mathcal{S}_k \to \mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n} \to \mathcal{Q}_{n-k} \to 0$$

is then simply written as $c \cdot c^\perp = 1$.

Proposition 5.3 *Let F be an arbitrary field. With the notation from Remark 5.2,*

$$\mathrm{CH}^\bullet(\mathrm{Gr}(k,n)) \cong \mathbb{Z}[c_1, \ldots, c_k, c_1^\perp, \ldots, c_{n-k}^\perp]/(c \cdot c^\perp = 1).$$

Proposition 5.4 *Let F be an arbitrary field. With the notation from Remark 5.2,*

(1) *there is a canonical isomorphism*

$$\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)) \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \ldots, \bar{c}_k, \bar{c}_1^\perp, \ldots, \bar{c}_{n-k}^\perp]/(\bar{c} \cdot \bar{c}^\perp = 1);$$

(2) *the Steenrod square is given by*

$$\mathrm{Sq}^2_{\mathcal{O}} \colon \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)), \quad \bar{c}_j^{(\perp)} \mapsto \bar{c}_1^{(\perp)} \bar{c}_j^{(\perp)} + (j-1)\bar{c}_{j+1}^{(\perp)};$$

(3) *the twisted Steenrod square is given by*

$$\mathrm{Sq}^2_{\det \mathcal{S}_k^\vee} \colon \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)), \quad \bar{c}_j^{(\perp)} \mapsto (j-1)\bar{c}_{j+1}^{(\perp)}.$$

The following consequence of the description of the Chow ring given in Proposition 5.4 will be relevant later.

Proposition 5.5 *Let $1 \le k < n$ and consider the ring*

$$A = \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \ldots, \bar{c}_k, \bar{c}_1^\perp, \ldots, \bar{c}_{n-k}^\perp]/(\bar{c} \cdot \bar{c}^\perp = 1).$$

(1) *The kernel of multiplication by $\bar{c}_{n-k}^\perp$ is the ideal $\langle \bar{c}_k \rangle \subseteq A$.*

(2) *The cokernel of multiplication by $\bar{c}_{n-k}^\perp$ is*

$$A/\langle \bar{c}_{n-k}^\perp \rangle \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \ldots, \bar{c}_k, \bar{c}_1^\perp, \ldots, \bar{c}_{n-k-1}^\perp]/(\bar{c} \cdot \bar{c}^\perp = 1).$$

Proof Statement (2) about the cokernel being $A/\langle \bar{c}_{n-k}^\perp \rangle$ is clear. The explicit description of the algebra also follows directly.

For (1), clearly $\langle \bar{c}_k \rangle \subseteq \ker \bar{c}_{n-k}^\perp$ since $\bar{c}_k \bar{c}_{n-k}^\perp = 0$ follows from the Whitney sum relation. The reverse inclusion can be seen, for example, by a dimension count in the kernel-cokernel exact sequence for multiplication by $\bar{c}_{n-k}^\perp$. □

Remark 5.6 This is also the formula for the mod 2 cohomology of the real Grassmannians; see eg [21]. The notation for the classes $\bar{c}_i$ and $\bar{c}_i^{\perp}$ is due to the fact that these are the mod 2 reductions of the Chern classes from the Chow ring. In the real realization these would go exactly to the corresponding Stiefel–Whitney classes.

5.3 Statement of the main results

Now we are ready to state the main results describing the $\boldsymbol{I}^{\bullet}$–cohomology of the finite Grassmannians. The lengthy formulation boils down to: "the characteristic classes of the tautological bundle and its complement generate the cohomology (except for a new class R in degree $(n-1,0)$ when $k(n-k)$ is odd) and the only new relations come from the Whitney sum formula".

As before in Remark 5.2, we will denote by $p = 1 + p_1 + \cdots + p_k$ and $p^{\perp} = 1 + p_1^{\perp} + \cdots + p_{n-k}^{\perp}$ for the total Pontryagin classes of the tautological subbundles and quotient bundles $\mathcal{S}_k$ and $\mathcal{Q}_{n-k}$, respectively.

Theorem 5.7 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$. The cohomology ring $\bigoplus_q H^q(\mathrm{Gr}(k,n), \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \mathcal{S}_k^{\vee}))$ is isomorphic to the $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$–graded $W(F)$–algebra generated by*

- (G1) *the Pontryagin classes $p_1, p_2, \ldots, p_k$ of the tautological rank k subbundle and the Pontryagin classes $p_1^{\perp}, p_2^{\perp}, \ldots, p_{n-k}^{\perp}$ of the tautological rank $n-k$ quotient bundle, where the class $p_i^{(\perp)}$ has degree $(2i, 0)$;*
- (G2) *the Euler classes e_k and $e_{n-k}^{\perp}$, having degrees $(k,1)$ and $(n-k,1)$, respectively;*
- (G3) *for every set $J = \{j_1, \ldots, j_l\}$ of natural numbers $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(k-1)\right]$, possibly empty, there are Bockstein classes $\beta_J = \beta_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ and $\tau_J = \beta_{\det \mathcal{S}_k^{\vee}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ in degrees $(d,0)$ and $(d,1)$, respectively, where $d = 1 + 2\sum_{i=1}^{l} j_i$;*
- (G4) *for every set $J = \{j_1, \ldots, j_l\}$ of natural numbers $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-k-1)\right]$, possibly empty, there are Bockstein classes $\beta_J^{\perp} = \beta_{\mathcal{O}}(\bar{c}_{2j_1}^{\perp} \cdots \bar{c}_{2j_l}^{\perp})$ and $\tau_J^{\perp} = \beta_{\det \mathcal{S}_k^{\vee}}(\bar{c}_{2j_1}^{\perp} \cdots \bar{c}_{2j_l}^{\perp})$ in degrees $(d,0)$ and $(d,1)$, respectively, where $d = 1 + 2\sum_{i=1}^{l} j_i$;*
- (G5) *if $k(n-k)$ is odd, there is a class R in degree $(n-1,0)$*

subject to the relations

- (R1) *the classes p_i, e_k, β_J and τ_J satisfy the relations holding in the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_k$, and the classes $p_i^{\perp}$, $e_{n-k}^{\perp}$, $\beta_J^{\perp}$ and $\tau_J^{\perp}$ satisfy the relations in the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_{n-k}$ (see Theorem 3.24);*[7]
- (R2) *$p \cdot p^{\perp} = 1$, ie the product of the total Pontryagin classes is 1;*
- (R3) *$e_k \cdot e_{n-k}^{\perp} = 0$;*
- (R4) *$\beta_{\mathcal{O}}(\bar{c} \cdot \bar{c}^{\perp}) = 1$ and $\beta_{\det \mathcal{S}_k^{\vee}}(\bar{c} \cdot \bar{c}^{\perp}) = \tau_{\varnothing} = \tau_{\varnothing}^{\perp}$, ie applying the (twisted) Bockstein to the product of the total Stiefel–Whitney classes in $\mathrm{Ch}^{\bullet}$ is trivial;*
- (R5) *$R^2 = 0$, and the product of R with an $I(F)$–torsion class α is zero if and only if $\bar{c}_{k-1}\bar{c}_{n-k}^{\perp}\rho(\alpha) = 0$.*

[7] In particular, the classes $\beta_{\varnothing} = \beta_{\varnothing}^{\perp} = 0$.

Proposition 5.8 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \le k < n$. The reduction homomorphism*

$$\rho\colon \bigoplus_q H^q(\mathrm{Gr}(k,n), \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \mathscr{S}_k)) \to \mathrm{Ch}^q(\mathrm{Gr}(k,n))$$

is given by

$$p_{2i}^{(\perp)} \mapsto (\bar{c}_{2i}^{(\perp)})^2, \quad e_k \mapsto \bar{c}_k, \quad e_{n-k}^{\perp} \mapsto \bar{c}_{n-k}^{\perp},$$
$$\beta_{\mathscr{L}}(\bar{c}_{2j_1}^{(\perp)} \cdots \bar{c}_{2j_l}^{(\perp)}) \mapsto \mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}^{(\perp)} \cdots \bar{c}_{2j_l}^{(\perp)}), \quad R \mapsto \bar{c}_{k-1}\bar{c}_{n-k}^{\perp} = \bar{c}_k\bar{c}_{n-k-1}^{\perp}.$$

The reduction homomorphism $\rho_{\mathscr{L}}$ is injective on the image of the Bockstein map $\beta_{\mathscr{L}}$.

Remark 5.9 This presentation gives a complete description of the cup product. To multiply two torsion classes, we first rewrite the complementary classes $\bar{c}_{2i}^{\perp}$ in terms of polynomials in the ordinary classes $\bar{c}_{2j}$. (It follows directly from the well-known presentation of $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ that it is generated by the classes $\bar{c}_i$ and the complementary classes $\bar{c}_j^{\perp}$ can be expressed in terms of these.) The product of classes of the form $\beta_{\mathscr{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ is then given by the relation in $H^\bullet(B\mathrm{GL}_k)$. Note also that the product of R with an even Pontryagin class is independent of the Pontryagin classes. The product of R with a torsion class is a torsion class, and so it can be determined by computation in $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$. More detailed descriptions of how to work out products can be found in [28].

Theorem 5.10 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \le k < n$. Then there is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras*

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), \mathscr{O} \oplus \det \mathscr{S}_k^\vee) \arrow[r] \arrow[d] & \ker \partial_{\mathscr{O}} \oplus \ker \partial_{\det \mathscr{S}_k^\vee} \arrow[d, "\mathrm{mod}\ 2"] \\
H^\bullet(\mathrm{Gr}(k,n), \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \mathscr{S}_k^\vee)) \arrow[r, "\rho"'] & \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))^{\oplus 2}
\end{tikzcd}$$

Here $\det \mathscr{S}_k^\vee$ is the determinant of the dual of the tautological rank k subbundle on $\mathrm{Gr}(k,n)$, and

$$\partial_{\mathscr{L}}\colon \mathrm{CH}^\bullet(\mathrm{Gr}(k,n)) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)) \xrightarrow{\beta_{\mathscr{L}}} H^{\bullet+1}(\mathrm{Gr}(k,n), \boldsymbol{I}^{\bullet+1}(\mathscr{L}))$$

is the (twisted) integral Bockstein map.

The kernel of the integral Bockstein map $\partial_{\mathscr{L}}$ is the preimage under reduction mod 2 of the subalgebra of $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ generated by $(c_i{}^{(\perp)})^2$, c_k, $c_{n-k}^{\perp}$, and $c_kc_{n-k-1}^{\perp}$ together with the image of $\mathrm{Sq}^2_{\mathscr{L}}$.

Proof This follows from [16, Proposition 2.11] since the Chow ring of $\mathrm{Gr}(k,n)$ is 2–torsion-free; see Proposition 5.3. The description of $\boldsymbol{I}^\bullet$–cohomology is given in Theorem 5.7, and the description of the reduction morphism ρ is given in Proposition 5.8. The description of the kernel of the boundary map follows directly from the definition and the Bär sequence, ie that the kernel of $\beta_{\mathscr{L}}$ is exactly the image of the reduction map $\rho_{\mathscr{L}}$. □

Remark 5.11 We can determine the images of Euler classes and Pontryagin classes in Chow theory using Theorem 3.27.

5.4 Examples

The following are two examples describing the $\boldsymbol{I}^\bullet$–cohomology of small Grassmannians. For alternative descriptions of the $\boldsymbol{I}^\bullet$–cohomology, using even Young diagrams for the $\boldsymbol{W}$–part and checkerboard fillings for Young diagrams for the image of β; see [28].

Example 5.12 Let us work out the components of the cartesian square of Theorem 5.10 in the example case $\mathrm{Gr}(2,4)$.

First, the mod 2 Chow ring $\mathrm{Ch}^\bullet(\mathrm{Gr}(2,4))$ is generated by the Stiefel–Whitney classes $\bar{c}_1^{(\perp)}$ and $\bar{c}_2^{(\perp)}$ of the tautological bundles, and the relations from the Whitney formula for Stiefel–Whitney classes are

$$\bar{c}_1=\bar{c}_1^{\perp},\quad \bar{c}_2+\bar{c}_1^2+\bar{c}_2^{\perp}=0,\quad \bar{c}_1\bar{c}_2^{\perp}+\bar{c}_2\bar{c}_1^{\perp}=\bar{c}_1^3=0,\quad \bar{c}_2^2+\bar{c}_2\bar{c}_1^2=0.$$

From these relations, the usual well-known description of the mod 2 Chow ring follows. The description of the integral Chow ring $\mathrm{CH}^\bullet(\mathrm{Gr}(2,4))$ is completely analogous, just in terms of Chern classes.

With the relations between the Stiefel–Whitney classes, we can now compute the twisted and untwisted Bocksteins of Stiefel–Whitney classes. This provides information both for the kernel of $\partial_{\mathbb{O}}$ and $\partial_{\det\mathscr{S}_2^\vee}$ in $\mathrm{CH}^\bullet(\mathrm{Gr}(2,4))$, and information on the torsion classes in $\boldsymbol{I}^\bullet$–cohomology. The first relation of Stiefel–Whitney classes above implies $\beta_{\mathbb{O}}(\bar{c}_i)=\beta_{\mathbb{O}}(\bar{c}_i^{\perp})$ for $i=1,2$. By the Wu formula, $\mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_2)=\bar{c}_1\bar{c}_2$ and therefore $\beta_{\mathbb{O}}(\bar{c}_1\bar{c}_2)=0$. Since Bocksteins of squares are trivial by the derivation property, this means that the only nontrivial untwisted Bockstein classes are $\beta_{\mathbb{O}}(\bar{c}_1)=p_1$ and $\beta_{\mathbb{O}}(\bar{c}_2)$.

With twisted coefficients, we have the class $\beta_{\det\mathscr{S}_2^\vee}(1)$. The other twisted Bockstein classes $\beta_{\det\mathscr{S}_2^\vee}(\bar{c}_i)$ are trivial: we can check on reduction mod 2, the case $i=1$ follows directly from the Wu formula and the case $i=2$ follows from $\bar{c}_3=0$. The other potential 2–torsion classes are products of the form $\tau_\varnothing\beta_{\mathbb{O}}$. We check on mod 2 reduction that $\rho(\tau_\varnothing\beta_{\mathbb{O}}(\bar{c}_1))=\bar{c}_1^3=0$ but $\rho(\tau_\varnothing\beta_{\mathbb{O}}(\bar{c}_2))=\bar{c}_1^2\bar{c}_2=\bar{c}_2^2\neq 0$. So the torsion classes with twisted coefficients are $\beta_{\det\mathscr{S}_2^\vee}(1)$ in degree 1 and $\beta_{\det\mathscr{S}_2^\vee}(1)\beta_{\mathbb{O}}(\bar{c}_2)$ in degree 4.

As a consequence of the above, we can now also determine the kernel of the boundary maps on the Chow ring. The Chern classes $c_1^{(\perp)}$ and $c_2^{(\perp)}$ have nontrivial Bocksteins and hence do not lift to the Chow–Witt ring. As noted above, the classes c_1^2, c_1c_2 and c_2^2 have trivial Steenrod squares and therefore

$$\ker\partial_{\mathbb{O}}=\mathbb{Z}[c_1c_2,c_1^2,c_2^2,(2)].$$

On the other hand, $\ker\partial_{\det\mathscr{S}_2^\vee}$ is the submodule generated by (2), c_1 and c_2.

Finally, we can turn to the description of $\boldsymbol{I}^\bullet$–cohomology. We already determined the nontrivial Bockstein classes above. In addition to these, the remaining characteristic classes for $\boldsymbol{I}^\bullet$–cohomology are the Pontryagin classes $p_0=1\in H^0(\mathrm{Gr}(2,4),\boldsymbol{I}^0)$,

$$p_1^{(\perp)}=\beta_{\mathbb{O}}(\bar{c}_1^{(\perp)})\in H^2(\mathrm{Gr}(2,4),\boldsymbol{I}^2)\quad\text{and}\quad p_2^{(\perp)}\in H^4(\mathrm{Gr}(2,4),\boldsymbol{I}^4)$$

and the Euler classes $e_2,e_2^{\perp}\in H^2(\mathrm{Gr}(2,4),\boldsymbol{I}^2(\det\mathscr{E}_2^\vee))$.

The relations encoded in the Whitney sum formula $p \cdot p^{\perp} = 1$ are

$$\beta(\bar{c}_1) = \beta(\bar{c}_1^{\perp}), \quad p_2 + \beta(\bar{c}_1)^2 + p_2^{\perp} = 0, \quad p_2^2 = 0.$$

There is also a relation $2p_2\beta(\bar{c}_1) = 0$ which is trivially satisfied. From $\bar{c}_1^3 = 0$ above we find that $\mathrm{Sq}^2(\bar{c}_1)^2 = \bar{c}_1^4 = 0$ and therefore $\beta(\bar{c}_1)^2 = 0$. In particular, $p_2 = -p_2^{\perp}$. Consequently, the only nontorsion classes are $p_0 = 1$, p_2, e_2 and $e_2^{\perp}$, with $e_2^2 = (e_2^{\perp})^2 = p_2$.

A posteriori, we can now note that $\mathrm{Gr}(2,4)$ is an orientable variety, and a Poincaré duality pattern as in singular cohomology is satisfied for the $\boldsymbol{I}^{\bullet}$–cohomology.

Example 5.13 As another example, we work out the Steenrod squares for $\mathrm{Gr}(2,5)$. The relevant relations arising from the Whitney sum formula are

$$\bar{c}_1 = \bar{c}_1^{\perp}, \quad \bar{c}_2^{\perp} = \bar{c}_2 + \bar{c}_1^2, \quad \bar{c}_3^{\perp} = \bar{c}_1^3, \quad \bar{c}_2^2 + \bar{c}_1^2\bar{c}_2 + \bar{c}_1^4 = \bar{c}_2\bar{c}_1^3 = 0.$$

Now we go through the individual degrees in $\mathrm{Ch}^{\bullet}(\mathrm{Gr}(2,5))$:

(1) Degree 1 has $\bar{c}_1$ and $\mathrm{Sq}^2(\bar{c}_1) = \bar{c}_1^2$; this class doesn't lift to $H^1(\mathrm{Gr}(2,5), \boldsymbol{I}^1)$.

(2) Degree 2 has $\bar{c}_2$ with $\mathrm{Sq}^2(\bar{c}_2) = \bar{c}_1\bar{c}_2$ and $\bar{c}_1^2$ with trivial Sq^2. So the latter class lifts to a torsion class $\beta(\bar{c}_1) \in H^2(\mathrm{Gr}(2,5), \boldsymbol{I}^2)$.

(3) Degree 3 has $\bar{c}_1^3$ with $\mathrm{Sq}^2(\bar{c}_1^3) = \bar{c}_1^4$ and $\bar{c}_1\bar{c}_2$ with trivial Sq^2. So the latter class lifts to a torsion class $\beta(\bar{c}_2) \in H^3(\mathrm{Gr}(2,5), \boldsymbol{I}^3)$.

(4) Degree 4 has $\bar{c}_1^4$ and $\bar{c}_1^2\bar{c}_2$, both with trivial Sq^2. The class $\bar{c}_2^2 = \bar{c}_1^4 + \bar{c}_1^2\bar{c}_2$ lifts to the Pontryagin class, and $\bar{c}_1^4$ lifts to $\beta(\bar{c}_1^3) \in H^4(\mathrm{Gr}(2,5), \boldsymbol{I}^4)$.

(5) Degree 5 has $\bar{c}_1^5$ with $\mathrm{Sq}^2(\bar{c}_1^5) = \bar{c}_1^6$ and consequently this class doesn't lift to $\boldsymbol{I}^5$–cohomology.

(6) Degree 6 has $\bar{c}_1^6$ with trivial Sq^2, and this class lifts to the integral class $\beta(\bar{c}_1^5) \in H^6(\mathrm{Gr}(2,5), \boldsymbol{I}^6)$.

This recovers exactly the pattern for integral cohomology as discussed in eg [9]. In addition to that, we can use the formulas from Theorem 1.1 to determine the cup product of the torsion classes. Computations as above could be done to determine the cohomology with twisted coefficients as well.

6 Chow–Witt rings of finite Grassmannians: proofs

In this section, we will now prove the claims about the structure of $\boldsymbol{I}^{\bullet}$–cohomology of the Grassmannians discussed in Section 4. The overall argument is again to use the decomposition of $\boldsymbol{I}$–cohomology into the image of β and the $\boldsymbol{W}$–cohomology. We compute the $\boldsymbol{W}$–cohomology using a version of the inductive procedure used by Sadykov to compute rational cohomology of the real Grassmannians; see [23]. The base case $k = 1$ is the case of projective space which basically follows from [14]. The inductive step compares the cohomology of the Grassmannians $\mathrm{Gr}(k-1,n)$ and $\mathrm{Gr}(k,n)$ via a space which appears as sphere bundle of the tautological quotient and subbundle over $\mathrm{Gr}(k-1,n)$ and $\mathrm{Gr}(k,n)$, respectively. The image of β is detected on the mod 2 Chow ring, which determines the multiplication with torsion classes.

6.1 Localization sequence for inductive proof

As a preparation for the inductive computation of $\boldsymbol{W}$–cohomology, we set up the relevant localization sequences which allow to compare cohomology of different Grassmannians.

Proposition 6.1 (1) *There are isomorphisms*

$$\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n).$$

(2) *Denote by* $q\colon \mathscr{S}_k \to \mathrm{Gr}(k,n)$ *and* $p\colon \mathscr{Q}_{n-k+1} \to \mathrm{Gr}(k-1,n)$ *the respective tautological bundles, and by* $z_q\colon \mathrm{Gr}(k,n) \to \mathscr{S}_k$ *and* $z_p\colon \mathrm{Gr}(k-1,n) \to \mathscr{Q}_{n-k+1}$ *the respective zero-sections. Then there is an* $\mathbb{A}^1$*–weak equivalence of associated sphere bundles*

$$\mathscr{S}_k \setminus z_q(\mathrm{Gr}(k,n)) \simeq \mathscr{Q}_{n-k+1} \setminus z_p(\mathrm{Gr}(k-1,n)).$$

Moreover, under this weak equivalence, we have a correspondence of pullbacks of tautological vector bundles $q^*\mathscr{S}_k \cong p^*\mathscr{S}_{k-1} \oplus \mathbb{O}$.

Proof (1) For a vector space V of dimension n, the natural bijection between k–dimensional subspaces of V and $(n-k)$–dimensional subspaces of its dual $V^\vee$ induces a natural isomorphism

$$\mathrm{Gr}(k,V) \cong \mathrm{Gr}(n-k,V^\vee).$$

Choosing an isomorphism $V \cong V^\vee$ induces an isomorphism $\mathrm{Gr}(n-k,V^\vee) \cong \mathrm{Gr}(n-k,V)$. This provides the claimed isomorphisms. Note that these are not natural.

(2) Since we only want to establish an $\mathbb{A}^1$–weak equivalence, we can replace the Grassmannians $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(k-1,n)$ by the quotients $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$ and $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$, respectively. The pullback of the vector bundle $\mathscr{S}_k$ over $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$ is the associated bundle for the Stiefel variety $\mathrm{GL}_n/\mathrm{GL}_{n-k}$ viewed as GL_k–torsor and the natural GL_k–representation on $\mathbb{A}^k$. As in the setup of the localization sequence before Proposition 3.1, the complement of the zero section is then, up to a torsor under a unipotent group, $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$ because GL_{k-1} is the stabilizer of a line in $\mathbb{A}^k$.

A similar argument works for $\mathrm{Gr}(k-1,n)$. The vector bundle $\mathscr{Q}_{n-k+1}$ over $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$ is the associated bundle for the Stiefel variety $\mathrm{GL}_n/\mathrm{GL}_{k-1}$ and the natural representation of GL_{n-k+1} on $\mathbb{A}^{n-k+1}$. The complement of the zero section can then, up to a torsor under a unipotent group, be identified with $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$. This yields the required $\mathbb{A}^1$–weak equivalence. By an argument similar to the one in the setup for the localization sequence before Proposition 3.1, the pullback of the universal bundle $\mathscr{S}_k$ to $\mathscr{S}_k \setminus \mathrm{Gr}(k,n)$ will split off a direct summand, and the remainder is the tautological rank $k-1$ bundle on $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$. On the other hand, the pullback of the tautological rank $k-1$ bundle on $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$ to $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$ will still be the tautological rank $k-1$ bundle. □

Remark 6.2 In abuse of notation, we will denote the total spaces of both sphere bundles

$$\mathscr{Q}_{n-k+1} \setminus z_p(\mathrm{Gr}(k-1,n)) \quad \text{and} \quad \mathscr{S}_k \setminus z_q(\mathrm{Gr}(k,n))$$

by $\mathscr{S}(k,n)$. This is justified by Proposition 6.1 because they are $\mathbb{A}^1$–equivalent and hence they will have isomorphic $\boldsymbol{W}$–cohomology.

We obtain two localization sequences, relating the Grassmannians $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(k-1,n)$ to their associated sphere bundles. This, combined with the equivalence between the sphere bundles, is the relevant input for the induction step for the computation of the $\boldsymbol{W}$–cohomology of the Grassmannians.

Proposition 6.3 (1) *For any line bundle $\mathscr{L}$ on $\mathrm{Gr}(k-1,n)$, there is a long exact localization sequence*

$$\cdots \xrightarrow{e^{\perp}_{n-k+1}} H^{\bullet}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathscr{L})) \xrightarrow{p^*} H^{\bullet}(\mathscr{S}(k,n),\boldsymbol{W}(\mathscr{L}))$$
$$\xrightarrow{\partial} H^{\bullet-n+k}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathscr{L}\otimes \det \mathscr{Q}_{n-k+1})) \xrightarrow{e^{\perp}_{n-k+1}} H^{\bullet+1}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathscr{L})) \to \cdots .$$

(2) *For any line bundle $\mathscr{L}$ on $\mathrm{Gr}(k,n)$, there is a long exact localization sequence*

$$\cdots \xrightarrow{e_k} H^{\bullet}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \xrightarrow{q^*} H^{\bullet}(\mathscr{S}(k,n),\boldsymbol{W}(\mathscr{L}))$$
$$\xrightarrow{\partial} H^{\bullet-k+1}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}\otimes \det \mathscr{S}_k)) \xrightarrow{e_k} H^{\bullet+1}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \to \cdots .$$

Similar localization sequences are true for the other cohomology theories considered in this paper, but we will not need those.

6.2 Inductive computation of $\boldsymbol{W}$–cohomology

We now determine the structure of the total $\boldsymbol{W}$–cohomology ring of $\mathrm{Gr}(k,n)$. The argument completely follows the computation of rational cohomology of $\mathrm{Gr}_k(\mathbb{R}^n)$ in [23]. Some formulas for oriented Grassmannians related to the ones below can already be found in Ananyevskiy's computation for η–inverted theories; see [1].

Theorem 6.4 *Let F be a perfect field of characteristic $\neq 2$ and let $1 \le k < n$. The total $\boldsymbol{W}$–cohomology ring $\bigoplus_{i,\mathscr{L}} H^i(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}))$ has the following presentation, as a commutative $\mathbb{Z}\oplus \mathrm{Pic}(\mathrm{Gr}(k,n))/2$–graded $W(F)$–algebra:*

(1) *For k and n even, the total $\boldsymbol{W}$–cohomology ring $\bigoplus_{j,\mathscr{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}))$ is isomorphic to*

$$\frac{W(F)[p_2,\dots,p_k,e_k,p^{\perp}_2,\dots,p^{\perp}_{n-k},e^{\perp}_{n-k}]}{(p\cdot p^{\perp}=1, e_k\cdot e^{\perp}_{n-k}=0, e_k^2=p_k, (e^{\perp}_{n-k})^2=(p^{\perp}_{n-k})^2)}.$$

(2) *If n is odd,*

$$\bigoplus_{j,\mathscr{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \cong \begin{cases} \frac{W(F)[p_2,\dots,p_k,e_k,p^{\perp}_2,\dots,p^{\perp}_{n-k-1}]}{(p\cdot p^{\perp}=1, e_k^2=p_k)} & \text{if } k \text{ is even,} \\ \frac{W(F)[p_2,\dots,p_{k-1},p^{\perp}_2,\dots,p^{\perp}_{n-k},e^{\perp}_{n-k}]}{(p\cdot p^{\perp}=1,(e^{\perp}_{n-k})^2=(p^{\perp}_{n-k})^2)} & \text{if } k \text{ is odd.} \end{cases}$$

(3) *For k and $n-k$ odd,*

$$\bigoplus_{j,\mathscr{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \cong \frac{W(F)[p_2,\dots,p_{k-1},p_2^\perp,\dots,p_{n-k-1}^\perp]}{(p\cdot p^\perp=1)} \otimes \bigwedge[R].$$

Here the notation is the one of Theorem 5.7, ie the bidegrees of the even Pontryagin classes p_{2i} are $(4i,0)$, the bidegrees of the Euler classes e_k and $e_{n-k}^\perp$ are $(k,1)$ and $(n-k,1)$, respectively, and the class R in the last case has bidegree $(n-1,0)$.

Remark 6.5 The description of the $\boldsymbol{I}^\bullet$–cohomology ring in Theorem 5.7 is compatible with the above claims via the natural projection $H^j(\mathrm{Gr}(k,n),\boldsymbol{I}^j(\mathscr{L}))\to H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}))$. Moreover, Theorem 5.7 implies Theorem 6.4.

The following is an analogue of Proposition 5.5 for the above cohomology; it will be used in the inductive proof of Theorem 6.4.

Proposition 6.6 *Let $1\le k<n$. Consider the morphism*

$$e_{n-k}^\perp\colon H^{\bullet-n+k}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}))\to H^\bullet(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}\otimes\det\mathscr{S}_{n-k}^\vee))$$

given by multiplication with the Euler class.

(1) *The cokernel is the quotient of the cohomology algebra modulo the ideal $\langle e_{n-k}^\perp\rangle$.*
(2) *If $k\equiv n-k\equiv 0 \bmod 2$, then the kernel of $e_{n-k}^\perp$ is the ideal $\langle e_k\rangle$. The cokernel is generated by the classes $p_2,\dots,p_k,e_k,p_2^\perp,\dots,p_{n-k-2}^\perp$ modulo the relations $p\cdot p^\perp=1$ and $e_k^2=p_k$. The classes in the kernel are products of e_k with a class in the cokernel.*
(3) *If $k+1\equiv n-k\equiv 0 \bmod 2$, then the cokernel is generated by the Pontryagin classes $p_2,\dots,p_{k-1}$, $p_2^\perp,\dots,p_{n-k-2}^\perp$ modulo the relation $p\cdot p^\perp=1$. The kernel is the ideal $\langle p_{k-1}e_{n-k}^\perp\rangle$.*
(4) *If $n-k\equiv 1 \bmod 2$, the multiplication map is 0. The kernel and cokernel are the whole cohomology algebra.*

Proof This follows directly from the explicit presentation of Theorem 6.4. □

Proof of Theorem 6.4 Fix a natural number n. The claim for $\mathrm{Gr}(k,n)$ is proved by induction on k.

The base case is the case $\mathbb{P}^{n-1}=\mathrm{Gr}(1,n)$, in which case the claim follows directly from the computations in [14] — realizing for instance $H^i(\mathbb{P}^n,\boldsymbol{W}(\mathscr{L}))\cong H^i(\mathbb{P}^n,\boldsymbol{I}^{i-1}(\mathscr{L}))$. In both cases there are only two nontrivial groups, one of them is $H^0(\mathbb{P}^{n-1},\boldsymbol{W})\cong W(F)$. If n is even, then $\mathbb{P}^{n-1}$ is orientable, and the other nontrivial cohomology groups is $H^{n-1}(\mathbb{P}^{n-1},\boldsymbol{W})\cong W(F)$ (nontwisted coefficients), generated by the orientation class R. If n is odd, the other nontrivial cohomology group is $H^{n-1}(\mathbb{P}^{n-1},\boldsymbol{W}(\det\mathscr{S}_1))\cong W(F)$, generated by $e_{n-1}^\perp$.[8]

[8]Note the similarity with the rational cohomology of the real projective space.

For the inductive step, assume that Theorem 6.4 holds for $\mathrm{Gr}(k-1,n)$. We have to make a case distinction depending on parities of k and n.

If $n-k+1$ and $k-1$ are even Then the Euler classes e_{k-1} and $e^{\perp}_{n-k+1}$ are nonzero. The kernel and cokernel of $e^{\perp}_{n-k+1}$ are described in parts (1) and (2) of Proposition 6.6. As an algebra over the image of the cokernel of $e^{\perp}_{n-k+1}$, the cohomology of $\mathcal{S}(k,n)=\mathcal{Q}_{n-k+1}\setminus z_p(\mathrm{Gr}(k-1,n))$ is an exterior algebra, generated by 1 and the class R in degree $(n-1,0)$ which is a lift of e_{k-1} along ∂. This follows from the localization sequence for the bundle $\mathcal{Q}_{n-k+1}$; see point (1) of Proposition 6.3.

For the second localization, for the bundle $\mathcal{S}_k$, we first note that the Euler classes e_k and $e^{\perp}_{n-k}$ are zero. We check what we can say about the map q^*: We have the Pontryagin classes $p_2,\dots,p_{k-1}$, and these are mapped to their counterparts in the cohomology of $\mathcal{S}_k\setminus z_q(\mathrm{Gr}(k,n))$, by Proposition 6.1. By exactness, all the classes in the image of the restriction morphism q^* will have trivial image under the boundary map ∂. Also, the class R from degree $(n-1,0)$ has image under ∂ in degree $(n-k,1)$; in the case at hand, $n-k$ is odd, so there are no nontrivial elements in this degree and therefore $\partial R=0$.

Now we need to determine which classes have nontrivial image under ∂. The class e_{k-1} from the cokernel of $e^{\perp}_{n-k+1}$ necessarily maps to 1 under ∂. By the derivation property, more generally a product $p\cdot e_{k-1}$ of the Euler class with a polynomial p in the Pontryagin classes $p_2,\dots,p_{k-1}$ will map under ∂ to p, viewed as an element of the cohomology of $\mathrm{Gr}(k,n)$.

At this point, we see that the $\boldsymbol{W}$–cohomology of $\mathrm{Gr}(k,n)$ is indeed generated by the characteristic classes listed in the theorem statement: the Pontryagin classes p_i and $p_i^{\perp}$ are mapped to their counterparts in the cohomology of $\mathcal{S}(k,n)$; and the same is true for the class R. The only missing generator of the cohomology of $\mathcal{S}(k,n)$ is the Euler class e_{k-1}, but we saw above that this class has nontrivial boundary. It follows similarly, that the only relation is given by the Whitney sum formula.

If both $n-k+1$ and $k-1$ are odd Then the Euler classes e_{k-1} and $e^{\perp}_{n-k+1}$ are zero. In particular, via the first localization sequence for the bundle $\mathcal{Q}_{n-k+1}\to\mathrm{Gr}(k-1,n)$, the cohomology of $\mathcal{S}(k,n)$ consists of two copies of the cohomology of $\mathrm{Gr}(k-1,n)$; one of the copies is obviously generated by 1 in degree $(0,0)$, the other generated by a class in bidegree $(n-k,1)$ which is a lift of $1\in H^0$ along the boundary map.

Now for the second bundle $\mathcal{S}_k\to\mathrm{Gr}(k,n)$, both Euler classes e_k and $e^{\perp}_{n-k+1}$ are nontrivial. We check what we can say about the restriction map q^* in the corresponding localization sequence. In the cohomology of $\mathrm{Gr}(k,n)$, we have the Pontryagin classes and these are mapped under q^* to their counterparts in the cohomology of $\mathcal{S}(k,n)$. The class in bidegree $(n-k,1)$ (which arose as lift of 1 in the first localization sequence) lifts to the Euler class $e^{\perp}_{n-k}$.

The class R from the cohomology of $\mathcal{S}(k,n)$ has nontrivial boundary; its degree is $(n-1,0)$ and its image under the boundary map has degree $(n-k,1)$, so the class R is mapped exactly to the Euler class $e^{\perp}_{n-k}$.

Consequently, this establishes the claimed presentation of the $\boldsymbol{W}$–cohomology ring.

If $n-k+1$ is even and $k-1$ is odd Then the Euler class $e^{\perp}_{n-k+1}$ is nontrivial. The kernel and cokernel of $e^{\perp}_{n-k+1}$ are described in Proposition 6.6. The cokernel is generated by the Pontryagin classes, and the kernel is the ideal $\langle p_{k-2}e^{\perp}_{n-k+1}\rangle$. The class $p_{k-2}e^{\perp}_{n-k+1}$ has degree $(n-k+3,1)$ and consequently lifts along the boundary map ∂ to a class in degree $(2n-3,0)$.

Now for the second bundle $\mathscr{S}_k \to \mathrm{Gr}(k,n)$, the Euler class e_k is also nontrivial. We check what happens in the associated localization sequence. For the moment, call the right-hand side of the isomorphism in (2) of the statement the "candidate presentation". The cokernel of the multiplication by e_k on the candidate presentation is generated by the Pontryagin classes which map to their counterparts in the cohomology of $\mathscr{S}(k,n)$. The kernel of the Euler class on the candidate presentation is the ideal generated by $e_k\, p^{\perp}_{n-k-1}$ in degree $(2n-k-2,1)$.

The Pontryagin classes in the cokernel all map to their counterparts under the restriction map q^*. The class in degree $(2n-3,0)$ (which arose as a lift of $p_{k-2}e^{\perp}_{n-k+1}$) maps to $e_k\, p^{\perp}_{n-k-1}$ in degree $(2n-k-2,1)$.

Consequently, we see that the description of the cohomology of $\mathrm{Gr}(k,n)$, given in Theorem 6.4, is true if and only if it is true for $\mathrm{Gr}(k-1,n)$. Therefore, this argument also settles the case where $n-k+1$ is odd and $k-1$ is even. □

6.3 Putting the pieces together

We are now in the position to prove the theorems about the structure of $\boldsymbol{I}$–cohomology of the Grassmannians $\mathrm{Gr}(k,n)$.

Proof of Proposition 5.8 For all characteristic classes except R the claims on their reductions follow from Theorem 3.24(2). The injectivity of ρ on the image of $\beta_{\mathscr{L}}$ follows from Theorem 6.4, in combination with Lemma 2.4, via the splitting of $\boldsymbol{I}$–cohomology as direct sum of $\boldsymbol{W}$–cohomology and the image of $\beta_{\mathscr{L}}$.

It remains to identify the reduction of R. This follows by tracing through the inductive proof of Theorem 5.7, noting that R arises via boundary maps from Euler classes. The key point to note is that the reduction of R in the mod 2 Chow ring must be divisible by both $\bar{c}_k$ and $\bar{c}^{\perp}_{n-k}$, which implies the claim. Alternatively, the identification can be deduced from [23, Remarks 4 and 5] using the real cycle class map isomorphisms of [17]. □

Proof of Theorem 5.7 Again, we use the splitting of $\boldsymbol{I}$–cohomology as direct sum of $\boldsymbol{W}$–cohomology and the image of $\beta_{\mathscr{L}}$ which follows from Theorem 6.4, in combination with Lemma 2.4.

Relations (1)–(4) claimed in the theorem are satisfied because they are already satisfied on the level of $B\mathrm{GL}_n$ by Theorem 3.24 and the Whitney sum formulas in Propositions 3.3 and 3.28. Relation (5) involving R has two components: the claim on multiplication with torsion classes follows from the injectivity of ρ on the image of β given by Proposition 5.8, and the claim $R^2=0$ in $\boldsymbol{W}$–cohomology follows from Theorem 6.4. The image of R^2 under the projection to $\mathrm{Im}\,\beta$ can be computed in mod 2 Chow theory, where we have $\rho(R^2)=\bar{c}_{k-1}\bar{c}^{\perp}_{n-k}\bar{c}_k\bar{c}^{\perp}_{n-k-1}=0$. In particular, we get a well-defined map from the candidate presentation (with generators listed in (G1)–(G5) and relations (R1)–(R5) of Theorem 5.7) to the total $\boldsymbol{I}$–cohomology ring of $\mathrm{Gr}(k,n)$.

To show that the generators listed in Theorem 5.7 generate the $\boldsymbol{I}$–cohomology ring we again first show that all the torsion classes in the image of β are accounted for. Knowing the mod 2 Chow ring of the Grassmannians — see Proposition 5.4 — this follows as in the proof of Theorem 3.24 by considering the image of $\mathrm{Sq}^2_{\mathscr{L}}$. Then the surjectivity for $\boldsymbol{W}$–cohomology follows from Theorem 6.4.

To show injectivity, ie that all relations in the cohomology ring are accounted for, we note that the $W(F)$–torsionfree part generated by the Pontryagin classes, as well as Euler classes or R (whenever applicable), has exactly the relations (2), (3) and (5), by Theorem 6.4. So it suffices to investigate relations among classes in the image of β. Since ρ is injective on the image of $\beta_{\mathscr{L}}$, it suffices to show that all relations appearing in $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ arise from those for $B\mathrm{GL}_n$ and the Whitney sum formulas. This follows from the presentation of the mod 2 Chow rings in Proposition 5.4 and Theorem 3.24. □

6.4 An example

We discuss the argument for nonorientable Grassmannians in the special case comparing $\mathbb{P}^4$ and $\mathrm{Gr}(2,5)$. The following computation also indicates how one may go about establishing the formulas for $\boldsymbol{I}$–cohomology directly without the β–$\boldsymbol{W}$–decomposition. A complete version of this argument can be found in the first version of the present paper on the arXiv [27].

First, we consider the localization sequence associated to the tautological rank 4 bundle on $\mathbb{P}^4$ which has the form

$$\cdots \to H^j(\mathbb{P}^4, \boldsymbol{I}^j(\mathscr{L})) \to H^j(T, \boldsymbol{I}^j(\mathscr{L})) \to H^{j-3}\big(\mathbb{P}^4, \boldsymbol{I}^{j-4}(\mathscr{L}(1))\big) \xrightarrow{e_4^\perp} \cdots,$$

where T is the complement of the zero section of the rank 4 bundle on $\mathbb{P}^4$. From the shape of the localization sequence, we see that there are isomorphisms

$$\begin{aligned} H^j(\mathbb{P}^4, \boldsymbol{I}^j(\mathscr{L})) &\cong H^j(T, \boldsymbol{I}^j(\mathscr{L})) && \text{for } j \le 2, \\ H^{j+3}(T, \boldsymbol{I}^{j+3}(\mathscr{L})) &\cong H^j\big(\mathbb{P}^4, \boldsymbol{I}^{j-1}(\mathscr{L}(1))\big) && \text{for } j \ge 2. \end{aligned}$$

The complicated bit is given by two exact sequences. First,

$$0 \to H^3(\mathbb{P}^4, \boldsymbol{I}^3(\mathscr{L})) \to H^3(T, \boldsymbol{I}^3(\mathscr{L})) \to H^0\big(\mathbb{P}^4, \boldsymbol{I}^{-1}(\mathscr{L}(1))\big) \to H^4(\mathbb{P}^4, \boldsymbol{I}^3(\mathscr{L})).$$

In the case where $\mathscr{L} = \mathbb{O}$, then the first and third terms in the exact sequence are trivial and so is $H^3(T, \boldsymbol{I}^3)$. In the case where $\mathscr{L} = \mathbb{O}(1)$, the third term is $W(F) \cdot 1$ and the last term is $W(F) \cdot e_4^\perp$, so multiplication with the Euler class $e_4^\perp$ is an isomorphism. Consequently, we have an isomorphism $H^3(T, \boldsymbol{I}^3(1)) \cong H^3(\mathbb{P}^4, \boldsymbol{I}^3(1)) \cong \mathbb{Z}/2\mathbb{Z}$, generated by e_1^3.

The second exact sequence is

$$H^0\big(\mathbb{P}^4, \boldsymbol{I}^0(\mathscr{L}(1))\big) \to H^4(\mathbb{P}^4, \boldsymbol{I}^4(\mathscr{L})) \to H^4(T, \boldsymbol{I}^4(\mathscr{L})) \to H^1\big(\mathbb{P}^4, \boldsymbol{I}^0(\mathscr{L}(1))\big) \to 0.$$

In the case where $\mathscr{L}=\mathbb{O}$, the first and last terms in the exact sequence are trivial and we get an isomorphism $H^4(T,\boldsymbol{I}^4)\cong H^4(\mathbb{P}^4,\boldsymbol{I}^4)\cong\mathbb{Z}/2\mathbb{Z}$, generated by e_1^4. In the case where $\mathscr{L}=\mathbb{O}(1)$, the first morphism is multiplication by the Euler class which is an isomorphism. In particular, we get an isomorphism $H^4(T,\boldsymbol{I}^4(1))\to H^1(\mathbb{P}^4,\boldsymbol{I}^0(1))\cong 0$.

Now we can consider the localization sequence for the tautological rank 2 bundle on $\mathrm{Gr}(2,5)$ which has the form

$$\cdots\xrightarrow{e_2} H^j(\mathrm{Gr}(2,5),\boldsymbol{I}^j(\mathscr{L}))\to H^j(T,\boldsymbol{I}^j(\mathscr{L}))\to H^{j-1}\big(\mathrm{Gr}(2,5),\boldsymbol{I}^{j-2}(\mathscr{L}(1))\big)\xrightarrow{e_2}\cdots.$$

Because of cohomology vanishing in negative degrees, $H^0(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\cong H^0(T,\boldsymbol{I}^0(\mathscr{L}))$, and we note that this is isomorphic to the respective cohomology of $\mathbb{P}^4$. Next, there is an exact sequence

$$0\to H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))\to H^1(T,\boldsymbol{I}^1(\mathscr{L})).$$

For $\mathscr{L}=\mathbb{O}$, the last group is trivial, implying triviality of $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))$. For $\mathscr{L}=\mathbb{O}(1)$, the last group is $\mathbb{Z}/2\mathbb{Z}$. The explicit generator $\beta_{\mathbb{O}(1)}(1)$ maps to a generator of the last group and this implies that $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))\cong\mathbb{Z}/2\mathbb{Z}$.

For H^2, we have an exact sequence

$$H^0\big(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}(1))\big)\to H^2(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\to H^2(T,\boldsymbol{I}^2(\mathscr{L}))\to H^1\big(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}(1))\big).$$

For $\mathscr{L}=\mathbb{O}$, the outer groups are both zero and hence $H^2(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\cong\mathbb{Z}/2\mathbb{Z}$. For $\mathscr{L}=\mathbb{O}(1)$, the first map is an isomorphism mapping 1 to e_2. Note that only using the localization sequence at this point would require knowledge of the restriction morphism $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^2(\mathscr{L}))\to H^1(T,\boldsymbol{I}^2(\mathscr{L}))$ to show that $H^2\big(\mathrm{Gr}(2,5),\boldsymbol{I}^2(\mathbb{O}(1))\big)$ is isomorphic to $W(F)$ and not a proper quotient.

The remaining cohomology groups can be computed similarly, producing exactly the results from Example 5.13.

6.5 Remarks on oriented Grassmannians

We briefly formulate the analogous results for the Chow–Witt rings of the oriented Grassmannians. Recall that the $\mathbb{A}^1$–fundamental group of the Grassmannians is $\pi_1^{\mathbb{A}^1}(\mathrm{Gr}(k,n))\cong\mathbb{G}_m$, since up to $\mathbb{A}^1$–weak equivalence the Grassmannians are GL_{n-k}–quotients of the Stiefel varieties $\mathrm{GL}_n/\mathrm{GL}_k$ which are highly $\mathbb{A}^1$–connected. The oriented Grassmannians $\widetilde{\mathrm{Gr}}(k,n)$ are the $\mathbb{A}^1$–universal covers of the Grassmannians $\mathrm{Gr}(k,n)$. Explicitly, they are given as the complement of the zero section of the line bundle $\det\mathscr{S}_k$. For the Chow–Witt rings of the oriented Grassmannians $\widetilde{\mathrm{Gr}}(k,n)$, we only have the trivial duality because they are $\mathbb{A}^1$–simply connected.

We can formulate a result analogous to Theorem 5.7 for the oriented Grassmannians. The proof of the result proceeds exactly along the lines of the proofs for $\mathrm{Gr}(k,n)$. Some results concerning the $\boldsymbol{W}$–cohomology of the oriented Grassmannians can be deduced from the work of Ananyevskiy in [1].

Theorem 6.7 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$.*

(1) *There is a cartesian square of $\mathbb{Z}$–graded* $\mathrm{GW}(F)$*–algebras,*

$$\begin{tikzcd} \widetilde{\mathrm{CH}}^\bullet(\widetilde{\mathrm{Gr}}(k,n)) \arrow[r] \arrow[d] & \ker \partial \arrow[d, "\mathrm{mod}\ 2"] \\ H^\bullet(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^\bullet) \arrow[r, "\rho"'] & \mathrm{Ch}^\bullet(\widetilde{\mathrm{Gr}}(k,n)) \end{tikzcd}$$

(2) *The cokernel of the Bockstein morphism*

$$\beta \colon \mathrm{CH}^j(\widetilde{\mathrm{Gr}}(k,n)) \to H^{j+1}(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^{j+1})$$

is described exactly as in Theorem 6.4, except that there is no additional $\mathbb{Z}/2\mathbb{Z}$–grading and the Euler classes are elements of the cohomology with trivial duality.

(3) *The reduction morphism*

$$\rho \colon H^{j+1}(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^{j+1}) \to \mathrm{CH}^{j+1}(\widetilde{\mathrm{Gr}}(k,n))$$

is injective on the image of the Bockstein morphism β. In particular, the image of Bockstein can be determined from the Wu formula for the Steenrod squares on the mod 2 Chow ring of $\widetilde{\mathrm{Gr}}(k,n)$.

Remark 6.8 A result like the above should be true for all flag varieties (at least in type A). The cokernel of the Bockstein should have the same presentation as the rational cohomology of the real realization (but of course as a $W(F)$–algebra). The Bockstein classes should all be detected on the mod 2 Chow ring so that the structure of the torsion can be determined just from the knowledge of the Steenrod squares.

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Higher chromatic Thom spectra via unstable homotopy theory

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We investigate implications of an old conjecture in unstable homotopy theory related to the Cohen–Moore–Neisendorfer theorem and a conjecture about the $\boldsymbol{E}[2]$–topological Hochschild cohomology of certain Thom spectra (denoted by A, B and $T(n)$) related to Ravenel's $X(p^n)$. We show that these conjectures imply that the orientations $M\mathrm{Spin} \to bo$ and $M\mathrm{String} \to \mathrm{tmf}$ admit spectrum-level splittings. This is shown by generalizing a theorem of Hopkins and Mahowald, which constructs $H\mathbb{F}_p$ as a Thom spectrum, to construct $\mathrm{BP}\langle n-1\rangle$, bo, and tmf as Thom spectra (albeit over $T(n)$, A, and B, respectively, and not over the sphere). This interpretation of $\mathrm{BP}\langle n-1\rangle$, bo, and tmf offers a new perspective on Wood equivalences of the form $bo \wedge C\eta \simeq bu$: they are related to the existence of certain EHP sequences in unstable homotopy theory. This construction of $\mathrm{BP}\langle n-1\rangle$ also provides a different lens on the nilpotence theorem. Finally, we prove a C_2–equivariant analogue of our construction, describing $\underline{H\mathbb{Z}}$ as a Thom spectrum.

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1 Introduction

1.1 Statement of the main results

One of the goals of this article is to describe a program to prove the following old conjecture (studied, for instance, by Laures and Schuster [60; 61], and discussed informally in many places, such as Section 7 of Mahowald and Rezk [75]):

Conjecture 1.1.1 *The Ando–Hopkins–Rezk orientation* (*see* [6]) $M\mathrm{String} \to \mathrm{tmf}$ *admits a spectrum-level splitting.*

The key idea in our program is to provide a universal property for mapping out of the spectrum tmf. We give a proof which is conditional on an old conjecture from unstable homotopy theory stemming from the Cohen–Moore–Neisendorfer theorem and a conjecture about the $\boldsymbol{E}_2$–topological Hochschild cohomology of certain Thom spectra (the latter of which simplifies the proof of the nilpotence theorem of Devinatz, Hopkins and Smith [38]). This universal property exhibits tmf as a certain Thom spectrum, similarly to the Hopkins–Mahowald construction of $H\mathbb{Z}_p$ and $H\mathbb{F}_p$ as Thom spectra.

To illustrate the gist of our argument in a simpler case, recall Thom's classical result from [92]: the unoriented cobordism spectrum MO is a wedge of suspensions of $H\mathbb{F}_2$. The simplest way to do so is to show that MO is an $H\mathbb{F}_2$–module, which in turn can be done by constructing an $\boldsymbol{E}_2$–map $H\mathbb{F}_2 \to MO$. The construction of such a map is supplied by the following theorem of Hopkins and Mahowald:

Theorem (Hopkins and Mahowald; see Mahowald [67] and Mahowald, Ravenel and Shick [73, Lemma 3.3]) *Let $\mu\colon \Omega^2 S^3 \to BO$ denote the real vector bundle over $\Omega^2 S^3$ induced by extending the map $S^1 \to BO$ classifying the Möbius bundle. Then the Thom spectrum of μ is equivalent to $H\mathbb{F}_2$ as an $\boldsymbol{E}_2$–algebra.*

Remark 1.1.2 The Thomification of the $\boldsymbol{E}_2$–map $\mu\colon \Omega^2 S^3 \to BO$ produces the desired $\boldsymbol{E}_2$–splitting $H\mathbb{F}_2 \to MO$.

Our argument for Conjecture 1.1.1 takes this approach: we shall show that an old conjecture from unstable homotopy theory and a conjecture about the $\boldsymbol{E}_2$–topological Hochschild cohomology of certain Thom spectra provide a construction of tmf (as well as bo and $\mathrm{BP}\langle n\rangle$) as a Thom spectrum, and utilize the resulting universal property of tmf to construct an (unstructured) map $\mathrm{tmf} \to M\mathrm{String}$.

Mahowald was the first to consider the question of constructing spectra like bo and tmf as Thom spectra (see [71]). Later work by Rudyak [88] sharpened Mahowald's results to show that bo and bu cannot appear as the Thom spectra of a p–complete spherical fibration. Angeltveit, Hill and Lawson [10] gave an alternative proof of this fact under the assumption that the p–complete spherical fibration is classified by a map of $\boldsymbol{E}_3$–spaces. Recently, Chatham [27] has shown that $\mathrm{tmf}_2^\wedge$ cannot appear as the Thom spectrum of a structured 2–complete spherical fibration over a loop space. Our goal is to argue that these issues are alleviated if we replace "spherical fibrations" with "bundles of R–lines" for certain well-behaved spectra R.

The first hint of tmf being a generalized Thom spectrum comes from a conjecture of Hopkins and Hahn regarding a construction of the truncated Brown–Peterson spectra $\mathrm{BP}\langle n\rangle$ as Thom spectra. To state this conjecture, we need to recall some definitions. Recall (see [38]) that $X(n)$ denotes the Thom spectrum of the map $\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU} \simeq BU$. Upon completion at a prime p, the spectra $X(k)$ for $p^n \le k \le p^{n+1}-1$ split as a direct sum of suspensions of certain homotopy commutative ring spectra $T(n)$, which in turn filter the gap between the p–complete sphere spectrum and BP (in the sense that $T(0) = \mathbb{S}$ and $T(\infty) = \mathrm{BP}$).

Conjecture 1.1.3 (Hahn and Hopkins, unpublished) *There is a map* $f\colon \Omega^2 S^{|v_n|+3} \to B\mathrm{GL}_1(T(n))$, *which detects an indecomposable element* $v_n \in \pi_{|v_n|} T(n)$ *on the bottom cell of the source, whose Thom spectrum is a form of* $\mathrm{BP}\langle n-1\rangle$.

The primary obstruction to proving that a map f as in Conjecture 1.1.3 exists stems from the failure of $T(n)$ to be an $\boldsymbol{E}_3$–ring (due to Lawson [62, Example 1.5.31]). If R is an $\boldsymbol{E}_1$– or $\boldsymbol{E}_2$–ring spectrum, let $\mathfrak{Z}_3(R)$ denote the $\boldsymbol{E}_2$–topological Hochschild cohomology of R (see Definition 3.3.2). Hahn suggested that one way to get past the failure of $T(n)$ to be an $\boldsymbol{E}_3$–ring would be via the following conjecture:

Conjecture 1.1.4 (Hahn) *There is an indecomposable element* $v_n \in \pi_{|v_n|} T(n)$ *which lifts to the* $\boldsymbol{E}_2$*–topological Hochschild cohomology* $\mathfrak{Z}_3(X(p^n))$ *of* $X(p^n)$.

We do not know how to prove this conjecture (and have no opinion on whether or not it is true). We shall instead show that Conjecture 1.1.3 is implied by the two conjectures alluded to above. We shall in a moment state these conjectures precisely as Conjectures D and E; let us first state our main results.

We need to introduce some notation. Let $y(n)$ (resp. $y_{\mathbb{Z}}(n)$) denote the Mahowald–Ravenel–Shick spectrum, constructed as a Thom spectrum over $\Omega J_{p^n-1}(S^2)$ (resp. $\Omega J_{p^n-1}(S^2)\langle 2\rangle$) introduced in [73] to study the telescope conjecture (resp. by Angelini-Knoll and Quigley [8] as $z(n)$). Let A denote the $\boldsymbol{E}_1$–quotient $\mathbb{S}/\!\!/\nu$ of the sphere spectrum by $\nu \in \pi_3(\mathbb{S})$; its mod 2 homology is $H_*(A) \cong \mathbb{F}_2[\zeta_1^4]$. The spectrum A has been intensely studied by Mahowald and his coauthors Davis and Unell in [67; 31; 69; 68; 70; 77], for instance, where it is often denoted by X_5. (See Remark 2.1.8 for motivation for the term "$\boldsymbol{E}_1$–quotient".) Let B denote the $\boldsymbol{E}_1$–ring we introduced in [34, Construction 3.1]; it has been briefly studied under the name $\overline{X}$ by Mahowald and Hopkins [72]. It may be constructed as the Thom spectrum of a vector bundle over an $\boldsymbol{E}_1$–space N which sits in a fiber sequence $\Omega S^9 \to N \to \Omega S^{13}$. The mod 2 homology of B is $H_*(B) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4]$.

We also need to recall some unstable homotopy theory. Cohen, Moore and Neisendorfer [29; 30; 81] constructed a map $\phi_n\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ whose composite with the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ is the degree p map. (The E stands for "Einhängung", which is German for "suspension".) Such a map was also constructed by Gray [44; 42]. In Section 4.1, we introduce the related notion of a *charming map* (Definition 4.1.1), one example of which is the Cohen–Moore–Neisendorfer map.

Our main result is then:

Theorem A *Suppose* R *is a base spectrum of height* n *as in the second line of Table 1. Let* K_{n+1} *denote the fiber of a charming map* $\Omega^2 S^{2p^{n+1}+1} \to S^{2p^{n+1}-1}$. *Then Conjectures D and E imply that there is a map* $\mu\colon K_{n+1} \to B\mathrm{GL}_1(R)$ *such that the mod* p *homology of the Thom spectrum* K_{n+1}^{μ} *is isomorphic to the mod* p *homology of the associated designer chromatic spectrum* $\Theta(R)$ *as a Steenrod comodule.*[1]

[1]We elected to use the symbol Θ because the first two letters of the English spelling of Θ and of Thom's name agree.

height	0	1	2	n	n	n
base spectrum R designer chromatic spectrum $\Theta(R)$	$\mathbb{S}_p^\wedge$ $H\mathbb{Z}_p$	A bo	B tmf	$T(n)$ $\mathrm{BP}\langle n\rangle$	$y(n)$ $k(n)$	$y_{\mathbb{Z}}(n)$ $k_{\mathbb{Z}}(n)$

Table 1: To go from a base spectrum "of height n", say R, in the second line to the third, one takes the Thom spectrum of a bundle of R–lines over K_{n+1}.

If R is any base spectrum other than B, the Thom spectrum K_{n+1}^μ is equivalent to $\Theta(R)$ upon p–completion for every prime p. If Conjecture F is true, then the same is true for B: the Thom spectrum K_{n+1}^μ is equivalent to $\Theta(B) = \mathrm{tmf}$ upon 2–completion.

Making sense of Theorem A relies on knowing that $T(n)$ admits the structure of an $\boldsymbol{E}_1$–ring; this is proved by Beardsley and Lawson [21]; see also Remark 3.1.6. Note that the spectra A, B, $y(n)$ and $y_{\mathbb{Z}}(n)$ all admit $\boldsymbol{E}_1$–structures by construction. In Remark 5.4.7, we sketch how Theorem A relates to the proof of the nilpotence theorem.

Although the form of Theorem A does not resemble Conjecture 1.1.3, we show that Theorem A implies the following result:

Corollary B *Conjectures D and E imply Conjecture 1.1.3.*

In the case $n = 0$, Corollary B recovers the Hopkins–Mahowald theorem constructing $H\mathbb{F}_p$. Moreover, Corollary B is true unconditionally when $n = 0, 1$.

Using the resulting universal property of tmf, we obtain a result pertaining to Conjecture 1.1.1.

Theorem C *Assume that the composite $\mathfrak{z}_3(B) \to B \to M\mathrm{String}$ is an $\boldsymbol{E}_3$–map. Then Conjectures D, E and F imply that there is a spectrum-level unital splitting of the Ando–Hopkins–Rezk orientation $M\mathrm{String}_{(2)} \to \mathrm{tmf}_{(2)}$.*

In particular, Conjecture 1.1.1 follows (at least after localizing at $p = 2$; a slight modification of our arguments should work at any prime). We believe that the assumption that the composite $\mathfrak{z}_3(B) \to B \to M\mathrm{String}$ is an $\boldsymbol{E}_3$–map is too strong: we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We stress that these splittings are unstructured; it seems unlikely that they can be refined to structured splittings. In [34], we showed (unconditionally) that the Ando–Hopkins–Rezk orientation $M\mathrm{String} \to \mathrm{tmf}$ induces a surjection on homotopy, a result which is clearly implied by Theorem C.

We remark that the argument used to prove Theorem C shows that, if the composite $\mathfrak{z}_3(A) \to A \to M\mathrm{Spin}$ is an $\boldsymbol{E}_3$–map, then Conjectures D and E imply that there is a spectrum-level unital splitting of the Atiyah–Bott–Shapiro orientation $M\mathrm{Spin} \to bo$. This splitting was originally proved unconditionally (ie without assuming Conjecture D or Conjecture E) by Anderson, Brown and Peterson [4] via a calculation with the Adams spectral sequence.

1.2 The statements of Conjectures D, E and F

We first state Conjecture D. The second part of this conjecture is a compilation of several old conjectures in unstable homotopy theory originally made by Cohen, Moore and Neisendorfer [29; 30; 81], Gray [44; 42] and Selick [89]. The statement we shall give differs slightly from the statements made in the literature; for instance, in Conjecture D(b), we demand a $\mathcal{Q}_1$–space splitting (Notation 2.2.6), rather than merely an H–space splitting.

Conjecture D *The following statements are true*:

(a) *The homotopy fiber of any charming map* (Definition 4.1.1) *is equivalent as a loop space to the loop space on an Anick space* (Example 4.1.3).

(b) *There exists a p–local charming map $f\colon \Omega^2 S^{2p^n+1} \to S^{2p^n-1}$ whose homotopy fiber admits a $\mathcal{Q}_1$–space retraction off of $\Omega^2(S^{2p^n}/p)$. There are also integrally defined maps $\Omega^2 S^9 \to S^7$ and $\Omega^2 S^{17} \to S^{15}$ whose composites with the double suspension on S^7 and S^{15}, respectively, are the degree 2 maps. Moreover, their homotopy fibers K_2 and K_3 (respectively) admit deloopings, and admit $\mathcal{Q}_1$–space retractions off of $\Omega^2(S^8/2)$ and $\Omega^2(S^{16}/2)$ (respectively).*

Next, we turn to Conjecture E. This conjecture is concerned with the $\boldsymbol{E}_2$–topological Hochschild cohomology of the Thom spectra $X(p^n-1)_{(p)}$, A and B introduced above.

Conjecture E *Let $n \geq 0$ be an integer. Let R denote $X(p^{n+1}-1)_{(p)}$, A (in which case $n=1$) or B (in which case $n=2$). Then the element $\sigma_n \in \pi_{|v_n|-1} R$ lifts to the $\boldsymbol{E}_2$–topological Hochschild cohomology $\mathfrak{Z}_3(R)$ of R, and is p–torsion in $\pi_*\mathfrak{Z}_3(R)$ if $R = X(p^{n+1}-1)_{(p)}$, and is 2–torsion in $\pi_*\mathfrak{Z}_3(R)$ if $R=A$ or B.*

Finally, we state Conjecture F. It is inspired by Adams and Priddy [2] and Angeltveit and Lind [12]. We believe this conjecture is the most approachable of the conjectures stated here.

Conjecture F *Suppose X is a spectrum which is bounded below and whose homotopy groups are finitely generated over $\mathbb{Z}_p$. If there is an isomorphism $H_*(X;\mathbb{F}_p) \cong H_*(\mathrm{tmf};\mathbb{F}_p)$ of Steenrod comodules, then there is a homotopy equivalence $X_p^\wedge \to \mathrm{tmf}_p^\wedge$ of spectra.*

After proving Theorems A and C, we explore relationships between the different spectra appearing on the second line of Table 1 in the remainder of the article. In particular, we prove analogues of Wood's equivalence $bo \wedge C\eta \simeq bu$ (see also Mathew [78]) for these spectra. We argue that these are related to the existence of certain EHP sequences.

Finally, we describe a C_2–equivariant analogue of Corollary B at $n=1$ as Theorem 7.2.1, independently of a C_2–equivariant analogue of Conjectures D and E. This result constructs $H\underline{\mathbb{Z}}$ as a Thom spectrum of an equivariant bundle of invertible $T(1)_{\mathbb{R}}$–modules over $\Omega^\rho S^{2\rho+1}$, where $T(1)_{\mathbb{R}}$ is the free $\boldsymbol{E}_\sigma$–algebra with a nullhomotopy of the equivariant Hopf map $\widetilde{\eta} \in \pi_\sigma(\mathbb{S})$, and ρ and σ are the regular and sign representations of C_2, respectively. This uses results of Behrens and Wilson [24] and Hahn and Wilson [48]. We believe there is a similar result at odd primes, but we defer discussion of this. We discuss why our methods do not work to yield $\mathrm{BP}\langle n\rangle_{\mathbb{R}}$ for $n \geq 1$ as in Corollary B.

Outline

Section 2 contains a review some of the theory of Thom spectra from the modern perspective, as well as the proof of the classical Hopkins–Mahowald theorem. The content reviewed in this section will appear in various guises throughout this project, hence its inclusion.

In Section 3, we study certain $\boldsymbol{E}_1$–rings; most of them appear as Thom spectra over the sphere. For instance, we recall some facts about Ravenel's $X(n)$ spectra, and then define and prove properties about the $\boldsymbol{E}_1$–rings A and B used in the statement of Theorem A. We state Conjecture E, and discuss (Remark 5.4.7) its relation to the nilpotence theorem.

In Section 4, we recall some unstable homotopy theory, such as the Cohen–Moore–Neisendorfer map and the fiber of the double suspension. These concepts do not show up often in stable homotopy theory, so we hope this section provides useful background to the reader. We state Conjecture D, and then explore properties of Thom spectra of bundles defined over Anick spaces.

In Section 5, we state and prove Theorem A and Corollary B, and state several easy consequences of Theorem A.

In Section 6, we study some applications of Theorem A. For instance, we use it to prove Theorem C, which is concerned with the splitting of certain cobordism spectra. In a previous version of this article, we had two subsections discussing Wood-like equivalences, and topological Hochschild homology of the chromatic Thom spectra of Table 1. However, while making revisions to this article, we decided to split these two sections off into separate articles [33; 36].

In Section 7, we prove an equivariant analogue of Corollary B at height 1. We construct equivariant analogues of $X(n)$ and A, and describe why our methods fail to produce an equivariant analogue of Corollary B at all heights, even granting an analogue of Conjectures D and E.

Finally, in Section 8, we suggest some directions for future research. There are also numerous interesting questions arising from our work, which we have indicated in the body of the article.

Conventions

Unless indicated otherwise, or if it goes against conventional notational choices, a Latin letter with a numerical subscript (such as x_5) denotes an element of degree given by its subscript. If X is a space and R is an $\boldsymbol{E}_1$–ring spectrum, then X^μ will denote the Thom spectrum of some bundle of invertible R–modules determined by a map $\mu\colon X \to B\mathrm{GL}_1(R)$. We shall often quietly localize or complete at an implicit prime p. Although we have tried to be careful, *all limits and colimits will be homotopy limits and colimits*; we apologize for any inconvenience this might cause.

We shall denote by $P^k(p)$ the mod p Moore space $S^{k-1} \cup_p e^k$ with top cell in dimension k. The symbols ζ_i and τ_i will denote the *conjugates* of the Milnor generators (commonly written nowadays as ξ_i and τ_i, although, as Haynes Miller pointed out to me, our notation for the conjugates was Milnor's original notation) in degrees $2(p^i-1)$ and $2p^i-1$ for $p>2$ and 2^i-1 (for ζ_i) at $p=2$. Unfortunately, we will use A to denote the $\boldsymbol{E}_1$–ring in appearing in Table 1, and write A_* to denote the dual Steenrod algebra. We hope this does not cause any confusion, since we will always denote the homotopy groups of A by π_*A and not A_*.

If $\mathcal{O}$ is an operad, we will simply write $\mathcal{O}$*–ring* to denote an $\mathcal{O}$–algebra object in spectra. A map of $\mathcal{O}$–rings respecting the $\mathcal{O}$–algebra structure will often simply be called a $\mathcal{O}$–map. Unless it is clear that we mean otherwise, all modules over non-$\boldsymbol{E}_\infty$–algebras will be left modules.

Hood Chatham pointed out to me that $S^3\langle 4\rangle$ would be the correct notation for what we denote by $S^3\langle 3\rangle = \mathrm{fib}(S^3 \to K(\mathbb{Z},3))$. Unfortunately, the literature seems to have chosen $S^3\langle 3\rangle$ as the preferred notation, so we stick to that in this project.

When we write that Theorem A, Corollary B or Theorem C implies a statement P, we mean that Conjectures D and E (and Conjecture F if the intended application is to tmf) imply P via Theorem A, Corollary B or Theorem C.

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2 Background, and some classical positive and negative results

2.1 Background on Thom spectra

In this section, we will recall some facts about Thom spectra and their universal properties; the discussion is motivated by [5].

Definition 2.1.1 Let A be an $\boldsymbol{E}_1$–ring and let $\mu\colon X \to B\mathrm{GL}_1(A)$ be a map of spaces. The Thom A–module X^μ is defined as the homotopy pushout

$$\begin{tikzcd} \Sigma^\infty_+ \mathrm{GL}_1(A) \arrow[r] \arrow[d] & \Sigma^\infty_+ \operatorname{fib}(\mu) \arrow[d] \\ A \arrow[r] & X^\mu \end{tikzcd}$$

Remark 2.1.2 Let A be an $\boldsymbol{E}_1$–ring and let $\mu\colon X \to B\mathrm{GL}_1(A)$ be a map of spaces. The Thom A–module X^μ is the homotopy colimit of the functor $X \xrightarrow{\mu} B\mathrm{GL}_1(A) \to \mathrm{Mod}(A)$, where we have abused notation by identifying X with its associated Kan complex. If A is an $\boldsymbol{E}_1$–R–algebra, then the R–module underlying X can be identified with the homotopy colimit of the composite functor

$$X \xrightarrow{\mu} B\mathrm{GL}_1(A) \to B\mathrm{Aut}_R(A) \to \mathrm{Mod}(R),$$

where we have identified X with its associated Kan complex. The space $B\mathrm{Aut}_R(A)$ can be regarded as the maximal subgroupoid of $\mathrm{Mod}(R)$ spanned by the object A.

The following is immediate from the description of the Thom spectrum as a Kan extension:

Proposition 2.1.3 *Let R and R' be $\boldsymbol{E}_1$–rings with an $\boldsymbol{E}_1$–ring map $R \to R'$ exhibiting R' as a right R–module. If $f\colon X \to B\mathrm{GL}_1(R)$ is a map of spaces, then the Thom spectrum of the composite $X \to B\mathrm{GL}_1(R) \to B\mathrm{GL}_1(R')$ is the base change $X^f \wedge_R R'$ of the (left) R–module Thom spectrum X^f.*

Corollary 2.1.4 *Let R and R' be $\boldsymbol{E}_1$–rings with an $\boldsymbol{E}_1$–ring map $R \to R'$ exhibiting R' as a right R–module. If $f\colon X \to B\mathrm{GL}_1(R)$ is a map of spaces such that the composite $X \to B\mathrm{GL}_1(R) \to B\mathrm{GL}_1(R')$ is null, then there is an equivalence $X^f \wedge_R R' \simeq R' \wedge \Sigma^\infty_+ X$.*

Moreover (see eg [15, Corollary 3.2]):

Proposition 2.1.5 *Let X be a k–fold loop space and let R be an $\boldsymbol{E}_{k+1}$–ring. Then the Thom spectrum of an $\boldsymbol{E}_k$–map $X \to B\mathrm{GL}_1(R)$ is an $\boldsymbol{E}_k$–R–algebra.*

We will repeatedly use the following classical result, which is again a consequence of the observation that Thom spectra are colimits, as well as the fact that total spaces of fibrations may be expressed as colimits; see also [19, Theorem 1]:

Proposition 2.1.6 *Let $X \xrightarrow{i} Y \to Z$ be a fiber sequence of k–fold loop spaces (where $k \geq 1$), and let R be an $\boldsymbol{E}_m$–ring for $m \geq k+1$. Suppose that $\mu\colon Y \to B\mathrm{GL}_1(R)$ is a map of k–fold loop spaces. Then there is a k–fold loop map $\phi\colon Z \to B\mathrm{GL}_1(X^{\mu\circ i})$ whose Thom spectrum is equivalent to Y^μ as $\boldsymbol{E}_{k-1}$–rings. Concisely, if arrows are labeled by their associated Thom spectra, then there is a diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "X^{\mu\circ i}"'] & Y \arrow[r] \arrow[d, "\mu"', "Y^\mu"] & Z \arrow[d, "\phi"', "Y^\mu = Z^\phi"] \\ & B\mathrm{GL}_1(R) \arrow[r] & B\mathrm{GL}_1(X^{\mu\circ i}) \end{tikzcd}$$

The argument to prove Proposition 2.1.6 also goes through with slight modifications when $k = 0$, and shows:

Proposition *Let $X \xrightarrow{i} Y \to Z$ be a fiber sequence of spaces with Z connected, and let R be an $\boldsymbol{E}_m$–ring for $m \geq 1$. Suppose that $\mu\colon Y \to B\mathrm{GL}_1(R)$ is a map of Kan complexes. Then there is a map $\phi\colon Z \to B\mathrm{Aut}_R(X^{\mu\circ i})$ such that the homotopy colimit (ie "Thom spectrum") Z^ϕ of the following composite is equivalent to Y^μ as an R–module:*

$$Z \xrightarrow{\phi} B\mathrm{Aut}_R(X^{\mu\circ i}) \subseteq \mathrm{Mod}_R. \tag{2-1}$$

We will abusively refer to this result in the sequel also as Proposition 2.1.6.

Proof of the second form of Proposition 2.1.6 It will be convenient to use the model for Thom spectra following [5]. Observe that a fibration $X \to Y \to Z$ implies (eg by [5, Remark 2.4]) that there is a functor $Z \to \mathrm{Top}$ whose homotopy colimit is Y, and whose fiber over any vertex of $z \in Z$ is X. Since X is connected, we may write $Y \simeq \mathrm{hocolim}_Z X$. The map $X \to Y$ is induced by the inclusion $\{z\} \hookrightarrow Z$. Since Y is a Kan complex, the Thom spectrum Y^μ can be identified (by [5, Definition 1.4]) with the homotopy colimit of the composite $Y \xrightarrow{\mu} B\mathrm{GL}_1(R) \simeq R\text{–line} \subseteq \mathrm{Mod}_R$ (which we will temporarily denote by $\mu\colon Y \to \mathrm{Mod}_R$). We will write this as $Y^\mu \simeq \mathrm{hocolim}_Y R$. The left Kan extension of the map $Y \to Z$ along the functor $\mu\colon Y \to \mathrm{Mod}_R$ defines a functor $\phi\colon Z \to \mathrm{Mod}_R$, which sends $z \in Z$ to $X^{\mu\circ i} \simeq \mathrm{hocolim}(X \to Y \xrightarrow{\mu} \mathrm{Mod}_R)$. Since Z is connected, this implies that $Y^\mu \simeq \mathrm{hocolim}_Y R$ is the homotopy colimit of the functor (2-1). □

The following is a slight generalization of [15, Theorem 4.10]:

Theorem 2.1.7 *Let R be an $\boldsymbol{E}_{k+1}$–ring for $k \geq 0$ and let $\alpha\colon Y \to B\mathrm{GL}_1(R)$ be a map from a pointed space Y. For any $0 \leq m \leq k$, let $\tilde{\alpha}\colon \Omega^m\Sigma^m Y \to B\mathrm{GL}_1(R)$ denote the extension of α. Then the Thom spectrum $(\Omega^m\Sigma^m Y)^{\tilde{\alpha}}$ is the free $\boldsymbol{E}_m$–R–algebra A for which the composite $Y \to B\mathrm{GL}_1(R) \to B\mathrm{GL}_1(A)$ is null. More precisely, if A is any $\boldsymbol{E}_m$–R–algebra, then*

$$\mathrm{Map}_{\mathrm{Alg}_R^{\boldsymbol{E}_m}}((\Omega^m\Sigma^m Y)^{\tilde{\alpha}}, A) \simeq \begin{cases} \mathrm{Map}_*(Y, \Omega^\infty A) & \text{if } \alpha\colon Y \to B\mathrm{GL}_1(R) \to B\mathrm{GL}_1(A) \text{ is null,} \\ \varnothing & \text{otherwise.} \end{cases}$$

Remark 2.1.8 Say $Y = S^{n+1}$, so α detects an element $\alpha \in \pi_n R$. Theorem 2.1.7 suggests interpreting the Thom spectrum $(\Omega^m S^{m+n+1})^{\widetilde{\alpha}}$ as an $\boldsymbol{E}_m$–quotient; to signify this, we will denote it by $R/\!\!/_{\boldsymbol{E}_m}\alpha$. If $m=1$, then we will simply denote it by $R/\!\!/\alpha$, while if $m=0$, then the $\boldsymbol{E}_m$–quotient is simply the ordinary quotient R/α. See [15, Definition 4.3], where the quotient $R/\!\!/_{\boldsymbol{E}_m}\alpha$ is called the *versal R–algebra of characteristic α*.

2.2 The Hopkins–Mahowald theorem

The primary motivation for this project is the following miracle (see [67] for $p=2$ and [73, Lemma 3.3] for $p>2$, as well as [15, Theorem 5.1] for a proof of the equivalence as one of $\boldsymbol{E}_2$–algebras):

Theorem 2.2.1 (Hopkins and Mahowald) *Let $\mathbb{S}_p^\wedge$ be the p–completion of the sphere at a prime p and let $f\colon S^1 \to B\mathrm{GL}_1(\mathbb{S}_p^\wedge)$ detect the element $1-p \in \pi_1 B\mathrm{GL}_1(\mathbb{S}_p^\wedge) \simeq \mathbb{Z}_p^\times$. Let $\mu\colon \Omega^2 S^3 \to B\mathrm{GL}_1(\mathbb{S}_p^\wedge)$ denote the $\boldsymbol{E}_2$–map extending by f; then there is a p–complete equivalence $(\Omega^2 S^3)^\mu \to H\mathbb{F}_p$ of $\boldsymbol{E}_2$–ring spectra.*

It is not too hard to deduce the following result from Theorem 2.2.1:

Corollary 2.2.2 *Let $S^3\langle 3\rangle$ denote the 3–connected cover of S^3. Then the Thom spectrum of the composite $\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \xrightarrow{\mu} B\mathrm{GL}_1(\mathbb{S}_p^\wedge)$ is equivalent to $H\mathbb{Z}_p$ as an $\boldsymbol{E}_2$–ring.*

Remark 2.2.3 Theorem 2.2.1 implies a restrictive version of the nilpotence theorem: if R is an $\boldsymbol{E}_2$–ring spectrum, and $x \in \pi_* R$ is a simple p–torsion element which has trivial $H\mathbb{F}_p$–Hurewicz image, then x is nilpotent. This is explained in [79, Proposition 4.19]. Indeed, to show that x is nilpotent, it suffices to show that the localization $R[1/x]$ is contractible. Since $px=0$, the localization $R[1/x]$ is an $\boldsymbol{E}_2$–ring in which $p=0$, so the universal property of Theorem 2.1.7 implies that there is an $\boldsymbol{E}_2$–map $H\mathbb{F}_p \to R[1/x]$. It follows that the unit $R \to R[1/x]$ factors through the Hurewicz map $R \to R \wedge H\mathbb{F}_p$. In particular, the multiplication-by-x map on $R[1/x]$ factors as the indicated dotted map:

$$\begin{tikzcd}
\Sigma^{|x|} R \arrow[r, "x"] \arrow[d] & R \arrow[r] \arrow[d] & R[1/x] \\
H\mathbb{F}_p \wedge \Sigma^{|x|} R \arrow[r, "x"] & R \wedge H\mathbb{F}_p \arrow[ur, dashed] &
\end{tikzcd}$$

However, the bottom map is null (because x has trivial $H\mathbb{F}_p$–Hurewicz image), so x must be null in $\pi_* R[1/x]$. This is possible if and only if $R[1/x]$ is contractible, as desired. See Proposition 5.4.1 for the analogous connection between Corollary 2.2.2 and nilpotence.

Since an argument similar to the proof of Theorem 2.2.1 will be necessary later in Step 2 of Section 5.2, we will recall a proof of this theorem. The key nonformal input is the following result of Steinberger's from [26, Theorems III.2.2 and III.2.3]:

Theorem 2.2.4 (Steinberger) *Let ζ_i denote the conjugate to the Milnor generators ξ_i of the dual Steenrod algebra, and similarly for τ_i at odd primes. Then*

$$Q^{p^i}\zeta_i = \zeta_{i+1}, \quad Q^{p^j}\tau_j = \tau_{j+1} \tag{2-2}$$

for $i, j+1 > 0$.

Proof of Theorem 2.2.1 By Theorem 2.1.7, the Thom spectrum $(\Omega^2 S^3)^\mu$ is the free $\boldsymbol{E}_2$–ring with a nullhomotopy of p. Since $H\mathbb{F}_p$ is an $\boldsymbol{E}_2$–ring with a nullhomotopy of p, we obtain an $\boldsymbol{E}_2$–map $(\Omega^2 S^3)^\mu \to H\mathbb{F}_p$. To prove that this map is a p–complete equivalence, it suffices to prove that it induces an isomorphism on mod p homology.

The mod p homology of $(\Omega^2 S^3)^\mu$ can be calculated directly via the Thom isomorphism $H\mathbb{F}_p \wedge (\Omega^2 S^3)^\mu \simeq H\mathbb{F}_p \wedge \Sigma^\infty_+ \Omega^2 S^3$. Note that this is *not* an equivalence as $H\mathbb{F}_p \wedge H\mathbb{F}_p$–comodules: the Thom twisting is highly nontrivial.

For simplicity, we will now specialize to the case $p=2$, although the same proof works at odd primes. The homology of $\Omega^2 S^3$ is classical: it is a polynomial ring generated by applying $\boldsymbol{E}_2$–Dyer–Lashof operations to a single generator x_1 in degree 1. Theorem 2.2.4 implies that the same is true for the mod 2 Steenrod algebra: it, too, is a polynomial ring generated by applying $\boldsymbol{E}_2$–Dyer–Lashof operations to the single generator $\zeta_1 = \xi_1$ in degree 1. Since the map $(\Omega^2 S^3)^\mu \to H\mathbb{F}_2$ is an $\boldsymbol{E}_2$–ring map, it preserves $\boldsymbol{E}_2$–Dyer–Lashof operations on mod p homology. By the above discussion, it suffices to show that the generator $x_1 \in H_*(\Omega^2 S^3)^\mu \cong H_*(\Omega^2 S^3)$ in degree 1 is mapped to $\zeta_1 \in H_* H\mathbb{F}_2$.

To prove this, note that x_1 is the image of the generator in degree 1 in homology under the double suspension $S^1 \to \Omega^2 S^3$ and that ζ_1 is the image of the generator in degree 1 in homology under the canonical map $\mathbb{S}/p \to H\mathbb{F}_p$. It therefore suffices to show that the Thom spectrum of the spherical fibration $S^1 \to B\mathrm{GL}_1(\mathbb{S}^\wedge_p)$ detecting $1-p$ is simply $\mathbb{S}/p$. This is an easy exercise. □

Remark 2.2.5 When $p=2$, one does not need to p–complete in Theorem 2.2.1: the map $S^1 \to B\mathrm{GL}_1(\mathbb{S}^\wedge_2)$ factors as $S^1 \to BO \to B\mathrm{GL}_1(\mathbb{S})$, where the first map detects the Möbius bundle over S^1 and the second map is the J–homomorphism.

Notation 2.2.6 Let $\mathcal{Q}_1$ denote the (operadic nerve of the) *cup*-1 *operad* from [62, Example 1.3.6]: this is the operad whose n^{th} space is empty unless $n=2$, in which case it is S^1 with the antipodal action of Σ_2. We will need to slightly modify the definition of $\mathcal{Q}_1$ when localized at an odd prime p: in this case, it will denote the operad whose n^{th} space is a point if $n<p$, empty if $n>p$, and, when $n=p$, is the ordered configuration space $\mathrm{Conf}_p(\mathbb{R}^2)$ with the permutation action of Σ_p. Any homotopy commutative ring admits the structure of a $\mathcal{Q}_1$–algebra at $p=2$, but at other primes it is slightly stronger to be a $\mathcal{Q}_1$–algebra than to be a homotopy commutative ring. If $k \ge 2$, any $\boldsymbol{E}_k$–algebra structure on a spectrum restricts to a $\mathcal{Q}_1$–algebra structure.

Remark 2.2.7 As stated in [62, Proposition 1.5.29], the operation Q_1 already exists in the mod 2 homology of any $\mathcal{Q}_1$*–ring* R, where $\mathcal{Q}_1$ is the cup-1 operad from Notation 2.2.6 — the entire $\boldsymbol{E}_2$–structure is not necessary. With our modification of $\mathcal{Q}_1$ at odd primes as in Notation 2.2.6, this is also true at odd primes.

Remark 2.2.8 We will again for the moment specialize to $p = 2$ for convenience. Steinberger's calculation in Theorem 2.2.4 can be rephrased as stating that $Q_1\zeta_i = \zeta_{i+1}$, where Q_1 is the *lower-indexed* Dyer–Lashof operation. (See [26, page 59] for this notation.) As in Remark 2.2.7, the operation Q_1 already exists in the mod p homology of any $\mathcal{Q}_1$–ring R. Since homotopy commutative rings are $\mathcal{Q}_1$–algebras in spectra, this observation can be used to prove results of Würgler [95, Theorem 1.1] and Pazhitnov and Rudyak [82, Theorem in Introduction].

Remark 2.2.9 The argument with Dyer–Lashof operations and Theorem 2.2.4 used in the proof of Theorem 2.2.1 will be referred to as the *Dyer–Lashof hopping argument*. It will be used again (in the same manner) in the proof of Theorem A.

Remark 2.2.10 Theorem 2.2.1 is *equivalent* to Steinberger's calculation (Theorem 2.2.4), as well as to Bökstedt's calculation of $\mathrm{THH}(\mathbb{F}_p)$ (as a ring spectrum, and not just the calculation of its homotopy). Let us sketch an argument. First, Theorem 2.2.4 implies Theorem 2.2.1 (by the proof above). The other direction (ie the calculation (2-2)) can be argued by observing that the Thom isomorphism $H\mathbb{F}_p \wedge H\mathbb{F}_p \simeq H\mathbb{F}_p \wedge \Sigma^\infty_+\Omega^2 S^3$ is an equivalence of $\boldsymbol{E}_2$–$H\mathbb{F}_p$–algebras, so that the Dyer–Lashof operations are determined by the operations in $H_*(\Omega^2 S^3; \mathbb{F}_p)$. But the Dyer–Lashof operations are *defined* by classes in $H_*(\Omega^2 S^3; \mathbb{F}_p)$, and Theorem 2.2.4 is a consequence of the fact that the iterates of Q_1 on the generator of $H_1(\Omega^2 S^3; \mathbb{F}_p)$ describe all the polynomial generators $H_*(\Omega^2 S^3; \mathbb{F}_p)$.

It remains to argue that Theorem 2.2.1 is equivalent to the calculation that $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[\Omega S^3]$ as an $\boldsymbol{E}_1$–$\mathbb{F}_p$–algebra. This is shown in [58, Remark 1.5].

2.3 No-go theorems for higher chromatic heights

In light of Theorem 2.2.1 and Corollary 2.2.2, it is natural to wonder if appropriate higher chromatic analogues of $H\mathbb{F}_p$ and $H\mathbb{Z}$, such as $\mathrm{BP}\langle n\rangle$, *bo* or tmf, can be realized as Thom spectra of spherical fibrations. The answer is known to be negative (see [71; 88; 27]) in many cases:

Theorem 2.3.1 (Mahowald, Rudyak and Chatham) *There is no space X with a spherical fibration $\mu\colon X \to B\mathrm{GL}_1(\mathbb{S})$ (even after completion) such that X^μ is equivalent to* $\mathrm{BP}\langle 1\rangle$ *or bo. Moreover, there is no 2–local loop space X' with a spherical fibration determined by an H–map $\mu\colon X' \to B\mathrm{GL}_1(\mathbb{S}^\wedge_2)$ such that X'^μ is equivalent to* $\mathrm{tmf}^\wedge_2$.

The proofs rely on calculations in the unstable homotopy groups of spheres.

Remark 2.3.2 Although not written down anywhere, a slight modification of the argument used by Mahowald to show that *bu* is not the Thom spectrum of a spherical fibration over a loop space classified by an H–map can be used to show that $\mathrm{BP}\langle 2\rangle$ at $p = 2$ (ie $\mathrm{tmf}_1(3)$) is not the Thom spectrum of a spherical fibration over a loop space classified by an H–map. We do not know a proof that $\mathrm{BP}\langle n\rangle$ is not the Thom spectrum of a spherical fibration over a loop space classified by an H–map for all $n \geq 1$ and all primes, but we strongly suspect this to be true.

Remark 2.3.3 A lesser-known no-go result, due to Priddy, appears in [64, Chapter 2.11], where it is shown that BP cannot appear as the Thom spectrum of a double loop map $\Omega^2 X \to B\mathrm{GL}_1(\mathbb{S})$. In fact, the argument shows that the same result is true with BP replaced by $\mathrm{BP}\langle n\rangle$ for $n \geq 1$; we had independently come up with this argument for $\mathrm{BP}\langle 1\rangle$ before learning about Priddy's argument. Since Lewis's thesis is not particularly easy to acquire, we give a sketch of Priddy's argument. By the Thom isomorphism and the calculation (see [63, Theorem 4.3] as well as [94, Proposition 1.7; 13, Proposition 5.3])

$$H_*(\mathrm{BP}\langle n-1\rangle; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^2, \dots, \zeta_{n-1}^2, \zeta_n^2, \zeta_{n+1}, \dots] & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(\tau_n, \tau_{n+1}, \dots) & \text{if } p > 2, \end{cases}$$

we find that the mod p homology of $\Omega^2 X$ would be isomorphic as an algebra to a polynomial ring on infinitely many generators, possibly tensored with an exterior algebra on infinitely many generators. The Eilenberg–Moore spectral sequence then implies that the mod p cohomology of X is given by

$$H^*(X; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[b_1, \dots, b_n, c_{n+1}, \dots] & \text{if } p = 2, \\ \mathbb{F}_p[b_1, b_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(c_{n+1}, \dots) & \text{if } p > 2, \end{cases}$$

where $|b_i| = 2p^i$ and $|c_i| = 2p^{i-1} + 1$. If p is odd, then, since $|b_1| = 2p$, we have $\mathrm{P}^p(b_1) = b_1^p$. Liulevicius's formula for P^1 in terms of secondary cohomology operations [65, Theorem 1] allows us to write $\mathrm{P}^p(b_1)$ as a sum $c_0\mathcal{R}(b_1) + \sum_\gamma c_{0,\gamma}\Gamma_\gamma(b_1)$, where $\mathcal{R}(b_1)$ is a coset in $H^{2p+4(p-1)}(X; \mathbb{F}_p)$ and Γ_γ is an operation of odd degree, so that $\Gamma_\gamma(b_1)$ is in odd degree. We will not need to know what exactly the sum is indexed by, or what any of these operations are. Observe that Γ_γ kills b_1 because everything is concentrated in even degrees in the relevant range, and $\mathcal{R}$ also kills b_1 since $|\mathcal{R}(b_1)| = 4(p-1) + 2p^i$ is never a sum of numbers of the form $2p^k$ when $p > 2$. Using this, one can conclude that $b_1^p = 0$, which is a contradiction. A similar calculation works at $p = 2$, using Adams' study of secondary mod 2 cohomology operations in [1].

Remark 2.3.4 Using the calculations of $\mathrm{THH}(bo)$ and $\mathrm{THH}(ku)$ from [11], Angeltveit, Hill and Lawson [10] show that neither bo nor ku can appear as the Thom spectrum of a double loop map $\Omega^2 X \to B\mathrm{GL}_1(\mathbb{S})$.

Our primary goal in this project is to argue that the issues in Theorem 2.3.1 are alleviated if we replace $B\mathrm{GL}_1(\mathbb{S})$ with the delooping of the space of units of an appropriate replacement of $\mathbb{S}$. In the next section, we will construct these replacements of $\mathbb{S}$.

3 Some Thom spectra

In this section, we introduce certain $\boldsymbol{E}_1$–rings; most of them appear as Thom spectra over the sphere. Table 2 summarizes the spectra introduced in this section and gives references to their locations in the text. The spectra A and B were introduced in [34].

Thom spectrum	definition	“height”	BP–homology
$T(n)$	Theorem 3.1.5	n	Theorem 3.1.5
$y(n)$ and $y_{\mathbb{Z}}(n)$	Definition 3.2.2	n	Proposition 3.2.3
A	Definition 3.2.8	1	Proposition 3.2.13
B	Definition 3.2.18	2	Proposition 3.2.21

Table 2: Certain Thom spectra and their homologies.

3.1 Ravenel's $X(n)$ spectra

The proof of the nilpotence theorem in [38; 54] crucially relied upon certain Thom spectra arising from Bott periodicity; these spectra first appeared in Ravenel's seminal paper [84].

Definition 3.1.1 Let $X(n)$ denote the Thom spectrum of the $\boldsymbol{E}_2$–map $\Omega\mathrm{SU}(n) \subseteq BU \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S})$, where the first map arises from Bott periodicity.

Example 3.1.2 The $\boldsymbol{E}_2$–ring $X(1)$ is the sphere spectrum, while $X(\infty)$ is MU. Since the map $\Omega\mathrm{SU}(n) \to BU$ is an equivalence in dimensions $\leq 2n-2$, the same is true for the map $X(n) \to MU$; the first dimension in which $X(n)$ has an element in its homotopy which is not detected by MU is $2n-1$.

Remark 3.1.3 The $\boldsymbol{E}_2$–structure on $X(n)$ does *not* extend to an $\boldsymbol{E}_3$–structure (see [62, Example 1.5.31]). If $X(n)$ admits such an $\boldsymbol{E}_3$–structure, then the induced map $H_*(X(n)) \to H_*(H\mathbb{F}_p)$ on mod p homology would commute with $\boldsymbol{E}_3$–Dyer–Lashof operations. However, we know that the image of $H_*(X(n))$ in $H_*(H\mathbb{F}_p)$ is $\mathbb{F}_p[\zeta_1^2, \dots, \zeta_n^2]$; since Steinberger's calculation (Theorem 2.2.4) implies that $Q_2(\zeta_i^2) = \zeta_{i+1}^2$ via the relation $Q_2(x^2) = Q_1(x)^2$, we find that the image of $H_*(X(n))$ in $H_*(H\mathbb{F}_p)$ cannot be closed under the $\boldsymbol{E}_3$–Dyer–Lashof operation Q_2.

Remark 3.1.4 The proof of the nilpotence theorem shows that each of the $X(n)$ detects nilpotence. However, it is known (see [84, Theorem 3.1]) that $\langle X(n)\rangle > \langle X(n+1)\rangle$.

After localizing at a prime p, the spectrum MU splits as a wedge of suspensions of BP; this splitting comes from the Quillen idempotent on MU. The same is true of the $X(n)$ spectra, as explained in [85, Section 6.5]: a multiplicative map $X(n)_{(p)} \to X(n)_{(p)}$ is determined by a polynomial $f(x) = \sum_{0\leq i\leq n-1} a_i x^{i+1}$ with $a_0 = 1$ and $a_i \in \pi_{2i}(X(n)_{(p)})$. One can use this to define a truncated form of the Quillen idempotent ϵ_n on $X(n)_{(p)}$ (see [50, Proposition 1.3.7]), and thereby obtain a summand of $X(n)_{(p)}$. We summarize the necessary results in the following theorem:

Theorem 3.1.5 *Let n be such that $p^n \leq k \leq p^{n+1}-1$. Then $X(k)_{(p)}$ splits as a wedge of suspensions of the spectrum $T(n) = \epsilon_{p^n} \cdot X(p^n)_{(p)}$.*

- *The map $T(n) \to \mathrm{BP}$ is an equivalence in dimensions $\leq |v_{n+1}|-2$, so there is an indecomposable element $v_i \in \pi_* T(n)$ which maps to an indecomposable element in $\pi_*\mathrm{BP}$ for $0 \leq i \leq n$.*

- *This map induces the inclusion* $\mathrm{BP}_*T(n) = \mathrm{BP}_*[t_1, \dots, t_n] \subseteq \mathrm{BP}_*(\mathrm{BP})$ *on* BP*–homology, and the inclusions* $\mathbb{F}_2[\zeta_1^2, \dots, \zeta_n^2] \subseteq \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots]$ *and* $\mathbb{F}_p[\zeta_1, \dots, \zeta_n] \subseteq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$ *on mod* 2 *and mod* p *homology.*
- $T(n)$ *is a homotopy associative and* $\mathcal{Q}_1$*–algebra spectrum.*

Remark 3.1.6 It is known that $T(n)$ admits the structure of an $\boldsymbol{E}_1$–ring (see [21, Section 7.5]). We will interpret the phrase "Thom spectrum X^μ of a map $\mu\colon X \to B\mathrm{GL}_1(T(n))$" where μ arises via a map $X \xrightarrow{\mu_0} B\mathrm{GL}_1(X(p^{n+1}-1))$ to mean the base change $X^{\mu_0} \wedge_{X(p^{n+1}-1)} T(n)$.

It is believed that $T(n)$ in fact admits more structure (see [9, Section 6] for some discussion):

Conjecture 3.1.7 *The* $\mathcal{Q}_1$*–ring structure on* $T(n)$ *extends to an* $\boldsymbol{E}_2$*–ring structure.*

Remark 3.1.8 This is true at $p = 2$ and $n = 1$. Indeed, in this case $T(1) = X(2)$ is the Thom spectrum of the bundle given by the 2–fold loop map $\Omega S^3 = \Omega^2 B\mathrm{SU}(2) \to BU$, induced by the inclusion $B\mathrm{SU}(2) \to B^3U = B\mathrm{SU}$.

Remark 3.1.9 Conjecture 3.1.7 is true at $p = 2$ and $n = 2$. The Stiefel manifold $V_2(\mathbb{H}^2)$ sits in a fiber sequence

$$S^3 \to V_2(\mathbb{H}^2) \to S^7.$$

There is an equivalence $V_2(\mathbb{H}^2) \simeq \mathrm{Sp}(2)$, so $\Omega V_2(\mathbb{H}^2)$ admits the structure of a double loop space. There is an $\boldsymbol{E}_2$–map $\mu\colon \Omega V_2(\mathbb{H}^2) \to BU$, given by taking double loops of the composite

$$B\mathrm{Sp}(2) \to B\mathrm{SU}(4) \to B\mathrm{SU} \simeq B^3U.$$

The map μ admits a description as the left vertical map in the map of fiber sequences

$$\begin{tikzcd} \Omega V_2(\mathbb{H}^2) \arrow[r] \arrow[d, "\mu"] & \Omega S^7 \arrow[r] \arrow[d] & S^3 \arrow[d] \\ BU \arrow[r] & * \arrow[r] & B^2U \end{tikzcd}$$

Here, the map $S^3 \to B^2U$ detects the generator of $\pi_2(BU)$ (which maps to $\eta \in \pi_2(B\mathrm{GL}_1(\mathbb{S}))$ under the J–homomorphism). The Thom spectrum $\Omega V_2(\mathbb{H}^2)^\mu$ is equivalent to $T(2)$, and it follows that $T(2)$ admits the structure of an $\boldsymbol{E}_2$–ring. We do not know whether $T(n)$ is the Thom spectrum of a p–complete spherical fibration over some space for $n \geq 3$.

It is possible to construct $X(n+1)$ as an $X(n)$–algebra (see also [20]):

Construction 3.1.10 There is a fiber sequence

$$\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega S^{2n+1}.$$

According to Proposition 2.1.6, the spectrum $X(n+1)$ is the Thom spectrum of an $\boldsymbol{E}_1$–map $\Omega S^{2n+1} \to B\mathrm{GL}_1(\Omega\mathrm{SU}(n))^\mu = B\mathrm{GL}_1(X(n))$. This $\boldsymbol{E}_1$–map is the extension of a map $S^{2n} \to B\mathrm{GL}_1(X(n))$ which detects an element $\chi_n \in \pi_{2n-1} X(n)$. This element is equivalently determined by the map $\Sigma^\infty_+ \Omega^2 S^{2n+1} \to X(n)$ given by the Thomification of the nullhomotopic composite

$$\Omega^2 S^{2n+1} \to \Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega\mathrm{SU} \simeq BU,$$

where the first two maps form a fiber sequence. By Proposition 2.1.6, $X(n+1)$ is the free $\boldsymbol{E}_1$–$X(n)$–algebra with a nullhomotopy of χ_n.

Remark 3.1.11 Another construction of the map $\chi_n \in \pi_{2n-1} X(n)$ from Construction 3.1.10 is as follows. There is a map $i : \mathbb{C}P^{n-1} \to \Omega\mathrm{SU}(n)$ given by sending a line $\ell \subseteq \mathbb{C}^n$ to the loop $S^1 \to \mathrm{SU}(n) = \mathrm{Aut}(\mathbb{C}^n, \langle\, , \rangle)$ defined as follows: $\theta \in S^1$ is sent to the (appropriate rescaling of the) unitary transformation of $\mathbb{C}^n$ sending a vector to its rotation around the line ℓ by the angle θ. The map i Thomifies to a stable map $\Sigma^{-2}\mathbb{C}P^n \to X(n)$. The map χ_n is then the composite

$$S^{2n-1} \to \Sigma^{-2}\mathbb{C}P^n \to X(n),$$

where the first map is the desuspension of the generalized Hopf map $S^{2n+1} \to \mathbb{C}P^n$ which attaches the top cell of $\mathbb{C}P^{n+1}$. The fact that this map is indeed χ_n follows immediately from the commutativity of the diagram

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & \mathbb{C}P^{n-1} \arrow[r] \arrow[d] & \mathbb{C}P^n \arrow[d] \\ \Omega^2 S^{2n+1} \arrow[r] & \Omega\mathrm{SU}(n) \arrow[r] & \Omega\mathrm{SU}(n+1) \end{tikzcd} \tag{3-1}$$

where the top row is a cofiber sequence and the bottom row is a fiber sequence.

An easy consequence of the observation in Construction 3.1.10 is the following lemma:

Lemma 3.1.12 *Let $\sigma_n \in \pi_{|v_{n+1}|-1} T(n)$ denote the element $\chi_{p^{n+1}-1}$. Then the Thom spectrum of the composite $\Omega S^{|v_{n+1}|+1} \to B\mathrm{GL}_1(X(p^{n+1}-1)) \to B\mathrm{GL}_1(T(n))$ is equivalent to $T(n+1)$.*

Example 3.1.13 The element $\sigma_0 \in \pi_{|v_1|-1} T(0) = \pi_{2p-3}\mathbb{S}_{(p)}$ is α_1.

Example 3.1.14 Let us specialize to $p = 2$. Theorem 3.1.5 implies that $H_* T(n) \cong \mathbb{F}_2[\zeta_1^2, \dots, \zeta_n^2]$. Using this, one can observe that the 6–skeleton of $T(1)$ is the smash product $C\eta \wedge C\nu$, and so $\sigma_1 \in \pi_5(C\eta \wedge C\nu)$. This element can be described very explicitly: the cell structure of $C\eta \wedge C\nu$ is shown in Figure 1, and the element σ_1 shown corresponds to the map defined by the relation $\eta\nu = 0$.

Example 3.1.15 The element σ_n in the Adams–Novikov spectral sequence for $T(n)$ is represented by the element $[t_{n+1}]$ in the cobar complex. See [86, Section 1], where σ_{n-1} is denoted by $\alpha(\hat{v}_1/p)$.

A calculation with the Adams–Novikov spectral sequence (as in [86, Theorem 3.17]) proves the following:

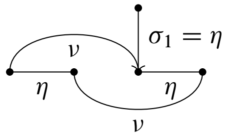

Figure 1: $C\eta \wedge C\nu$ shown horizontally, with 0–cell on the left. The element σ_1 is given by the map η on the 4–cell defined by a nullhomotopy of $\eta\nu = 0 \in \pi_4(S^0)$, as indicated in (3-1).

Lemma 3.1.16 *The class σ_{n-1} is killed by p in $\pi_{|v_n|-1} X(p^n - 1)$.*

Proof The argument is essentially the same as the classical observation that $\alpha_1 \in \pi_{2p-3}(S^0)$ is simple p–torsion. As mentioned in Example 3.1.15, $\sigma_{n-1} = \alpha(\hat{v}_1/p)$ in the notation of [86]. If

$$\Gamma(n+1) = \mathrm{BP}_*(\mathrm{BP})/(t_1, \dots, t_n)$$

denotes the Hopf algebroid of [86] — so that $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}\big(\mathrm{BP}_*, \mathrm{BP}_*(T(n))\big) \cong \mathrm{Ext}_{\Gamma(n+1)}(\mathrm{BP}_*, \mathrm{BP}_*)$ — then α is the connecting homomorphism in cohomology over $\Gamma(n+1)$ for the short exact sequence

$$0 \to \mathrm{BP}_* \to p^{-1}\mathrm{BP}_* \to p^{-1}\mathrm{BP}_*/\mathrm{BP}_* \to 0.$$

Since $\hat{v}_1/p$ is of order p in $p^{-1}\mathrm{BP}_*/\mathrm{BP}_*$, we see that $\alpha(\hat{v}_1/p)$ is of order p in the E_2–page of the Adams–Novikov spectral sequence computing $\pi_* T(n)$. The class $\alpha(\hat{v}_1/p)$ survives to the E_∞–page; one observes there are no possible additive extensions, so $p\sigma_{n-1} = 0 \in \pi_* T(n)$. □

In particular, the element $\sigma_{n-1} = \chi_{p^n-1} \in \pi_{|v_n|-1} X(p^n - 1)$ is p–torsion, and the following is a consequence of Example 3.1.15:

Proposition 3.1.17 *The class $\sigma_{n-1} \in \pi_{|v_n|-1} X(p^n - 1)$ is null in $\pi_* X(p^n)$, and the Toda bracket $\langle p, \sigma_{n-1}, 1_{X(p^n)} \rangle$ in $\pi_{|v_n|} X(p^n)$ contains an indecomposable v_n.*

Corollary 3.1.18 *The element $\sigma_{n-1} \in \pi_{|v_n|-1} X(p^n - 1)$ lifts to $\pi_{|v_n|+1}(\mathbb{C}P^{|v_n|/2})$ along the map $\Sigma^{-2}\mathbb{C}P^{|v_n|/2} \to X(p^n - 1)$.*

Proof By Remark 3.1.11, the map $\sigma_{n-1} \colon S^{|v_n|-1} \to X(p^n - 1)$ is given by the composite of the generalized Hopf map $S^{|v_n|-1} \to \Sigma^{-2}\mathbb{C}P^{p^n-1}$ with the map $\Sigma^{-2}\mathbb{C}P^{p^n-1} \to X(p^n - 1)$. Moreover, this generalized Hopf map is the desuspension of the unstable generalized Hopf map $S^{|v_n|+1} \to \mathbb{C}P^{p^n-1}$, and so σ_{n-1} lifts to an element of the unstable homotopy group $\pi_{|v_n|+1}(\mathbb{C}P^{|v_n|/2})$. □

3.2 Related Thom spectra

We now introduce several Thom spectra related to the $\boldsymbol{E}_1$–rings $T(n)$ described in the previous section; some of these were introduced in [34]. (Relationships to $T(n)$ will be further discussed in Section 6.2.) For the reader's convenience, we have included a table of the spectra introduced below with internal references to their definitions at the beginning of this section.

Remark 3.2.1 Recall (eg from [85, Section 4.4]) that, under the map

$$\mathrm{BP}_*(\mathrm{BP}) \cong \mathrm{BP}_*[t_1, t_2, \dots] \to H_*(\mathrm{BP}; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots] & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots] & \text{if } p > 2, \end{cases}$$

the class t_i is sent to ζ_i^2 (resp. ζ_i) modulo decomposables when $p = 2$ (resp. $p > 2$). Moreover, under the map

$$H_*(\mathrm{BP}; \mathbb{F}_p) \to H_*(H\mathbb{F}_p; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1, \zeta_2, \dots] & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots] \otimes E(\tau_0, \tau_1, \dots) & \text{if } p > 2, \end{cases}$$

the classes ζ_{i+1} (resp. τ_i) at $p = 2$ (resp. $p > 2$) detect a nullhomotopy of $v_i \in \pi_{2p^i-2}\mathrm{BP}$ in $H\mathbb{F}_p \otimes H\mathbb{F}_p$. This implies, for instance, that, if X is a spectrum such that $\mathrm{BP}_*(X) \simeq \mathrm{BP}_*/(p, \dots, v_{j-1})[t_1, \dots, t_m]$ with $j \leq m$, then

$$H_*(X; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[\zeta_1, \dots, \zeta_m] \otimes E(\tau_1, \dots, \tau_{j-1}) & \text{if } p > 2, \\ \mathbb{F}_2[\zeta_1^2, \dots, \zeta_j^2, \zeta_{j+1}, \dots, \zeta_m] & \text{if } p = 2. \end{cases}$$

The following Thom spectrum was introduced in [73]:

Definition 3.2.2 Let $y(n)$ denote the Thom spectrum of the composite

$$\Omega J_{p^n-1}(S^2) \to \Omega^2 S^3 \xrightarrow{1-p} B\mathrm{GL}_1(\mathbb{S}_p^\wedge).$$

If $J_{p^n-1}(S^2)\langle 2\rangle$ denotes the 2–connected cover of $J_{p^n-1}(S^2)$, then let $y_{\mathbb{Z}}(n)$ denote the Thom spectrum of the composite

$$\Omega J_{p^n-1}(S^2)\langle 2\rangle \to \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \xrightarrow{1-p} B\mathrm{GL}_1(\mathbb{S}_p^\wedge),$$

so that both $y(n)$ and $y_{\mathbb{Z}}(n)$ admit the structure of $\boldsymbol{E}_1$–rings via [15, Corollary 3.2].

Proposition 3.2.3 *As* $\mathrm{BP}_*\mathrm{BP}$*–comodules, we have*

$$\mathrm{BP}_*(y(n)) \cong \mathrm{BP}_*/I_n[t_1, \dots, t_n], \quad \mathrm{BP}_*(y_{\mathbb{Z}}(n)) \cong \mathrm{BP}_*/(v_1, \dots, v_{n-1})[t_1, \dots, t_n],$$

where I_n denotes the invariant ideal $(p, v_1, \dots, v_{n-1})$.

Proof The claim for $y(n)$ is [73, equation 2.8]. There is an equivalence $y_{\mathbb{Z}}(n)/p \simeq y(n)$, so that $\mathrm{BP}_*(y_{\mathbb{Z}}(n))/p \simeq \mathrm{BP}_*(y(n))$. The Bockstein spectral sequence collapses, and the extensions on the E_∞–page simply place p in filtration 1. This implies the second equivalence. □

One corollary is the following; this can be deduced from Proposition 3.2.3 using Remark 3.2.1. We also refer to [8, Lemma 2.3] for a direct proof.

Corollary 3.2.4 *As A_*–comodules, we have*

$$H_*(y(n); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1, \zeta_2, \dots, \zeta_n] & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_0, \dots, \tau_{n-1}) & \text{if } p \geq 3, \end{cases}$$

and

$$H_*(y_{\mathbb{Z}}(n); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots, \zeta_n] & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_1, \dots, \tau_{n-1}) & \text{if } p \geq 3. \end{cases}$$

We will now relate $y(n)$ and $y_{\mathbb{Z}}(n)$ to $T(n)$.

Construction 3.2.5 Let $m \le n$, and let I_m be the ideal generated by $p, v_1, \dots, v_{m-1}$, where the v_i are some choices of indecomposables in $\pi_{|v_i|}(T(n))$ which form a regular sequence. Inductively define $T(n)/I_m$ as the cofiber of the map

$$T(n)/I_{m-1} \xrightarrow{v_m \wedge 1} T(n) \wedge T(n)/I_{m-1} \to T(n)/I_{m-1}.$$

The BP–homology of $T(n)/I_m$ is $\mathrm{BP}_*/I_m[t_1, \dots, t_n]$. The spectrum $T(n)/(v_1, \dots, v_{m-1})$ is defined similarly.

Proposition 3.2.6 *Let $p > 2$. There is an equivalence between $T(n)/I_n$ (resp. $T(n)/(v_1, \dots, v_{n-1})$) and the spectrum $y(n)$ (resp. $y_{\mathbb{Z}}(n)$) of Definition 3.2.2.*

Proof We will prove the result for $y(n)$; the analogous proof works for $y_{\mathbb{Z}}(n)$. By [43], the space $\Omega J_{p^n-1}(S^2)$ is homotopy commutative (since $p > 2$). Moreover, the map $\Omega J_{p^n-1}(S^2) \to \Omega^2 S^3$ is an H–map, so $y(n)$ is a homotopy commutative $\boldsymbol{E}_1$–ring spectrum. It is known (see [85, Section 6.5]) that homotopy commutative maps $T(n) \to y(n)$ are equivalent to partial complex orientations of $y(n)$, ie factorizations

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[dr, "1"'] & \Sigma^{-2}\mathbb{C}P^{p^n-1} \arrow[d, dashed, "\gamma_n"] \\ & y(n) \end{tikzcd}$$

Such a γ_n indeed exists by obstruction theory: Suppose $k < p^n - 1$ and we have a map $\Sigma^{-2}\mathbb{C}P^k \to y(n)$. Since there is a cofiber sequence

$$S^{2k-1} \to \Sigma^{-2}\mathbb{C}P^k \to \Sigma^{-2}\mathbb{C}P^{k+1}$$

of spectra, the obstruction to extending along $\Sigma^{-2}\mathbb{C}P^{k+1}$ is an element of $\pi_{2k-1}y(n)$. However, the homotopy of $y(n)$ is concentrated in even degrees in the appropriate range, so a choice of γ_n does indeed exist. Moreover, this choice can be made such that they fit into a compatible family in the sense that there is a commutative diagram

$$\begin{tikzcd} \Sigma^{-2}\mathbb{C}P^{p^n-1} \arrow[r] \arrow[d, "\gamma_n"'] & \Sigma^{-2}\mathbb{C}P^{p^{n+1}-1} \arrow[d, "\gamma_{n+1}"] \\ y(n) \arrow[r] & y(n+1) \end{tikzcd}$$

The formal group law over $H\mathbb{F}_p$ has infinite height; this forces the elements $p, v_1, \dots, v_{n-1}$ (defined for the "(p^n-1)–bud" on $\pi_* y(n)$) to vanish in the homotopy of $y(n)$. It follows that the orientation $T(n) \to y(n)$ constructed above factors through the quotient $T(n)/I_n$. The induced map $T(n)/I_n \to y(n)$ can be seen to be an isomorphism on homology (via, for instance, Definition 3.2.2 and Construction 3.2.5). □

Remark 3.2.7 Since $y(n)$ has a v_n–self-map, we can form the spectrum $y(n)/v_n$; its mod p homology is

$$H_*(y(n)/v_n; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbb{F}_2}(\zeta_{n+1}) & \text{if } p = 2, \\ \mathbb{F}_p[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \dots, \tau_{n-1}, \tau_n) & \text{if } p \geq 3. \end{cases}$$

It is in fact possible to give a construction of $y(1)/v_1$ as a spherical Thom spectrum. We will work at $p = 2$ for convenience. Define Q to be the fiber of the map $2\eta\colon S^3 \to S^2$. There is a map of fiber sequences

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & S^3 \arrow[r, "2\eta"] \arrow[d] & S^2 \arrow[d] \\ B\mathrm{GL}_1(\mathbb{S}) \arrow[r] & * \arrow[r] & B^2\mathrm{GL}_1(\mathbb{S}) \end{tikzcd}$$

By [31, Theorem 3.7], the Thom spectrum of the leftmost map is $y(1)/v_1$.

We end this section by recalling the definition of two Thom spectra which, unlike $y(n)$ and $y_{\mathbb{Z}}(n)$, are not indexed by integers (we will see that they are only defined at "heights 1 and 2"). These were both studied in [34].

Definition 3.2.8 Let $S^4 \to B\mathrm{Spin}$ denote the generator of $\pi_4 B\mathrm{Spin} \cong \mathbb{Z}$, and let $\Omega S^5 \to B\mathrm{Spin}$ denote the extension of this map, which classifies a real vector bundle of virtual dimension zero over ΩS^5. Let A denote the Thom spectrum of this bundle.

Remark 3.2.9 As mentioned in the introduction, the spectrum A has been intensely studied by Mahowald and his coauthors in (for instance) [67; 31; 69; 68; 70; 77], where it is often denoted by X_5.

Remark 3.2.10 The map $\Omega S^5 \to B\mathrm{Spin}$ is one of $\boldsymbol{E}_1$–spaces, so the Thom spectrum A admits the structure of an $\boldsymbol{E}_1$–ring with an $\boldsymbol{E}_1$–map $A \to M\mathrm{Spin}$.

Remark 3.2.11 There are multiple equivalent ways to characterize this Thom spectrum. For instance, the J–homomorphism $B\mathrm{Spin} \to B\mathrm{GL}_1(\mathbb{S})$ sends the generator of $\pi_4 B\mathrm{Spin}$ to $\nu \in \pi_4 B\mathrm{GL}_1(\mathbb{S}) \cong \pi_3\mathbb{S}$. The universal property of Thom spectra in Theorem 2.1.7 shows that A is the free $\boldsymbol{E}_1$–ring $\mathbb{S}/\!\!/\nu$ with a nullhomotopy of ν. Note that A is defined integrally, and not just p–locally for some prime p.

Remark 3.2.12 There is a canonical map $A \to T(1)$ of $\boldsymbol{E}_1$–rings, constructed as follows. By the universal property of A, it suffices to prove that the unit $\mathbb{S} \to T(1)$ extends along the inclusion $\mathbb{S} \to C\nu$, ie that $\nu = 0 \in \pi_3 T(1)$ up to units. To see this, let us compute $\pi_3 C\eta$ via the exact sequence

$$\pi_3 S^1 \xrightarrow{\eta} \pi_3 S^0 \to \pi_3 C\eta \to \pi_2 S^0 \xrightarrow{\eta} \pi_1 S^0.$$

This can be identified with

$$\mathbb{Z}/2\{\eta^2\} \xrightarrow{\eta} \mathbb{Z}/8\{\nu\} \to \pi_3 C\eta \to \mathbb{Z}/2\{\eta\} \xrightarrow{\eta} \mathbb{Z}/2\{\eta^2\};$$

the final map is an isomorphism and the first map sends $\eta^2 \mapsto \eta^3 = 4\nu$. Therefore, $\pi_3 C\eta \cong \mathbb{Z}/4\{\nu\}$. Now, since the class in $H_4(T(1); \mathbb{F}_2)$ is detected by a nontrivial Sq^4, the attaching map of the 4–cell

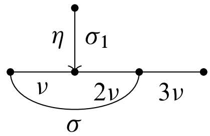

Figure 2: 15–skeleton of A at the prime 2 shown horizontally, with 0–cell on the left. The element σ_1 given by the map η on the 4–cell, as indicated in the diagram above.

in $T(1)$ must be $\pm\nu$. Therefore, one of $\pm\nu$ must be null in $T(1)$, which implies that there must be a map $C\nu \to T(1)$ (or $C(-\nu) \to T(1)$), as claimed.

The following result is [34, Proposition 2.7]; it is proved there at $p=2$, but the argument clearly works for $p=3$ too:

Proposition 3.2.13 *There is an isomorphism* $\mathrm{BP}_*(A) \cong \mathrm{BP}_*[y_2]$, *where* $|y_2| = 4$. *There is a map* $A_{(p)} \to \mathrm{BP}$. *Under the induced map on* BP*–homology*, y_2 *maps to* t_1^2 *mod decomposables at* $p=2$, *and to* t_1 *mod decomposables at* $p=3$.

Remark 3.2.14 For instance, when $p=2$, we have $\mathrm{BP}_*(A) \cong \mathrm{BP}_*[t_1^2 + v_1 t_1]$.

One corollary (using Remark 3.2.1) is the following:

Corollary 3.2.15 *As* A_**–comodules, we have*

$$H_*(A;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^4] & \text{if } p=2, \\ \mathbb{F}_3[\zeta_1] & \text{if } p=3, \\ \mathbb{F}_p[x_4] & \text{if } p \geq 5, \end{cases}$$

where x_4 *is a polynomial generator in degree* 4.

Example 3.2.16 Let us work at $p=2$ for convenience. Example 3.1.14 showed that σ_1 is the element in $\pi_5(C\eta \wedge C\nu)$ given by the lift of η to the 4–cell (which is attached to the bottom cell by ν) via a nullhomotopy of $\eta\nu$. In particular, σ_1 already lives in $\pi_5(C\nu)$ and, as such, defines an element of $\mathbb{S}/\!\!/\nu = A$ (by viewing $C\nu$ as the 4–skeleton of A); note that, by construction, this element is 2–torsion. The image of $\sigma_1 \in \pi_5(A)$ under the canonical map of Remark 3.2.12 is its namesake in $\pi_5(T(1))$. See Figure 2.

Remark 3.2.17 The element $\sigma_1 \in \pi_5(A_{(2)})$ defined in Example 3.2.16 in fact lifts to an element of $\pi_5(A)$, because the relation $\eta\nu = 0$ is true integrally and not just 2–locally. An alternative construction of this map is the following: The Hopf map $\eta_4\colon S^5 \to S^4$ (which lives in the stable range) defines a map $S^5 \to S^4 \to \Omega S^5$ whose composite to $B\mathrm{Spin}$ is null (since $\pi_5(B\mathrm{Spin}) = 0$). Upon Thomification of the composite $S^5 \to \Omega S^5 \to B\mathrm{Spin}$, one therefore gets a map $S^5 \to A$ whose composite with $A \to M\mathrm{Spin}$ is null. The map $S^5 \to A$ is the element $\sigma_1 \in \pi_5(A)$.

Finally, we have:

Definition 3.2.18 Let BN be the space defined by the homotopy pullback

$$\begin{tikzcd} BN \arrow[r] \arrow[d] & S^{13} \arrow[d, "f"] \\ BO(9) \arrow[r] & BO(10) \end{tikzcd}$$

where the map $f\colon S^{13} \to BO(10)$ detects an element of $\pi_{12}O(10) \cong \mathbb{Z}/12$. There is a fiber sequence

$$S^9 \to BO(9) \to BO(10),$$

and the image of f under the boundary map in the long exact sequence of homotopy detects $2\nu \in \pi_{12}(S^9) \cong \mathbb{Z}/24$. In particular, there is a fiber sequence

$$S^9 \to BN \to S^{13}.$$

If N is defined to be ΩBN, then there is a fiber sequence

$$N \to \Omega S^{13} \to S^9.$$

Define a map $N \to B\mathrm{String}$ via the map of fiber sequences

$$\begin{tikzcd} N \arrow[r] \arrow[d] & \Omega S^{13} \arrow[r] \arrow[d] & S^9 \arrow[d] \\ B\mathrm{String} \arrow[r] & * \arrow[r] & B^2\mathrm{String} \end{tikzcd}$$

where the map $S^9 \to B^2\mathrm{String}$ detects a generator of $\pi_8 B\mathrm{String}$. Let B denote the Thom spectrum of the induced bundle over N.

Remark 3.2.19 The map $N \to B\mathrm{String}$ is in fact one of $\boldsymbol{E}_1$–spaces, so B admits the structure of an $\boldsymbol{E}_1$–ring. To prove this, it suffices to show that there is a map $BN \to B^2\mathrm{String}$. Recall that $B\mathrm{String} = \tau_{\geq 8}\Omega^\infty KO$, so the desired map is the same as a class in $KO^1(BN)$. Using the Serre spectral sequence for the fiber sequence defining BN, one can calculate that there is a class in $KO^1(BN)$ which lifts the generator of $KO^1(S^9) \cong \pi_8 KO \cong \mathbb{Z}$.

We introduced the spectrum B and studied its Adams–Novikov spectral sequence in [34]. The Steenrod module structure of the 20–skeleton of B is shown in [34, Figure 1], and is reproduced here as Figure 3. As mentioned in the introduction, the spectrum B has been briefly studied under the name $\overline{X}$ in [72].

Remark 3.2.20 As with A, there are multiple different ways to characterize B. There is a fiber sequence

$$\Omega S^9 \to N \to \Omega S^{13},$$

and the map $\Omega S^9 \to N \to B\mathrm{String}$ is an extension of the map $S^8 \to B\mathrm{String}$ detecting a generator. Under the J–homomorphism $B\mathrm{String} \to B\mathrm{GL}_1(\mathbb{S})$, this generator maps to $\sigma \in \pi_8 B\mathrm{GL}_1(\mathbb{S}) \cong \pi_7\mathbb{S}$, so

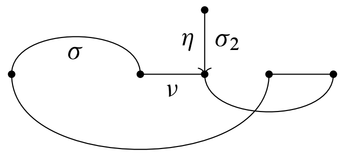

Figure 3: Steenrod module structure of the 20–skeleton of B; the bottom cell (in dimension 0) is on the left; straight lines are Sq^4, and curved lines correspond to Sq^8 and Sq^{16}, in order of increasing length. The bottom two attaching maps of B are labeled. The map σ_2 is shown.

the Thom spectrum of the bundle over ΩS^9 determined by the map $\Omega S^9 \to B\mathrm{String}$ is the free $\boldsymbol{E}_1$–ring $\mathbb{S}/\!\!/\sigma$ with a nullhomotopy of ν. Proposition 2.1.6 now implies that N is the Thom spectrum of a map $\Omega S^{13} \to B\mathrm{GL}_1(\mathbb{S}/\!\!/\sigma)$. While a direct definition of this map is not obvious, we note that the restriction to the bottom cell S^{12} of the source detects an element $\widetilde{\nu}$ of $\pi_{12} B\mathrm{GL}_1(\mathbb{S}/\!\!/\sigma) \cong \pi_{11}\mathbb{S}/\!\!/\sigma$. This in turn factors through the 11–skeleton of $\mathbb{S}/\!\!/\sigma$, which is the same as the 8–skeleton of $\mathbb{S}/\!\!/\sigma$ (namely $C\sigma$). This element is precisely a lift of the map $\nu\colon S^{11} \to S^8$ to $C\sigma$ determined by a nullhomotopy of $\sigma\nu$ in $\pi_*\mathbb{S}$. Although $\widetilde{\nu} \in \pi_{11} C\sigma$ does not come from a class in $\pi_{11}\mathbb{S}$, its representative in the Adams spectral sequence for $C\sigma$ is the image of h_{22} in the Adams spectral sequence for the sphere.

The following result is [34, Proposition 3.2]; it is proved there at $p = 2$, but the argument clearly works for $p \geq 3$ too:

Proposition 3.2.21 *The* BP_**–algebra* $\mathrm{BP}_*(B)$ *is isomorphic to a polynomial ring* $\mathrm{BP}_*[b_4, y_6]$, *where* $|b_4| = 8$ *and* $|y_6| = 12$. *There is a map* $B_{(p)} \to \mathrm{BP}$. *On* BP_**–homology, the elements* b_4 *and* y_6 *map to* t_1^4 *and* t_2^2 *mod decomposables at* $p = 2$, *and* y_6 *maps to* t_1^3 *mod decomposables at* $p = 3$.

One corollary (using Remark 3.2.1) is the following:

Corollary 3.2.22 *As* A_**–comodules, we have*

$$H_*(B; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^8, \zeta_2^4] & \text{if } p = 2, \\ \mathbb{F}_3[\zeta_1^3, b_4] & \text{if } p = 3, \\ \mathbb{F}_5[\zeta_1, x_{12}] & \text{if } p = 5, \\ \mathbb{F}_p[x_8, x_{12}] & \text{if } p \geq 7, \end{cases}$$

where x_8 *and* x_{12} *are polynomial generators in degrees* 8 *and* 12, *and* b_4 *is an element in degree* 8.

Example 3.2.23 For simplicity, let us work at $p = 2$. There is a canonical ring map $B \to T(2)$, and the element $\sigma_2 \in \pi_{13} T(2)$ lifts to B. We can be explicit about this: the 12–skeleton of B is shown in Figure 3, and σ_2 is the element of $\pi_{13}(B)$ that exists thanks to the relation $\eta\nu = 0$ and the fact that the Toda bracket $\langle \eta, \nu, \sigma \rangle$ contains 0. This also shows that $\sigma_2 \in \pi_{13}(B)$ is 2–torsion.

Remark 3.2.24 The element $\sigma_2 \in \pi_{13}(B_{(2)})$ defined in Example 3.2.23 in fact lifts to an element of $\pi_{13}(B)$, because the relations $\nu\sigma = 0$, $\eta\nu = 0$ and $0 \in \langle \eta, \nu, \sigma \rangle$ are all true integrally and not just 2–locally. An alternative construction of this map $S^{13} \to B$ is the following: The Hopf map $\eta_{12}\colon S^{13} \to S^{12}$ (which lives in the stable range) defines a map $S^{13} \to S^{12} \to \Omega S^{13}$. Moreover, the composite $S^{13} \to \Omega S^{13} \to S^9$ is null, since it detects an element of $\pi_{13}(S^9) = 0$; choosing a nullhomotopy of this composite defines a lift $S^{13} \to N$. (In fact, this comes from a map $S^{14} \to BN$.) The composite $S^{13} \to N \to B\mathrm{String}$ is null (since $\pi_{13}(B\mathrm{String}) = 0$). Upon Thomification, we obtain a map $S^{13} \to B$ whose composite with $B \to M\mathrm{String}$ is null; the map $S^{13} \to B$ is the element $\sigma_2 \in \pi_{13}(B)$.

The following theorem packages some information contained in this section:

Theorem 3.2.25 *Let R denote any of the spectra in Table 2, and let n denote its "height". If $R = T(n)$, $y(n)$ or $y_{\mathbb{Z}}(n)$, then there is a map $T(n) \to R$ and, if $R = A$ (resp. B), then there is a map from R to $T(1)$ (resp. $T(2)$). In the first three cases, there is an element $\sigma_n \in \pi_{|v_{n+1}|-1}R$ coming from $\sigma_n \in \pi_{|v_{n+1}|-1}T(n)$ and, in the cases $R = A$ and B, there are elements $\sigma_1 \in \pi_5(A)$ and $\sigma_2 \in \pi_{13}(B)$ mapping to the corresponding elements in $T(1)_{(2)}$ and $T(2)_{(2)}$, respectively. Moreover, σ_n is p–torsion in $\pi_* R$; similarly, σ_1 and σ_2 are 2–torsion in $\pi_* A_{(2)}$ and $\pi_* B_{(2)}$.*

Proof The existence statement for $T(n)$ is contained in Theorem 3.1.5, while the torsion statement is the content of Lemma 3.1.16. The claims for $y(n)$ and $y_{\mathbb{Z}}(n)$ now follow from Proposition 3.2.6. The existence and torsion statements for A and B are contained in Examples 3.2.16 and 3.2.23. □

The elements in Theorem 3.2.25 can in fact be extended to infinite families; this is discussed in Section 5.4.

3.3 Centers of Thom spectra

In this section, we review some of the theory of $\boldsymbol{E}_k$–centers and state Conjecture E. We begin with the following important result, and refer to [40; 66, Section 5.5.4] for proofs:

Theorem 3.3.1 [66, Example 5.5.4.16; 40, Definition 2.5] *Let $\mathcal{C}$ be a symmetric monoidal presentable ∞–category and let A be an $\boldsymbol{E}_k$–algebra in $\mathcal{C}$. Then the category of $\boldsymbol{E}_k$–A–modules is equivalent to the category of left modules over the factorization homology $U(A) = \int_{S^{k-1}\times\mathbb{R}} A$* (known as the *enveloping algebra* of A), *which is an $\boldsymbol{E}_1$–algebra in $\mathcal{C}$.*

Definition 3.3.2 The $\boldsymbol{E}_{k+1}$*–center* $\mathfrak{Z}(A)$ of an $\boldsymbol{E}_k$–algebra A in $\mathcal{C}$ is the ($\boldsymbol{E}_{k+1}$–)Hochschild cohomology $\mathrm{End}_{U(A)}(A)$, where A is regarded as a left module over its enveloping algebra via Theorem 3.3.1.

Remark 3.3.3 We are using slightly different terminology than that used by Lurie [66, Section 5.3]: our $\boldsymbol{E}_{k+1}$–center is his $\boldsymbol{E}_k$–center. In other words, Lurie's terminology expresses the structure on the input, while our terminology expresses the structure on the output.

The following proposition summarizes some results from [40; 66, Section 5.3]:

Proposition 3.3.4 [66, Theorem 5.3.2.5; 40, Theorem 1.1] *The $\boldsymbol{E}_{k+1}$–center $\mathfrak{Z}(A)$ of an $\boldsymbol{E}_k$–algebra A in a symmetric monoidal presentable ∞–category $\mathcal{C}$ exists, and satisfies the following properties:*

(a) *$\mathfrak{Z}(A)$ is the universal $\boldsymbol{E}_k$–algebra of $\mathcal{C}$ which fits into a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr, equal] & A \otimes \mathfrak{Z}(A) \arrow[d] \\ & A \end{tikzcd}$$

in $\mathrm{Alg}_{\boldsymbol{E}_k}(\mathcal{C})$.

(b) *The $\boldsymbol{E}_k$–algebra $\mathfrak{Z}(A)$ of $\mathcal{C}$ defined via this universal property in fact admits the structure of an $\boldsymbol{E}_{k+1}$–algebra in $\mathcal{C}$.*

(c) *There is a fiber sequence*

$$\mathrm{GL}_1(\mathfrak{Z}(A)) \to \mathrm{GL}_1(A) \to \Omega^{k-1}\,\mathrm{End}_{\mathrm{Alg}_{\boldsymbol{E}_k}(\mathcal{C})}(A)$$

of k–fold loop spaces.

In the sequel, we will need a more general notion:

Definition 3.3.5 Let $m \geq 1$. The *$\boldsymbol{E}_{k+m}$–center* $\mathfrak{Z}_{k+m}(A)$ of an $\boldsymbol{E}_k$–algebra A in a presentable symmetric monoidal ∞–category $\mathcal{C}$ with all limits is defined inductively as the $\boldsymbol{E}_{k+m}$–center of the $\boldsymbol{E}_{k+m-1}$–center $\mathfrak{Z}_{k+m-1}(A)$. In other words, it is the universal $\boldsymbol{E}_{k+m}$–algebra of $\mathcal{C}$ which fits into a commutative diagram

$$\begin{tikzcd} \mathfrak{Z}_{k+m-1}(A) \arrow[r] \arrow[dr, equal] & \mathfrak{Z}_{k+m-1}(A) \otimes \mathfrak{Z}_{k+m}(A) \arrow[d] \\ & \mathfrak{Z}_{k+m-1}(A) \end{tikzcd}$$

in $\mathrm{Alg}_{\boldsymbol{E}_{k+m-1}}(\mathcal{C})$.

Proposition 3.3.4 gives:

Corollary 3.3.6 *Let $m \geq 1$. The $\boldsymbol{E}_{k+m-1}$–algebra $\mathfrak{Z}_{k+m}(A)$ associated to an $\boldsymbol{E}_k$–algebra object A of $\mathcal{C}$ exists and, in fact, admits the structure of an $\boldsymbol{E}_{k+m}$–algebra in $\mathcal{C}$.*

We can now finally state Conjecture E:

Conjecture E *Let $n \geq 0$ be an integer. Let R denote $X(p^{n+1}-1)_{(p)}$, A (in which case $n = 1$) or B (in which case $n = 2$). Then the element $\sigma_n \in \pi_{|\sigma_n|} R$ lifts to the $\boldsymbol{E}_3$–center $\mathfrak{Z}_3(R)$ of R, and is p–torsion in $\pi_* \mathfrak{Z}_3(R)$ if $R = X(p^{n+1}-1)_{(p)}$, and is 2–torsion in $\pi_* \mathfrak{Z}_3(R)$ if $R = A$ or B.*

Remark 3.3.7 If R is A or B, then $\mathfrak{Z}_3(R)$ is the $\boldsymbol{E}_3$–center of the $\boldsymbol{E}_2$–center of R. This is a rather unwieldy object, so it would be quite useful to show that the $\boldsymbol{E}_1$–structure on A or B admits an extension to an $\boldsymbol{E}_2$–structure; we do not know if such extensions exist. Since neither ΩS^5 nor N admits the structure of a double loop space, such an $\boldsymbol{E}_2$–structure would not arise from their structure as Thom spectra. In any case, if such extensions do exist, then $\mathfrak{Z}_3(R)$ in Conjecture E should be interpreted as the $\boldsymbol{E}_3$–center of the $\boldsymbol{E}_2$–ring R. However, we showed in [35, Theorem 4.2] that $(\mathrm{tmf} \wedge A)[x_2]$ admits an $\boldsymbol{E}_2$–algebra structure, where $|x_2| = 2$.

Remark 3.3.8 In the introduction, we stated Conjecture 1.1.4, which instead asked about whether $v_n \in \pi_{|v_n|} X(p^n)$ lifts to $\pi_* \mathfrak{Z}_3(X(p^n))$. It is natural to ask about the connection between Conjectures E and 1.1.4. Proposition 3.1.17 implies that, if $\mathfrak{Z}_3(X(p^n))$ admitted an $X(p^n-1)$–orientation factoring the canonical $X(p^n-1)$–orientation $X(p^n-1) \to X(p^n)$, and $\sigma_{n-1} \in \pi_{|v_n|-1} X(p^n-1)$ was killed by the map $X(p^n-1) \to \mathfrak{Z}_3(X(p^n))$, then Conjecture E would imply Conjecture 1.1.4. However, we do not believe that either of these statements are true.

Remark 3.3.9 One of the main results of [56] implies that the $\boldsymbol{E}_3$–center of $X(n)$ — which, recall, is the Thom spectrum of a bundle over $\Omega^2 B\mathrm{SU}(n)$ — is $\mathrm{Hom}_{\mathrm{SU}(n)_+}(\mathbb{S}, X(n)) \simeq X(n)^{h\mathrm{SU}(n)}$, where $\mathrm{SU}(n)$ acts on $X(n)$ by a Thomification of the conjugation action on $\Omega\mathrm{SU}(n)$.

Remark 3.3.10 The conjugation action of $\mathrm{SU}(n)$ on $X(n)$ can be described very explicitly, via a concrete model for $\Omega\mathrm{SU}(n)$. As explained in [83; 96], if G is a reductive linear algebraic group over $\mathbb{C}$, the loop space $\Omega G(\mathbb{C})$ of its complex points (viewed as a complex Lie group) is equivalent to the homogeneous space $G(\mathbb{C}((t)))/G(\mathbb{C}[\![t]\!])$; this is also commonly studied as the complex points of the affine Grassmannian Gr_G of G. The conjugation action of $G(\mathbb{C})$ on $\Omega G(\mathbb{C})$ arises by restricting the descent (to $G(\mathbb{C}((t)))/G(\mathbb{C}[\![t]\!])$) of the translation action by $G(\mathbb{C}[\![t]\!])$ on $G(\mathbb{C}((t)))$ to the subgroup $G(\mathbb{C}) \subseteq G(\mathbb{C}[\![t]\!])$. Setting $G = \mathrm{SL}_n$ gives a description of the conjugation action of $\mathrm{SU}(n)$ on $\Omega\mathrm{SU}(n)$. In light of its connections to geometric representation theory, we believe that there may be an algebrogeometric approach to proving that χ_n is $\mathrm{SU}(n)$–trivial in $X(n)$ and in $\Omega\mathrm{SU}(n)$.

Example 3.3.11 The element $\chi_2 \in \pi_3 X(2)$ is central. To see this, note that $\alpha \in \pi_* R$ (where R is an $\boldsymbol{E}_k$–ring) is in the $\boldsymbol{E}_{k+1}$–center of R if and only if α is in the $\boldsymbol{E}_{k+1}$–center of $R_{(p)}$ for all primes $p \geq 0$. It therefore suffices to show that χ_2 is central after p–localizing for all p. First, note that χ_2 is torsion, so it is nullhomotopic (and therefore central) after rationalization. Next, if $p > 2$, then $X(2)_{(p)}$ splits as a wedge of suspensions of spheres. If χ_2 is detected in π_3 of a sphere living in dimension 3, then it could not be torsion, so it must be detected in π_3 of a sphere living in dimension $3-k$ for some $0 \leq k \leq 2$. If $k = 1$ or 2, then $\pi_3(S^{3-k})$ is either $\pi_1(S^0)$ or $\pi_2(S^0)$, but both of these groups vanish for $p > 2$. Therefore, χ_2 must be detected in π_3 of the sphere in dimension 0, ie in $\pi_3 X(1)$. This group vanishes for $p > 3$, and when $p = 3$, it is isomorphic to $\mathbb{Z}/3$ (generated by α_1). Since $X(1) = S^0$ is an $\boldsymbol{E}_\infty$–ring, we conclude that χ_2 is central in $X(2)_{(p)}$ for all $p > 2$.

At $p=2$, we know the cell structure of $X(2)$ in the bottom few dimensions (see Example 3.1.14; note that σ_1 is *not* χ_2). In dimensions ≤ 3, it is equivalent to $C\eta$, so $\pi_3 X(2) \cong \pi_3 C\eta$. However, it is easy to see that the canonical map $\pi_3 \mathbb{S} \simeq \mathbb{Z}/8\{\nu\} \to \pi_3 C\eta$ is surjective and exhibits an isomorphism $\pi_3 C\eta \cong \mathbb{Z}/4\{\nu\}$. Therefore, χ_2 is in the image of the unit $\mathbb{S} \to X(2)$, and is therefore vacuously central. We conclude from the above discussion that χ_2 is indeed central in $X(2)$.

4 Review of some unstable homotopy theory

4.1 Charming and Gray maps

A major milestone in unstable homotopy theory was Cohen, Moore and Neisendorfer's result on the p–exponent of unstable homotopy groups of spheres from [29; 30; 81]. They defined for all $p > 2$ and $k \geq 1$ a map $\phi_n\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ (the integer k is assumed implicit) such that the composite of ϕ_n with the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ is homotopic to the $(p^k)^{\text{th}}$ power map. By induction on n, they concluded via a result of Selick's (see [89]) that p^n kills the p–primary component of the homotopy of S^{2n+1}. Such maps will be important in the rest of this article, so we will isolate their desired properties in the definition of a *charming map*, inspired by [90]. (Our choice of terminology is nonstandard, and admittedly horrible, but it does not seem like the literature has chosen any naming convention for the sort of maps we desire.)

Definition 4.1.1 A p–local map $f\colon \Omega^2 S^{2np+1} \to S^{2np-1}$ is called a *Gray map* if the composite of f with the double suspension E^2 is the degree p map, and the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

is nullhomotopic. Moreover, a p–local map $f\colon \Omega^2 S^{2np+1} \to S^{2np-1}$ is called a *charming map* if the composite of f with the double suspension E^2 is the degree p map, the fiber of f admits the structure of a $\mathcal{Q}_1$–space, and there is a space BK which sits in a fiber sequence

$$S^{2np-1} \to BK \to \Omega S^{2np+1}$$

such that the boundary map $\Omega^2 S^{2np+1} \to S^{2np-1}$ is homotopic to f.

Remark 4.1.2 If f is a charming map, then the fiber of f is a loop space. Indeed, $\mathrm{fib}(f) \simeq \Omega BK$.

Example 4.1.3 Let f denote the Cohen–Moore–Neisendorfer map with $k=1$. Anick proved (see [14; 45]) that the fiber of f admits a delooping, ie there is a space $T^{2np+1}(p)$ (now known as an *Anick space*) which sits in a fiber sequence

$$S^{2np-1} \to T^{2np+1}(p) \to \Omega S^{2np+1}.$$

It follows that f is a charming map.

Remark 4.1.4 We claim that $T^{2p+1}(p) = \Omega S^3\langle 3\rangle$, where $S^3\langle 3\rangle$ is the 3–connected cover of S^3. To prove this, we will construct a p–local fiber sequence

$$S^{2p-1} \to \Omega S^3\langle 3\rangle \to \Omega S^{2p+1}.$$

This fiber sequence was originally constructed by Toda [93]. To construct this fiber sequence, we first note that there is a p–local fiber sequence

$$S^{2p-1} \to J_{p-1}(S^2) \to \mathbb{C}P^\infty,$$

where the first map is the factorization of $\alpha_1\colon S^{2p-1} \to \Omega S^3$ through the $2(p-1)$–skeleton of ΩS^3, and the second map is the composite $J_{p-1}(S^2) \to \Omega S^3 \to \mathbb{C}P^\infty$. This fiber sequence is simply an odd primary version of the Hopf fibration $S^3 \to S^2 \to \mathbb{C}P^\infty$; the identification of the fiber of the map $J_{p-1}(S^2) \to \mathbb{C}P^\infty$ is a simple exercise with the Serre spectral sequence. Next, we have the EHP sequence

$$J_{p-1}(S^2) \to \Omega S^3 \to \Omega S^{2p+1}.$$

Since $\Omega S^3\langle 3\rangle$ is the fiber of the map $\Omega S^3 \to \mathbb{C}P^\infty$, the desired fiber sequence is obtained by taking vertical fibers in the map of fiber sequences

$$\begin{tikzcd}
J_{p-1}(S^2) \arrow[r] \arrow[d] & \Omega S^3 \arrow[r] \arrow[d] & \Omega S^{2p+1} \arrow[d] \\
\mathbb{C}P^\infty \arrow[r, equal] & \mathbb{C}P^\infty \arrow[r] & *
\end{tikzcd}$$

Example 4.1.5 Let W_n denote the fiber of the double suspension $S^{2n-1} \to \Omega^2 S^{2n+1}$. Gray [44; 42] proved that W_n admits a delooping BW_n, and that, after p–localization, there is a fiber sequence

$$BW_n \to \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

for some map f. As suggested by the naming convention, f is a Gray map.

As proved in [90], Gray maps satisfy an important rigidity property:

Proposition 4.1.6 (Selick and Theriault) *The fiber of any Gray map admits an H–space structure, and is H–equivalent to BW_n.*

Remark 4.1.7 It has been conjectured by Cohen, Moore, Neisendorfer and Gray in the papers cited above that there is an equivalence $BW_n \simeq \Omega T^{2np+1}(p)$, and that $\Omega T^{2np+1}(p)$ retracts off of $\Omega^2 P^{2np+1}(p)$ as an H–space, where $P^k(p)$ is the mod p Moore space $S^{k-1} \cup_p e^k$ with top cell in dimension k. For our purposes, we shall require something slightly stronger; namely, the retraction should be one of $\mathcal{Q}_1$–spaces. The first part of this conjecture would follow from Proposition 4.1.6 if the Cohen–Moore–Neisendorfer map were a Gray map. In [3], it is shown that the existence of p–primary elements of Kervaire invariant one would imply equivalences of the form $BW_{p^{n-1}} \simeq \Omega T^{2p^n+1}(p)$.

Motivated by Remark 4.1.7 and Proposition 4.1.6, we state the following conjecture; it is slightly weaker than the conjecture mentioned in Remark 4.1.7, and is an amalgamation of slight modifications of conjectures of Cohen, Moore, Neisendorfer, Gray and Mahowald in unstable homotopy theory, as well as an analogue of Proposition 4.1.6. (For instance, we strengthen having an H–space retraction to having a $\mathcal{Q}_1$–space retraction.)

Conjecture D *The following statements are true*:

(a) *The homotopy fiber of any charming map is equivalent as a loop space to the loop space on an Anick space.*

(b) *There exists a p–local charming map $f\colon \Omega^2 S^{2p^n+1} \to S^{2p^n-1}$ whose homotopy fiber admits a $\mathcal{Q}_1$–space retraction off of $\Omega^2 P^{2p^n+1}(p)$. There are also integrally defined maps $\Omega^2 S^9 \to S^7$ and $\Omega^2 S^{17} \to S^{15}$ whose composite with the double suspension on S^7 and S^{15}, respectively, is the degree 2 map, whose homotopy fibers K_2 and K_3 (respectively) admit deloopings, and which admits a $\mathcal{Q}_1$–space retraction off of $\Omega^2 P^9(2)$ and $\Omega^2 P^{17}(2)$ (respectively).*

Remark 4.1.8 Conjecture D is already not known when $n=1$. In this case, it asserts that $\Omega^2 S^3\langle 3\rangle$ retracts off of $\Omega^2 P^{2p+1}(p)$. A theorem of Selick's states that $\Omega^2 S^3\langle 3\rangle$ retracts off of $\Omega^2 S^{2p+1}\{p\}$ for p odd, where $\Omega^2 S^{2p+1}\{p\}$ is the fiber of the degree p map on $\Omega^2 S^{2p+1}$. This implies that $\Omega^2 S^3\langle 3\rangle$ retracts off of $\Omega^3 P^{2p+2}(p)$. In [28, Observation 9.2], the question of whether $\Omega^2 S^3\langle 3\rangle$ retracts off of $\Omega^2 P^{2p+1}(p)$ was shown to be equivalent to the question of whether there is a map $\Sigma^2\Omega^2 S^3\langle 3\rangle \to P^{2p+1}(p)$ which is onto in homology. Some recent results regarding Conjecture D for $n=1$ can be found in [23].

It follows that a retraction of $\Omega^2 S^3\langle 3\rangle$ off of $\Omega^2 P^{2p+1}(p)$ will be compatible with the canonical map $\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3$ in the following manner. The p–torsion element $\alpha_1 \in \pi_{2p}(S^3)$ defines a map $P^{2p-1}(p) \to \Omega^2 S^3$, which extends to an $\boldsymbol{E}_2$–map $\Omega^2 P^{2p+1}(p) \to \Omega^2 S^3$. We will abusively denote this extension by α_1. The resulting composite

$$\Omega^2 S^3\langle 3\rangle \to \Omega^2 P^{2p+1}(p) \xrightarrow{\alpha_1} \Omega^2 S^3$$

is homotopic to the canonical map $\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3$.

The element $\alpha_1 \in \pi_{2p-3}(\mathbb{S}_{(p)})$ defines a map $S^{2p-2} \to B\mathrm{GL}_1(\mathbb{S}_{(p)})$ and, since it is p–torsion, admits an extension to a map $P^{2p-1}(p) \to B\mathrm{GL}_1(\mathbb{S}_{(p)})$. (This extension is in fact unique, because $\pi_{2p-1}(B\mathrm{GL}_1(\mathbb{S}_{(p)})) \cong \pi_{2p-2}(\mathbb{S}_{(p)})$ vanishes.) Since $B\mathrm{GL}_1(\mathbb{S}_{(p)})$ is an infinite loop space, this map further extends to a map $\Omega^2 P^{2p+1}(p) \to B\mathrm{GL}_1(\mathbb{S}_{(p)})$. The discussion in the previous paragraph implies that, if Conjecture D is true for $n=1$, then the map $\mu\colon \Omega^2 S^3\langle 3\rangle \to B\mathrm{GL}_1(\mathbb{S}_{(p)})$ from Corollary 2.2.2 is homotopic to the composite

$$\Omega^2 S^3\langle 3\rangle \to \Omega^2 P^{2p+1}(p) \to B\mathrm{GL}_1(\mathbb{S}_{(p)}).$$

4.2 Fibers of charming maps

We shall need the following proposition:

Proposition 4.2.1 *Let $f\colon \Omega^2 S^{2p^n+1} \to S^{2p^n-1}$ be a charming map. Then there are isomorphisms of coalgebras*

$$H_*(\mathrm{fib}(f);\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[x^2_{2^{n+1}-1}] \otimes \bigotimes_{k>1} \mathbb{F}_2[x_{2^{n+k}-1}] & \text{if } p=2, \\ \bigotimes_{k>0} \mathbb{F}_p[y_{2(p^{n+k}-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbb{F}_p}[x_{2p^{n+j}-1}] & \text{if } p>2. \end{cases}$$

Proof This is an easy consequence of the Serre spectral sequence coupled with the well-known coalgebra isomorphisms

$$H_*(\Omega^2 S^{2n+1};\mathbb{F}_p) \cong \begin{cases} \bigotimes_{k>0} \mathbb{F}_2[x_{2^k n-1}] & \text{if } p=2, \\ \bigotimes_{k>0} \mathbb{F}_p[y_{2(np^k-1)}] \otimes \bigotimes_{j\geq 0} \Lambda_{\mathbb{F}_p}[x_{2np^j-1}] & \text{if } p>2, \end{cases}$$

where these classes are generated by the one in dimension $2n-1$ via the single Dyer–Lashof operation (coming already from the cup-1 operad; see Remark 2.2.8). □

Remark 4.2.2 The Anick spaces $T^{2np+1}(p)$ from Example 4.1.3 sit in fiber sequences

$$S^{2np-1} \to T^{2np+1}(p) \to \Omega S^{2np+1}$$

and are homotopy commutative H–spaces. A Serre spectral sequence calculation gives an identification of coalgebras

$$H_*(T^{2np+1}(p);\mathbb{F}_p) \cong \mathbb{F}_p[a_{2np}] \otimes \Lambda_{\mathbb{F}_p}[b_{2np-1}]$$

with $\beta(a_{2np}) = b_{2np-1}$, where β is the Bockstein homomorphism. An argument with the bar spectral sequence recovers the result of Proposition 4.2.1 in this particular case.

Remark 4.2.3 Suppose that X is a space which sits in a fiber sequence

$$S^{2np-1} \to X \to \Omega S^{2np+1}$$

such that the boundary map $\Omega^2 S^{2np+1} \to S^{2np-1}$ has degree p^j on the bottom cell of the source. The Serre spectral sequence then only has a differential on the E_{2np-1}–page, and

$$H_i(BK;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i=0, \\ \mathbb{Z}/p^j k & \text{if } i = 2npk-1, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude this section by investigating Thom spectra of bundles defined over fibers of charming maps. Let R be a p–local $\boldsymbol{E}_1$–ring and let $\mu\colon K \to B\mathrm{GL}_1(R)$ denote a map from the fiber K of a charming map $f\colon \Omega^2 S^{2np+1} \to S^{2np-1}$. There is a fiber sequence $\Omega S^{2np-1} \to K \to \Omega^2 S^{2np+1}$ of loop spaces, so we obtain a map $\Omega S^{2np-1} \to B\mathrm{GL}_1(R)$. Such a map gives an element $\alpha \in \pi_{2np-3} R$ via the effect on the bottom cell S^{2np-2}.

Theorem 2.1.7 implies that the Thom spectrum of the map $\Omega S^{2np-1} \to B\mathrm{GL}_1(R)$ should be thought of as the $\boldsymbol{E}_1$–quotient $R/\!/\alpha$, although this may not make sense if R is not at least $\boldsymbol{E}_2$. However, in many cases (such as the ones we are considering here), the Thom R–module $R/\!/\alpha$ is in fact an $\boldsymbol{E}_1$–ring such that the map $R \to R/\!/\alpha$ is an $\boldsymbol{E}_1$–map. By Proposition 2.1.6, there is an induced map $\phi\colon \Omega^2 S^{2np+1} \to B\mathrm{GL}_1(R/\!/\alpha)$ whose Thom spectrum is equivalent as an $\boldsymbol{E}_1$–ring to K^μ. We would like to determine the element[2] of $\pi_* R/\!/\alpha$ detected by the restriction to the bottom cell S^{2np-1} of the source of ϕ. First, we note:

Lemma 4.2.4 *The element $\alpha \in \pi_{2np-3} R$ is p–torsion.*

Proof Since f is a charming map, the composite $S^{2np-1} \to \Omega^2 S^{2np-1} \xrightarrow{f} S^{2np-1}$ is the degree p map. Therefore, the element $p\alpha \in \pi_{2np-3} R$ is detected by the composite

$$S^{2np-2} \to \Omega S^{2np-1} \to \Omega^3 S^{2np-1} \xrightarrow{\Omega f} \Omega S^{2np-1} \to K \xrightarrow{\mu} B\mathrm{GL}_1(R).$$

But there is a fiber sequence $\Omega^2 S^{2np-1} \xrightarrow{f} S^{2np-1} \to BK$ by the definition of a charming map, so the composite detecting $p\alpha$ is null, as desired. $\square$

There is now a square

$$\begin{tikzcd}
S^{2np-2}/p \arrow[r] \arrow[d] & S^{2np-1} \arrow[d] \\
K \arrow[r] \arrow[d, "\alpha"'] & \Omega^2 S^{2np+1} \arrow[d] \\
B\mathrm{GL}_1(R) \arrow[r] & B\mathrm{GL}_1(R/\!/\alpha)
\end{tikzcd}$$

and the following result is a consequence of the lemma and the definition of Toda brackets:

Lemma 4.2.5 *The element in $\pi_{2np-2}(R/\!/\alpha)$ detected by the vertical map $S^{2np-1} \to B\mathrm{GL}_1(R/\!/\alpha)$ lives in the Toda bracket $\langle p, \alpha, 1_{R/\!/\alpha}\rangle$.*

The upshot of this discussion is the following:

Proposition 4.2.6 *Let R be a p–local $\boldsymbol{E}_1$–ring and let $\mu\colon K \to B\mathrm{GL}_1(R)$ denote a map from the fiber K of a charming map $f\colon \Omega^2 S^{2np+1} \to S^{2np-1}$, providing an element $\alpha \in \pi_{2np-3} R$. Assume that the Thom spectrum $R/\!/\alpha$ of the map $\Omega S^{2np-1} \to B\mathrm{GL}_1(R)$ is an $\boldsymbol{E}_1$–R–algebra. Then there is an element $v \in \langle p, \alpha, 1_{R/\!/\alpha}\rangle$ such that K^μ is equivalent to the Thom spectrum of the map $\Omega^2 S^{2np+1} \xrightarrow{v} B\mathrm{GL}_1(R/\!/\alpha)$.*

[2]Technically, this is bad terminology: there are multiple possibilities for the map ϕ, and each gives rise to a map $S^{2np-1} \to B\mathrm{GL}_1(R/\!/\alpha)$. The elements in $\pi_{2np-2}(R/\!/\alpha)$ determined in this way need not agree, but they are the same modulo the indeterminacy of the Toda bracket $\langle p, \alpha, 1_{R/\!/\alpha}\rangle$.

Remark 4.2.7 Let R be an $\boldsymbol{E}_1$–ring and let $\alpha \in \pi_d R$. Then α defines a map $S^{d+1} \to B\mathrm{GL}_1(R)$, and it is natural to ask when α extends along $S^{d+1} \to \Omega S^{d+2}$, or at least along $S^{d+1} \to J_k(S^{d+1})$ for some k. This is automatic if R is an $\boldsymbol{E}_2$–ring, but not necessarily so if R is only an $\boldsymbol{E}_1$–ring. Recall that there is a cofiber sequence

$$S^{(k+1)(d+1)-1} \to J_k(S^{d+1}) \to J_{k+1}(S^{d+1}),$$

where the first map is the $(k+1)$–fold iterated Whitehead product $[\iota_{d+1}, [\dots, [\iota_{d+1}, \iota_{d+1}]], \dots]$. In particular, the map $S^{d+1} \to B\mathrm{GL}_1(R)$ extends along the map $S^{d+1} \to J_k(S^{d+1})$ if and only if there are compatible nullhomotopies of the n–fold iterated Whitehead products $[\alpha, [\dots, [\alpha, \alpha]], \dots] \in \pi_* B\mathrm{GL}_1(R)$ for $n \le k$. These amount to properties of Toda brackets in the homotopy of R. We note, for instance, that the Whitehead bracket $[\alpha, \alpha] \in \pi_{2d+1} B\mathrm{GL}_1(R) \cong \pi_{2d} R$ is the element $2\alpha^2$; therefore, the map $S^{d+1} \to B\mathrm{GL}_1(R)$ extends to $J_2(S^{d+1})$ if and only if $2\alpha^2 = 0$.

Remark 4.2.8 Let R be a p–local $\boldsymbol{E}_2$–ring and let $\alpha \in \pi_d(R)$ with d even. Then α defines an element $\alpha \in \pi_{d+2} B^2\mathrm{GL}_1(R)$. The p–fold iterated Whitehead product $[\alpha, \dots, \alpha] \in \pi_{p(d+2)-(p-1)} B^2\mathrm{GL}_1(R) \cong \pi_{pd+(p-1)} R$ is given by $p!Q_1(\alpha)$ modulo decomposables. This is in fact true more generally. Let R be an $\boldsymbol{E}_n$–ring and suppose $\alpha \in \pi_d(R)$. Let $i < n$, so α defines an element $\alpha \in \pi_{d+i} B^i\mathrm{GL}_1(R)$. The p–fold iterated Whitehead product $[\alpha, \dots, \alpha] \in \pi_{p(d+i)-(p-1)} B^i\mathrm{GL}_1(R) \cong \pi_{pd+(i-1)(p-1)} R$ is given by $p!Q_{i-1}(\alpha)$ modulo decomposables.

We will describe this in detail in forthcoming work; the basic idea is to reduce to the universal example of an $\boldsymbol{E}_n$–ring, and relate Whitehead products on $\pi_*(S^n)$ to the $\boldsymbol{E}_d$–Browder bracket on $\Omega^d S^n_+$ (where $d \ge n$). Recall the isomorphism $\pi_j S^n \cong \pi_{j-d}\Omega^d S^n$. If $\alpha \in \pi_i S^n$ and $\beta \in \pi_j S^n$, then we will show in future work that the stabilization of the Whitehead product $[\alpha, \beta] \in \pi_{i+j-1} S^n \cong \pi_{i+j-d}\Omega^d S^n$ is closely related to the $\boldsymbol{E}_d$–Browder bracket $[\alpha, \beta]_{\boldsymbol{E}_d}$.

5 Chromatic Thom spectra

5.1 Statement of the theorem

To state the main theorem of this section, we set some notation. Fix an integer $n \ge 1$ and work in the p–complete stable category. For each Thom spectrum R of height $n-1$ in Table 1, let $\sigma_{n-1} \colon S^{|\sigma_{n-1}|} \to B\mathrm{GL}_1(R)$ denote a map detecting $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|}(R)$ (which exists by Theorem 3.2.25). Let K_n denote the fiber of a p–local charming map $\Omega^2 S^{2p^n+1} \to S^{2p^n-1}$ satisfying the hypotheses of Conjecture D, and let K_2 (resp. K_3) denote the fiber of an integrally defined charming map $\Omega^2 S^9 \to S^7$ (resp. $\Omega^2 S^{17} \to S^{15}$) satisfying the hypotheses of Conjecture D.

Then:

Theorem A *Let R be a height $n-1$ spectrum as in the second line of Table 1. Then Conjectures D and E imply that there is a map $K_n \to B\mathrm{GL}_1(R)$ such that the mod p homology of the Thom spectrum K_n^μ is isomorphic to the mod p homology of the associated designer chromatic spectrum $\Theta(R)$ as a Steenrod comodule.*

If R is any base spectrum other than B, the Thom spectrum K_n^μ is equivalent to $\Theta(R)$ upon p–completion for every prime p. If Conjecture F is true, then the same is true for B: the Thom spectrum K_n^μ is equivalent to $\Theta(B) = \mathrm{tmf}$ upon 2–completion.

We emphasize again that naively making sense of Theorem A relies on knowing that $T(n)$ admits the structure of an $\boldsymbol{E}_1$–ring; we shall interpret this phrase as in Remark 3.1.6.

Remark 5.1.1 Theorem A is proved independently of the nilpotence theorem. (In fact, it is even independent of Quillen's identification of $\pi_* MU$ with the Lazard ring provided one regards the existence of designer chromatic spectra as being independent of Quillen's identification.) We shall elaborate on the connection between Theorem A and the nilpotence theorem in future work; a sketch is provided in Remark 5.4.7.

Remark 5.1.2 Theorem A is true unconditionally when $n = 1$, since that case is simply Corollary 2.2.2.

Remark 5.1.3 Table 2 implies that the homology of each of the Thom spectra in Table 1 are given by the Q_0–Margolis homology of their associated designer chromatic spectra. In particular, the map $R \to \Theta(R)$ is a rational equivalence.

Before we proceed with the proof of Theorem A, we observe some consequences.

Corollary B *Conjectures D and E imply Conjecture 1.1.3.*

Proof This follows from Theorem A and Propositions 4.2.6 and 3.1.17. □

Remark 5.1.4 Corollary B is true *unconditionally* when $n = 1$, since Theorem A is true unconditionally in that case by Remark 5.1.2. See also Remark 4.1.4.

Remark 5.1.5 We can attempt to apply Theorem A for $R = A$ in conjunction with Proposition 4.2.6. Theorem A states that Conjectures D and E imply that there is a map $K_2 \to B\mathrm{GL}_1(A)$ whose Thom spectrum is equivalent to bo. There is a fiber sequence

$$\Omega S^7 \to K_2 \to \Omega^2 S^9,$$

so we obtain a map $\mu\colon \Omega S^7 \to K_2 \to B\mathrm{GL}_1(A)$. The proof of Theorem A shows that the bottom cell S^6 of the source detects $\sigma_1 \in \pi_5(A)$. A slight variation of the argument used to establish Proposition 4.2.6 supplies a map $\Omega^2 S^9 \to B\mathrm{Aut}((\Omega S^7)^\mu)$ whose Thom spectrum is bo. The spectrum $(\Omega S^7)^\mu$ has mod 2 homology $\mathbb{F}_2[\zeta_1^4, \zeta_2^2]$. However, unlike A, it does not naturally arise an $\boldsymbol{E}_1$–Thom spectrum over the sphere spectrum; this makes it unamenable to study via techniques of unstable homotopy.

More precisely, $(\Omega S^7)^\mu$ is not the Thom spectrum of an $\boldsymbol{E}_1$–map $X \to B\mathrm{GL}_1(\mathbb{S})$ from a loop space X which sits in a fiber sequence

$$\Omega S^5 \to X \to \Omega S^7$$

of loop spaces. Indeed, BX would be a S^5–bundle over S^7, which, by [71, Lemma 4], implies that X is then equivalent as a loop space to $\Omega S^5 \times \Omega S^7$. The resulting $\boldsymbol{E}_1$–map $\Omega S^7 \to B\mathrm{GL}_1(\mathbb{S})$ is specified by an element of $\pi_5(\mathbb{S}) \cong 0$, so $(\Omega S^7)^\mu$ must then be equivalent as an $\boldsymbol{E}_1$–ring to $A \wedge \Sigma^\infty_+ \Omega S^7$. In particular, $\sigma_1 \in \pi_5(A)$ would map nontrivially to $(\Omega S^7)^\mu$, which is a contradiction.

The proof of Theorem A will also show:

Corollary 5.1.6 *Let R be a height $n-1$ spectrum as in the second line of Table 1, and assume Conjecture F if $R = B$. Let M be an $\boldsymbol{E}_3$–R–algebra. Conjectures D and E imply that if*

(a) *the composite $\mathfrak{Z}_3(R) \to R \to M$ is an $\boldsymbol{E}_3$–algebra map,*
(b) *the element σ_{n-1} in $\pi_* M$ is nullhomotopic, and*
(c) *the bracket $\langle p, \sigma_{n-1}, 1_M \rangle$ contains zero,*

then there is a unital map $\Theta(R) \to M$.

5.2 The proof of Theorem A

This section is devoted to giving a proof of Theorem A, dependent on Conjectures D and E. The proof of Theorem A will be broken down into multiple steps. The result for $y(n)$ and $y_{\mathbb{Z}}(n)$ follow from the result for $T(n)$ by Proposition 3.2.6, so we shall restrict ourselves to the cases of R being $T(n)$, A and B.

Fix $n \geq 1$. If R is A or B, we will restrict to $p = 2$, and let K_2 and K_3 denote the integrally defined spaces from Conjecture D. By Remarks 3.2.17 and 3.2.24, the elements $\sigma_1 \in \pi_5(A)$ and $\sigma_2 \in \pi_{13}(B)$ are defined integrally. We will write σ_{n-1} to generically denote this element, and will write it as living in degree $|\sigma_{n-1}|$. We shall also write R to denote $X(p^n - 1)$ and *not* $T(n)$; this will be so that we can apply Conjecture D. We apologize for the inconvenience, but hope that this is worth circumventing the task of having to read through essentially the same proofs for these slightly different cases.

Step 1 We begin by constructing a map $\mu\colon K_n \to B\mathrm{GL}_1(R)$ as required by the statement of Theorem A; the construction in the case $n = 1$ follows Remark 4.1.8. By Conjecture D, the space K_n splits off of $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ (if $R = T(n)$, then $|\sigma_{n-1}| + 4 = |v_n| + 3$). We are therefore reduced to constructing a map $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \to B\mathrm{GL}_1(R)$. Theorem 3.2.25 shows that the element $\sigma_{n-1} \in \pi_* R$ is p–torsion, so the map $S^{|\sigma_{n-1}|+1} \to B\mathrm{GL}_1(R)$ detecting σ_{n-1} extends to a map

$$S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \to B\mathrm{GL}_1(R). \tag{5-1}$$

Since $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \simeq \Omega^2 \Sigma^2 P^{|\sigma_{n-1}|+2}(p)$, we would obtain an extension $\widetilde{\mu}$ of this map through $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ if R admits an $\boldsymbol{E}_3$–structure.

Unfortunately, this is not true; but this is where Conjecture E comes in: it says that the element $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|} R$ lifts to the $\boldsymbol{E}_3$–center $\mathfrak{Z}_3(R)$, where it has the same torsion order as in R. (Here, we are abusively writing $\mathfrak{Z}_3(T(n-1))$ to denote the $\boldsymbol{E}_3$–center of $X(p^n-1)_{(p)}$.) The lifting of σ_{n-1} to $\pi_{|\sigma_{n-1}|}\mathfrak{Z}_3(R)$ provided by Conjecture E gives a factorization of the map from (5-1) as

$$S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \to B\mathrm{GL}_1(\mathfrak{Z}_3(R)) \to B\mathrm{GL}_1(R).$$

Since $\mathfrak{Z}_3(R)$ is an $\boldsymbol{E}_3$–ring, $B\mathrm{GL}_1(\mathfrak{Z}_3(R))$ admits the structure of an $\boldsymbol{E}_2$–space. In particular, the map $P^{|\sigma_{n-1}|+2}(p) \to B\mathrm{GL}_1(\mathfrak{Z}_3(R))$ factors through $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$, as desired. We let $\widetilde{\mu}$ denote the resulting composite

$$\widetilde{\mu}\colon \Omega^2 P^{|\sigma_{n-1}|+4}(p) \to B\mathrm{GL}_1(\mathfrak{Z}_3(R)) \to B\mathrm{GL}_1(R).$$

Step 2 Theorem A asserts that there is an identification between the Thom spectrum of the induced map $\mu\colon K_n \to B\mathrm{GL}_1(R)$ and the associated designer chromatic spectrum $\Theta(R)$ via Table 1. We shall identify the Steenrod comodule structure on the mod p homology of K_n^μ, and show that it agrees with the mod p homology of $\Theta(R)$.

In Table 3, we have recorded the mod p homology of the designer chromatic spectra in Table 1 (see [63, Theorem 4.3] for $\mathrm{BP}\langle n-1\rangle$). It follows from Proposition 4.2.1 that there is an isomorphism

$$H_*(K_n^\mu) \cong \begin{cases} H_*(R) \otimes \mathbb{F}_2[x_{2^{n+1}-1}^2] \otimes \bigotimes_{k>1} \mathbb{F}_2[x_{2^{n+k}-1}] & \text{if } p=2, \\ H_*(R) \otimes \bigotimes_{k>0} \mathbb{F}_p[y_{2(p^{n+k}-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbb{F}_p}[x_{2p^{n+j}-1}] & \text{if } p>2. \end{cases}$$

Combining this isomorphism with Theorem 3.1.5 and Corollaries 3.2.4, 3.2.15 and 3.2.22, we find that there is an *abstract* equivalence between the mod p homology of K_n^μ and the mod p homology of $\Theta(R)$.

We shall now work at $p=2$ for the remainder of the proof; the same argument goes through with slight modifications at odd primes. We now identify the Steenrod comodule structure on $H_*(K_n^\mu)$. Recall that $\widetilde{\mu}$ is the map $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \to B\mathrm{GL}_1(R)$ from Step 1. By construction, there is a map $K_n^\mu \to \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}$. The map Φ factors through a map $\widetilde{\Phi}\colon \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}} \to \Theta(R)$. The Thom spectrum $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}$ admits the structure of a $\mathfrak{Q}_1$–ring. Indeed, it is the smash product $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\phi} \wedge_{\mathfrak{Z}_3(R)} R$, where $\phi\colon \Omega^2 P^{|\sigma_{n-1}|+4}(p) \to B\mathrm{GL}_1(\mathfrak{Z}_3(R))$; it therefore suffices to observe that the Thom spectrum $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\phi}$ admits the structure of an $\boldsymbol{E}_1 \otimes \mathfrak{Q}_1$–ring. (Here, $\boldsymbol{E}_1 \otimes \mathfrak{Q}_1$ denotes the Boardman–Vogt tensor product of the $\boldsymbol{E}_1$– and $\mathfrak{Q}_1$–operads.) Since there is a map $\mathfrak{Q}_1 \to \boldsymbol{E}_2$ of ∞–operads, this is a consequence of the fact that ϕ is a double loop map, and hence an $\boldsymbol{E}_1 \otimes \mathfrak{Q}_1$–algebra map. Moreover, the image of $H_*(K_n^\mu)$ in $H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}})$ is generated under the single Dyer–Lashof operation (arising from the cup-1 operad; see Remark 2.2.8) by the indecomposables in the image of the map $H_*(R) \to H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}})$.

The Postnikov truncation map $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}} \to H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}})$ is one of $\mathfrak{Q}_1$–rings. Since $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ is highly connected, $\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}) \cong \pi_0(R)$. In particular, there is an $\boldsymbol{E}_\infty$–map $H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}) \to H\mathbb{F}_p$. The composite

$$\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}} \to H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}) \to H\mathbb{F}_p$$

designer chromatic spectrum	mod p homology	
$\mathrm{BP}\langle n-1\rangle$	$p=2$	$\mathbb{F}_2[\zeta_1^2,\dots,\zeta_{n-1}^2,\zeta_n^2,\zeta_{n+1},\dots]$
	$p>2$	$\mathbb{F}_p[\zeta_1,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_p}(\tau_n,\tau_{n+1},\dots)$
$k(n-1)$	$p=2$	$\mathbb{F}_2[\zeta_1,\dots,\zeta_{n-1},\zeta_n^2,\zeta_{n+1},\dots]$
	$p>2$	$\mathbb{F}_p[\zeta_1,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_p}(\tau_0,\dots,\tau_{n-2},\tau_n,\tau_{n+1},\dots)$
$k_{\mathbb{Z}}(n-1)$	$p=2$	$\mathbb{F}_2[\zeta_1^2,\zeta_2,\dots,\zeta_{n-1},\zeta_n^2,\zeta_{n+1},\dots]$
	$p>2$	$\mathbb{F}_p[\zeta_1,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_p}(\tau_1,\dots,\tau_{n-2},\tau_n,\tau_{n+1},\dots)$
bo	$p=2$	$\mathbb{F}_2[\zeta_1^4,\zeta_2^2,\zeta_3,\dots]$
	$p>2$	$\mathbb{F}_p[x_4]/v_1\otimes\mathbb{F}_p[\zeta_1,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_p}(\tau_2,\tau_3,\dots)$
tmf	$p=2$	$\mathbb{F}_2[\zeta_1^8,\zeta_2^4,\zeta_3^2,\zeta_4,\dots]$
	$p=3$	$\Lambda_{\mathbb{F}_3}(b_4)\otimes\mathbb{F}_3[\zeta_1^3,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_3}(\tau_3,\tau_4,\dots)$
	$p\geq 5$	$\mathbb{F}_p[c_4,c_6]/(v_1,v_2)\otimes\mathbb{F}_p[\zeta_1,\zeta_2,\dots]\otimes\Lambda_{\mathbb{F}_p}(\tau_3,\tau_4,\dots)$

Table 3: The mod p homology of designer chromatic spectra. See [63, Theorem 4.3], as well as [94, Proposition 1.7; 13, Proposition 5.3] for a proof of the statement for $H_*(\mathrm{BP}\langle n-1\rangle;\mathbb{F}_p)$; this implies the calculations of $H_*(k(n-1);\mathbb{F}_p)$ and $H_*(k_{\mathbb{Z}}(n-1);\mathbb{F}_p)$. See [13, Proposition 6.1] for a proof of the statements for $H_*(bo;\mathbb{F}_2)$ and $H_*(\mathrm{tmf};\mathbb{F}_2)$, and [87, Theorem 21.5] for $H_*(\mathrm{tmf};\mathbb{F}_p)$ for any p. For odd p, $bo_{(p)}$ is a sum of shifts of $\mathrm{BP}\langle 1\rangle$, which implies the statement about $H_*(bo;\mathbb{F}_p)$.

is therefore a $\mathcal{Q}_1$–algebra map. Moreover, the composite

$$R\to\Omega^2P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}}\to H\pi_0(\Omega^2P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}})\to H\mathbb{F}_p$$

is simply the Postnikov truncation for R. It follows that the indecomposables in $H_*(\Omega^2P^{|\sigma_{n-1}|+4}(p)^{\widetilde{\mu}})$ which come from the indecomposables in $H_*(R)$ are sent to the indecomposables in $H_*(H\mathbb{F}_p)$. Using the discussion in the previous paragraph, Steinberger's calculation (Theorem 2.2.4) and the Dyer–Lashof hopping argument of Remark 2.2.9, we may conclude that the Steenrod comodule structure on $H_*(K_n^{\mu})$ (which, recall, is abstractly isomorphic to $H_*(\Theta(R))$) agrees with the Steenrod comodule structure on $H_*(\Theta(R))$.

Step 3 By Step 2, the mod p homology of the Thom spectrum K_n^{μ} is isomorphic to the mod p homology of the associated designer chromatic spectrum $\Theta(R)$ as a Steenrod comodule. The main results of [12; 2, Theorem 1.1] now imply that, unless $R=B$, the Thom spectrum K_n^{μ} is equivalent to $\Theta(R)$ upon p–completion for every prime p. Finally, if Conjecture F is true, then the same conclusion can be drawn for B: the Thom spectrum K_n^{μ} is equivalent to $\Theta(B)=\mathrm{tmf}$ upon p–completion for every prime p.

This concludes the proof of Theorem A.

5.3 Remark on the proof

Before proceeding, we note the following consequence of the proof of Theorem A:

Proposition 5.3.1 *Let p be an odd prime. Assume Conjectures D and E. Then the composite*

$$g_2\colon \Omega^2 S^{|\sigma_{n-1}|+3} \to \Omega^2 P^{|\sigma_{n-1}|+4}(p) \xrightarrow{\tilde{\mu}} B\mathrm{GL}_1(X(p^n-1)) \to B\mathrm{GL}_1(\mathrm{BP}\langle n-1\rangle)$$

is null.

Proof Let $R = X(p^n-1)$ and $\Theta(R) = \mathrm{BP}\langle n-1\rangle$. The map g_2 is the composite of $B\mathrm{GL}_1(\mathfrak{Z}_3(R)) \to B\mathrm{GL}_1(\Theta(R))$ with the extension of the map

$$\sigma_{n-1}\colon S^{|\sigma_{n-1}|+1} \to B\mathrm{GL}_1(\mathfrak{Z}_3(R))$$

along the double suspension $S^{|\sigma_{n-1}|+1} \to \Omega^2 S^{|\sigma_{n-1}|+3}$. Since σ_{n-1} is null in $\pi_*\Theta(R)$, we would be done if g_2 were homotopic to the dotted extension

$$\begin{tikzcd} S^{|\sigma_{n-1}|+1} \arrow[r, "\sigma_{n-1}"] \arrow[d] & B\mathrm{GL}_1(\Theta(R)) \\ \Omega^2 S^{|\sigma_{n-1}|+3} \arrow[ur, dashed, "g_2'"'] & \end{tikzcd}$$

The potential failure of these maps to be homotopic stems from the fact that the composite $\mathfrak{Z}_3(R) \to R \to \Theta(R)$ need not be a map of $\boldsymbol{E}_3$–rings. It is, however, a map of $\boldsymbol{E}_2$–rings; therefore, the maps

$$g_1\colon \Omega S^{|\sigma_{n-1}|+2} \to B\mathrm{GL}_1(\mathfrak{Z}_3(R)) \to B\mathrm{GL}_1(\Theta(R)) \quad \text{and} \quad g_1'\colon \Omega S^{|\sigma_{n-1}|+2} \to B\mathrm{GL}_1(\Theta(R))$$

obtained by extending along the suspension $S^{|\sigma_{n-1}|+1} \to \Omega S^{|\sigma_{n-1}|+2}$ *are* homotopic. We now utilize the following result of Serre's:

Proposition 5.3.2 (Serre [91, page 281]) *Let p be an odd prime. Then the suspension $S^{2n-1} \to \Omega S^{2n}$ splits upon p–localization: there is a p–local equivalence*

$$E \times \Omega[\iota_{2n}, \iota_{2n}]\colon S^{2n-1} \times \Omega S^{4n-1} \to \Omega S^{2n}.$$

This implies that the suspension map $\Omega S^{|\sigma_{n-1}|+2} \to \Omega^2 S^{|\sigma_{n-1}|+3}$ admits a splitting as loop spaces. In particular, this implies that the map g_2 is homotopic to the composite

$$\Omega^2 S^{|\sigma_{n-1}|+3} \to \Omega S^{|\sigma_{n-1}|+2} \xrightarrow{g_1} B\mathrm{GL}_1(\mathfrak{Z}_3(R)) \to B\mathrm{GL}_1(\Theta(R)),$$

and similarly for g_2'. Since g_1 and g_1' are homotopic and g_1' (and hence g_2') is null, we find that g_2 is also null, as desired. □

5.4 Infinite families and the nilpotence theorem

We now briefly discuss the relationship between Theorem A and the nilpotence theorem. We begin by describing a special case of this connection. Recall from Remark 2.2.3 that Theorem 2.2.1 implies that, if

R is an $\boldsymbol{E}_2$–ring spectrum and $x \in \pi_* R$ is a simple p–torsion element which has trivial MU–Hurewicz image, then x is nilpotent. A similar argument implies the following:

Proposition 5.4.1 *Assume Conjecture D when $n = 1$. Then Corollary 2.2.2* (ie Theorem A when $n = 1$) *implies that, if R is a p–local $\boldsymbol{E}_3$–ring spectrum and $x \in \pi_* R$ is a class with trivial $H\mathbb{Z}_p$–Hurewicz image such that*

- *$\alpha_1 x = 0$ in $\pi_* R$, and*
- *the Toda bracket $\langle p, \alpha_1, x\rangle$ contains zero,*

then x is nilpotent.

Proof We claim that the composite

$$\Omega^2 S^3\langle 3\rangle \to BGL_1(\mathbb{S}_{(p)}) \to BGL_1(R[1/x]) \tag{5-2}$$

is null. Remark 4.1.8 implies that Conjecture D for $n = 1$ reduces us to showing that the composite

$$\Omega^2 P^{2p+1}(p) \xrightarrow{\alpha_1} BGL_1(\mathbb{S}_{(p)}) \to BGL_1(R[1/x])$$

is null. Since this composite is one of double loop spaces, it further suffices to show that the composite

$$P^{2p-1}(p) \to BGL_1(\mathbb{S}_{(p)}) \to BGL_1(R[1/x]) \tag{5-3}$$

is null. The bottom cell S^{2p-2} of $P^{2p-1}(p)$ maps trivially to $BGL_1(R[1/x])$, because the bottom cell detects α_1 (by Remark 4.1.8) and α_1 is nullhomotopic in $R[1/x]$. Therefore, the map (5-3) factors through the top cell S^{2p-1} of $P^{2p-1}(p)$. The resulting map

$$S^{2p-1} \to BGL_1(\mathbb{S}_{(p)}) \to BGL_1(R[1/x])$$

detects an element of the Toda bracket $\langle p, \alpha_1, x\rangle$, but this contains zero by hypothesis, so is nullhomotopic.

Since the map (5-2) is null, Corollary 2.2.2 and Theorem 2.1.7 imply that there is a ring map $H\mathbb{Z}_p \to R[1/x]$. In particular, the composite of the map $x\colon \Sigma^{|x|} R \to R$ with the unit $R \to R[1/x]$ factors as

$$\begin{tikzcd}
\Sigma^{|x|} R \arrow[r, "x"] \arrow[d] & R \arrow[r] \arrow[d] & R[1/x] \\
H\mathbb{Z}_p \wedge \Sigma^{|x|} R \arrow[r, "x"] & H\mathbb{Z}_p \wedge R \arrow[ur, dashed] &
\end{tikzcd}$$

The bottom map, however, is null, because x has zero $H\mathbb{Z}_p$–Hurewicz image. Therefore, the element $x \in \pi_* R[1/x]$ is null, and hence $R[1/x]$ is contractible. □

Remark 5.4.2 One can prove by a different argument that Proposition 5.4.1 is true without the assumption that Conjecture D holds when $n = 1$. At $p = 2$, this was shown by Astey [18, Theorem 1.1].

To discuss the relationship between Theorem A for general n and the nilpotence theorem (which we will expand upon in future work), we embark on a slight digression. The following proposition describes the construction of some infinite families:

Proposition 5.4.3 *Let R be a height $n-1$ spectrum as in the second line of Table 2, and assume Conjecture E if $R = A$ or B. Then there is an infinite family $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}(R)$. Conjecture E implies that σ_{n-1,p^k} lifts to $\pi_{p^k|v_n|-1}(\mathfrak{z}_3(R))$, where $\mathfrak{z}_3(R)$ abusively denotes the $\boldsymbol{E}_3$–center of $X(p^n-1)$ if $R = T(n-1)$.*

Proof We construct this family by induction on k. The element $\sigma_{n-1,1}$ is just σ_{n-1}, so assume that we have defined σ_{n-1,p^k}. The element $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}R$ defines a map $\sigma_{n-1,p^k} \colon S^{p^k|v_n|} \to B\mathrm{GL}_1(R)$. When $R = T(n-1)$, Lemma 3.1.12 (and the inductive hypothesis) implies that the map defined by σ_n factors through the map $B\mathrm{GL}_1(X(p^n-1)) \to B\mathrm{GL}_1(T(n-1))$. When $R = A$ or B, Conjecture E (and the inductive hypothesis) implies that the map defined by σ_n factors through the map $B\mathrm{GL}_1(\mathfrak{z}_3(R)) \to B\mathrm{GL}_1(R)$. This implies that, for all R as in the second line of Table 2, the map $\sigma_{n-1,p^k} \colon S^{p^k|v_n|} \to B\mathrm{GL}_1(R)$ factors through an $\boldsymbol{E}_1$–space, which we shall just denote by $\mathcal{Z}_R$ for the purpose of this proof. If we assume Conjecture E, then we may take $\mathcal{Z}_R = B\mathrm{GL}_1(\mathfrak{z}_3(R))$.

Therefore, we get a map $\sigma_{n-1,p^k} \colon \Omega S^{p^k|v_n|+1} \to B\mathrm{GL}_1(R)$ via the composite

$$\Omega S^{p^k|v_n|+1} \to \mathcal{Z}_R \to B\mathrm{GL}_1(R).$$

Since $\mathcal{Z}_R$ is an $\boldsymbol{E}_1$–space, the map $\Omega S^{p^k|v_n|+1} \to \mathcal{Z}_R$ is adjoint to a map

$$\bigvee_{j \geq 1} S^{jp^k|v_n|+1} \simeq \Sigma\Omega S^{p^k|v_n|+1} \to B\mathcal{Z}_R;$$

the source splits as indicated via the James splitting. These splittings are given by Whitehead products; in particular, the map $S^{p^{k+1}|v_n|+1} = S^{p(p^k|v_n|+1)-(p-1)} \to B\mathcal{Z}_R$ is given by the p–fold Whitehead product $[\sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k}]$. This is divisible by p, so it yields a map $S^{p^{k+1}|v_n|} \to \mathcal{Z}_R$, and hence a map $S^{p^{k+1}|v_n|} \to B\mathrm{GL}_1(R)$ given by composing with the map $\mathcal{Z}_R \to B\mathrm{GL}_1(R)$. This defines the desired element $\sigma_{n-1,p^{k+1}} \in \pi_{p^{k+1}|v_n|-1}(R)$. As the construction makes clear, assuming Conjecture E and taking $\mathcal{Z}_R = B\mathrm{GL}_1(\mathfrak{z}_3(R))$ implies that σ_{n-1,p^k} lifts to $\pi_{p^k|v_n|-1}(\mathfrak{z}_3(R))$. □

Remark 5.4.4 This infinite family is detected in the 1–line of the ANSS for R by $\delta(v_n^k)$, where δ is the boundary map induced by the map $\Sigma^{-1}R/p \to R$. This is a consequence of the geometric boundary theorem (see [85, Theorem 2.3.4]) applied to the cofiber sequence $R \xrightarrow{p} R \to R/p$.

Remark 5.4.5 The element $\sigma_{n-1,1} \in \pi_{2p^n-3}(R)$ is precisely σ_{n-1}.

Remark 5.4.6 When $n = 1$, the ring R is the (p–local) sphere spectrum. The infinite family σ_{n-1,p^k} is the Adams–Toda α–family; namely, $\alpha_{p^k} \in \pi_{2p^k(p-1)-1}(\mathbb{S})$ maps to $\sigma_{0,p^k} \in \pi_{2p^k(p-1)-1}X(p-1)$ under the unit map $\mathbb{S} \to X(p-1)$.

We now briefly sketch an argument relating Theorem A to the proof of the nilpotence theorem; we shall elaborate on this discussion in forthcoming work.

Remark 5.4.7 The heart of the nilpotence theorem is what is called Step III in [38]; this step amounts to showing that certain self-maps of $T(n-1)$–module skeleta (denoted by G_k in [38]) of $T(n)$ are nilpotent. Let us assume that $p > 2$ for simplicity. Then these self-maps are given by multiplication by the p–fold Toda bracket $b_{n,k} = \langle \sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k} \rangle$ at an odd prime p; this lives in degree $p|\sigma_{n-1,p^k}| + p - 2 = 2p^k(p^n-1) - 2$. (When $p = 2$, the desired element σ_{n-1,p^k} is denoted by h in [51, Theorem 3].) It therefore suffices to establish the nilpotency of the $b_{n,k}$.

This can be proven through Theorem A via induction on k; we shall assume Conjectures D and E for the remainder of this discussion. The motivation for this approach stems from the observation that, if R is any $\boldsymbol{E}_3$–$\mathbb{F}_2$–algebra and $x \in \pi_*(R)$, then there is a relation $Q_1(x)^2 = Q_2(x^2)$ (at odd primes, one has a relation involving the p–fold Toda bracket $\langle Q_1(x), \dots, Q_1(x) \rangle$). In our setting, Proposition 5.4.3 implies that the elements $\sigma_{n-1,k}$ lift to $\pi_* \mathfrak{Z}_3(X(p^n-1))$. At $p = 2$, one can prove (in the same way that the Cartan relation $Q_1(x)^2 = Q_2(x^2)$ is proven) that the construction of this infinite family implies that $\sigma^2_{n-1,p^{k+1}}$ can be described in terms of $Q_2(\sigma^2_{n-1,p^k})$. At odd primes, there is a similar relation involving the p–fold Toda bracket defining $b_{n,k}$. In particular, induction on k implies that the $b_{n,k}$ are all nilpotent in $\pi_* \mathfrak{Z}_3(X(p^n-1))$ if $b_{n,1}$ is nilpotent. Note that $|b_{n,1}| = 2p^{n+1} - 2p - 2$.

To argue that $b_{n,1}$ is nilpotent, one first observes that $\sigma_{n-1} b^p_{n,1} = 0$ in $\pi_* \mathfrak{Z}_3(X(p^n-1))$; when $n = 0$, this follows from the statement that $\alpha_1 \beta_1^p = 0$ in the sphere. To show that $b_{n,1}$ is nilpotent, it suffices to establish that $\mathfrak{Z}_3(X(p^n-1))[1/b^p_{n,1}]$ is contractible; when $n = 1$, this follows from Proposition 5.4.1. We give a very brief sketch of this nilpotence for general n, by arguing as in Proposition 5.4.1, and with a generous lack of precision which will be remedied in forthcoming work.

For notational convenience, we now write $d_{n,1} = b^p_{n,1}$, so that $|d_{n,1}| = 2p^{n+2} - 2p^2 - 2p$. It suffices to show that the multiplication-by-$d_{n,1}$ map

$$d_{n,1} \colon \Sigma^{|d_{n,1}|} \mathfrak{Z}_3(X(p^n-1)) \to \mathfrak{Z}_3(X(p^n-1))[1/d_{n,1}]$$

is nullhomotopic. Since σ_{n-1} kills $d_{n,1}$, we know that σ_{n-1} is nullhomotopic in $\mathfrak{Z}_3(X(p^n-1))[1/d_{n,1}]$. Moreover, the bracket $\langle p, \sigma_{n-1}, 1_{\mathfrak{Z}_3(X(p^n-1))[1/d_{n,1}]} \rangle$ contains zero. By arguing as in Proposition 5.4.1, we can conclude that the composite

$$K_n \to \Omega^2 P^{|\sigma_{n-1}|+4} \xrightarrow{\phi} B\mathrm{GL}_1\big(\mathfrak{Z}_3(X(p^n-1))\big) \to B\mathrm{GL}_1\big(\mathfrak{Z}_3(X(p^n-1))[1/d_{n_0}]\big)$$

is nullhomotopic, where the map ϕ is as constructed in Step 1 of the proof of Theorem A. (Recall that the proof of Theorem A shows that the Thom spectrum $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$ is an $\boldsymbol{E}_1 \otimes \mathbb{Q}_1$–$\mathfrak{Z}_3(X(p^n-1))$–algebra such that $\mathrm{BP}\langle n-1 \rangle$ splits off its base change along the map $\mathfrak{Z}_3(X(p^n-1)) \to T(n-1)$.) It follows from Theorem 2.1.7 that the multiplication-by-$d_{n,1}$ map factors as

$$\begin{tikzcd} \Sigma^{|d_{n,1}|}\mathfrak{Z}_3(X(p^n-1)) \arrow[r, "d_{n,1}"] \arrow[d] & \mathfrak{Z}_3(X(p^n-1)) \arrow[r] \arrow[d] & \mathfrak{Z}_3(X(p^n-1))[1/d_{n,1}] \\ \Sigma^{|d_{n,1}|}(\Omega^2 P^{|\sigma_{n-1}|+4})^{\phi} \arrow[r, "d_{n,1}"] & (\Omega^2 P^{|\sigma_{n-1}|+4})^{\phi} \arrow[ur, dashed] & \end{tikzcd}$$

To show that the top composite is null, it therefore suffices to show that the self-map of K_n^{ϕ} defined by $d_{n,1}$ is nullhomotopic. This essentially follows from the fact that $(\Omega^2 P^{|\sigma_{n-1}|+4})^{\phi}$ is an $\boldsymbol{E}_1\otimes\mathcal{Q}_1$–$\mathfrak{Z}_3(X(p^n-1))$–algebra: multiplication by $d_{n,1}$ is therefore null on K_n^{ϕ}, because $d_{n,1}$ is built from σ_{n-1} (which is null in $(\Omega^2 P^{|\sigma_{n-1}|+4})^{\phi}$) via $\boldsymbol{E}_1$–power operations.

6 Applications

6.1 Splittings of cobordism spectra

The goal of this section is to prove the following:

Theorem C *Assume that the composite $\mathfrak{Z}_3(B)\to B\to M\mathrm{String}_{(2)}$ is an $\boldsymbol{E}_3$–map. Then Conjectures D, E and F imply that there is a unital splitting of the Ando–Hopkins–Rezk orientation $M\mathrm{String}_{(2)}\to \mathrm{tmf}_{(2)}$.*

Remark 6.1.1 We believe that the assumption that the composite $\mathfrak{Z}_3(B)\to B\to M\mathrm{String}_{(2)}$ is an $\boldsymbol{E}_3$–map is too strong: we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We only construct unstructured splittings; it seems unlikely that they can be refined to structured splittings. A slight modification of our arguments should work at any prime.

Remark 6.1.2 In fact, the same argument used to prove Theorem C shows that, if the composite $\mathfrak{Z}_3(A)\to A\to M\mathrm{Spin}_{(2)}$ is an $\boldsymbol{E}_3$–map, then Conjectures D and E imply that there are unital splittings of the Atiyah–Bott–Shapiro orientation $M\mathrm{Spin}_{(2)}\to bo_{(2)}$. This splitting was originally proved unconditionally (ie without assuming Conjecture D or Conjecture E) by Anderson, Brown and Peterson [4] via a calculation with the Adams spectral sequence.

Remark 6.1.3 The inclusion of the cusp on $\overline{\mathcal{M}}_{\mathrm{ell}}$ defines an $\boldsymbol{E}_\infty$–map $c:\mathrm{tmf}\to bo$ as in [63, Theorem 1.2]. The resulting diagram

$$\begin{tikzcd} M\mathrm{String}_{(2)} \arrow[r] \arrow[d] & M\mathrm{Spin}_{(2)} \arrow[d] \\ \mathrm{tmf}_{(2)} \arrow[r, "c"] & bo_{(2)} \end{tikzcd}$$

commutes (see eg [34, Lemma 6.4]). The splitting $s\colon \mathrm{tmf}_{(2)} \to M\mathrm{String}_{(2)}$ of Theorem C defines a composite
$$\mathrm{tmf}_{(2)} \xrightarrow{s} M\mathrm{String}_{(2)} \to M\mathrm{Spin}_{(2)} \to bo_{(2)}$$
which agrees with c.

Remark 6.1.4 The Anderson–Brown–Peterson splitting implies that, if X is any compact space, then the Atiyah–Bott–Shapiro $\widehat{A}$–genus (ie the index of the Dirac operator in families) $M\mathrm{Spin}^*(X) \to bo^*(X)$ is surjective. Similarly, if the composite $\mathfrak{z}_3(B) \to B \to M\mathrm{String}$ is an $\boldsymbol{E}_3$–map, then Conjectures D, E and F imply that the Ando–Hopkins–Rezk orientation (ie the Witten genus in families) $M\mathrm{String}^*(X) \to \mathrm{tmf}^*(X)$ is also surjective.

Remark 6.1.5 In [34], we proved (unconditionally) that the map $\pi_* M\mathrm{String} \to \pi_*\mathrm{tmf}$ is surjective. Our proof proceeds by showing that the map $\pi_* B \to \pi_*\mathrm{tmf}$ is surjective via arguments with the Adams–Novikov spectral sequence and by exploiting the $\boldsymbol{E}_1$–ring structure on B to lift the powers of Δ living in $\pi_*\mathrm{tmf}$.

The discussion preceding [75, Remark 7.3] implies that, for a particular model of $\mathrm{tmf}_0(3)$, we have:

Corollary 6.1.6 *Assume that the composite $\mathfrak{z}_3(B) \to B_{(2)} \to M\mathrm{String}_{(2)}$ is an $\boldsymbol{E}_3$–map. Then Conjectures D, E and F imply that $\Sigma^{16}\mathrm{tmf}_0(3)^\wedge_2$ is a summand of $M\mathrm{String}^\wedge_2$.*

We now turn to the proof of Theorem C.

Proof of Theorem C First, note that such a splitting exists after rationalization. Indeed, it suffices to check that this is true on rational homotopy; since the orientations under considerations are $\boldsymbol{E}_\infty$–ring maps, the induced map on homotopy is one of rings. It therefore suffices to lift the generators.

We now show that the generators of $\pi_*\mathrm{tmf} \otimes \mathbb{Q} \cong \mathbb{Q}[c_4, c_6]$ lift to $\pi_* M\mathrm{String} \otimes \mathbb{Q}$. Although one can argue this by explicitly constructing manifold representatives (as is done for c_4 in [34, Corollary 6.3]), it is also possible to provide a more homotopy-theoretic proof: The elements c_4 and c_6 live in dimensions 8 and 12, respectively, and the map $M\mathrm{String} \to \mathrm{tmf}$ is known to be an equivalence in dimensions ≤ 15 by [49, Theorem 2.1]. It follows that the same is true rationally, so c_4 and c_6 indeed lift to $\pi_* M\mathrm{String} \otimes \mathbb{Q}$, as desired.

We will now construct a splitting after p–completion, where $p = 2$. By Corollary 5.1.6, we obtain a unital map $\mathrm{tmf} \simeq \Theta(B) \to M\mathrm{String}$ upon p–completion which splits the orientation $M\mathrm{String} \to \Theta(B)$ because

(a) the map $\mathfrak{z}_3(B) \to B \to M\mathrm{String}$ is an $\boldsymbol{E}_3$–ring map (by assumption),

(b) the element σ_2 vanishes in $\pi_{13} M\mathrm{String}_{(2)}$ (because $\pi_{13} M\mathrm{String}_{(2)} \cong \pi_{13}\mathrm{tmf}_{(2)} \cong 0$), and

(c) the Toda bracket $\langle 2, \sigma_2, 1_{M\mathrm{String}_{(2)}}\rangle \subseteq \pi_* M\mathrm{String}_{(2)}$ contains zero because $\pi_{14} M\mathrm{String}_{(2)} \cong \pi_{14}\mathrm{tmf}_{(2)}$, and the corresponding bracket $\langle 2, \sigma_2, 1_{M\mathrm{String}_{(2)}}\rangle \subseteq \pi_{14}\mathrm{tmf}_{(2)}$ detects v_3, and hence contains zero.

To obtain a map $\mathrm{tmf}_{(p)} \to M\mathrm{String}_{(p)}$, we need to show that the induced map $\mathrm{tmf} \otimes \mathbb{Q} \to \mathrm{tmf}_p^\wedge \otimes \mathbb{Q} \to M\mathrm{String}_p^\wedge \otimes \mathbb{Q}$ agrees with the rational splitting constructed in the previous paragraph. However, this is immediate from the fact that the splittings $\mathrm{tmf}_p^\wedge \to M\mathrm{String}_p^\wedge$ are constructed to be equivalences in dimensions ≤ 15, and the fact that the map out of $\mathrm{tmf} \otimes \mathbb{Q}$ is determined by its effect on the generators c_4 and c_6. $\square$

Remark 6.1.7 The proof recalled in Remark 1.1.2 of Thom's splitting of MO proceeded essentially unstably: there is an $\boldsymbol{E}_2$–map $\Omega^2 S^3 \to BO$ of spaces over $B\mathrm{GL}_1(\mathbb{S})$, whose Thomification yields the desired $\boldsymbol{E}_2$–map $H\mathbb{F}_2 \to MO$. This argument also works for $M\mathrm{SO}$: there is an $\boldsymbol{E}_2$–map $\Omega^2 S^3\langle 3\rangle \to B\mathrm{SO}$ of spaces over $B\mathrm{GL}_1(\mathbb{S})$, whose Thomification yields the desired $\boldsymbol{E}_2$–map $H\mathbb{Z} \to M\mathrm{SO}$. One might hope for the existence of a similar unstable map which would yield Theorem C. We do not know how to construct such a map. To illustrate the difficulty, let us examine how such a proof would work; we will specialize to the case of $M\mathrm{String}$, but the discussion is the same for $M\mathrm{Spin}$.

According to Theorem A, Conjectures D and E imply that there is a map $K_3 \to B\mathrm{GL}_1(B)$ whose Thom spectrum is equivalent to tmf. There is a map $BN \to B^2\mathrm{String}$, whose fiber we will denote by Q. Then there is a fiber sequence

$$N \to B\mathrm{String} \to Q,$$

and so Proposition 2.1.6 implies that there is a map $Q \to B\mathrm{GL}_1(B)$ whose Thom spectrum is $M\mathrm{String}$. Theorem C would follow if there was a map $f\colon K_3 \to Q$ of spaces over $B\mathrm{GL}_1(B)$, since Thomification would produce a map $\mathrm{tmf} \to M\mathrm{String}$.

Conjecture D reduces the construction of f to the construction of a map $\Omega^2 P^{17}(2) \to Q$. This map would in particular imply the existence of a map $P^{15}(2) \to Q$ (and would be equivalent to the existence of such a map if Q was a double loop space), which in turn stems from a 2–torsion element of $\pi_{14}(Q)$. The long exact sequence on homotopy runs

$$\cdots \to \pi_{14}(B\mathrm{String}) \to \pi_{14}(Q) \to \pi_{13}(N) \to \pi_{13}(B\mathrm{String}) \to \cdots.$$

Bott periodicity states that $\pi_{13} B\mathrm{String} \cong \pi_{14} B\mathrm{String} \cong 0$, so we find that $\pi_{14}(Q) \cong \pi_{13}(N)$. The desired 2–torsion element of $\pi_{14}(Q)$ is precisely the element of $\pi_{13}(N)$ described in Remark 3.2.24. Choosing a particular nullhomotopy of twice this 2–torsion element of $\pi_{14}(Q)$ produces a map $g\colon P^{15}(2) \to Q$. To extend this map over the double suspension $P^{15}(2) \to \Omega^2 P^{17}(2)$, it would suffice to show that there is a double loop space $\tilde{Q}$ with a map $\tilde{Q} \to Q$ such that g factors through $\tilde{Q}$.

Unfortunately, we do not know how to prove such a result; this is the unstable analogue of Conjecture E. In fact, such an unstable statement would bypass the need for Conjecture E in Theorem A. (One runs into the same obstruction for $M\mathrm{Spin}$, except with the fiber of the map $S^5 \to B^2\mathrm{Spin}$.) These statements are reminiscent of the conjecture (see Section 4.1) that the fiber $W_n = \mathrm{fib}(S^{2n-1} \to \Omega^2 S^{2n+1})$ of the double suspension admits the structure of a double loop space.

Remark 6.1.8 The following application of Theorem C was suggested by Mike Hopkins. In [53], the Anderson–Brown–Peterson splitting is used to show that the Atiyah–Bott–Shapiro orientation $M\mathrm{Spin} \to KO$ induces an isomorphism

$$M\mathrm{Spin}_*(X) \otimes_{M\mathrm{Spin}_*} KO_* \xrightarrow{\cong} KO_*(X)$$

of KO_*–modules for all spectra X. In future work, we shall show that Theorem C can be used to prove the following height 2 analogue of this result: Conjectures D, E and F imply that the Ando–Hopkins–Rezk orientation $M\mathrm{String} \to \mathrm{Tmf}$ induces an isomorphism

$$M\mathrm{String}_*(X) \otimes_{M\mathrm{String}_*} \mathrm{Tmf}_* \xrightarrow{\cong} \mathrm{Tmf}_*(X) \tag{6-1}$$

of Tmf_*–modules for all spectra X. The $K(1)$–analogue of this isomorphism was obtained by Laures [60].

6.2 Wood equivalences

The Wood equivalence states that $bo \wedge C\eta \simeq bu$. There are generalizations of this equivalence to tmf (see [78]); for instance, there is a 2–local 8–cell complex DA_1 whose cohomology is isomorphic to the double of $A(1)$ as an $A(2)$–module such that $\mathrm{tmf}_{(2)} \wedge DA_1 \simeq \mathrm{tmf}_1(3) \simeq \mathrm{BP}\langle 2\rangle$. Similarly, if X_3 denotes the 3–local 3–cell complex $S^0 \cup_{\alpha_1} e^4 \cup_{2\alpha_1} e^8$, then $\mathrm{tmf}_{(3)} \wedge X_3 \simeq \mathrm{tmf}_1(2) \simeq \mathrm{BP}\langle 2\rangle \vee \Sigma^8 \mathrm{BP}\langle 2\rangle$. We will use the umbrella term "Wood equivalence" to refer to equivalences of this kind.

Our goal in this section is to revisit these Wood equivalences using the point of view stemming from Theorem A. In particular, we propose that these equivalences are suggested by the existence of certain EHP sequences; we will greatly expand on this in a forthcoming document. We find this to be a rather beautiful connection between stable and unstable homotopy theory.

The first Wood-style result was proved in Proposition 3.2.6. The next result, originally proved in [67, Section 2.5; 31, Theorem 3.7], is the simplest example of a Wood-style equivalence which is related to the existence of certain EHP sequences.

Proposition 6.2.1 *Let $\mathbb{S}/\!/\eta = X(2)$ (resp. $\mathbb{S}/\!/2$) denote the $\boldsymbol{E}_1$–quotient of $\mathbb{S}$ by η (resp. 2). If $Y = C\eta \wedge \mathbb{S}/2$ and A_1 is a spectrum whose cohomology is isomorphic to $A(1)$ as a module over the Steenrod algebra, then there are equivalences*

$$A \wedge C\eta \simeq \mathbb{S}/\!/\eta, \quad A \wedge Y \simeq \mathbb{S}/\!/2, \quad A \wedge A_1 \simeq y(1)/v_1$$

of A–modules.

Remark 6.2.2 Proposition 6.2.1 implies the Wood equivalence $bo \wedge C\eta \simeq bu$. Although this implication is already true before 2–completion, we will work in the 2–complete category for convenience. Recall that Theorem A states that Conjectures D and E imply that there is a map $\mu\colon K_2 \to B\mathrm{GL}_1(A)$ whose Thom spectrum is equivalent to bo (as left A–modules). Moreover, the Thom spectrum of the composite

$K_2 \xrightarrow{\mu} B\mathrm{GL}_1(A) \to B\mathrm{GL}_1(T(1))$ is equivalent to $\mathrm{BP}\langle 1\rangle$. Since this Thom spectrum is the base change $K_2^\mu \wedge_A T(1)$, and Proposition 6.2.1 implies that $T(1) = X(2) \simeq A \wedge C\eta$, we find that

$$\mathrm{BP}\langle 1\rangle \simeq K_2^\mu \wedge_A (A \wedge C\eta) \simeq K_2^\mu \wedge C\eta \simeq bo \wedge C\eta,$$

as desired. Similarly, noting that $\mathbb{S}/\!/2 = y(1)$, we find that Proposition 6.2.1 also proves the equivalence $bo \wedge Y \simeq k(1)$.

Remark 6.2.3 The argument of Remark 6.2.2 in fact proves that Theorem A for A implies Theorem A for $T(1)$, $y_{\mathbb{Z}}(1)$ and $y(1)$.

Proof of Proposition 6.2.1 For the first two equivalences, it suffices to show that $A \wedge C\eta \simeq \mathbb{S}/\!/\eta$ and that $\mathbb{S}/\!/\eta \wedge \mathbb{S}/2 \simeq \mathbb{S}/\!/2$. We will prove the first statement; the proof of the second statement is exactly the same. There is a map $C\eta \to \mathbb{S}/\!/\eta$ given by the inclusion of the 2–skeleton. There is also an $\boldsymbol{E}_1$–ring map $A \to \mathbb{S}/\!/\eta$ given as follows. The multiplication on $\mathbb{S}/\!/\eta$ defines a unital map $C\eta \wedge C\eta \to \mathbb{S}/\!/\eta$. But, since the Toda bracket $\langle \eta, 2, \eta\rangle$ contains ν, there is a unital map $C\nu \to C\eta \wedge C\eta$. This supplies a unital map $C\nu \to \mathbb{S}/\!/\eta$, which, by the universal property of $A = \mathbb{S}/\!/\nu$ (via Theorem 2.1.7), extends to an $\boldsymbol{E}_1$–ring map $A \to \mathbb{S}/\!/\eta$.

For the final equivalence, it suffices to construct a map $A_1 \to y(1)/v_1$ for which the induced map $A \wedge A_1 \to y(1)/v_1$ gives an isomorphism on mod 2 homology. Since A_1 may be obtained as the cofiber of a v_1–self-map $\Sigma^2 Y \to Y$, it suffices to observe that the diagram

$$\begin{tikzcd}
\Sigma^2 Y \arrow[r] \arrow[d, "v_1"] & \Sigma^2 y(1) \arrow[d, "v_1"] \\
Y \arrow[r] & y(1)
\end{tikzcd}$$

commutes; our desired map is the induced map on vertical cofibers. □

Remark 6.2.4 There are EHP sequences

$$S^1 \to \Omega S^2 \to \Omega S^3, \quad S^2 \to \Omega S^3 \to \Omega S^5.$$

Recall that $\mathbb{S}/2$, $C\eta$, $\mathbb{S}/\!/2$, $\mathbb{S}/\!/\eta = X(2)$ and A are Thom spectra over S^1, S^2, ΩS^2, ΩS^3 and ΩS^5, respectively. Proposition 2.1.6 therefore implies that there are maps $f : \Omega S^3 \to B\mathrm{Aut}(\mathbb{S}/2)$ and $g : \Omega S^5 \to B\mathrm{Aut}(C\eta)$ whose Thom spectra are equivalent to $\mathbb{S}/\!/2$ and $\mathbb{S}/\!/\eta$, respectively. The maps f and g define local systems of spectra over ΩS^3 and ΩS^5 whose fibers are equivalent to $\mathbb{S}/2$ and $C\eta$ (respectively), and one interpretation of Proposition 6.2.1 is that these local systems in fact factor as

$$\Omega S^3 \xrightarrow{\eta} B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{Aut}(\mathbb{S}/2), \quad \Omega S^5 \xrightarrow{\nu} B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{Aut}(C\eta).$$

Proposition 6.2.1 is an immediate consequence of these factorizations. We argue this for the first case in Remark 6.2.5, and for the second in Remark 6.2.6, thereby giving an alternative EHP-based argument for Proposition 6.2.1.

Remark 6.2.5 The first EHP sequence in Remark 6.2.4 splits via the Hopf map $S^3 \to S^2$. The map $f\colon \Omega S^3 \to B\mathrm{Aut}(\mathbb{S}/2)$ in fact factors through the dotted map in the diagram

$$\begin{tikzcd}
S^2 \arrow[r] & \Omega S^2 \arrow[r] \arrow[d] & \Omega S^3 \arrow[d] \arrow[dl, dashed] \\
& B\mathrm{GL}_1(\mathbb{S}) \arrow[r] & B\mathrm{Aut}(\mathbb{S}/2)
\end{tikzcd}$$

Indeed, the composite $\Omega S^3 \to \Omega S^2 \to B\mathrm{GL}_1(\mathbb{S})$ is a loop map and, therefore, is determined by the composite $\phi\colon S^3 \to S^2 \to B^2\mathrm{GL}_1(\mathbb{S})$. Since the map $S^2 \to B^2\mathrm{GL}_1(\mathbb{S})$ detects the element $-1 \in \pi_0(\mathbb{S})^\times$, the map ϕ does in fact determine a unit multiple of η. This implies the desired claim.

Remark 6.2.6 The map $g\colon \Omega S^5 \to B\mathrm{Aut}(C\eta)$ from Remark 6.2.4 factors through $B\mathrm{GL}_1(\mathbb{S})$. To see this, let us begin with the following observation: View BU and $B\mathrm{SU}$ as H–spaces via the tensor product of vector bundles. Then the map $B\mathrm{SU} \times \mathbb{C}P^\infty \to BU$ classifying $\mathcal{V} \boxtimes \mathcal{L}$, with $\mathcal{V}$ the universal SU–bundle over $B\mathrm{SU}$ and $\mathcal{L}$ the universal line bundle over BU, is an equivalence of H–spaces. In particular, there is a fiber sequence

$$\mathbb{C}P^\infty \to BU \to B\mathrm{SU}.$$

The map $\Omega S^3 \to B\mathrm{GL}_1(\mathbb{S})$ defining $T(1)$ factors as

$$\Omega S^3 \to BU \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S});$$

similarly, the map $\Omega S^5 \to B\mathrm{GL}_1(\mathbb{S})$ defining A factors as

$$\Omega S^5 \to B\mathrm{SU} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}).$$

These factorizations make the following diagram of fiber sequences commute:

$$\begin{tikzcd}
S^2 \arrow[r] \arrow[d] & \Omega S^3 \arrow[r] \arrow[d] & \Omega S^5 \arrow[d] \\
\mathbb{C}P^\infty \arrow[r] & BU \arrow[r] & B\mathrm{SU}
\end{tikzcd}$$

The map $\Omega S^5 \to B\mathrm{Aut}(C\eta)$ was defined using Proposition 2.1.6. It then follows from the splitting of the bottom fiber sequence in the above diagram that the dotted map exists in the diagram

$$\begin{tikzcd}
S^2 \arrow[r] \arrow[d] & \Omega S^3 \arrow[r] \arrow[d] & \Omega S^5 \arrow[d] \\
\mathbb{C}P^\infty \arrow[r] \arrow[dr] & BU \arrow[r] \arrow[d, "J"] & B\mathrm{SU} \arrow[d] \arrow[dl, dashed, "J"] \\
& B\mathrm{GL}_1(\mathbb{S}) \arrow[r] & B\mathrm{Aut}(C\eta)
\end{tikzcd}$$

The composite

$$\Omega S^5 \to B\mathrm{SU} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{Aut}(C\eta)$$

is g, giving our desired factorization.

Next, we have the following result at height 2:

Proposition 6.2.7 *Let DA_1 denote the double of A_1* (see [78]). *There are 2–complete equivalences*

$$B \wedge DA_1 \simeq T(2), \quad B \wedge Z \simeq y(2), \quad B \wedge A_2 \simeq y(2)/v_2,$$

where Z is the spectrum "$\frac{1}{2}A_2$" from [76; 25][3] *and A_2 is a spectrum whose cohomology is isomorphic to $A(2)$ as a module over the Steenrod algebra.*

Remark 6.2.8 Arguing as in Remark 6.2.2 shows that Proposition 6.2.7 and Theorem A imply the Wood equivalences

$$\mathrm{tmf} \wedge DA_1 \simeq \mathrm{tmf}_1(3) = \mathrm{BP}\langle 2\rangle, \quad \mathrm{tmf} \wedge Z \simeq k(2), \quad \mathrm{tmf} \wedge A_2 \simeq H\mathbb{F}_2.$$

Remark 6.2.9 Exactly as in Remark 6.2.3, the argument of Remark 6.2.8 in fact proves that Theorem A for B implies Theorem A for $T(2)$, $y_{\mathbb{Z}}(2)$ and $y(2)$.

Remark 6.2.10 The telescope conjecture [84, Conjecture 10.5], which we interpret as stating that L_n–localization is the same as L_n^f–localization, is known to be true at height 1. For odd primes, it was proved by Miller [80], and at $p = 2$ it was proved by Mahowald [68; 70]. Mahowald's approach was to calculate the telescopic homotopy of the type 1 spectrum Y. In [73], Mahowald, Ravenel and Shick proposed an approach to *dis*proving the telescope conjecture at height 2: they suggest that, for $n \geq 2$, the L_n–localization and the v_n–telescopic localization of $y(n)$ have different homotopy groups. They show, however, that the L_1–localization and the v_1–telescopic localization of $y(1)$ agree, so this approach (thankfully) does not give a counterexample to the telescope conjecture at height 1.

Motivated by Mahowald's approach to the telescope conjecture, Behrens, Beaudry, Bhattacharya, Culver and Xu study the v_2–telescopic homotopy of Z in [22], with inspiration from the Mahowald–Ravenel–Shick approach. Propositions 6.2.1 and 6.2.7 can be used to relate these two (namely, the finite spectrum and the Thom spectrum) approaches to the telescope conjecture. As in Section 6.1, we will let R denote A or B. Moreover, let F denote Y or Z (depending on what R is) and let R' denote $y(1)$ or $y(2)$ (again depending on what R is), so that $R \wedge F = R'$ by Propositions 6.2.1 and 6.2.7. Then:

Corollary 6.2.11 *If the telescope conjecture is true for F (and hence any type 1 or 2 spectrum) or R, then it is true for R'.*

[3] In the former source, Z is denoted by $\overline{M}$.

Proof Since L_n– and L_n^f–localizations are smashing, we find that, if the telescope conjecture is true for F or R, then Propositions 6.2.1 and 6.2.7 yield equivalences

$$L_n^f R' \simeq R \wedge L_n^f F \simeq R \wedge L_n F \simeq L_n R'. \qquad \square$$

Finally, we prove Proposition 6.2.7.

Proof of Proposition 6.2.7 We first construct maps $B \to T(2)$ and $DA_1 \to T(2)$. The top cell of DA_1 is in dimension 12, and the map $T(2) \to \mathrm{BP}$ is an equivalence in dimensions ≤ 12. It follows that constructing a map $DA_1 \to T(2)$ is equivalent to constructing a map $DA_1 \to \mathrm{BP}$. However, both BP and DA_1 are concentrated in even degrees, so the Atiyah–Hirzebruch spectral sequence collapses, and we find that $\mathrm{BP}^*(DA_1) \cong H^*(DA_1; \mathrm{BP}_*)$. The generator in bidegree $(0,0)$ produces a map $DA_1 \to T(2)$; its effect on homology is the additive inclusion $\mathbb{F}_2[\zeta_1^2, \zeta_2^2]/(\zeta_1^8, \zeta_2^4) \to \mathbb{F}_2[\zeta_1^2, \zeta_2^2]$.

The map $B \to T(2)$ may be defined via the universal property of Thom spectra from Section 2.1 and Remark 3.2.20. Its effect on homology is the inclusion $\mathbb{F}_2[\zeta_1^8, \zeta_2^4] \to \mathbb{F}_2[\zeta_1^2, \zeta_2^2]$. We obtain a map $B \wedge DA_1 \to T(2)$ via the multiplication on $T(2)$, and this induces an isomorphism in mod 2 homology.

For the second equivalence, we argue similarly: The map $B \to T(2)$ defines a map $B \to T(2) \to y(2)$. Next, recall that Z is built through iterated cofiber sequences:

$$\Sigma^2 Y \xrightarrow{v_1} Y \to A_1, \qquad \Sigma^5 A_1 \wedge Cv \xrightarrow{\sigma_1} A_1 \wedge Cv \to Z.$$

As an aside, we note that the element σ_1 is intimately related to the element discussed in Example 3.1.14; namely, it is given by the self-map of $A_1 \wedge Cv$ given by smashing A_1 with the diagram

$$\begin{tikzcd} \Sigma^5 Cv \arrow[r] \arrow[drrr, dashed, "\sigma_1"'] & \Sigma^5 A \arrow[r, "\sigma_1 \wedge \mathrm{id}"] & A \wedge A \arrow[r] & A \\ & & & Cv \arrow[u] \end{tikzcd}$$

Using these cofiber sequences and Proposition 3.2.6, one obtains a map $Z \to y(2)$, which induces the additive inclusion $\mathbb{F}_2[\zeta_1, \zeta_2]/(\zeta_1^8, \zeta_2^4) \to \mathbb{F}_2[\zeta_1, \zeta_2]$ on mod 2 homology. The multiplication on $y(2)$ defines a map $B \wedge Z \to y(2)$, which induces an isomorphism on mod 2 homology.

For the final equivalence, it suffices to construct a map $A_2 \to y(2)/v_2$ for which the induced map $B \wedge A_2 \to y(2)/v_2$ gives an isomorphism on mod 2 homology. Since A_2 may be obtained as the cofiber of a v_2–self-map $\Sigma^6 Z \to Z$, it suffices to observe that the diagram

$$\begin{tikzcd} \Sigma^6 Z \arrow[r] \arrow[d, "v_2"] & \Sigma^6 y(2) \arrow[d, "v_2"] \\ Z \arrow[r] & y(2) \end{tikzcd}$$

commutes; our desired map is the induced map on vertical cofibers. $\square$

Arguing exactly as in the proof of Proposition 6.2.7 shows the following result at the prime 3:

Proposition 6.2.12 *Let X_3 denote the 8–skeleton of $T(1) = \mathbb{S}/\!/\alpha_1$. There are 3–complete equivalences*

$$B \wedge X_3 \simeq T(2) \vee \Sigma^8 T(2), \quad B \wedge X_3 \wedge \mathbb{S}/(3, v_1) \simeq y(2) \vee \Sigma^8 y(2).$$

In forthcoming work, we will discuss the relation between Proposition 6.2.7 and EHP sequences, along the lines of Remark 6.2.4.

7 C_2–equivariant analogue of Corollary B

Our goal in this section is to study a C_2–equivariant analogue of Corollary B at height 1. The odd primary analogue of this result is deferred to the future; it is considerably more subtle.

7.1 C_2–equivariant analogues of Ravenel's spectra

In this section, we construct the C_2–equivariant analogue of $T(n)$ for all n. We 2–localize everywhere until mentioned otherwise. There is a C_2–action on $\Omega\mathrm{SU}(n)$ given by complex conjugation, and the resulting C_2–space is denoted by $\Omega\mathrm{SU}(n)_{\mathbb{R}}$. Real Bott periodicity gives a C_2–equivariant map $\Omega\mathrm{SU}(n)_{\mathbb{R}} \to BU_{\mathbb{R}}$ whose Thom spectrum is the (genuine) C_2–spectrum $X(n)_{\mathbb{R}}$. This admits the structure of an $\boldsymbol{E}_\rho$–ring, since it is the Thom spectrum of an $\boldsymbol{E}_\rho$–map $\Omega^\rho B^\sigma \mathrm{SU}(n)_{\mathbb{R}} \to \Omega^\rho B^\rho BU_{\mathbb{R}} \simeq \Omega^\rho B\mathrm{SU}_{\mathbb{R}}$. As in the nonequivariant case, the equivariant Quillen idempotent on $MU_{\mathbb{R}}$ restricts to one on $X(m)_{\mathbb{R}}$, and therefore defines a summand $T(n)_{\mathbb{R}}$ of $X(m)_{\mathbb{R}}$ for $2^n \leq m \leq 2^{n+1} - 1$. Again, this summand admits the structure of an $\boldsymbol{E}_1$–ring.

Construction 7.1.1 There is an equivariant fiber sequence

$$\Omega\mathrm{SU}(n)_{\mathbb{R}} \to \Omega\mathrm{SU}(n+1)_{\mathbb{R}} \to \Omega S^{n\rho+1},$$

where ρ is the regular representation of C_2; the equivariant analogue of Proposition 2.1.6 then shows that there is a map $\Omega S^{n\rho+1} \to B\mathrm{GL}_1(X(n)_{\mathbb{R}})$ (detecting an element $\chi_n \in \pi_{n\rho-1} X(n)_{\mathbb{R}}$) whose Thom spectrum is $X(n+1)_{\mathbb{R}}$. Here, $B\mathrm{GL}_1(X(n)_{\mathbb{R}})$ is the delooping of the $\boldsymbol{E}_\rho$–space $\mathrm{GL}_1(X(n)_{\mathbb{R}})$, and the C_2–equivariant notion of Thom spectrum is taken in the sense of [46, Theorem 3.2]. (The constructions from loc. cit. can be verified to go through for equivariant maps to $B\mathrm{GL}_1(X(n)_{\mathbb{R}})$; for example, when $n = \infty$, the idea of taking Thom spectra for an equivariant map to $B\mathrm{GL}_1(MU_{\mathbb{R}})$ was already used in [47, Section 3].)

If $\widetilde{\sigma}_n$ denotes the image of the element $\chi_{2^{n+1}\rho-1}$ in $\pi_{(2^{n+1}-1)\rho-1} T(n)_{\mathbb{R}}$, then we have a C_2–equivariant analogue of Lemma 3.1.12:

Lemma 7.1.2 *The Thom spectrum of the map*

$$\Omega S^{(2^{n+1}-1)\rho+1} \to B\mathrm{GL}_1(X(2^{n+1}-1)_{\mathbb{R}})$$

detecting $\widetilde{\sigma}_n$ is a direct sum of shifts of $T(n+1)_{\mathbb{R}}$.

Example 7.1.3 For instance, $T(1)_{\mathbb{R}} = X(2)_{\mathbb{R}}$ is the Thom spectrum of the map $\Omega S^{\rho+1} \to BU_{\mathbb{R}}$; upon composing with the equivariant J–homomorphism $BU_{\mathbb{R}} \to B\mathrm{GL}_1(\mathbb{S})$, this detects the element $\widetilde{\eta} \in \pi_\sigma \mathbb{S}$, and the extension of the map $S^\rho \to B\mathrm{GL}_1(\mathbb{S})$ to $\Omega S^{\rho+1}$ uses the $\boldsymbol{E}_1$–structure on $B\mathrm{GL}_1(\mathbb{S})$. The case of $X(2)_{\mathbb{R}}$ exhibits a curious property: $S^{\rho+1}$ is the loop space $\Omega^\sigma \mathbb{H}P^\infty_{\mathbb{R}}$, and there are equivalences (see [48, Propositions 3.4 and 3.6])

$$\Omega S^{\rho+1} \simeq \Omega^{\sigma+1} \mathbb{H}P^\infty_{\mathbb{R}} \simeq \Omega^\sigma (\Omega \mathbb{H}P^\infty_{\mathbb{R}}).$$

However, $\Omega \mathbb{H}P^\infty_{\mathbb{R}} \simeq S^{\rho+\sigma}$, so $\Omega S^{\rho+1} = \Omega^\sigma S^{\rho+\sigma}$. The map $\Omega^\sigma S^{\rho+\sigma} \to B\mathrm{GL}_1(\mathbb{S})$ still detects the element $\widetilde{\eta} \in \pi_\sigma \mathbb{S}$ on the bottom cell, but the extension of the map $S^\rho \to B\mathrm{GL}_1(\mathbb{S})$ to $\Omega^\sigma S^{\rho+\sigma}$ is now defined via the $\boldsymbol{E}_\sigma$–structure on $B\mathrm{GL}_1(\mathbb{S})$. The upshot of this discussion is that $X(2)_{\mathbb{R}}$ is not only the free $\boldsymbol{E}_1$–ring with a nullhomotopy of $\widetilde{\eta}$, but also the free $\boldsymbol{E}_\sigma$–algebra with a nullhomotopy of $\widetilde{\eta}$.

Warning 7.1.4 Unlike the nonequivariant setting, the element $\widetilde{\eta} \in \pi_\sigma \mathbb{S}$ is neither torsion nor nilpotent. This is because its geometric fixed points is $\Phi^{C_2} \widetilde{\eta} = 2 \in \pi_0 \mathbb{S}$; see [39, Proposition C.5], although note that the orientations chosen there are the opposite of ours. Briefly, the map $\widetilde{\eta}$ is obtained by ρ–desuspending the unstable equivariant Hopf map $S^{\rho+\sigma} = \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1 = S^\rho$, whose homotopy fiber is S^σ. In other words, there is a fiber sequence $S^\sigma \to S^{\rho+\sigma} \xrightarrow{\widetilde{\eta}} S^\rho$. On geometric fixed points, this produces the fiber sequence $S^0 = C_2 \to S^1 \to S^1$, which forces the map $\Phi^{C_2}\widetilde{\eta}$ to have degree 2 (or -2, depending on the choice of orientation).

Example 7.1.5 Consider the element $\widetilde{\sigma}_1 \in \pi_{3\rho-1} T(1)_{\mathbb{R}}$. The underlying nonequivariant element of $\pi_5 T(1)_{\mathbb{R}}$ is simply σ_1. To determine $\Phi^{C_2}\widetilde{\sigma}_1 \in \pi_2 \Phi^{C_2} T(1)_{\mathbb{R}}$, we first note that $\Phi^{C_2} T(1)_{\mathbb{R}}$ is the Thom spectrum of the map $\Phi^{C_2}\widetilde{\eta}\colon \Phi^{C_2} \Omega S^{\rho+1} \to B\mathrm{GL}_1(\mathbb{S})$. Since $\Phi^{C_2}\Omega S^{\rho+1} = \Omega S^2$ and $\Phi^{C_2}\widetilde{\eta} = 2$, we find that $\Phi^{C_2} T(1)_{\mathbb{R}}$ is the $\boldsymbol{E}_1$–quotient $\mathbb{S}/\!\!/2$. The element $\Phi^{C_2}\widetilde{\sigma}_1 \in \pi_2 \mathbb{S}/\!\!/2 \cong \pi_2 \mathbb{S}/2$ is simply a map $S^2 \to \mathbb{S}/2$ which is η on the top cell. Such a map exists because $2\eta = 0$.

As an aside, we mention that there is a C_2–equivariant lift of the spectrum A:

Definition 7.1.6 Let A_{C_2} denote the Thom spectrum of the map $\Omega S^{2\rho+1} \to B\mathrm{GL}_1(\mathbb{S})$ defined by the extension of the map $S^{2\rho} \to B\mathrm{GL}_1(\mathbb{S})$ which detects the equivariant Hopf map $\widetilde{\nu} \in \pi_{2\rho-1}\mathbb{S}$.

Remark 7.1.7 The underlying spectrum of A_{C_2} is A. To determine the geometric fixed points of A_{C_2}, $\Phi^{C_2} A_{C_2}$ is the Thom spectrum of the map $\Phi^{C_2}\widetilde{\nu}\colon \Phi^{C_2}\Omega S^{2\rho+1} \to B\mathrm{GL}_1(\mathbb{S})$. We claim that $\widetilde{\Phi}^{C_2}\widetilde{\nu} = \eta$; indeed, the map $\widetilde{\nu}$ is obtained by 2ρ–desuspending the unstable equivariant map $S^{4\rho-1} = \mathbb{H}^2 - \{0\} \to \mathbb{H}P^1 = S^{2\rho}$. The homotopy fiber of this map is $S^{2\rho-1} = S^{\rho+\sigma}$, so that there is an equivariant fiber sequence $S^{\rho+\sigma} \to S^{4\rho-1} \to S^{2\rho}$. On geometric fixed points, we obtain a fiber sequence $S^1 \to S^3 \to S^2$, which implies that $\Phi^{C_2}\widetilde{\nu}$ be identified with the Hopf fibration $S^3 \to S^2$. Now, since $\Phi^{C_2}\Omega S^{2\rho+1} = \Omega S^3$, we find that $\Phi^{C_2} A_{C_2} = T(1)$. In particular, A_{C_2} may be thought of as the free C_2–equivariant $\boldsymbol{E}_1$–ring with a nullhomotopy of $\widetilde{\nu}$.

Example 7.1.8 The element $\widetilde{\sigma}_1$ lifts to $\pi_{3\rho-1}A_{C_2}$. Indeed, Remark 3.2.17 works equivariantly too: the equivariant Hopf map $S^{3\rho-1} \to S^{2\rho}$ defines a composite $S^{3\rho-1} \to S^{2\rho} \to \Omega S^{2\rho+1}$. The composite $S^{3\rho-1} \to \Omega S^{2\rho+1} \to B\mathrm{SU}_{\mathbb{R}}$ is null, since $\pi_{3\rho-1}B\mathrm{SU}_{\mathbb{R}} = 0$. It follows that, upon Thomification, the map $S^{3\rho-1} \to \Omega S^{2\rho+1}$ defines an element $\widetilde{\sigma}_1'$ of $\pi_{3\rho-1}A_{C_2}$. In order to show that this element indeed deserves to be called $\widetilde{\sigma}_1$, we use Proposition 7.1.9. The map $A_{C_2} \to T(1)_{\mathbb{R}}$ from the proposition induces a map $\pi_{3\rho-1}A_{C_2} \to \pi_{3\rho-1}T(1)_{\mathbb{R}}$, and we need to show that the image of $\widetilde{\sigma}_1' \in \pi_{3\rho-1}A_{C_2}$ is in fact $\widetilde{\sigma}_1$. By Example 7.1.5, it suffices to observe that the underlying nonequivariant map corresponding to $\widetilde{\sigma}_1' \in \pi_{3\rho-1}T(1)_{\mathbb{R}}$ is σ_1, and that the geometric fixed point $\Phi_{C_2}\widetilde{\sigma}_1' \in \pi_2\mathbb{S}/\!\!/2$ is the lift of η appearing in Example 7.1.5.

We now prove the proposition used above.

Proposition 7.1.9 *There is a genuine C_2–equivariant $\boldsymbol{E}_1$–map $A_{C_2} \to T(1)_{\mathbb{R}}$.*

Proof By Remark 7.1.7, it suffices to show that $\widetilde{\nu} = 0 \in \pi_{3\rho-1}T(1)_{\mathbb{R}}$. The underlying map is null, because $\nu = 0 \in \pi_5 T(1)$. The geometric fixed points are also null, because $\Phi^{C_2}\widetilde{\nu} = \eta$ is null in $\pi_2\Phi^{C_2}T(1)_{\mathbb{R}} = \pi_2\mathbb{S}/\!\!/2$. Therefore, $\widetilde{\nu}$ is null in $\pi_{3\rho-1}T(1)_{\mathbb{R}}$. □

In fact, it is easy to prove the following analogue of Proposition 6.2.1:

Proposition 7.1.10 *There is a C_2–equivariant equivalence $A_{C_2} \wedge C\widetilde{\eta} \simeq T(1)_{\mathbb{R}}$.*

Proof There are maps $A_{C_2} \to T(1)_{\mathbb{R}}$ and $C\widetilde{\eta} \to T(1)_{\mathbb{R}}$, which define a map $A_{C_2} \wedge C\widetilde{\eta} \to T(1)_{\mathbb{R}}$ via the multiplication on $T(1)_{\mathbb{R}}$. This map is an equivalence on underlying spaces by Proposition 6.2.1, and on geometric fixed points induces the map $T(1) \wedge \mathbb{S}/2 \to \mathbb{S}/\!\!/2$. This was also proved in the course of Proposition 6.2.1. □

Remark 7.1.11 As in Remark 6.2.2, one might hope that this implies the C_2–equivariant Wood equivalence $bo_{C_2} \wedge C\widetilde{\eta} \simeq bu_{\mathbb{R}}$ via some equivariant analogue of Theorem A.

Remark 7.1.12 The equivariant analogue of Remark 6.2.4 remains true: the equivariant Wood equivalence of Proposition 7.1.10 stems from the EHP sequence $S^\rho \to \Omega S^{\rho+1} \to \Omega S^{2\rho+1}$. To prove the existence of such a fiber sequence, we use [37, Construction 4.26] to get the Hopf map $h\colon \Omega S^{\rho+1} \to \Omega S^{2\rho+1}$, as well as a nullhomotopy of the composite $S^\rho \to \Omega S^{\rho+1} \to \Omega S^{2\rho+1}$. In particular, if $F = \mathrm{fib}(h)$, there is an equivariant map $S^\rho \to F$. We claim that this map is an equivalence: it suffices to prove that $S^\rho \to F$ is an equivalence on underlying and on geometric fixed points, since these functors preserve homotopy limits and colimits, and these functors are jointly conservative. The desired equivalence on underlying spaces follows from the classical EHP sequence $S^2 \to \Omega S^3 \to \Omega S^5$, and the equivalence on geometric fixed points follows from the splitting $\Omega S^2 \simeq S^1 \times \Omega S^3$.

7.2 The C_2–equivariant analogue of Corollary B at $n = 1$

Recall (see [55]) that there are indecomposable classes $\bar{v}_n \in \pi_{(2^n-1)\rho}\mathrm{BP}_{\mathbb{R}}$; as in Theorem 3.1.5, these lift to classes in $\pi_\star T(m)_{\mathbb{R}}$ if $m \geq n$. The main result of this section is the following:

Theorem 7.2.1 *There is a map $\Omega^\rho S^{2\rho+1} \to B\mathrm{GL}_1(T(1)_{\mathbb{R}})$ detecting an indecomposable in $\pi_\rho T(1)_{\mathbb{R}}$ on the bottom cell, whose Thom spectrum is $\underline{H\mathbb{Z}}$.*

Note that, as with Corollary B at $n = 1$, this result is *unconditional*. The argument is exactly as in the proof of Corollary B at $n = 1$, with practically no modifications. We need the following analogue of Theorem 2.2.1, originally proved in [24; 48]:

Proposition 7.2.2 (Behrens and Wilson; Hahn and Wilson) *Let p be any prime, and let λ denote the 2–dimensional standard representation of C_p on $\mathbb{C}$. The Thom spectrum of the map $\Omega^\lambda S^{\lambda+1} \to B\mathrm{GL}_1(S^0)$ extending the map $1-p\colon S^1 \to B\mathrm{GL}_1(S^0)$ is equivalent to $\underline{H\mathbb{F}_p}$ as an $\boldsymbol{E}_\lambda$–ring. Moreover, if $S^{\lambda+1}\langle\lambda+1\rangle$ denotes the $(\lambda+1)$–connected cover of $S^{\lambda+1}$ (ie the fiber of the map $S^{\lambda+1} \to \Omega^\infty\Sigma^{\lambda+1}\underline{H\mathbb{Z}}$), then the Thom spectrum of the induced map $\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle \to B\mathrm{GL}_1(S^0)$ is equivalent to $\underline{H\mathbb{Z}}$ as an $\boldsymbol{E}_\lambda$–ring.*

Proof of Theorem 7.2.1 In [48], the authors prove that there is an equivalence of C_2–spaces between $\Omega^\lambda S^{\lambda+1}$ and $\Omega^\rho S^{\rho+1}$, and that $\underline{H\mathbb{F}_2}$ is in fact the Thom spectrum of the induced map $\Omega^\rho S^{\rho+1} \to B\mathrm{GL}_1(S^0)$ detecting -1. Since both $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle$ and $\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle$ are defined as fibers of maps to S^1 which are degree one on the bottom cell, Hahn and Wilson's equivalence lifts to a C_2–equivariant equivalence $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle \simeq \Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle$, and we find that $\underline{H\mathbb{Z}}$ is the Thom spectrum of the map $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle \to B\mathrm{GL}_1(S^0)$.

Since $T(1)_{\mathbb{R}}$ is the Thom spectrum of the composite map $\Omega S^{\rho+1} \to \Omega^\rho S^{\rho+1}\langle\rho+1\rangle \to B\mathrm{GL}_1(S^0)$ detecting $\tilde{\eta}$ on the bottom cell of the source, it follows from the C_2–equivariant analogue of Proposition 2.1.6 and the above discussion that it is sufficient to define a fiber sequence

$$\Omega S^{\rho+1} \to \Omega^\rho S^{\rho+1}\langle\rho+1\rangle \to \Omega^\rho S^{2\rho+1},$$

and check that the induced map $\Omega^\rho S^{2\rho+1} \to B\mathrm{GL}_1(T(1)_{\mathbb{R}})$ detects an indecomposable element of $\pi_\rho T(1)_{\mathbb{R}}$. See Remark 4.1.4 for the nonequivariant analogue of this fiber sequence.

Since there is an equivalence $\Omega S^{\rho+1} \simeq \Omega^\sigma S^{\rho+\sigma}$, it suffices to prove that there is a fiber sequence

$$S^{\rho+\sigma} \to \Omega S^{\rho+1}\langle\rho+1\rangle \to \Omega S^{2\rho+1}; \tag{7-1}$$

taking σ–loops produces the desired fiber sequence. The fiber sequence (7-1) can be obtained by taking vertical fibers in the map of fiber sequences

$$\begin{tikzcd} S^\rho \arrow[r] \arrow[d] & \Omega S^{\rho+1} \arrow[r] \arrow[d] & \Omega S^{2\rho+1} \arrow[d] \\ \mathbb{C}P^\infty_{\mathbb{R}} \arrow[r, equal] & \mathbb{C}P^\infty_{\mathbb{R}} \arrow[r] & * \end{tikzcd}$$

Here, the top horizontal fiber sequence is the EHP fiber sequence

$$S^\rho \to \Omega S^{\rho+1} \to \Omega S^{2\rho+1}.$$

To identify the fibers, note that there is the Hopf fiber sequence

$$S^{\rho+\sigma} \xrightarrow{\widetilde{\eta}} S^\rho \to \mathbb{C}P^\infty_{\mathbb{R}}.$$

The fiber of the middle vertical map is $\Omega S^{\rho+1}\langle\rho+1\rangle$ via the definition of $S^{\rho+1}\langle\rho+1\rangle$ as the homotopy fiber of the map $S^{\rho+1} \to B\mathbb{C}P^\infty_{\mathbb{R}}$.

It remains to show that the map $\Omega^\rho S^{2\rho+1} \to B\mathrm{GL}_1(T(1)_{\mathbb{R}})$ detects an indecomposable element of $\pi_\rho T(1)_{\mathbb{R}}$. Indecomposability in $\pi_\rho T(1)_{\mathbb{R}} \cong \pi_\rho \mathrm{BP}_{\mathbb{R}}$ is the same as not being divisible by 2, so we just need to show that the dotted map in the following diagram does not exist:

$$\begin{tikzcd}
S^{\rho+1} \arrow[d, "E^2"'] \arrow[dr, "2"] & \\
\Omega^\rho S^{2\rho+1} \arrow[r] \arrow[d] & S^{\rho+1} \arrow[dl, dashed] \\
B\mathrm{GL}_1(T(1)_{\mathbb{R}}) &
\end{tikzcd}$$

If this factorization existed, there would be an orientation $\underline{H\mathbb{Z}} \to T(1)_{\mathbb{R}}$, which is absurd. $\square$

We now explain why we do not know how to prove the equivariant analogue of Corollary B at higher heights. One could propose an equivariant analogue of Conjecture D, and such a conjecture would obviously be closely tied with the existence of some equivariant analogue of the work of Cohen, Moore and Neisendorfer. We do not know if any such result exists, but it would certainly be extremely interesting.

Suppose that one wanted to prove a result like Corollary B, stating that the equivariant analogues of Conjectures D and E imply that there is a map $\Omega^\rho S^{2^n\rho+1} \to B\mathrm{GL}_1(T(n)_{\mathbb{R}})$, detecting an indecomposable in $\pi_{(2^n-1)\rho} T(n)_{\mathbb{R}}$ on the bottom cell, whose Thom spectrum is $\mathrm{BP}\langle n-1\rangle_{\mathbb{R}}$. One could then try to run the same proof as in the nonequivariant case by constructing a map from the fiber of a charming map $\Omega^\rho S^{2^n\rho+1} \to S^{(2^n-1)\rho+1}$ to $B\mathrm{GL}_1(T(n-1)_{\mathbb{R}})$, but the issue comes in replicating Step 1 of Section 5.2: there is *no* analogue of Lemma 3.1.16, since the equivariant element $\widetilde{\sigma}_n \in \pi_\star T(n)$ is neither torsion nor nilpotent. See Warning 7.1.4. This is intimately tied with the failure of an analogue of the nilpotence theorem in the equivariant setting. In future work, we shall describe a related project connecting the $T(n)$ spectra to the Andrews–Gheorghe–Miller w_n–periodicity in $\mathbb{C}$–motivic homotopy theory (see [7; 41; 57]).

However, since there is a map $\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle \to B\mathrm{GL}_1(\mathbb{S})$ as in Proposition 7.2.2, there may nevertheless be a way to construct a suitable map from the fiber of a charming map $\Omega^\rho S^{2^n\rho+1} \to S^{(2^n-1)\rho+1}$ to $B\mathrm{GL}_1(T(n-1)_{\mathbb{R}})$. Such a construction would presumably provide a more elegant construction of the nonequivariant map used in the proof of Theorem A.

8 Future directions

In this section, we suggest some directions for future investigation. This is certainly not an exhaustive list; there are numerous questions we do not know how to address that are scattered throughout this document, but we have tried to condense some of them into the list below. We have tried to order the questions in order of our interest in them. We have partial progress on many of these questions.

(a) Some obvious avenues for future work are the conjectures studied in this article: Conjectures D, E, F and 3.1.7. Can the $\boldsymbol{E}_3$–assumption in the statement of Theorem C be removed?

(b) One of the main goals of this project is to rephrase the proof of the nilpotence theorem from [38; 54]. As mentioned in Remark 2.2.3, the Hopkins–Mahowald theorem for $H\mathbb{F}_p$ immediately implies the nilpotence theorem for simple p–torsion classes in the homotopy of a homotopy commutative ring spectrum (see also [50]). We will expand on the relation between the results of this article and the nilpotence theorem in forthcoming work; see Remark 5.4.7 for a sketch.

From this point of view, Theorem A is very interesting: it connects torsion in the unstable homotopy groups of spheres (via Cohen, Moore and Neisendorfer) to nilpotence in the stable homotopy groups of spheres. We are not sure how to do so, but could the Cohen–Moore–Neisendorfer bound for the exponents of unstable homotopy groups of spheres be used to obtain bounds for the nilpotence exponent of the stable homotopy groups of spheres?

(c) It is extremely interesting to contemplate the interaction between unstable homotopy theory and chromatic homotopy theory apparent in this article. Connections between unstable homotopy theory and the chromatic picture have appeared elsewhere in the literature (eg in [17; 16; 70; 76]), but their relationship to the content of this project is not clear to me. It would be interesting to have this clarified. One naive hope is that such a connection could stem from a construction of a charming map (such as the Cohen–Moore–Neisendorfer map) via Weiss calculus.

(d) Let R denote $\mathbb{S}$ or A. The map $R \to \Theta(R)$ is an equivalence in dimensions $< |\sigma_n|$. Moreover, the $\Theta(R)$–based Adams–Novikov spectral sequence has a vanishing line of slope $1/|\sigma_n|$ (see [68] for the case $R = A$). Can another proof of this vanishing line be given using general arguments involving Thom spectra? We have some results in this direction which we shall address in future work.

(e) The unit maps from each of the Thom spectra in the second line of Table 1 to the corresponding designer spectrum on the third line are surjective on homotopy. In the case of tmf, this requires some computational effort to prove, and has been completed in [34]. This behavior is rather unexpected: in general, the unit map from a structured ring to some structured quotient will not be surjective on homotopy. Is there a conceptual reason for this surjectivity?

(f) In [22], the tmf resolution of a certain type 2 spectrum Z is studied. Mahowald uses the Thom spectrum A to study the bo resolution of the sphere in [68], so perhaps the spectrum B could be used to study the tmf resolution of Z. This is work in progress. See also Corollary 6.2.11 and the discussion preceding it.

(g) Is there an equivariant analogue of Theorem A at higher heights and other primes? Currently, we have such an analogue at height 1 and at $p = 2$; see Section 7.

(h) The Hopkins–Mahowald theorem may used to define Brown–Gitler spectra. Theorem A produces "relative" Brown–Gitler spectra for $\mathrm{BP}\langle n\rangle$, bo and tmf. In future work, we will study these spectra and show how they relate to the Davis–Mahowald *nonsplitting* of $\mathrm{tmf} \wedge \mathrm{tmf}$ as a wedge of shifts of bo–Brown–Gitler spectra smashed with tmf from [32].

(i) The story outlined in the introduction above could fit into a general framework of "fp Mahowaldean spectra" (for "finitely presented Mahowaldean spectrum", inspired by [74]), of which A, B, $T(n)$ and $y(n)$ would be examples. One might then hope for a generalization of Theorem A which relates fp Mahowaldean spectra to fp spectra. It would also be interesting to prove an analogue of Mahowald–Rezk duality for fp Mahowaldean spectra which recovers their duality for fp spectra upon taking Thom spectra as above.

(j) One potential approach to the question about surjectivity raised above is as follows. The surjectivity claim at height 0 is the (trivial) statement that the unit map $\mathbb{S} \to H\mathbb{Z}$ is surjective on homotopy. The Kahn–Priddy theorem, stating that the transfer $\lambda\colon \Sigma^\infty \mathbb{R}P^\infty \to \mathbb{S}$ is surjective on $\pi_{*\geq 1}$, can be interpreted as stating that $\pi_* \Sigma^\infty \mathbb{R}P^\infty$ contains those elements of $\pi_*\mathbb{S}$ which are *not* detected by $H\mathbb{Z}$. One is then led to wonder: for each of the Thom spectra R in the second line of Table 1, is there a spectrum P along with a map $\lambda_R\colon P \to R$ such that each $x \in \pi_* R$ in the kernel of the map $R \to \Theta(R)$ lifts along λ_R to $\pi_* P$? (The map $R \to \Theta(R)$ is an equivalence in dimensions $< |\sigma_n|$ (if R is of height n), so P would have bottom cell in dimension $|\sigma_n|$.)

Since $\Sigma^\infty \mathbb{R}P^\infty \simeq \Sigma^{-1} \mathrm{Sym}^2(\mathbb{S})/\mathbb{S}$, the existence of such a result is very closely tied to an analogue of the Whitehead conjecture (see [59]; the Whitehead conjecture implies the Kahn–Priddy theorem). In particular, one might expect the answer to the question posed above to admit some interaction with Goodwillie calculus.

(k) Let $p \geq 5$. Is there a p–primary analogue of B which would provide a Thom spectrum construction (via Table 1) of the conjectural spectrum eo_{p-1}? Such a spectrum would be the Thom spectrum of a p–complete spherical fibration over a p–local space built via $p-1$ fiber sequences from the loop spaces $\Omega S^{2k(p-1)+1}$ for $2 \leq k \leq p$.

(l) The spectra $T(n)$ and $y(n)$ have algebrogeometric interpretations: the stack $\mathcal{M}_{T(n)}$ (see [52]; this stack is well defined since $T(n)$ is homotopy commutative) associated to $T(n)$ classifies p–typical formal groups with a coordinate up to degree $p^{n+1}-1$, while $y(n)$ is the closed substack of $\mathcal{M}_{T(n)}$ defined by the vanishing locus of $p, v_1, \ldots, v_{n-1}$. What are the moduli problems classified by A and B? We do not know if this question even makes sense at $p = 2$, since A and B are a priori only $\boldsymbol{E}_1$–rings. Nonetheless, in [35], we provide a description of $\mathrm{tmf} \wedge A$ in terms of the Hodge filtration of the universal elliptic curve (even at $p = 2$); we also showed that $(\mathrm{tmf} \wedge A)[x_2]$ admits an $\boldsymbol{E}_2$–algebra structure, where $|x_2| = 2$.

(m) Theorem A shows that the Hopkins–Mahowald theorem for $H\mathbb{Z}_p$ can be generalized to describe forms of $\mathrm{BP}\langle n\rangle$; at least for small n, these spectra have associated algebrogeometric interpretations (see [52]). What is the algebrogeometric interpretation of Theorem A?

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The deformation space of nonorientable hyperbolic 3–manifolds

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We consider nonorientable hyperbolic 3–manifolds of finite volume M^3. When M^3 has an ideal triangulation Δ, we compute the deformation space of the pair (M^3, Δ) (its Neumann–Zagier parameter space). We also determine the variety of representations of $\pi_1(M^3)$ in $\mathrm{Isom}(\mathbb{H}^3)$ in a neighborhood of the holonomy. As a consequence, when some ends are nonorientable, there are deformations from the variety of representations that cannot be realized as deformations of the pair (M^3, Δ). We also discuss the metric completion of these structures and we illustrate the results on the Gieseking manifold.

57K32; 57K35, 57Q99

1 Introduction

Let M^3 be a complete noncompact hyperbolic three-manifold of finite volume. Assume first that M^3 is orientable. Assume also that M^3 has a geometric ideal triangulation Δ, defined by Neumann and Zagier [16]. Following Thurston's construction for the figure eight knot exterior in [18], Neumann and Zagier defined in [16] a deformation space of the pair (M^3, Δ) by considering the set of parameters of the ideal simplices of Δ subject to compatibility equations. We denote the Neumann–Zagier parameter space by $\mathrm{Def}(M^3, \Delta)$. It is proved in [16] that it is homeomorphic to an open subset of $\mathbb{C}^l$, where l is the number of ends of M^3.

Another approach to deformations is based on $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$, the variety of conjugacy classes of representations of $\pi_1(M^3)$ in $\mathrm{Isom}(\mathbb{H}^3)$. It is proved, for instance by Kapovich in [14], that a neighborhood of the holonomy of M^3 is bianalytic to an open subset of $\mathbb{C}^l$.

Both approaches to deformations can be used to prove the hyperbolic Dehn filling theorem (even if it is still an open question whether an orientable M^3 admits a geometric ideal triangulation). Among other things, one has to take into account that $\mathrm{Def}(M^3, \Delta)$ is a 2^l to 1 branched covering of the neighborhood in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. When M^3 is orientable, both approaches yield the same deformation space.

We investigate the nonorientable setting, that is, M^3 is a connected nonorientable hyperbolic 3–manifold of finite volume. When it has an ideal triangulation Δ, we define a deformation space of the pair $\mathrm{Def}(M^3, \Delta)$ à la Neumann and Zagier. Here is our main result (for simplicity, we assume that M^3 has a single end, which is nonorientable):

Theorem 1.1 *Let M^3 be a complete nonorientable hyperbolic 3–manifold of finite volume with a single end, which is nonorientable.*

(a) *If M^3 admits a geometric ideal triangulation Δ, then $\mathrm{Def}(M^3, \Delta) \cong (-1, 1)$, where the parameters $\pm t \in (-1, 1)$ correspond to the same structure.*

(b) *A neighborhood of the holonomy in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to an interval $(-1, 1)$.*

Furthermore, the holonomy map $\mathrm{Def}(M^3, \Delta) \to \mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ folds the interval $(-1, 1)$ at 0 and its image is the half-open interval $[0, 1)$, where 0 corresponds to the complete structure.

The version of this theorem with *several cusps* is Theorem 3.11.

For M^3 as in the theorem, structures in the subinterval $[0, 1) \subset (-1, 1)$ in the variety of representations are realized by $\mathrm{Def}(M^3, \Delta)$, but structures in $(-1, 0)$ are not. This corresponds to two different kinds of representations of the Klein bottle, which we call type I when realized, and II when not. These are described in Section 4.

Deformations of the complete structure are noncomplete, and therefore, for a deformation of the holonomy, the hyperbolic structure is not unique. Deformations of type I can be realized by ideal triangulations. Hence there is a natural choice of structure, and we prove in Theorem 5.15 that its metric completion consist in adding a singular geodesic so that it is the core of a solid Klein bottle and it has a singularity of cone angle in a neighborhood of zero. For deformations of type II, we prove that there is a natural choice of structure (radial), and the metric completion consists in adding a singular interval, also in Theorem 5.15. This singular interval is the soul of a twisted disc orbibundle over an interval with mirror boundary. Topologically, a neighborhood of this interval is the disc sum of two cones on a projective plane. Metrically, it is a conifold, with cone angle at the interior of the interval in a neighborhood of zero.

The paper is organized as follows. In Section 2 we describe $\mathrm{Def}(M^3, \Delta)$ and in Section 3 we describe $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. Section 4 is devoted to representations of the Klein bottle. Metric completions are described in Section 5, and finally, in Section 6, we describe in detail the deformation space(s) of the Gieseking manifold.

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2 Deformation space from ideal triangulations

Before discussing nonorientable manifolds, we recall first the orientable case. The first example was constructed by Thurston in his notes [18, Chapter 4] for the figure eight knot exterior, and the general case was constructed by Neumann and Zagier in [16]. We point the reader to these references for the upcoming exposition.

From the point of view of a triangulation, the deformation of the hyperbolic structure on a manifold with a given *geometric ideal* triangulation is the space of parameters of ideal tetrahedra, subject to compatibility equations.

A *geometric ideal* tetrahedron is a geodesic tetrahedron of $\mathbb{H}^3$ with all of its vertices in the ideal sphere $\partial_\infty \mathbb{H}^3$. We say that a hyperbolic 3–manifold *admits a geometric ideal triangulation* if it is the union of such tetrahedra, along the geodesic faces. Though it has been established in many cases, it is still an open problem to decide whether every orientable hyperbolic three-manifold of finite volume admits a geometric ideal triangulation.

Given an ideal tetrahedron in $\mathbb{H}^3$, up to isometry, we may assume that its ideal vertices in $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$ are 0, 1, ∞ and $z \in \mathbb{C}$. The idea of Thurston is to equip the (unoriented) edge between 0 and ∞ with the complex number z, called the *edge invariant*. The edge invariant determines the isometry class of the tetrahedron, and for different edges the corresponding invariants satisfy some relations, called *tetrahedron relations*:

- Opposite edges have the same invariant.
- Given three edges with a common endpoint and invariants z_1, z_2 and z_3, indexed following the right-hand rule towards the common ideal vertex, they are related to z_1 by $z_2 = 1/(1 - z_1)$ and $z_3 = (z_1 - 1)/z_1$.

Let M^3 be a possibly nonorientable complete hyperbolic 3–manifold of finite volume, which admits a geometric ideal triangulation $\Delta = \{A_1, \dots, A_n\}$. As we have stated before, up to (oriented) isometry the hyperbolic structure of each tetrahedron can be determined by a single edge invariant, thus the usual parametrization of the triangulation goes as follows: fix an edge e_i in each tetrahedron A_i, and consider its edge invariant z_i. Hence, the hyperbolic structure of M^3 can be parametrized by n parameters (one for each tetrahedron) and we will denote the parameters of the complete triangulation by $\{z_1^0, \dots, z_n^0\}$. The *deformation space of M^3 with respect to* Δ, $\mathrm{Def}(M^3, \Delta)$, is defined as the set of parameters $\{z_1, \dots, z_n\}$ in a small enough neighborhood of the complete structure for which the gluing bestows a hyperbolic structure on M^3. However, we find that the equations defining the deformation space are easier to work with if we use $3n$ parameters (one for each edge after taking into account the duplicity in opposite edges) and ask them to satisfy the second tetrahedron relation too.

When M^3 is orientable, in order for the gluing to be geometric, it is necessary and sufficient that around each edge cycle $[e] = \{e_{i_1,j_1}, \dots, e_{i_n,j_n}\}$ the following two compatibility conditions are satisfied:

$$\prod_{l=1}^{n} z(e_{i_l,j_l}) = 1, \tag{1}$$

$$\sum_{l=1}^{n} \arg(z(e_{i_l,j_l})) = 2\pi. \tag{2}$$

Geometrically, if we try to realize in $\mathbb{H}^3$ the tetrahedra around the edge cycle $[e]$, (1) means that the triangulation must "close up", and (2) means that the angle around $[e]$ must be precisely 2π (instead of a multiple). The parameters of the complete hyperbolic structure are denoted by $\{z^0(e_{1,1}),\dots,z^0(e_{n,3})\}$. In a small enough neighborhood of $\{z^0(e_{1,1}),\dots,z^0(e_{n,3})\}$, fulfillment of (1) implies (2). We end the overview of the orientable case with the theorem we want to extend to the nonorientable case:

Theorem 2.1 (Neumann and Zagier [16]) *Let M^3 be connected oriented hyperbolic of finite volume with l cusps. Then $\mathrm{Def}(M^3,\Delta)$ is biholomorphic to an open set of $\mathbb{C}^l$.*

When we deal with nonorientable manifolds, again the problem of the gluing being geometric lives within a neighborhood of the edges. The compatibility equations in this case carry the same geometric meaning as in (1) and (2), while accounting for the possible change of orientation of the tetrahedra.

Proposition 2.2 *Let M^3 be a nonorientable manifold that is triangulated by a finite number of ideal tetrahedra A_i, which bestows a hyperbolic structure around the edge cycle $[e]=\{e_{i_1,j_1},\dots,e_{i_n,j_n}\}$ if and only if the following compatibility equations are satisfied*:

$$\prod_{l=1}^{n}\frac{z(e_{i_l,j_l})^{\epsilon_l}}{\overline{z(e_{i_l,j_l})}^{1-\epsilon_l}}=1, \tag{3}$$

$$\sum_{l=1}^{n}\arg(z(e_{i_l,j_l}))=2\pi. \tag{4}$$

Here $z(e_{i_l,j_l})$ is the edge invariant of e_{i_l,j_l}, and $\epsilon_l=0,1$ in such a way that, in the gluing around the edge cycle $[e]$, a coherent orientation of the tetrahedra is obtained by gluing a copy of A_{i_l} with its orientation reversed if $\epsilon_l=0$ (or preserved if $\epsilon_l=1$), and with the initial condition that the orientation of the tetrahedron A_{i_1} is kept as given.

Proof When we follow a cycle of side identifications around an edge, we can always reorient the tetrahedra (maybe more than once) so that the gluing is done by orientable isometries. The compatibility equations for the orientable case can be then applied and hence, for the neighborhood of the edge cycle to inherit a hyperbolic structure, (1) must be satisfied, with the corresponding edge invariants.

Now, let us consider an edge $e_{i,j}\in A_i$ with parameter $z(e_{i_j})$. To see how the edge invariant changes under a nonorientable isometry, we can assume that A_i has vertices 0, 1, $z(e_{i,j})$ and ∞ in the upper space model, and consider the isometry c, the Poincaré extension of complex conjugation in $\partial_\infty\mathbb{H}^3\cong\mathbb{C}\cup\{\infty\}$. Then, the edge invariant of $c(e_{i,j})\in c(A_i)$ is $1/\overline{z(e_{i,j})}$.

Thus, the proposition follows with ease after changing the orientation of some tetrahedra. □

Definition 2.3 Let M^3 be a connected complete nonorientable hyperbolic 3–manifold of finite volume. Let Δ be an ideal triangulation of M^3. The *deformation space of M^3 related to the triangulation* Δ is the set $\mathrm{Def}(M^3, \Delta)$ consisting of those $(z_{1,1}, \dots, z_{n,3}) \in U \cap \mathbb{C}^{3n}$ satisfying the compatibility equations (3) and (4) and the tetrahedron relations, where U is a small enough neighborhood of the parameters $(z^0_{i,j})$ of the complete structure.

Let M^3_+ be the orientation covering of M^3. The ideal triangulation on M^3, Δ, can be lifted to an ideal triangulation Δ_+ on M^3_+. There is an orientation-reversing homeomorphism ι acting on M^3_+ such that $M^3 = M^3_+/\iota$ and $\iota^2 = \mathrm{Id}$. The triangulation on M^3_+ is constructed in the usual way: for every tetrahedron A_i we take another tetrahedron with the opposite orientation, $\iota(A_i)$, and glue them so that the orientation is coherent. For every edge $e_{i,j} \in A_i$, let $z(e_{i,j})$ or $z_{i,j}$ denote its edge invariant. Analogously, $w(\iota(e_{i,j}))$ or $w_{i,j}$ will denote the edge invariant of $\iota(e_{i,j}) \in \iota(A_i)$.

Remark 2.4 The compatibility equations (3) and (4) around $[e] \in M^3$ are precisely the (orientable) compatibility equations in any lift of $[e]$ to the orientation covering.

The orientation-reversing homeomorphism acts on $\mathrm{Def}(M^3_+, \Delta_+)$ by pulling back (equivalently, pushing forward) the associated hyperbolic metric on each tetrahedron. Combinatorially, the action is described in the following lemma:

Lemma 2.5 *Let $M^3 = M^3_+/\iota$, where ι is an orientation-reversing homeomorphism. Let M^3 admit an ideal triangulation Δ. Then ι acts on $\mathrm{Def}(M^3_+, \Delta_+)$ as*

$$\iota_*((z_{i,j}, w_{i,j})) = \left(\frac{1}{\overline{w}_{i,j}}, \frac{1}{\overline{z}_{i,j}}\right). \tag{5}$$

Proof The proof follows easily from the fact that ι permutes the edges and, for $e_{i,j} \in A_i$ with invariant $z(e_{i,j})$, the edge invariant of $c(e_{i,j}) \in c(A_i)$ is $1/\overline{z(e_{i,j})}$, where c is the Poincaré extension of complex conjugation. □

Remark 2.6 Metrics on tetrahedra are considered up to isotopy.

Corollary 2.7 *The map defined by $(z_{i,j}) \in \mathrm{Def}(M^3, \Delta) \mapsto (z_{i,j}, 1/\bar{z}_{i,j}) \in \mathrm{Def}(M^3_+, \Delta_+)^\iota$ is a real analytic isomorphism.*

Proof This follows from Remark 2.4 and Lemma 2.5. □

Our goal is to use Corollary 2.7 and Theorem 2.1 to identify the deformation space of M^3 with the fixed points under an action on $\mathbb{C}^k$. Let us suppose for the time being that M^3 has only one cusp which is nonorientable. The section of this cusp must be a Klein bottle. In order to define the biholomorphism through generalized Dehn filling coefficients, we must first fix a longitude–meridian pair in the peripheral torus in the orientation covering M^3_+. As we will see, there is a canonical choice. Afterwards, following Thurston, we will compute the derivative of the holonomy, hol', and translate the action of ι over there, and finally to the generalized Dehn filling coefficients.

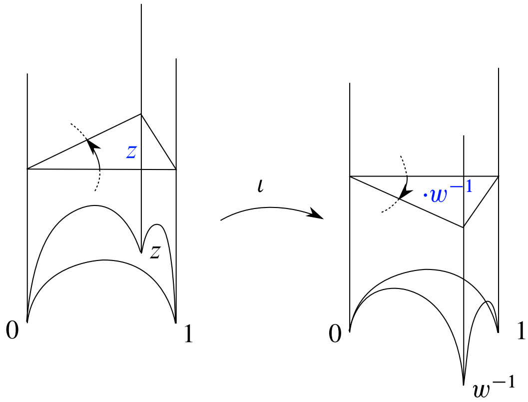

Figure 1: Change under the action of ι.

Fixing a longitude–meridian pair Letting K^2 be the Klein bottle, its fundamental group admits a presentation

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements a^2 and b in the orientation covering T^2 are generators of $\pi_1(T^2)$ and are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves). From now on, we will choose as longitude–meridian pair the elements:

$$l := a^2, \quad m := b.$$

Definition 2.8 The previous generators of $\pi_1(T^2)$ are called *distinguished* elements.

Lemma 2.9 *Let* $[\alpha] \in \pi_1(T)$*, and let* ι *be the involution in the orientation covering* M^3_+*, that is,* $M^3 \cong M^3_+/\iota$*. We also denote by* ι *the restriction of* ι *to the peripheral torus* T*. If*

$$\mathrm{hol}'(\alpha) = \prod_{r \in I} z(e_{i_r,j_r})^{\epsilon_r} \prod_{s \in J} w(\iota(e_{i_s,j_s}))^{\epsilon_s}, \tag{6}$$

where $\epsilon_r, \epsilon_s \in \{\pm 1\}$*, then*

$$\mathrm{hol}'(\iota(\alpha)) = \prod_{r \in I} w(\iota(e_{i_r,j_r}))^{-\epsilon_r} \prod_{s \in J} z(e_{i_s,j_s})^{-\epsilon_s}. \tag{7}$$

When we compute the derivative of the holonomy of an element, $\mathrm{hol}'(\gamma)$, we assume that $\mathrm{hol}(\gamma)$ fixes ∞.

Proof We use Thurston's method for computing the holonomy through the developing of triangles in $\mathbb{C}$; see [18]. Thus, the factor that each piece of path adds to the derivative of the holonomy changes, as in Figure 1, under the action of ι. □

Proposition 2.10 *For the chosen longitude–meridian pair, the action of* ι *on* $\mathrm{Im}(\mathrm{hol}') \subset \mathbb{C}^2$ *is*

$$\iota_*(L, M) = (\bar{L}, \overline{M}^{-1}), \tag{8}$$

where $L = \mathrm{hol}'(l)$*,* $M = \mathrm{hol}'(m)$*.*

Proof The action of ι on the longitude–meridian pair is $\iota_*(l)=l$ and $\iota_*(m)=m^{-1}$. Hence, the previous lemma implies that the derivative holonomy of the longitude and the meridian has the following features:

$$\begin{aligned}\mathrm{hol}'(m)&=\prod_{r\in I} z(e_{i_r,j_r})^{\epsilon_r}\prod_{s\in J} w(\iota(e_{i_s,j_s}))^{\epsilon_s}=\prod_{r\in I} w(\iota(e_{i_r,j_r}))^{\epsilon_r}\prod_{s\in J} z(e_{i_s,j_s})^{\epsilon_s},\\ \mathrm{hol}'(l)&=\prod_{r\in I}(z(e_{i_r,j_r})w(\iota(e_{i_r,j_r}))^{-1})^{\epsilon_r}.\end{aligned}$$ □

Remark 2.11 Following the notation of Proposition 2.10, $(L,M)\in(\mathbb{C}^2)^\iota$ if and only if $L\in\mathbb{R}$ and $|M|=1$.

Let us denote by $u:=\log\mathrm{hol}'(l)$ and $v:=\log\mathrm{hol}'(m)$ the *generalized Dehn coefficients*. These are the solutions in $\mathbb{R}^2\cup\{\infty\}$ to *Thurston's equation*

$$pu+qv=2\pi i. \tag{9}$$

Indeed, Neumann and Zagier's Theorem 2.1 (see also [18]) states that, for M^3 orientable, the map $(z_{i,j})\in\mathrm{Def}(M^3,\Delta)\mapsto(p_k,q_k)$ is a biholomorphism and the image is a neighborhood of $(\infty,\dots,\infty)\in\overline{\mathbb{C}}^l$, where l is the number of cusps of M^3.

Proposition 2.12 *The action of ι on $(p,q)\in U\cap\mathbb{R}^2\cup\{\infty\}$, where (p,q) are the generalized Dehn coefficients, is*

$$\iota_*(p,q)=(-p,q). \tag{10}$$

Proof The action of ι can be translated through the logarithm to (u,v) from the action on the holonomy (8) as $\iota_*(u,v)=(\bar u,-\bar v)$. Then, to find the action on generalized Dehn coefficients, we have to solve Thurston's equation (9) with $\bar u$ and $-\bar v$, that is,

$$p'\bar u-q'\bar v=2\pi i, \tag{11}$$

where $\iota_*(p,q)=(p',q')\in\mathbb{R}^2\cup\{\infty\}$. It is straightforward to check that $(p',q')=(-p,q)$ is the solution. □

Corollary 2.13 *The fixed points under ι, which are in correspondence with* $\mathrm{Def}(M^3,\Delta)$, *are those whose generalized Dehn filling coefficients are of type* $(0,q)$.

Theorem 2.14 *Let M^3 be a connected complete nonorientable hyperbolic* 3*–manifold of finite volume. Let M^3 have k nonorientable cusps and l orientable ones, and let it admit an ideal triangulation Δ. Then* $\mathrm{Def}(M^3,\Delta)$ *is real bianalytic to an open set of $\mathbb{R}^{k+2l}$.*

Proof We have already proved the theorem for $k=1$ and $l=0$.

Let $k=0$ and $l=1$. Any peripheral torus on M^3 is lifted to two peripheral tori, T_1 and T_2, on M^3_+. Here ι acts by permutation. More precisely, we can fix any longitude–meridian pair in one, $l_1,m_1\in\pi_1(T_1)$, and choose the longitude–meridian pair in the second torus as $l_2 := \iota_*(l_1), m_2 := \iota_*(m_1) \in \pi_1(T_2)$. The same arguments as in Proposition 2.10 show that $\iota_*(p_1, q_1, p_2, q_2) = -(p_2, q_2, p_1, q_1)$, and hence the fixed points have generalized coefficients $(p, q, -p, -q)$, $p, q \in \mathbb{R}$.

Finally, in general the action of ι on $\mathrm{Im}(\mathrm{hol}') \subset \mathbb{C}^{k+2l}$ can be understood as a product of $k+l$ actions $\iota_1 \times \cdots \times \iota_l$, the first k, ι_i for $i = 1, \dots, k$, acting on $\mathbb{C}$ as in the case for a Klein bottle cusp, and the subsequent l, ι_j for $j = k+1, \dots, k+l$, acting on $\mathbb{C}^2$ as in the case for a peripheral torus. □

3 Varieties of representations

The group of isometries of hyperbolic space is denoted by G, and we will use the well-known isomorphisms

$$G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PO}(3,1) \cong \mathrm{PSL}(2,\mathbb{C}) \rtimes \mathbb{Z}_2$$

in order to identify elements of G with elements of $\mathrm{PSL}(2,\mathbb{C}) \rtimes \mathbb{Z}_2$. The group G has two connected components, according to whether the isometries preserve or reverse the orientation.

For a finitely generated group Γ, the variety of representations of Γ in G is denoted by

$$\hom(\Gamma, G).$$

As G is algebraic, it has a natural structure of an algebraic set (see Johnson and Millson [13]), but we consider only its topological structure. We are interested in the set of conjugacy classes of representations

$$\mathcal{R}(\Gamma, G) = \hom(\Gamma, G)/G.$$

When M^3 is hyperbolic, we write $\Gamma = \pi_1(M^3)$. The holonomy of M^3

$$\mathrm{hol}\colon \Gamma \to G$$

is well defined up to conjugacy, and hence $[\mathrm{hol}] \in \mathcal{R}(\Gamma, G)$. To understand deformations, we analyze a neighborhood of the holonomy in $\mathcal{R}(\Gamma, G)$. The main result of this section is:

Theorem 3.1 *Let M^3 be a hyperbolic manifold of finite volume. Assume that it has k nonorientable cusps and l orientable cusps. Then there exists a neighborhood of $[\mathrm{hol}]$ in $\mathcal{R}(\Gamma, G)$ homeomorphic to $\mathbb{R}^{k+2l}$.*

When M^3 is orientable this result is well known (see for instance Boileau and Porti [4] or Kapovich [14]), and hence we assume that M^3 is nonorientable. We will prove a more precise result in Theorem 3.10, as for our purposes it is relevant to describe local coordinates in terms of the geometry of holonomy structures at the ends.

Before starting the proof, we need a lemma on varieties of representations. The projection to the quotient $\pi\colon \hom(\Gamma, G) \to \mathcal{R}(\Gamma, G)$ can have quite bad properties. For instance, even if $\hom(\Gamma, G)$ is Hausdorff, in general $\mathcal{R}(\Gamma, G)$ is not. But, in a neighborhood of the holonomy:

Lemma 3.2 *There exists a neighborhood $V \subset \mathcal{R}(\Gamma, G)$ of* [hol] *such that*:

(a) *If $[\rho] = [\rho'] \in V$, then the matrix $A \in G$ satisfying $A\rho(\gamma)A^{-1} = \rho'(\gamma)$ for all $\gamma \in \Gamma$ is **unique**.*

(b) *V is Hausdorff and the projection $\pi : \pi^{-1}(V) \to V$ is open.*

(c) *If $[\rho] \in V$, then for all $\gamma \in \Gamma$, $\rho(\gamma)$ preserves the orientation of $\mathbb{H}^3$ if and only if γ is represented by a loop that preserves the orientation of M^3.*

Assertions (a) and (b) are proved, for instance, by Johnson and Millson in [13]. They define the property of *good* representation, that is, open in $\mathcal{R}(\Gamma, G)$; it implies assertions (a) and (b), and it is satisfied by the conjugacy class of the holonomy. Assertion (c) is clear by continuity and the decomposition of G into two components, according to the orientation.

To describe the neighborhood of the holonomy in $\mathcal{R}(\Gamma, G)$ we use the orientation covering.

3.1 Orientation covering and the involution on representations

As mentioned, we assume M^3 is nonorientable. Let

$$M^3_+ \to M^3$$

denote the orientation covering, with fundamental group $\Gamma_+ = \pi_1(M^3_+)$. In particular, we have a short exact sequence:

$$1 \to \Gamma_+ \to \Gamma \to \mathbb{Z}_2 \to 1.$$

Definition 3.3 For $\zeta \in \Gamma \setminus \Gamma_+$, define the group automorphism

$$\sigma_* \colon \Gamma_+ \to \Gamma_+, \quad \gamma \mapsto \zeta\gamma\zeta^{-1}.$$

The automorphism σ_* depends on the choice of $\zeta \in \Gamma \setminus \Gamma_+$: automorphisms corresponding to different choices of ζ differ by composition (or precomposition) with an inner automorphism of Γ_+. Furthermore, σ_*^2 is an inner automorphism because $\zeta^2 \in \Gamma_+$. This automorphism σ_* is the map induced by the deck transformation of the orientation covering $M^3_+ \to M^3$.

The map induced by σ_* in the variety of representations is denoted by

$$\sigma^* \colon \mathcal{R}(\Gamma, G) \to \mathcal{R}(\Gamma, G), \quad [\rho] \mapsto [\rho \circ \sigma_*],$$

and σ^* does not depend on the choice of ζ because σ_* is well defined up to inner automorphism. Furthermore σ^* is an involution, $(\sigma^*)^2 = \mathrm{Id}$.

Consider the restriction map

$$\mathrm{res} \colon \mathcal{R}(\Gamma, G) \to \mathcal{R}(\Gamma_+, G)$$

that maps the conjugacy class of a representation of Γ to the conjugacy class of its restriction to Γ_+.

Lemma 3.4 *There exist* $U \subset \mathcal{R}(\Gamma, G)$ *a neighborhood of* [hol] *and* $V \subset \mathcal{R}(\Gamma_+, G)$ *a neighborhood of* res([hol]) *such that*

$$\mathrm{res}\colon U \xrightarrow{\cong} \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

is a homeomorphism.

Proof We show first that $\mathrm{res}(\mathcal{R}(\Gamma, G)) \subset \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}$. If $\rho_+ = \mathrm{res}(\rho)$, then for all $\gamma \in \Gamma_+$,

$$\sigma^*(\rho_+)(\gamma) = \rho_+(\sigma_*(\gamma)) = \rho_+(\zeta\gamma\zeta^{-1}) = \rho(\zeta)\rho_+(\gamma)\rho(\zeta)^{-1}.$$

Hence $\sigma^*([\mathrm{res}(\rho)]) = [\mathrm{res}(\rho)]$.

Next, given $[\rho_+] \in \mathcal{R}(\Gamma_+, G)$ satisfying $\sigma^*([\rho_+]) = [\rho_+]$, by construction there exists $A \in G$ that conjugates ρ_+ and $\rho_+ \circ \sigma_*$. We chose the neighborhood V so that Lemma 3.2 applies, and hence such an $A \in G$ is unique. From uniqueness (of A and A^2), it follows easily that, if $\zeta \in \Gamma \setminus \Gamma_+$ is the element such that σ_* is conjugation by ζ, then, by choosing $\rho(\zeta) = A$, ρ_+ extends to $\rho\colon \Gamma \to G$. Hence

$$\mathrm{res}(\mathcal{R}(\Gamma, G)) = \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}.$$

Let $U = \mathrm{res}^{-1}(V)$. With this choice of U and V,

$$\mathrm{res}\colon U \to \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

is a continuous bijection.

Finally we establish continuity of res^{-1} using a slice. The existence of a slice $S \subset \mathcal{R}(\Gamma_+, G)$ at res(hol) is proved by Johnson and Millson in [13, Theorem 1.2], who point to Borel and Wallach [5, IX.5.3] for a definition of slice. From the properties of the slice, and as the stabilizer of $\mathrm{hol}|_{\Gamma_+}$ is trivial, the natural map $G \times S \to \mathcal{R}(\Gamma_+, G)$, that maps $(g, s) \in G \times S$ to gsg^{-1}, yields a homeomorphism between $G \times S$ and a neighborhood of the orbit of res(hol), and the projection induces a homeomorphism $S \cong V$. It follows from the product structure that the $A \in G$ that conjugates ρ_+ and $\rho_+ \circ \sigma_*$ is continuous on ρ_+, so the extension of ρ_+ to a representation of the whole Γ is continuous on ρ_+. Then continuity of res^{-1} follows by composing the homeomorphism $V \cong S$ (restricted to the fixed point set of σ^*) with the extension from Γ_+ to Γ, and projecting to $U \subset \mathcal{R}(\Gamma, G)$. □

As Γ_+ preserves the orientation, next we use the complex structure of the identity component $G_0 = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$.

3.2 Representations in $\mathrm{PSL}(2, \mathbb{C})$

The holonomy of the orientation covering M^3_+ is contained in $\mathrm{PSL}(2, \mathbb{C})$, and it is well defined up to the action of $G = \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ by conjugation. If we furthermore choose an orientation on M^3_+, then the holonomy is unique up to the action by conjugacy of $G_0 = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, and complex conjugation corresponds to changing the orientation. We call the conjugacy class in $\mathrm{PSL}(2, \mathbb{C})$ of the holonomy of M^3_+ the *oriented holonomy*.

We consider

$$\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})) = \hom(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))/\,\mathrm{PSL}(2,\mathbb{C}).$$

Its local structure is well known:

Theorem 3.5 *A neighborhood of the oriented holonomy of M^3_+ in $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ has a natural structure of a $\mathbb{C}$–analytic variety defined over $\mathbb{R}$.*

The fact that it is $\mathbb{C}$–analytic follows, for instance, from [13] or [14]. In Theorem 3.9 we precisely describe $\mathbb{C}$–analytic coordinates; for the moment this is sufficient for our purposes.

Lemma 3.6 *Let hol_+ be the oriented holonomy of M^3_+. Then*

$$[\mathrm{hol}_+] \neq [\overline{\mathrm{hol}_+}] \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})).$$

Namely, the oriented holonomy and its complex conjugate are not conjugate by a matrix in $\mathrm{PSL}(2,\mathbb{C})$.

Proof For contradiction, assume that hol_+ and $\overline{\mathrm{hol}_+}$ are conjugate by a matrix in $\mathrm{PSL}(2,\mathbb{C})$, so there exists an orientation-preserving isometry $A \in \mathrm{PSL}(2,\mathbb{C})$ such that

$$A\,\mathrm{hol}_+(\gamma)A^{-1} = \overline{\mathrm{hol}_+(\gamma)} \quad \text{for all } \gamma \in \Gamma_+.$$

Consider the orientation-reversing isometry $B = c \circ A$, where c is the isometry with Möbius transformation complex conjugation, $z \mapsto \bar{z}$. The previous equation is equivalent to

$$B\,\mathrm{hol}_+(\gamma)B^{-1} = \mathrm{hol}_+(\gamma) \quad \text{for all } \gamma \in \Gamma_+. \tag{12}$$

Brouwer's fixed point theorem yields that the fixed point set of B in the ball compactification $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ is nonempty:

$$\mathrm{Fix}(B) = \{x \in \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \mid B(x) = x\} \neq \varnothing.$$

By (12), $\mathrm{hol}_+(\Gamma_+)$ preserves $\mathrm{Fix}(B)$. Thus, by minimality of the limit set of a Kleinian group, since $\mathrm{Fix}(B) \neq \varnothing$ is closed and $\mathrm{hol}_+(\Gamma_+)$–invariant, it contains the whole ideal boundary: $\partial_\infty \mathbb{H}^3 \subset \mathrm{Fix}(B)$. Hence B is the identity, contradicting that B reverses the orientation. □

From Lemma 3.6 and Theorem 3.5, we obtain:

Corollary 3.7 *There exists a neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ of the conjugacy class of the oriented holonomy of M_+ that is disjoint from its complex conjugate:*

$$\overline{W} \cap W = \varnothing.$$

By choosing the neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ sufficiently small, we may assume that its projection to $\mathcal{R}(\Gamma_+, G)$ is contained in V, as in Lemma 3.4. The neighborhood V can also be chosen smaller, to be equal to the projection of W, as this map is open. Namely the neighborhoods can be chosen so that $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C})) \to \mathcal{R}(\Gamma_+, G)$ restricts to a homeomorphism between W (or $\overline{W}$) and V. In particular, we can lift to W the restriction map from U to V:

$$\begin{tikzcd} & W \arrow[d, "\cong"] \\ U \arrow[ur, dotted, "\widetilde{\mathrm{res}}"] \arrow[r, "\mathrm{res}"] & V \end{tikzcd}$$

Lemma 3.8 *For $U \subset \mathcal{R}(\Gamma, G)$ and $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ as above, the lift of the restriction map yields a homeomorphism*

$$\widetilde{\mathrm{res}}\colon U \xrightarrow{\cong} \{[\rho] \in W \mid [\rho \circ \sigma_*] = [\bar{\rho}]\}.$$

This lemma has same proof as Lemma 3.4, just taking into account that $\rho(\zeta) \in G$ reverses the orientation, for $[\rho] \in U$ and $\zeta \in \Gamma \setminus \Gamma_+$.

3.3 Local coordinates

Here we give the local coordinates of Theorem 3.5 and prove a stronger version of Theorem 3.1.

For $\gamma \in \Gamma_+$ and $[\rho] \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$, as defined by Culler and Shalen in [7],

$$I_\gamma([\rho]) = (\mathrm{trace}(\rho(\gamma)))^2 - 4. \tag{13}$$

Thus I_γ is a function from $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ to $\mathbb{C}$, which plays a role in the generalization of Theorem 3.1.

Theorem 3.9 *Let M^3_+ be as above, and assume that it has n cusps. Chose $\gamma_1, \dots, \gamma_n \in \Gamma_+$ a nontrivial element for each peripheral subgroup. Then, for a neighborhood $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2,\mathbb{C}))$ of the oriented holonomy,*

$$(I_{\gamma_1}, \dots, I_{\gamma_n})\colon W \to \mathbb{C}^n$$

defines a bianalytic map between W and a neighborhood of the origin.

This theorem holds for any orientable hyperbolic manifold of finite volume, though we only use it for the orientation covering. Again, see [4; 14] for a proof. As explained in these references, this is the algebraic part of the proof of Thurston's hyperbolic Dehn filling theorem using varieties of representations.

For a Klein bottle K^2, in Definition 2.8 we considered the presentation of its fundamental group

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements a^2 and b are called *distinguished* elements. Recall that, in terms of paths, those are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves).

Here we prove the following generalization of Theorem 3.1:

Theorem 3.10 *Let M^3 be a nonorientable manifold of finite volume with k nonorientable cusps and l orientable cusps. For each horospherical Klein bottle K^2_i, chose $\gamma_i \in \pi_1(K^2_i)$ distinguished for $i = 1, \dots, k$. For each horospherical torus T^2_j, chose a nontrivial $\mu_j \in \pi_1(T^2_j)$ for $j = 1, \dots, l$.*

There exists a neighborhood $U \subset \mathcal{R}(\Gamma, G)$ of the holonomy of M^3 such that the map

$$(I_{\gamma_1}, \dots, I_{\gamma_k}, I_{\mu_1}, \dots, I_{\mu_l}) \circ \widetilde{\widetilde{\mathrm{res}}}\colon U \to \mathbb{R}^k \times \mathbb{C}^l$$

defines a homeomorphism between U and a neighborhood of the origin in $\mathbb{R}^k \times \mathbb{C}^l$.

Proof Let $M^3_+ \to M^3$ be the orientation covering. By construction, by the choice of distinguished elements in the peripheral Klein bottles, $\gamma_i \in \Gamma_+$. Furthermore, as the peripheral tori are orientable, $\mu_j \in \Gamma_+$. Hence

$$\{\gamma_1, \dots, \gamma_k, \mu_1, \dots, \mu_l, \sigma_*(\mu_1), \dots, \sigma_*(\mu_l)\}$$

gives a nontrivial element for each peripheral subgroup of Γ_+, where σ_* is the group automorphism from Definition 3.3. We apply Theorem 3.9, which gives that

$$I = (I_{\gamma_1}, \dots, I_{\gamma_k}, I_{\mu_1}, \dots, I_{\mu_l}, I_{\sigma_*(\mu_1)}, \dots, I_{\sigma_*(\mu_l)})\colon W \to \mathbb{C}^{k+2l}$$

is a bianalytic map with a neighborhood of the origin. Furthermore, as $\sigma_*(\gamma_i) = \gamma_i^{\pm 1}$ and $(\sigma^*)^2 = \mathrm{Id}$,

$$I \circ \sigma^* \circ I^{-1}(x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l) = (x_1, \dots, x_k, z_1, \dots, z_l, y_1, \dots, y_l).$$

In addition, by construction I commutes with complex conjugation. Hence, by Lemma 3.8, the image $(I \circ \widetilde{\widetilde{\mathrm{res}}})(U)$ is the subset of a neighborhood of the origin in $\mathbb{C}^{k+2l}$ defined by

$$\begin{cases} x_i = \bar{x}_i & \text{for all } i = 1, \dots, k, \\ z_j = \bar{y}_j & \text{for all } j = 1, \dots, l. \end{cases}$$

Finally, by combining Theorem 3.9 and Lemma 3.8, the map $I \circ \widetilde{\widetilde{\mathrm{res}}}$ is a homeomorphism between U and its image. □

We now state the generalization of Theorem 1.1 to several cusps. Here $D(1) \subset \mathbb{C}$ denotes a disk of radius 1.

Theorem 3.11 *Let M^3 be a complete nonorientable hyperbolic 3–manifold of finite volume with k nonorientable cusps and l orientable cusps.*

(a) *If M^3 admits a geometric ideal triangulation Δ, then $\mathrm{Def}(M^3, \Delta) \cong (-1, 1)^k \times D(1)^l$. The parameters $(\pm t_1, \dots, \pm l_k, \pm u_1, \dots, \pm u_l) \in (-1, 1)^k \times D(1)^l$ correspond to the same structure.*

(b) *A neighborhood of the holonomy in $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to $(-1, 1)^k \times D(1)^l$.*

Furthermore, the holonomy map $\mathrm{Def}(M^3, \Delta) \to \mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$ is written, in coordinates, as

$$(-1, 1)^k \times D(1)^l \to (-1, 1)^k \times D(1)^l,$$

$$(t_1, \dots, t_k, v_1, \dots, v_l) \mapsto (t_1^2, \dots, {t_k}^2, {v_1}^2, \dots, {v_l}^2).$$

Namely, each interval $(-1, 1)$ is folded along 0 and has image $[0, 1)$, and disks $D(1)$ are mapped to disks by a $2:1$ branched covering.

Proof Assertion (a) is Theorem 2.14, and assertion (b) is Theorem 3.10. To describe the holonomy map in coordinates, for each cusp (orientable or not) choose an orientation-preserving peripheral element m and let v be the logarithm of the holonomy of m, defined as in (9), in a neighborhood of the origin in $\mathbb{C}$ (with $v \in i\mathbb{R}$ in the nonorientable case). In particular v is a component of the local coordinates of $\mathrm{Def}(M^3, \Delta)$. Furthermore, the holonomy of m is conjugate to

$$\pm \begin{pmatrix} e^{v/2} & 1 \\ 0 & e^{-v/2} \end{pmatrix}.$$

So it has trace $\pm 2\cosh\left(\frac{1}{2}v\right)$, which is a component of the local coordinates of $\mathcal{R}(\pi_1(M^3), \mathrm{Isom}(\mathbb{H}^3))$. Then the assertion follows from applying a suitable coordinate change. $\square$

4 Representations of the Klein bottle

Let $\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ be a presentation of the fundamental group of the Klein bottle, and $G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$. The variety of representations $\hom(\pi_1(K^2), G)$ is identified as

$$\hom(\pi_1(K^2), G) \cong \{A, B \in G \mid ABA^{-1} = B^{-1}\}.$$

Topologically we can expect to have at least four (possibly empty) connected components according to the orientable nature of A and B. We are interested in studying one of them.

Definition 4.1 A representation $\rho \in \hom(\pi_1(K^2), G)$ is said to *preserve the orientation type* if, for every $\gamma \in \pi_1(K^2)$, $\rho(\gamma)$ is an orientation-preserving isometry if and only if γ is represented by an orientation-preserving loop of K^2. We denote this subspace of representations by

$$\hom_+(\pi_1(K^2), G).$$

Let $T^2 \to K^2$ be the orientation covering. The restriction map on the varieties of representations (without quotienting by conjugation) is

$$\mathrm{res}\colon \hom(\pi_1(K^2), G) \to \hom(\pi_1(T^2), \mathrm{PSL}(2, \mathbb{C})).$$

Theorem 4.2 *Let $\rho \in \hom_+(\pi_1(K^2), G)$ preserve the orientation type and let $\rho(b) \neq \mathrm{Id}$. By writing $A = \rho(a)$ and $B = \rho(b)$ as Möbius transformations, up to conjugation one of the following holds*:

(a) $A(z) = \bar{z} + 1$ *and* $B(z) = z + \tau i$, *with* $\tau \in \mathbb{R}_{>0}$.

(a′) $A(z) = \bar{z}$ *and* $B(z) = z + \tau i$, *with* $\tau \in \mathbb{R}_{>0}$.

(b) $A(z) = e^l \bar{z}$ *and* $B(z) = e^{\alpha i} z$, *with* $l \in \mathbb{R}_{\geq 0}$ *and* $\alpha \in (0, \pi]$.

(c) $A(z) = e^{\alpha i}/\bar{z}$ *and* $B(z) = e^l z$, *with* $l \in \mathbb{R}_{>0}$ *and* $\alpha \in [0, \pi]$.

Proof Let $G^0 = \mathrm{PSL}(2, \mathbb{C}) \lhd G$ be the connected component of the identity. The variety of representations $\hom(\pi_1(T^2), G^0)/G^0$ is well known. A representation $[\rho_0]$ in this variety is the class of a parabolic representation with $\rho_0(l)(z) = z + 1$, $\rho_0(m)(z) = z + \tau$ and $\tau \in \mathbb{C}$, a parabolic degenerated one with $\rho_0(l)(z) = z$, $\rho_0(m)(z) = z + \tau$ and $\tau \in \mathbb{C}$, or a hyperbolic one with $\rho_0(l)(z) = \lambda z$, $\rho_0(m)(z) = \mu z$ and $\lambda, \mu \in \mathbb{C}$, where $\pi_1(T^2) = \langle l, m \mid lm = ml \rangle$.

For $\rho_0 = \mathrm{res}(\rho)$, let $A = \rho(a)$ and $B = \rho(b)$, where a and b are generators of $\pi_1(K^2)$, and $L = \rho(l)$ and $M = \rho(m)$. Then

$$\begin{aligned} (A^2, B) &= (L, M) && \text{(restriction of a representation to the torus),} \\ ABA^{-1} &= B^{-1} && \text{(Klein bottle relation).} \end{aligned}$$

In fact, in order for ρ_0 to be a restriction, there must be A and B satisfying the previous conditions. We prove the theorem using these equations.

If $[\rho]$ is in the parabolic case, by hypothesis $\tau \neq 0$. Then the solution is unique, $A(z) = \bar{z} + 1$, $B(z) = z + \tau i$ and $\tau \in \mathbb{R} \setminus \{0\}$, and hence $L(z) = z + 2$, $M(z) = z + \tau i$. Similarly, for the degenerated parabolic case, $A(z) = \bar{z}$, $B(z) = z + \tau i$ and $\tau \in \mathbb{R} \setminus \{0\}$.

On the other hand, for $[\rho]$ hyperbolic, either L corresponds to a real dilation and M to a rotation, or the other way around. In the case $L(z) = e^{2l} z$, $M(z) = e^{\alpha i} z$, $l \in \mathbb{R}$ and $\alpha \in (-\pi, \pi]$, the representation can be written as the restriction of several representations of the Klein bottle, but all of them are conjugated to $A(z) = e^l \bar{z}$ and $B(z) = e^{\alpha i} z$. A similar situation happens when $L(z) = e^{2\alpha i} z$, $M(z) = e^l z$, $l \in \mathbb{R}$ and $\alpha \in (-\pi, \pi]$, obtaining $A(z) = e^{\alpha i}/\bar{z}$ and $B(z) = e^l z$. However, in the last case we should note down that, for every such representation $[\rho]$, we get two nonconjugated representations $[\rho_1]$ and $[\rho_2]$ such that $[\rho_0] = \mathrm{res}([\rho_1]) = \mathrm{res}([\rho_2])$, where they differ in that $A_1(z) = e^{\alpha i}/\bar{z}$ and $A_2(z) = e^{(\alpha+\pi)i}/\bar{z} = -e^{\alpha i}/\bar{z}$.

Thus, we obtain a classification of representations in $\hom(\pi_1(K^2), G)/G^0$. To get the classification quotienting by the whole group $\hom(\pi_1(K^2), G)/G$, we only have to see how complex conjugation c acts by conjugation on each representation. In (a) and (a′), c maps $z + \tau i$ to $z - \tau i$, in (b) $e^{\alpha i} zo$ to $e^{-\alpha i} z$, and in (c) $e^{\alpha i}/\bar{z}$ to $e^{-\alpha i}/\bar{z}$. The choice $\alpha > 0, l > 0$ in (b) and (c) is obtained by taking into account that $[\rho] = [\rho^{-1}]$. □

Definition 4.3 According to the cases in Theorem 4.2, a representation $\rho \in \hom_+(\pi_1(K^2), G)$ is called

- *parabolic nondegenerate* in case (a) and *parabolic degenerate* in case (a′),
- *type I* in case (b), and
- *type II* in case (c).

Further, types I and II are called nondegenerate if $l \neq 0$ or $\alpha \neq 0$, respectively, and degenerate otherwise.

Remark 4.4 The holonomy of a nonorientable cusp restricts to a representation of the Klein bottle that preserves the orientation type and is parabolic nondegenerate.

Furthermore, deformations of this representation still preserve the orientation type and are nondegenerate (possibly of type I or II), by continuity.

For $\gamma \in \pi_1(T^2) \lhd \pi_1(K^2)$, recall from (13) that

$$I_\gamma \colon \hom(\pi_1(K^2), G) \to \mathbb{C}, \quad \rho \mapsto (\mathrm{trace}_{\mathrm{PSL}(2,\mathbb{C})}(\rho(\gamma)))^2 - 4,$$

where $\mathrm{trace}_{\mathrm{PSL}(2,\mathbb{C})}$ means the trace as a matrix in $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 4.5 *Let $\rho \in \hom(\pi_1(K^2), G)$ preserve the orientation type and $\rho(b) \neq \mathrm{Id}$.*

- *If ρ is parabolic, then $I_\gamma(\rho) = 0$ for all $\gamma \in \pi_1(T^2)$.*
- *If ρ is of type I, then $I_{a^2}(\rho) \geq 0$ and $I_b(\rho) < 0$.*
- *If ρ is of type II, then $I_{a^2}(\rho) \leq 0$ and $I_b(\rho) > 0$.*

Proof This is a straightforward computation from Theorem 4.2. □

Corollary 4.6 (a) *The holonomy of a representation in $\mathrm{Def}(M, \Delta)$ is of type I.*

(b) *Representations in a neighborhood of $[\mathrm{hol}]$ in $\mathcal{R}(M^3, G)$ are of both type I and II.*

(c) *In particular, the holonomy map $\mathrm{Def}(M, \Delta) \to \mathcal{R}(M^3, G)$ is not surjective in a neighborhood of the holonomy.*

Proof Assertion (a) follows from Remark 2.11 and (b) from Theorem 3.10, both using Lemma 4.5. □

5 Metric completion

As we deform noncompact manifolds, the deformations into noncomplete manifolds are not unique (eg one can consider proper open subsets of a noncomplete manifold). We are not discussing the different issues related to this nonuniqueness, just the existence of a deformation into a metric that can be complete as a conifold (see below).

The main result of this section is Theorem 5.15. In the orientable case, the metric completion after deforming an orientable cusp is a singular space with a singularity called of *Dehn type* (this includes nonsingular manifolds); see Hodgson's thesis [11] and Boileau and Porti [4, Appendix B]. In the nonorientable case, the singularity is more specific, a so-called conifold.

5.1 Conifolds and cylindrical coordinates

A *conifold* is a metric length space locally isometric to the metric cone of constant curvature on a spherical conifold of dimension one less; see for instance [3]. When, as topological space, a conifold is homeomorphic to a manifold, it is called a *cone manifold*, but in general it is only a pseudomanifold. In dimension two, conifolds are also cone manifolds, but in dimension three there may be points with a neighborhood homeomorphic to the cone on a projective plane P^2.

We are interested in three local models of singular spaces as conifolds:

- The first is the hyperbolic cone over a round sphere S^2. This corresponds to a point with a nonsingular hyperbolic metric.
- The second is the hyperbolic cone over $S^2(\alpha, \alpha)$, the sphere with two cone points of angle α, that is the spherical suspension of a circle of perimeter α. This corresponds to a singular axis of angle α.

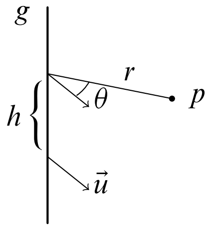

Figure 2: Cylindrical coordinates.

- The third is the hyperbolic cone over $P^2(\alpha)$, the projective plane with a cone point of angle α. This is the quotient of the previous one by a metric involution, which is the antipodal map on each concentric sphere.

Next we describe metrically those local models by using cylindrical coordinates in the hyperbolic space. These coordinates are defined from a geodesic line g in $\mathbb{H}^3$, and we fix a point in the unit normal bundle to g, ie a vector $\vec{u}$ of norm 1 and perpendicular to g. Cylindrical coordinates give a diffeomorphism

$$\mathbb{H}^3 \setminus g \xrightarrow{\cong} (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}, \qquad p \mapsto (r,\theta,h),$$

where r is the distance between g and p, θ is the angle parameter (the angle between the parallel transport of $\vec{u}$ and the tangent vector to the orthogonal geodesic from g to p) and h is the arc parameter of g, the signed distance between the base point of $\vec{u}$ and the orthogonal projection from p to g; see Figure 2.

In the upper half-space model of $\mathbb{H}^3$, if g is the geodesic from 0 and ∞, then there exists a choice of coordinates (a choice of $\vec{u}$) such that the projection from g to the ideal boundary $\partial_\infty \mathbb{H}^3$ maps a point with cylindrical coordinates (r,θ,h) to $e^{h+i\theta} \in \mathbb{C}$; see Figure 3. A different choice of $\vec{u}$ would yield instead $\lambda e^{h+i\theta} \in \mathbb{C}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

The hyperbolic metric on $\mathbb{H}^3$ in these coordinates is

$$dr^2 + \sinh^2(r)\,d\theta^2 + \cosh^2(r)\,dh^2.$$

More precisely, $\mathbb{H}^3$ is the metric completion of $(0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ with this metric.

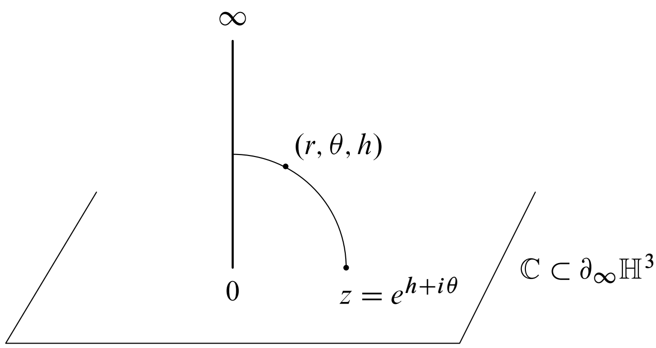

Figure 3: Orthogonal projection to $\partial_\infty \mathbb{H}^3$ with g the geodesic with ideal endpoints 0 and ∞.

Definition 5.1 For $\alpha \in (0, 2\pi)$, $\mathbb{H}^3(\alpha)$ is the metric completion of $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ for the metric

$$ds^2 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 \sinh^2(r)\, d\theta^2 + \cosh^2(r)\, dh^2.$$

The metric space $\mathbb{H}^3(\alpha)$ may be visualized by taking a sector in $\mathbb{H}^3$ of angle α and identifying its sides by a rotation. Alternatively, with the change of coordinates $\tilde{\theta} = \alpha/(2\pi)\theta$, the metric completion of $(0, +\infty) \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}$ is $\mathbb{H}^3(\alpha)$, for the metric $dr^2 + \sinh^2(r)\, d\tilde{\theta}^2 + \cosh^2(r)\, dh^2$.

Remark 5.2 The metric models are

- $\mathbb{H}^3$ for the nonsingular case (the cone on the round sphere),
- $\mathbb{H}^3(\alpha)$ for the singular axis (the cone on $S^2(\alpha, \alpha)$),
- the quotient
$$\mathbb{H}^3(\alpha)/(r, \theta, h) \sim (r, -\theta, -h)$$
for the cone on $P^2(\alpha)$.

5.2 Conifolds bounded by a Klein bottle

We keep the notation of Section 5.1, with cylindrical coordinates. Before discussing conifolds bounded by a Klein bottle, we describe a cone manifold bounded by a torus.

Definition 5.3 A *solid torus with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric action of $\mathbb{Z}$ generated by

$$(r, \theta, h) \mapsto (r, \theta + \tau, h + L)$$

for $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ and $L > 0$.

The space $\mathbb{H}^3(\alpha)/\sim$ is a solid torus of infinite radius with singular soul of cone angle α, length of the singularity $L > 0$ and torsion parameter $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ (the rotation angle induced by parallel transport along the singular geodesic is $\alpha/(2\pi)\tau \in \mathbb{R}/\alpha\mathbb{Z}$).

By considering the metric neighborhood of radius $r_0 > 0$ on the singular soul, we get a compact solid torus, bounded by a 2–torus. This compact solid torus depicts a tubular neighborhood of a component of the singular locus of a cone manifold (compare Hodgson and Kerckhoff [12] and Hodgson's thesis [11]).

We describe two conifolds bounded by a Klein bottle that are a quotient of this solid torus by an involution.

Definition 5.4 A *solid Klein bottle with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric action of $\mathbb{Z}$ generated by

$$(r, \theta, h) \mapsto (r, -\theta, h + L)$$

for $L > 0$.

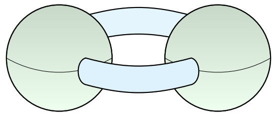

Figure 4: A solid torus as two 3–balls joined by two 1–handles.

The space $\mathbb{H}^3(\alpha)/\sim$ is a solid Klein bottle of infinite radius with singular soul of cone angle α, and length of the singularity $L > 0$. We may consider a metric tubular neighborhood of radius r_0, bounded by a Klein bottle. Its orientation cover is a solid torus with singular soul, cone angle α, length of the singularity $2L$ and torsion parameter $\tau = 0$.

Definition 5.5 The *disc orbibundle with singular soul* is $\mathbb{H}^3(\alpha)/\sim$, where $\sim$ is the relation induced by the isometric involutions

$$(r,\theta,h) \mapsto (r,\theta+\pi,-h) \quad \text{and} \quad (r,\theta,h) \mapsto (r,\theta+\pi,2L-h)$$

for $L > 0$.

To describe this space, it is useful first to look at the action on the preserved geodesic, corresponding to $r = 0$. These involutions map $h \in \mathbb{R}$ to $-h$ and to $2L-h$, respectively. Thus it is the action of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ on a line generated by two reflections. Its orientation-preserving subgroup is $\mathbb{Z}$ acting by translations on $\mathbb{R}$. Thus $\mathbb{R}/\mathbb{Z}$ is a circle, and $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$ is an orbifold. The solid torus is a disc bundle over the circle, and our space is an orbifold-bundle over $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$ with fiber a disc.

This space is the quotient of an involution on the solid torus. View the solid torus as two 3–balls joined by two 1–handles; see Figure 4. On each 3–ball, apply the antipodal involution (on each concentric sphere of given radius), and extend this involution by permuting the 1–handles. The quotient of each ball is the (topological) cone on P^2, and hence our space is the result of joining two cones on P^2 by a 1–handle. Its boundary is the connected sum $P^2 \# P^2 \cong K^2$.

The singular locus of the disc orbibundle $\mathbb{H}^3(\alpha)/\sim$ is an interval (the underlying space of the orbifold bundle) of length L. The interior points of the singular locus have cone angle α, and the boundary points of the interval are precisely the points where it is not a topological manifold.

Again $\mathbb{H}^3(\alpha)/\sim$ has radius ∞, and the metric tubular neighborhood of radius r of the singularity is bounded by a Klein bottle. It is the quotient of a solid torus of length $2L$ and torsion parameter $\tau = 0$ by an isometric involution with two fixed points (thus, as an orbifold, its orientation orbicovering is a solid torus).

Remark 5.6 The boundaries of both the solid Klein bottle and the disc orbibundle are Klein bottles. In both cases the holonomy preserves the orientation type, but the type of the presentation as in Definition 4.3 is different:

(a) The holonomy of the boundary of a solid Klein bottle with singular soul is a representation of type I.

(b) The holonomy of the boundary of a disc orbibundle over a singular interval is of type II.

For a nonorientable cusp, the holonomy of the peripheral torus is either parabolic nondegenerate, of type I or of type II, also nondegenerate (Remark 4.4). The aim of next section is to prove that the deformations can be defined so that the metric completion is either a solid Klein bottle with singular soul or a disc orbibundle with singular soul, according to the type. This is the content of Theorem 5.15, which we prove at the end of the section.

5.3 The radial structure

Let M^3 be a noncompact hyperbolic 3–manifold of finite volume. We deform its holonomy representation and accordingly we deform its hyperbolic metric. Nonetheless, incomplete metrics are not unique, so here we give a statement about the existence of a maximal structure, which corresponds to the one completed in Theorem 5.15.

Let $[\rho] \in \mathcal{R}(\pi_1(M^3), G)$ be a deformation of its complete structure. There is some nuance in associating to $[\rho]$ a hyperbolic structure which is made explicit by Canary, Epstein and Green in [6]. Here, the authors conclude that deformations with a given holonomy representation are related by an isotopy of the inclusion of M^3 in some fixed thickening $(M^3)^*$, where a thickening is just another hyperbolic 3–manifold containing ours.

We will start by making clear what we mean by a maximal structure.

Definition 5.7 Let M be a manifold with an analytic (G, X)–structure. We say that M^* is an *isotopic thickening of* M if it is a thickening and there is a isotopy i' of the inclusion $i : M \hookrightarrow M^*$ such that $i'(M) = M^*$.

Given two isotopic thickenings of M, we say that $M_1^* \leq M_2^*$ if there is a (G, X) isomorphism from M_1^* to some subset of M_2^* extending the identity on M. Hence, we say that an isotopic thickening is maximal if it is maximal with respect the partial order relation we have just defined.

In general, it is not clear whether maximal isotopic thickenings exist, nor under which circumstances they do exist. However, we will construct in our situation an explicit maximal thickening.

Lemma 5.8 *Let $\mathrm{inj}_{M^3}(x)$ denote the injectivity radius at a point $x \in M^3$. Then a necessary condition for a nontrivial thickening of M^3 to exist is that there must exist a sequence $\{x_n\} \subset M^3$ with $\mathrm{inj}_{M^3}(x_n) \to 0$.*

Proof Let us suppose a thickening $(M^3)^*$ exists. Then take a point $x \in \partial((M^3)^* \setminus M^3)$. Any sequence $\{x_n\} \subset M^3$ such that $x_n \to x$ satisfies $\mathrm{inj}_{M^3}(x_n) \to 0$. □

The purpose of Lemma 5.8 is twofold. First, it gives a condition for a thickening to be maximal (in the sense of the partial order relation we just defined), and second, it shows where a manifold could possibly be thickened. Taking into account a thick–thin decomposition of the manifold, the thickening can only be done in the deformed cusps.

Each cusp of M^3 is diffeomorphic to either $T^2 \times [0,\infty)$ or $K^2 \times [0,\infty)$. Let us consider a proper product compact subset $K^2 \times [0,\lambda]$ or $T^2 \times [0,\lambda]$ of an end, for some $\lambda > 0$, and let us denote by D_ρ the developing map of a structure with holonomy ρ in the equivalence class $[\rho] \in \mathcal{R}(\Gamma, G)$.

Lemma 5.9 *The image of the proper product subset under the developing map, $D_\rho(\widetilde{K}^2 \times [0,\lambda])$ or $D_\rho(\widetilde{T}^2 \times [0,\lambda])$, lies within two tubular neighborhoods of a geodesic $\gamma \in M^3$, that is, in $N_{\epsilon_2}(\gamma) \setminus N_{\epsilon_1}(\gamma)$, where $N_\epsilon(\gamma) = \{x \in \mathbb{H}^3 \mid d(x,\gamma) < \epsilon\}$. Moreover, for every geodesic ray exiting orthogonally from γ, the intersection of the ray with $D_\rho(\widetilde{K}^2 \times [0,\lambda])$ is nonempty and transverse to any section $D_\rho(\widetilde{K}^2 \times \{\mu\})$, $\mu \in [0,\lambda]$, and analogously for an orientable end.*

Proof We use a modified argument of Thurston (see his notes [18, Chapters 4 and 5]) to prove the lemma for a nonorientable end (the same idea goes for an orientable one). The original argument of Thurston shows that, in an ideal triangulated manifold, the image of the universal cover of the end under the developing map is the whole tubular neighborhood except the geodesic. Let $[\rho_0]$ be the parabolic representation corresponding to the complete structure. Then $D_{\rho_0}(\widetilde{K}^2 \times [0,\lambda])$ is the region between two horospheres centered at an ideal point $p_\infty \in \partial_\infty \mathbb{H}^3$. Let $K \subset \widetilde{K}^2 \times [0,\lambda]$ denote a fundamental domain of $K^2 \times [0,\lambda]$. The domain K can be taken so that $D_{\rho_0}(\overline{K})$ is a rectangular prism between two horospheres. We want to deform $D_{\rho_0}(\overline{K})$ as we deform ρ_0 to ρ. We do that by deforming the horosphere centered at p_∞ to surfaces equidistant to the geodesics γ_ρ invariant by the holonomy of the peripheral subgroup $\rho(\pi_1(K^2))$. The deformation of the horosphere to equidistant surfaces is described in [18, Section 4.4] in the half-space model of $\mathbb{H}^3$; see also Benedetti and Petronio [2, Section E.6.iv]. Alternatively, we can view the deformation of the horosphere to the equidistant surfaces as follows. Considering $\mathbb{Z}^2 < \pi_1(K^2)$ the orientation-preserving subgroup of index 2, $\rho(\mathbb{Z}^2)$ is contained in a unique one-complex parameter subgroup $U_\rho \subset \mathrm{PSL}(2,\mathbb{C})$ — ie U_ρ is the exponential image of a $\mathbb{C}$–line in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. This U_ρ depends continuously on ρ, and given $x \in \mathbb{H}^3$ the orbit $U_\rho(x) = \{g(x) \mid g \in U_\rho\}$ is a surface containing x such that when $\rho = \rho_0$, $U_\rho(x)$ is a horosphere centered at p_∞ and when $\rho \neq \rho_0$, $U_\rho(x)$ is a surface equidistant to the geodesic γ_ρ. Using this construction, the image of the domain $D_{\rho_0}(\overline{K})$ deforms to $D_\rho(\overline{K})$ with the required properties by following the equidistant surfaces for the factor $\widetilde{K}^2$ and the geodesics orthogonal to these surfaces for the factor $[0,\lambda]$. □

Definition 5.10 The geodesic of Lemma 5.9 is called the *soul* of the end.

Remark 5.11 The face of the section of the proper product subset of the cusp $K^2 \times [0,\lambda]$ or $T^2 \times [0,\lambda]$ that is glued to the thick part of the manifold is the section of the cusp which is further away from the geodesic. Hence, we will only consider thickenings "towards" the soul.

Let x be a point in a cusp of the manifold and consider the image under the developing map $y = D_\rho(\tilde{x})$ of any lift $\tilde{x}$. There is only one geodesic segment in $\mathbb{H}^3$ such that $\gamma(0) = y$ and that goes towards the soul orthogonally. In cylindrical coordinates, if $y = (r,\theta,h)$, the image of the geodesic consists of $\{(t,\theta,h) \mid t \in [0,r]\}$. Let us denote by γ_x the corresponding geodesic in M^3.

Theorem 5.12 *There exists a maximal thickening M^* of a half-open product $M = K^2 \times [0,\lambda)$ or $T^2 \times [0,\mu)$. It is characterized by the following property: for every point $x \in M$, the geodesic γ_x can be extended in M^* so that $D_\rho(\tilde{\gamma}_x)$ is the geodesic whose cylindrical coordinates with respect to the soul are $\{(t,\theta,h) \mid t \in (0,r]\}$.*

Proof Given a cusp section $S := K^2$ or T^2, a product subset of the end $K := S \times [0,\lambda]$, a fixed fundamental domain K_0 of K and a small neighborhood $N(K_0)$ of K_0, the set

$$T := \{t \in \mathrm{Deck}(\widetilde{K}/K) \mid tN(K_0) \cap N(K_0) \neq \varnothing\}$$

is finite, where $\mathrm{Deck}(\widetilde{K}/K)$ denotes the group of covering transformations of the universal cover. Hence, we can suppose that $D_{\rho|(T\overline{K}_0)}$ is an embedding.

Let $\mathcal{U}$ be an open cover of K by simply connected charts. For each U, take a lift $U_0 \in \widetilde{\mathcal{U}}$ such that $U_0 \cap K_0 \neq \varnothing$ and consider $D_\rho(U_0)$. Given such a lift U_0, the other possible lifts that could have nonempty intersection with K_0 are tU, for $t \in T$. Furthermore, we can always assume that the chart U coincides with the image of U_0 under the developing map $D_\rho(U_0)$. Thus, we can identify

$$K \cong \Big(\bigcup_{U \in \mathcal{U}} D_\rho(U_0)\Big)\Big/\sim,$$

where the equivalence relation is by the action of $\mathrm{hol}(t)$, for $t \in T$.

Each $U \in \mathcal{U}$ can be thickened by identifying U with $D_\rho(U_0)$ and considering, in cylindrical coordinates, the set of rays $R(U) = \{(t,\theta,h) \in \mathbb{H}^3 \setminus \{\text{soul}\} \mid \text{there exists } (t_0,\theta,h) \in U, t < t_0\}$. Given two lifts of two thickened charts $R(U_1)$ and $R(U_2)$ with nonempty intersection with K_0, we glue them together at the points corresponding to $\mathrm{hol}(t)(R(U_1)) \cap R(U_2)$, where $t \in T$. This defines a thickening of the cusp K^*.

We have yet to show that this is isotopic to the original (half-open) product subset. Let us consider the section $S \times \{0\}$ of the cusp; the radial geodesics γ_x for $x \in S \times \{0\}$ define a foliation of K^* of finite length. Moreover, due to Lemma 5.9, the foliation is transversal to $S \times \{0\}$. By reparametrizing the foliation and considering its flow, we obtain a trivialization of the cusp, $K^* \cong S \times [0,\mu)$. Similarly, $K^* \setminus K$ is also a product. This let us construct an isotopy from K^* to K.

For this thickening, clearly $\gamma_x \subset K^*$ can be extended so that $D_\rho(\tilde{\gamma}_x) = \{(t,\theta,h) \mid t \in (0,r]\}$. By taking geodesics γ_x to geodesics through the developing map, it is clear our thickening can be mapped into every other thickening satisfying this property. Furthermore, if we consider the thickenings to be isotopic, we obtain an embedding.

Regarding the maximality, we will differentiate between an orientable end and a nonorientable one. The general idea will be the same: for another isotopic thickening $(K)^{**}$ to include ours, the developing map should map some open set V into a ball W around a point y_0 in the soul, which will lead to a contradiction.

If K is nonorientable, let us denote the distinguishable generators of $\pi_1(K^2)$ by a and b, with $aba^{-1} = b^{-1}$. If $[\rho]$ is type I, y_0 is fixed by $\rho(b)$. Let $y \in W \setminus \{\text{soul}\}$ and $x \in V$ be its preimage. W is invariant by $\rho(b)$

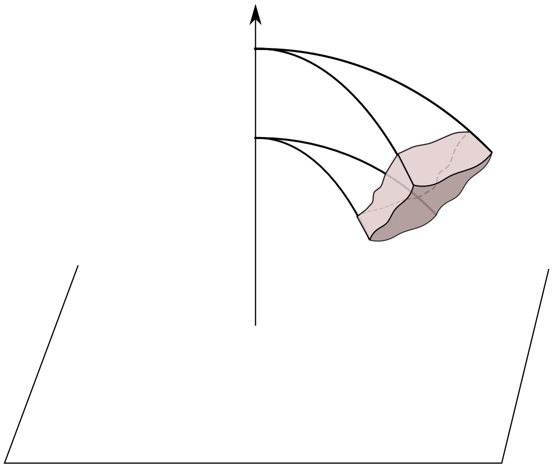

Figure 5: The radial thickening.

and, in addition, both x and $b \cdot x$ belong to V. Now take the geodesic $\gamma\colon I \mapsto \widetilde{(K)}^{**}$ from x to x_0 which corresponds to the geodesic from y to y_0. By equivariance and continuity, $x_0 = \lim \gamma(t) = \lim b\gamma(t) = bx_0$. This contradicts b being a covering transformation. If $[\rho]$ is type II, the previous argument with a^2 holds.

If K is orientable, we will follow the same arguments leading to the completion of the cusp (for more details see, for instance, [2]). the deformation $[\rho]$ is characterized in terms of its generalized Dehn filling coefficients $\pm(p,q)$. The cases $p = 0$ or $q = 0$ are solved as in the nonorientable cusp, so we have the two usual cases, $p/q \in \mathbb{Q}$ or $p/q \in \mathbb{I}$. For $p/q \in \mathbb{Q}$, there exists $k > 0$ such that $k(p,q) \in \mathbb{Q}^2$ and $(kp)a + (kq)b$ is a trivial loop in the new thickening. If $p/q \in \mathbb{I}$, then y_0 is dense in $\{\text{soul}\} \cap V$, which is a contradiction. $\square$

Definition 5.13 We call the previous thickening the *radial thickening* of the cusp.

Remark 5.14 If the manifold M^3 admits an ideal triangulation, the canonical structure coming from the triangulation is precisely the radial thickening of the cusp.

Theorem 5.15 *For a deformation of the holonomy M^3, the corresponding deformation of the metric can be chosen so that on a nonorientable end:*

- *It is a cusp (a metrically complete end) if the peripheral holonomy is parabolic.*
- *The metric completion is a solid Klein bottle with singular soul if the peripheral holonomy is of type I.*
- *The metric completion is a disc orbibundle with singular soul if it is of type II.*

Furthermore, the cone angle α and the length L of the singular locus are described by the peripheral boundary, so that those parameters start from $\alpha = L = 0$ for the complete structure and grow continuously when deforming in either direction.

Proof The proof uses the orientation covering and equivariance. More precisely, the deformation is constructed in the complete case for the orientation covering and it can be made equivariant. The holonomy of a torus restricted from a Klein bottle is either parabolic or the holonomy of a solid torus with singular soul (and $\tau = 0$). In particular, the holonomy of a Klein bottle is parabolic if and only if its restriction to the orientable covering is parabolic. Furthermore, by using the description in cylindrical coordinates (and using Figure 3) and as $\tau = 0$, the solid torus is equivariant by the action of $\pi_2(K^2)/\pi_1(T^2) \cong \mathbb{Z}_2$. $\square$

6 Example: the Gieseking manifold

We use the Gieseking manifold to illustrate our results. In particular the difference between deformation spaces obtained from ideal triangulations and from the variety of representations.

The Gieseking manifold M is a nonorientable hyperbolic 3–manifold with finite volume and one cusp, with horospherical section a Klein bottle. It has an ideal triangulation with a single tetrahedron. The orientation cover of the Gieseking manifold is the figure eight knot exterior, and the ideal triangulation with one simplex lifts to Thurston's ideal triangulation with two ideal simplices; see Thurston's notes [18].

This manifold M was constructed by Gieseking in his thesis in 1912; here we follow the description of Magnus in [15], using the notation of Alperin, Dicks and Porti [1]. Start with the regular ideal vertex Δ in $\mathbb{H}^3$, with vertices $\{0, 1, \infty, \frac{1}{2}(1 - i\sqrt{3})\}$; see Figure 6. The side identifications are the nonorientable isometries defined by the Möbius transformations

$$U(z) = \frac{1}{1 + \frac{1}{2}(1 + i\sqrt{3})\bar{z}} \quad \text{and} \quad V(z) = -\tfrac{1}{2}(1 + i\sqrt{3})\bar{z} + 1.$$

The identifications of the faces are defined by their action on vertices:

$$U : \left(\tfrac{1}{2}(1 - i\sqrt{3}), 0, \infty\right) \mapsto \left(\tfrac{1}{2}(1 - i\sqrt{3}), 1, 0\right) \quad \text{and} \quad V : (1, 0, \infty) \mapsto \left(\tfrac{1}{2}(1 - i\sqrt{3}), 1, \infty\right).$$

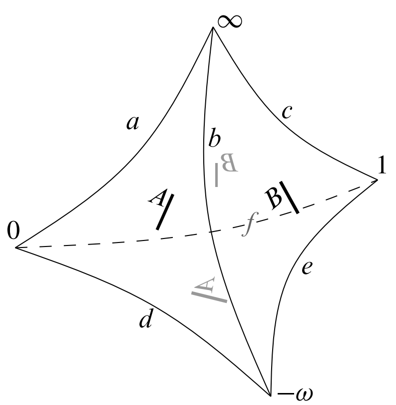

Figure 6: Gieseking manifold with labeled edges.

By applying Poincaré's fundamental theorem,

$$\pi_1(M) \cong \langle U, V \mid VU = U^2V^2 \rangle. \tag{14}$$

The relation $VU = U^2V^2$ corresponds to a cycle of length six around the edge.

6.1 The deformation space $\mathbf{Def}(M, \Delta)$

We compute the deformation space of the triangulation with a single tetrahedron, as in Section 2.

For any ideal tetrahedron in $\mathbb{H}_3$, we set its ideal vertices as 0, 1, ∞ and $-\omega$, where ω is in $\mathbb{C}_+$, the upper half-space of $\mathbb{C}$. The role played by $-\omega$ will be that of $\frac{1}{2}(1-i\sqrt{3})$ in the complete structure. For any such ω it is possible to glue the faces of the tetrahedron in the same pattern as in the Gieseking manifold via two orientation-reversing hyperbolic isometries, which we will likewise call U and V.

For the gluing to follow the same pattern, it must map

$$U\colon (-\omega, 0, \infty) \to (-\omega, 1, 0) \quad \text{and} \quad V\colon (1, 0, \infty) \to (-\omega, 1, \infty).$$

The orientation-reversing isometries U and V satisfying this are

$$U(z) = \frac{1}{((1+\omega)/|\omega|^2)\bar{z}+1} \quad \text{and} \quad V(z) = -(1+\omega)\bar{z}+1.$$

Although it is always possible to glue the faces in the same pattern as in the Gieseking manifold, the gluing will not always have a hyperbolic structure.

Let us label the edges as in Figure 6. For the topological manifold to be geometric, we only have to check that the pairing is proper; see [17]. In this case, the only condition which we need to satisfy is that the isometry that goes through the only edge cycle is the identity. This is given by

$$a \xrightarrow{V} c \xrightarrow{V} b \xrightarrow{U} d \xrightarrow{U} e \xrightarrow{V^{-1}} f \xrightarrow{U^{-1}} a,$$

and therefore we will have a hyperbolic structure if and only if $U^{-1}V^{-1}U^2V^2 = \mathrm{Id}$. Doing this computation, we obtain the equation

$$|\omega(1+\omega)| = 1. \tag{15}$$

Let us show that this equation matches the one obtained from Definition 2.3. If we denote by $z(a)$ the edge invariant of a and analogously for the rest of the edges, we have that the equation describing the deformation space of the manifold in terms of this triangulation is

$$\frac{z(a)z(b)z(e)}{\overline{z(c)z(d)z(f)}} = 1.$$

Writing down all of the edge invariants in terms of $z(a)$ by means of the tetrahedron relations results in the equation

$$\frac{z(a)^2\overline{z(a)}^2}{(1-z(a))(1-\overline{z(a)})} = \frac{|z(a)|^4}{|1-z(a)|^2} = 1. \tag{16}$$

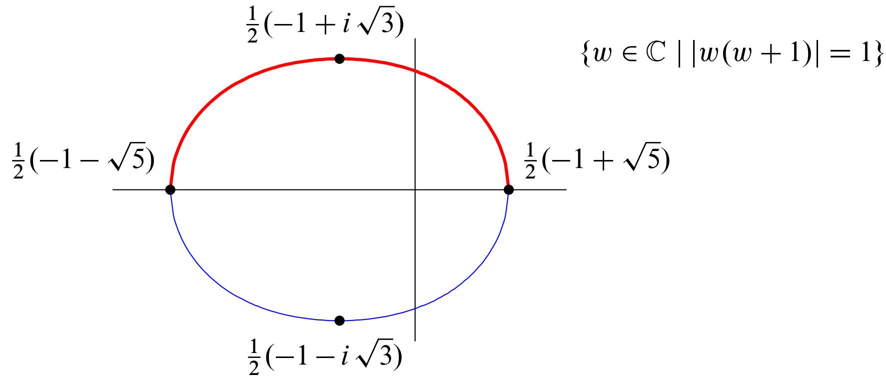

Figure 7: The set of solutions of the compatibility equations and $\mathrm{Def}(M, \Delta)$ (the top half).

If we substitute $z(a) = -1/\omega$, we obtain

$$\frac{1}{\omega\overline{\omega}(\omega+1)(\overline{\omega}+1)} = 1,$$

which is equivalent to (15).

Remark 6.1 The set $\{w \in \mathbb{C} \mid |w(1+w)| = 1\}$ is homeomorphic to S^1, and the deformation space $\{w \in \mathbb{C} \mid |w(1+w)| = 1 \text{ and } \mathrm{Im}(w) > 0\}$ is homeomorphic to an open interval; see Figure 7.

We justify the remark and Figure 7. Firstly, to prove that the set of algebraic solutions is homeomorphic to a circle, we write the defining equation $|w(1+w)| = 1$ as

$$\left|\left(w+\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right| = 1.$$

Thus $\left(w+\frac{1}{2}\right)^2$ lies in the circle of center $\frac{1}{4}$ and radius 1. As this circle separates 0 from ∞, the equation defines a connected covering of degree two of the circle. Secondly, the set of algebraic solutions is invariant by the involutions $w \mapsto \overline{w}$ and $w \mapsto -1-w$, and hence symmetric with respect to the real line and the line defined by the set of points with real part equal to $-\frac{1}{2}$. Furthermore, it intersects the real line at $w = \frac{1}{2}(-1 \pm \sqrt{5})$ and the line with real part $-\frac{1}{2}$ at $\frac{1}{2}(-1 \pm i\sqrt{3})$.

Let us construct the link of the cusp. We denote the link of each cusp point as in Figure 8, left, and glue them to obtain the link as in Figure 8, right, which is a Klein bottle.

Now we take two tetrahedra and construct the orientation covering of M (the figure eight knot exterior). For the first tetrahedron, we will define $z_1 := z(a)$, and define z_2 and z_3 so that they follow the cyclic order described in the tetrahedron relations. Afterwards, in the second tetrahedron, we denote by w_i the edge invariant of the corresponding edge after applying an orientation-reversing isometry to the tetrahedron, that is, $w_i = 1/\bar{z}_i$.

We consider the link of the orientation covering. The derivatives of the holonomy of the two loops in the link of the orientation covering l_1 and l_2, depicted in Figure 9, left (which are free homotopic), are $w_1/z_1 = 1/|w_1|^2$ and $w_3/z_3 = 1/|w_3|^2$. For the manifold to be complete we need $\mathrm{hol}'(l_i) = 1$ for $i = 1, 2$,

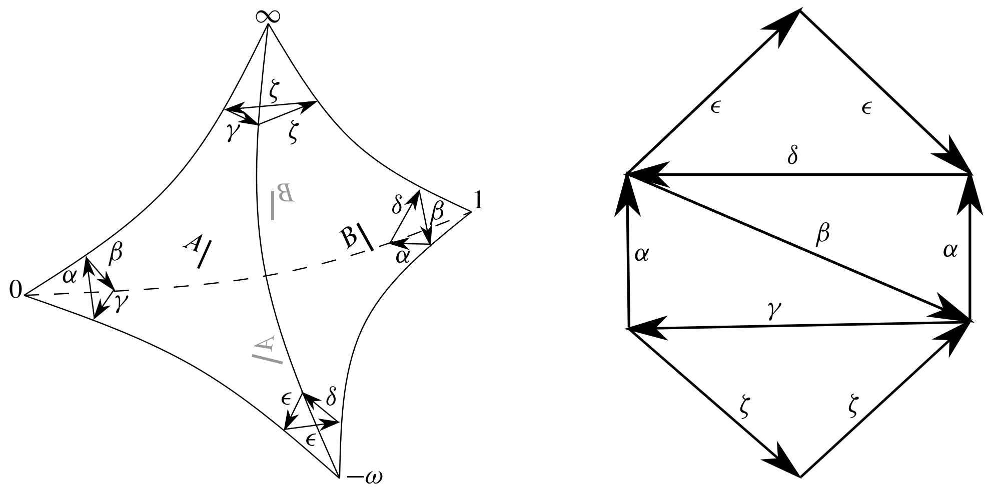

Figure 8: Left: Gieseking manifold with link. Right: link of the cusp point.

which happens if and only if $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$. This corresponds to the regular ideal tetrahedron, which, as expected, is the manifold originally given by Gieseking. Notice that the upper loop (the one going through the side ϵ) can be taken as a *distinguished* longitude. A suitable meridian is drawn in Figure 9, right.

Let us check that both the longitude and the meridian satisfy the conditions we stated for their holonomy in Remark 2.11, that is, $\mathrm{hol}'(l) \in \mathbb{R}$ and $|\mathrm{hol}'(m)| = 1$. We have already shown it for the longitude. Regarding the meridian,

$$\mathrm{hol}'(m) = \frac{z_2 z_3 w_2 w_3}{w_2 z_1 z_2 w_1} = \frac{z_1}{z_3}\frac{w_1}{w_3} = \frac{z_1}{\bar{z}_1}\frac{\bar{z}_3}{z_3},$$

therefore $|\mathrm{hol}'(m)| = 1$. This leads to the result that the generalized Dehn filling coefficients of a lifted structure have the form $(0, q)$, after an appropriate choice of longitude–meridian pair.

The last result could also have been obtained from Thurston's triangulation. By rotating the tetrahedra, our triangulation could be related with his, and the parameters identified. We can then check that, in his choice of longitude and meridian, the holonomy has the same features if the structure is a lift from the Gieseking manifold.

6.2 The Gieseking manifold as a punctured torus bundle

The Gieseking manifold M is fibered over the circle with fiber a punctured torus $T^2 \setminus \{*\}$. We use this structure to compute the variety of representations. The monodromy of the fibration is an automorphism

$$\phi \colon T^2 \setminus \{*\} \to T^2 \setminus \{*\}.$$

The map ϕ is the restriction of a map of the compact torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ that lifts to the linear map of $\mathbb{R}^2$ with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

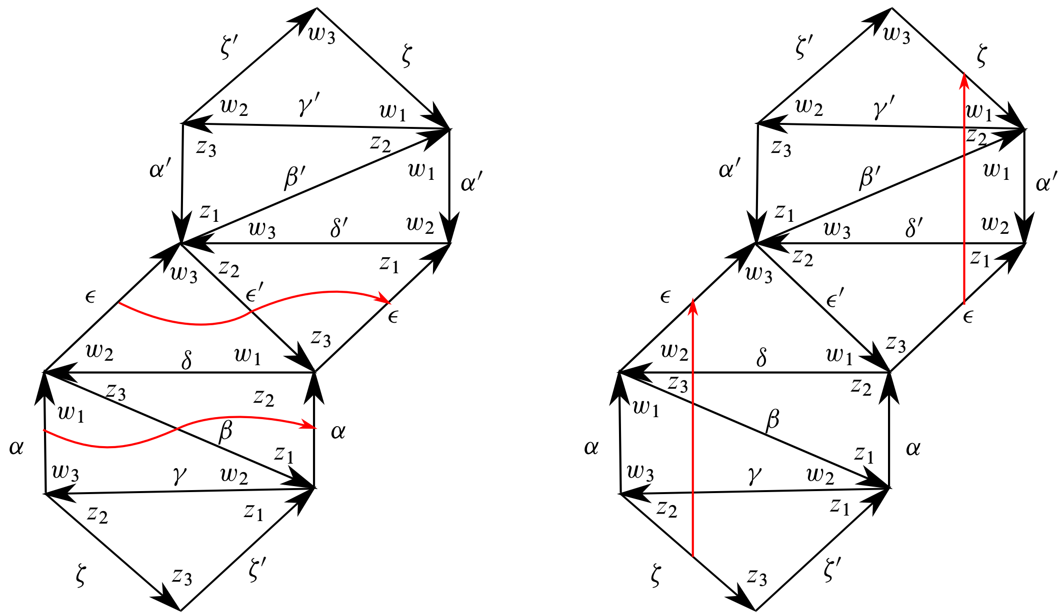

Figure 9: Left: two free homotopic loops. Right: meridian in the link of the cover.

This matrix also describes the action on the first homology group $H_1(T^2 \setminus \{*\}, \mathbb{Z}) \cong \mathbb{Z}^2$. The map ϕ is orientation reversing (the matrix has determinant -1) and ϕ^2 is the monodromy of the orientation covering of M, the figure eight knot exterior.

The fibration induces a presentation of the fundamental group of M

$$\pi_1(M) \cong \langle r, s, t \mid trt^{-1} = \phi(r),\ tst^{-1} = \phi(s) \rangle,$$

where $\langle r, s \mid \rangle = \pi_1(T^2 \setminus \{*\}) \cong F_2$, and

$$\phi_* \colon F_2 \to F_2, \qquad r \mapsto s, \quad s \mapsto rs,$$

is the algebraic monodromy, the map induced by ϕ on the fundamental group.

The relationship with the presentation (14) of $\pi_1(M)$ from the triangulation is given by

$$r = UV, \quad s = VU, \quad t = U^{-1}.$$

Furthermore, a peripheral group is given by $\langle rsr^{-1}s^{-1}, t \rangle$, which is the group of the Klein bottle.

We use this fibered structure to compute the variety of conjugacy classes of representations. Set

$$G = \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PO}(3,1) \cong \mathrm{PSL}(2,\mathbb{C}) \rtimes \mathbb{Z}_2,$$

and let

$$\hom^{\mathrm{irr}}(\pi_1(M), G)$$

denote the space of *irreducible* representations (ie representations that have no invariant line in $\mathbb{C}^2$). As we are interested in deformations, we restrict to representations ρ that preserve the orientation type: $\rho(\gamma)$

is an orientation-preserving isometry if and only if $\gamma \in \pi_1(M)$ is represented by a loop that preserves the orientation of M for all $\gamma \in \pi_1(M)$. We denote the subspace of representations that preserve the orientation type by

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G).$$

Let

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$$

be their the space of their conjugacy classes.

Proposition 6.2 *We have a homeomorphism, via the trace of $\rho(s)$,*

$$\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G \to (\{x \in \mathbb{C} \mid |x-1| = 1 \text{ and } x \neq 2\})/\sim, \quad [\rho] \mapsto \operatorname{trace}(\rho(s)),$$

where $\sim$ is the relation given by complex conjugation.

In particular, $\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$ is homeomorphic to a half-open interval.

Proof Let $\rho\colon \pi_1(M) \to G$ be an irreducible representation. The fiber $T^2 \setminus \{*\}$ is orientable, so the restriction of ρ to the free group $\langle r, s \mid \rangle \cong F_2$ is contained in $\mathrm{PSL}(2,\mathbb{C})$. Furthermore, as $\langle r, s \mid \rangle$ is the commutator subgroup, we may assume that $\rho(\langle r, s \mid \rangle) \subset \mathrm{SL}(2,\mathbb{C})$; see Heusener–Porti [9].

We consider the variety of characters $X(F_2, \mathrm{SL}(2,\mathbb{C}))$ and the action of the algebraic monodromy ϕ_* on the variety of characters:

$$\phi^*\colon X(F_2, \mathrm{SL}(2,\mathbb{C})) \to X(F_2, \mathrm{SL}(2,\mathbb{C})), \quad \chi \mapsto \chi \circ \phi_*.$$

Lemma 6.3 *The restriction of $\hom_+^{\mathrm{irr}}(\pi_1(M), G)/G$ to $X(F_2, \mathrm{SL}(2,\mathbb{C}))$ is contained in*

$$\{\chi \in X(F_2, \mathrm{SL}(2,\mathbb{C})) \mid \phi^*(\chi) = \bar{\chi}\}.$$

Proof Let $\rho \in \hom^{\mathrm{irr}}(\pi_1(M), G)$. If we write $\rho(t) = A \circ c$ for $A \in \mathrm{PSL}(2,\mathbb{C})$ and c complex conjugation, from the relation

$$t\gamma t^{-1} = \phi_*(\gamma) \quad \text{for all } \gamma \in F_2$$

we get

$$A\overline{\rho(\gamma)}A^{-1} = \rho(\phi_*(\gamma)) \quad \text{for all } \gamma \in F_2.$$

Hence, if ρ_0 denotes the restriction of ρ to F_2, $\bar{\rho}_0$ and $\rho_0 \circ \phi_*$ are conjugate, so they have the same character and the lemma follows. □

Lemma 6.3 motivates the following computation:

Lemma 6.4 *We have a homeomorphism*

$$\{\chi_\rho \in X(F_2, \mathrm{SL}(2,\mathbb{C})) \mid \phi^*(\chi_\rho) = \bar{\chi}_\rho\} \cong \{x \in \mathbb{C} \mid |x-1| = 1\}$$

by setting $x = \operatorname{trace}(\rho(s)) = \chi_\rho(s)$.

Proof First we describe coordinates for $X(F_2, \mathrm{SL}(2,\mathbb{C}))$. Let τ_r, τ_s and τ_{rs} denote the trace functions, ie $\tau_r(\chi_\rho) = \chi_\rho(r) = \mathrm{trace}(\rho(r))$, and similarly for s and rs. The Fricke–Klein theorem yields an isomorphism

$$(\tau_r, \tau_s, \tau_{rs})\colon X(F^2, \mathrm{SL}(2,\mathbb{C})) \cong \mathbb{C}^3$$

(see Goldman [8] for a proof). From the relations

$$\phi_*(r) = s, \quad \phi_*(s) = rs, \quad \phi_*(rs) = srs,$$

the equality $\phi^*(\chi_\rho) = \bar{\chi}_\rho$ is equivalent to

$$\bar{\tau}_r = \tau_s, \quad \bar{\tau}_s = \tau_{rs}, \quad \bar{\tau}_{rs} = \tau_{srs} = \tau_s\tau_{rs} - \tau_r.$$

In the expression for τ_{srs} we have used the relation $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$ for $A, B \in \mathrm{SL}(2,\mathbb{C})$. Taking $x = \tau_r = \tau_{rs}$ and $\tau_s = \bar{x}$, the defining equation is $x + \bar{x} = x\bar{x}$. Namely, the circle $|x-1| = 1$. □

To prove Proposition 6.2, we need to know which conjugacy classes of representations of F^2 are irreducible. By Culler and Shalen [7], a character χ_ρ in $X(F^2, \mathrm{SL}(2,\mathbb{C}))$ is reducible if and only if $\chi_\rho([r,s]) = \mathrm{tr}(\rho([r,s])) = 2$, and a straightforward computation shows that this happens in the circle $|x-1| = 1$ precisely when $x = 2$. Now, let ρ be a representation of F^2 in $\mathrm{SL}(2,\mathbb{C})$ whose character χ_ρ satisfies $\phi^*(\chi_\rho) = \bar{\chi}_\rho$. Assume ρ is irreducible. Then $\rho\circ\phi_*$ and $\bar{\rho}$ are conjugate by a unique matrix $A \in \mathrm{PSL}(2,\mathbb{C})$,

$$Ac\rho(\gamma)cA^{-1} = A\overline{\rho(\gamma)}A^{-1} = \rho(\phi_*(\gamma)) \quad \text{for all } \gamma \in F_2,$$

where c means complex conjugation. Thus, defining $\rho(t) = A\circ c$ gives a unique way to extend ρ to $\pi_1(M)$.

When χ_ρ is reducible, $x = 2$ and the character χ_ρ is trivial. Then either ρ is trivial or parabolic. In any case, it is easy to check that all possible extensions to $\pi_1(M)$ yield reducible representations. □

6.3 Comparing both ways of computing deformation spaces

We relate both ways of computing deformation spaces, via the ideal simplex and via the fibration:

Lemma 6.5 *Given a triangulated structure with parameter w as in (15), the parameter x of its holonomy as in Proposition 6.2 is*

$$x = 1 + w + |w|^2$$

(*or* $x = 1 + \bar{w} + |w|^2$, *because x is only defined up to complex conjugation*).

Proof As $r = UV$, a straightforward computation yields

$$\rho(r) = \begin{pmatrix} 0 & |w|^2 \\ -1/|w|^2 & 1 + w + |w|^2 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

Then the lemma follows from $x = \mathrm{trace}(\rho(r))$. □

The fact that not all deformations are obtained from triangulations (Corollary 4.6) is illustrated in the following remark, whose proof is an elementary computation.

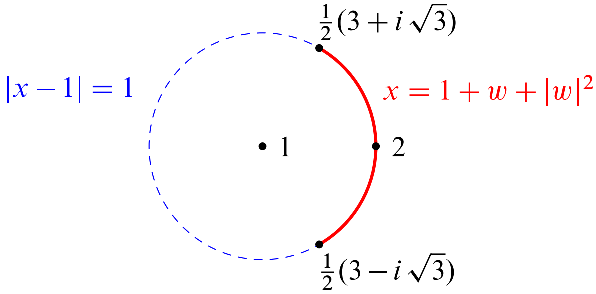

Figure 10: The image of $x=1+w+|w|^2$ in the circle $|x-1|=1$.

Remark 6.6 The image of the map

$$\{w\in\mathbb{C}\mid |w(1+w)|=1\}\to\{x\in\mathbb{C}\mid |x-1|=1\},\quad w\mapsto x=1+w+|w|^2,$$

is $\{|x-1|=1\}\cap\left\{\mathrm{Re}(x)\geq\frac{3}{2}\right\}$, ie the arc of a circle bounded by the image of the holonomy structure (and its complex conjugate); see Figure 10.

To be precise on the type of structures at the peripheral Klein bottle, we compute the trace of the peripheral element $[r,s]$ for each method and apply Lemma 4.5:

- We compute it from the variety of representations, ie from x. Using the notation of the proof of Proposition 6.2:

  $$\tau_{[r,s]}=x_1^2+x_2^2+x_3^2-x_1x_2x_3-2=(x+\bar{x})^2-3(x+\bar{x})-2=(x+\bar{x})((x+\bar{x})-3)-2.$$

  The complete hyperbolic structure corresponds to $x+\bar{x}=3$; hence, by deforming x we may have either $\tau_{[r,s]}>-2$ or $\tau_{[r,s]}<-2$.

- Next we compute it from the ideal triangulation, ie from w. As $x=1+w+|w|^2$, we get

  $$\tau_{[r,s]}=2\,\mathrm{Re}(w+w^2)\geq -2$$

  because $|w+w^2|=1$.

Remark 6.7 Finally, the path of deformations of the Gieseking manifold lifts to a path of deformations of the figure eight knot exterior that is the same as the one considered by Hilden, Lozano and Montesinos in [10] by deforming polyhedra. The transition from type I to type II of the Gieseking manifold corresponds to the *spontaneous surgery* in [10].

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Realization of Lie algebras and classifying spaces of crossed modules

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The category of complete differential graded Lie algebras provides nice algebraic models for the rational homotopy types of nonsimply connected spaces. In particular, there is a realization functor, $\langle - \rangle$, of any complete differential graded Lie algebra as a simplicial set. In a previous article, we considered the particular case of a complete graded Lie algebra, L_0, concentrated in degree 0 and proved that $\langle L_0 \rangle$ is isomorphic to the usual bar construction on the Maltsev group associated to L_0.

Here we consider the case of a complete differential graded Lie algebra, $L = L_0 \oplus L_1$, concentrated in degrees 0 and 1. We establish that the category of such two-stage Lie algebras is equivalent to explicit subcategories of crossed modules and Lie algebra crossed modules, extending the equivalence between pronilpotent Lie algebras and Maltsev groups. In particular, there is a crossed module $\mathscr{C}(L)$ associated to L. We prove that $\mathscr{C}(L)$ is isomorphic to the Whitehead crossed module associated to the simplicial pair $(\langle L \rangle, \langle L_0 \rangle)$. Our main result is the identification of $\langle L \rangle$ with the classifying space of $\mathscr{C}(L)$.

17B55, 55P62; 55U10

Introduction

In this text, we pursue the study of the rational homotopy type of spaces with models in the category **cdgl** of complete differential graded Lie algebras, as developed by the authors with Buijs and Murillo [4]. We emphasize that in this approach, there are no requirements concerning simple connectivity or nilpotency. In particular, to any finite simplicial complex is associated a cdgl M_X whose homology in degree 0 is the Maltsev completion of $\pi_1(X)$ [4, Theorem 10.5].

One of the main tools in this theory is a cosimplicial cdgl $\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n \geq 0}$, where $\mathfrak{L}_0$ is the free Lie algebra on a Maurer–Cartan element in degree -1, and $\mathfrak{L}_1$ is the Lawrence–Sullivan interval (see below for more details). This cosimplicial cdgl plays a role similar to the simplicial algebra of PL–forms on $\underline{\Delta}^\bullet$. It enables us to construct a realization functor from the category of complete differential graded Lie algebras to the category of simplicial sets, $\langle - \rangle \colon \mathbf{cdgl} \to \mathbf{Sset}$, defined by

$$\langle L \rangle_\bullet := \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

If a Lie algebra L is concentrated in degree 0, we proved in [6, Theorem 0.1] that its realization $\langle L \rangle$ is *isomorphic* to the usual bar construction on the group $\exp L$, constructed on the set L with the Baker–Campbell–Hausdorff product.

Here we consider the next step: L is a connected cdgl with nontrivial homology only in degrees 0 and 1. Geometrically, this corresponds to the notion of homotopy 2–types and, by analogy, a connected cdgl L such that $H_*L = H_0L \oplus H_1L$ is called a 2*–type cdgl*. First of all, if $L = L_{\geq 0}$ and $H_{\geq 2}L = 0$, then the Lie subalgebra $I = L_{\geq 2} \oplus dL_2$ is an ideal because if $a \in L_0$ and $b \in L_2$, then $da = 0$ and $[a, d(b)] = d[a, b]$. Moreover I is acyclic, and the quotient map is a quasi-isomorphism,

$$\varphi\colon (L, d) \xrightarrow{\simeq} (L/I, \bar{d}).$$

Since the realization functor $\langle - \rangle$ preserves quasi-isomorphisms of connected cdgls [4, Corollary 8.2 and Remark 8.6], we get a weak homotopy equivalence

$$\langle \varphi \rangle\colon \langle L, d \rangle \xrightarrow{\simeq} \langle L/I, \bar{d} \rangle.$$

We have thus reduced the problem to considering only cdgls L of the form $L = L_0 \oplus L_1$ and denote by $\mathbf{cdgl}_{\leq 1}$ the corresponding subcategory of **cdgl**. We associate to such L a natural crossed module $\mathscr{C}(L)$ and denote by **CrMod** the category of crossed modules. Our main result, which extends [6, Theorem 0.1], can be formulated as follows.

Theorem 1 *If L is a complete differential graded Lie algebra such that $L = L_0 \oplus L_1$, then its geometric realization $\langle L \rangle$ is naturally isomorphic to the classifying simplicial set $B\mathscr{C}(L)$; ie the diagram*

$$\begin{tikzcd} \mathbf{cdgl}_{\leq 1} \arrow[r, "\langle - \rangle"] \arrow[d, "\mathscr{C}"'] & \mathbf{Sset} \\ \mathbf{CrMod} \arrow[ur, "B"'] & \end{tikzcd}$$

commutes up to natural isomorphisms.

This theorem shows that the functor $\langle - \rangle$ generalizes many classical constructions.

Geometrically, crossed modules appear in the work of Whitehead [14]. If (X, A) is a pair of topological spaces, based in A, Whitehead proved that the boundary map $d\colon \pi_2(X, A) \to \pi_1(A)$, together with the action of $\pi_1(A)$ on $\pi_2(X, A)$, defines a crossed module. Then, in [11], Mac Lane and Whitehead showed that the spaces X with $\pi_q(X) = 0$, for $q \geq 2$, are determined by the crossed module of the pair (X, X_1), where X_1 is the 1–dimensional skeleton of X. For any cdgl $L = L_0 \oplus L_1$, the geometric realization $\langle L \rangle$ is determined by the crossed module associated to the pair $(\langle L \rangle, \langle L_0 \rangle)$. Our second main result identifies this crossed module with $\mathscr{C}(L)$.

Theorem 2 *The Whitehead crossed module associated to the simplicial pair $(\langle L \rangle, \langle L_0 \rangle)$ is isomorphic to the crossed module $\mathscr{C}(L)$ introduced above.*

In short, these two theorems unify the geometric realizations of complete differential graded Lie algebras of the form $L = L_0 \oplus L_1$ and of crossed modules. In the last section, we extend the correspondence between Maltsev groups and pronilpotent Lie algebras to crossed modules. We introduce the categories of Maltsev crossed modules and of pronilpotent Lie algebra crossed modules and prove an isomorphism of categories.

Theorem 3 *The following three categories are isomorphic*:

(1) *the category of pronilpotent differential graded Lie algebras of the form* $L = L_0 \oplus L_1$,

(2) *the category of pronilpotent Lie algebra crossed modules*,

(3) *the category of Maltsev crossed modules.*

Moreover, *the equivalence between* (1) *and* (3) *is given by the functor* $\mathscr{C}$.

As a next step for the future, we can consider a connected cdgl L such that $H_{\geq n+1}L = 0$ for some $n \geq 1$. Using the ideal $J = L_{\geq n+1} \oplus dL_{n+1}$, the same argument used above gives a weak homotopy equivalence

$$\langle \varphi \rangle \colon \langle L, d \rangle \xrightarrow{\simeq} \langle L/J, \bar{d} \rangle.$$

We conjecture that the differential d defines an n–cat-group structure on $\mathscr{C}(L)$ — in the sense of Loday in [10] — and that the geometric realization $\langle L/J, \bar{d} \rangle$ is isomorphic to the realization of this n–cat-group.

Our program is carried out in Sections 1–7 below, whose headings are self-explanatory.

Conventions and notation

In a graded Lie algebra L, the group of elements of degree i is denoted by L_i. A Lie algebra differential decreases the degree by 1, ie $dL_i \subset L_{i-1}$. If $x \in L$, we denote by ad_x the Lie derivation of L defined by $\mathrm{ad}_x(y) = [x, y]$.

If there is no ambiguity, the product of two elements m and m' of a group M is denoted by mm'. Sometimes, if several laws are involved, we can use some specific notation, such as $m \perp m'$ or $m * m'$, to avoid confusion. An action of a group N on a group M is always a left action and is denoted by $(n, m) \mapsto {}^n m$. We denote then by $M \rtimes N$ the semidirect product whose multiplication law is defined by

$$(m, n)(m', n') = (m\, {}^n m', nn').$$

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1 Background on Lie models

A *complete differential graded Lie algebra* (henceforth cdgl) is a differential graded Lie algebra L equipped with a decreasing filtration of differential Lie ideals such that $F^1 = L$, $[F^p L, F^q L] \subset F^{p+q} L$ and

$$L = \varprojlim_n L/F^n L.$$

If no filtration is specified, it is understood that we consider the lower central series.

Let $V=\bigoplus_{i\in\mathbb{Z}} V_i$ be a rational graded vector space. We denote by $\mathbb{L}(V)$ the free graded Lie algebra on V, and by $\mathbb{L}^{\geq n}(V)$ the ideal of $\mathbb{L}(V)$ generated by the brackets of length greater than or equal to n. The *completion of* $\mathbb{L}(V)$ is the inverse limit

$$\widehat{\mathbb{L}}(V)=\varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V).$$

This is a cdgl for the filtration given by the ideals $G^n=\ker(\widehat{\mathbb{L}}(V)\to\mathbb{L}(V)/\mathbb{L}^{>n}(V))$. The correspondence $V\to\widehat{\mathbb{L}}(V)$ gives a left adjoint to the forgetful functor to graded rational vector spaces [4, Proposition 3.10]. We call $\widehat{\mathbb{L}}(V)$ the *free complete graded Lie algebra on V*.

If θ is a derivation of degree 0 on a cgl L, the exponential map e^θ is a cgl automorphism of L defined by

$$e^\theta=\sum_{i\geq 0}\frac{\theta^i}{i!}.$$

In particular, for any $x\in L_0$, $e^{\operatorname{ad}x}$ is a cgl automorphism of L. Therefore, in any cgl L, the Lie subalgebra L_0 admits a group structure whose multiplication law $*$ is given by the Baker–Campbell–Hausdorff product [1, Chapter II.6, Proposition 4; 13, Section 3.4] and characterized by

$$e^{\operatorname{ad}_{x*y}}=e^{\operatorname{ad}_x}\circ e^{\operatorname{ad}_y}.$$

Now we recall the first properties of the cosimplicial cdgl $\mathfrak{L}_\bullet$ [4, Chapter 6]. Denote as usual by $\underline{\Delta}^n$ the simplicial set in which $\underline{\Delta}^n_p$ is the set of $(p+1)$–tuples of integers $(j_0,\dots,j_p)$ such that $0\leq j_0\leq\cdots\leq j_p\leq n$. We also denote by Δ^n the simplicial complex formed by the nonempty subsets of $\{0,\dots,n\}$. The subcomplex $\dot{\Delta}^n$ of Δ^n is the simplicial complex containing the proper nonempty subsets of $\{0,\dots,n\}$.

Finally $s^{-1}C_*\underline{\Delta}^n$ denotes the desuspension of the simplicial chain complex on $\underline{\Delta}^n$ and $s^{-1}C_*\Delta^n$ the desuspension of the complex of simplicial chains on Δ^n, which is isomorphic to $s^{-1}N_*\underline{\Delta}^n$, the complex of nondegenerate chains on $\underline{\Delta}^n$. Then, as a graded Lie algebra (without differential), we set

$$\mathfrak{L}_n=\widehat{\mathbb{L}}(s^{-1}C_*\Delta^n).$$

In other words, $\mathfrak{L}_n$ is the free complete graded Lie algebra on elements $a_{i_0\dots i_k}$ of degree $|a_{i_0\dots i_k}|=k-1$, for all $0\leq i_0<\cdots<i_k\leq n$. For instance, we have $|a_i|=-1$ and $|a_{i_0i_1}|=0$.

The family $\underline{\Delta}^\bullet=\{\underline{\Delta}^n\}_{n\geq 0}$ is a cosimplicial object in the category of simplicial sets. It follows that the family $s^{-1}N_*\underline{\Delta}^\bullet$ is a cosimplicial object in the category of chain complexes. The identification $s^{-1}C_*\Delta^n\cong s^{-1}N_*\underline{\Delta}^n$ makes $s^{-1}C_*\Delta^n$ a cosimplicial object in the category of chain complexes. The extension of the cofaces and codegeneracies as morphisms of Lie algebras gives morphisms of complete graded Lie algebras $\delta^i\colon\mathfrak{L}_n\to\mathfrak{L}_{n+1}$ and $\sigma^i\colon\mathfrak{L}_n\to\mathfrak{L}_{n-1}$. More precisely,

$$\delta^i(a_{j_0\dots j_p})=a_{r_0\dots r_p}\qquad\text{with } r_k=\begin{cases} j_k & \text{if } j_k<i,\\ j_k+1 & \text{if } j_k\geq i,\end{cases}$$

$$\sigma^i(a_{j_0\dots j_p})=a_{r_0\dots r_p}\qquad\text{with } r_k=\begin{cases} j_k & \text{if } j_k\leq i,\\ j_k-1 & \text{if } j_k>i,\end{cases}$$

if $r_0<\cdots<r_p$. Otherwise, $\sigma^i(a_{j_0\dots j_p})=0$.

Proposition 1.1 [4, Theorem 6.1] *Each $\mathfrak{L}_n$ can be endowed with a differential d satisfying the following properties.*

(i) *The linear part d_1 of d is given by*
$$d_1 a_{i_0\dots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0\dots\hat{i}_j\dots p}.$$

(ii) *The generators a_i are Maurer–Cartan elements; ie $da_i = -1/2[a_i, a_i]$.*

(iii) *The cofaces δ^i and the codegeneracies σ^i are cdgl morphisms.*

(iv) *For $n \geq 2$,*
$$da_{0\dots n} = [a_0, a_{0\dots n}] + \Phi,$$
with $\Phi \in \widehat{\mathbb{L}}(s^{-1} C_* \dot{\Delta}^n)$.

Thus, in particular, the family $\mathfrak{L}_\bullet$ is a cosimplicial cdgl.

Let us specify the cdgl $\mathfrak{L}_n$ in low dimensions.

- $\mathfrak{L}_0 = (\mathbb{L}(a_0), d)$ is the free Lie algebra on a Maurer–Cartan element a_0.
- $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$ is the Lawrence–Sullivan interval — see [9] — with
$$da_{01} = [a_{01}, a_1] + \frac{\mathrm{ad}_{a_{01}}}{e^{\mathrm{ad}_{a_{01}}} - 1}(a_1 - a_0).$$
- $\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d)$ is a model of the triangle — see [4, Proposition 5.14] — with the differential

$$d(a_{012}) = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}]. \tag{1-1}$$

The cosimplicial cdgl $\mathfrak{L}_\bullet$ leads naturally to the definition of cdgl models for any simplicial set and to a geometric realization for any given cdgl; see [4, Chapter 7]. For our purpose, we only need the realization of a cdgl L, defined as the simplicial set
$$\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L),$$
which satisfies properties of the classical Quillen realization. For instance, for any $n \geq 1$, $\pi_n\langle L\rangle = H_{n-1}L$, where the group law of $H_0 L$ is the BCH product; see [4, Section 4.2] or [1, Chapter II.6.4].

2 Crossed modules and cdgls

For general background on crossed modules, we refer the reader to the historical papers of Whitehead [11; 14] or to more modern presentations, such as [2; 3; 10]. We recall only the basics we need.

Definition 2.1 A crossed module $\mathscr{C} = (d\colon M \to N)$ is a morphism of groups d together with an action of N on M, given by group automorphisms $n \mapsto (m \mapsto {}^n m)$ satisfying two conditions:

(1) For all $m \in M$ and $n \in N$, $d({}^n m) = n d(m) n^{-1}$.

(2) For all $m \in M$, $m' \in M$, ${}^{d(m)}m' = m m' m^{-1}$.

If the group N acts on itself by conjugation, the first property means that d is compatible with the N–action. It also implies that the group $d(M)$ is a normal subgroup of N and that $\ker d$ is an N–submodule of M.

On the other hand, we remark that if $d(m) = 1$, the second property implies $mm' = m'm$ which means that $\ker d$ is included in the center of M. The same property shows that $\operatorname{Im} d$ acts trivially on $\ker d$ and induces thus an action of $\operatorname{coker} d$ on $\ker d$.

Now let $L = L_0 \oplus L_1$ be a cdgl. In what follows L_0 is always considered as a group equipped with the BCH product denoted by $*$. We will prove that $d\colon L_1 \to L_0$ is a crossed module. The first step consists in defining a group structure on L_1. This construction was originally carried out in [4, Definition 6.14].

Proposition 2.2 *For any cdgl (L, d) such that $L = L_0 \oplus L_1$, L_1 admits a natural product $\perp$ for which the differential $d\colon (L_1, \perp) \to (L_0, *)$ is a group morphism. Moreover, $a \perp b = a + b$ if a and b are cycles.*

Proof The different possibilities for a definition of this law are described in [4, Section 6.5]. We recall here the construction for the convenience of the reader, beginning with the "universal" example, the cdgl $L' = \widehat{\mathbb{L}}(u_1, u_2, du_1, du_2)$, with u_i in degree 1. Since $HL' = 0$ there is an element ω in L'_1 such that

$$d\omega = du_1 * du_2. \tag{2-1}$$

Of course such an element is not unique. If ω' is another element satisfying (2-1), the difference $\omega - \omega'$ is a boundary since $H_{\geq 1}L' = 0$. This shows that the class of ω is well defined in the cdgl quotient $(L'/(L'_{\geq 2} \oplus dL'_2), \bar{d})$. We denote this class by $u_1 \perp u_2$. By construction, it satisfies

$$\bar{d}(u_1 \perp u_2) = du_1 * du_2.$$

Among all the different possible choices for ω, one starts with the Baker–Campbell–Hausdorff series for $du_1 * du_2$. Replacing in each term one and only one du_i by u_i, we get an element ω with $d\omega = du_1 * du_2$. This gives

$$\omega = u_1 + u_2 + \tfrac{1}{2}[u_1, du_2] + \tfrac{1}{12}[du_1, [du_1, u_2]] - \tfrac{1}{12}[du_2, [du_1, u_2]] + \cdots . \tag{2-2}$$

Now, let L be a cdgl with $L = L_0 \oplus L_1$, $e_1, e_2 \in L_1$, and $f\colon L' \to L$ the unique cdgl map sending u_i to e_i. Then the element $e_1 \perp e_2 := f(u_1 \perp u_2)$ is a well-defined element in L_1. By construction, if e_1 and e_2 are cycles, using the image of the formula (2-2) in L, we have $e_1 \perp e_2 = e_1 + e_2$.

For the associativity of $\perp$, we consider $L'' = \widehat{\mathbb{L}}(u_1, u_2, u_3, du_1, du_2, du_3)$ and observe that in L''_1/dL''_2 we have $(u_1 \perp u_2) \perp u_3 = u_1 \perp (u_2 \perp u_3)$ because both have the same boundary. The same is thus true in L_1. □

With this group structure on L_1 we can now prove that $L = L_0 \oplus L_1$ is a crossed module.

Proposition 2.3 *Let (L, d) be a connected complete differential graded Lie algebra such that $L = L_0 \oplus L_1$. Then $d\colon (L_1, \perp) \to (L_0, *)$ is a crossed module.*

Proof Recall from [4, Definition 12.40] that the group L_0 acts on L_1 by

$$ {}^{x}z = e^{\mathrm{ad}_x}(z), \quad \text{for all } x \in L_0, z \in L_1. $$

From [4, Corollary 4.12] it follows that, for any $x \in L_0$, $y \in L_0$ and $z \in L_1$,

$$ {}^{(x*y)}z = e^{\mathrm{ad}_{x*y}}(z) = e^{\mathrm{ad}_x}(e^{\mathrm{ad}_y} z) = {}^{x}({}^{y}z). $$

To prove that the function $y \mapsto {}^{x}y$ is a group homomorphism, as in Proposition 2.2, we consider a universal example. Let $E = \widehat{\mathbb{L}}(x, z, t, dz, dt)$ with x in degree 0, z and t in degree 1, and $dx = 0$. Since the injection $\mathbb{L}(x) \to E$ is a quasi-isomorphism, we have $H_{\geq 1}(E) = 0$. Observe that in $E/(E_{\geq 2} \oplus dE_2)$,

$$ \begin{aligned} d({}^{x}(z \perp t)) = e^{\mathrm{ad}_x}(d(z \perp t)) = e^{\mathrm{ad}_x}(dz * dt) \\ &= x * dz * dt * x^{-1} \\ &= x * dz * x^{-1} * x * dt * x^{-1} \\ &= e^{\mathrm{ad}_x}(dz) * e^{\mathrm{ad}_x}(dt) = d(e^{\mathrm{ad}_x} z) * d(e^{\mathrm{ad}_x} t) = d({}^{x}z \perp {}^{x}t). \end{aligned} $$

Thus, in E_1/dE_2, we get

$$ {}^{x}(z \perp t) = {}^{x}z \perp {}^{x}t. $$

The same is therefore also true in L_1.

As x is a cycle, by [4, Propositions 4.10 and 4.13],

$$ d({}^{x}z) = e^{\mathrm{ad}_x}(dz) = x * dz * x^{-1}, $$

and property (1) of Definition 2.1 is satisfied. For property (2), we use once again the universal example $L' = \widehat{\mathbb{L}}(u_1, u_2, du_1, du_2)$ already considered in the proof of Proposition 2.2. Since in L'_1/dL'_2 we have

$$ d({}^{du_1}u_2) = du_1 * du_2 * du_1^{-1} = d(u_1 \perp u_2 \perp u_1^{-1}), $$

we deduce that

$$ {}^{du_1}u_2 = u_1 \perp u_2 \perp u_1^{-1}, $$

and thus the same is true in L_1. □

Remark 2.4 By Proposition 2.2, under the hypotheses of Proposition 2.3, we deduce that the group structures $\perp$ and $+$ coincide on $H_1 L = \ker d$.

We have thus defined a functor $\mathscr{C}\colon \mathbf{cdgl}_{\leq 1} \to \mathbf{CrMod}$.

3 The crossed module of a realization and Theorem 2

In this section, in the case $L = L_0 \oplus L_1$, we establish the isomorphism between $\mathscr{C}(L)$ and the Whitehead crossed module of $(\langle L \rangle, \langle L_0 \rangle)$.

Proof of Theorem 2 The realization $\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ of a cdgl $L = L_0 \oplus L_1$ is a Kan complex [4, Proposition 7.13]. We first compute $\pi_1(\langle L_0 \rangle)$ and $\pi_2(\langle L \rangle, \langle L_0 \rangle)$, and for that we use the homotopy relation introduced in [12, Section 3].

Since $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$, the map $f \mapsto f(a_{01})$ induces an isomorphism of sets

$$\langle L_0 \rangle_1 = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_1, L_0) \xrightarrow{\cong} L_0.$$

Since $\partial_i f = 0$, for $i = 0, 1$, each element of L_0 defines an element of $\pi_1(\langle L_0 \rangle)$. Now, two such 1–simplices, g and f, are homotopic in $\langle L_0 \rangle$ if there exists a map $h\colon \mathfrak{L}_2 \to L_0$ such that $\partial_1 h = g$, $\partial_2 h = f$ and $\partial_0 h = 0$. The simplex h is called a homotopy from f to g.

In the particular case $g = 0$, from the simplicial structure of the realization, we get $h(a_{02}) = h(a_{12}) = 0$ and $h(a_{01}) = f(a_{01})$. Since $h(a_{012}) = 0$, we have an equivalence

$$f \sim 0 \iff 0 = dh(a_{012}) = h(a_{01} * a_{12} * a_{02}^{-1}) = f(a_{01}).$$

Therefore $\pi_1 \langle L_0 \rangle = L_0$.

To compute the relative homotopy group $\pi_2(\langle L \rangle, \langle L_0 \rangle)$, we consider the set

$$K = \bigl\{ f \in \langle L \rangle_2 = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_2, L) \mid \partial_i f = 0 \text{ for } i = 1,\, 2 \text{ and } \partial_0 f \in \langle L_0 \rangle \bigr\}.$$

If $f \in K$, we have $\partial_0 f(a_{01}) = f(\delta^0(a_{01})) = f(a_{12}) = f(da_{012}) = df(a_{012})$ and thus the correspondence $K \to L_1$ which maps f to $f(a_{012})$ is an isomorphism. By [12, Definitions 3.3 and 3.6], two simplices, f and g, of K are homotopic rel $\langle L_0 \rangle$ if $\partial_0 f \sim \partial_0 g$ in $\langle L_0 \rangle$ by a homotopy h, and there exists a 3–simplex $\omega\colon \mathfrak{L}_3 \to L$ such that $\partial_0 \omega = h$, $\partial_2 \omega = f$, $\partial_3 \omega = g$ and $\partial_1 \omega = 0$.

For getting an expression of these conditions at the level of cdgls, we recall [4, Proposition 6.16] the differential d of $\mathfrak{L}_3$, which uses the operation $\perp$ introduced in the proof of Proposition 2.2,

$$d(a_{0123}) = e^{\mathrm{ad}_{a_{01}}} a_{123} - (a_{012} \perp a_{023} \perp a_{013}^{-1}). \tag{3-1}$$

From $L_{\geq 2} = 0$, we deduce $\omega(a_{0123}) = 0$. By the definition of K, $\omega(a_{123}) = \partial_0 \omega(a_{012}) = h(a_{012}) = 0$ since L_0 has no element of degree 1. We also have

$$\omega(a_{012}) = \partial_3 \omega(a_{012}) = g(a_{012}), \quad \omega(a_{013}) = \partial_2 \omega(a_{012}) = f(a_{012}), \quad \omega(a_{023}) = \partial_1 \omega(a_{012}) = 0.$$

Thus, by applying ω to both sides of (3-1), we obtain

$$0 = 0 - g(a_{012}) \perp 0 \perp f(a_{012})^{-1},$$

ie $0 = g(a_{012}) \perp f(a_{012})^{-1}$. This implies $0 = dg(a_{012}) * df(a_{012})^{-1}$ and $df(a_{012}) = dg(a_{012})$.

It remains to describe $g(a_{012}) \perp f(a_{012})^{-1}$. From the compatibility of the differential with Lie bracket and the fact that L_1 is an abelian Lie algebra, we get $[g(a_{012}), dg(a_{012})] = -\frac{1}{2} d[g(a_{012}), g(a_{012})] = 0$. In the BCH product $df(a_{012}) * dg(a_{012})$, all terms except the linear ones contain a bracket $[df(a_{012}), dg(a_{012})]$ which becomes $[df(a_{012}), g(a_{012})] = [dg(a_{012}), g(a_{012})] = 0$ in the formula (2-2). We thus obtain

$$g \perp f^{-1} = g - f.$$

We have proven $\pi_2(\langle L\rangle, \langle L_0\rangle) \cong L_1$ and $\pi_1(\langle L_0\rangle) \cong L_0$. We also showed that the connecting map $\partial\colon \pi_2(\langle L\rangle, \langle L_0\rangle) \to \pi_1(\langle L_0\rangle)$, given by $[f] \mapsto [\partial_0 f]$, corresponds to $df(a_{012})$ in the previous isomorphisms.

Consider now the action of $\pi_1(\langle L_0\rangle) = L_0$ on $\pi_2(\langle L\rangle, \langle L_0\rangle) = L_1$. Let $a \in L_0$, $b \in L_1$ and ${}^a b$ the element of L_1 corresponding to this action. Recall [4, Lemma 4.23] that $y = e^{\mathrm{ad}_a} b$ is also an element of L_1 such that $dy = a * db * a^{-1}$. Both constructions, ${}^a b$ and $e^{\mathrm{ad}_a} b$, are natural, so to prove ${}^a b = e^{\mathrm{ad}_a} b$, we have only to prove it for the cdgl L'' quotient of $L' = \widehat{\mathbb{L}}(a, u, du)$, with $\deg u = 1$, by the ideal $L'_{\geq 2} \oplus dL'_2$. The required identification follows from $d({}^a u) = d(e^{\mathrm{ad}_a} u)$ and the injectivity of $d\colon L''_1 \to L''_0$. We have thus recovered the crossed module $\mathscr{C}(L)$. $\square$

4 The classifying space of a crossed module

By definition, the classifying space of a crossed module $\mathscr{C}$ is the classifying space of the nerve of the categorical group associated to $\mathscr{C}$. Let us specify this association.

Recall that a categorical group is a group object in the category of groups (see [10, Section 1.1]),

$$G \overset{s}{\underset{t}{\rightrightarrows}} N,$$

where N is a subgroup of G, s and t are homomorphisms such that $s|_N = t|_N = \mathrm{id}_N$ and $[\ker s, \ker t] = 1$.

In [10], J L Loday defines a categorical group associated to a crossed module $\mathscr{C} = (d\colon (M, \perp) \to (N, *))$ as follows:

- $G = M \rtimes N$ is the product $M \times N$ with the semidirect product given by the action of N on M. Thus, the product of (m', n') and (m, n) in G is
$$(m', n') \bullet (m, n) = (m' \perp {}^{n'} m, n' * n).$$
- An element (m, n) of G has for source and target, respectively,
$$s(m, n) = dm * n \quad \text{and} \quad t(m, n) = n.$$

Thus, the group N is interpreted as the group of objects viewed in G as $\{1\} \times N$. The group $G = M \rtimes N$ is the group of arrows with the morphisms s and t giving the source and the target. Two elements (m', n') and (m, n) are composable if

$$n' = t(m', n') = s(m, n) = dm * n.$$

In this case the composition is defined by

$$(m', n') \circ (m, n) = (m' \perp m, n).$$

We deduce easily from property (1) of Definition 2.1 that s and t are group homomorphisms. We also verify that the source of a composite is the source of the first factor and the target is the target of the second factor:

$$s(m' \perp m, n) = d(m' \perp m) * n = dm' * dm * n = dm' * n' = s(m', n'), \quad t(m' \perp m, n) = n = t(m, n).$$

Finally, composition is a group homomorphism; see [10, Lemma 2.2].

The usual nerve of a category is a simplicial set. When the category is a categorical group, we obtain naturally a simplicial group. Let us describe the nerve of the categorical group associated to a crossed module $\mathscr{C} = (d\colon (M, \perp) \to (N, *))$. We have

$$\mathrm{Ner}_1 = M \rtimes N \overset{d_1}{\underset{d_0}{\rightrightarrows}} \mathrm{Ner}_0 = N$$

with $d_0(m, n) = t(m, n) = n$, $d_1(m, n) = s(m, n) = dm * n$ and $s_0\colon \mathrm{Ner}_0 \to \mathrm{Ner}_1$ is the canonical injection $N \to M \rtimes N$.

An element of Ner_k is a sequence $(m_i, n_i)_{1 \le i \le k}$ such that

$$n_i = t(m_i, n_i) = s(m_{i-1}, n_{i-1}) = dm_{i-1} * n_{i-1}.$$

As the n_i, for $i \ge 2$, are determined by n_1 and the family $(m_i)_{1 \le i \le k}$, the sequence $(m_i, n_i)_{i \le k}$ can be identified with the sequence

$$(m_k, m_{k-1}, \dots, m_1, n_1) \in M^k \times N.$$

In particular,

$$\mathrm{Ner}_k = M^k \times N. \tag{4-1}$$

Each Ner_k is a group, the multiplication being given component wise. With the identification (4-1), this product is given by

$$((m_i)_{1 \le i \le k}, n) \bullet ((m_i')_{1 \le i \le k}, n') = ((m_i \perp^{d(\perp_{j=1}^{i-1} m_j) * n} m_i')_{1 \le i \le k}, n * n').$$

The boundary and degeneracy maps of Ner_* are morphisms of groups defined as usual by

$$\begin{aligned} d_0(m_k, \dots, m_1, n) &= (m_k, \dots, m_2, d(m_1) * n), \\ d_i(m_k, \dots, m_1, n) &= (m_k, \dots, m_{i+1} \perp m_i, \dots, m_1, n), \quad 0 < i < k, \\ d_k(m_k, \dots, m_1, n) &= (m_{k-1}, \dots, m_1, n), \\ s_i(m_k, \dots, m_1, n) &= (m_k, \dots, m_i, 1, m_{i-1}, \dots, m_1, n), \quad 0 \le i \le k. \end{aligned}$$

The identity $e_k \in \mathrm{Ner}_k$ is the element $(1, \dots, 1, 1)$.

Recall from [5, Definition 3.20] or [7, page 255] the classifying functor $\overline{W}$ which goes from the category of simplicial groups to the category of reduced simplicial sets. The classifying space $B\mathscr{C}$ of the crossed module $\mathscr{C}$ is the space obtained by composing Ner_* with $\overline{W}$,

$$B\mathscr{C} = \overline{W}(\mathrm{Ner}_*).$$

By definition of $\overline{W}$,

$$(B\mathscr{C})_k = \{(h_{k-1}, \dots, h_0) \mid h_i \in \mathrm{Ner}_i\}.$$

The boundaries and degeneracies are given by

$$\begin{aligned}
d_0(h_{k-1}, \dots, h_0) &= (h_{k-2}, \dots, h_0),\\
d_i(h_{k-1}, \dots, h_0) &= (d_{i-1}h_{k-1}, \dots, d_0 h_{k-i} \bullet h_{k-i-1}, h_{k-i-2}, \dots, h_0), \quad 0 < i < k,\\
d_k(h_{k-1}, \dots, h_0) &= (d_{k-1}h_{k-1}, \dots, d_1 h_1),\\
s_0(h_{k-1}, \dots, h_0) &= (1, h_{k-1}, \dots, h_0),\\
s_i(h_{k-1}, \dots, h_0) &= (s_{i-1}h_{k-1}, \dots, s_0 h_{k-i}, 1, h_{k-i-1}, \dots, h_0), \quad 0 < i \le k.
\end{aligned}$$

In particular, in low dimensions,

$$B\mathscr{C}_0 = 1, \quad B\mathscr{C}_1 = N, \quad B\mathscr{C}_2 = (M \rtimes N) \times N, \quad B\mathscr{C}_3 = (M^2 \rtimes N) \times (M \rtimes N) \times N.$$

5 The classifying space functor $\overline{W}$ and twisting functions

Let A_* be a simplicial set. By [12, Corollary 27.2], there is a bijective correspondence between morphisms of simplicial sets $\varphi\colon A_* \to \overline{W} \circ \mathrm{Ner}_* = B\mathscr{C}$ and twisting functions

$$\tau = \{\tau_k\colon A_k \to \mathrm{Ner}_{k-1}\}_{k \ge 1}.$$

Recall [12, Definition 18.3] that a twisting function τ is a family of maps $\tau_k\colon A_k \to \mathrm{Ner}_{k-1}$ satisfying, for $x \in A_k$,

$$\begin{aligned}
d_0 \tau x &= \tau d_1 x \bullet (\tau d_0 x)^{-1},\\
d_i \tau x &= \tau d_{i+1} x, \quad i > 0,\\
s_i \tau x &= \tau s_{i+1} x, \quad i \ge 0,\\
\tau s_0 x &= e_k \in \mathrm{Ner}_k.
\end{aligned}$$

The simplicial map $\varphi_k\colon A_k \to (B\mathscr{C})_k$ associated to the twisting function τ is given by

$$\varphi_k x = (\tau x, \tau d_0 x, \dots, \tau d_0^{k-1} x). \tag{5-1}$$

Conversely [12, page 88], the twisting function τ associated to a simplicial morphism $\varphi\colon A_* \to \overline{W}(\mathrm{Ner}_*)$ is defined by

$$\tau = \tau(\mathrm{Ner}_*) \circ \varphi,$$

where $\tau(\mathrm{Ner}_*)$ is the twisting function associated to the identity on $\overline{W}(\mathrm{Ner}_*)$,

$$\tau(\mathrm{Ner}_*)\colon \overline{W}(\mathrm{Ner}_*)_k \to \mathrm{Ner}_{k-1},$$

defined by

$$\tau(\mathrm{Ner}_*)(g_{n-1}, \dots, g_0) = g_{n-1}.$$

6 Proof of Theorem 1

First we compute the simplicial set $\langle L\rangle_\bullet = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ in the case $L = L_0 \oplus L_1$. By $L_{\geq 2} = 0$ and [4, Corollary 6.5], we have isomorphisms

$$\mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \cong \mathrm{Hom}_{\mathbf{cdgl}}((\widehat{\mathbb{L}}((s^{-1}\Delta^k)_{\leq 2}), d), L) \cong \mathrm{Hom}_{\mathbf{cdgl}}((\widehat{\mathbb{L}}((s^{-1}\Lambda^k_0)_{\leq 2}), d), L).$$

Since any morphism of codomain L vanishes on elements of negative degree, we can quotient by the differential ideal generated by the generators of degree -1. This gives as free cgl

$$\overline{\mathfrak{L}}_k = (\widehat{\mathbb{L}}(a_{ij}, a_{0st}), d) \quad \text{with } 0 \leq i < j \leq k \text{ and } 0 < s < t \leq k.$$

Finally, in view of the differential in $\mathfrak{L}_2$, recalled in (1-1), the differential of $\overline{\mathfrak{L}}_k$ satisfies

$$da_{ij} = 0 \quad \text{and} \quad da_{0st} = a_{0s} * a_{st} * a_{0t}^{-1}.$$

In the rest of this text, we will use that, for all k, there exists an isomorphism

$$\langle L\rangle_k = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) = \mathrm{Hom}_{\mathbf{cdgl}}(\overline{\mathfrak{L}}_k, L).$$

Proposition 6.1 *If $L = L_0 \oplus L_1$, then the morphism*

$$\Psi\colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \to L_0^k \times L_1^{k(k-1)/2}$$

given by $\Psi(f) = \big((f(a_{r\,r+1}))_{0\leq r<k}, (f(a_{r,r+1,s}))_{r+1<s\leq k}\big)$ is an isomorphism.

Proof For the sake of simplicity write for $i < j$, $a_{ji} = a_{ij}^{-1}$, and for $0 \leq i < j < r \leq k$,

$$\begin{aligned} a_{irj} &= a_{ijr}^{-1}, \\ a_{rij} &= {}^{a_{ri}}a_{ijr} = {}^{a_{ir}^{-1}}a_{ijr}, \\ a_{jir} &= {}^{a_{ji}}a_{irj} = {}^{a_{ij}^{-1}}a_{ijr}^{-1}, \\ a_{jri} &= {}^{a_{ji}}a_{ijr} = {}^{a_{ij}^{-1}}a_{ijr}, \\ a_{rji} &= {}^{a_{ri}}a_{irj} = {}^{a_{ir}^{-1}}a_{ijr}^{-1}. \end{aligned}$$

With this notation, when the integers i, j and r are all different from each other and between 0 and k,

$$da_{ijr} = a_{ij} * a_{jr} * a_{ri}.$$

Suppose that the elements $f(a_{r,r+1})$ and $f(a_{r,r+1,t})$, with $r+1<t$, are defined. Then the other elements, $f(a_{r,r+s})$ and $f(a_{r,r+s,t})$ with $r+s<t$, can be derived by induction on s from the formulas

$$\begin{aligned} f(a_{r,r+s+1}) &= df(a_{r,r+1,r+s+1})^{-1} * f(a_{r,r+1}) * f(a_{r+1,r+s+1}), \\ f(a_{r,r+s+1,t}) &= {}^{f(a_{r,r+1})}\big(f(a_{r+1,r,r+s+1}) \perp f(a_{r+1,r+s+1,t}) \perp f(a_{r+1,t,r})\big). \end{aligned}$$

This shows that Ψ is injective. The same construction process shows that Ψ is also surjective. $\square$

The isomorphism of our main theorem is based on a family τ of maps

$$\tau_k \colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \to \mathrm{Ner}_{k-1}, \quad k \geq 1,$$

defined by

$$\tau_k f = (m_{k-1}, \dots, m_1, n) \in M^{k-1} \times N,$$

with $n = f(a_{01})$, $m_1 = (f(a_{012}))^{-1}$ and $m_i = (f(a_{01(i+1)}))^{-1} \perp f(a_{01i})$, for $i \geq 2$. In low dimensions, this gives

$$\begin{aligned}
\tau_1 f &= f(a_{01}) \in N,\\
\tau_2 f &= (f(a_{012})^{-1}, f(a_{01})) \in M \times N,\\
\tau_3 f &= (f(a_{013})^{-1} \perp f(a_{012}), f(a_{012})^{-1}, f(a_{01})) \in M^2 \times N.
\end{aligned}$$

Proposition 6.2 *The family τ is a twisting function.*

Proof Observe that $m_{i+1} \perp m_i = f(a_{01(i+2)})^{-1} \perp f(a_{01i})$. Thus, the index $i+1$ disappears in the expression of $d_i \tau_k f$ and we get $d_i \tau_k f = \tau_{k-1} d_{i+1} f$ for $0 < i < k-1$. A similar argument gives also the result for d_{k-1}. We have reduced the problem to proving the more subtle equality involving d_0. We use an induction, supposing the result is true for τ_j, with $j < k$, and considering τ_k. Due to the inductive step, we can concentrate the computations on the left-hand factor. From the definitions,

$$\begin{aligned}
\tau_{k-1} d_1 f &= (f(a_{02k}))^{-1} \perp f(a_{02(k-1)}), \dots, f(a_{02})),\\
\tau_{k-1} d_0 f &= ((f(a_{12k})^{-1} \perp f(a_{12(k-1)}), \dots, f(a_{12})),\\
d_0 \tau_k f &= ((f(a_{01k})^{-1} \perp f(a_{01(k-1)}), \dots, (df(a_{012}))^{-1} * f(a_{01})).
\end{aligned}$$

We determine the product of the two last terms,

$$d_0 \tau_k f \bullet \tau_{k-1} d_0 f = (f(a_{01k})^{-1} \perp f(a_{01(k-1)}) \perp {}^{\gamma}(f(a_{12k})^{-1} \perp f(a_{12(k-1)})), \dots),$$

where $\gamma = dm_{k-2} * dm_{k-1} * \cdots * dm_1 * n = f(a_{0(k-1)}) * (f(a_{1(k-1)}))^{-1}$. To obtain the equality with $\tau_{k-1} d_1 f$, we consider the computation in $\overline{\mathfrak{L}}_k$,

$$\begin{aligned}
d(a_{01k}^{-1} \perp a_{01(k-1)} \perp {}^{a_{0(k-1)} * a_{1(k-1)}^{-1}}(a_{12k}^{-1} \perp a_{12(k-1)})) &= a_{0k} * a_{2k}^{-1} * a_{2(k-1)} * a_{0(k-1)}^{-1}\\
&= d(a_{02k}^{-1} \perp a_{02(k-1)}).
\end{aligned}$$

Similar computations give the corresponding equalities for degeneracy maps. □

Denote by φ the morphism of simplicial sets induced by the previous twisting function τ,

$$\varphi \colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}, L) \to B\mathscr{C}(L).$$

The following result finishes the proof of the theorem.

Proposition 6.3 *The morphism φ is an isomorphism of simplicial sets.*

Proof Recall from (5-1) that
$$\varphi_k f = (\tau f, \tau d_0 f, \dots, \tau d_0^{k-1} f).$$
Moreover, using $d_0 f = f\delta^0$, we get $\tau d_0 f = (m'_{k-2}, \dots, m'_1, n')$, with $n' = f(a_{12})$, $m'_1 = f(a_{123})^{-1}$ and for $i > 1$, $m'_i = f(a_{12(i+2)})^{-1} \perp f(a_{12(i+1)})$. By iteration from $(d_0)^\ell f = f(\delta^0)^\ell$, we deduce that the image of φ_k is the linear subspace generated by the elements $f(a_{r,r+1})$, for $0 \le r < k$, and $f(a_{r,r+1,s})$, for $r+1 < s \le k$. The result thus follows from Proposition 6.1. □

7 Maltsev crossed modules and Theorem 3

In this section, we establish an isomorphism of categories between $\mathbf{cdgl}_{\le 1}$ and a subcategory of crossed modules. We use the Lie algebra crossed modules introduced by Kassel and Loday in [8]. We begin with a reminder of [8].

In Definition 2.1, the group action of N on M corresponds to a homomorphism from N in the group of automorphisms of M. For Lie algebras, $\mathfrak{n}$ and $\mathfrak{m}$, an action of $\mathfrak{n}$ on $\mathfrak{m}$ corresponds to a Lie morphism $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ in the Lie algebra of derivations of $\mathfrak{m}$. The action of $n \in \mathfrak{n}$ on $m \in \mathfrak{m}$ is denoted $\mathfrak{v}(n).m$. We can now state [8, Définition A.1].

Definition 7.1 A Lie algebra crossed module is a morphism of Lie algebras, $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$, together with an action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$, satisfying two conditions:

(1) For all $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$, $\mathfrak{u}(\mathfrak{v}(n).m) = [n, \mathfrak{u}(m)]$.

(2) For all $m \in \mathfrak{m}$, $m' \in \mathfrak{m}$, $\mathfrak{v}(\mathfrak{u}(m)).m' = [m, m']$.

We now introduce the "rational" versions of crossed modules. If G is a group, $G^k = [G, G^{k-1}]$ denotes the lower central series of G.

Definition 7.2 (1) A group G is a *Maltsev group* (or prounipotent rational group) if each G^k/G^{k+1} is a $\mathbb{Q}$–vector space, $\dim G/G^2 < \infty$ and $G = \varprojlim_k G/G^k$.

(2) A crossed module $d\colon M \to N$ is a *Maltsev crossed module* if M and N are Maltsev groups and the action of N on M satisfies $({}^n m)m^{-1} \in M^{k+1}$ for all $m \in M^k$ and $n \in N$.

If $\mathfrak{m}$ is a Lie algebra, $\mathfrak{m}^k = [\mathfrak{m}, \mathfrak{m}^{k-1}]$ denotes the lower central series of $\mathfrak{m}$.

Definition 7.3 (1) A Lie algebra $\mathfrak{m}$ is *pronilpotent* if $\dim \mathfrak{m}/\mathfrak{m}^2 < \infty$ and $\mathfrak{m} = \varprojlim_k \mathfrak{m}/\mathfrak{m}^k$.

(2) A Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ is *pronilpotent* if $\mathfrak{m}$ and $\mathfrak{n}$ are pronilpotent Lie algebras and the action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ satisfies $\mathfrak{v}(\mathfrak{n}).\mathfrak{m}^k \subset \mathfrak{m}^{k+1}$.

Remark 7.4 The completion of a Lie algebra $\mathfrak{m}$ satisfying $\dim \mathfrak{m}/\mathfrak{m}^2 < \infty$ is the Lie algebra

$$\widehat{\mathfrak{m}} = \lim_k \mathfrak{m}/\mathfrak{m}^k.$$

This is a pronilpotent Lie algebra since $\widehat{\mathfrak{m}} = \lim_k \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^k$.

If a Lie algebra $\mathfrak{m}$ acts on a vector space V, we denote by V^k the sequence of subspaces $V^0 = V$, $V^k = \mathfrak{m}.V^{k-1}$.

Definition 7.5 The action of $\mathfrak{m}$ on V is *pronilpotent* if $V = \varprojlim_k V^k$. In particular, a cdgl $L = L_0 \oplus L_1$ is *pronilpotent* if the Lie algebra L_0 is pronilpotent and if the adjoint action of L_0 on L_1 is pronilpotent.

Proof of Theorem 3 We only define the correspondences for objects, the extension to morphisms being immediate.

To show that (1) implies (2), we start with a pronilpotent cdgl $L = L_0 \oplus L_1$ and we construct a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$. We denote d the differential of L and $[-,-]$ its bracket.

We set $\mathfrak{n} = L_0$, $\mathfrak{m} = L_1$. The bracket on $\mathfrak{n}$ is the bracket of L_0 and the bracket on $\mathfrak{m}$ is defined by

$$[a,b]' = [da,b] \quad \text{for } a,b \in L_1.$$

We check that $[-,-]'$ is an (ungraded) Lie bracket. Since $[a,b] = 0$, the antisymmetry follows from

$$0 = d[a,b] = [da,b] + [db,a] = [a,b]' + [b,a]'.$$

The proof is similar for the Jacobi identity. The morphism $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ is the differential d; this is a Lie algebra morphism,

$$\mathfrak{u}([a,b]') = d[da,b] = [da,db] = [\mathfrak{u}(a),\mathfrak{u}(b)] \quad \text{for all } a,b \in \mathfrak{m}.$$

The action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ is given by the adjoint action, $\mathfrak{v}(x) = \mathrm{ad}_x$. The formulas (1) and (2) of Definition 7.1 also follow immediately: letting $a,b \in \mathfrak{m} = L_1$ and $x \in \mathfrak{n} = L_0$,

$$\mathfrak{u}(\mathfrak{v}(x).a) = d(\mathrm{ad}_x(a)) = d[x,a] = [x,da] = [x,\mathfrak{u}(a)], \quad \mathfrak{v}(\mathfrak{u}(a)).b = \mathrm{ad}_{da}(b) = [da,b] = [a,b]'.$$

By definition, since L is pronilpotent the associated Lie algebra crossed module is also pronilpotent.

To show that (2) implies (1), we start with a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ and we construct a pronilpotent cdgl $L = L_0 \oplus L_1$. We define $L_0 = \mathfrak{n}$ as a Lie algebra and $L_1 = \mathfrak{m}$ as a vector space. For $a \in L_1$ and $x \in L_0$, we set $[x,a] = \mathfrak{v}(x).a$ and $d = \mathfrak{u}$. We check easily that d is a derivation and $L = L_0 \oplus L_1$ is pronilpotent.

The associations (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) give the desired isomorphism of categories for the two first points of the statement.

To show that (2) implies (3), we start with a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ and we construct a Maltsev crossed module $d\colon M \to N$. We define M and N to be the vector spaces $\mathfrak{m}$ and $\mathfrak{n}$ respectively, with the group structure given by the Baker–Campbell–Hausdorff product, and set $d = \mathfrak{u}$. The action $\mathfrak{v}$ extends in an action by $e^{\mathfrak{v}}$: for $n \in N = \mathfrak{n}$ and $m \in M = \mathfrak{m}$, we set

$$ {}^{n}m = e^{\mathfrak{v}(n)}(m). $$

As $\mathfrak{v}$ is a morphism of Lie algebras, we have $\mathfrak{v}[n,n'] = [\mathfrak{v}(n), \mathfrak{v}(n')]$ for all $n, n' \in N$, and so the Baker–Campbell–Hausdorff formula implies $\mathfrak{v}(n * n') = \mathfrak{v}(n) * \mathfrak{v}(n')$ and

$$ {}^{(n*n')}m = e^{\mathfrak{v}(n*n')}(m) = e^{\mathfrak{v}(n)}(e^{\mathfrak{v}(n')}(m)). $$

Thus, we have a group action. The two additional properties of Maltsev crossed modules are easily deduced from the corresponding properties of Lie algebra crossed modules as well as the pronilpotency conditions.

To show that (3) implies (2), as we do for the cases (1) and (2), the previous process are reversed. We associate a pronilpotent Lie algebra to a Maltsev group, replacing the exponential by the functor $L \mapsto \log(1+L)$. The only significant point is the construction of the Lie algebra action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ from the group action $\nu\colon N \to \mathrm{Aut}(M)$; this is done by

$$ \mathfrak{v}(n).m = \log(1+\nu(n))(m). $$

We end with the study of the composition (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). We start with $L = L_0 \oplus L_1$ and in step (2) we define a bracket on L_1 by $[a,b]' = [da,b]$. Then, in the second implication, we endow L_1 with a group law coming from the Baker–Campbell–Hausdorff formula, $a * b = \log(e^a e^b)$. This formula can be written as

$$ \begin{aligned} a * b &= a + b + \tfrac{1}{2}[a,b]' + \tfrac{1}{12}[a,[a,b]']' - \tfrac{1}{12}b,[a,b]']' + \cdots \\ &= a + b + \tfrac{1}{2}[da,b] + \tfrac{1}{12}[da,[da,b]] - \tfrac{1}{12}db,[da,b]] + \cdots . \end{aligned} $$

This is exactly the expression of $a \perp b$ given in the formula (2-2). We recover the group law on L_1 in $\mathscr{C}(L)$. The rest of the verification is straightforward. □

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Knot Floer homology, link Floer homology and link detection

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We give new link detection results for knot and link Floer homology, inspired by recent work on Khovanov homology. We show that knot Floer homology detects $T(2,4)$, $T(2,6)$, $T(3,3)$, $L7n1$ and the link $T(2,2n)$ with the orientation of one component reversed. We show link Floer homology detects $T(2,2n)$ and $T(n,n)$, for all n. Additionally, we identify infinitely many pairs of links such that both links in the pair are each detected by link Floer homology but have the same Khovanov homology and knot Floer homology. Finally, we use some of our knot Floer detection results to give topological applications of annular Khovanov homology.

57K10, 57K18

1 Introduction

Knot and link Floer homology are invariants of links in S^3; see Ozsváth and Szabó [31; 32] and Rasmussen [34]. There are a number of formal similarities between these Floer theoretic invariants and the combinatorial Khovanov homology. Recently, Khovanov homology has been shown to detect a number of simple links; see Baldwin, Dowlin, Levine, Lidman and Sazdanovic [2], Li, Xie and Zhang [23], Martin [26] and Xie and Zhang [38; 40]. Some of these detection results have used knot and link Floer homology without going so far as to determine whether knot or link Floer homology detects the relevant link. Inspired by this work, we give such detection results for knot and link Floer homology. We remind the reader that the knot Floer homology of a link L is computed using an associated knot, called the knotification of L, in a connected sum of copies of $S^1 \times S^2$, while the link Floer homology of L is computed directly from the link L in S^3.

Previously it was known that knot Floer homology detects the unknot (see Ozsváth and Szabó [30]), the trefoil (see Ghiggini [7]), the figure eight knot [7], the Hopf link (see Ni [28] and [30]) and the unlink (see Hedden and Watson [15] and Ni [29]). Link Floer homology was known to detect the trivial n–braid together with its braid axis (see Baldwin and Grigsby [3]) and determine if a link is split; see Wang [37]. It was also known that a stronger version of link Floer homology, CFL^∞, detects the Borromean rings and the Whitehead link; see Gorsky, Lidman, B Liu and Moore [11].

We prove the following knot Floer homology detection results:

Theorem 4.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$, then L is isotopic to $T(2,4)$.*

Throughout, we take the links $T(2,2n)$ to be oriented as the closure of the 2–braids σ_1^{2n}.

Theorem 5.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,6))$, *then* L *is isotopic to* $T(2,6)$.

Let J_n be the link obtained from $T(2,2n)$ by reversing the orientation on one of the components. Then:

Theorem 3.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ *for some* n, *then* L *is isotopic to* J_n.

Theorem 7.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$, *then* L *is isotopic to* $T(3,3)$.

Theorem 8.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L7n1)$, *then* L *is isotopic to* $L7n1$.

We also prove the following link Floer homology detection results:

Theorem 3.2 *If* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,2n))$ *for some* n, *then* L *is isotopic to* $T(2,2n)$.

Theorem 6.1 *If* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$, *then* L *is isotopic to* $T(n,n)$.

Proposition 9.2 *Suppose link Floer homology detects a link* L, *and that if permuting some collection of Alexander gradings of* $\widehat{\mathrm{HFL}}(L)$ *induces an isomorphism on* $\widehat{\mathrm{HFL}}(L)$ *then there is a symmetry of* L *that exchanges the corresponding components. Then link Floer homology detects* $L \# H$ *for each choice of component of* L *to connect sum with.*

Throughout, we view the Hopf link as $T(2,2)$ and endow it with the associated orientation. A consequence of these detection results is that every link currently known to be detected by Khovanov homology is also detected by either knot or link Floer homology. This leads to the following natural question:

Question 1.1 Is there a link which Khovanov homology detects but which neither knot nor link Floer homology detects?

On the other hand, we show that there are infinitely many links detected by link Floer homology but which are detected by neither Khovanov homology nor knot Floer homology.

Theorem 9.4 *There exist infinitely many pairs of links* (L, L') *such that link Floer homology detects* L *and* L' *but* $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$ *and* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L')$.

Finally, we use some of our torus link detection results to derive applications to annular Khovanov homology. Annular Khovanov homology is an invariant of links in the thickened annulus $A \times I$, sometimes thought of as $S^3 \setminus U$ where U is an unknot or the annular axis. To do this, we utilize a generalization of the Ozsváth–Szabó spectral sequence, which relates annular Khovanov homology and knot Floer homology of the lift of the annular axis $\widetilde{U}$ in $\Sigma(L)$, the double branched cover of L; see Grigsby and Wehrli [14] and Roberts [35].

Theorem 10.4 *Let* $L \subseteq A \times I \subseteq S^3$ *be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, *then* L *is isotopic to* $\widehat{\sigma_1\sigma_2}$ *in* $A \times I$.

Theorem 10.6 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A \times I$.*

Theorem 10.7 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3}$ in $A \times I$.*

Theorem 10.8 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ in $A \times I$.*

Notice that $\widehat{\sigma_1\sigma_2}$, $\widehat{\sigma_1\sigma_2^{-1}}$, $\widehat{\sigma_1\sigma_2\sigma_3}$ and $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ all represent the unknot when considered in S^3, but these braid closures are all nontrivial knots in $A \times I$.

This paper is organized as follows. In Section 2 we briefly review knot and link Floer homology. In Section 3 we prove that knot Floer homology detects J_n, and link Floer homology detects $T(2,2n)$. In Section 4 we prove that knot Floer homology detects $T(2,4)$. In Section 5 we prove that knot Floer homology detects $T(2,6)$. In Section 6 we prove that link Floer homology detects $T(n,n)$. In Section 7 we prove that knot Floer homology detects $T(3,3)$. In Section 8 we prove that knot Floer homology detects $L7n1$. In Section 9 we prove that there are infinite families of links detected by link Floer homology that also have the same Khovanov homology and knot Floer homology. Finally, in Section 10 we prove the annular Khovanov homology results using some of our knot Floer detection results.

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2 Knot Floer homology and link Floer homology

Knot Floer homology and link Floer homology are invariants of links in S^3, defined using a version of Lagrangian Floer homology [31; 32; 34]. They are categorifications of the single variable and multivariable Alexander polynomials, respectively. Here we briefly highlight the key features of knot Floer homology and link Floer homology that we use to obtain our detection results. We work with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Let L be an oriented link in S^3, with components $L_1, L_2, \ldots, L_n$. The link Floer homology of L is a multigraded vector space

$$\widehat{\mathrm{HFL}}(L) = \bigoplus_{d, A_1, \ldots A_n} \widehat{\mathrm{HFL}}_d(L; A_1, \ldots A_n).$$

The grading denoted by "d" above is called the Maslov or algebraic grading, while the A_i gradings are called the Alexander gradings. Each A_i satisfies $2A_i + \ell\mathrm{k}(L_i, L - L_i) \in 2\mathbb{Z}$.

The knot Floer homology of L is a vector space bigraded by a Maslov grading and a single Alexander grading. The knot Floer homology of L can be obtained by projecting the link Floer homology onto the diagonal of the multi-Alexander gradings, which becomes the Alexander grading, and adding $\frac{1}{2}(n-1)$ to the Maslov grading.

We use a number of formal properties of knot and link Floer homology in proving our link detection results. The first of these is that link Floer homology has a symmetry relating the component of the homology supported in grading $(m, A_1, \dots, A_n)$ with the component of the homology supported in grading $\left(m - 2\sum_{i=1}^n A_i, -A_1, \dots, -A_n\right)$. Knot Floer homology enjoys the same symmetry property, since it can be defined by projecting the multi-Alexander gradings onto the diagonal. There is also a Künneth formula for computing the link Floer homology of a connected sum in terms of a tensor product of link Floer homologies.

The main formal property we will use, however, is that the link Floer homology of L admits spectral sequences to the link Floer homologies of its sublinks [3, Lemmas 2.4 and 2.5; 32]. In particular, when L_i is a sublink of L, there is a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L - L_i) \otimes V^{|L_i|}$ shifting each Alexander grading by $\frac{1}{2}\,\ell\mathrm{k}(L - L_i, L_i)$. It follows that there is also a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$, or equivalently that there is a spectral sequence from $\widehat{\mathrm{HFK}}(L)$ to $\widehat{\mathrm{HF}}\left(\#^{n-1}(S^1 \times S^2)\right)$. Here V is the multigraded vector space $\mathbb{F} \oplus \mathbb{F}$ with nonzero Maslov gradings 0 and -1 and multi-Alexander grading $(0, \dots, 0)$.

In addition to enjoying the above algebraic properties, $\widehat{\mathrm{HFK}}(L)$ and $\widehat{\mathrm{HFL}}(L)$ are known to reflect a number of topological properties of L. For starters, there are a number of things we can say about the number of components of L. Since $\widehat{\mathrm{HFL}}(L)$ admits a spectral sequence to V^{n-1}, a link is a knot if and only if $\mathrm{rank}(\widehat{\mathrm{HFK}}(L))$ is odd. Since the Maslov grading for $\widehat{\mathrm{HFL}}(L)$ is integer valued, while the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are $\mathbb{Z} + \frac{1}{2}(n-1)$ valued, it follows that if the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are contained in $\mathbb{Z} + \frac{1}{2}$ then L has an odd number of components, while if the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are contained in grading $\mathbb{Z}$ then L has an even number of components. Finally, since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\left(\#^{n-1}(S^1 \times S^2)\right)$ — which has a generator of Maslov grading $\frac{1}{2}(n-1)$ — we have that $n \le 1 + 2\max\{m : \mathrm{rank}(\widehat{\mathrm{HFK}}_m(L)) \ne 0\}$.

Moreover, since knot Floer homology categorifies the Alexander–Conway polynomial, and the Alexander–Conway polynomial detects the linking number of 2–component links by a result of Hoste [16], it follows that knot Floer homology detects the linking number of 2–component links.

We will also make use of the fact that the link Floer homology of L yields information about the topology of $S^3 - L$; in particular, that link Floer homology detects the Thurston norm of $S^3 - L$ [33]. Finally, if L is not a split link and has a component L_i which is fibered, then the top Alexander grading associated to L_i determines if $L - L_i$ is a braid in the complement of L_i. Specifically, this happens exactly when the rank in maximal nonzero Alexander grading is 2^{n-1} [26, Proposition 1].

3 Knot Floer homology detects J_n

Given that knot Floer homology detects the maximal Euler characteristic of oriented links, it is natural to try and detect links with large Euler characteristic. The unique 1–component link of maximal Euler characteristic is the unknot, which knot Floer homology is known to detect. Links with two components that bound annuli, ie 2–cable links, have high Euler characteristic. The simplest of these are 2–cables of unknots, ie the links $T(2,2n)$ with the orientation of one component reversed. We call these links J_n. In this section we show that knot Floer homology detects each J_n.

Theorem 3.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ for some n, then L is isotopic to J_n.*

For reference, we note that, when n is positive, $\widehat{\mathrm{HFK}}(J_n) \cong \mathbb{F}^n_{3/2}[1] \oplus \mathbb{F}^{2n}_{1/2}[0] \oplus \mathbb{F}^n_{-1/2}[-1]$, where the subscript denotes the Maslov grading of the generator, and $[i]$ denotes the Alexander grading of a summand. When n is negative, $\widehat{\mathrm{HFK}}(J_n)$ can be computed from the above formula using an understanding of how knot Floer homology is affected by mirroring.

A consequence of this detection result is that link Floer homology detects the links $T(2,2n)$. This follows from Theorem 3.1 by considering how link Floer homology changes under reversing the orientation of a single component.

Theorem 3.2 *If $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,2n))$ for some n, then L is isotopic to $T(2,2n)$.*

To prove Theorem 3.1, we will first prove that L is a 2–component link and that both of the components of L are unknots. Then we will use the fact that knot Floer homology detects genus to show that J_n is detected among 2–component links with unknotted components. The topological argument used here was communicated to the authors by Eugene Gorsky (2020), and also appears in Liu's classification of the links $T(2,2n)$ in terms of surgery to a Heegaard Floer L–space [25].

Lemma 3.3 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ for some n, then L is a 2–component link, and both of the components are unknots.*

Proof First we show that L is a 2–component link. Notice that the parity of the rank of $\widehat{\mathrm{HFK}}(L)$ rules out the case that L is a knot. If L is an n–component link, then there is a spectral sequence from $\widehat{\mathrm{HFK}}(L)$ to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1\times S^2)\big)$. Because $\widehat{\mathrm{HFK}}(L)$ is only nonzero in Maslov gradings $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$, this spectral sequence can only exist for $n=2$.

To see that both components of L are unknotted, we consider the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFK}}(K)\otimes V$ where K is a component of L. From this spectral sequence we see that $\widehat{\mathrm{HFK}}(K)$ is 0 in all Maslov gradings, except possibly 0 and 1. Considering how the Maslov grading changes under the symmetry of the Alexander grading for knot Floer homology, we can see that $\widehat{\mathrm{HFK}}(K)$ can only be supported in Alexander grading 0, so K is an unknot. □

With Lemma 3.3 we can now prove Theorem 3.1. The key step is to deduce that J_n is a cable of the unknot.

Proof of Theorem 3.1 Suppose $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,2n))$. By Lemma 3.3, L is a 2–component link. If $n = 0$ then the maximal Alexander grading of $\widehat{\mathrm{HFK}}(L)$ is 0, whence L bounds two disjoint disks and L is the 2–component unlink, as desired.

We now consider the $n \neq 0$ case. Here the maximal Alexander grading of $\widehat{\mathrm{HFK}}(L)$ is 1, and we see that the two components of L bound a surface of Euler characteristic 0. Note that the linking number of L, which knot Floer homology detects, is nonzero, so L cannot bound the disjoint union of a disk and a punctured torus. Thus L bounds an annulus, ie it is the twisted 2–cable of some knot. Each component of L is isotopic to the knot that was cabled, and so L is a twisted 2–cable of the unknot by Lemma 3.3. This means that L is J_m for some m. Finally, a simple computation of the respective ranks in each Maslov grading shows that $\widehat{\mathrm{HFK}}(J_m) \cong \widehat{\mathrm{HFK}}(J_n)$ if and only if $m = n$. Thus L is isotopic to J_n. □

4 Knot Floer homology detects $T(2,4)$

Here we will utilize the results of the previous section to obtain a detection result for the torus link $T(2,4)$. The link Floer homology of $T(2,4)$ is shown in Table 1, for reference.

Theorem 4.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$, then L is isotopic to $T(2,4)$.*

To prove this, we show the following lemma:

Lemma 4.2 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$ then L consists of two components, L_1 and L_2, such that each $\widehat{\mathrm{HFK}}(L_i)$ has a unique Maslov index 0 generator. Moreover, that generator is supported in Alexander grading 0 in $\widehat{\mathrm{HFK}}(L_i)$.*

We then show that L has the same link Floer homology as $T(2,4)$, using structural properties of link Floer homology, and apply Theorem 3.2 to complete the proof.

The following lemma will be useful in proving Lemma 4.2:

Lemma 4.3 *Suppose K is a component of a link L such that $\widehat{\mathrm{HFL}}(L)$ is supported in Maslov gradings at most 0 with a unique Maslov grading 0 generator. Then there is a unique Maslov index grading 0 generator in $\widehat{\mathrm{HFK}}(K)$, and it is of nonnegative Alexander grading.*

1		$\mathbb{F}_{-1}$	$\mathbb{F}_0$
0	$\mathbb{F}_{-3}$	$\mathbb{F}^2_{-2}$	$\mathbb{F}_{-1}$
−1	$\mathbb{F}_{-4}$	$\mathbb{F}_{-3}$	
	−1	0	1

Table 1: The link Floer homology of $T(2,4)$. The coordinates give the multi-Alexander grading; the subscript gives the Maslov grading.

Proof The Maslov grading 0 generator must persist under the spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$, as else it cannot persist to $\widehat{\mathrm{HF}}(S^3) \otimes V^{|L|-1}$. If this generator sat in a negative Alexander grading then the symmetry properties of knot Floer homology would imply that there is a positive Maslov index generator in $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$. However there are no positive Maslov index generators in $\widehat{\mathrm{HFL}}(L)$, and so there are none in $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$. □

Proof of Lemma 4.2 Suppose L is an n–component link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$. Then $n \leq 2$ since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$. Indeed, since $\mathrm{rank}(\widehat{\mathrm{HFK}}(L))$ is odd for knots, $n = 2$. Since knot Floer homology detects the linking number of 2–component links, it follows that $\ell k(L_1, L_2) = 2$.

There is only one generator in Maslov grading 0 and it must survive in the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_i) \otimes V$. We call this generator θ_0. The bi-Alexander grading for θ_0 is then $\big(A_1 + \frac{1}{2}l, A_2 + \frac{1}{2}l\big)$, where A_i is the Alexander grading of the generator in Maslov grading 0 in $\widehat{\mathrm{HFK}}(L_i)$ and l is the linking number between the components. Since $A_1 + \frac{1}{2}l + A_2 + \frac{1}{2}l = 2$ and $l = 2$, it follows that $A_1 + A_2 = 0$. By Lemma 4.3, $A_1 = A_2 = 0$, as desired. □

To complete our proof of Theorem 4.1 we show that if L has the same knot Floer homology as $T(2,4)$ then L also has the same link Floer homology as $T(2,4)$. This result, combined with Theorem 3.2, proves Theorem 4.1.

Proof of Theorem 4.1 Suppose L is a link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$. We seek to understand $\widehat{\mathrm{HFL}}(L)$. From the argument in the proof of Lemma 4.2, the only Maslov grading 0 generator of $\widehat{\mathrm{HFL}}(L)$ sits in bi-Alexander grading $(1,1)$.

Since there are spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFK}}(K_i) \otimes V$ for each i, there are also generators of $\widehat{\mathrm{HFL}}(L)$ in (A_1, A_2) gradings $(1,0)$ and $(0,1)$. The symmetry of $\widehat{\mathrm{HFL}}(L)$ gives generators at $(-1,-1)$, $(-1,0)$ and $(0,-1)$ as well. With these 6 generators determined, there are now 2 more generators to add so that the link Floer homology has rank 8. To maintain an even rank in each A_i grading, they both must be added at the same bigrading. The only way to do this and maintain symmetry is to add them at $(0,0)$, so that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,4))$, and Theorem 3.2 shows L is isotopic to $T(2,4)$. □

5 Knot Floer homology detects $T(2,6)$

In the previous section we showed that knot Floer homology detects the torus link $T(2,4)$. The torus link $T(2,6)$ is then a natural candidate for detection results. In this section we show that knot Floer homology indeed detects $T(2,6)$.

Theorem 5.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,6))$, then L is isotopic to $T(2,6)$.*

Note that the maximal Maslov grading of $\widehat{\mathrm{HFK}}(L)$ is $\frac{1}{2}$, while $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$ has a Maslov grading $\frac{1}{2}(n-1)$ generator. Since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$, where n is the number of components of L, L has at most 2 components. Indeed, as $\operatorname{rank}(\widehat{\mathrm{HFK}}(L))$ is even, L has exactly two components. Since knot Floer homology detects the linking number of 2–component links, the linking number is 3.

From here, the proof of Theorem 5.1 amounts to an algebraic argument showing that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFK}}(L)$, and applying Theorem 3.2.

For reference, after renormalizing the Maslov gradings to agree with the link Floer homology, the knot Floer homology of $T(2,6)$ is: rank one in (M, A) gradings $(0,3)$ and $(-6,-3)$; rank two in (M, A) gradings $(-1,2)$, $(-2,1)$, $(-3,0)$, $(-4,-1)$ and $(-5,-2)$; and rank 0 in all other bigradings.

Proof of Theorem 5.1 Suppose that L has the same knot Floer homology as $T(2,6)$. As in the proof that knot Floer homology detects $T(2,4)$, we have that $A_1 + A_2 + \ell\mathrm{k}(L_1, L_2) = 3$, where each A_i is the Alexander grading of the Maslov index 0 generator in $\widehat{\mathrm{HFK}}(L_i)$. Thus $A_1 + A_2 = 0$, and Lemma 4.3 implies that $A_1 = A_2 = 0$.

We now show that L has the same link Floer homology as $T(2,6)$, so it follows from Theorem 3.2 that L is isotopic to $T(2,6)$.

Since the linking number is 3 and the Maslov index 0 generator sits in Alexander grading 0 in the knot Floer homology of each component, it follows that, in $\widehat{\mathrm{HFL}}(L)$, the Maslov index 0 generator sits in Alexander bigrading $\left(\frac{3}{2}, \frac{3}{2}\right)$. The Maslov index -1 generators in $\widehat{\mathrm{HFK}}(L)$ must be in bi-Alexander gradings $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}\right)$. There is also a Maslov index -2 generator in Alexander grading $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consider the remaining Maslov index -2 generator. Suppose it sits in Alexander grading $(y, 1-y)$. Observe that there must be Maslov index -3 generators sitting in Alexander gradings $(y-1, 1-y)$ and $(y, -y)$. The symmetry property of $\widehat{\mathrm{HFK}}(L)$ then implies that $y = \frac{1}{2}$. The symmetry properties of $\widehat{\mathrm{HFL}}$ then show that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,6))$, and so by Theorem 3.2 L is isotopic to $T(2,6)$, as desired. □

6 Link Floer homology detects $T(n,n)$

In the previous section we showed that link Floer homology detects the $T(2,2n)$ torus links, motivated by detection results for $T(2,2)$, $T(2,4)$ and $T(2,6)$. The torus link $T(2,2)$ can also be viewed as one of the simplest links in the family of $T(n,n)$ torus links. In this section we show that link Floer homology detects the links $T(n,n)$. We use a characterization of $T(n+1,n+1)$ as an n–braid for $T(n,n)$ union the braid axis.

J Licata gave a computation of $\widehat{\mathrm{HFL}}(T(n,n))$ without the Maslov gradings of certain generators [24]. The computation of the Maslov gradings of these generators was subsequently completed by Gorsky and Hom [10]. We prove that link Floer homology detects $T(n,n)$ using only certain structural properties of the link Floer homology. It follows from this that there are many graded vector spaces that do not arise as the link Floer homology of any link. In particular, we will be interested in multigraded vector spaces B_n exhibiting the following four properties:

(1) There is a unique Maslov grading 0 generator.

(2) The multi-Alexander grading of the Maslov grading 0 generator is $\left(\frac{1}{2}(n-1), \frac{1}{2}(n-1), \ldots, \frac{1}{2}(n-1)\right)$.

(3) B_n has support contained only in multi-Alexander gradings $(A_1, A_2, \ldots, A_n)$ satisfying $A_i \leq \frac{1}{2}(n-1)$ for all i.

(4) B_n has rank 2^{n-1} in A_i grading $\frac{1}{2}(n-1)$.

Note that $\widehat{\mathrm{HFL}}(T(n,n))$ satisfies these properties. Observe that if L is any link whose link Floer homology satisfies all of the above conditions, then L is not a split link, so each component L_i of L is a braid axis for $L - L_i$.

Theorem 6.1 *If $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$, then L is isotopic to $T(n,n)$.*

The main ingredient of this proof is a result stating that, under certain circumstances, if the link Floer homology of a link has certain algebraic properties then the linking numbers of certain components with the rest of the link are positive.

Lemma 6.2 *Let L be a link with components L_i for $1 \leq i \leq n$. Suppose that $\widehat{\mathrm{HFL}}(L)$ has a unique generator of Maslov index 0 with A_i grading $x \geq 0$. Suppose $\widehat{\mathrm{HFL}}(L)$ is supported in A_i gradings at most x. Then $\ell\mathrm{k}(L_i, L_j) \geq 0$ for all j.*

Proof Let θ_0 denote the unique Maslov index 0 generator. The vector space $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$ is nonzero in Maslov grading 0, so all other intermediate vector spaces with spectral sequences fitting between $\widehat{\mathrm{HFL}}(L)$ and $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$ must also be nonzero in this Maslov grading. Because θ_0 is the only generator in this Maslov grading, it must survive in every such spectral sequence.

Consider the spectral sequence to $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ obtained by forgetting the component L_j. The A_i grading on $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will be shifted by $\frac{1}{2}\,\ell\mathrm{k}(L_i, L_j)$ for $i \neq j$. We will show that this shift must be nonnegative.

Because θ_0 survives this spectral sequence, $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will have top A_i grading $\frac{1}{2}(n-1)$. Considering the A_i grading on $\widehat{\mathrm{HFL}}(L)$, we see that $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will have bottom A_i grading no smaller than $\frac{1}{2}(-n+1)$. Since the A_i grading on $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ must be symmetric about a nonnegative number, the shift applied to the Alexander grading must be nonnegative, so $\ell\mathrm{k}(L_i, L_j) \geq 0$. $\square$

With this result on the nonnegativity of linking numbers, we can proceed with the proof of Theorem 6.1. We will proceed by induction, using the characterization of $T(n+1, n+1)$ as the link consisting of the unique n–braid for $T(n,n)$ together with the braid axis.

Proof of Theorem 6.1 Suppose that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$. Lemma 6.2 tells us that $\ell\mathrm{k}(L_i, L_j) \geq 0$ for every distinct i and j. Moreover, because L is not split and each component L_i of L is a braid axis for $L - L_i$, we have $\ell\mathrm{k}(L_i, L_j) \neq 0$.

The top nonzero A_i grading is $\frac{1}{2}(n-1)$. The relationship between the top nonzero A_i grading and the Seifert genus of L_i implies that

$$\tfrac{1}{2}(n-1) \geq g(L_i) + \sum_{j \neq i} \tfrac{1}{2}\,\ell\mathrm{k}(L_i, L_j)2.$$

However, because $\ell\mathrm{k}(L_i, L_j) > 0$,

$$g(L_i) + \sum_{j \neq i} \tfrac{1}{2}\,\ell\mathrm{k}(L_i, L_j)2 \geq g(L_i) + \tfrac{1}{2}n - 1,$$

with equality when $\ell\mathrm{k}(L_i, L_j) = 1$ for all j. Combining these inequalities gives that $g(L_i) = 0$, and $\ell\mathrm{k}(L_i, L_j) = 1$ for all i, j.

We now know that L is an n–component link where each component is an unknot, each component is a braid axis for the rest of the link, and the linking number between any two components is 1. The torus link $T(n,n)$ is the only n–component link satisfying all of these conditions. This can be verified by induction on n. Specifically, check explicitly that $T(2,2)$ is the only such 2–component link, then view L as a braid axis of some n–braid representing an n–component link satisfying the same properties. □

7 Knot Floer homology detects $T(3,3)$

In previous sections we showed that, for some of the first members of the family of $T(2,2n)$ torus links, the link Floer homology detection results can be strengthened to knot Floer homology detection results. In this section we do the same for $T(3,3)$, the third member of the $T(n,n)$ family. The knot Floer homology of $T(3,3)$ is given as follows:

$$\widehat{\mathrm{HFK}}(T(3,3), i) = \begin{cases} \mathbb{F}_1 & \text{for } i = 3, \\ \mathbb{F}_0^3 & \text{for } i = 2, \\ \mathbb{F}_{-1}^3 & \text{for } i = 1, \\ \mathbb{F}_{-1} \oplus \mathbb{F}_{-2}^3 & \text{for } i = 0, \\ \mathbb{F}_{-3}^3 & \text{for } i = -1, \\ \mathbb{F}_{-4}^3 & \text{for } i = -2, \\ \mathbb{F}_{-5} & \text{for } i = -3, \\ 0 & \text{otherwise.} \end{cases}$$

See [24] for a discussion of this result.

Theorem 7.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$, *then* L *is isotopic to* $T(3,3)$.

To prove this, we will use various spectral sequence arguments to show that L has the same link Floer homology as $T(3,3)$. The above theorem then follows immediately from Theorem 6.1.

Proof Suppose L is an n–component link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$.

We first argue that $n = 3$. Note that the maximal Maslov grading of a generator of $\widehat{\mathrm{HFK}}(L)$ is 1. Thus $n \leq 3$, as else $\widehat{\mathrm{HFK}}(L)$ would not admit a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$. Also $n \neq 2$, since the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are supported in integer gradings. Moreover L cannot be a knot, since $\mathrm{rank}(\widehat{\mathrm{HFK}})(L)$ is even. Thus $n = 3$.

Let L_1, L_2 and L_3 be the components of L. We now seek to determine the structure of $\widehat{\mathrm{HFL}}(L)$.

The symmetry of $\widehat{\mathrm{HFL}}(L)$ implies that the unique generator in Maslov grading -2 and Alexander grading 0 sits in multi-Alexander grading $(0,0,0)$. Similarly the symmetry implies that at least one of the Maslov grading -3 generators also sits at multigrading $(0,0,0)$.

Since the Maslov grading 0 generator in knot Floer homology is of Alexander grading 3, the Maslov grading 0 generator in link Floer homology sits in Alexander multigrading $(x, y, 3-x-y)$ for some pair of integers (x, y). In order that the link Floer homology admits the requisite spectral sequences, there are Maslov grading -1 generators in multi-Alexander gradings $(x, y, 2-x-y)$, $(x-1, y, 3-x-y)$ and $(x, y-1, 3-x-y)$.

Now, observe that each Maslov grading -1 generator has at least one distinct Alexander grading from the unique Maslov grading 0 generator. In order to admit the requisite spectral sequences, there must be Maslov grading -2 generators with $(A_1, A_2) = (x, y-1), (x-1, y)$, $(A_1, A_3) = (x-1, 3-x-y), (x, 2-x-y)$ and $(A_2, A_3) = (y-1, 3-x-y), (y, 2-x-y)$. A direct computation shows that at most one of these corresponds to the generator in multigrading $(0,0,0)$. Thus there are Maslov index -2 generators in Alexander gradings $(x-1, y-1, 3-x-y)$, $(x, y-1, 2-x-y)$ and $(x-1, y, 2-x-y)$.

By a similar argument, we can see that there is a Maslov index -3 generator in multi-Alexander grading $(x-1, y-1, 2-x-y)$. If $(x, y) \neq (1,1)$ this determines the entire link Floer homology of L. If $x = y = 1$ then the remaining Maslov index -3 generators must be of multi-Alexander grading $(0,0,0)$ to ensure that each (A_i, A_j) grading is of even rank, so again the link Floer homology of L is determined.

Consider the Maslov index -3 generator in multi-Alexander grading $(0,0,0)$, θ_{-3}. Since θ_{-3} does not persist under the spectral sequence to $\widehat{\mathrm{HFL}}(L_i) \otimes V^{\otimes 2}$ for any i, there must be a Maslov index -2 generator in nonzero Alexander grading with each Alexander grading at least 0, or a Maslov index -4 generator with each Alexander grading at most 0. Observe that these two conditions are equivalent by the symmetry of the link Floer homology of L. Thus $x-1, y-1, 3-x-y \geq 0$, $x, y-1, 2-x-y \geq 0$, or $x-1, y-1, 2-x-y \geq 0$. By permuting the components, we may take $x-1, y-1, 3-x-y \geq 0$. There are only three solutions: $(x, y) = (1,1)$, $(x, y) = (2,1)$ or $(x, y) = (1,2)$. If $(x, y) = (1,1)$, then $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(3,3))$.

We complete the proof by excluding the cases $(x, y) = (2,1)$ and $(x, y) = (1,2)$. After permuting components, we may take $x = 1$ and $y = 2$ without loss of generality.

Since $\mathrm{rank}(\widehat{\mathrm{HFK}}(L)) = 18$, we have $\mathrm{rank}(\widehat{\mathrm{HFK}}(L_i)) \leq \frac{9}{2}$. It follows that each component is an unknot or a trefoil. Observe that if a component is a trefoil then it must be $T(2,3)$, as there are no positive Maslov index generators in $\widehat{\mathrm{HFL}}(L)$. Indeed there can be no $T(2,3)$ component, as this would require there to be an Alexander grading 2 less than an Alexander grading of the Maslov index 0 generator containing a summand $\mathbb{F}_{-2} \oplus \mathbb{F}_{-3}^2 \oplus \mathbb{F}_{-4}$, which does not occur. Thus each component is an unknot. From here we can compute the linking numbers from the Alexander gradings of the Maslov grading 0 generator.

We find $\ell k(L_1, L_3) = -1 = -\ell k(L_2, L_3)$, and $\ell k(L_1, L_2) = 3$. Since L is not a split link, $L - L_3$ is a 2–braid in the complement of L_3. Each of L_1 and L_2 are unknots and $\ell k(L_1, L_2) = 3$, so $L - L_3$ is $T(2,6)$ as an unoriented link. However, $\operatorname{rank}(\widehat{\mathrm{HFK}}(T(2,6) \otimes V)) = 24$, so $T(2,6)$ cannot be a sublink of L and $(x, y) \neq (1, 2)$. □

8 Knot Floer homology detects $L7n1$

We have now shown that knot Floer homology detects a number of the low crossing number links that Khovanov homology is known to detect. In this section we continue this task, showing that knot Floer homology detects the link $L7n1$.

Theorem 8.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L7n1)$, then L is isotopic to $L7n1$.*

Our proof relies on the observation that $L7n1$ can be realized as a 2–braid representing $T(2,3)$ together with the braid axis.

Note that $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$ — where n is the number of components of L — and that the knot Floer homology of a knot is of odd rank. It follows that L has two components. Since knot Floer homology detects the linking number of 2–component knots, it follows that the linking number of L is two. From here we break up the proof of Theorem 8.1 into the following lemmas:

Lemma 8.2 *Suppose L is a 2–component link such that $\widehat{\mathrm{HFK}}(L7n1) \cong \widehat{\mathrm{HFK}}(L)$. Then $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L7n1)$.*

Lemma 8.3 *Suppose L satisfies $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L7n1)$. Then L is isotopic to $L7n1$.*

The combination of these lemmas immediately gives the proof of Theorem 8.1.

L7n1 has homology as computed in [32] and shown in Table 2.

Lemma 8.2 is proven by combining the symmetry and parity properties of link Floer homology.

Proof of Lemma 8.2 Since L has 2 components, $\widehat{\mathrm{HFL}}(L)$ has exactly 2 Alexander gradings.

2		$\mathbb{F}_{-1}$	$\mathbb{F}_0$
1		$\mathbb{F}_{-2}$	$\mathbb{F}_{-1}$
0		$\mathbb{F}_{-2} \oplus \mathbb{F}_{-3}$	
-1	$\mathbb{F}_{-5}$	$\mathbb{F}_{-4}$	
-2	$\mathbb{F}_{-6}$	$\mathbb{F}_{-5}$	
	-1	0	1

Table 2: The link Floer homology of L7n1.

Let θ_0 be the Maslov grading 0 generator. This generator θ_0 has bi-Alexander grading $\left(\frac{3}{2}+x, \frac{3}{2}-x\right)$ for some x. Indeed, there must be generators sitting in gradings $\left(\frac{1}{2}+x, \frac{3}{2}-x\right)$ and $\left(\frac{3}{2}+x, \frac{1}{2}-x\right)$, each of Maslov index -1. Together with the symmetry properties of link Floer homology, this determines the Alexander bigradings of 6 generators. The same symmetry properties also imply that the 2 generators in Alexander grading 0 must have bi-Alexander grading $(0,0)$. Thus, up to choice of x, we need only specify the location of 1 more generator to determine the link Floer homology of L. Since each Alexander grading needs to be of even rank, the remaining Maslov grading -2 element must be in bi-Alexander grading $\left(\frac{1}{2}+x, \frac{1}{2}-x\right)$. Moreover, since the Maslov grading -3 component, θ_{-3}, cannot persist in the spectral sequence to $\widehat{\mathrm{HF}}(S^3)\otimes V$, it follows that $x \in \left\{\frac{1}{2}, 0, -\frac{1}{2}\right\}$, for the Alexander gradings obstruct the existence of generators y with $\langle \partial y, \theta_{-3}\rangle \neq 0$ and $\langle \partial\theta_{-3}, y\rangle \neq 0$ unless x is in this range. Indeed, $x \neq 0$, since otherwise we would have an element with Alexander grading in $\mathbb{Z}$, and another with Alexander grading in $\mathbb{Z}+\frac{1}{2}$. If $x=\frac{1}{2}$ then we have the link Floer homology of $L7n1$, while if $x=-\frac{1}{2}$ we can switch the components and thereby obtain the link Floer homology of $L7n1$. □

We complete the proof of Theorem 8.1 by showing that link Floer homology detects the link $L7n1$. We use the fact that $L7n1$ is the closure of a braid for $T(2,3)$ together with its braid axis.

Proof of Lemma 8.3 Suppose a link L, with components L_1 and L_2, satisfies $\widehat{\mathrm{HFL}}(L)\cong\widehat{\mathrm{HFL}}(L7n1)$. The rank in each of the maximal A_i gradings is 2, so L is either split or the L is exchangeably braided. L cannot be split, as if it were, then at least 1 component of L would be an unknot, for reasons of rank. But if this were the case, then $\widehat{\mathrm{HFL}}(L)$ would be supported in A_i grading 0 for at least one i, which is not the case. Thus each component is a braid axis for the other, and in particular $\ell k(L_1, L_2)\neq 0$.

Observe that $\operatorname{rank}(\widehat{\mathrm{HFK}}(L_i)) \leq 5$, with equality if and only if the spectral sequence corresponding to L_i collapses on the E_1 page. If the spectral sequence collapses on the E_1 page, then $\widehat{\mathrm{HFK}}(L_i)$ would have no shift applied to its Alexander grading as it is already symmetric around grading 0. Therefore, if $\operatorname{rank}(\widehat{\mathrm{HFK}}(L_i))=5$, then $\ell k(L_1, L_2)=0$, a contradiction.

Thus $\operatorname{rank}(\widehat{\mathrm{HFK}}(L_i))<5$, and the link has components that are either unknots or trefoils. Observe that any trefoil component must be $T(2,3)$, as there are no generators of positive Maslov grading.

Suppose L_1 and L_2 are both unknots. The shifts in Alexander grading coming from the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_i)\otimes V$ would imply that $\ell k(L_1, L_2)=4$ and $\ell k(L_2, L_1)=2$, a contradiction.

Observe that L_1 cannot be a trefoil, for there is no Maslov grading -2 generator in an A_1 grading 2 less than the A_1 grading of the unique grading 0 element. Thus L_2 is $T(2,3)$ and $\ell k(L_2, L_1)=2$. Since L_2 is a 2–braid closure in the complement of L_1 and L_2 is $T(2,3)$, the link L must be L7n1. □

9 Connected sums with a Hopf link

In this section we deduce some properties of link Floer homology under the operation of taking a connected sum with a Hopf link. We then explore some applications of these properties to the question of link detection. Our main application is to find two infinite families of links which are not detected by Khovanov homology or knot Floer homology, but which are detected by link Floer homology. Throughout this section we let H denote the Hopf link.

Proposition 9.1 *A link L can be expressed as $L' \# H$ if and only if there is an Alexander grading in $\widehat{\mathrm{HFL}}(L)$ where the span of its nonzero grading levels is $\{\frac{-1}{2}, \frac{1}{2}\}$.*

Proof This observation follows directly from the connection between link Floer homology and the Thurston norm. A component has a span of its nonzero grading levels $\{\frac{-1}{2}, \frac{1}{2}\}$ if and only if that component bounds a disk which intersects the rest of the link in a single point. This is equivalent to expressing L as $L' \# H$. □

This observation has the following immediate consequence:

Proposition 9.2 *Suppose link Floer homology detects a link L, and that if permuting some collection of Alexander gradings of $\widehat{\mathrm{HFL}}(L)$ induces an isomorphism on $\widehat{\mathrm{HFL}}(L)$, then there is a symmetry of L that exchanges the corresponding components. Then link Floer homology detects $L \# H$ for each choice of component of L to connect sum with.*

Proof Suppose L' is a link such that $\widehat{\mathrm{HFL}}(L') \cong \widehat{\mathrm{HFL}}(L \# H)$. Consider the span of the Alexander grading associated to the new unknotted component. Proposition 9.1 implies that $L' = L'' \# H$ for some link L'' and some choice of component of L'' to connect sum onto. It follows from the Künneth formula that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L'')$, whence L'' is L by assumption. Indeed, we have also assumed that if permuting some collection of Alexander gradings of $\widehat{\mathrm{HFL}}(L)$ induces an isomorphism of $\widehat{\mathrm{HFL}}(L)$, then there is a symmetry of L that exchanges the corresponding components. Thus if different choices of component on which to connect sum H give the same links with the same link Floer homology, the resulting links are isotopic. It follows that link Floer homology detects $L \# H$ irrespective of which component of L is used for the connected sum. □

Remark 9.3 While Proposition 9.1 allows one to determine if a link L can be decomposed as a smaller link L' connect sum with a Hopf link, there is in general an issue with determining where the connected sum is occurring. An instructive example is the case that L' is the disjoint union of two different knots with the same knot Floer homology. In this case the two choices of where to connect sum with a Hopf link produce topologically distinct links which have the same link Floer homology. While in this example link Floer homology does not detect L', we cannot rule out the possibility that something similar could occur for links detected by link Floer homology.

Combining Proposition 9.2 and previous knot Floer detection results immediately gives some new detection results for link Floer homology.

We now provide two infinite families of links which are detected by link Floer homology but are not detected by Khovanov homology or knot Floer homology.

Theorem 9.4 *There exist infinitely many pairs of links* (L, L') *such that link Floer homology detects* L *and* L' *but* $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$ *and* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L')$.

To prove Theorem 9.4 we introduce two families of links and show that every link in either of these families is detected by link Floer homology. This is the content of Theorems 9.5 and 9.7. We then highlight explicit examples within these families that neither Khovanov homology nor knot Floer homology can distinguish.

Both families of links are trees of unknots. The first family consists of links L_n given by the tree of unknots corresponding to the graph with $n-1$ vertices, each connected to a fixed vertex. For the second family, let $L_{(a,b)}$ be the tree of unknots corresponding to the graph with $a+b+2$ vertices $\{x_1, x_2, \dots x_a, x, y_1, y_2, \dots y_b, y\}$. Each x_i has a unique edge, connecting it to x. Each y_i has a unique edge, connecting it to y. Finally there is a unique edge connecting x and y.

Theorem 9.5 *For each* $n \geq 2$, *if* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L_n)$, *then* L *is isotopic to the link* L_n.

Remark 9.6 The links L_n can be viewed as the trivial $(n-1)$–braid together with its braid axis. So Theorem 9.5 was already known, because link Floer homology detects braid closures [26, Proposition 1] and detects the trivial braid amongst braid closures [3, Theorem 3.1]. However, we provide a different proof of Theorem 9.5 because it is a simpler case of the ideas used in the proof of Theorem 9.7.

Proof of Theorem 9.5 Suppose L has the same link Floer homology as L_n. First notice that L cannot be a split link because its Alexander polynomial is nonzero. By the observation that L is not split, we see that each of these $n-1$ components must bound a disk which only intersects the final component of L. Then L must be built from a knot K by connect summing K with $n-1$ Hopf links. It follows from the Künneth formula and the fact that knot Floer homology detects the unknot that L is isotopic to the link L_n. □

Theorem 9.7 *For every pair* (a, b) *with* a *and* b *positive, if* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L_{(a,b)})$, *then* L *is isotopic to the link* $L_{(a,b)}$.

Proof First notice that link Floer homology detects the link $L_{(0,b)} = L_{b+1}$. We will now proceed by induction on a.

Suppose that L has the same link Floer homology as $L_{(a,b)}$. First notice that L cannot be a split link because its Alexander polynomial is nonzero. By the observation that L is not split, we see that each of these $a+b$ components must bound a disk which only intersects one of the final two components of L. Call these final components X and Y, based on if their Alexander gradings agree with the Alexander gradings associated to the component in the tree of unknots for the vertex x or y, respectively. Without loss of generality, at least one component bounds a disk that intersects X in a single point. Then L can be written as $L' \# H$, where the connect sum is taken along the component X. A quick computation shows that $\widehat{\mathrm{HFL}}(L') \cong \widehat{\mathrm{HFL}}(L_{(a-1,b)})$. By induction, L' is isotopic to $L_{(a-1,b)}$, whence L is isotopic to $L_{(a,b)}$. □

With these detection results in place, we are now ready to prove Theorem 9.4.

Proof of Theorem 9.4 Consider the links L_n and $L_{(a,b)}$ with $a+b+1=n$. These links are detected by link Floer homology. We now check that $\mathrm{Kh}(L_n) \cong \mathrm{Kh}(L_{(a,b)})$ and $\widehat{\mathrm{HFK}}(L_n) \cong \widehat{\mathrm{HFK}}(L_{(a,b)})$.

Both links can be constructed by starting with an unknot and connect summing a Hopf link n times in total. A simple computation shows that Khovanov homology and knot Floer homology of $L \# H$ do not depend on which component of L the Hopf link is connect summed onto. This shows $\mathrm{Kh}(L_n) \cong \mathrm{Kh}(L_{(a,b)})$ and $\widehat{\mathrm{HFK}}(L_n) \cong \widehat{\mathrm{HFK}}(L_{(a,b)})$. □

10 Applications to annular Khovanov homology

Annular Khovanov homology was defined by Asaeda, Przytycki and Sikora [1] as a categorification of the Kauffman bracket skein module of the thickened annulus. The resulting theory is an invariant of links in the thickened annulus $A \times I$, or alternatively the complement of an unknot in the 3–sphere $S^3 \setminus U$. In particular, annular Khovanov homology is well suited to studying braid closures [3; 13; 17; 18].

In this section we apply some of our earlier knot Floer detection results to show that annular Khovanov homology detects certain braid closures. The proofs will rely on the spectral sequence from annular Khovanov homology of a link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ [13; 35].

Let $\beta_n := \sigma_1\sigma_2\ldots\sigma_{n-1}$. We use knot Floer detection results for $T(2,3)$, $T(2,4)$ and $T(2,6)$ to show that annular Khovanov homology detects the closure of the braids β_3, β_4 and β_6. The structure of each proof is similar. First we use properties of annular Khovanov homology to deduce necessary topological properties of the annular knot, like braidedness or unknottedness. Then we use a knot Floer detection result to show that the lift of the annular axis is $T(2,3)$, $T(2,4)$ or $T(2,6)$, respectively. Finally, we translate this into information about the annular link. In this section we use the following result for mapping class groups:

Proposition 10.1 *Suppose γ is an n–braid and β is a periodic n–braid. If $BH(\gamma)$ and $BH(\beta)$ are conjugate then so too are γ and β.*

Remark 10.2 An alternative proof of this proposition was originally communicated to the authors by Marissa Loving and Dan Margalit (2020).

Proof Let β and γ be as in the statement of the proposition. Note that both conjugation and the Birman–Hilden correspondence preserve the Nielsen–Thurston classification, so we know that γ is periodic as well. That is, a power of γ is some power of the full twist Δ^2. Thus there are numbers N and M such that $\beta^N = \gamma^M$.

Now we consider the fractional Dehn twist coefficients of β and γ. We know that $\mathrm{FDTC}(\beta) = k/m$ for some fixed k and m. The Birman–Hilden correspondence either preserves the fractional Dehn twist coefficient of n–braids or halves it, depending on the parity of n. The fractional Dehn twist coefficient is preserved under conjugation by a combination of [20, Corollary 4.17] and [8, Proposition 5.3], so

$\mathrm{FDTC}(BH(\beta)) = \mathrm{FDTC}(BH(\gamma))$. Thus $\mathrm{FDTC}(\beta) = \mathrm{FDTC}(\gamma) = k/m$. The fractional Dehn twist coefficient is multiplicative under exponentiation, so $\mathrm{FDTC}(\beta^N) = kN/m$ and $\mathrm{FDTC}(\gamma^M) = kM/m$, but $\beta^N = \gamma^M$ so we must have that $M = N$. Finally, N^{th} roots are unique up to conjugation in the braid group [9], so that β and γ are conjugate. □

The spectral sequence from the annular Khovanov homology of an annular link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ is defined with $\mathbb{Z}/2\mathbb{Z}$ coefficients. At times, however, we will work with annular Khovanov homology over $\mathbb{C}$, because with these coefficients annular Khovanov homology has the structure of an $\mathfrak{sl}_2(\mathbb{C})$ representation [12, Proposition 3].

For the readers convenience we recall, from [12, Proposition 14], that

$$\mathrm{AKh}^i(\beta_n, \mathbb{C}) = \begin{cases} V_{(n)}\{n-1\} & \text{for } i = 0, \\ V_{(n-2)}\{n+1\} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here $V_{(m)}$ is the $(m+1)$–dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. We now study annular Khovanov homology with $\mathbb{Z}/2$ coefficients. Note that $\mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{C})) = 2n$. Now, $T(2,n)$ can be thought of as a 2–periodic knot with quotient β_n. It follows from [36] that $\mathrm{rank}(\mathrm{Kh}(T(2,n); \mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2))$. Indeed, $\mathrm{rank}(\mathrm{Kh}(T(2,n); \mathbb{Z}/2)) = 2n$ by the universal coefficient theorem and [21, Proposition 26], so in fact $\mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2)) = 2n$, and the above description of annular Khovanov homology is equally valid for $\mathbb{Z}/2$ coefficients.

More is known about the knot Floer homology of genus-1 fibered knots, so we are able to prove a stronger result for the closure of the 3–braid $\beta_3 = \sigma_1\sigma_2$ than β_4 and β_6. We are also able to use the classification of 3–braids representing the unknot to show that annular Khovanov homology also detects the closure of the 3–braid $\sigma_1\sigma_2^{-1}$.

Theorem 10.3 *If L is a 3–braid closure and $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) = 6$, then L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ in $A \times I$.*

Proof The lift $\widetilde{U}_L$ of the braid axis U in $\Sigma(L)$ is a genus-1 fibered knot.

The manifold $\Sigma(L)\setminus\widetilde{U}_L$ is naturally a sutured manifold, where the sutures on $S^3\setminus\widetilde{U}_L$ are two distinct pairs of meridional sutures lifted into the double branched cover from the product sutures on $A \times I$. There is a spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{SFH}}(-\Sigma(L) \setminus \widetilde{U}_L, \mathbb{Z}/2\mathbb{Z}) \cong \widehat{\mathrm{HFK}}(\widetilde{U}_L, -\Sigma(K), \mathbb{Z}/2\mathbb{Z}) \otimes V$, where V is a 2–dimensional vector space supported in bigradings $(0,0)$ and $(-1,-1)$. Furthermore, the k grading in AKh corresponds to the Alexander grading on $\widehat{\mathrm{SFH}}$ or $\widehat{\mathrm{HFK}}$; see [14, Theorem 2.1; 35, Theorem 1.1 and Proposition 5.3].

From this spectral sequence we can see that $\widehat{\mathrm{HFK}}(\widetilde{U}_K, -\Sigma(K), \mathbb{Z}/2\mathbb{Z})$ has rank no larger than 3. Every genus-1 fibered knot has knot Floer homology at least rank 3 and there are only four genus-1 fibered knots with rank 3 knot Floer homologies. They are the left- and right-handed trefoils in S^3 and two knots in the Poincaré homology sphere [5, Corollary 1.6].

The monodromies of fibered knots are unique up to conjugation. The monodromy of a fibered knot in $\Sigma(L)$ is the image of a braid representing L in $\mathrm{Mod}(S_1^1)$ under the Birman–Hilden correspondence. Finally, because $B_3 \cong \mathrm{Mod}(S_1^1)$, or by Proposition 10.1, conjugate monodromies must come from conjugate braids, so L must be isotopic to the closure of one of the 4–braids on this list that corresponds to one of these four possible fibered knots:

(1) $\sigma_1\sigma_2$,
(2) $\sigma_1^{-1}\sigma_2^{-1}$,
(3) $(\sigma_1\sigma_2)^{-6}\sigma_1\sigma_2$,
(4) $(\sigma_1\sigma_2)^{6}\sigma_1^{-1}\sigma_2^{-1}$.

A computation shows that the ranks of the annular Khovanov homologies of the last two braid closures are larger than 6 [19].

Therefore L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ in $A \times I$. □

The detection result in Theorem 10.4 follows immediately from Theorem 10.3 and previous results about annular Khovanov homology.

Theorem 10.4 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2}$ in $A \times I$.*

Proof If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to a 3–braid closure [13, Corollary 1.2; 39, Corollary 8.4]. The careful reader may worry about coefficients because we are working over $\mathbb{Z}/2\mathbb{Z}$ while [39, Corollary 8.4] is stated for coefficients over $\mathbb{C}$, but the corollary is also true over $\mathbb{Z}/2\mathbb{Z}$. This is because the universal coefficient theorem ensures that, in each annular grading, the rank of annular Khovanov homology over $\mathbb{C}$ is bounded above by the rank over $\mathbb{Z}/2\mathbb{Z}$ and the ranks will have the same parity. In particular, if the annular Khovanov homology in an annular grading is rank 1 over $\mathbb{Z}/2\mathbb{Z}$ then it is also rank 1 over $\mathbb{C}$.

From Theorem 10.3 we have that L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$. A simple computation shows that $\mathrm{AKh}(\widehat{\sigma_1^{-1}\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z}) \ncong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, so L must be isotopic to $\widehat{\sigma_1\sigma_2}$ in $A \times I$. □

One interpretation of Theorem 10.3 is that $\widehat{\sigma_1\sigma_2}$ and $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ are the simplest 3–braids from the point of view of annular Khovanov homology.

Proposition 10.5 *If L is isotopic to a 3–braid closure in $A \times I$, then $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq 6$.*

Proof From the universal coefficient theorem it follows that $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq \dim(\mathrm{AKh}(L, \mathbb{C}))$, so it suffices to show that $\dim(\mathrm{AKh}(L, \mathbb{C})) \geq 6$. Because L is a 3–braid closure, $\mathrm{AKh}(L, \mathbb{C})$ has dimension one in grading $k = 3$. The $\mathfrak{sl}_2(\mathbb{C})$ action on $\mathrm{AKh}(L, \mathbb{C})$ and the fact that the k grading gives the $\mathfrak{sl}_2(\mathbb{C})$ weights implies that $\mathrm{AKh}(L, \mathbb{C})$ contains an irreducible weight-3 representation of $\mathfrak{sl}_2(\mathbb{C})$, showing that $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq 4$.

The $\mathfrak{sl}_2(\mathbb{C})$ action gives a symmetry in the k gradings. Since $\mathrm{AKh}(L,\mathbb{C})$ is 0 in $k=0$, the dimension of $\mathrm{AKh}(L,\mathbb{C})$ must be even. It thus only remains to rule out the case that $\dim(\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z}))=4$.

If $\dim(\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z}))=4$, then $\mathrm{AKh}(L,\mathbb{C})$ consists only of an irreducible weight-3 representation of $\mathfrak{sl}_2(\mathbb{C})$ which must live in a single homological grading. Because $\mathrm{AKh}(L,\mathbb{C})$ is supported in a single homological grading, the spectral sequence from $\mathrm{AKh}(L,\mathbb{C})$ to $\mathrm{Kh}(L,\mathbb{C})$ collapses. The proof of Theorem 3.1(a) in [3] shows that the only braid closures for which this spectral sequence collapses immediately are closures of trivial braids. A computation shows that $\dim(\mathrm{AKh}(\hat{1}_3,\mathbb{C}))>6$, so there is no 3–braid closure with $\dim(\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z}))=4$. □

Because the 3–braid closures that are unknotted in S^3 are completely classified, we are able to use the previous results to show that annular Khovanov homology detects the closure of the 3–braid $\sigma_1\sigma_2^{-1}$ in $A\times I$.

Theorem 10.6 *Let $L\subseteq A\times I\subseteq S^3$ be an annular link. If $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})\cong\mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}},\mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A\times I$.*

Proof If $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})\cong\mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}},\mathbb{Z}/2\mathbb{Z})$ then L is isotopic to a 3–braid closure [13, Corollary 1.2; 39, Corollary 8.4]. Computing that the ungraded Euler characteristic of $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})$ is 2 shows that L is a knot, because the ungraded Euler characteristic is $2^{|L|}$, where $|L|$ is the number of components of L.

Because the rank of $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})$ is 10 and L is not the trivial braid, $\mathrm{Kh}(L,\mathbb{Z}/2\mathbb{Z})$ has rank strictly smaller than 10. The only knots in S^3 with the rank of $\mathrm{Kh}(L,\mathbb{Z}/2\mathbb{Z})$ strictly less than 10 are the unknot and the trefoils [4; 22]. Also, $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})$ is not nonzero in the correct bigradings for L to be one of the trefoils, so L must be the unknot in S^3.

Up to conjugation, the only 3–braids that represent the unknot are $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}$ [27, Theorem 12.1]. The annular Khovanov homology shows that L is not the closure of $\sigma_1\sigma_2$ or $\sigma_1^{-1}\sigma_2^{-1}$, so L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A\times I$. □

We now consider the annular Khovanov homology of the braids $\beta_n=\sigma_1\sigma_2\dots\sigma_{n-1}$ in $A\times I$ more generally. We will first show that any braid closure with the same annular Khovanov homology as $\hat{\beta}_n$ must represent an unknot in S^3. We then show that the lift of the braid axis for β_{2n} has the same knot Floer homology as $T(2,2n)$.

The next two results then follow from the fact that knot Floer homology detects $T(2,4)$ and $T(2,6)$.

Theorem 10.7 *Let $L\subseteq A\times I\subseteq S^3$ be an annular link. If $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})\cong\mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3},\mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3}$ in $A\times I$.*

Theorem 10.8 *Let $L\subseteq A\times I\subseteq S^3$ be an annular link. If $\mathrm{AKh}(L,\mathbb{Z}/2\mathbb{Z})\cong\mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5},\mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ in $A\times I$.*

To prove these theorems we first prove two general lemmas.

Lemma 10.9 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\hat{\beta}_n, \mathbb{Z}/2\mathbb{Z})$, then L is an unknot in S^3.*

Proof We first compute $\mathrm{AKh}(L, \mathbb{C})$ from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$. Throughout, we will use that the dimension of annular Khovanov homology over $\mathbb{C}$ can be no larger than that over $\mathbb{Z}/2\mathbb{Z}$. Because L is an n–braid closure, $\mathrm{AKh}(L, \mathbb{C})$ must contain a weight-n irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation in grading $i = 0$, of dimension n. Thus $\mathrm{AKh}(L, \mathbb{C})$ must consist only of this representation in grading $i = 0$, because $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ has dimension n in homological grading 0. We therefore have that all of the generators in grading $i = 1$ for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ must correspond to generators of $\mathrm{AKh}(L, \mathbb{C})$. If not, they would correspond to 2–torsion in $\mathrm{AKh}(L, \mathbb{Z})$, but the torsion contributes dimension in two different homological gradings by the universal coefficient theorem.

A simple computation of annular Khovanov homology verifies that L is not the trivial braid. Thus by [3, Theorem 3.1], we know that the differential ∂_- on $\mathrm{AKh}(L, \mathbb{C})$ inducing the spectral sequence to $\mathrm{Kh}(L)$ must send the highest-weight generator in the grading $i = 0$ to something nonzero. The only generator in the correct quantum grading is the highest-weight generator in the grading $i = 1$, so that must be the image of the highest-weight generator in the grading $i = 0$ under ∂_-. The action ∂_- is part of the action of $\mathfrak{sl}_2(\wedge)$ on $\mathrm{AKh}(L, \mathbb{C})$ and commutes up to sign with the lowering operator f [12, Theorem 1]. This means that the image of ∂_- is spanned by all generators in grading $i = 1$. Thus $\mathrm{Kh}(L)$ is dimension 2, and L is the unknot. □

Lemma 10.10 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\hat{\beta}_{2n}, \mathbb{Z}/2\mathbb{Z})$, then $\widehat{\mathrm{HFK}}(\widetilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$.*

Proof For this computation, we will use the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ and compute the Maslov gradings of the generators of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$.

From the construction of the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ as an iterated mapping cone, it follows that in each i grading on $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ the relative Maslov grading of any two generators agrees with half the difference of the quantum gradings of the generators. It remains to relate the relative Maslov gradings for generators in i grading 0 and 1, and then upgrade this information to an absolute Maslov grading.

The induced differential ∂_- giving the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ is part of the total differential on the iterated mapping cone induced by counting pseudoholomorphic polygons. Thus ∂_- lowers the Maslov grading by one. This implies that, for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$, generators in the same k grading live in the same relative Maslov grading. Since generators in the same k grading or $2A$ grading also have the same Maslov grading, the spectral sequence to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ collapses as all differentials preserve the k grading or $2A$ grading and change the Maslov grading.

To upgrade the above to a statement about the absolute Maslov grading, notice that there are only two generators which survive in the spectral sequence to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$, namely the generators that sit in the

k gradings $-2n$ and $2-2n$. These generators must then be in Maslov gradings 0 and 1, respectively. From here we can pin down the Maslov gradings of the remaining generators of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$.

The claim $\widehat{\mathrm{HFK}}(\widetilde{U}) \cong \widehat{\mathrm{HFK}}(T(2,2n))$ then follows from the fact that $\widehat{\mathrm{HFK}}(-\widetilde{U}) \cong (\widehat{\mathrm{HFK}}(\widetilde{U}))^*$, with the appropriate change in gradings. □

Proof of Theorem 10.7 Suppose L is as in the statement of the theorem. Lemma 10.10 implies that $\widehat{\mathrm{HFK}}(\widetilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2,4))$, whence $\widetilde{U}$ is $T(2,4)$ by Theorem 4.1, which is fibered of genus 2. Up to isotopy, fibered link exteriors have unique fibrations by Seifert surfaces; for instance see [6, Chapter 1.4]. Note that the exterior of a link may fiber in different ways if one does not require the fibers to be Seifert surfaces. The monodromies of fibered links are unique up to conjugation. The monodromy of a fibered link in $\Sigma(L)$ is the image of a braid representing L in $\mathrm{Mod}(S_2^2)$ under the Birman–Hilden correspondence. Finally, by Proposition 10.1 conjugate monodromies must come from conjugate braids, so L must be isotopic to β_4. □

The proof of Theorem 10.8 is identical so we omit it.

Remark 10.11 Similar techniques could be applied to show that annular Khovanov homology detects the closure of the braid $\sigma_1 \in B_2$. However, the fact that annular Khovanov homology detects this braid closure already follows from the fact that annular Khovanov homology detects braid closures and the braid index [13, Corollary 1.2; 39, Corollary 8.4], combined with the computations of the annular Khovanov homologies of all 2–braid closures [12, Proposition 15].

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Models for knot spaces and Atiyah duality

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Let $\mathrm{Emb}(S^1, M)$ be the space of smooth embeddings from the circle to a closed manifold M. We introduce a new spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$ for a simply connected closed manifold M of dimension 4 or more, which has an explicit E_1–page and a computable E_2–page. As applications, we compute some part of the cohomology for $M = S^k \times S^l$ with some conditions on the dimensions k and l, and prove that the inclusion $\mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ to the immersions induces an isomorphism on π_1 for some simply connected 4–manifolds. This gives a restriction on a question posed by Arone and Szymik. The idea to construct the spectral sequence is to combine a version of Sinha's cosimplicial model for the knot space and a spectral sequence for a configuration space by Bendersky and Gitler. The cosimplicial model consists of configuration spaces of points (with a tangent vector) in M. We use Atiyah duality to transfer the structure maps on the configuration spaces to maps on Thom spectra of the quotient of a direct product of M by the fat diagonal. This transferred structure is the key to defining our spectral sequence, and is also used to show that Sinha's model can be resolved into simpler pieces in a stable category.

18M75, 55P43, 55T99, 57R40; 18N40

1 Introduction

In [36; 37] Sinha constructed cosimplicial models of spaces of knots in a manifold of dimension ≥ 4, based on Goodwillie–Weiss embedding calculus; see Goodwillie and Klein [17], Goodwillie and Weiss [18], and Weiss [41]. The model was crucially used in the affirmative solution to Vassiliev's conjecture for a spectral sequence for the space of long knots in $\mathbb{R}^d$ (with $d \geq 4$) for rational coefficient by Lambrechts, Turchin and Volić in [25] (see Boavida de Brito and Horel [5] for other coefficients). We study a version of Sinha's model in stable categories.

Let $\mathrm{Emb}(S^1, M)$ be the space of smooth embeddings from the circle S^1 to a manifold M (without any basepoint condition) endowed with the C^∞–topology. The space $\mathrm{Emb}(S^1, M)$ is studied by Arone and Szymik [1] and Budney [8], and study of embedding spaces including the knot space is a motivation of Campos and Willwacher [10] and Idrissi [22]

In the rest of the paper, M denotes a connected closed smooth manifold of dimension d. Our knot space $\mathrm{Emb}(S^1, M)$ is slightly different from the one considered by Sinha, but we can construct a cosimplicial model similar to Sinha's, which is called *Sinha's cosimplicial model* and denoted by $\mathcal{C}^\bullet(M)$. Its n^{th} space is homotopy equivalent to the configuration space of $n+1$ ordered points in M with a unit tangent vector.

To state our first main theorem, we need some notation. Let SM be the tangent sphere bundle of M. Fix an embedding $e_0\colon SM \to \mathbb{R}^K$, and a tubular neighborhood ν of the image $e_0(SM)$ in $\mathbb{R}^K$. Let $\mathcal{D}$ be the little interval operad. We use a notion of a $\mathcal{D}$–comodule, which plays a role similar to a simplicial object but is homotopically more flexible. We work with the category of symmetric spectra $\mathcal{SP}$. For a manifold N and an integer $n \geq 1$, N^n denotes the direct product of n copies of N. The fat diagonal of M^n is by definition the union of all the diagonals of M^n. We regard the product ν^n as a disk bundle over SM^n via the obvious identification $(e_0(SM))^n = SM^n$. The following theorem gives a dual equivalence between the configuration spaces and quotients by a fat diagonal, which preserves structure necessary to recover (some part of) the knot space.

Theorem 1.1 (Theorem 4.4 and Lemma 4.7) *Under the above notation, there exists a zigzag of weak equivalences of left $\mathcal{D}$–comodules of nonunital commutative symmetric ring spectra*

$$(\mathcal{C}_M)^\vee \simeq \mathcal{T}_M,$$

where $(\mathcal{C}_M)^\vee$ is a comodule whose n^{th} object is the Spanier–Whitehead dual of the configuration space of n points with a tangent vector in M, and $\mathcal{T}_M$ is a comodule whose n^{th} object is a natural model of the Thom spectrum

$$\Sigma^{-nK}\,\mathrm{Th}(\nu^n)/\mathrm{Th}(\nu^n|_{\mathrm{FD}_n}).$$

Here

- *Σ denotes the suspension equivalence and $\mathrm{Th}(-)$ denotes the associated Thom space,*
- *FD_n is the preimage of the fat diagonal by (the product of) the projection $SM^n \to M^n$, and*
- *$\nu^n|_{\mathrm{FD}_n}$ denotes the restriction of the base to FD_n.*

See Section 2.1 and Definitions 2.10, 4.1, 4.3 and 4.5 for details of the notation. Theorem 1.1 is a structured version of the Poincaré–Lefschetz duality

(1-1) $$H^*(\mathcal{C}^{n-1}(M)) \cong H_*(SM^n, \mathrm{FD}_n),$$

deduced from a homotopy equivalence $\mathcal{C}^{n-1}(M) \simeq SM^n - \mathrm{FD}_n$. (We are loose on degrees.) If we do not consider the (nonunital) commutative multiplications, an analogue of Theorem 1.1 holds in the category of prespectra (in the sense of Mandell, May, Schwede and Shipley [28]), a more naive, nonsymmetric monoidal category of spectra, and it is enough to prove Theorem 1.2, but the multiplications may be useful for future study and our construction hardly becomes easier for prespectra.

To state the second main theorem, we need additional notation. For a positive integer n, let $\mathsf{G}(n)$ be the set of graphs G with set of vertices $V(G) = \underline{n} = \{1, \dots, n\}$ and set of edges $E(G) \subset \{(i, j) \mid i, j \in \underline{n} \text{ with } i < j\}$. Let D_G be the subspace of SM^n consisting of elements whose image by the projection to M^n has the same i^{th} and j^{th} components if i and j are connected by an edge of G ($i, j \in \underline{n}$). The space FD_n in Theorem 1.1 is the union of the spaces D_G whose graph G has at least one edge. D_G is a rather comprehensible space compared to the space $\mathcal{C}^{n-1}(M)$. For example, its cohomology ring is computed in Lemmas 6.5 and 6.6 under some assumptions. Throughout this paper, we fix a coefficient ring k and suppose k is either of a subring of the rationals $\mathbb{Q}$ or the field $\mathbb{F}_{\mathfrak{p}}$ of $\mathfrak{p}$ elements for a prime $\mathfrak{p}$. All normalized singular (co)chains C^* and C_* and singular (co)homology H^* and H_* are supposed to have coefficients in k, unless otherwise stated. As an application of Theorem 1.1, we introduce a new spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$.

Theorem 1.2 (Theorems 5.16, 5.17 and 6.11) *Suppose M is simply connected and of dimension $d \geq 4$. There exists a second-quadrant spectral sequence $\{\check{\mathbb{E}}_r^{pq}\}_r$ converging to $H^{p+q}(\mathrm{Emb}(S^1, M))$ such that:*

1. *Its E_2–page is isomorphic to the total homology of the normalization of a simplicial commutative differential bigraded algebra $A_\bullet^{\star *}(M)$ which is defined in terms of the cohomology ring $H^*(D_G)$ for various graphs G and maps between them,*
$$\check{\mathbb{E}}_2^{pq} \cong H(NA_\bullet^{\star *}(M)) \Rightarrow H^{p+q}(\mathrm{Emb}(S^1, M)),$$
where the bidegree is given by $* = p$ and $\star - \bullet = q$.
2. *If $H^*(M)$ is a free k–module, and the Euler number $\chi(M)$ is zero or invertible in k, the object $A_\bullet^{\star *}(M)$ is determined by the ring $H^*(M)$.*

We call this spectral sequence the *Čech spectral sequence*, or in short, the *Čech s.s.* A feature of this spectral sequence is that its E_1 page and differential d_1 are explicitly determined by the cohomology of M. As spectral sequences for $H^*(\mathrm{Emb}(S^1, M))$ we have the Bousfield–Kan type cohomology spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$, see Definition 2.7, and Vassiliev's spectral sequence [40] converging to the relative cohomology $H^*(\Omega_f(M), \mathrm{Emb}(S^1, M))$, where $\Omega_f(M)$ is the space of smooth maps $S^1 \to M$. But no small (ie degreewise finite-dimensional) page of these spectral sequences has been computed in general. The E_1–page of the Bousfield–Kan type s.s. is described by the cohomology of the ordered configuration spaces of points with a vector in M, which is difficult to compute; Vassiliev's first term is also interesting but complicated. By this feature, we can compute examples; see Section 7. We obtain new computational results in the case of the product of two spheres. While we only do elementary computation in the present paper, one of potential merits of Čech s.s. is that computation of higher differentials will be relatively accessible since we deal with the fat diagonals and Čech complex instead of configuration spaces. The other is that we will be able to enrich it with operations such as the cup product and square, and relate them to those on $H^*(M)$. We will deal with these subjects in future work. Precisely speaking, we can also construct the Čech spectral sequence in the 3–dimensional or nonsimply connected case, where it does not converge to $H^*(\mathrm{Emb}(S^1, M))$ but might have some information about the knot space; see Remark 5.18.

Arone and Szymik studied $\mathrm{Emb}(S^1, M)$ for the case of dimension $d = 4$ in [1]. Let $\mathrm{Imm}(S^1, M)$ be the space of smooth immersions $S^1 \to M$ with the C^∞–topology and $i_M : \mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ be the inclusion. Among other results, they proved that i_M is 1–connected, so in particular surjective on π_1 in general. (They proved interesting results for the nonsimply connected case $M = S^1 \times S^3$; see also Budney and Gabai [9].) They asked whether there is a simply connected 4–manifold M such that i_M has nontrivial kernel on π_1. Using Theorem 1.2, we give a restriction to this question:

Corollary 1.3 *Suppose that M is simply connected, of dimension 4 and satisfies $H_2(M;\mathbb{Z}) \neq 0$, and that the intersection form on $H_2(M;\mathbb{F}_2)$ is represented by a matrix whose inverse has at least one nonzero diagonal component. Let $i_M : \mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ be the inclusion to the space of immersions. Then the map i_M induces an isomorphism on π_1. In particular, $\pi_1(\mathrm{Emb}(S^1, M)) \cong H_2(M;\mathbb{Z})$.*

The assumption does not depend on the choice of matrix. For example, $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, the connected sum of complex projective planes, satisfies the assumption, while $M = S^2 \times S^2$ does not. For the case of $H_2(M) = 0$, by Proposition 5.2 of [1], $\mathrm{Emb}(S^1, M)$ is simply connected. We can also prove this similarly to Corollary 1.3. The case of all of the diagonal components of the matrix being zero is unclear by our method.

Remark 1.4 In the recent preprint [23], Kosanović gave a proof of a complete answer to the question, which states that the inclusion i_M induces an isomorphism of π_{d-1} if M is simply connected and of dimension $d \geq 4$ by an independent method.

Sinha's cosimplicial model can be considered as a resolution of $\mathrm{Emb}(S^1, M)$ into simpler spaces. We resolve it into further simpler pieces in the category of chain complexes as an application of Theorems 1.1 and 1.2. To state the result, we need additional notation. We consider a category Ψ of planar rooted trees and edge contractions. It is equipped with a functor $\mathcal{G} \circ \mathcal{F} : \Psi \to \Delta$, where Δ is the category of the standard simplices. We also use a category $\mathsf{G}(n)^+$. Roughly speaking, the objects of $\mathsf{G}(n)^+$ are a symbol $*$ and the graphs in $\mathsf{G}(n)$, and the morphisms are the inclusions (of edge sets) and formal arrows $* \to G$ to the graphs having at least one edge. Let $\widetilde{\Psi}$ be the Grothendieck construction of a functor from Ψ sending a tree T to the category $\mathsf{G}(|v_r| - 1)^+$, where $|v_r|$ denotes the valence of the root vertex of T. So an object of $\widetilde{\Psi}$ is a pair (T, G) of a tree T and a graph G with exactly $|v_r| - 1$ vertices (or the symbol $*$). Let $\eta\colon \widetilde{\Psi} \to \Psi$ be the projection given by $\eta(T, G) = T$.

Theorem 1.5 (Theorem 8.4) *Under the above notation, there exists a functor $\mathsf{T}_M : \widetilde{\Psi}^{\mathrm{op}} \to \mathcal{SP}$ satisfying the following conditions:*

(1) *Its value on $(T, G) \in \widetilde{\Psi}$ is a natural model of the Thom spectrum*

$$\Sigma^{-mK}\,\mathrm{Th}(\nu^m|_{D_G}) \quad \text{with } m = |v_r| - 1$$

if G is a graph, and the basepoint if $G = *$.

(2) *There exists a zigzag of weak equivalences of functors*

$$(\mathcal{G} \circ \mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee) \simeq \mathbb{L}\eta_! \mathsf{T}_M : \Psi^{\mathrm{op}} \to \mathcal{SP}.$$

Here the dual of the cosimplicial model is regarded as a functor from Δ^{op} and $\mathbb{L}\eta_!$ is the (derived) left Kan extension along η.

(3) *Suppose M is simply connected and of dimension $d \geq 4$. There exists a zigzag of quasi-isomorphisms of chain complexes*

$$C^*(\mathrm{Emb}(S^1, M)) \simeq \operatorname*{hocolim}_{\widetilde{\Psi}^{\mathrm{op}}} C_* \circ \mathsf{T}_M.$$

Here hocolim *denotes the homotopy colimit, and C_* on the right-hand side is a certain singular chain functor from spectra to chain complexes.*

See Section 2.1 and Definitions 5.1 and 8.1 for details of the notation. We give an intuitive explanation for this theorem. We regard $\mathsf{G}(n)$ as the full subcategory of $\mathsf{G}(n)^+$. Let $\varnothing$ denote the graph with no edges. There is a standard quasi-isomorphism $C_*(\mathrm{FD}_n) \simeq \mathrm{hocolim}_{G \in C_1} C_*(D_G)$, where $C_1 = \mathsf{G}(n)^{\mathrm{op}} - \{\varnothing\}$. Since the relative complex $C_*(SM^n, \mathrm{FD}_n)$ is the homotopy cofiber of the inclusion $C_*(\mathrm{FD}_n) \to C_*(SM^n) = C_*(D_\varnothing)$, we have quasi-isomorphisms

$$C^*(\mathcal{C}^{n-1}(M)) \simeq C_*(SM^n, \mathrm{FD}_n) \simeq \operatorname*{hocolim}_{G \in C_2} C_*(D_G),$$

where we set $C_2 = (\mathsf{G}(n)^+)^{\mathrm{op}}$ and $C_*(D_G) = 0$ for $G = *$. We regard this presentation as a resolution of $C^*(\mathcal{C}^{n-1}(M))$. A category of planar rooted trees is a lax analogue of the category of the standard simplices. Actually, homotopy limits over these categories are weakly equivalent. So, intuitively speaking, existence of the functor T_M means potential compatibility of the resolution and the cosimplicial structure.

We shall explain why we use spectra, which also serves as an outline of our arguments. Our motivation is to derive a new spectral sequence from Sinha's cosimplicial model. The idea is to combine the cosimplicial model and a procedure of constructing a spectral sequence for the cohomology of the configuration space due to Bendersky and Gitler [3]. So we consider the above duality (1-1), and describe the chain complex $C_*(SM^n, \mathrm{FD}_n)$ by an augmented Čech complex as follows. Consider

$$C_*(D_\varnothing) \xleftarrow{\partial} \bigoplus_{G \in \mathsf{G}(n,1)} C_*(D_G) \xleftarrow{\partial} \bigoplus_{G \in \mathsf{G}(n,2)} C_*(D_G) \xleftarrow{\partial} \bigoplus_{G \in \mathsf{G}(n,3)} C_*(D_G) \xleftarrow{\partial} \cdots,$$

where $\mathsf{G}(n,p) \subset \mathsf{G}(n)$ denotes the subset of graphs with exactly p edges. We want to extend this to the following commutative diagram of semisimplicial chain complexes by defining suitable face maps d_i:

$$\begin{tikzcd}
C^*(\mathcal{C}^n(M)) \arrow[d, "(d^i)^*"] & C_*(D_\varnothing) \arrow[l, "\mathrm{PD}"'] \arrow[d, "d_i"] & \bigoplus_{G\in\mathsf{G}(n+1,1)} C_*(D_G) \arrow[l] \arrow[d, "d_i"] & \bigoplus_{G\in\mathsf{G}(n+1,2)} C_*(D_G) \arrow[l] \arrow[d, "d_i"] & \cdots \arrow[l] \\
C^*(\mathcal{C}^{n-1}(M)) & C_*(D_\varnothing) \arrow[l, "\mathrm{PD}"'] & \bigoplus_{G\in\mathsf{G}(n,1)} C_*(D_G) \arrow[l] & \bigoplus_{G\in\mathsf{G}(n,2)} C_*(D_G) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{1-2}$$

Here d^i is the coface map of $\mathcal{C}^\bullet(M)$, and PD actually denotes the zigzag

$$C^*(\mathcal{C}^n(M)) \to C_*(D_\varnothing, \mathrm{FD}_n) \leftarrow C_*(D_\varnothing)$$

of the cap product with the fundamental class and the quotient map. If we could construct a semisimplicial double complex in the right-hand side of PD in (1-2), by taking the total complex, we would have a certain triple complex $C_{\bullet\star *}$, where $\bullet$ (resp. $\star$, $*$) denotes the cosimplicial (resp. Čech, singular) degree. Then by filtering with $\star + \bullet$, we would obtain a spectral sequence as in Theorem 1.2.

Unfortunately, it is difficult to define degeneracy maps d_i fitting into (1-2). This difficulty is essentially analogous to the one in the construction of a certain chain-level intersection product on $C_*(M)$. We shall explain this point more precisely. The coface map $d^i : \mathcal{C}^n(M) \to \mathcal{C}^{n+1}(M)$ is a deformed diagonal, and the usual diagonal induces the intersection product on homology. So the maps d_i should be something like a deformed intersection product. The simplicial identities for d_i are analogous to the associativity of an intersection product. In addition, the map $(d^i)^*$ on the cochain is analogous to the cup product. So construction of d_i is analogous to construction of a chain-level intersection product which is associative and compatible with the cup product through the duality. We could not find such a product in the literature.

A nice solution is found in a construction due to R Cohen and Jones [11; 12] in string topology. They used spectra to give a homotopy theoretic realization of the loop product, which led to a proof of an isomorphism between the loop product and a product on Hochschild cohomology (see Moriya [30] for a detailed account). Their key notion is the Atiyah duality, which is an equivalence between the Spanier–Whitehead dual $M^\vee$ and the Thom spectrum $M^{-TM} = \Sigma^{-K}\,\mathrm{Th}(\nu)$. To prove their isomorphism, Cohen [11] introduced a model of M^{-TM} in the category $\mathcal{SP}$, and refined the duality to an equivalence of (nonunital) commutative symmetric ring spectra. This equivalence can be regarded as a multiplicative version of the Poincaré duality. In fact, the multiplication on the model of M^{-TM} works as an analogue of a chain level intersection product in their theory. So is efficient to construct necessary semisimplicial objects and their equivalence in $\mathcal{SP}$, then take chain complexes of them, and derive a spectral sequence. This is why we use spectra.

Even if we use spectra, the (co)simplicial object is too rigid, and we use a laxer notion of a left comodule over an A_∞–operad.

As we demonstrate, the duality is very useful to transfer structures on the configuration space to the Thom spectrum of the quotient by the fat diagonal, which is homotopically more accessible, and may be applied in much research on configuration spaces. In future work, we will study collapse of Sinha's (or Vassiliev's) spectral sequence for the space of long knots in $\mathbb{R}^d$ [36] using the duality.

The organization of the paper is as follows. In Section 2, we introduce basic notions. We define a version of Sinha's cosimplicial model and show that its homotopy limit is equivalent to the space $\mathrm{Emb}(S^1, M)$. We define the notions of a (co)module and Hochschild complex of a comodule over the associahedral operad. These notions are minor variations of ones given by others. Section 3 is the technical heart of this paper. We introduce a version of Cohen's model of Thom spectra and use it to construct the comodule $\mathcal{T}_M$ in Theorem 1.1. We take care about definitions of parameters such as the radius of tubular neighborhoods to make structure maps of a comodule compatible with the diagonals. In Section 4, we prove Theorem 1.1. In Sections 5 and 6, we prove Theorem 1.2. These two sections have a homotopical and algebraic nature compared to the previous sections, where we give detailed space level constructions. In Section 5, we define a chain functor for symmetric spectra and construct the spectral sequence filtering Hochschild complex of the chains of a resolution of the comodule $\mathcal{T}_M$. We prove that the E_1–page of the Čech spectral sequence is quasi-isomorphic to the total complex of a simplicial differential bigraded algebra, and prove the convergence of the Čech spectral sequence. In Sections 3–5 we mainly deal with comodules, but we need the cosimplicial model in the proof of convergence since we deduce it from a theorem of Bousfield. In Section 6, we compute the cohomology rings $H^*(D_G)$ and maps between them, and give a description of the simplicial algebra in terms of the cohomology ring $H^*(M)$ under some assumptions. The computation is standard work based on Serre spectral sequences. In Section 7, we compute examples and prove Corollary 1.3. In Section 8, we prove Theorem 1.5.

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2 Preliminaries

In this section, we fix notation and introduce basic notions. Nothing is essentially new.

2.1 Notation and terminology

- We denote by Δ the category of standard simplices. Its objects are the finite ordered sets $[n]=\{0,\dots,n\}$ for $n\geq 0$ and its morphisms are the weakly order-preserving maps. We denote by Δ_n the full subcategory of Δ that consists of the objects $[k]$ with $k\leq n$. We define a category (or poset) P_n as follows. The objects are the nonempty subsets S of $\underline{n}$, and there is a unique morphism $S\to S'$ if and only if $S\subset S'$.

$\mathcal{G}_n : \mathrm{P}_{n+1} \to \Delta_n$ denotes the functor given in [37, Definition 6.3]. It sends a set S to $[\#S-1]$ and an inclusion $S \subset S'$ to the composition $[\#S-1] \cong S \subset S' \cong [\#S'-1]$, where $\cong$ denotes the order-preserving bijection.

- For a category $\mathcal{C}$, a morphism of $\mathcal{C}$ is also called a *map of* $\mathcal{C}$. A *symmetric sequence in* $\mathcal{C}$ is a sequence $\{X_k\}_{k\geq 0}$ (or $\{X(k)\}_{k\geq 1}$) of objects in $\mathcal{C}$ equipped with an action of the k^{th} symmetric group Σ_k on X_k (or $X(k)$) for each k. The group Σ_k acts from the right throughout this paper.

- Let $\mathsf{G}(n)$ be the set of graphs defined in Section 1. For a graph $G \in \mathsf{G}(n)$, we regard $E(G)$ as an ordered set with the lexicographical order. To ease notation, we write (i,j) with $i > j$ to denote the edge (j,i) of a graph in $\mathsf{G}(n)$. For a map $f : \underline{n} \to \underline{m}$ of finite sets, we denote by the same symbol f the map $\mathsf{G}(n) \to \mathsf{G}(m)$ defined by

$$E(f(G)) = \{(f(i), f(j)) \mid (i,j) \in E(G) \text{ with } f(i) \neq f(j)\}.$$

Also, f denotes the natural map $\pi_0(G) \to \pi_0(f(G))$ between the connected components.

- Our notion of a *model category* is that of [21]. $\mathbf{Ho}(\mathcal{M})$ denotes the homotopy category of a model category $\mathcal{M}$.

- We will denote by $\mathcal{CG}$ the category of all compactly generated spaces and continuous maps (see [21, Definition 2.4.21]), by $\mathcal{CG}_*$ the category of pointed compactly generated spaces and pointed maps, and by $\wedge$ the smash product of pointed spaces.

- For a category $\mathcal{C}$, a *cosimplicial object* $X^\bullet$ *in* $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. A map of cosimplicial objects is a natural transformation. X^n denotes the object of $\mathcal{C}$ at $[n]$. We define maps

$$d^i : [n] \to [n+1] \quad \text{for } 0 \leq i \leq n+1 \qquad \text{and} \qquad s^i : [n] \to [n-1] \quad \text{for } 0 \leq i \leq n-1$$

by

$$d^i(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geq i, \end{cases} \qquad \text{and} \qquad s^i(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i. \end{cases}$$

Here $d^i, s^i : X^n \to X^{n\pm 1}$ denote the maps corresponding to the same symbols. As is well known, a cosimplicial object $X^\bullet$ is identified with a sequence of objects $X_0, X_1, \ldots, X_n, \ldots$ equipped with a family of maps $\{d^i, s^i\}$ satisfying the cosimplicial identity; see [16]. We call a cosimplicial object in $\mathcal{CG}$ a *cosimplicial space*. Similarly, a simplicial object $X_\bullet$ in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. We denote by $d_i, s_i : X_{n\pm 1} \to X_n$ the maps corresponding to d^i and s^i.

- Our notion of a *symmetric spectrum* is that of Mandell, May, Schwede and Shipley [28]. A symmetric spectrum consists of a symmetric sequence $\{X_k\}_{k\geq 0}$ in $\mathcal{CG}_*$ and a map $\sigma_X : S^1 \wedge X_k \to X_{k+1}$ for each $k \geq 0$ which is subject to certain conditions. The category of symmetric spectra is denoted by $\mathcal{SP}$. We denote by $\wedge = \wedge_S$ the canonical symmetric monoidal product on $\mathcal{SP}$ given in [28], and by $\mathbb{S}$ the sphere spectrum, the unit for $\wedge$. Henceforth the term "spectrum" means symmetric spectrum. For a spectrum, we refer to the numbering of the underlying sequence as the *level*.

- For $K \in \mathcal{CG}$ and $X \in \mathcal{SP}$, we define a tensor $K \mathbin{\hat{\otimes}} X \in \mathcal{SP}$ by $(K \mathbin{\hat{\otimes}} X)_k = (K_+) \wedge X_k$, where K_+ is K with disjoint basepoint. This tensor is extended to a functor $\mathcal{CG} \times \mathcal{SP} \to \mathcal{SP}$ in an obvious manner. For $K, L \in \mathcal{CG}$ and $X, Y \in \mathcal{SP}$, we call the natural isomorphisms
$$K \mathbin{\hat{\otimes}} (L \mathbin{\hat{\otimes}} X) \cong (K \times L) \mathbin{\hat{\otimes}} X \quad \text{and} \quad K \mathbin{\hat{\otimes}} (X \wedge Y) \cong (K \mathbin{\hat{\otimes}} X) \wedge Y,$$
the *associativity isomorphisms*. A natural isomorphism $(K \times L) \mathbin{\hat{\otimes}} (X \wedge Y) \cong (K \mathbin{\hat{\otimes}} X) \wedge (L \mathbin{\hat{\otimes}} Y)$ is defined by successive compositions of the associativity isomorphisms and the symmetry one for $\wedge$. We define a mapping object $\mathrm{Map}(K, X) \in \mathcal{SP}$ by $\mathrm{Map}(K, X)_k = \mathrm{Map}_*(K_+, X_k)$, where the right-hand side is the usual internal hom object (mapping space) of $\mathcal{CG}_*$. This defines a functor $(\mathcal{CG})^{\mathrm{op}} \times \mathcal{SP} \to \mathcal{SP}$. The functors $K \mathbin{\hat{\otimes}} (-)$ and $\mathrm{Map}(K, -)$ form an adjoint pair. We set $K^\vee = \mathrm{Map}(K, \mathbb{S})$ for $K \in \mathcal{CG}$.
- We use the *stable model structure on $\mathcal{SP}$*; see [28]. This is only used in Section 5.1 and Section 8. Weak equivalences in this model structure are called *stable equivalences*. *Level equivalences* and *π_*–isomorphisms* are more restricted classes of maps in $\mathcal{SP}$; see [28]. The former are the levelwise weak homotopy equivalences and the latter are the maps which induce an isomorphism between (naive) homotopy groups defined as the colimit of the sequence of canonical maps $\iota_k : \pi_*(X_k) \to \pi_{*+1}(X_{k+1})$.
- We say a spectrum X is *semistable* if there exists a number $\alpha > 1$ such that, for any sufficiently large l, the map $\iota_l : \pi_k(X_l) \to \pi_{k+1}(X_{l+1})$ is an isomorphism for each $k \le \alpha l$. Semistability in this sense implies semistability in the sense of [34], so a stable equivalence between semistable spectra (in our sense) is a π_*–isomorphism.
- A *nonunital commutative symmetric ring spectrum* (in short, *NCRS*) is a spectrum A with a commutative associative multiplication $A \wedge A \to A$ (but possibly without a unit). A map of NCRS is a map of spectra preserving the multiplication.
- $\mathcal{CH}_{\mathsf{k}}$ denotes the category of (possibly unbounded) chain complexes over k and chain maps. Differentials raise the degree (see the next item for our degree rule). We endow $\mathcal{CH}_{\mathsf{k}}$ with the model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections. We denote by $\otimes = \otimes_k$ the standard tensor product of complexes.
- We deal with modules with multiple degrees (or gradings). For modules having superscript(s) and/or subscript(s), their total degree is given by the formula
$$(\text{total degree}) = (\text{sum of superscripts}) - (\text{sum of subscripts}).$$
For example, singular chains in $C_p(M)$ have degree $-p$, and the total degree of a triply graded module $A^{\star *}_{\bullet}$ is $* + \star - \bullet$. We denote by $|a|$ the (bi)degree of a. We sometimes omit super- or subscripts if unnecessary.
- For a simplicial chain complex $C^*_\bullet$ (ie a functor $\Delta^{\mathrm{op}} \to \mathcal{CH}_{\mathsf{k}}$), the *normalized complex* (or *normalization*) $NC^*_\bullet$ is a double complex defined by taking the normalized complex of a simplicial k–module in each chain degree.
- For a small category C and a cofibrantly generated model category $\mathcal{M}$ (in the sense of [21]), we denote by $\mathcal{F}un(C, \mathcal{M})$ the category of functors $C \to \mathcal{M}$ and natural transformations, which is endowed with

the projective model structure; see [20]. The colimit functor $\mathrm{colim}_C\colon \mathcal{F}un(C,\mathcal{M})\to\mathcal{M}$ is a left Quillen functor. Its left derived functor is denoted by $\mathrm{hocolim}_C$ and called the *homotopy colimit* over C.

- A *commutative differential bigraded algebra* (in short, *CDBA*) is a bigraded module $A^{\star *}$ equipped with a unital multiplication which is graded commutative for the total degree and preserves the bigrading, and a differential $\partial\colon A^{\star *}\to A^{\star+1,*}$ which satisfies the Leibniz rule for the total degree. A *map* of CDBA is a map of differential graded algebras preserving bigrading.

2.2 Čech complex and homotopy colimit

Definition 2.1 Let $\mathcal{M}$ be a cofibrantly generated model category. We define a functor

$$\check{\mathrm{C}}\colon \mathcal{F}un(\mathrm{P}_{n+1}^{\mathrm{op}},\mathcal{M})\to\mathcal{F}un(\Delta^{\mathrm{op}},\mathcal{M})\quad\text{by}\ \check{\mathrm{C}}X[k]=\coprod_{f\colon[k]\to\underline{n+1}}X_{f([k])},$$

where f runs through the weakly order-preserving maps. For an order-preserving $\alpha\colon[l]\to[k]\in\Delta$, the map $\check{\mathrm{C}}X[k]\to\check{\mathrm{C}}X[l]$ is the sum of the maps $X_{f([k])}\to X_{f\circ\alpha([l])}$ induced by the inclusion $f\circ\alpha([l])\subset f([k])$.

Lemma 2.2 *We use the notation of Definition 2.1. Let $X\in\mathcal{F}un(\mathrm{P}_{n+1}^{\mathrm{op}},\mathcal{M})$ be a functor.*

(1) *There exists an isomorphism* $\mathrm{hocolim}_{\mathrm{P}_{n+1}^{\mathrm{op}}}X\cong\mathrm{hocolim}_{\Delta^{\mathrm{op}}}\check{\mathrm{C}}X$ *in* $\mathbf{Ho}(\mathcal{M})$ *which is natural for X.*

(2) *X is cofibrant in $\mathcal{F}un(\mathrm{P}_{n+1}^{\mathrm{op}},\mathcal{M})$ if the following canonical map is a cofibration in $\mathcal{M}$ for each $S\in\mathrm{P}_{n+1}$:*

$$\operatorname*{colim}_{S'\supsetneq S}X_{S'}\to X_S.$$

Proof Let $(i_n\circ\mathcal{G}_n)^*\colon\mathcal{F}un(\Delta^{\mathrm{op}},\mathcal{M})\to\mathcal{F}un(\mathrm{P}_{n+1}^{\mathrm{op}},\mathcal{M})$ be the pullback by the composition of $\mathcal{G}_n$ and the inclusion $i_n\colon\Delta_n\to\Delta$. Clearly the pair $(\check{\mathrm{C}},(i_n\circ\mathcal{G}_n)^*)$ is a Quillen adjoint pair, and it is also clear that $\mathrm{colim}_{\mathrm{P}_{n+1}^{\mathrm{op}}}X$ and $\mathrm{colim}_{\Delta^{\mathrm{op}}}\check{\mathrm{C}}X$ are naturally isomorphic. Part (1) follows from these observations. Part (2) is a special case of [21, Theorem 5.1.3]. □

2.3 Goodwillie–Weiss embedding calculus and Sinha's cosimplicial model

In this subsection, we give the definition of the cosimplicial space $\mathcal{C}^\bullet(M)$ modeling $\mathrm{Emb}(S^1,M)$, and state its property. This is a minor variation of the model given in [37]. In [37], models of a space of embeddings from the interval $[0,1]$ to a manifold with some endpoint condition, while we consider embeddings $S^1\to M$ without any basepoint condition. The difference which needs care is that the homotopy limit of our cosimplicial model on the subcategory Δ_n need not to be weak homotopy equivalent to the n^{th} stage of the corresponding Taylor tower, while Sinha's original one is. At the ∞–stage, they are equivalent, which is sufficient for our purpose. We begin with an analogue of the punctured knot model in [37, Definition 3.4], which is an intermediate object between $\mathrm{Emb}(S^1,M)$ and $\mathcal{C}^\bullet(M)$.

Definition 2.3 • Let $S^1=[0,1]/0\sim 1$ and $J_i\subset S^1$ be the image of the interval $(1-1/2^i-1/10^i,1-1/2^i)$ by the quotient map $[0,1]\to S^1$.

- We fix an embedding $M \to \mathbb{R}^{N+1}$ for sufficiently large N. We endow M with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^{N+1}$ via this embedding. Let SM denote the total space of the unit tangent sphere bundle of M.
- For a subset $S \subset \underline{n+1}$, let $E_S(M)$ be the space of embeddings $S^1 - \bigcup_{i \in S} J_i \to M$ of constant speed.
- Define a functor $\mathcal{E}_n(M) \colon \mathrm{P}_{n+1} \to \mathcal{CG}$ by assigning to a subset S the space $E_S(M)$, and set

$$P_n \operatorname{Emb}(S^1, M) := \operatorname*{holim}_{\mathrm{P}_{n+1}} \mathcal{E}_n(M).$$

Let $\alpha_n \colon \operatorname{Emb}(S^1, M) \to P_n \operatorname{Emb}(S^1, M)$ be the map induced by restriction of the domain. The category P_n is regarded as a subcategory of P_{n+1} via the standard inclusion $\underline{n} \to \underline{n+1}$. By our choice of J_i, we have a canonical restriction map $r_n \colon P_n \operatorname{Emb}(S^1, M) \to P_{n-1} \operatorname{Emb}(S^1, M)$. The maps α_n induce a map

$$\alpha_\infty \colon \operatorname{Emb}(S^1, M) \to \operatorname*{holim}_n P_n \operatorname{Emb}(S^1, M),$$

where the right side is the homotopy limit of the tower $\cdots \xrightarrow{r_{n+1}} P_n \operatorname{Emb}(S^1, M) \xrightarrow{r_n} P_n \operatorname{Emb}(S^1, M) \xrightarrow{r_{n-1}} \cdots \xrightarrow{r_2} P_1 \operatorname{Emb}(S^1, M)$.

Remark 2.4 Our choice of J_i is different from [37], since we adopt the reverse labeling of coface and codegeneracy maps of the cosimplicial model to [37], for the author's preference. This does not cause any new problem.

Lemma 2.5 *Suppose $d \geq 4$. The map $\alpha_n \colon \operatorname{Emb}(S^1, M) \to P_n \operatorname{Emb}(S^1, M)$ is $(n-1)(d-3)$–connected. In particular, α_∞ is a weak homotopy equivalence.*

Proof Let $p \colon \operatorname{Emb}(S^1, M) \to SM$ be the evaluation of value and tangent vector at $0 \in S^1$. As is well known, p is a fibration. Let D be a closed subset on M diffeomorphic to a closed d–dimensional disk. Let $\operatorname{Emb}([0,1], M - \operatorname{Int}(D))$ be the space of embeddings $[0,1] \to M - \operatorname{Int}(D)$ whose value and tangent vector at endpoints are a fixed value in ∂D and vector. If we take a point of SM, for some choice of the disk D, fixed endpoints and embedded path between the points in D, we have the inclusion from $\operatorname{Emb}([0,1], M - \operatorname{Int}(D))$ to the fiber of p at the point. This inclusion is a weak homotopy equivalence. Its homotopy inverse is given by shrinking the disk D to the point. Thus, we have a homotopy fiber sequence

$$\operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \to \operatorname{Emb}(S^1, M) \to SM.$$

Restricting the domain, we have a similar fiber sequence $E_S(M - \operatorname{Int}(D)) \to E_S(M) \to SM$, where the left-hand side is the space defined in [37, Definition.3.1] with the obvious modification for J_i. (In [37], M denotes a manifold with boundary, so we apply the definitions to $M - \operatorname{Int}(D)$ instead of our closed M.) Passing to homotopy limits, we have the diagram

$$\begin{tikzcd}
\operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \arrow[r] \arrow[d] & \operatorname{Emb}(S^1, M) \arrow[r] \arrow[d, "\alpha_n"] & SM \arrow[d, "\mathrm{id}"] \\
P_n \operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \arrow[r] & P_n \operatorname{Emb}(S^1, M) \arrow[r] & SM
\end{tikzcd}$$

where both horizontal sequence are homotopy fiber sequences and the left bottom corner is the punctured knot model in [37, Definition.3.4] (with the obvious modification for J_i). As in [37, Theorem.3.5], by theorems of Goodwillie, Klein, and Weiss, the left vertical arrow is $(n-1)(d-3)$–connected, and so is the middle. □

Remark 2.6 Let $T_n \operatorname{Emb}(S^1, M)$ be the n^{th} stage of the Taylor tower (or polynomial approximation). Restriction of the domain induces a map $P_n \operatorname{Emb}(S^1, M) \to T_n \operatorname{Emb}(S^1, M)$ which is compatible with canonical maps from $\operatorname{Emb}(S^1, M)$, but the author does not know whether this map is a weak homotopy equivalence.

Our cosimplicial space is analogous to the well-known cosimplicial model of a free loop space, just like Sinha's original space is analogous to that of a based loop space. So the space $\mathcal{C}^n(M)$ is related to a configuration space of $n+1$ points (not n points).

Definition 2.7 Let $\|-\|$ denote the standard Euclidean norm in $\mathbb{R}^{N+1}$.

- Let $C_n(M) = \{(x_0, \dots, x_{n-1}) \in M^n \mid x_k \neq x_l \text{ if } k \neq l\}$ be the ordered configuration space of n points in M. Similarly, we set $C_2([n]) = \{(k,l) \in [n]^{\times 2} \mid k \neq= l\}$.

- Let $\overline{C}_n(M)$ be the closure of the image of the map

$$C_n(M) \to M^n \times (S^N)^{\times C_2([n-1])}, \quad (x_k)_k \mapsto (x_k, u_{kl})_{kl},$$

where $u_{kl} = (x_l - x_k)/\|x_l - x_k\|$. $\overline{C}_n(M)$ is the same as the space in Definition 4.1(6) of [37], though our labeling of points begins with 0. Define a space $\mathcal{C}^n(M)$ by the following pullback diagram:

$$\begin{tikzcd} \mathcal{C}^n(M) \arrow[r] \arrow[d] & SM^{n+1} \arrow[d] \\ \overline{C}_{n+1}(M) \arrow[r] & M^{n+1} \end{tikzcd}$$

Here the right vertical arrow is the product of standard projection and the bottom horizontal one is the composition of the canonical inclusion $\overline{C}_{n+1}(M) \to M^{\times n+1} \times (S^N)^{\times C_2([n])}$ and the projection.

- Let $\tau \colon T_x M \to \mathbb{R}^{N+1}$ be the linear monomorphism from the tangent space induced by the differential of the embedding fixed in Definition 2.3 and the identification $T_x\mathbb{R}^{N+1} \cong \mathbb{R}^{N+1}$ by the standard basis. Set $A'_{n+1}(M) := M^{\times n+1} \times (S^N)^{\times([n]^{\times 2})}$. Let $\beta'_{n+1} \colon \mathcal{C}^n(M) \to A'_{n+1}(M)$ be the map given by

$$\beta'_{n+1}(x_k, u_{kl}, y_k) = (x_k, u'_{kl}) \qquad \text{and} \qquad u'_{kl} = \begin{cases} u_{kl} & \text{if } k \neq l, \\ \tau(y_k) & \text{if } k = l, \end{cases}$$

where y_k is a unit tangent vector at x_k. This is clearly a monomorphism. For an integer i with $0 \leq i \leq n+1$, we define a map $d_i \colon [n+1] \to [n]$ by

$$d_i(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases} \quad \text{for } 0 \leq i \leq n \qquad \text{and} \qquad d_{n+1} = d_0 \circ \sigma,$$

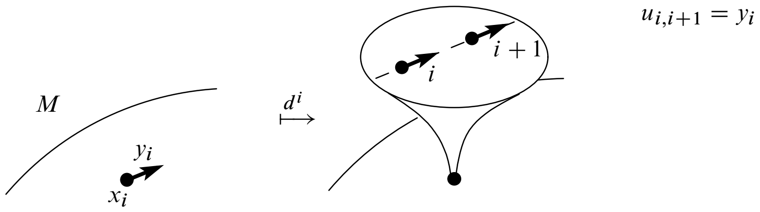

Figure 1: Intuition of the coface map d^i. Here y_i is the vector at x_i.

where σ is the cyclic permutation $\sigma(k) = k+1 \pmod{n+2}$. (This d_i is the same as s^i in Section 2.1, but we use the different notation to avoid confusion.) We define a map $d^i : A'_{n+1}(M) \to A'_{n+2}(M)$ by

$$d^i(x_k, u_{kl})_{0\le k,l\le n} = (x_{f(k)}, u_{f(k),f(l)})_{0\le k,l\le n+1} \quad \text{with } f = d_i.$$

This map restricts to the map $d^i : \mathcal{C}^n(M) \to \mathcal{C}^{n+1}(M)$ via $\beta'_{n+1}, \beta'_{n+2}$. Similarly, we define a map $s^i : \mathcal{C}^n(M) \to \mathcal{C}^{n-1}(M)$ for $0 \le i \le n-1$ as the pullback by the map

$$s_i : [n-1] \to [n], \qquad s_i(k) = \begin{cases} k & \text{if } k \le i, \\ k+1 & \text{if } k > i. \end{cases}$$

The collection $\mathcal{C}^\bullet(M) = \{\mathcal{C}^n(M), d^i, s^i\}$ forms a cosimplicial space. Well-definedness of this is verified in Lemma 2.8.

- We call the Bousfield–Kan type cohomology spectral sequence associated to $\mathcal{C}^\bullet(M)$ the *Sinha spectral sequence for M*, in short, the *Sinha s.s.*, and denote it by $\{\mathbb{E}_r\}_r$.

Intuitively, an element of $\overline{C}_n(M)$ is a configuration of n points in M, some points of which are allowed to collide, or in other words, to be infinitesimally close, and the direction of collision is recorded as the unit vector u_{kl} if the k^{th} and l^{th} points collide. An element of $\mathcal{C}^n(M)$ is an element of $\overline{C}_{n+1}(M)$, each point of which has a unit tangent vector. For $0 \le i \le n$, the map d^i replaces the i^{th} point in a configuration with the two points colliding at the point along its vector. These points are labeled by i and $i+1$. Their vectors are copies of the original vector (see Figure 1). The map d^{n+1} replaces the 0^{th} points with two points similarly, and labels them by $n+1$ and 0 (and slides other labels). The map s^i forgets the $(i+1)^{\text{th}}$ point and vector.

Lemma 2.8 (1) *The map $C_n(M) \to M^n \times (S^N)^{\times C_2([n-1])}$ given in Definition 2.7 restricts to a homotopy equivalence $C_n(M) \to \overline{C}_n(M)$.*

(2) *The cosimplicial space $\mathcal{C}^\bullet(M)$ is well defined.*

Proof Part (1) is proved in [35, Corollary 4.5 and Theorem. 5.10]. For (2), by [35, Proposition 6.6] the image of d^i and s^i is contained in $\mathcal{C}^{n\pm1}(M)$ — $C'_n\langle[M]\rangle$ in the proposition is the same as $\mathcal{C}^{n-1}(M)$ in our notation. Confirmation of the cosimplicial identities is routine work. For example, to confirm $d^{n+2}d^i = d^i d^{n+1} : \mathcal{C}^n(M) \to \mathcal{C}^{n+2}(M)$ for $i < n+2$, it is enough to confirm the dual identity

$d_i d_{n+2} = d_{n+1} d_i : [n+2] \to [n]$. Both sides are equal to the map

$$k \mapsto \begin{cases} k & \text{if } k \le i, \\ k-1 & \text{if } i < k < n+2, \\ 0 & \text{if } k = n+2, \end{cases} \quad \text{if } i < n+1, \qquad k \mapsto \begin{cases} k & \text{if } k \le n, \\ 0 & \text{if } k = n+1, n+2, \end{cases} \quad \text{if } i = n+1.$$

□

Lemma 2.9 *Let $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ be the composition functor $\mathrm{P}_{n+1} \xrightarrow{\mathcal{G}_n} \Delta_n \xrightarrow{\mathcal{C}^\bullet(M)} \mathcal{CG}$.*

1. *The homotopy limits of $\mathcal{E}_n(M)$ and $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $\underline{n} \to \underline{n+1}$.*
2. *The homotopy limit of $\mathcal{C}^\bullet(M)$ over Δ_n and that of $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ over P_{n+1} are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $\underline{n} \to \underline{n+1}$.*
3. *If $d \ge 4$, the homotopy limit of $\mathcal{C}^\bullet(M)$ over Δ and $\mathrm{Emb}(S^1, M)$ are connected by a zigzag of weak homotopy equivalences.*

Proof The proof of (1) is completely analogous to the proof of [37, Lemma 5.19] so we omit details. The idea of the proof is to consider the two space $\mathcal{C}^{\#S-1}(M)$ and $E_S(M)$ as subspaces of a common space, where one can "shrink components of embeddings until they become tangent vectors", as in [37, Definition 5.14]. The space is a subspace of the space of compact subspaces of $\mathcal{C}^{\#S-1}(M)$ with the Hausdorff metric. This space and the inclusions can be chosen to be compatible with maps in P_{n+1}. For example, the restriction $E_S(M) \to E_{S'}(M)$ corresponding to the inclusion $S = \underline{n+1} \subset S' = \underline{n+2}$ divides the component including the image of $0 \in S^1$ into two components, since the image of J_{n+2} is removed. At the limit of shrinking components, this is consistent with the coface map d^{n+1}. These inclusions to the common space give rise to a zigzag of natural transformations which is a weak homotopy equivalence at each set $S \subset \underline{n+1}$. This induces the claimed zigzag. Part (2) follows from the fact that the functor $\mathcal{G}_n$ is left cofinal; see Theorem 6.7 of [37]. Part (3) follows from (1), (2) and Lemma 2.5. □

2.4 Operads, comodules and the Hochschild complex

The term *operad* means *nonsymmetric* (or *non–Σ*) operad; see [24; 31]. An operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n \ge 1}$ in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is a sequence of objects equipped with maps

$$(- \circ_i -) : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m+n-1) \quad \text{for } 1 \le i \le m$$

in $\mathcal{C}$, called *partial compositions*, which are subject to certain conditions. $\mathcal{O}(n)$ is called the object at *arity* n. More precisely, our notion of an operad is different from the one in [24; 31] only in that we do not consider the object at arity 0, so conditions on partial compositions given in [24; 31] are imposed only in the ranges of all involved arities being 1 or more. We mainly consider operads in $\mathcal{CG}$ (resp. in $\mathcal{CH}_{\mathsf{k}}$), which are called topological operads (resp. chain operads), where the monoidal product is the standard cartesian product (resp. tensor product). Let $\mathcal{O}$ be a topological operad. $C_*(\mathcal{O})$ denotes the chain operad given by $C_*(\mathcal{O})(n) = C_*(\mathcal{O}(n))$ with the induced structure. We equip the sequence $\{\mathcal{O}(n) \hat{\otimes} \mathbb{S}\}_n$ of spectra with a structure of an operad in $\mathcal{SP}$ as follows. The i^{th} partial composition is given by

$$(\mathcal{O}(m)\hat{\otimes}\mathbb{S})\wedge(\mathcal{O}(n)\hat{\otimes}\mathbb{S})\cong(\mathcal{O}(m)\times\mathcal{O}(n))\hat{\otimes}(\mathbb{S}\wedge\mathbb{S})\cong(\mathcal{O}(m)\times\mathcal{O}(n))\hat{\otimes}\mathbb{S}\xrightarrow{(-\circ_i-)\hat{\otimes}\mathrm{id}}\mathcal{O}(m+n-1)\hat{\otimes}\mathbb{S}.$$

See Section 2.1 for the isomorphisms. The action of Σ_n is the naturally induced action. We denote this operad by the same symbol, $\mathcal{O}$. We let $\mathcal{A}$ denote both of the (discrete) topological and k–linear versions of the associative operad by abuse of notation. For the k–linear version, we fix a generator $\mu\in\mathcal{A}(2)$ throughout this paper. $\mathcal{K}$ denotes the Stasheff associahedral operad, and $\mathcal{A}_\infty$ the cellular chain operad of $\mathcal{K}$. Precisely speaking, $\mathcal{A}_\infty$ is generated by a set $\{\mu_k\in\mathcal{A}_\infty(k)\}_{k\geq 2}$ with $|\mu_k|=-k+2$, with partial compositions. The differential is given by the formula

$$d\mu_k=\sum_{\substack{l,p,q\\ l+q=k+1}}(-1)^\zeta\mu_l\circ_{p+1}\mu_q,$$

where $\zeta=\zeta(l,p,q)=p+q(l-p-1)$.

In the following definition, we adopt the point–set description, as if a category $\mathcal{C}$ were the category of sets, for simplicity.

Definition 2.10 • Let $\mathcal{O}$ be an operad over a symmetric monoidal category $\mathcal{C}$. A (*left*) *$\mathcal{O}$–comodule in $\mathcal{C}$* is a symmetric sequence $X=\{X(n)\}_{n\geq 1}$ in $\mathcal{C}$ equipped with maps

$$(-\circ_i -)\colon \mathcal{O}(m)\otimes X(m+n-1)\to X(n)\in\mathcal{C}$$

for $m\geq 1$, $n\geq 1$ and $1\leq i\leq n$, called *partial compositions*, which satisfy the following conditions:

(1) For $a\in\mathcal{O}(m)$, $b\in\mathcal{O}(l)$ and $x\in X(l+m+n-2)$,

$$a\circ_i(b\circ_j x)=\begin{cases} b\circ_j(a\circ_{i+l-1}x) & \text{if } j<i,\\ (a\circ_{j-i+1}b)\circ_i x & \text{if } i\leq j\leq i+m-1,\\ b\circ_{j-m+1}(a\circ_i x) & \text{if } i+m-1<j.\end{cases}$$

(2) For the unit $1\in\mathcal{O}(1)$ and $x\in X(n)$, we have $1\circ_i x=x$.

(3) For $a\in\mathcal{O}(m)$, $x\in X(m+n-1)$ and $\sigma\in\Sigma_n$,

$$(a\circ_i x)^\sigma=a\circ_{\sigma^{-1}(i)}(x^{\sigma_1}),$$

where $\sigma_1\in\Sigma_{m+n-1}$ is the permutation induced by σ, replacing the letter $\sigma^{-1}(i)$ with the m letters $\sigma^{-1}(i),\dots,\sigma^{-1}(i)+m-1$. In other words,

$$\sigma_1(k)=\begin{cases}\sigma(k) & \text{if } k<\sigma^{-1}(i) \text{ and } \sigma(k)<i,\\ \sigma(k)+m-1 & \text{if } k<\sigma^{-1}(i) \text{ and } \sigma(k)>i,\\ i+k-\sigma^{-1}(i) & \text{if } \sigma^{-1}(i)\leq k\leq\sigma^{-1}(i)+m-1,\\ \sigma(k-m+1) & \text{if } k>\sigma^{-1}(i)+m-1 \text{ and } \sigma(k-m+1)<i,\\ \sigma(k-m+1)+m-1 & \text{if } k>\sigma^{-1}(i)+m-1 \text{ and } \sigma(k-m+1)>i.\end{cases}$$

A *map $f\colon X_1\to X_2$ of $\mathcal{O}$–comodules* is a sequence of maps in $\mathcal{C}$ $\{f_n\colon X_1(n)\to X_2(n)\}_n$ which is compatible with the actions of symmetric groups and the partial compositions.

- A *(right) $\mathcal{O}$–module in $\mathcal{C}$* is a symmetric sequence $Y=\{Y(n)\}_{n\geq 1}$ equipped with a set of partial compositions $Y(n)\otimes\mathcal{O}(m)\to Y(m+n-1)$ which satisfy the following conditions:

  (1) For $a\in\mathcal{O}(m)$, $b\in\mathcal{O}(l)$ and $y\in y(n)$,
$$(y\circ_j a)\circ_i b=\begin{cases}(y\circ_i b)\circ_{j+l-1} a & \text{if } i<j,\\ y\circ_j(a\circ_{i-j+1} b) & \text{if } j\leq i\leq j+m-1,\\ (y\circ_{i+m-1} b)\circ_j a & \text{if } i>j+m-1.\end{cases}$$

  (2) For the unit $1\in\mathcal{O}(1)$ and $y\in X(n)$, we have $y\circ_i 1=y$.

  (3) For $a\in\mathcal{O}(m)$, $y\in X(n)$ and $\sigma\in\Sigma_n$,
$$y^{\sigma}\circ_i a=(y\circ_{\sigma(i)} a)^{\sigma_2},$$
  where $\sigma_2\in\Sigma_{m+n-1}$ is the permutation induced by σ, replacing the letter i with the m letters $i,\dots,i+m-1$. In other words,
$$\sigma_2(k)=\begin{cases}\sigma(k) & \text{if } k<i \text{ and } \sigma(k)<\sigma(i),\\ \sigma(k)+m-1 & \text{if } k<i \text{ and } \sigma(k)>\sigma(i),\\ \sigma(i)+k-i & \text{if } i\leq k\leq i+m-1,\\ \sigma(k-m+1) & \text{if } k>i+m-1 \text{ and } \sigma(k-m+1)<\sigma(i),\\ \sigma(k-m+1)+m-1 & \text{if } k>i+m-1 \text{ and } \sigma(k-m+1)>\sigma(i).\end{cases}$$

A *map of modules* is defined similarly to that of comodules.

- For a topological operad $\mathcal{O}$ (regarded as an operad in $\mathcal{SP}$), an *$\mathcal{O}$–comodule of NCRS* is an $\mathcal{O}$–comodule X in $\mathcal{SP}$ such that each $X(n)$ is equipped with a structure of an NCRS and the action of Σ_n on $X(n)$ and the partial composition $(a\circ_i -)\colon X(n+m-1)\to X(n)$ is a map of NCRS for each $a\in\mathcal{O}(m)$. A *map of comodules of NCRS* is a map of comodules which is also a map of NCRS at each arity.

- For a topological operad $\mathcal{O}$ and an $\mathcal{O}$–module Y, we define an $\mathcal{O}$–comodule $Y^{\vee}$ of NCRS as follows:

  (1) We set $Y^{\vee}(n)=Y(n)^{\vee}$ (see Section 2.1).

  (2) For $f\in Y^{\vee}(n)$ and $\sigma\in\Sigma_n$, we define an action f^{σ} by $f^{\sigma}(y)=f(y^{\sigma^{-1}})$ for each $y\in Y(n)$.

  (3) For $a\in\mathcal{O}(m)$ and $f\in Y^{\vee}(m+n-1)$, we define a partial composition $a\circ_i f$ by $a\circ_i f(y)=f(y\circ_i a)$ for each $y\in Y(n)$.

  (4) We define a multiplication $Y^{\vee}(n)\wedge Y^{\vee}(n)\to Y^{\vee}(n)$ as the pushforward by the multiplication of $\mathbb{S}$. (This is actually unital.)

This construction is natural for maps of $\mathcal{O}$–modules.

- An *$\mathcal{A}$–comodule X of CDBA* is an $\mathcal{A}$–comodule (in $\mathcal{CH}_{\mathsf{k}}$) such that each $X(n)$ is a CDBA, and the partial composition $\mu\circ_i(-)\colon X(n)\to X(n-1)$ — with the fixed generator $\mu\in\mathcal{A}(2)$ — and action of $\sigma\in\Sigma_n$ preserve the differential, bigrading, multiplication and unit.

The axioms for the partial compositions of modules (Definition 2.10) are the standard ones, which are naturally interpreted in terms of concatenation of trees. The action of $\sigma\in\Sigma_n$ is interpreted as replacement of labels i on leaves with labels $\sigma^{-1}(i)$, and the axiom is the natural one with this interpretation. The axioms for a comodule are simply dual to those for a module. The comodule in Example 2.14 may give some intuition for it.

Remark 2.11 The notion of a right module in Definition 2.10 is similar to the one in [26]. A right $\mathcal{O}$–module is also essentially the same as a topological contravariant functor from the PROP of $\Sigma\mathcal{O}$ to spaces (or spectra), and a left $\mathcal{O}$–comodule is a covariant functor. Here $\Sigma\mathcal{O}$ is the standard symmetrization of $\mathcal{O}$, ie $\Sigma\mathcal{O}(n) = \mathcal{O}(n) \times \Sigma_n$; see [29].

Composing the unity and associativity isomorphisms, we get a natural isomorphism $K \mathbin{\hat{\otimes}} X \cong (K \mathbin{\hat{\otimes}} \mathbb{S}) \wedge X$ in $\mathcal{SP}$. Let $\mathcal{O}$ be a topological operad. Via this isomorphism, a structure of an $\mathcal{O}$–comodule in $\mathcal{SP}$ on a symmetric sequence X is equivalent to a set of maps

$$\mathcal{O}(m) \mathbin{\hat{\otimes}} X(m+n-1) \to X(n)$$

which satisfy conditions completely similar to those given in Definition 2.10. We also call these maps partial compositions, and henceforth will define comodules in $\mathcal{SP}$ with these maps.

Remark 2.12 Precisely speaking, comodules in Definition 2.10 should be called contracomodules, because our comodules are to modules as contramodules are to comodules in [32], but for simplicity we adopt our terminology.

The following definition is essentially due to [16], though we adopt a different sign rule.

Definition 2.13 Let X^* be an $\mathcal{A}_\infty$–comodule in $\mathcal{CH}_{\mathsf{k}}$. We define a chain complex $(\mathrm{CH}_\bullet\, X^*, \tilde{d})$, called the *Hochschild complex of* X, as follows. Set $\mathrm{CH}_n\, X^* = X^*(n+1)$. By our convention, the total degree is $*-\bullet$. The differential $\tilde{d}$ is given as a map

$$\tilde{d} = d - \delta\colon \bigoplus_{a-n=k} \mathrm{CH}_n\, X^a \to \bigoplus_{a-n=k+1} \mathrm{CH}_n\, X^a.$$

Here d is the internal (original) differential on $X^a(n+1)$ and δ is given by

$$\delta(x) = \sum_{i=0}^{n} \sum_{k=2}^{n-i+1} (-1)^{\epsilon} \mu_k \circ_{i+1} x + \sum_{s=1}^{n} \sum_{k=s+1}^{n+1} (-1)^{\theta} \mu_k \circ_1 x^s$$

for $x \in X^a(n+1)$, where $\epsilon = \epsilon(a,i,k) = (a+i)(k+1)$, $\theta = \theta(s,n,k,a) = sn + (k+1)a$ and x^s denotes the image of x by the action of the permutation in Σ_{n+1} which transposes the first $n-s+1$ letters and the last s letters.

The following example gives some intuition for the definitions of a comodule and the Hochschild complex, but is not used later.

Example 2.14 Let $\mathcal{C}$ be the category of k–modules and A be a k–algebra. Let $m_n \in \mathcal{A}(n)$ be the element defined by successive partial compositions of the generator $\mu \in \mathcal{A}(2)$. Define an $\mathcal{A}$–comodule X_A by

$$X_A(n) = A^{\otimes n}, \quad m_k \circ_i (x_1 \otimes \cdots \otimes x_{k+n-1}) = x_1 \otimes \cdots \otimes x_{i-1} \otimes (x_i \cdots x_{i+k-1}) \otimes x_{i+k} \otimes \cdots \otimes x_{k+n-1},$$

where $x_i \cdots x_{i+k-1}$ is the product in A. We regard X_A as an $\mathcal{A}_\infty$–comodule via a map $\mathcal{A}_\infty \to \mathcal{A}$ of operads. The Hochschild complex of X_A is the usual (unnormalized) Hochschild complex of the associative algebra A.

Lemma 2.15 *With the notation of Definition 2.13*, $(\tilde{d})^2 = 0$.

Proof Roughly,

$$\begin{aligned}
(\tilde{d})^2(x) &= \tilde{d}(dx - \delta x) = ddx - d\delta x - \delta dx - \delta\delta x \\
&= d(\mu_k \circ_{i+1} x + \mu_k \circ_1 x^s) + (\mu_k \circ_{i+1} dx + \mu_k \circ_1 dx^s) \\
&\qquad - \mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_l \circ (\mu_k \circ_1 x^s) + \mu_l \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_l \circ_1 (\mu_k \circ_1 x^s)^t \\
&= (d\mu_k) \circ_{i+1} x + (d\mu_k) \circ_1 x^s \\
&\qquad - \mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_l \circ (\mu_k \circ_1 x^s) + \mu_l \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_l \circ_1 (\mu_k \circ_1 x^s)^t.
\end{aligned}$$

(Here we already canceled the terms containing dx, since the cancellation of signs is obvious.) So we have six types of terms. To see which terms cancel with each other, we divide these terms into the following smaller classes:

(1) $(d\mu_k) \circ_{i+1} x$, $d\mu_k = \sum \mu_l \circ_{p+1} \mu_q$,

(2) $(d\mu_k) \circ_1 x^s$, $d\mu_k = \sum \mu_l \circ_{p+1} \mu_q$:
   - (a) $s < p+1$,
   - (b) $p + q \le s$,
   - (c) $p = 0$ and $q > s$,
   - (d) $p > 0$ and $p + q > s \ge p + 1$,

(3) $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x)$:
   - (a) $i < j$,
   - (b) $j + l - 1 < i$,
   - (c) $j \le i \le j + l - 1$,

(4) $\mu_l \circ_{j+1} (\mu_k \circ_1 x^s)$:
   - (a) $j = 0$,
   - (b) $j > 0$,

(5) $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t$:
   - (a) $i + 1 < n - k - t + 3$ and $l < s + i + 1$,
   - (b) $i + 1 < n - k - t + 3$ and $l \ge s + i + 1$,
   - (c) $i + 1 \ge n - k - t + 3$,

(6) $\mu_l \circ_1 (\mu_k \circ_1 x^s)^t$.

Now we claim that the terms in (1) cancel with the terms in (3c), (2a) with (5b), (2b) with (5c), (2c) with (4a), (2d) with (6), (3a) with (3b) and (4b) with (5a).

We shall verify the first and third parts of the claim. Other verification is similar and omitted. For the first one, the coefficient of a term $(\mu_l \circ_{p+1} \mu_q) \circ_{i+1} x$ in (1) is $(-1)^{\alpha_1}$, where

$$\alpha_1 = \zeta(l, p, q) + \epsilon(a, i, l+q+1) + 1.$$

For a term in (3-c), by the rules of the partial composition, $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_l \circ_{i-j+1} \mu_k) \circ_{j+1} x$. In order to match this term with a term in (1), we set $q' = k,\ p' + 1 = i - j + 1$ and $i' + 1 = j + 1$. This change of subscripts implies $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_l \circ_{p'+1} \mu_{q'}) \circ_{i'+1} x$. Clearly $j = i'$ and $i = p' + i'$. The coefficient of $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x)$ in (3-c) is $(-1)^{\alpha_2}$, where

$$\alpha_2 = \epsilon(a, i, k) + 1 + \epsilon(a + k - 2, j, l) + 1 = \epsilon(a, p' + i', q') + \epsilon(a - q' + 2, i', l) + 2.$$

When we substitute $q' = q,\ p' = p$ and $i' = i$ in the last expression, elementary computation shows $\alpha_1 + \alpha_2 \equiv 1 \pmod 2$. Thus the terms in (1) cancel with the terms in (3-c).

For the third part, the coefficient of a term $(\mu_l \circ_{p+1} \mu_q) \circ_1 x^s$ in (2-b) is $(-1)^{\beta_1}$, where

$$\beta_1 = \zeta(l, p, q) + \theta(s, n, l + q - 1, a) + 1.$$

For a term in (5-c), the condition $i + 1 \geq n - k - t + 3$ implies that μ_k acts on a part of the last t letters. By this, and the rule of the partial composition, we have

$$\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t = \mu_l \circ_1 (\mu_k \circ_{i-n+k+t-1} (x^{t+k-1})) = (\mu_l \circ_{i-n+k+t-1} \mu_k) \circ_1 x^{t+k-1}.$$

In order to match this term with a term in (2-b), we set $p' + 1 = i - n + k + t - 1$, $q' = k$ and $s' = t + k - 1$. This change of subscripts implies $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t = (\mu_l \circ_{p'+1} \mu_{q'}) \circ_1 x^{s'}$. Clearly $t = s' - q' + 1$ and $i = p' + n - s' + 1$. The coefficient of $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t$ is $(-1)^{\beta_2}$, where

$$\begin{aligned}\beta_2 &= \epsilon(a, i, k) + 1 + \theta(t, a - k + 2, n - k + 1, l) + 1 \\ &= \epsilon(a, p' + n - s' + 1, q') + \theta(s' - q' + 1, n - q' + 1, a - q' + 2, l) + 2.\end{aligned}$$

When we substitute $q' = q,\ p' = p$ and $s' = s$ in the last expression, elementary computation shows $\beta_1 + \beta_2 \equiv 1 \pmod 2$. Thus the terms in (2-b) cancel with the terms in (5-c). □

3 The comodule $\mathcal{T}_M$

The purpose of this section is to define the comodule $\mathcal{T}_M$.

3.1 A model of a Thom spectrum

We introduce a model of a Thom spectrum in the category of symmetric spectra. This model is essentially due to Cohen [11], and is slightly different from Cohen's original nonunital model, mainly in that we use expanding embeddings.

Definition 3.1 Let N be a closed manifold. We fix a Riemannian metric on N and denote by $d_N(-,-)$ the distance on N induced by the metric. The standard Euclidean norm on $\mathbb{R}^k$ is denoted by $\|-\|$. The distance in $\mathbb{R}^k$ is induced by $\|-\|$.

- For a smooth embedding $e\colon N \to L$ to a Riemannian manifold L, we set a number

$$r(e) = \inf\left\{ \frac{d_L(e(x), e(y))}{d_N(x, y)} \;\middle|\; x, y \in N \text{ with } x \neq y \right\}.$$

It is easy to see $r(e) > 0$. We say e is *expanding* if the inequality $r(e) \geq 1$ holds. $\mathrm{Emb}^{\mathrm{ex}}(N, L)$ denotes the space of all expanding embeddings from N to L with the topology induced by the C^∞–topology.

- For a smooth embedding $e\colon N \to \mathbb{R}^k$, we define a number $|e|$ by

$$|e| = \sum_{i=1}^{k} \max\{|e^i(y)| \mid y \in N\},$$

where $e^i : N \to \mathbb{R}$ is the i^{th} component of e and $|-|$ is the absolute value.

- Let $e\colon N \to \mathbb{R}^k$ be a smooth embedding. For $\epsilon > 0$, we denote by $\nu_\epsilon(e)$ the open subset of $\mathbb{R}^k$ consisting of the points whose Euclidean distance from $e(N)$ is smaller than ϵ. Let $L(e)$ denote the minimum of 1 and the least upper bound of $\epsilon > 0$ such that there exists a retraction $\pi_e\colon \nu_\epsilon(e) \to e(N)$ satisfying the following conditions:

  - For any $u \in \nu_\epsilon(e)$ and any $y \in N$ we have $\|\pi_e(u) - u\| \leq \|e(y) - u\|$, and equality holds if and only if $\pi_e(u) = e(y)$.
  - For any $y \in N$ we have $\pi_e^{-1}(\{e(y)\}) = B_\epsilon(e(y)) \cap (e(y) + (T_y N)^\perp)$. Here $B_\epsilon(e(y))$ is the open ball with center $e(y)$ and radius ϵ.
  - The closure $\bar{\nu}_\epsilon(e)$ of $\nu_\epsilon(e)$ is a smooth submanifold of $\mathbb{R}^k$ with boundary.

(Such a retraction exists for a sufficiently small $\epsilon > 0$ by a version of the tubular neighborhood theorem; see [27].) The retraction π_e satisfying the above conditions is unique. We regard the map $\pi_e\colon \nu_\epsilon(e) \to e(N)$ as a disk bundle over N, identifying N and $e(N)$ via e.

- Let $\widetilde{N}_k^{-\tau}$ be the subspace of $\mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \times \mathbb{R} \times \mathbb{R}^k$ consisting of the triples (e, ϵ, u) with $0 < \epsilon < L(e)$. Define a subspace $\partial \widetilde{N}_k^{-\tau} \subset \widetilde{N}_k^{-\tau}$ by $(e, \epsilon, u) \in \partial \widetilde{N}_k^{-\tau}$ if and only if $u \notin \nu_\epsilon(e)$. We put

$$N_k^{-\tau} = \widetilde{N}_k^{-\tau} / \partial \widetilde{N}_k^{-\tau}.$$

We define a structure of a symmetric spectrum on $N^{-\tau}$ as follows:

  - We let Σ_k act on $\mathbb{R}^k$ and $\mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k)$ by the standard permutation on components. The action of Σ_k on $N_k^{-\tau}$ is given by $[e, \epsilon, u]^\sigma = [e^\sigma, \epsilon, u^\sigma]$.
  - The map $S^1 \wedge N_k^{-\tau} \to N_{k+1}^{-\tau}$ is given by $t \wedge [e, \epsilon, u] \mapsto [0 \times e, \epsilon, (t, u)]$, where we regard S^1 as $\mathbb{R} \cup \{\infty\}$, and $0 \times e\colon M \to \mathbb{R}^{k+1}$ is given by $(0 \times e)(x) = (0, e(x))$.

• We shall define a structure of NCRS on $N^{-\tau}$. An element of $(N^{-\tau} \wedge N^{-\tau})_k$ is represented by data $\langle [e_1, \epsilon_1, u_1], [e_2, \epsilon_2, u_2]; \sigma\rangle$ consisting of $[e_i, \epsilon_i, u_i] \in N^{-\tau}_{k_i}$ for $i = 1, 2$ and $k_1 + k_2 = k$, and $\sigma \in \Sigma_k$. We define a commutative associative multiplication $\mu\colon N^{-\tau} \wedge N^{-\tau} \to N^{-\tau}$ by

$$\mu(\langle [e_1, \epsilon_1, u_1], [e_2, \epsilon_2, u_2]; \sigma\rangle) = [e_{12}, \epsilon_{12}, (u_1, u_2)]^{\sigma}.$$

Here we set $e_{12} = (e_1 \times e_2) \circ \Delta$, where $\Delta\colon N \to N \times N$ is the diagonal map, and set

$$\epsilon_{12} = \min\left\{ \frac{\epsilon_1}{8^{|e_2|}}, \frac{\epsilon_2}{8^{|e_1|}}, L(e_{12}), \frac{L(e'_1)}{8^{|e_{12}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{12}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1\colon N \to \mathbb{R}^{l_1}, \dots, e'_m\colon N \to \mathbb{R}^{l_m} \right\},$$

where the finite sequence $(e'_1, \dots, e'_m)$ runs through the sequences of expanding embeddings satisfying $(e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{12})^{\tau}$ for a permutation $\tau \in \Sigma_{k_1+k_2}$ and the diagonal map $\Delta^m\colon N \to N^m$.

Lemma 3.2 *The structure of NCRS on $N^{-\tau}$ given in Definition 3.1 is well defined*

Proof Most of the proof is the same as the proof of [11, Theorem 3]. We shall only verify the associativity of the number ϵ_{12}. Let $[e_i, \epsilon_i, u_i]$ be an element of $N^{-\tau}_{k_i}$ for $i = 1, 2, 3$. We denote by $\epsilon_{(12)3}$ (resp. $\epsilon_{1(23)}$) the number in the second entry of the product of the three elements where the elements labeled by $i = 1, 2$ (resp. $i = 2, 3$) are multiplied at first. By definition,

$$\epsilon_{(12)3} = \min\left\{ \frac{\epsilon_{12}}{8^{|e_3|}}, \frac{\epsilon_3}{8^{|e_{12}|}}, L(e_{123}), \frac{L(e'_1)}{8^{|e_{123}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{123}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1, \dots, e'_m \right\},$$

where $e_{123} = (e_1 \times e_2 \times e_3) \circ \Delta^3$, and the finite sequence $(e'_1, \dots, e'_m)$ runs through the sequences of expanding embeddings satisfying $(e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{123})^{\tau}$ for some $\tau \in \Sigma_{k_1+k_2+k_3}$. By the obvious equality $|e_{12}| = |e_1| + |e_2|$, we have

$$\epsilon_{(12)3} = \min\left\{ \frac{\epsilon_1}{8^{|e_2|+|e_3|}}, \frac{\epsilon_2}{8^{|e_1|+|e_3|}}, \frac{\epsilon_3}{8^{|e_1|+|e_2|}}, L(e_{123}), \frac{L(e'_1)}{8^{|e_{123}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{123}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1, \dots, e'_m \right\},$$

where the finite sequence $(e'_1, \dots, e'_m)$ runs through the same set as above. The number $\epsilon_{1(23)}$ is also seen to be equal to the value of the right-hand side. □

3.2 Construction of a comodule $\widetilde{\mathcal{T}}_M$

Definition 3.3 • For a closed interval $c = [a, b]$, we set $|c| = b - a$, and call the point $\frac{1}{2}(a + b) \in c$ the *center of* c.

• We define a version of the little interval operad, denoted by $\mathcal{D}$, as follows. For $n \geq 1$, let $\mathcal{D}(n)$ be the set of n–tuples $(c_1, c_2, \dots, c_n)$ of closed subintervals $c_i \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ such that $c_1 \cup \cdots \cup c_n = [-\frac{1}{2}, \frac{1}{2}]$ and $c_i \cap c_j$ is a one-point set, or empty if $i \neq j$, and the labeling of $1, \dots, n$ is consistent with the usual order of the real line $\mathbb{R}$ (so $-\frac{1}{2} \in c_1$ and $\frac{1}{2} \in c_n$). $\mathcal{D}(1)$ is understood as the one-point set consisting of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We topologize $\mathcal{D}(n)$ as a subspace of $\mathbb{R}^n$ by the inclusion sending each interval to its center. The partial composition is given in a way that is completely analogous to the usual little interval operad.

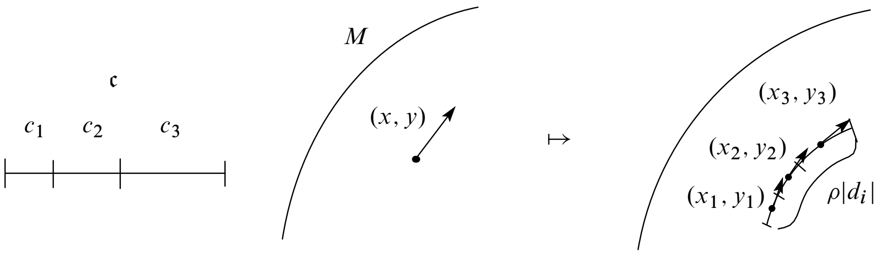

Figure 2: The map Δ'. The geodesic segment is divided into the pieces of rate of length $|c_1| : |c_2| : |c_3|$.

- We identify $H_0(\mathcal{D}(2))$ with $\mathcal{A}(2)$ by sending the generator represented by a topological point to the generator μ.

Recall that we fixed a Riemannian metric on M in Definition 2.3. Henceforth we equip the space SM with the Sasaki metric, and the product SM^n of n copies of SM with the product metric. We assume the maximum of the distance between two points in SM is larger than 1. This is clearly possible by modifying the embedding used in the definition of the metric on M. This assumption is used in the proof Lemma 3.11(2). We fix a positive number ρ small enough that a geodesic of length ρ exists for any initial value in M. After Lemma 3.7, we impose an additional assumption on ρ.

Definition 3.4 We define a map

$$\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i] \colon SM \to SM^m$$

for each $\mathfrak{d} = (d_1, \dots, d_n) \in \mathcal{D}(n)$, $\mathfrak{c} = (c_1, \dots, c_m) \in \mathcal{D}(m)$ and $1 \le i \le n$. Let (x, y) denote a point of SM with $x \in M$ and $y \in S_xM$, where S_xM denotes the fiber of the sphere bundle over x. Let $s \colon \left[-\frac{1}{2}\rho, \frac{1}{2}\rho\right] \to M$ denote the geodesic segment with length parameter such that $s(0) = x$ and the tangent vector of s at 0 is y. Let $t_j \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ be the center of c_j, put $x_j = s(\rho \cdot |d_i| \cdot t_j)$ and set y_j to be the tangent vector of s at $\rho \cdot |d_i| \cdot t_j$. We set $\Delta'(x, y) = ((x_1, y_1), \dots, (x_m, y_m))$; see Figure 2.

The following lemma is clear from the definition of $\Delta[\mathfrak{d}, \mathfrak{c}; i]$.

Lemma 3.5 *For any configurations $\mathfrak{d}$, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and numbers i and j, the following equality holds*:

$$\Delta[\mathfrak{d}, \mathfrak{c}_1 \circ_j \mathfrak{c}_2; i] = (1_{j-1} \times \Delta[\mathfrak{d} \circ_i \mathfrak{c}_1, \mathfrak{c}_2; i + j - 1] \times 1_{m-j}) \circ \Delta[\mathfrak{d}, \mathfrak{c}_1; i].$$

Here m is the arity of $\mathfrak{c}_1$, and 1_l is the identity on SM^l. □

Lemma 3.6 *For any sufficiently small positive number ρ, the map $\Delta[\mathfrak{d}, \mathfrak{c}; i]$ is expanding for any numbers $n \ge 1$, $m \ge 1$ and i with $1 \le i \le n$, and elements $\mathfrak{d} \in \mathcal{D}(n)$ and $\mathfrak{c} \in \mathcal{D}(m)$.*

Proof It is enough to prove the case of $m = 2$, since for $m \ge 3$, Δ' is equal to a successive composition of copies of Δ' of arity 2 by Lemma 3.5. We set $\rho_0 = |d_i|\rho$. We shall consider the case that M is a metric vector space V as a local model. Take points $(x, y), (v, w) \in \hat{V} = V \times SV$, where SV is the unit sphere in V. Put $\mathfrak{c} = (c_1, c_2)$. Let $-s$ and t be the centers of c_1 and c_2, respectively, with $0 < s, t < \frac{1}{2}$

and $s+t=\frac{1}{2}$. By definition, $\Delta'(x,y)=[(x-\rho_0 sy,y),(x+\rho_0 ty,y)]$. When we set $a=\|x-v\|$ and $b=\|y-w\|$, we easily see

$$\begin{aligned}\|\Delta'(x,y)-\Delta'(v,w)\|^2 &\geq 2a^2-\rho_0|s-t|ab+\bigl\{\tfrac{1}{4}\rho_0^2(s^2+t^2)+2\bigr\}b^2\\ &\geq 2a^2-\tfrac{1}{2}\rho_0|s-t|(a^2+b^2)+\bigl\{\tfrac{1}{4}\rho_0^2(s^2+t^2)+2\bigr\}b^2.\end{aligned}$$

So

$$\frac{\|\Delta'(x,y)-\Delta'(v,w)\|}{\|(x,y)-(v,w)\|}\geq\frac{\sqrt{7}}{2}\quad\text{for }\rho<1. \tag{3-1}$$

We shall consider the case of a general manifold M. There exists a number $r>0$ such that, for sufficiently small ρ, for any point $p\in M$ and any pair $(x,y),(v,w)\in T_pM\times ST_pM$ with $\|x\|,\|v\|\leq r$, we have the inequality

$$\frac{d(\Delta'_M(\exp x,\exp' y),\Delta'_M(\exp v,\exp' w))}{d(\Delta'_{T_pM}(x,y),\Delta'_{T_pM}(v,w))}>1-\tfrac{1}{100}, \tag{3-2}$$

where exp is the exponential map at p and $\exp'$ is its differential. Combining (3-1) and (3-2), for $(x,y),(v,w)\in SM$, we see $d_{SM^2}(\Delta'(x,y),\Delta'(v,w))>d_{SM}((x,y),(v,w))$ if $d_M(x,v)\leq r$. For the case of $d_M(x,v)>r$, if we take ρ sufficiently small relative to r, the following inequality holds:

$$\frac{d(\Delta'(x,y),\Delta'(v,w))}{d(\Delta(x,y),\Delta(v,w))}>1-\tfrac{1}{100}\quad\text{for }(x,y),(v,w)\in SM\text{ with }d(x,v)>r.$$

Here $\Delta\colon SM\to SM^{\times 2}$ is the usual diagonal. Then, if $d_M(x,v)>r$, we have the inequality

$$d(\Delta'(x,y),\Delta'(v,w))>\bigl(1-\tfrac{1}{100}\bigr)\sqrt{2}\,d((x,y),(v,w)).$$

Thus, we have shown the lemma. □

The following lemma is an exercise of Riemannian geometry:

Lemma 3.7 *For any sufficiently small positive number ρ, the following condition holds. For any $n\geq 2$, $G\in\mathsf{G}(n)$ and set of positive numbers $\{\epsilon_{ij}\mid i<j\text{ for }(i,j)\in E(G)\}$ satisfying $\sum_{(i,j)\in E(G)}\epsilon_{ij}<\rho$, the inclusion of subspaces of M^n*

$$\{(x_1,\dots,x_n)\mid\forall(i,j)\in E(G),x_i=x_j\}\to\{(x_1,\dots,x_n)\mid\forall(i,j)\in E(G),d(x_i,x_j)\leq\epsilon_{ij}\}$$

is a homotopy equivalence. □

Assumption In the rest of paper, we fix the number ρ so that Lemmas 3.6 and 3.7 hold.

We define a $\mathcal{D}$–comodule $\widetilde{\mathcal{T}}_M$ of NCRS. We set

$$SM^{-\tau}(n)=(SM^n)^{-\tau};$$

see Definition 3.1. We first define a subspectrum $\widetilde{\mathcal{T}}_M(\mathfrak{c})\subset SM^{-\tau}(n)$ as follows:

$$\widetilde{\mathcal{T}}_M(\mathfrak{c})_k=\bigl\{[e,\epsilon,u]\in SM^{-\tau}(n)_k\mid\epsilon<\tfrac{1}{2}\rho\min\{|c_1|,\dots,|c_n|\}\bigr\}.$$

We define a subspectrum $\widetilde{\mathcal{T}}_M(n) \subset \mathrm{Map}(\mathcal{D}(n), SM^{-\tau}(n))$ as follows:

$$\phi \in \widetilde{\mathcal{T}}_M(n)_k \quad \Longleftrightarrow \quad \phi(\mathfrak{c}) \in \widetilde{\mathcal{T}}_M(\mathfrak{c})_k \quad \text{for all } \mathfrak{c} \in \mathcal{D}(n).$$

It is clear that the inclusion $\widetilde{\mathcal{T}}_M(n) \to \mathrm{Map}(\mathcal{D}(n), SM^{-\tau}(n))$ is a level-equivalence for any $n \geq 1$. We denote the sequence $\{\widetilde{\mathcal{T}}_M(n)\}$ by $\widetilde{\mathcal{T}}_M$.

We shall define an action of Σ_n on $\widetilde{\mathcal{T}}_M(n)$, with which we regard $\widetilde{\mathcal{T}}_M$ as a symmetric sequence. For $\mathfrak{c} = (c_1, \ldots, c_n) \in \mathcal{D}(n)$ and $\sigma \in \Sigma_n$, we define $\mathfrak{c}^\sigma \in \mathcal{D}(n)$ to be the configuration of the subintervals of length $|c_{\sigma(1)}|, |c_{\sigma(2)}|, \ldots, |c_{\sigma(n)}|$ placed from the side of $-\frac{1}{2}$ to the side of $\frac{1}{2}$. For $[e, \epsilon, u] \in SM^{-\tau}(n)_k$ and $\sigma \in \Sigma_n$, we set $[e, \epsilon, u]_\sigma = [e \circ \underline{\sigma}, \epsilon, u]$ where $\underline{\sigma}\colon SM^n \to SM^n$ is given by $(z_1, \ldots, z_n) \mapsto (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})$. (To distinguish the action of Σ_k which is a part of the structure of the spectrum, we use the subscript $[-]_\sigma$.)

Definition 3.8 With the above notation, for $\phi \in \widetilde{\mathcal{T}}_M(n)_k$ and $\sigma \in \Sigma_n$ we define an element $\phi^\sigma \in \widetilde{\mathcal{T}}_M(n)_k$ by

$$\phi^\sigma(\mathfrak{c}) = \{\phi(\mathfrak{c}^{\sigma^{-1}})\}_\sigma.$$

Clearly $\phi \mapsto \phi^\sigma$ gives a Σ_n–action on $\widetilde{\mathcal{T}}_M(n)$.

In order to define a partial composition on $\widetilde{\mathcal{T}}_M$, we shall define a map

$$\Xi = \Xi[\mathfrak{d}, \mathfrak{c}; i]\colon SM^{-\tau}(n+m-1) \to SM^{-\tau}(n).$$

For an element $[e, \epsilon, u] \in SM^{-\tau}(n+m-1)_k$, we put

- $e' = e \circ (1_{i-1} \times \Delta' \times 1_{n-i})\colon SM^n \to \mathbb{R}^k$, where $\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i]$ and 1_l is the identity on SM^l, and
- $\epsilon' = (1/8^{m-1}) \min\{\epsilon, L(e, \mathfrak{d} \circ_i \mathfrak{c})\}$, where $L(e, \mathfrak{c}')$ is the minimum of the numbers $L(e \circ \Delta[\mathfrak{c}_1, \mathfrak{c}_2; j])$ over all triples $(\mathfrak{c}_1, \mathfrak{c}_2, j)$ satisfying $\mathfrak{c}' = (\mathfrak{c}_1 \circ_j \mathfrak{c}_2) \circ_l \mathfrak{c}_3$ for some configuration $\mathfrak{c}_3$ and number l.

By Lemma 3.6, e' is expanding. We set $\Xi([e, \epsilon, u]) = [e', \epsilon', u]$. Clearly Ξ is a well-defined map of spectra.

Definition 3.9 Using the above notation:

- We define a partial composition

  $$(- \circ_i -)\colon \mathcal{D}(m) \mathbin{\hat{\otimes}} \widetilde{\mathcal{T}}_M(n+m-1) \to \widetilde{\mathcal{T}}_M(n)$$

  on $\widetilde{\mathcal{T}}_M$ by setting

  $$(\mathfrak{c} \circ_i \phi)(\mathfrak{d}) = \Xi(\phi(\mathfrak{d} \circ_i \mathfrak{c})) \quad \text{where } \Xi = \Xi[\mathfrak{d}, \mathfrak{c}; i],$$

  for elements $\phi \in \widetilde{\mathcal{T}}_M(n+m-1)$, $\mathfrak{c} \in \mathcal{D}(m)$ and $\mathfrak{d} \in \mathcal{D}(n)$.
- We define a multiplication $\tilde{\mu}\colon \widetilde{\mathcal{T}}_M(n) \wedge \widetilde{\mathcal{T}}_M(n) \to \widetilde{\mathcal{T}}_M(n)$ by

  $$\tilde{\mu}(\langle \phi_1, \phi_2; \sigma \rangle)(\mathfrak{d}) = \mu(\langle \phi_1(\mathfrak{d}), \phi_2(\mathfrak{d}); \sigma \rangle),$$

  where μ denotes the multiplication given in Definition 3.1.

With these operations and the action of Σ_n in Definition 3.8, we regard $\widetilde{\mathcal{T}}_M$ as a $\mathcal{D}$–comodule of NCRS.

Lemma 3.10 *The structure of a $\mathcal{D}$–comodule of NCRS on $\widetilde{\mathcal{T}}_M$ given in Definition 3.9 is well defined.*

Proof By Lemma 3.5, we see the equality in Definition 2.10(1) holds. The equality in (2) in the same definition is clear.

We shall prove the equality in (3). Take elements $\mathfrak{c} \in \mathcal{D}(m)$, $\mathfrak{d} \in \mathcal{D}(n)$, $\phi \in \widetilde{\mathcal{T}}_M(m+n-1)$ and $\sigma \in \Sigma_n$. By definition,

$$(\mathfrak{c} \circ_i \phi)^\sigma(\mathfrak{d}) = \{\mathfrak{c} \circ_i \phi(\mathfrak{d}^{\sigma^{-1}})\}_\sigma = \{\Xi_1(\phi(\mathfrak{d}^{\sigma^{-1}} \circ_i))\}_\sigma,$$
$$\mathfrak{c} \circ_{\sigma^{-1}(i)} (\phi^{\sigma_1})(\mathfrak{d}) = \Xi_2\{\phi((\mathfrak{d} \circ_{\sigma^{-1}(i)} \mathfrak{c})^{\sigma_1^{-1}})_{\sigma_1}\},$$

where $\Xi_1 = \Xi[\mathfrak{d}^{\sigma^{-1}}, \mathfrak{c}; i]$ and $\Xi_2 = \Xi[\mathfrak{d}, \mathfrak{c}; \sigma^{-1}(i)]$. It is easy to check the equalities

$$\mathfrak{d}^{\sigma^{-1}} \circ_i \mathfrak{c} = (\mathfrak{d} \circ_{\sigma^{-1}(i)} \mathfrak{c})^{\sigma_1^{-1}} \quad \text{and} \quad \{\Xi_1(x)\}_\sigma = \Xi_2(x_{\sigma_1}).$$

These verify the desired equality. Compatibility of the multiplication with the partial composition is obvious. □

3.3 Construction of the comodule $\mathcal{T}_M$

Let p and q be two different integers with $1 \le p, q \le n$, and $\mathfrak{c} \in \mathcal{D}(n)$ be an element. We set a number $\delta_{pq}(\mathfrak{c}, \epsilon)$ by

$$\delta_{pq}(\mathfrak{c}, \epsilon) = \tfrac{1}{2}\rho(|c_p| + |c_q|) - \epsilon$$

for a number ϵ. We define a subspectrum $\mathcal{T}_{pq}(\mathfrak{c}) \subset \widetilde{\mathcal{T}}_M(\mathfrak{c})$ by the following equivalence. For each $k \ge 0$,

$$[e, \epsilon, u] \in \mathcal{T}_{pq}(\mathfrak{c})_k \quad \Longleftrightarrow \quad [e, \epsilon, u] = * \quad \text{or} \quad d_M(x_p, x_q) \le \delta_{pq}(\mathfrak{c}, \epsilon),$$

where $x_i \in M$ is the image of the i^{th} component of $\pi_e(u)$ by the standard projection $SM \to M$ for $i = p, q$. On the right-hand side, $\delta_{pq}(\mathfrak{c}, \epsilon)$ is positive by the definition of $\widetilde{\mathcal{T}}_M(\mathfrak{c})$. Define a subspectrum $\mathcal{T}_{pq}(n) \subset \widetilde{\mathcal{T}}_M(n)$ by

$$\phi \in \mathcal{T}_{pq}(n)_k \quad \Longleftrightarrow \quad \phi(\mathfrak{c}) \in \mathcal{T}_{pq}(\mathfrak{c})_k \quad \text{for all } \mathfrak{c} \in \mathcal{D}(n).$$

Clearly we have $\mathcal{T}_{pq}(n) = \mathcal{T}_{qp}(n)$. The following lemma is the key to defining the comodule $\mathcal{T}_M$. Most of the preceding technical definitions are necessary to make this lemma hold.

Lemma 3.11 (1) *For any numbers $n \ge 1$ and $m \ge 2$ and element $\mathfrak{c} \in \mathcal{D}(m)$, let $\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1) \subset \widetilde{\mathcal{T}}_M(n)$ denote the image of $\mathcal{T}_{pq}(n+m-1)$ by the map $\mathfrak{c} \circ_i (-)$. We have the following inclusion at each level k:*

$$\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1) \subset \begin{cases} \{*\} & \text{if } i \le p < q \le i+m-1, \\ \mathcal{T}_{pi}(n) & \text{if } p < i \le q \le i+m-1, \\ \mathcal{T}_{p,q-m+1}(n) & \text{if } p < i, i+m-1 < q, \\ \mathcal{T}_{i,q-m+1}(n) & \text{if } i \le p \le i+m-1 < q, \\ \mathcal{T}_{p-m+1,q-m+1}(n) & \text{if } i+m-1 < p < q. \end{cases}$$

More precisely, for example, the second inclusion means $\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1)_k \subset \mathcal{T}_{pi}(n)_k$ for each k.

(2) *The image of $\mathcal{T}_{pq}(n) \wedge \widetilde{\mathcal{T}}_M(n)$ by the multiplication $\tilde{\mu}$ given in Definition 3.9 is contained in $\mathcal{T}_{pq}(n)$.*

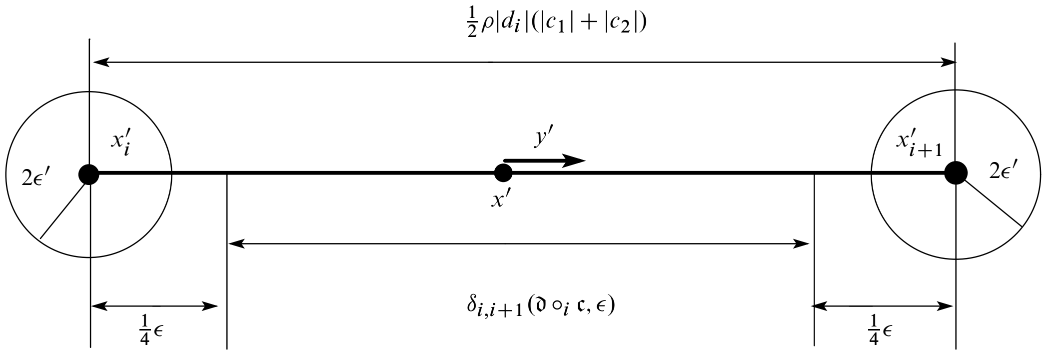

Figure 3: The first inclusion of Lemma 3.11(1) with $n=2$. The bold line is a part of the geodesic segment used to define Δ', (x',y') is the i^{th} component of $\pi_{e'}(u)\in SM^n$, and x_i and x_{i+1} exist in the interior of the disks at x'_i and x'_{i+1} if $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$.

Proof We shall show (1). Let $\mathfrak{c}\in\mathcal{D}(m)$, $\mathfrak{d}\in\mathcal{D}(n)$ and $\phi\in\mathcal{T}_{pq}(n+m-1)_k$ be elements. Let (e,ϵ,u) be a representative of $\phi(\mathfrak{d}\circ_i\mathfrak{c})$. Write

$$\pi_e(u)=((x_1,y_1),\dots,(x_{n+m-1},y_{n+m-1})),$$
$$\{(1_{i-1})\times\Delta'\times(1_{n-i})\}(\pi_{e'}(u))=((x'_1,y'_1),\dots,(x'_{n+m-1},y'_{n+m-1})),$$

with $x_j,x'_j\in M$, $y_j\in S_{x_j}M$ and $y'_j\in S_{x'_j}M$. Here we use the notation given in the paragraph above Definition 3.9. We shall show the first inclusion, the case of $i\le p<q\le i+m-1$.

The situation of the case $n=2$ is as in Figure 3 (so $p=i$ and $q=i+1$). We first give a sketch of the proof for $n=2$. We suppose $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$ and will show a contradiction. Since the map Δ' arranges points along a geodesic and the length of the geodesic segment between x'_i and x'_{i+1} is $\frac{1}{2}\rho|d_i|(|c_1|+|c_2|)$, we have $d_M(x'_i,x'_{i+1})>\delta(\mathfrak{d}\circ_i\mathfrak{c},\epsilon)$. As we have taken ϵ' in the definition of Ξ sufficiently small, x_i and x'_i (resp. x_{i+1} and x'_{i+1}) are sufficiently close. These observations imply $d_M(x_i,x_{i+1})>\delta(\mathfrak{d}\circ_i\mathfrak{c},\epsilon)$, or, equivalently, $\phi(\mathfrak{d}\circ_i\mathfrak{c})\notin\mathcal{T}_{pq}(\mathfrak{d}\circ_i\mathfrak{c})$.

We shall give the formal proof. We assume $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$. Since the image of e' is contained in the image of e and the map π_e sends u to its closest point in $e(M)=M$, we have

$$\|u-e(\pi_e(u))\|\le\|u-e'(\pi_{e'}(u))\|<\epsilon'.$$

So

$$\|e'(\pi_{e'}(u))-e(\pi_e(u))\|\le\|e'(\pi_{e'}(u))-u\|+\|u-e(\pi_e(u))\|<2\epsilon'.$$

As $e'=e\circ(1_{i-1})\times\Delta'\times(1_{n-i})$ and e is expanding,

$$d\{((x'_1,y'_1),\dots,(x'_{n+m-1},y'_{n+m-1})),((x_1,y_1),\dots,(x_{n+m-1},y_{n+m-1}))\}<2\epsilon',$$

where d denotes the distance in SM^{n+m-1}. So

$$d_M(x_j,x'_j)\le d_{SM}((x_j,y_j),(x'_j,y'_j))<2\epsilon'\quad\text{for } j=1,\dots,n+m-1.$$

By this inequality, and the definition of the map Δ', we have the inequality

$$\begin{aligned} d_M(x_p, x_q) &\geq d_M(x'_p, x'_q) - d_M(x_p, x'_p) - d_M(x_q, x'_q) \geq d_M(x'_p, x'_q) - 4\epsilon' \\ &\geq \tfrac{1}{2}\rho|d_i|(|c_{p-i+1}| + |c_{q-i+1}|) - 4\epsilon' = \tfrac{1}{2}\rho(|(\mathfrak{d} \circ_i \mathfrak{c})_p| + |(\mathfrak{d} \circ_i \mathfrak{c})_q|) - 4\epsilon' \\ &\geq \tfrac{1}{2}\rho(|(\mathfrak{d} \circ_i \mathfrak{c})_p| + |(\mathfrak{d} \circ_i \mathfrak{c})_q|) - \epsilon/2 > \delta_{pq}(\mathfrak{d} \circ_i \mathfrak{c}, \epsilon). \end{aligned}$$

This inequality implies $\phi(\mathfrak{d} \circ_i \mathfrak{c}) \notin \mathcal{T}_{pq}(\mathfrak{d} \circ_i \mathfrak{c})$, which is a contradiction. So $(\mathfrak{c} \circ_i \phi)(\mathfrak{d}) = *$, and we have proved the first inclusion.

We shall show the second inclusion, the case of $p < i \leq q \leq i + m - 1$. Let $(x', y') \in SM$ be the i^{th} component of $\pi_{\epsilon'}(u)$. Clearly,

$$((x'_i, y'_i), \dots, (x'_{i+m-1}, y'_{i+m-1})) = \Delta'(x', y').$$

By an argument similar to the above, we have the inequality

$$\begin{aligned} d_M(x'_p, x') &\leq d_M(x'_p, x_p) + d_M(x_p, x_q) + d_M(x_q, x'_q) + d_M(x'_q, x') \\ &\leq 2\epsilon' + \delta_{pq}(\mathfrak{d} \circ_i \mathfrak{c}, \epsilon) + 2\epsilon' + \tfrac{1}{2}\rho|d_i|(1 - |c_{q-i+1}|) \\ &= \tfrac{1}{2}\rho(|d_p| + |d_i||c_{q-i+1}|) - \epsilon + 4\epsilon' + \tfrac{1}{2}\rho|d_i|(1 - |c_{q-i+1}|) \leq \tfrac{1}{2}\rho(|d_p| + |d_i|) - \tfrac{1}{2}\epsilon < \delta_{pq}(\mathfrak{d}, \epsilon'). \end{aligned}$$

This implies the second inclusion. The other cases are similar to the first and second cases. The proof of (2) is similar in view of the assumption on the metric given in the paragraph after Definition 3.3, and so is omitted. □

Let $\mathcal{T}_{\text{fat}}(n)$ be the subspectrum of $\widetilde{\mathcal{T}}_M(n)$ whose space at level k is given by

$$\mathcal{T}_{\text{fat}}(n)_k = \bigcup_{1 \leq p < q \leq n} \mathcal{T}_{pq}(n)_k.$$

Since $\{\mathcal{T}_{pq}(n)\}^\sigma = \mathcal{T}_{\sigma^{-1}(p), \sigma^{-1}(q)}(n)$, we have that $\mathcal{T}_{\text{fat}}(n)$ is stable under the action of Σ_n. By Lemma 3.11, the sequence $\{\mathcal{T}_{\text{fat}}(n)\}_{n \geq 0}$ is stable under partial compositions and is an ideal for the multiplication $\tilde{\mu}$. So the sequence $\{\mathcal{T}_{\text{fat}}(n)\}_{n \geq 0}$ inherits a structure of a comodule from $\widetilde{\mathcal{T}}_M$, and we can define the quotient comodule as follows:

Definition 3.12 We define a spectrum $\mathcal{T}_M(n)$ by the quotient (collapsing to $*$)

$$\mathcal{T}_M(n)_k = \widetilde{\mathcal{T}}_M(n)_k / \mathcal{T}_{\text{fat}}(n)_k$$

for each $k \geq 0$ and $n \geq 2$, and by $\mathcal{T}_M(1) = \widetilde{\mathcal{T}}_M(1)$. We regard the sequence $\mathcal{T}_M = \{\mathcal{T}_M(n)\}_{n \geq 1}$ as a comodule of NCRS with the structure induced by that on $\widetilde{\mathcal{T}}_M$.

4 Atiyah duality for comodules

Definition 4.1 We define the following zigzag consisting of $\mathcal{D}$–comodules of NCRS and maps between them:

$$(\mathcal{C}_M)^\vee \xleftarrow{(i_0)^\vee} (\widetilde{F}_M)^\vee \xrightarrow{(i_1)^\vee} (F_M)^\vee \xleftarrow{q_*} F'_M \xrightarrow{p_*} F^\dagger_M \xleftarrow{\Phi} \mathcal{T}_M.$$

• Set $\mathcal{C}_M(n) = \mathcal{C}^{n-1}(M)$. When we regard a configuration as an element of $\mathcal{C}_M(n)$, we label its points by $1, \dots, n$ instead of $0, \dots, n-1$. We give the sequence $\mathcal{C}_M = \{\mathcal{C}_M(n)\}_{n\geq 1}$ a structure of an $\mathcal{A}$–module as follows. For the unique element $\mu \in \mathcal{A}(2)$ and an element $x \in \mathcal{C}_M(n)$, we set $x \circ_i \mu = d^{i-1}(x)$, where d^{i-1} is the coface operator of $\mathcal{C}^\bullet(M)$. The action of Σ_n on $\mathcal{C}_M(n)$ is given by permutation of labels and $(\mathcal{C}_M)^\vee$ is the $\mathcal{A}$–comodule of NCRS given in Definition 2.10. By pulling back the action by the unique operad morphism $\mathcal{D} \to \mathcal{A}$, we also regard $(\mathcal{C}_M)^\vee$ as a $\mathcal{D}$–comodule.

• Let $F_M(n)$ be the subspace of $\mathcal{D}(n) \times SM^n$ defined by the following condition. For an element $(\mathfrak{c}; (x_1, y_1), \dots, (x_n, y_n)) \in \mathcal{D}(n) \times SM^n$ with $x_i \in M$ and $y_i \in S_{x_i} M$,

$$(\mathfrak{c}; (x_1, y_1), \dots, (x_n, y_n)) \in F_M(n) \quad \Longleftrightarrow \quad d(x_i, x_j) \geq \tfrac{1}{2}\rho(|c_i|+|c_j|) \quad \text{for each pair } (i,j) \text{ with } i \neq j,$$

where ρ is the number fixed in Section 3.2.

• The sequence $\{F_M(n)\}$ has a structure of a $\mathcal{D}$–module. For $\mathfrak{c} \in \mathcal{D}(n)$ and $(\mathfrak{d}; z_1, \dots, z_n) \in F_M(n)$, we set $(\mathfrak{d}; z_1, \dots, z_n) \circ_i \mathfrak{c} = (\mathfrak{d} \circ_i \mathfrak{c}; z_1, \dots, \Delta'(z_i), \dots, z_n)$, where $\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i]$ is given in Definition 3.4. The symmetric group acts on $F_M(n)$ by permutation of little intervals and components. The $\mathcal{D}$–comodule of NCRS $(F_M)^\vee$ is the one induced by F_M.

• We shall define a symmetric sequence of spectra $\{\mathbb{S}_M(n)\}_n$. Set $\widetilde{\mathbb{S}}_M(n)_k = \widetilde{N}_k^{-\tau}$ for $N = SM^n$ (see Definition 3.1). Define a subspace $\partial(\widetilde{\mathbb{S}}_M(n))_k \subset \widetilde{\mathbb{S}}_M(n)_k$ by $(e, \epsilon, v) \in \partial\widetilde{\mathbb{S}}_M(n)_k$ if and only if $\|v\| \geq \epsilon$. We put

$$\mathbb{S}_M(n)_k = \widetilde{\mathbb{S}}_M(n)_k / \partial\widetilde{\mathbb{S}}_M(n)_k.$$

We regard $\mathbb{S}_M(n)$ as an NCRS by a multiplication defined similarly to that of $N^{-\tau}$, given in Definition 3.1.

• Set $F_M^\dagger(n) := \mathrm{Map}(F_M(n), \mathbb{S}_M(n))$. We give the sequence $\{F_M^\dagger(n)\}_n$ a structure of a $\mathcal{D}$–comodule as follows. For $\mathfrak{c} \in \mathcal{D}(n)$ and $f \in F_M^\dagger(n+m-1)$, set $\mathfrak{c} \circ_i f$ to be the composition

$$F_M(m) \xrightarrow{(-\circ_i \mathfrak{c})} F_M(n+m-1) \xrightarrow{f} \mathbb{S}_M(n+m-1) \xrightarrow{\alpha} \mathbb{S}_M(n).$$

Here α is given by

$$\alpha([e, \epsilon, v]) = [e', \epsilon', v],$$

where e' and ϵ' are as defined in the paragraph above Definition 3.9. Similarly to $(\mathcal{C}_M)^\vee$, we define a multiplication on $F_M^\dagger(n)$ as the pushforward by the multiplication on $\mathbb{S}_M(n)$.

• We define a map $\widetilde{\Phi}_n \colon \widetilde{\mathcal{T}}_M(n) \to F_M^\dagger(n)$ of spectra by

$$\widetilde{\Phi}_n(\phi)((\mathfrak{c}; z_1, \dots, z_n)) = [e, \bar{\epsilon}, u - e(z_1, \dots, z_n)].$$

Here we write $\phi(\mathfrak{c}) = [e, \epsilon, u]$ and we set $\bar{\epsilon} = \frac{1}{4}\epsilon$. Lemma 4.2 proves that $\widetilde{\Phi}_n$ induces a morphism $\Phi_n \colon \mathcal{T}_M(n) \to F_M^\dagger(n)$ which forms a morphism of comodules.

• We shall define a $\mathcal{D}$–module $\widetilde{F}_M$. Set

$$\widetilde{F}_{M,1}(n) = [0,1] \times \mathcal{D}(n) \times \mathcal{C}_M(n)/\sim,$$

where the equivalence relation is generated by the relation $(t,\mathfrak{c},z)\sim(s,\mathfrak{d},z')$ if and only if $s=t=0$ and $z=z'$. $\widetilde{F}_M(n)$ is the subspace of $\widetilde{F}_{M,1}(n)$ consisting of elements $(t,\mathfrak{c},z)$ with $z=(x_k,u_{kl},y_k)$ satisfying

$$t\neq 0 \quad\Longrightarrow\quad z\in \mathrm{Int}(\mathcal{C}_M(n)) \quad\text{and}\quad d_M(x_i,x_j)\geq t\cdot\tfrac{1}{2}\rho(|c_i|+|c_j|).$$

Here $\mathrm{Int}(\mathcal{C}_M(n))$ is the subspace consisting of the elements (x_k,u_{kl},y_k) such that $x_k\neq x_l$ if $k\neq l$, or equivalently, (x_k,u_{kl}) belongs to $C_n(M)$ via the canonical inclusion $C_n(M)\subset \overline{C}_n(M)$. We endow the sequence $\{\widetilde{F}_M(n)\}_n$ with a structure of a $\mathcal{D}$–module analogous to that of F_M. The difference is that we use the number $t\rho$ instead of ρ in the definition of Δ' for $t>0$, and use the module structure on $\mathcal{C}_M$ for $t=0$. The obvious inclusions $i_0\colon\mathcal{C}_M(n)\to\widetilde{F}_M(n)$ and $i_1\colon F_M(n)\to\widetilde{F}_M(n)$ to $t=0,1$ give rise to morphisms of $\mathcal{D}$–modules $i_0\colon\mathcal{C}_M\to\widetilde{F}_M$ and $i_1\colon F_M\to\widetilde{F}_M$.

- To define F'_M, p_* and q_*, we shall define a symmetric sequence of spectra $\{\mathbb{S}'_M(n)\}_n$. Let $\widetilde{\mathbb{S}}'_M(n)$ be the subspace of $\mathrm{Emb}((SM)^n,\mathbb{R}^k)\times\mathbb{R}\times S^k$ consisting of triples (e,ϵ,v) with $0<\epsilon<L(e)$. We put
$$\mathbb{S}'_M(n)_k=\widetilde{\mathbb{S}}'_M(n)_k/\{(e,\epsilon,\infty)\mid e,\epsilon\text{ arbitrary}\},$$
where we regard $S^k=\mathbb{R}^k\cup\{\infty\}$. We regard $\mathbb{S}'_M(n)$ as a spectrum analogously to $\mathbb{S}_M(n)$. Let $p\colon\mathbb{S}'_M(n)\to\mathbb{S}_M(n)$ be the map induced by the collapsing map $S^k\to\mathbb{R}^k/\{v\mid\|v\|\geq\epsilon\}$ and $q\colon\mathbb{S}'_M\to\mathbb{S}$ be the map forgetting the data (e,ϵ). Set $F'_M(n)=\mathrm{Map}(F_M(n),\mathbb{S}'_M(n))$. We regard $\{F'_M(n)\}$ as a $\mathcal{D}$–comodule of NCRS analogously to $F^\dagger_M$. The pushforwards p_* and q_* are clearly morphisms of comodules of NCRS.

Verification of well-definedness of the objects defined in Definition 4.1 is routine work. For example, the associativity of the composition of $\mathcal{C}_M$ follows from the cosimplicial identities of $\mathcal{C}^\bullet(M)$, and that of F_M can be verified similarly to the associativity of little cubes operads. We omit details.

Remark Right modules similar to F_M are used in [2; 6].

Lemma 4.2 *The map $\widetilde{\Phi}_n$ uniquely factors through a map $\Phi_n\colon\mathcal{T}_M(n)\to F^\dagger_M(n)$, and the sequence $\{\Phi_n\}$ is a map of $\mathcal{D}$–comodules of NCRS.*

Proof We shall show that $\widetilde{\Phi}_n(\phi)=*$ for any element $\phi\in\mathcal{T}_{pq}(n)$. Suppose that there exists an element $(\mathfrak{c};z_1,\dots,z_n)\in F_M(n)$ such that $\widetilde{\Phi}_n(\phi)(\mathfrak{c};z_1,\dots,z_n)\neq *\in\mathbb{S}_M(n)$. If we put $\phi(\mathfrak{c})=[e,\epsilon,u]$, the inequality $\|u-e(z_1,\dots,z_n)\|<\frac14\epsilon$ holds. So $\|u-e(\pi_e u)\|<\frac14\epsilon$. Thus,

$$\|e(\pi_e u)-e(z_1,\dots,z_n)\|\leq\|e(\pi_e u)-u\|+\|u-e(z_1,\dots,z_n)\|<\tfrac12\epsilon.$$

As e is expanding, we have $d(\pi_e(u),(z_1,\dots,z_n))<\frac12\epsilon$ where d denotes the distance in SM^n. If we write $z_i=(x_i,y_i)$ and $\pi_e(u)=((\bar{x}_1,\bar{y}_1),\dots,(\bar{x}_n,\bar{y}_n))$ as pairs of a point of M and a tangent vector, it follows that $d_M(\bar{x}_i,x_i)<\frac12\epsilon$, and

$$d(\bar{x}_p,\bar{x}_q)\geq d(x_p,x_q)-d(x_p,\bar{x}_p)-d(x_q,\bar{x}_q)>\tfrac12\rho(|c_p|+|c_q|)-\epsilon=\delta_{pq}(\mathfrak{c},\epsilon).$$

This inequality contradicts the assumption $\phi\in\mathcal{T}_{pq}(n)$. Thus we have proved $\widetilde{\Phi}_n(\mathcal{T}_{pq}(n))=*$. This implies the former part of the lemma. The latter part is obvious. □

Definition 4.3 A $\mathcal{D}$–comodule of NCRS is *semistable* if the spectrum $X(n)$ is semistable for each n. A map $f : X \to Y$ of $\mathcal{D}$–comodules of NCRS is a *π_*–isomorphism* if each map $f_n : X(n) \to Y(n)$ is a π_*–isomorphism (see Section 2.1).

The notion of a π_*–isomorphism in Definition 4.3 is what we call "weak equivalence" in Theorem 1.1. Since a π_*–isomorphism of spectra is a stable equivalence, a π_*–isomorphism of $\mathcal{D}$–comodules gives a stable equivalence at each arity. The following is a version of Atiyah duality which respects our comodules. We devote the rest of this section to its proof.

Theorem 4.4 *As $\mathcal{D}$–comodules of nonunital commutative symmetric ring spectra, $(\mathcal{C}_M)^\vee$ and $\mathcal{T}_M$ are π_*–isomorphic. Precisely speaking, all the comodules in the zigzag in Definition 4.1 are semistable and all the maps in the same zigzag are π_*–isomorphisms.*

Definition 4.5 • For $G \in \mathsf{G}(n)$ and $\mathfrak{c} \in \mathcal{D}(n)$, we define two subspectra $\mathcal{T}_G(\mathfrak{c}), \mathcal{T}_{\mathrm{fat}}(\mathfrak{c}) \subset \widetilde{\mathcal{T}}_M(\mathfrak{c})$ by

$$\mathcal{T}_G(\mathfrak{c}) = \begin{cases} \bigcap_{(p,q)\in E(G)} \mathcal{T}_{pq}(\mathfrak{c}) & \text{if } G \neq \varnothing, \\ \widetilde{\mathcal{T}}_M(\mathfrak{c}) & \text{if } G = \varnothing, \end{cases} \qquad \text{and} \qquad \mathcal{T}_{\mathrm{fat}}(\mathfrak{c}) = \bigcup_{1\le p<q\le n} \mathcal{T}_{pq}(\mathfrak{c}).$$

Similarly, we define a subspectrum $\mathcal{T}_G \subset \widetilde{\mathcal{T}}_M(n)$ by

$$\mathcal{T}_G = \begin{cases} \bigcap_{(p,q)\in E(G)} \mathcal{T}_{pq} & \text{if } G \neq \varnothing, \\ \widetilde{\mathcal{T}}_M(n) & \text{if } G = \varnothing. \end{cases}$$

Here the union and intersections are taken in the levelwise manner.

- We fix an expanding embedding $e_0 : SM \to \mathbb{R}^K$, a positive number $\epsilon_0 < L(e_0)$ and a configuration $\mathfrak{c}_0 \in \mathcal{D}(n)$ such that $\epsilon_0 < \frac{1}{4}\min\{|c_1|, \dots, |c_n|\}$. We set $\nu = \nu_{\epsilon_0}(e_0)$. We impose an additional condition on ϵ_0 in Definition 5.8, which is satisfied by any sufficiently small ϵ_0, and we will assume K is a multiple of 4 in the proof of Theorem 5.16. (We may impose the assumption on K from the beginning, but for the convenience of verification of signs we do not do so.)

- For a graph $G \in \mathsf{G}(n)$, let $M^{\pi_0(G)}$ be the space of maps $\pi_0(G) \to M$ with the product topology, where $\pi_0(G)$ is the set of connected components of G. Let D_G be the pullback of the diagram

$$SM^n \xrightarrow{\text{projection}} M^n \leftarrow M^{\pi_0(G)},$$

where the right arrow is the pullback by the quotient map $\underline{n} \to \pi_0(G)$. D_G is naturally regarded as a subspace of SM^n via the projection of the pullback. This subspace is the same as the one given in Section 1. We define the subspace $\mathrm{FD}_n \subset SM^n$ as the unions of the spaces D_G whose graph G has at least one edge.

- Consider $\nu^n \subset \mathbb{R}^{nK}$ as a disk bundle over SM^n and denote by ν_G be the preimage of D_G by the projection $\nu^n \to SM^n$. Let $\lambda_G : \mathrm{Th}(\nu_G) \to \mathcal{T}_G(\mathfrak{c}_0)_{nK}$ be the map $[u] \mapsto [(e_0)^n, \epsilon_0, u]$. Then λ_G induces a morphism $\lambda_G : \Sigma^{nK}\,\mathrm{Th}(\nu_G) \to \mathcal{T}_G(\mathfrak{c}_0)$ in $\mathbf{Ho}(\mathcal{SP})$, where Σ denotes the suspension.

Lemma 4.6 *For a closed smooth manifold N and $k \geq 1$, the inclusion $I : \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \to \mathrm{Emb}(N, \mathbb{R}^k)$ is a homotopy equivalence.*

Proof Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be a C^∞–function which satisfies the following inequalities:

$$f(x) > \frac{1}{x} \quad \text{for } x < 1, \qquad f(x) \geq 1 \quad \text{for } x \geq 1.$$

We define a continuous map $F\colon \mathrm{Emb}(N, \mathbb{R}^k) \to \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k)$ by $e \mapsto f(r(e)) \cdot e$, where $r(e)$ is the number given in Definition 3.1, and $\cdot$ denotes componentwise scalar multiplication. A homotopy from $F \circ I$ to id is given by $(t, e) \mapsto \{t + (1-t) f(r(e))\} \cdot e$, and a homotopy from $I \circ F$ to id is also given by the same formula. □

Lemma 4.7 *We use the notation in Definition 4.5. For each $n \geq 1$ and $G \in \mathsf{G}(n)$, $\mathcal{T}_M(n)$ and $\mathcal{T}_G$ are semistable, and each map in the following zigzags in* $\mathbf{Ho}(\mathcal{SP})$ *is an isomorphism.*

$$\Sigma^{nK}\,\mathrm{Th}(\nu_G) \xrightarrow{\lambda_G} \mathcal{T}_G(\mathfrak{c}_0) \leftarrow \mathcal{T}_G,$$

$$\Sigma^{nK}\{\mathrm{Th}(\nu^n)/\mathrm{Th}(\nu^n|_{\mathrm{FD}_n})\} \xrightarrow{\lambda_G} \mathcal{T}_\varnothing(\mathfrak{c}_0)/\{\mathcal{T}_{\mathrm{fat}}(\mathfrak{c}_0)\} \leftarrow \mathcal{T}_M(n).$$

Here, see Section 1 for FD_n*, and the right maps are the evaluations at* $\mathfrak{c}_0$.

Proof For simplicity, we shall prove the claim for the maps in the first line for the case of $G = \varnothing$. The same proof works for general G thanks to the assumptions on ρ given in Section 3.2. Set $N = (SM)^n$. The evaluation at $\mathfrak{c}_0$ and the inclusion $\mathcal{T}_\varnothing(\mathfrak{c}_0) \subset N^{-\tau}$ are clearly level equivalences. So all we have to prove is that $\mathcal{T}_\varnothing$ is semistable and that the composition of λ_G and the inclusion, which is also denoted by $\lambda_G\colon \Sigma^{nK}\,\mathrm{Th}(\nu_G) \to N^{-\tau}$, is an isomorphism in $\mathbf{Ho}(\mathcal{SP})$. We define a space $\mathcal{E}_k$ by

$$\mathcal{E}_k = \{(e, \epsilon) \mid e \in \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \text{ and } 0 < \epsilon < L(e)\}.$$

By Lemma 4.6 and Whitney's theorem, $\mathcal{E}_k$ is $\bigl(\frac{1}{2}k - n(2d-1) - 1\bigr)$–connected. Let $P\colon \bar{N}_k^{-\tau} \to \mathcal{E}_k$ be the fiber bundle obtained from the obvious projection $\widetilde{N}_k^{-\tau} \to \mathcal{E}_k$ by collapsing the complements of the $\nu_\epsilon(e)$ in a fiberwise manner (see Definition 3.1). So each fiber of the map P is a Thom space homeomorphic to $\mathrm{Th}(\nu_G)$. P has a section $s\colon \mathcal{E}_k \to \bar{N}_k^{-\tau}$ to the basepoints, and there is an obvious homeomorphism

$$\bar{N}_k^{-\tau}/s(\mathcal{E}_k) \cong N_k^{-\tau}.$$

With this, by observing the Serre spectral sequence for P, we see that the composition

$$S^{k-nK} \wedge \mathrm{Th}(\nu_G) \xrightarrow{\lambda_G} S^{k-nK} \wedge N_{nK}^{-\tau} \xrightarrow{\text{action of } \mathbb{S}} N_k^{-\tau}$$

is $\bigl(\frac{3}{2}k - 2n(2d-1) - 2\bigr)$–connected. This implies $N^{-\tau}$ is semistable and λ_G is an isomorphism. □

Proof of Theorem 4.4 Similarly to the proof of Lemma 4.7, it is easy to see $\mathbb{S}_M$ and $\mathbb{S}'_M$ are semistable, which implies each comodule in the zigzag in Definition 4.1 is semistable, combined with the fact that the spaces $F_M(n)$, $\widetilde{F}_M(n)$ and $\mathcal{C}_M(n)$ have homotopy types of finite CW complexes. It is clear that p and q are π_*–isomorphisms, and so are p_* and q_*. Then i_0 and i_1 are homotopy equivalences for each n, since $\widetilde{F}_M(n)$ is homotopy equivalent to the mapping cylinder of the inclusion $C_n(M) \subset \bar{C}_n(M)$, which is also a homotopy equivalence. So $(i_0)^\vee$ and $(i_1)^\vee$ are π_*–isomorphisms. Finally Φ_n is a π_*–isomorphism since it reduces to the equivalence of the original Atiyah duality in the (homotopy) category of classical spectra via Lemma 4.7; see [7]. □

5 Spectral sequences

5.1 A chain functor

Definition 5.1 • For a chain complex C_*, $C[k]_*$ is the chain complex given by $C[k]_l = C_{k+l}$ with the same differential as C_* (without extra sign).

• Fix a fundamental cycle $w_{S^1} \in C_1(S^1)$. Let $\overline{C}_*(U)$ denote the reduced singular chain complex of a pointed space U. We shall define a chain complex $C_*(X)$ for a spectrum X. Define a chain map $i^X_k : \overline{C}_*(X_k)[k] \to \overline{C}_*(X_{k+1})[k+1]$ by $i^X_k(x) = (-1)^l \sigma_*(w_{S^1} \times x)$ for $x \in \overline{C}_l(X_k)$, where $\sigma : S^1 \wedge X_k \to X_{k+1}$ is the structure map of X. We define $C_*(X)$ as the colimit of the sequence $\{\overline{C}_*(X_k)[k]; i^X_k\}_{k\geq 0}$. Clearly the procedure $X \mapsto C_*(X)$ is extended to a functor $\mathcal{SP} \to \mathcal{CH}_\mathsf{k}$ in an obvious manner.

• For a spectrum X, we denote by $H_*(X)$ the homology group of $C_*(X)$.

• Let $f\mathcal{CW}$ denote the full subcategory of $\mathcal{CG}$ spanned by finite CW complexes. We define a functor $C^*_S : (f\mathcal{CW})^{\mathrm{op}} \to \mathcal{CH}_\mathsf{k}$ by $C^q_S(X) = C_{-q}(X^\vee)$.

The proofs of the following two lemmas are very standard, so we omit them.

Lemma 5.2 *If $f : X \to Y$ is a stable equivalence between semistable spectra, the induced map $f_* : C_*(X) \to C_*(Y)$ is a quasi-isomorphism.* □

Lemma 5.3 *There exists a zigzag of natural transformations between C^* and $C^*_S : (f\mathcal{CW})^{\mathrm{op}} \to \mathcal{CH}_\mathsf{k}$, in which each natural transformation is an objectwise quasi-isomorphism.* □

Remark 5.4 The functor C_* does not have any compatibility with symmetry isomorphisms of the monoidal products $\wedge$ in $\mathcal{SP}$ and $\otimes_\mathsf{k}$ in $\mathcal{CH}_\mathsf{k}$, so the multiplication on $\mathcal{T}_M(n)$ defined in Section 3 does not straightforwardly induce a multiplication on $C_*(\mathcal{T}_M(n))$. To enrich the Čech spectral sequence with multiplicative operations, we will need some extra work as in [33], which is not dealt with here. The E_2–term of the spectral sequence has a multiplication induced by a simplicial CDBA given in Definition 5.14, but its topological meaning is unclear at present.

The functor $C_* : \mathcal{SP} \to \mathcal{CH}_\mathsf{k}$ has some compatibility with the tensor $\hat{\otimes}$ with a space.

Lemma 5.5 (1) *For $U \in \mathcal{CG}$ and $X \in \mathcal{SP}$, the collection of Eilenberg–Zilber shuffle maps*

$$\{EZ : C_*(U) \otimes \overline{C}_*(X_k)[k] \to \overline{C}_*((U_+) \wedge X_k)[k]\}_k$$

induces a quasi-isomorphism

$$C_*(U) \otimes C_*(X) \to C_*(U \hat{\otimes} X).$$

(2) *Let $\mathcal{O}$ be a topological operad and Y be an $\mathcal{O}$–comodule in $\mathcal{SP}$. A natural structure of a chain $C_*(\mathcal{O})$–comodule on the collection $C_*Y = \{C_*(Y(n))\}_n$ is defined as follows. The partial composition is given by the composition*

$$C_*(\mathcal{O}(m)) \otimes C_*(Y(m+n-1)) \to C_*(\mathcal{O}(m) \mathbin{\hat{\otimes}} Y(m+n-1)) \to C_*(Y(n)),$$

where the left map is the one defined in (1) *and the right map is induced by the partial composition on Y. The action of Σ_n on $C_*(Y)(n)$ is the one induced naturally.*

Proof The cross product $w_{S^1} \times x$ is equal to $EZ(w_{S^1} \otimes x)$ by definition, and the shuffle maps are associative and compatible with the symmetry isomorphisms of monoidal products without any chain homotopy for normalized singular chains, so the maps EZ are compatible with the maps i_k^X in Definition 5.1 (the sign commuting an element of $C_*(U)$ and w_{S^1} is canceled with the sign attached in the definition of i_k^X). This implies the first part. The second part follows from commutativity of the following diagram, which is clear from the property of the shuffle map mentioned above:

$$\begin{tikzcd}
C_*(U) \otimes C_*(V) \otimes C_*(X) \arrow[r] \arrow[d] & C_*(U) \otimes C_*(V \mathbin{\hat{\otimes}} X) \arrow[d] \\
C_*(U \times V) \otimes C_*(X) \arrow[r] & C_*((U \times V) \mathbin{\hat{\otimes}} X)
\end{tikzcd}$$

Here $U, V \in \mathcal{CG}$, $X \in \mathcal{SP}$, the left vertical arrow is induced by the EZ shuffle map and other arrows are given by (1). □

5.2 Construction of the Čech spectral sequence

Definition 5.6 We define a $C_*(\mathcal{D})$–*comodule* $\check{\mathrm{T}}^M_{\star\star}$ *of double complexes* consisting of the following data:

- a sequence of double complexes $\{\check{\mathrm{T}}^M_{\star\star}(n)\}_{n\geq 1}$ with two differentials d and ∂ of degree $(0,1)$ and $(1,0)$, respectively,
- an action of Σ_n on $\check{\mathrm{T}}^M_{\star\star}(n)$ which preserves the bigrading, and
- a partial composition $(-\circ_i -)\colon C_k(\mathcal{D}(m)) \otimes \check{\mathrm{T}}^M_{\star\star}(m+n-1) \to \check{\mathrm{T}}^M_{\star,\ast+k}(n)$.

These satisfy the following compatibility conditions in addition to the conditions in Definition 2.10:

$$d\partial = \partial d, \quad d(\alpha \circ_i x) = d\alpha \circ_i x + (-1)^{|\alpha|}\alpha \circ_i dx, \quad \partial(\alpha \circ_i x) = \alpha \circ_i \partial x.$$

We define the double complex $\check{\mathrm{T}}^M_{\star\star}(n)$ by

$$\check{\mathrm{T}}^M_{p*}(n) = \bigoplus_{G \in \mathsf{G}(n,p)} C_*(\mathcal{T}_G)$$

for $p \geq 0$ and $\check{\mathrm{T}}^M_{p,*}(n) = 0$ for $p < 0$, where $\mathsf{G}(n,p) \subset \mathsf{G}(n)$ is the set of graphs with exactly p edges (see Definition 4.5 for $\mathcal{T}_G$). The differential d is the original differential of $C_*(\mathcal{T}_G)$. The other differential ∂ is given by the signed sum

$$\partial = \sum_{t=1}^{p} (-1)^{t+1} \partial_t,$$

where ∂_t is the standard pushforward by the inclusion $\mathcal{T}_G \to \mathcal{T}_{G_t}$ where the graph G_t is defined by removing the t^{th} edge from G (in the lexicographical order). The action of σ on $\widetilde{\mathcal{T}}_M(n)$ restricts to a map $\sigma : \mathcal{T}_G \to \mathcal{T}_{\sigma^{-1}(G)}$; see Section 2.1 for $\sigma^{-1}(G)$. This map induces a chain map $\sigma_* : C_*(\mathcal{T}_G) \to C_*(\mathcal{T}_{\sigma^{-1}(G)})$ by the pushforward of chains. For $G \in G(n,p)$, let $\sigma_G \in \Sigma_p$ denote the composition

$$\underline{p} \cong E(\sigma^{-1}(G)) \to E(G) \cong \underline{p},$$

where $\cong$ denotes the order-preserving bijection and the middle map is given by $(i,j) \mapsto (\sigma(i), \sigma(j))$. We define the action of σ on $\check{\mathrm{T}}^M(n)$ as $\mathrm{sgn}(\sigma_G) \cdot \sigma_*$ on each summand. We now define the partial composition. Let $f_i : \underline{m+n-1} \to \underline{n}$ be the order-preserving surjection which satisfies $f_i(i+t) = i$ for $t = 1, \dots, m-1$. For elements $\alpha \in C_*(\mathcal{D}(m))$ and $x \in C_*(\mathcal{T}_G)$ with $G \in \mathsf{G}(n+m-1)$, if $\#E(f_i(G)) = \#E(G)$ then the partial composition $\alpha \circ_i x \in C_*(\mathcal{T}_{f_i G})$ is defined similarly to Lemma 5.5 with the map $(- \circ_i -) : \mathcal{D}(m) \hat{\otimes} \mathcal{T}_G \to \mathcal{T}_{f_i G}$, and if $\#E(f_i(G)) < \#E(G)$ then $\alpha \circ_i x$ is zero. This partial composition is well defined by Lemma 3.11. The compatibility between d, ∂ and $(- \circ_i -)$ is obvious. We have completed the definition of $\check{\mathrm{T}}^M$.

Let $\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *}(n)$ denote the total complex. Its differential is given by $d + (-1)^q \partial$ on $\check{\mathrm{T}}^M_{\star q}(n)$. We regard the sequence $\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *} = \{\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *}(n)\}_n$ as a chain $C_*(\mathcal{D})$–comodule with the induced structure. We fix an operad map $f : \mathcal{A}_\infty \to C_*(\mathcal{D})$, and regard $\mathrm{Tot}\, \check{\mathrm{T}}^M$ as an $\mathcal{A}_\infty$–comodule by pulling back the partial compositions by f. We consider the Hochschild complex $\mathrm{CH}_\bullet(\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *})$ associated to this $\mathcal{A}_\infty$–comodule; see Definition 2.13. The total degree of elements of $\mathrm{CH}_\bullet(\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *})$ is $-*-\star-\bullet$. We define two filtrations $\{F^{-p}\}$ and $\{\bar{F}^{-p}\}$ on this complex as follows. F^{-p} (resp. $\bar{F}^{-p}$) is generated by the homogeneous parts whose degree satisfies $\star + \bullet \le p$ (resp. $\bullet \le p$). We call the spectral sequence associated to $\{F^{-p}\}$ the *Čech spectral sequence*, in short, *Čech s.s.*, and denote it by $\{\check{\mathbb{E}}_r^{-p,q}\}_r$. The spectral sequence associated to $\{\bar{F}^{-p}\}$ is denoted by $\{\bar{\mathbb{E}}_r^{-p,q}\}_r$.

Lemma 5.7 *The spectral sequence $\bar{\mathbb{E}}_r$ in Definition 5.6 and Sinha spectral sequence $\mathbb{E}_r$ in Definition 2.7 are isomorphic after the E_1–page.*

Proof Put $N_0 = \#\{(i,j) \mid i, j \in \underline{n} \text{ with } i < j\}$ and let $X : \mathrm{P}_{N_0} = \mathsf{G}(n) - \{\varnothing\} \to \mathcal{SP}$ be the functor given by $X_G = \mathcal{T}_G$. By applying Lemma 2.2 to this functor, we see that the map $\mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *}(n) \to \mathcal{C}_*(\mathcal{T}_M(n))$ induced by the collapsing (quotient) map $\widetilde{\mathcal{T}}_M(n) \to \mathcal{T}_M(n)$ is a quasi-isomorphism. Combining this with Theorem 4.4 and Lemma 5.2, the two comodules $C_*(\mathcal{C}_M^\vee)$ and $\mathrm{Tot}\, \check{\mathrm{T}}_{\star *}$ are quasi-isomorphic. Clearly $\mathrm{CH}_\bullet\, C_*(\mathcal{C}_M)$ is quasi-isomorphic to the normalized complex of $C_*(\mathcal{C}^\bullet(M)^\vee)$, which is quasi-isomorphic to the normalized total complex of $C^*(\mathcal{C}^\bullet(M))$ by Lemma 5.3. Thus, $\mathrm{CH}_\bullet\, \mathrm{Tot}\, \check{\mathrm{T}}^M_{\star *}$ and the normalized total complex of $C^*(\mathcal{C}^\bullet(M))$ are connected by a zigzag of quasi-isomorphisms which preserve the filtration. This zigzag induces a zigzag of morphisms of spectral sequences which are isomorphisms after the E_1–page because the homology of $\mathrm{Tot}\, \check{\mathrm{T}}_{\star *}(n+1)$ is isomorphic to $H^*(\mathcal{C}^n(M))$ under the zigzag. $\square$

5.3 Convergence

In this subsection, we assume M is orientable. We shall prepare some notation and terminology which is necessary to analyze the E_1–page of the Čech s.s.

Definition 5.8 • We fixed an embedding $e_0\colon SM \to \mathbb{R}^K$ and a number ϵ_0 in Definition 4.5. We also fix an isotopy $\iota_t\colon SM \to \mathbb{R}^{2K}$ with $\iota_0 = 0 \times e_0$ and $\iota_1 = \Delta_{\mathbb{R}^K} \circ e_0$, where $0 \times e_0\colon SM \to \mathbb{R}^{2K}$ is given by $(0 \times e_0)(z) = (0, e_0(z))$ and $\Delta_{\mathbb{R}^K}$ is the diagonal map on $\mathbb{R}^K$. We impose the additional condition that ϵ_0 is smaller than $\min\{L(\iota_t) \mid 0 \le t \le 1\}$. We also fix a 1–parameter family of bundle maps $\kappa_t\colon \nu_{\epsilon_0}(0 \times e_0) \to \nu_{\epsilon_0}(\iota_t)$ with $\kappa_0 = \mathrm{id}$.

- We fix the following classes:

$$\hat{w} \in H_{2d-1}(SM), \quad \omega_\Delta \in H^{2d-1}(SM \times SM, \Delta(SM)^c), \quad w_{S^K} \in H_K(S^K), \quad \omega_{S^K} \in H^K(S^K),$$
$$\omega_\nu \in H^{K-2d+1}(\mathrm{Th}(\nu)), \quad \omega(n) \in H^{n(K-2d+1)}(\mathrm{Th}(\nu^n)), \quad \gamma \in H^d(SM \times SM, (SM \times_M SM)^c).$$

Here $\hat{w}$ is a fundamental class of SM, $\Delta(SM)^c$ is the complement of the tubular neighborhood of the (standard, nondeformed) diagonal, ω_Δ is the diagonal class satisfying the equality

$$(\hat{w} \times \hat{w}) \cap \omega_\Delta = \Delta_*(\hat{w}) \in H_{2d-1}(SM^2),$$

w_{S^K} is the cross product $(w_{S^1})^{\times n}$ of K copies of the class w_{S^1} fixed in Definition 5.1, ω_{S^K} is the class such that $w_{S^K} \cap \omega_{S^K}$ is the class represented by a point, and ω_ν is the Thom class satisfying the equality

$$\kappa_1^*(\omega_\Delta \cdot (\omega_\nu \times \omega_\nu)) = \omega_{S^K} \times \omega_\nu.$$

Here $\omega_\Delta \cdot (\omega_\nu \times \omega_\nu)$ is naturally regarded as a Thom class for the bundle $\nu_{\epsilon_0}(\Delta_{\mathbb{R}^K} \circ e_0)$. We set $\omega(n) = \omega_\nu^{\times n}$. The class γ is a Thom class of a tubular neighborhood of $SM \times_M SM$ in $SM \times SM$.

- We call a graph in $\mathsf{G}(n)$ which does not contain a cycle (a closed path) a *tree*. For a graph $G \in \mathsf{G}(n)$, vertices i and j are said to be *disconnected* in G if i and j belong to different connected components of G.

- For $i < j$, let $\pi_{ij}\colon SM^n \to SM^{\times 2}$ be the projection given by $\pi_{ij}(z_1, \dots, z_n) = (z_i, z_j)$. Set $D_{ij} = D_G$ for $E(G) = \{(i, j)\}$, and

$$\gamma_{ij} = \pi_{ij}^*(\gamma) \in H^d(SM^n, (D_{ij})^c).$$

For a tree $G \in \mathsf{G}(n)$, write $E(G)$ as $\{(i_1, j_1) < \dots < (i_r, j_r)\}$ with $i_t < j_t$ for $t = 1, \dots, r$. We put

$$w_G = \hat{w}^{\times n} \cap \gamma_{i_1,j_1} \cdots \gamma_{i_r,j_r} \in H_{n(2d-1)-rd}(D_G).$$

Clearly w_G is a fundamental class of D_G.

- Let $G \in \mathsf{G}(n, r)$ be a tree. Suppose i and $i+1$ are disconnected in G. Let $d_i\colon \underline{n} \to \underline{n-1}$ be the map given by

$$d_i(j) = \begin{cases} j & \text{if } j \le i, \\ j-1 & \text{if } j \ge i+1, \end{cases}$$

and set $H = d_i(G) \in \mathsf{G}(n-1)$. We define maps

$$\phi_G\colon \overline{H}_*(\mathrm{Th}(\nu_G)) \to H_{*-nK}(\mathcal{T}_G), \quad \zeta_G\colon H_*(\mathcal{T}_G) \to H^{-*-dr}(D_G),$$
$$\mu_i\colon H_*(\mathcal{T}_G) \to H_*(\mathcal{T}_H), \qquad m_i\colon H^*(D_G) \to H^*(D_H).$$

The map ϕ_G is the composition

$$\overline{H}_*(\mathrm{Th}(\nu_G)) \xrightarrow{(\lambda_G)_*} \overline{H}_*(\mathcal{T}_G(\mathfrak{c}_0)_{nK}) \to H_{*-nK}(\mathcal{T}_G(\mathfrak{c}_0)) \to H_{*-nK}(\mathcal{T}_G),$$

where λ_G is the map defined in Definition 4.5, the second map is the canonical one and the third is the inverse of evaluation at $\mathfrak{c}_0$. Clearly ϕ_G is an isomorphism. The map ζ_G is the composition $(w_G \cap -)^{-1} \circ (- \cap \omega(n)) \circ \phi_G^{-1}$ consisting of

$$H_*(\mathcal{T}_G) \xrightarrow{\phi_G^{-1}} \overline{H}_{*+nK}(\mathrm{Th}(\nu_G)) \xrightarrow{-\cap\omega(n)} H_{*+n(2d-1)}(D_G) \xrightarrow{(w_G\cap -)^{-1}} H^{-*-dr}(D_G).$$

The map μ_i is induced by the partial composition $\mu \circ_i -$, where $\mu \in H_0(\mathcal{D}(2)) = \mathcal{A}(2)$ is the fixed generator. The map m_i is given by $(-1)^A \Delta_i^*$, where $A = * + dr + n$ with $r = \#E(G)$, and Δ_i^* denotes the pullback by the restriction to D_H of the diagonal

$$\Delta_i : SM^{n-1} \to SM^n, \quad (z_1, \dots, z_{n-1}) \mapsto (z_1, \dots, z_i, z_i, \dots, z_{n-1}).$$

- We denote by $H\check{\mathrm{T}}^M_{\star *}(n)$ the bigraded chain complex obtained by taking the homology of $\check{\mathrm{T}}^M_{\star *}(n)$ for the differential d; see Definition 5.6. Its differential is induced by the differential $(-1)^q \partial$ on $\check{\mathrm{T}}^M_{\star q}(n)$. We regard the collection $H\check{\mathrm{T}}^M = \{H\check{\mathrm{T}}^M(n)\}$ as an $\mathcal{A}$–comodule with the structure induced by $\check{\mathrm{T}}^M$. As a k–module, $H\check{\mathrm{T}}^M(n)$ is the direct sum $\bigoplus_{G \in \mathsf{G}(n)} H_*(\mathcal{T}_G)$. We denote by aG the element of $H\check{\mathrm{T}}^M(n)$ corresponding to $a \in H_*(\mathcal{T}_G)$.

- The homology of the Hochschild complex $\mathrm{CH}_\bullet(H\check{\mathrm{T}}^M_{\star *})$ has the bidegree $(-\bullet-\star, -*)$. We denote the homogeneous part of bidegree (p,q) by $H_{-p,-q}(\mathrm{CH}(H\check{\mathrm{T}}^M))$.

- For two graphs $G, H \in \mathsf{G}(n)$ with $E(G) \cap E(H) = \varnothing$, the product $GH \in \mathsf{G}(n)$ denotes the graph with $E(GH) = E(G) \cup E(H)$. Let $i, j, k \in \underline{n}$ be distinct vertices, and $[ijk] \in \mathsf{G}(n)$ denote the graph with $E([ijk]) = \{(i,j),(j,k)\}$. For a graph $G \in \mathsf{G}(n)$, the products $G[ijk]$, $G[jki]$ and $G[kij]$ have the same connected component (if they are defined), so $\nu_{G[ijk]} = \nu_{G[jki]} = \nu_{G[kij]}$. Using these equalities, and the isomorphisms $\phi_{G'}$ for $G' = G[ijk], G[jki]$ and $G[kij]$, we identify the three groups $H_*(\mathcal{T}_{GH[ijk]})$, $H_*(\mathcal{T}_{G[jki]})$ and $H_*(\mathcal{T}_{G[kij]})$ with one another. Under this identification, let $I(n) \subset H\check{\mathrm{T}}^M(n)$ be the submodule generated by
  - summands of graphs which are not trees, and
  - elements of the form $aG[jki] + (-1)^s aG[ijk] + (-1)^{s+t} aG[kij]$ for $(i,j),(j,k),(i,k) \notin E(G)$, where $a \in H_*(\mathcal{T}_{G[ijk]})$, $s+1$ is the number of edges of G between (i,j) and (i,k), and $t+1$ is the number of edges between (i,k) and (j,k).

- We say a graph $G \in \mathsf{G}(n)$ with an edge set $E(G) = \{(i_1, j_1) < \cdots < (i_r, j_r)\}$ is *distinguished* if the following inequalities hold:

$$i_1 < j_1, \dots, i_r < j_r, \quad i_1 < \cdots < i_r.$$

We denote by $\mathsf{G}(n)^{\mathrm{dis}} \subset \mathsf{G}(n)$ the subset of the distinguished graphs.

The following lemma is obvious by the definition of the Čech s.s.

Lemma 5.9 *With the notation in Definition 5.8, the E_2–page of Čech s.s. is isomorphic to the homology of the Hochschild complex of $H\check{\mathrm{T}}^M_{\star\star}$. More precisely, there exists an isomorphism of* k*–modules*

$$\check{\mathbb{E}}_2^{pq} \cong H_{-p,-q}(\mathrm{CH}(H\check{\mathrm{T}}^M)) \quad \textit{for each} \quad (p,q). \qquad \square$$

Lemma 5.10 *With the notation in Definition 5.8, $I(n)$ is acyclic, ie $H_\partial(I(n)) = 0$, and the sequence $\{I(n)\}_n$ is closed under the partial compositions and symmetric group actions.*

Proof Since $\mathsf{G}(n)^{\mathrm{dis}}$ is stable under removing edges, the submodule $\bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_*(\mathcal{T}_G)$ of $H\check{\mathrm{T}}^M(n)$ is a subcomplex. By an argument similar to (the dual of) [14], the inclusion

$$\check{\mathrm{T}}(\mathsf{G}(n)^{\mathrm{dis}}) := \bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_*(\mathcal{T}_G) \subset H\check{\mathrm{T}}^M(n)$$

is a quasi-isomorphism. We easily see that the map $\check{\mathrm{T}}(\mathsf{G}(n)^{\mathrm{dis}}) \to \check{\mathrm{T}}(n)/I(n)$ induced by the inclusion is an isomorphism (see the proof of Lemma 6.9). $\square$

Lemma 5.11 *Let $\bar{e}_t : SM \to \mathbb{R}^{2K}$ be an isotopy with $\bar{e}_0 = 0\times e_0$ and $\bar{e}_1 = e_0\times 0$, and $F_t : \nu_{\epsilon_0}(\bar{e}_0) \to \nu_{\epsilon_0}(\bar{e}_t)$ be an isotopy which is also a bundle map covering $\bar{e}_t$ satisfying $F_0 = \mathrm{id}$. Then*

$$(F_1)^*(\omega_\nu \times \omega_{S^K}) = (-1)^K \omega_{S^K} \times \omega_\nu.$$

Here $\omega_\nu \times \omega_{S^K}$ is considered as a class of $H^{2K-2d+1}(\mathrm{Th}(\nu_{\epsilon_0}(\bar{e}_1)))$ via the map collapsing the subset $\nu_{\epsilon_0}(e)\times\mathbb{R}^K - \nu_{\epsilon_0}(\bar{e}_1)$, and $\omega_{S^K}\times\omega_\nu$ is similarly understood.

Proof Since the only problem is the orientation, it is enough to see a variation of a basis via a local model. Let $e_0 : \mathbb{R}^{2d-1} \to \mathbb{R}^K$ be the inclusion to the subspace of elements with the last $K-2d+1$ coordinates being zero. A covering isotopy is given by $F_t(u,v) = ((1-t)u - tv, tu + (1-t)v)$ for $u, v \in \mathbb{R}^K$. Since $F_1(u,v) = (-v,u)$, the derivative $(F_1)_*$ maps a basis $\{\boldsymbol{a},\boldsymbol{b}\}$ of the tangent space of $\mathbb{R}^{2K}$ to $\{\boldsymbol{b},-\boldsymbol{a}\}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ denote bases of $T\mathbb{R}^K\times 0$ and $0\times T\mathbb{R}^K$, respectively. This implies $(F_1)^*(\omega_\nu\times\omega_{S^K}) = (-1)^K(-1)^{K(K-2d+1)}\omega_{S^K}\times\omega_\nu = (-1)^K\omega_{S^K}\times\omega_\nu$. $\square$

Lemma 5.12 *We use the notation in Definition 5.8. Let $G\in\mathsf{G}(n,r)$ be a tree whose vertices i and $i+1$ are disconnected in G. Set $H = d_i(G)\in\mathsf{G}(n-1)$. Then the diagram*

$$\begin{tikzcd} H_*(\mathcal{T}_G) \arrow[r, "\mu_i"] \arrow[d, "\zeta_G"] & H_*(\mathcal{T}_H) \arrow[d, "\varepsilon_1\zeta_H"] \\ H^{-*-dr}(D_G) \arrow[r, "m_i"] & H^{-*-dr}(D_H) \end{tikzcd}$$

is commutative, where $\varepsilon_1 = (-1)^B$ with $B = K\big(*+1+\frac{1}{2}(K-1)\big)$.

Proof The claim follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
H_*(\mathcal{T}_G) \arrow[r, "\mu_i"] & H_*(\mathcal{T}_H) & \\
\overline{H}_{*+nK}(\mathrm{Th}(\nu_G)) \arrow[u, "\phi_G"] \arrow[r, "\mu'"] \arrow[d, "\omega(n)"] & \overline{H}_{*+nK}(\mathrm{Th}(\nu')) \arrow[u, "\phi'_H"] \arrow[d, "\omega'"] & \overline{H}_{*+(n-1)K}(\mathrm{Th}(\nu_H)) \arrow[l, "\alpha"'] \arrow[ul, "\phi_H"'] \arrow[dl, "\varepsilon_1\omega(n-1)"] \\
H_{*+n(2d-1)}(D_G) \arrow[r, "\mu''"] & H_{*+(n-1)(2d-1)}(D_H) & \\
H^{-*-dr}(D_G) \arrow[u, "w_G"] \arrow[r, "m_i"] & H^{-*-dr}(D_H) \arrow[u, "w_H"] &
\end{tikzcd}
\tag{5-1}
$$

Here:

- ν' is the disk bundle over D_H of fiber dimension $nK-(n-1)(2d-1)$ defined by
$$\nu' = \nu_{\epsilon_0}(e_0^n \circ \Delta_i)|_{D_H},$$
where the restriction is taken as a disk bundle over SM^{n-1}; see Definition 5.8 for Δ_i.
- $\omega' \in \overline{H}^{nK-(n-1)(2d-1)}(\mathrm{Th}(\nu'))$ is given by
$$\omega' = (-1)^C(\omega_\nu)^{\times i-1} \times \omega_\Delta \cdot (\omega_\nu \times \omega_\nu) \times (\omega_\nu)^{\times n-i-1} \quad \text{with } C = (n+i+1)K.$$
- ϕ'_H is defined by using the following map λ'_H similarly to ϕ_H:
$$\lambda'_H : \nu^n \ni u \mapsto (e_0^n \circ \Delta_i, \epsilon_0, u) \in \mathcal{T}(\mathfrak{c}_0)_{nK}.$$
- μ' is the map collapsing the subset $\nu_G - \nu'$, where ν' and ν_G are regarded as subsets in $\mathbb{R}^{nK}$.
- μ'' is the composition
$$H_*(D_G) \to H_*(D_G, \Delta_i(D_H)^c) \to H_{*-2d+1}(\Delta_i(D_H)) \cong H_{*-2d+1}(D_H),$$
where the first map is the standard quotient map, the third is the inverse of the diagonal and the second is the cap product with the class
$$(-1)^{i+1+n} 1 \times \cdots \times \omega_\Delta \times \cdots \times 1 \quad \text{with } \omega_\Delta \text{ in the } i^{\text{th}} \text{ factor}.$$
- α is the composition $(1\times\kappa_1\times 1)_* \circ T \circ (\varepsilon_2 w_{S^K} \times -)$ of the maps
$$\overline{H}_{*'}(\mathrm{Th}(\nu_H)) \xrightarrow{\varepsilon_2 w_{S^K}\times -} \overline{H}_{*'+K}(S^K \wedge \mathrm{Th}(\nu_H)) \xrightarrow{T} \overline{H}_{*'+K}(\mathrm{Th}(\nu'')) \xrightarrow{(1\times\kappa_1\times 1)_*} \overline{H}_{*'+K}(\mathrm{Th}(\nu')),$$
where ν'' is the disk bundle over D_H of the same fiber dimension as ν' given by
$$\nu'' = \nu_{\epsilon_0}(e'')|_{D_H} \quad \text{with } e'' = e_0^{\times i-1} \times (0\times e) \times e_0^{\times n-i} : SM^{n-1} \to \mathbb{R}^{nK},$$
T is the composition of the transposition of S^K from the first to the i^{th} component with the map induced by the map collapsing the subset $(\nu^{\times i-1} \times \mathbb{R}^K \times \nu^{\times n-i+1})|_{D_H} - \nu''$,
$$\varepsilon_2 = (-1)^D, \quad D = K\big(*' + \tfrac{1}{2}(K-1) + i + 1\big),$$

and $1\times\kappa_1\times 1$ is induced by the restriction of the product map

$$1\times\kappa_1\times 1\colon \mathbb{R}^{(i-1)K}\times\nu_{\epsilon_0}(0\times e_0)\times\mathbb{R}^{(n-i-1)K}\to\mathbb{R}^{(i-1)K}\times\nu_{\epsilon_0}(\Delta_{\mathbb{R}^K}\circ e_0)\times\mathbb{R}^{(n-i-1)K}$$

with κ_1 in the i^{th} component.

- The arrows with a (co)homology class denote the map given by taking the cap product with the class. For example, the right vertical arrow of the middle square denotes the map $x\mapsto x\cap\omega'$.

Our sign rules for graded products are the usual graded commutativity, except for the compatibility of cross and cap products, for which we use the rule

$$(a\times b)\cap(x\times y)=(-1)^{(|a|-|x|)|y|}(a\cap x)\times(b\cap y).$$

These are the rules based on the definitions in [19]. More precisely, we use the homology cross product induced by the simplicial cross product in [19, page 277] (or equivalently, the Eilenberg–Zilber shuffle map) and the cohomology cross product defined by $a\times b=p_1^*a\cup p_2^*b$ where p_i is the projection to the i^{th} component of the product and the cup product is given in [19, page 215]. We also use the cap product given in [19, page 239]. (This irregular sign rule is caused by absence of sign in the definition of cup product, as is standard.) With these rules, the commutativity of the squares in (5-1) is clear since the map Δ' defined in Section 3.2 is isotopic to the usual diagonal. We shall prove commutativity of the two triangles. The commutativity of the upper triangle follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\overline{H}_l(\mathrm{Th}(\nu)_H) \arrow[r, "T\circ(\varepsilon_2 w_{S^K}\times -)"] \arrow[d, "(\lambda_G)_*"] & \overline{H}_{l+K}(\mathrm{Th}(\nu'')) \arrow[r, "(1\times\kappa_1\times 1)_*"] \arrow[dr, "(\lambda''_H)_*"] & \overline{H}_{l+K}(\mathrm{Th}(\nu')) \arrow[d, "(\lambda'_H)_*"] \\
\overline{H}_l(\mathcal{T}_H(\mathfrak{c}_0)_{(n-1)K}) \arrow[rr, "\varepsilon_2 w_{S^K}\times"] & & H_{l+K}(\mathcal{T}_H(\mathfrak{c}_0)_{nK})
\end{tikzcd}$$

Here λ''_H is given by $u\mapsto(e_0^{\times i-1}\times(0\times e_0)\times e_0^{\times n-i},\epsilon_0,u)$. Commutativity of the left trapezoid follows from Lemma 5.11 (the sign ε_2 is the product of the sign in i_k^X in Definition 5.1 and the sign in Lemma 5.11), and that of the right triangle follows from the homotopy between $\lambda'_H\circ\kappa_1$ and λ''_H constructed from the isotopy κ_t in Definition 5.6. We shall show that the lower triangle is commutative. We see

$$\begin{aligned}
\varepsilon_1\alpha(x)\cap\omega' &= \{(\kappa_1)_*T_*(w_{S^K}\times x)\}\cap(\omega\times\cdots\times\omega_\Delta(\omega\times\omega)\times\cdots\times\omega)\\
&= \{(\kappa_1)_*T_*(w_{S^K}\times x)\}\cap(\omega\times\cdots\times(\kappa_1^{-1})^*(\omega_{S^K}\times\omega)\times\cdots\times\omega)\\
&= (\kappa_1)_*\{T_*(w_{S^K}\times x)\cap(\omega\times\cdots\times(\omega_{S^K}\times\omega)\times\cdots\times\omega)\}\\
&= (\kappa_1)_*T_*\{(w_{S^K}\times x)\cap T^*(\omega\times\cdots\times(\omega_{S^K}\times\omega)\times\cdots\times\omega)\}\\
&= (\kappa_1)_*T_*\{(w_{S^K}\times x)\cap\omega_{S^K}\times\omega\times\cdots\times\omega)\}\\
&= (w_{S^K}\times x)\cap(\omega_{S^K}\times\omega\times\cdots\times\omega)=x\cap\omega(n-1).
\end{aligned}$$

Here $(\kappa_1)_*$ is an abbreviation of $(1\times\kappa_1\times 1)_*$ and ω of ω_ν. All the capped classes are considered as elements of the homology of the base space D_H of involved disk bundles by projections. The second equality follows from the definition of ω_ν. As endomorphisms on the base space, T_* and $(1\times\kappa_1\times 1)_*$ are the identity, and hence the sixth equality holds. ◻

The following lemma is easily verified and a proof is omitted.

Lemma 5.13 *Let $G \in \mathsf{G}(n,r)$ be a tree and $K \in \mathsf{G}(n,r-1)$ be the tree made by removing the t^{th} edge (i,j) from G. Under the notation in Definition 5.8, the diagram*

$$\begin{tikzcd} H_*(\mathcal{T}_G) \arrow[r] \arrow[d, "\zeta_G"] & H_*(\mathcal{T}_K) \arrow[d, "\zeta_K"] \\ H^{-*-dr}(D_G) \arrow[r] & H^{-*-d(r-1)}(D_K) \end{tikzcd}$$

is commutative, where the top horizontal arrow is induced by the inclusion and the bottom one is given by $(-1)^{(r-t)d}\Delta^!_{ij}$ with $\Delta^!_{ij}(x) = \gamma_{ij}\cdot x$. □

Definition 5.14 • In the following, for a module X with a decomposition $X = \bigoplus_{G\in\mathsf{G}(n)} X_G$, we denote by $X^{\mathrm{tr}} \subset X$ the direct sum of the summands X_G labeled by a tree G.

• We define an $\mathcal{A}$–comodule $A^{\star *}_M$ of CDBA (see Definition 2.10). Put $H^*_G = H^*(D_G)$. Let $\wedge(g_{ij})$ be the free bigraded commutative algebra generated by elements g_{ij} for $1 \le i < j \le n$, with bidegree $(-1,d)$. For notational convenience, we set $g_{ij} = (-1)^d g_{ji}$ for $i > j$ and $g_{ii} = 0$. For $G \in \mathsf{G}(n)$ with $E(G) = \{(i_1,j_1) < \cdots < (i_r,j_r)\}$, we set $g_G = g_{i_1,j_1}\cdots g_{i_r,j_r}$. Put

$$\widetilde{A}^{\star *}_M(n) = \bigoplus_{G\in\mathsf{G}(n)} H^*_G g_G.$$

Here $H^*_G g_G$ is a copy of H^*_G with degree shift. For $G \in \mathsf{G}(n,r)$ and $a \in H^l_G$, the bidegree of the element $a g_G \in \widetilde{A}_M(n)$ is $(-r, l+dr)$. We give a graded commutative multiplication on $\widetilde{A}_M(n)$ as follows. For $a \in H^l_G$ and $b \in H^m_H$, we set

$$(ag_G)\cdot(bg_H) = \begin{cases} (-1)^{mr(d-1)+s}(a\cdot b)g_{GH} \in H^{l+m}_{GH} g_{GH} & \text{if } E(G)\cap E(H) = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Here we set $r = \#E(G)$, a is regarded as an element of H^*_{GH} by pulling back by the map $i_G : \Delta_{GH} \to D_G$ induced by the quotient map $\pi_0(G) \to \pi_0(GH)$, and similarly for b, and the product $a\cdot b$ is taken in H^*_{GH}. Also, s is the number determined by the equality $g_G \cdot g_H = (-1)^s g_{GH}$ for the product in $\wedge(g_{ij})$.

Let $J(n) \subset \widetilde{A}_M(n)$ be the ideal generated by the elements

$$a(g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij})g_G \quad \text{and} \quad bg_K,$$

where $G, K \in \mathsf{G}(n)$, $a \in H^*_{G[ijk]}$ and $b \in H^*_K$ are elements such that $(i,j),(j,k),(k,i) \notin E(G)$, and K is not a tree. Here by definition, D_G depends only on $\pi_0(G)$, so $\Delta_{G[ijk]} = \Delta_{G[jki]} = \Delta_{G[kij]}$. With these identities, we regard a as an element of $H_{G[jki]} = H_{G[kij]}$, and the first type of generators as elements of

$$H_{G[ijk]}g_{G[ijk]} \oplus H_{G[jki]}g_{G[jki]} \oplus H_{G[kij]}g_{G[kij]}.$$

We define an algebra $A^{\star *}_M(n)$ as the following quotient:

$$A^{\star *}_M(n) = \widetilde{A}^{\star *}_M(n)/J(n).$$

Since the restriction of the quotient map $\widetilde{A}_M(n)^{\mathrm{tr}} \to A_M(n)$ is surjective, we may define a differential, a partial composition and an action of Σ_n on the sequence $A_M = \{A_M(n)\}_n$ through $\widetilde{A}_M(n)^{\mathrm{tr}}$. We define a map $\tilde{\partial}\colon \widetilde{A}_M(n)^{\mathrm{tr}} \to \widetilde{A}_M(n)^{\mathrm{tr}}$ by

$$\tilde{\partial}(ag_G) = \sum_{t=1}^{r} (-1)^{(l+t-1)(d-1)} \Delta^!_{i_t,j_t}(a) g_{i_1,j_1} \cdots \check{g}_{i_t,j_t} \cdots g_{i_r,j_r} \quad \text{for } G \in \mathsf{G}(n) \text{ and } a \in H^l_G,$$

where $\Delta^!_{ij}(a) = \gamma_{ij} \cdot a$ and $\check{g}_{ij}$ means removing g_{ij}. It is easy to see $\tilde{\partial}(\widetilde{A}_M(n)^{\mathrm{tr}} \cap J(n)) \subset \widetilde{A}_M(n)^{\mathrm{tr}} \cap J(n)$. We define the differential ∂ on $A_M(n)$ to be the map induced by $\tilde{\partial}$. For the generator $\mu \in \mathcal{A}(2)$ fixed in Definition 5.8 and an element $ag_G \in \widetilde{A}_M(n)^{\mathrm{tr}}$, we define the partial composition $\mu \circ_i (ag_G)$ by

$$\mu \circ_i (ag_G) = \begin{cases} \Delta^*_i(a) g_H & \text{if } i \text{ and } i+1 \text{ are disconnected in } G, \\ 0 & \text{otherwise,} \end{cases}$$

where $H = d_i(G)$; see Definition 5.8. The action of $\sigma \in \Sigma_n$ on $\widetilde{A}_M(n)^{\mathrm{tr}}$ is given by $(ag_G)^\sigma = a^\sigma (g_G)^\sigma$, where a^σ is the pullback of a by $(\sigma_G)^{-1}$ (see Definition 5.6) and $(g_G)^\sigma$ denotes $g_{\tau(i_1)\tau(j_1)} \cdots g_{\tau(i_r)\tau(j_r)}$ with $\tau = \sigma^{-1}$. The partial composition and the action of Σ_n on $\{\widetilde{A}_M(n)^{\mathrm{tr}}\}_n$ are easily seen to preserve the submodule $\{J(n) \cap \widetilde{A}_M(n)^{\mathrm{tr}}\}_n$ and induce a structure of an $\mathcal{A}$–comodule on A_M.

- Let $s_i : \underline{n} \to \underline{n+1}$ denote the order-preserving monomorphism skipping $i+1$ for $1 \le i \le n$. Then s_i naturally induces a monomorphism $s_i : \pi_0(G) \to \pi_0(s_i G)$ (see Section 2.1), which in turn induces $(s_i)^*\colon D_{s_i G} \to D_G$. Let s_i also denote the induced map $(s_i^*)^*\colon H^*(D_G) \to H^*(D_{s_i G})$. By further abuse of notation, we also denote by s_i the map $A_M(n) \to A_M(n+1)$ given by $s_i(ag_G) = s_i(a) g_{s_i G}$.
- Define a simplicial CDBA $A^{\star *}_\bullet(M)$ (a functor from Δ^{op} to the category of CDBAs) as follows. We set

$$A^{\star *}_n(M) = A^{\star *}_M(n+1).$$

If we consider an element of $A_M(n+1)$ as an element of $A_n(M)$, we relabel its subscripts with $0, 1, \ldots, n$ instead of $1, 2, \ldots, n+1$. For example, $g_{01} \in A_n(M)$ corresponds to $g_{12} \in A_M(n+1)$. The partial compositions and the maps s_i (defined in the previous item) are also considered as beginning with $(- \circ_0 -)$ and s_0 (originally written as $(- \circ_1 -)$ and s_1). The face map $d_i : A_n(M) \to A_{n-1}(M)$ for $0 \le i \le n$ is given by $d_i = \mu \circ_i (-)$ for $i < n$ and $d_n = \mu \circ_0 (-)^\sigma$, where $\sigma = (n, 0, 1, \ldots, n-1)$. The degeneracy map $s_i : A_n(M) \to A_{n+1}(M)$ for $0 \le i \le n$ is the map defined in the previous item.

Lemma 5.15 *Let i, j and k be numbers with $i < j < k$. The equalities $\gamma_{ij}\gamma_{ik} = \gamma_{ij}\gamma_{jk} = \gamma_{ik}\gamma_{jk}$ hold.*

Proof The three classes are Thom classes in $H^*(SM^n, \Delta^c_{[ijk]})$. So to prove the equality, it is enough to identify the corresponding orientations. This is easily done by observing the corresponding bases. □

Theorem 5.16 *Suppose M is orientable.*

1. *The two $\mathcal{A}$–comodules $H\check{\mathrm{T}}^M_{\star\star}$ and $A^{\star\star}_M$ of differential bigraded k–modules are quasi-isomorphic in a manner where $H\check{\mathrm{T}}^M_{-p,-q}$ and $A^{p,q}_M$ correspond for integers p and q. (For $H\check{\mathrm{T}}^M$, see Definition 5.8.)*
2. *The E_2–page of the Čech s.s. in Definition 5.6 is isomorphic to the total homology of the normalized complex $NA^{\star *}_\bullet(M)$. Under this isomorphism, the homogeneous part $\check{\mathbb{E}}^{pq}_2$ consists of the classes*

represented by a sum of elements in the complex, whose triple degree $(-\bullet, \star, *)$ *satisfies* $p = \star - \bullet$ *and* $q = *$.

The latter part of (2) of this theorem may need some care. It does not mean that the E_2–page is generated by the classes which are represented by elements of $NA(M)$ which are homogeneous for each of the three degrees, since the differential of the E_1–page of the Čech s.s. corresponds to the total differential of $NA(M)$ and changes both of the degrees $\star$ and $\bullet$.

Proof For (1), we consider the composition

$$H_{-*}(\mathcal{T}_G) \xrightarrow{\zeta_G} H_G^{*-dr} \to H_G^{*-dr} g_G.$$

The right map is given by $a \mapsto \varepsilon_3 a g_G$ with the sign

$$\varepsilon_3 = (-1)^E \quad \text{where } E = E(*', n, r) = *'(n+dr) + drn + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}dr(r+1)$$

on $H_G^{*'}$. This composition defines an isomorphism as bigraded $\Bbbk$–modules between $H\check{\mathrm{T}}^M(n)^{\mathrm{tr}}$ and $\widetilde{A}_M(n)^{\mathrm{tr}}$. By Lemma 5.15, this isomorphism maps $H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}} \cap I(n)$ into $\widetilde{A}_M^{\star *}(n)^{\mathrm{tr}} \cap J(n)$ isomorphically. A quasi-isomorphism $H\check{\mathrm{T}}^M(n) \to A_M(n)$ is defined by the composition

$$\begin{aligned} H\check{\mathrm{T}}^M_{-\star,-*}(n) \to H\check{\mathrm{T}}^M_{-\star,-*}(n)/I(n) &\cong H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}}/\{H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}} \cap I(n)\} \\ &\cong \widetilde{A}_M^{\star *}(n)^{\mathrm{tr}}/\{\widetilde{A}_M^{\star *}(n)^{\mathrm{tr}} \cap J(n)\} \cong A_M^{\star *}(n), \end{aligned}$$

where the first map is the quotient map, which is a quasi-isomorphism by Lemma 5.10, the second and fourth maps are induced by inclusions, and the third map is the isomorphism defined above. For the above number E, we see

$$E(*', n-1, r) - E(*', n, r) \equiv *' + dr + n \quad \text{and} \quad E(*'+d, n, r-1) - E(*', n, r) \equiv (*'+1)d$$

modulo 2. Now we may assume the integer K is a multiple of 4. With this assumption and the above equalities for E, compatibility of the quasi-isomorphism with the partial composition follows from Lemma 5.12 as $\varepsilon_1 = 1$. Compatibility with the (Čech) differentials follows from Lemma 5.13. Compatibility with the actions of Σ_n is clear. The sign $\mathrm{sgn}(\sigma_G)$ in Definition 5.6, the sign occurring in permutations of γ_{ij} and the sign occurring in permutations of g_{ij} are canceled. Thus the isomorphism is a morphism of $\mathcal{A}$–comodules. For (2), by (1), the E_2–page is isomorphic to the homology of the Hochschild complex $\mathrm{CH}_\bullet(A_M)$, which is isomorphic to the unnormalized total complex of $A_\bullet(M)$, and so is quasi-isomorphic to the normalized complex. □

Sinha proved the convergence of his spectral sequences using the Cohen–Taylor spectral sequence. Here we prove the convergence of the Čech and Sinha spectral sequences simultaneously by an independent method.

Theorem 5.17 *If M is simply connected and of dimension $d \geq 4$, both the Čech s.s. and Sinha s.s. for M converge to* $H^*(\mathrm{Emb}(S^1, M))$.

Proof We set a number s_d by $s_d = \min\{\frac{1}{3}d, 2\}$. If $d \geq 4$, clearly $s_d > 1$. Recall that $\{\check{\mathbb{E}}_r\}_r$ denotes the Čech s.s. By Lemma 5.7, we identify the Sinha s.s. with the spectral sequence $\overline{\mathbb{E}}_r$. We shall first show the claim that $\check{\mathbb{E}}_2^{-p,q} = 0$ if $q/p < s_d$. If a graph $G \in \mathsf{G}(n+1)$ has k discrete vertices, $H^*(D_G)$ is isomorphic to $H^*(SM)^{\otimes k} \otimes H^*(D_{G'}) \otimes \{\text{torsions}\}$, where $G' \in \mathsf{G}(n+1-k)$ is the graph made by removing discrete vertices. With this observation, and simple connectivity of M, we see that generators of the normalization $NA_n(M)$ are presented as $a_1 \cdots a_k b g_G$ where G is a graph in $\mathsf{G}(n+1)$ with r edges and k discrete vertices except for the vertex 0, a_t belongs to the t^{th} discrete tensor factor $H^{\geq 2}(SM)$, and $b \in H^*_{G'}$. We may ignore the torsion part in estimation of degree by the universal coefficient theorem. The bidegree $(-p,q)$ of this element satisfies $p = n+r$ and $q \geq 2k + rd$. Clearly we have $k + 2r \geq n + \epsilon$, with $\epsilon = 0$ or 1 according to whether the vertex 0 has valence 0 in G. With this, if $d \leq 5$, we have the following estimate:

$$\frac{q}{p} - \tfrac{1}{3}d \geq \frac{6k + (3r-p)d}{3(n+r)} \geq \frac{(6-d)k + d\epsilon}{3(n+r)} \geq 0.$$

If $d \geq 6$, we have the following estimate:

$$\frac{q}{p} - 2 = \frac{2\epsilon + (d-6)r}{n+r} \geq 0.$$

We have shown the claim. Since the filtration $\{F^{-p}\}$ of the Čech s.s. is exhaustive, and the total homology of each F^{-p} is of finite type, the Čech s.s. $\{\check{\mathbb{E}}_r\}_r$ converges to the total homology $H(NA_\bullet(M))$ of the normalized complex. By the same reasoning, $\{\bar{\mathbb{E}}_r\}$ also converges to $H(NA_\bullet(M))$. We shall show $\bar{\mathbb{E}}_r^{-p,q} = 0$ if $q/p < s_d$, for sufficiently large r. Suppose there exists a nonzero element $x \in \bar{\mathbb{E}}_\infty^{-p,q}$ with $q/p < s_d$. We consider x as an element of $(\bar{F}^{-p}/\bar{F}^{-p+1})H(NA_\bullet(M))$. Take a class x' in $\bar{F}^{-p}H(NA_\bullet(M))$ representing x. Take the smallest p' such that $F^{-p'}H(NA_\bullet(M))$ contains x'. Then $\check{\mathbb{E}}_\infty^{-p',q+p'-p}$ is not zero and $p' \geq p$ as $\bar{F}^{-p} \supset F^{-p}$. In the coordinate plane of bidegree, x' and x are on the same line of the form $-p+q = \text{constant}$. This and $p' \geq p$ imply that the "slope" of x' is smaller than s_d, which contradicts to the claim. This vanishing result on $\bar{\mathbb{E}}_r$ and (the cohomology version of) [4, Theorem 3.4] imply the convergence of $\bar{\mathbb{E}}_r$ and $\check{\mathbb{E}}_r$ to $H^*(\mathrm{Emb}(S^1, M))$. □

Remark 5.18 If the dimension of the target manifold M is 3, or if M is not simply connected, the Čech s.s. does not converge to the cohomology of the knot space but it does to the same target as the Sinha s.s. (see the proof of Theorem 5.17). The diagonal of the Sinha s.s. for long knots converges to the universal finite type invariants at least in the rational coefficient. So the Čech s.s. in dimension 3 may contain some information about knot invariants.

6 Algebraic presentations of the E_2–page of the Čech spectral sequence

In this section, we assume M is oriented and simply connected and $H^*(M)$ is a free k–module.

Definition 6.1 • A *Poincaré algebra of dimension d* is a graded commutative algebra $\mathcal{H}^*$ with a linear isomorphism $\epsilon\colon \mathcal{H}^d \to \mathsf{k}$ such that the bilinear form defined as the composition

$$\mathcal{H}^* \otimes \mathcal{H}^* \xrightarrow{\text{multiplication}} \mathcal{H}^* \xrightarrow{\text{projection}} \mathcal{H}^d \xrightarrow{\epsilon} \mathsf{k}$$

induces an isomorphism $\mathcal{H}^* \cong (\mathcal{H}^{d-*})^\vee$. We call ϵ the *orientation of* $\mathcal{H}$.

- For a Poincaré algebra $\mathcal{H}^*$, we denote by $\Delta_{\mathcal{H}}$ the *diagonal class for* $\mathcal{H}^*$ given by

$$\sum_i (-1)^{|a_i^*|} a_i \otimes a_i^* \in (\mathcal{H} \otimes \mathcal{H})^d,$$

where $\{a_i\}$ and $\{a_i^*\}$ are two bases of $\mathcal{H}^*$ such that $\epsilon(a_i \cdot a_j^*) = \delta_{ij}$, the Kronecker delta. This definition does not depend on a choice of a basis $\{a_i\}$.

- Let $\mathcal{H}$ be a Poincaré algebra $\mathcal{H}$ of dimension d with $\mathcal{H}^1 = 0$. We set $\mathcal{H}^{\leq d-2} = \bigoplus_{p \leq d-2} \mathcal{H}^p$ and $\mathcal{H}^{\geq 2} = \bigoplus_{p \geq 2} \mathcal{H}^p$, and define a graded k–module $\mathcal{H}^{\geq 2}[d-1]$ by $(\mathcal{H}^{\geq 2}[d-1])^p = X^{p-d+1}$ with $X^* = \mathcal{H}^{\geq 2}$. We denote by $\bar{a}$ the element in $(\mathcal{H}^{\geq 2}[d-1])^p$ corresponding to $a \in \mathcal{H}^{p-d+1}$. We define a Poincaré algebra $S\mathcal{H}$ of dimension $2d-1$ as follows. As a graded k–module, we set

$$S\mathcal{H}^* = \mathcal{H}^{\leq d-2} \oplus \mathcal{H}^{\geq 2}[d-1].$$

For $a, b \in \mathcal{H}^{\leq d-2}$, the multiplication $a \cdot b$ in $S\mathcal{H}$ is the one in $\mathcal{H}$ except for the case $|a| + |b| = d$, in which we set $a \cdot b = 0$. We set $a \cdot \bar{b} = \overline{ab}$ for $a \in \mathcal{H}^{\leq d-2}$ and $b \in \mathcal{H}^{\geq 2}$, and $\bar{a} \cdot \bar{b} = 0$ for $a, b \in \mathcal{H}^{\geq 2}$. We give the same orientation on $S\mathcal{H}$ as the one on $\mathcal{H}$ via the identity $S\mathcal{H}^{2d-1} = \mathcal{H}^d$.

- We regard $\mathcal{H} = H^*(M)$ as a Poincaré algebra with the orientation

$$H^d(M) \xrightarrow{w_M \cap} H_0(M) \cong \mathsf{k},$$

where w_M is the fundamental class of M determined by the orientation on M, and the isomorphism sends the class represented by a point to 1.

The following lemma is obvious:

Lemma 6.2 *With the notation of Definition 6.1, let $(b_{ij})_{ij}$ denote the inverse of the matrix $(\epsilon(a_i \cdot a_j))_{ij}$. Then*

$$\Delta_{\mathcal{H}} = \sum_{i,j} (-1)^{|a_j|} b_{ji} a_i \otimes a_j. \qquad \square$$

Under some assumptions, $S\mathcal{H}$ is isomorphic to $H^*(SM)$ (see the proof of Lemma 6.6), and the algebras $A^*_{\mathcal{H},G}$ and $B^*_{\mathcal{H},G}$ defined as follows are isomorphic to $H^*(D_G)$.

Definition 6.3 For a Poincaré algebra $\mathcal{H}$ of dimension d and graph $G \in \mathsf{G}(n)$, define a graded commutative algebra $A_{\mathcal{H},G}$ by

$$A_{\mathcal{H},G} = \mathcal{H}^{\otimes \pi_0(G)} \otimes \bigwedge \{y_1, \dots, y_n\}, \quad \deg y_i = d-1.$$

Here we regard $\pi_0(G)$ as an ordered set by the minimum in each component, and the tensor product is taken in this order. Furthermore, we also define a graded commutative algebra $B_{\mathcal{H},G}$ by

$$B_{\mathcal{H},G} = S\mathcal{H}^{\otimes n} \otimes \bigwedge \{y_{ij} \mid 1 \leq i, j \leq n \text{ and } i \sim_G j\}/J_G, \quad \deg y_{ij} = d-1.$$

Here $i \sim_G j$ means that the vertices i and j belong to the same connected component of G, and J_G is the ideal generated by the following relation:

$$\{e_i(a)-e_j(a), e_i(\bar{a})-e_j(\bar{a})-ay_{ij}, e_i(\bar{b})-e_j(\bar{b}), y_{ii}, y_{ij}+y_{jk}-y_{ik} \\ \mid a \in \mathcal{H}^{\leq d-2}, b \in \mathcal{H}^d, 1 \leq i,j,k \leq n, i \sim_G j \sim_G k\}.$$

Here $e_j(\bar{a})$ is regarded as 0 if $a \in \mathcal{H}^0$.

For $i < j$, let $f_{ij}\colon \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes n}$ denote the map given by

$$f_{ij}(a \otimes b) = 1 \otimes \cdots \otimes a \otimes \cdots \otimes b \otimes \cdots \otimes 1$$

(a is the i^{th} factor, b is the j^{th} factor and the other factors are 1). We set

$$\Delta_{\mathcal{H}}^{ij} = f_{ij}(\Delta_{\mathcal{H}}) \in \mathcal{H}^{\otimes n}.$$

We sometimes regard $\Delta_{\mathcal{H}}^{ij}$ as an element of $(S\mathcal{H})^{\otimes n}$ via the projection and inclusion $\mathcal{H} \to \mathcal{H}^{\leq d-1} \subset S\mathcal{H}$. We also regard it as an element of $A_{\mathcal{H},G}$ for a graph G via the map $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes \pi_0(G)}$ given by multiplication of factors in the same components with the standard commuting signs. We also set

$$\Delta_{S\mathcal{H}}^{ij} = f'_{ij}(\Delta_{S\mathcal{H}}) \in S\mathcal{H}^{\otimes n},$$

where $\Delta_{S\mathcal{H}}$ is the diagonal class for the Poincaré algebra $S\mathcal{H}$ and $f'_{ij}\colon S\mathcal{H}^{\otimes 2} \to S\mathcal{H}^{\otimes n}$ is the map defined by the same formula as f_{ij}. We regard $\Delta_{\mathcal{H}}^{ij}$ and $\Delta_{S\mathcal{H}}^{ij}$ as elements of $B_{\mathcal{H},G}$, similarly to the case of $A_{\mathcal{H},G}$.

As a graded algebra, $B^*_{\mathcal{H},G}$ is isomorphic to $(S\mathcal{H})^{\otimes \pi_0(G)} \bigwedge\{y_{ij} \mid i \sim_G j\}/(y_{ii}, y_{ij}+y_{jk}-y_{ik})$, but we need the presentation to describe maps induced by identifying vertices and removing edges.

The proof of the following lemma is easy and omitted.

Lemma 6.4 *Consider the Serre spectral sequence for a fibration*

$$F \to E \to B$$

with the base simply connected and the cohomology groups of the fiber and base finitely generated in each degree. If for each k there is at most a single p such that $E_\infty^{p,k-p} \neq 0$, the quotient map $F^p \to F^p/F^{p+1}$ has a unique section which preserves cohomological degree. Gathering these sections for all p, one can define an isomorphism of graded algebra $E_\infty \to H^*(E)$, which we call the ***canonical isomorphism****. The canonical isomorphisms are natural for maps between fibrations satisfying the above assumption.* □

Henceforth we regard the Euler number $\chi(M)$ as an element of the base ring k via the ring map $\mathbb{Z} \to \mathsf{k}$, and $\mathsf{k}^\times \subset \mathsf{k}$ denotes the subsets of the invertible elements.

Lemma 6.5 *We use the notation d_i, Δ_i, s_i and $\Delta^!_{ij}$ given in Definitions 5.8 and 5.14. Suppose $\chi(M) = 0 \in \mathsf{k}$. Set $\mathcal{H}^* = H^*(M)$. There exists a family of isomorphisms of graded algebras*

$$\{\varphi_G\colon A_{\mathcal{H},G} \xrightarrow{\cong} H^*(D_G) \mid n \geq 1, G \in \mathsf{G}(n)\}$$

which satisfies the following conditions:

(1) *Let $G \in \mathsf{G}(n)$ be a tree with i and $i+1$ disconnected. Set $H = d_i(G)$. The following diagram is commutative*:

$$\begin{tikzcd} A^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_i"] & H^*(D_G) \arrow[d, "\Delta_i^*"] \\ A^*_{\mathcal{H},H} \arrow[r, "\varphi_H"] & H^*(D_H) \end{tikzcd}$$

Here the algebra map $\bar{\Delta}_i$ is defined as follows. For $a_1 \otimes \cdots \otimes a_p \in \mathcal{H}^{\otimes \pi_0(G)}$, we set

$$\bar{\Delta}_i(a_1 \otimes \cdots \otimes a_p) = \pm a_1 \otimes \cdots \otimes a_s \cdot a_t \otimes \cdots \otimes a_p \qquad \text{and} \qquad \bar{\Delta}_i(y_j) = y_{j'} \quad \text{with } j' = d_i(j).$$

Here $s, t \in \pi_0(G)$ are the connected components containing i and $i+1$, respectively, and $\pm$ is the standard sign in transposing graded elements.

(2) *For a graph $G \in \mathsf{G}(n)$, set $S = s_i(G)$. The following diagram is commutative*:

$$\begin{tikzcd} A^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{s}_i"] & H^*(D_G) \arrow[d, "s_i"] \\ A^*_{\mathcal{H},S} \arrow[r, "\varphi_S"] & H^*(D_S) \end{tikzcd}$$

Here $\bar{s}_i$ is given by inserting the unit 1 as the factor of $H^{\otimes \pi_0(G)}$ which corresponds to the component containing $i+1$, and by skipping the subscript $i+1$, ie by the equality $\bar{s}_i(y_j) = y_{s_i(j)}$.

(3) *For a graph $G \in \mathsf{G}(n)$ and a permutation $\sigma \in \Sigma_n$, the following diagram is commutative*:

$$\begin{tikzcd} A^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\sigma}"] & H^*(D_G) \arrow[d, "\sigma"] \\ A^*_{\mathcal{H},\tau(G)} \arrow[r, "\varphi_{\tau(G)}"] & H^*(D_{\tau(G)}) \end{tikzcd}$$

Here $\tau = \sigma^{-1}$, the right vertical arrow is induced by the natural permutation of factors of the product $D_{\tau(G)} \to D_G$ and the left vertical arrow $\bar{\sigma}$ is the algebra map given by the permutation of tensor factors and subscripts.

(4) *For an edge (i, j) of a tree $G \in \mathsf{G}(n)$ with $i < j$, we define $K \in \mathsf{G}(n)$ by $E(K) = E(G) - \{(i, j)\}$. The following diagram is commutative*:

$$\begin{tikzcd} A^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_{ij}"] & H^*(D_G) \arrow[d, "\Delta^!_{ij}"] \\ A^{*+d}_{\mathcal{H},K} \arrow[r, "\varphi_K"] & H^{*+d}(D_K) \end{tikzcd}$$

Here $\bar{\Delta}_{ij}$ is the right $A^*_{\mathcal{H},K}$–module homomorphism determined by $\bar{\Delta}_{ij}(1) = \Delta^{ij}_{\mathcal{H}}$, and $A^*_{\mathcal{H},G}$ is considered as an $A^*_{\mathcal{H},K}$–module via the natural algebra map $A^*_{\mathcal{H},K} \to A^*_{\mathcal{H},G}$.

Proof In this proof we fix a generator y of $H^{d-1}(S^{d-1})$, and we denote by y_i (or $\bar{y}_i$) the image of y by the inclusion to the i^{th} factor, $H^{d-1}(S^{d-1}) \to H^{d-1}(S^{d-1})^{\otimes n}$. We consider Serre spectral sequence for the fibration

$$(S^{d-1})^n \to D_G \to M^{\pi_0(G)},$$

where the projection is the restriction of that of the tangent sphere bundle. The first possibly nontrivial differential is $d_d\colon H^{d-1}((S^{d-1})^n) = E_d^{0,d-1} \to E_d^{d,0} = H^d(M)$, where d in the super- and subscripts is the dimension of M. This differential takes y_i to the generator of $H^d(M)$ multiplied by $\chi(M)$. As $\chi(M) = 0$, we have $d_d = 0$. Since the this differential on y_i is zero for degree reasons, y_i survives eternally, which implies $E_2 \cong E_\infty$. Clearly E_∞ satisfies the assumption of Lemma 6.4. We define φ_G as the composition

$$A_{\mathcal{H},G} \to E_2 = E_\infty \to H^*(D_G),$$

where the left map is the isomorphism given by identifying y_i in both of the sides and $\mathcal{H}^{\otimes \pi_0(G)}$ with $H^*(M^{\times \pi_0(G)})$ by the Künneth isomorphism, and the right map is the canonical isomorphism defined in Lemma 6.4. Parts (1), (2) and (3) obviously follow from naturality of the canonical isomorphisms. For (4), $H^*(D_G)$ is regarded as a $H^*(D_K)$–module via the pullback $\Delta_{ij}^*\colon H^*(D_K) \to H^*(D_G)$ by the inclusion $D_G \to \Delta_K$. This module structure is compatible with the $A^*_{\mathcal{H},K}$–module structure on $A^*_{\mathcal{H},G}$ via φ_G and φ_K by naturality of the canonical isomorphism. By a general property of a shriek map, the map $\Delta^!_{ij}$ is a $H^*(D_K)$–module homomorphism. So to prove the compatibility, we have only to check the image of 1. For simplicity, we may assume $n = 2$ and $(i, j) = (1, 2)$. We may write D_G as $SM \times_M SM$. The diagram

$$\begin{tikzcd}
H_{d-*}(M) \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_*"] & H^*(M) \arrow[r, "\mathrm{proj}^*"] \arrow[d, "\Delta^!"] & H^*(SM \times_M SM) \arrow[d, "\Delta^!_{12}"] \\
H_{d-*}(M \times M) \arrow[r, "\mathrm{PD}"] & H^{*+d}(M \times M) \arrow[r, "\mathrm{proj}^*"] & H^{*+d}(SM \times SM)
\end{tikzcd}$$

is commutative, where PD denotes the cap product with the fundamental class. By the commutativity of the left square, we see that $\Delta^!(1)$ is the Poincaré dual class in $H^*(M \times M)$ of the diagonal $\Delta(M)$, which corresponds to $\Delta_{\mathcal{H}}$ by the Künneth isomorphism. By the commutativity of the right square, we see that $\Delta^!_{12}(1)$ corresponds to $f_{ij}\Delta_{\mathcal{H}}$. This completes the proof. $\square$

Lemma 6.6 *We use the notation d_i, Δ_i, s_i and $\Delta^!_{ij}$ given in Definitions 5.8 and 5.14. Suppose $\chi(M) \in \mathsf{k}^\times$. Set $\mathcal{H} = H^*(M)$. There exists a family of isomorphisms of graded algebras*

$$\{\varphi_G\colon B_{\mathcal{H},G} \xrightarrow{\cong} H^*(D_G) \mid n \geq 1, G \in \mathsf{G}(n)\}$$

which satisfies the following conditions:

(1) *Let G and H be trees given in Lemma 5.12*(1). *The following diagram is commutative*:

$$\begin{tikzcd}
B_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_i"] & H^*(D_G) \arrow[d, "\Delta_i^*"] \\
B_{\mathcal{H},H} \arrow[r, "\varphi_H"] & H^*(D_H)
\end{tikzcd}$$

Here $\bar{\Delta}_i$ is defined by

$$\bar{\Delta}_i(e_j(x)) = e_{j'}(x) \quad \text{for } x \in S\mathcal{H} \qquad \text{and} \qquad \bar{\Delta}_i(y_{jk}) = y_{j'k'},$$

where we set $j' = d_i(j)$ and $k' = d_i(k)$.

(2) *For a graph $G \in \mathsf{G}(n)$, set $S = s_i(G)$. The following diagram is commutative:*

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{s}_i"] & H^*(D_G) \arrow[d, "s_i"] \\ B^*_{\mathcal{H},S} \arrow[r, "\varphi_S"] & H^*(D_S) \end{tikzcd}$$

Here $\bar{s}_i$ is given by inserting 1 in the $(i+1)^{th}$ factor of $S\mathcal{H}^{\otimes n}$ and skipping the subscript $i+1$.

(3) *For a graph $G \in \mathsf{G}(n)$ and a permutation $\sigma \in \Sigma_n$, the following diagram is commutative:*

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\sigma}"] & H^*(D_G) \arrow[d, "\sigma"] \\ B^*_{\mathcal{H},\tau(G)} \arrow[r, "\varphi_{\tau(G)}"] & H^*(D_{\tau(G)}) \end{tikzcd}$$

Here τ and the right vertical arrow are defined as in Lemma 6.5, and $\bar{\sigma}$ is the algebra homomorphism defined by the permutation of the tensors and subscripts.

(4) *For an edge $(i,j) \in E(G)$ of a tree $G \in \mathsf{G}(n)$ with $i < j$, define $K \in \mathsf{G}(n)$ by $E(K) = E(G) - \{(i,j)\}$. The following square is commutative:*

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_{ij}"] & H^*(D_G) \arrow[d, "\Delta^!_{ij}"] \\ B^{*+d}_{\mathcal{H},K} \arrow[r, "\varphi_K"] & H^{*+d}(D_K) \end{tikzcd}$$

Here $\bar{\Delta}_{ij}$ is the right $B^*_{\mathcal{H},K}$–module homomorphism determined by $\bar{\Delta}_{ij}(1) = \Delta^{ij}_{\mathcal{H}}$ and $\bar{\Delta}_{ij}(y_{ij}) = \Delta^{ij}_{S\mathcal{H}}$, and $B^*_{\mathcal{H},G}$ is considered as a $B^*_{\mathcal{H},K}$–module via the algebra map $f^G_K : B^*_{\mathcal{H},K} \to B^*_{\mathcal{H},G}$ given by

$$f^G_K(e_k(x)) = e_k(x) \quad \text{for } x \in S\mathcal{H} \qquad \text{and} \qquad f^G_K(y_{kl}) = \begin{cases} 0 & \text{if } (k,l) = (i,j), \\ y_{kl} & \text{if otherwise.} \end{cases}$$

Proof As in the proof of Lemma 6.5, we fix a generator $y \in H^{d-1}(S^{d-1})$. Note that d is even as $\chi(M) \neq 0$. We first show an isomorphism of algebras $S\mathcal{H}^* \cong H^*(SM)$. Consider the Serre spectral sequence for the tangent sphere fibration

$$S^{d-1} \to SM \to M.$$

The only nontrivial differential is $d_d \colon E^{0,d-1} = H^{d-1}(S^{d-1}) \to H^d(M)$. As $\chi(M)$ is invertible, d_d is an isomorphism. Since all other differentials vanish by degree reasons, $E_\infty \cong E_{d+1} \cong S\mathcal{H}$, where the second isomorphism is given by $E^{p,0}_{d+1} = H^p(M) \subset \mathcal{H}^{\leq d-2} \subset S\mathcal{H}$ for $p \leq d-2$ and $E^{p,d-1} = H^{d-1}(S^{d-1}) \otimes H^p(M) \ni y \otimes a \mapsto \bar{a} \in S\mathcal{H}$ for $p \geq 2$. Since $H^1(M) = 0$ and $H^*(M)$ is free, $H^{d-1}(M)=0$, which implies the fibration satisfies the conditions of Lemma 6.4. Composing this isomorphism with the canonical isomorphism $E_\infty \to H^*(SM)$, we have an isomorphism

$$S\mathcal{H}^* \cong H^*(SM). \tag{6-1}$$

If necessary, we modify y so that the composition $S\mathcal{H}^{2d-1} \to H^{2d-1}(SM) \to \mathsf{k}$ of (6-1) and the cap product with the fundamental class $\hat{w}$ in Definition 5.8 coincides with the orientation given in Definition 6.1 by multiplying by a scalar.

We shall define the isomorphism φ_G. We may assume that $G \in \mathsf{G}(n)$ is connected, as in the disconnected case everything involved is a tensor product of the objects corresponding to connected subgraphs. Consider the Serre spectral sequence for the fibration

$$(S^{d-1})^{n-1} \to D_G \to SM$$

given by projection to the first component. As $E_2^{d,0} = S\mathcal{H}^d = 0$, elements in $E_2^{0,d-1} \cong H^{d-1}(S^{d-1})^{\otimes n-1}$ survive eternally. As in the proof of Lemma 6.5, y_j denotes the copy of y living in the j^{th} factor of $H^*(S^{d-1})^{\otimes n-1}$, which is also regarded as a generator of $E_2^{0,d-1}$. We construct an isomorphism $\psi_G\colon S\mathcal{H}^* \otimes \bigwedge(y_1,\dots,y_{n-1}) \cong E_\infty \cong H^*(D_G)$ using (6-1) similarly to the construction of (6-1). Consider the Serre spectral sequence $\{\bar{E}_r^{p,q}\}$ for the fibration

$$(S^{d-1})^n \to D_G \to M$$

given by the projection of the sphere bundle. Let $\bar{y}_j$ be the copy of y in the j^{th} factor of $\bar{E}_2^{0,d-1} \cong (H^*(S^{d-1})^{\otimes n})^{*=d-1}$. For any i and j, since $d_d(\bar{y}_i) = d_d(\bar{y}_j) = (\text{a multiple of})\chi(M)w_M$, the element $\bar{y}_i - \bar{y}_j$ survives eternally by degree reasons. Clearly $\bar{E}_\infty$ satisfies the assumption of Lemma 6.4, so we can take the canonical isomorphism $\bar{E}_\infty^{*,*} \to H^*(D_G)$. We define an algebra map

$$\varphi'_G\colon (S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij} \mid 1 \le i,j \le n\} \to \bar{E}_\infty^{*,*}$$

by $e_i(a) \mapsto a \in E_\infty^{*,0}$ for $a \in \mathcal{H}^{\le d-2}$, $e_i(\bar{b}) \mapsto b\bar{y}_i \in E_\infty^{*,d-1}$ for $b \in \mathcal{H}^{\ge 2}$, and $y_{ij} \mapsto \bar{y}_i - \bar{y}_j$. We see $\varphi'_G(J_G) = 0$, where J_G is the ideal in Definition 6.3. For example, since $d_d(\bar{y}_i\bar{y}_j) = \chi(M)(\bar{y}_j - \bar{y}_i)w_M$ (up to $\mathsf{k}^\times$) and $\chi(M)$ is invertible, $(\bar{y}_i - \bar{y}_j)w_M = 0$ in $\bar{E}_{d+1}^{d,d-1}$, which implies $\varphi'_G(e_i(\bar{b}) - e_j(\bar{b})) = 0$ for $b \in \mathcal{H}^d$. Annihilation of other elements in J_G is obvious. We define φ_G to be the unique map which makes the following diagram commutative:

$$\begin{tikzcd}
(S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij}\} \arrow[r] \arrow[d, "\varphi'_G"] & (S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij}\}/J_G \arrow[r, equal] & B^*_{\mathcal{H},G} \arrow[d, "\varphi_G"] \\
\bar{E}_\infty^{*,*} \arrow[rr, "\text{canonical isomorphism}"] & & H^*(D_G)
\end{tikzcd}$$

Since G is connected, $e_1\colon S\mathcal{H} \to S\mathcal{H}^{\otimes n}$ induces an isomorphism $\alpha_G\colon S\mathcal{H} \otimes \bigwedge\{y_{12},\dots,y_{1n}\} \cong B^*_{\mathcal{H},G}$. It is easy to see that the composition

$$S\mathcal{H} \otimes \bigwedge\{y_{12},\dots,y_{1n}\} \overset{\alpha_G}{\cong} B^*_{\mathcal{H},G} \xrightarrow{\varphi_G} H^*(D_G) \overset{\psi_G^{-1}}{\cong} S\mathcal{H} \otimes \bigwedge\{y_1,\dots,y_n\}$$

identifies the subalgebra $S\mathcal{H}$ in both sides and the sub-k–module $\mathsf{k}\langle y_{12},\dots,y_{1n}\rangle$ with $\mathsf{k}\langle y_1,\dots,y_n\rangle$ (since these are both isomorphic to $H^{d-1}(D_G)$), which implies the composition is an isomorphism and we conclude that φ_G is an isomorphism.

Parts (1), (2) and (3) obviously follow from naturality of the canonical isomorphism. We shall show (4). Since φ_G is an isomorphism, we may define $\bar{\Delta}_{ij}$ to be the map which makes the square in (4) commute. As in the proof of Lemma 6.5, $\bar{\Delta}_{ij}$ is a $B^*_{\mathcal{H},K}$–module homomorphism and we have $\bar{\Delta}_{ij}(1)=f_{ij}(\Delta_{\mathcal{H}})$. We shall show the equality $\bar{\Delta}_{ij}(y_{ij})=f_{ij}(\Delta_{S\mathcal{H}})$. We may assume $n=2$ and $G=(1,2)$. In this case, clearly $D_G=SM\times_M SM$. We consider the commutative diagram

$$\begin{tikzcd}
H^0(S^{d-1}) \arrow[d, "\Delta_1^!"] & H^0(SM) \arrow[l] \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_2^!"] & H_{2d-1}(SM) \arrow[d, "(\Delta_2)_*"] \\
H^{d-1}(S^{d-1}\times S^{d-1}) & H^{d-1}(SM\times_M SM) \arrow[l] \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_{12}^!"] & H_{2d-1}(SM\times_M SM) \arrow[d, "(\Delta_{12})_*"] \\
 & H^{2d-1}(SM\times SM) \arrow[r, "\mathrm{PD}"] & H_{2d-1}(SM\times SM)
\end{tikzcd}$$

where the left horizontal arrows are induced by the fiber restriction, the right ones are capping with the fixed fundamental classes, and $\Delta_1^!$ and $\Delta_2^!$ are the shriek maps induced by the diagonals. As d is even, $\Delta_1^!(1)=\bar{y}_1-\bar{y}_2$. As $\bar{y}_1-\bar{y}_2$ coincides with the image of $\varphi_G(y_{12})$ by the fiber restriction which induces an isomorphism in degree $d-1$, we have $\Delta_2^!(1)=\varphi_G(y_{12})$. So $\Delta_{12}^!(\varphi_G(y_{12}))=(\Delta_{12}\circ\Delta_2)^!(1)$. By the commutativity of the right-hand square, $(\Delta_{12}\circ\Delta_2)^!(1)$ is the diagonal class for SM. Thanks to the modification of y after the definition of (6-1), the diagonal class corresponds to $\Delta_{S\mathcal{H}}$ by φ_G. This implies $\bar{\Delta}_{12}(y_{12})=\Delta_{S\mathcal{H}}$. $\square$

Definition 6.7 Let $\mathcal{H}$ be a Poincaré algebra of dimension d.

- We define a CDBA $A^{\star *}_{\mathcal{H}}(n)$ by the equality

$$A^{\star *}_{\mathcal{H}}(n)=\mathcal{H}^{\otimes n}\otimes\bigwedge\{y_i,g_{ij}\mid 1\le i,j\le n\}/\mathcal{I}.$$

Here, for the bidegrees, we set $|a|=(0,l)$ for $a\in(\mathcal{H}^{\otimes n})^{*=l}$, $|y_i|=(0,d-1)$ and $|g_{ij}|=(-1,d)$. The ideal $\mathcal{I}$ is generated by the elements

$$g_{ij}-(-1)^d g_{ji},\ (g_{ij})^2,\ g_{ii},\ (e_i(a)-e_j(a))g_{ij},\ g_{ij}g_{jk}+g_{jk}g_{ki}+g_{ki}g_{ij}\quad \text{for } 1\le i,j,k\le n \text{ and } a\in\mathcal{H}.$$

We call the last relation the 3*–term relation for* g_{ij}. The differential is given by $\partial(a)=0$ for $a\in\mathcal{H}^{\otimes n}$ and $\partial(g_{ij})=\Delta^{ij}_{\mathcal{H}}$; see Definition 6.3.

- Suppose d is even. We define a CDBA $B^{\star *}_{\mathcal{H}}(n)$ by the equality

$$B^{\star *}_{\mathcal{H}}(n)=(S\mathcal{H})^{\otimes n}\otimes\bigwedge\{g_{ij},h_{ij}\mid 1\le i,j\le n\}/\mathcal{J}$$

Here, for the bidegrees, we set $|a| = (0, l)$ for $a \in (\mathcal{H}^{\otimes n})^{*=l}$, $|g_{ij}| = (-1, d)$ and $|h_{ij}| = (-1, 2d-1)$. The ideal $\mathcal{J}$ is generated by the elements

$$g_{ij} - g_{ji}, (g_{ij})^2, g_{ii}, h_{ij} + h_{ji}, (h_{ij})^2, h_{ii}, e_{ij}(a)g_{ij}, e_{ij}(a)h_{ij}, e_{ij}(\bar{b})g_{ij} - e_i(b)h_{ij}, e_{ij}(\bar{b})h_{ij},$$
$$g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij}, h_{ij}h_{jk} + h_{jk}h_{ki} + h_{ki}h_{ij}, (h_{ij} + h_{ki})g_{jk} - (h_{ij} + h_{jk})g_{ki}$$

for $1 \le i, j, k \le n$, $a \in \mathcal{H}^{\le d-2}$ and $b \in \mathcal{H}^{\ge 2}$, where we regard $e_i(b)$ as 0 for $b \in \mathcal{H}^d$, and $e_{ij} : S\mathcal{H} \to (S\mathcal{H})^{\otimes n}$ is the map given by $e_{ij} = e_i - e_j$. The differential is given by $\partial(x) = 0$ for $x \in S\mathcal{H}^{\otimes n}$, $\partial(g_{ij}) = \Delta^{ij}_{\mathcal{H}}$ and $\partial(h_{ij}) = \Delta^{ij}_{S\mathcal{H}}$; see Definition 6.3.

- We equip the sequences $A_{\mathcal{H}} = \{A_{\mathcal{H}}(n)\}_n$ and $B_{\mathcal{H}} = \{B_{\mathcal{H}}(n)\}_n$ with the structures of $\mathcal{A}$–comodules of CDBA as follows. For $B_{\mathcal{H}}$, we define a partial composition and an action of Σ_n by the equalities

$$\mu \circ_i e_j(x) = e_{j'}(x), \quad \mu \circ_i (h_{jk}) = h_{j'k'}, \quad \mu \circ_i (g_{jk}) = g_{j'k'}, \quad e_j(x)^\sigma = e_{\tau(j)}(x),$$
$$h^\sigma_{jk} = h_{\tau(j),\tau(k)}, \quad g^\sigma_{jk} = g_{\tau(j),\tau(k)}, \quad \text{for } x \in S\mathcal{H} \text{ and } \sigma \in \Sigma_n,$$

where j' and k' are the numbers given by $j' = d_i(j)$ and $k' = d_i(k)$, and we set $\tau = \sigma^{-1}$ (see Definition 5.8 for d_i and μ). The definition of $A_{\mathcal{H}}$ is similar.

- We define simplicial CDBAs $A^{\star\star}_\bullet(\mathcal{H})$ and $B^{\star\star}_\bullet(\mathcal{H})$ as follows. For $B^{\star\star}_\bullet(\mathcal{H})$, we set

$$B^{\star\star}_n(\mathcal{H}) = B^{\star\star}_{\mathcal{H}}(n+1).$$

As in Definition 5.14, we relabel the involved subscripts with $0, \dots, n$. The face map $d_i : B^{\star\star}_n(\mathcal{H}) \to B^{\star\star}_{n-1}(\mathcal{H})$ is given by $d_i = \mu \circ_i (-)$ for $i < n$ and $d_n = \mu \circ_0 (-)^\sigma$ where $\sigma = (n, 0, 1, \dots, n-1)$. The degeneracy map $s_i : B^{\star\star}_n(\mathcal{H}) \to B^{\star\star}_{n+1}(\mathcal{H})$ is given by inserting 1 as the $(i+1)^{\text{th}}$ factor of $S\mathcal{H}^{\otimes n+1}$ and skipping the subscript $i+1$. $A^{\star\star}_\bullet(\mathcal{H})$ is defined similarly using $A^{\star\star}_{\mathcal{H}}$.

Remark 6.8 An algebra similar to the algebras $A^{\star\star}_{\mathcal{H}}(n)$ and $B^{\star\star}_{\mathcal{H}}(n)$ has already appeared as the E_2–page of Totaro's spectral sequence defined in [39].

In the rest of this section, we prove that $A_{\mathcal{H}}$ and $B_{\mathcal{H}}$ are isomorphic to A_M as $\mathcal{A}$–comodules of CDBA under different assumptions, and also prove similar statements for the simplicial CDBAs. We mainly deal with the case of $B_{\mathcal{H}}$. The case of $A_{\mathcal{H}}$ is similar.

Lemma 6.9 *The map*

$$\bigoplus_{G \in \mathsf{G}(n)^{\mathrm{dis}}} H^*_G g_G \to A_M$$

defined by the composition of the inclusion and quotient map is an isomorphism of k*–modules* (see Definition 5.8 for $\mathsf{G}(n)^{\mathrm{dis}}$).

Proof Let Π be the set of partitions of $\underline{n}$. The ideal $J(n)$ in Definition 5.14 has a decomposition $J(n) = \bigoplus_{\pi \in \Pi} J(n)_\pi$ such that $J(n)_\pi \subset \bigoplus_{\pi_0(G)=\pi} H_G$, since generators of $J(n)$ are sums of monomials which have the same connected components. If $\pi_0(G) = \pi_0(H) = \pi$, clearly $H^*_G = H^*_H$. We denote this module by H^*_π. We have $\bigoplus_{\pi_0(G)=\pi} H_G g_G = H_\pi \otimes (\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G)$. Similarly $J(n)_\pi = H_\pi \otimes J(n)'_\pi$

where $J(n)'_\pi$ is the sub-k–module of $\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G$ generated by multiples of 3–term relations, g_{ij}^2 and $g_{ij} - (-1)^d g_{ji}$. We have

$$A_M^* = \bigoplus_{\pi\in\Pi} \Big\{ \Big(\bigoplus_{\pi_0(G)=\pi} H_G g_G \Big) / J(n)_\pi \Big\} = \bigoplus_{\pi\in\Pi} H_\pi \otimes \Big\{ \Big(\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G \Big) / J(n)'_\pi \Big\}.$$

Note that $\bigoplus_{\pi\in\Pi}\{(\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G)/J(n)'_\pi\}$ is isomorphic to the cohomology group of the configuration space $H^*(C_n(\mathbb{R}^d))$, whose basis is $\{g_G \mid G \in \mathsf{G}(n)^{\mathrm{dis}}\}$. So then $(\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G)/J(n)'_\pi$ has a basis $\{g_G \mid G \in \mathsf{G}(n)^{\mathrm{dis}}, \pi_0(G) = \pi\}$, which implies the lemma. □

Under the assumptions and notation of Lemma 6.6, we identify H_G^* with $B_{\mathcal{H},G}$ by the isomorphism φ_G, so $A_M^*(n)$ is regarded as a quotient of $\bigoplus_{G\in\mathsf{G}(n)} B^*_{\mathcal{H},G} g_G$. With this identification, we set $\bar{h}_{ij} = y_{ij} g_{ij} \in A_M(n)$. $A_M(n)$ contains $S\mathcal{H}^{\otimes n}$ as the subalgebra $H_\varnothing g_\varnothing$, the summand corresponding to the graph $\varnothing \in \mathsf{G}(n)$. We regard $A_M(n)$ as a left $S\mathcal{H}^{\otimes n}$–module via the multiplication by $H_\varnothing g_\varnothing$. In the following lemma and its proof, h_G, $\bar{h}_G$ and y_G are defined similarly to g_G. For example, $h_G = h_{i_1,j_1} \cdots h_{i_r,j_r}$ for $E(G) = \{(i_1, j_1) < \cdots < (i_r, j_r)\}$.

Lemma 6.10 *Under the assumptions of Lemma 6.6 and the above notation, as an $S\mathcal{H}^{\otimes n}$–module, $A_M(n)$ is generated by the set $S = \{g_G \bar{h}_H \mid G, H \in \mathsf{G}(n), E(G) \cap E(H) = \varnothing, GH \in \mathsf{G}(n)^{\mathrm{dis}}\}$, and $B_\mathcal{H}(n)$ is generated by the set $S' = \{g_G h_H \mid G, H \in \mathsf{G}(n), E(G) \cap E(H) = \varnothing, GH \in \mathsf{G}(n)^{\mathrm{dis}}\}$.*

Proof $A_M(n)$ is generated by the elements $y_H g_G$, for graphs G and H, such that each connected component of H is contained in some connected component of G. We can express g_G as a sum of monomials g_{G_1} with $G_1 \in \mathsf{G}(n)^{\mathrm{dis}}$ and $\pi_0(G) = \pi_0(G_1)$ using the 3–term relation and the relation $g_{ij} = g_{ji}$ (this is standard procedure in the computation of $H^*(C_n(\mathbb{R}^d))$. So we may assume G is distinguished. For a sequence of edges $(i, k_1), (k_1, k_2), \ldots, (k_s, j)$ in G, we have $y_{ij} = y_{i,k_1} + \cdots + y_{k_s,j}$. By successive application of this equality, y_H is expressed as a sum of monomials y_{H_1} with H_1 being a subgraph of G. Thus any element of $A_M(n)$ is expressed as a $S\mathcal{H}^{\otimes n}$–linear combination of monomials $y_H g_G$ with $G \in \mathsf{G}(n)^{\mathrm{dis}}$ and $E(H) \subset E(G)$. Clearly $y_H g_G = \pm g_{G-H} \bar{h}_H$. Thus the set S generates $A_M(n)$. A proof for the assertion for $B_\mathcal{H}(n)$ is similar when one use 3–term relations for g_{ij} and h_{ij}, and the last relation for g_{ij} and h_{ij} in the ideal $\mathcal{J}$ in Definition 6.7. □

To prove that $B_\mathcal{H}(n)$ and $A_M(n)$ are isomorphic, we define a structure of a $B_{\mathcal{H},G}$–module on $B_\mathcal{H}(n)$ as follows. We first define two graded algebras $\widetilde{B}_{\mathcal{H},G}$ and $\widetilde{B}_\mathcal{H}(n)$. For a graph $G \in \mathsf{G}(n)$, we set

$$\widetilde{B}_{\mathcal{H},G} = S\mathcal{H}^{\otimes n} \otimes T\{y_{ij} \mid i < j \text{ and } i \sim_G j\} \quad \text{and} \quad \widetilde{B}_\mathcal{H}(n) = S\mathcal{H}^{\otimes n} \otimes \bigwedge\{g_{ij}, h_{ij} \mid 1 \le i < j \le n\},$$

where $T\{y_{ij}\}$ denotes the tensor algebra generated by the y_{ij}. For convenience, we set $y_{ij} = -y_{ji}$, $g_{ij} = g_{ji}$ and $h_{ij} = -h_{ji}$ for $i > j$. The degrees are the same as the elements of the same symbols in $B_{\mathcal{H},G}$ and $B_\mathcal{H}(n)$. We shall define a map of graded k–modules

$$(-\cdot-)\colon \widetilde{B}_{\mathcal{H},G} \otimes_\mathsf{k} \widetilde{B}_\mathcal{H}(n) \to B_\mathcal{H}(n).$$

We define $y_{ij}\cdot xg_Gh_H$ for $x\in S\mathcal{H}^{\otimes n}$ and $G,H\in\mathsf{G}(n)$ as follows. If $E(G)\cap E(H)\neq\varnothing$, we set $y_{ij}\cdot xg_Gh_H=0$. Suppose $E(G)\cap E(H)=\varnothing$. If $(i,j)\in E(G)$ is the t^{th} edge (in the lexicographical order), we set $y_{ij}\cdot xg_Gh_H=(-1)^{t+1+|x|}h_{ij}xg_Kh_H$ with $E(K)=E(G)-\{(i,j)\}$. If $(i,j)\in E(H)$ is an edge, we set $y_{ij}\cdot xg_Gh_H=0$. If $i\sim_{GH}j$, we take a sequence of edges $(k_0,k_1),\dots,(k_s,k_{s+1})$ of GH with $k_0=i$ and $k_{s+1}=j$ and set $y_{ij}\cdot xg_Gh_H=\sum_{l=0}^{s}y_{k_l,k_{l+1}}\cdot xg_Gh_H$. This does not depend on the choice of the sequence, because $g_Gh_H=0$ if GH is not a tree, which is proved by using the last three relations in the definition of $\mathcal{J}$ in Definition 6.7. If i and j are disconnected in GH, we set $y_{ij}\cdot xg_Gh_H=0$. For $z\in S\mathcal{H}^{\otimes n}$, we set $z\cdot xg_Gh_H=zxg_Gh_H$, the multiplication in $B_{\mathcal{H}}(n)$. We shall show that the map $(-\cdot-)$ annihilates the elements of $\mathcal{J}$ (we regard $\mathcal{J}$ as an ideal in $\widetilde{B}_{\mathcal{H}}(n)$). Direct computation shows that the generators of $\mathcal{J}$ are annihilated by any elements of $\widetilde{B}_{\mathcal{H},G}$. For example, $y_{ij}\cdot(g_{ij}g_{jk}+g_{jk}g_{ki}+g_{ki}g_{ij})=(h_{ij}+h_{ik})g_{jk}-(h_{ij}+h_{jk})g_{ki}=0$ and $y_{jk}\cdot\{(h_{ij}+h_{ki})g_{jk}-(h_{ij}+h_{jk})g_{ki}\}=h_{ij}h_{jk}+h_{jk}h_{ki}+h_{ki}h_{ij}=0$. We also easily see $y_{ij}\cdot xg_Gh_H=\pm(y_{ij}\cdot xg_{G'}h_{H'})g_{G-G'}h_{H-H'}$ for subgraphs $G'\subset G$ and $H'\subset H$ such that $i\sim_{G'H'}j$. These observations imply the assertion, and we see that $(-\cdot-)$ factors through $\widetilde{B}_{\mathcal{H},G}\otimes_{\mathsf{k}}B_{\mathcal{H}}(n)\to B_{\mathcal{H}}(n)$, which is also denoted by $(-\cdot-)$. Clearly the map $(-\cdot-)$ annihilates J_G in the definition of $B_{\mathcal{H},G}$. It also annihilates the commutativity relation $y_{ij}y_{kl}+y_{kl}y_{ij}$. If two paths connecting i and j or k and l have a common edge, both of the actions of $y_{ij}y_{kl}$ and $y_{kl}y_{ij}$ are zero, and otherwise the commutativity in $B_{\mathcal{H}}(n)$ implies the annihilation. Annihilation of these relations implies that the map $(-\cdot-)$ factors through a map $(-\cdot-)\colon B_{\mathcal{H},G}\otimes B_{\mathcal{H}}(n)\to B_{\mathcal{H}}(n)$, which defines a structure of $B_{\mathcal{H},G}$–module on $B_{\mathcal{H}}(n)$.

Theorem 6.11 *Suppose M is simply connected and oriented, and $H^*(M)$ is a free k–module. Set $\mathcal{H}=H^*(M)$.*

1. *Suppose $\chi(M)=0\in\mathsf{k}$. The two $\mathcal{A}$–comodules of CDBA $A_M^{\star *}$ and $A_{\mathcal{H}}^{\star *}$ are isomorphic, and the two simplicial CDBAs $A_\bullet^{\star *}(M)$ and $A_\bullet^{\star *}(\mathcal{H})$ are isomorphic. In particular, the E_2–page of the Čech s.s. is isomorphic to the total homology of the normalization $NA_\bullet^{\star *}(\mathcal{H})$ as a bigraded k–module. The bigrading is given by $(\star-\bullet,*)$.*
2. *Suppose $\chi(M)\in\mathsf{k}^{\times}$. The two $\mathcal{A}$–comodules of CDBA $A_M^{\star *}$ and $B_{\mathcal{H}}^{\star *}$ are isomorphic, and the two simplicial CDBAs $A_\bullet^{\star *}(M)$ and $B_\bullet^{\star *}(\mathcal{H})$ are isomorphic. In particular, the E_2–page of the Čech s.s. is isomorphic to the total homology of the normalization $NB_\bullet^{\star *}(\mathcal{H})$ as a bigraded k–module. The bigrading is given by $(\star-\bullet,*)$.*

Proof Part (1) obviously follows from Theorem 5.16 and Lemma 6.5. We shall prove (2). We define a map $\Phi_n\colon B_{\mathcal{H}}(n)\to A_M(n)$ of algebras by identifying the subalgebra $S\mathcal{H}^{\otimes n}$ and elements g_{ij} in both sides, and taking h_{ij} to $\bar{h}_{ij}$ (see the paragraph above Lemma 6.10). We easily verify that Φ_n is well defined. Then Φ_n fits into the following commutative diagram:

$$\begin{tikzcd}
\bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_G g_G \arrow[d] \arrow[dr] & \\
B_{\mathcal{H}}(n) \arrow[r, "\Phi_n"] & A_M(n)
\end{tikzcd}$$

Here the vertical arrow is induced by the inclusion of a submodule $H_G g_G = B_{\mathcal{H},G} g_G \subset B_{\mathcal{H}}(n)$ given by the isomorphism φ_G in Lemma 6.6 and the module structure defined above, and the slanting arrow is given in Lemma 6.9. The vertical arrow and Φ_n are epimorphisms by Lemma 6.10, and the slanting arrow is an isomorphism by Lemma 6.9, so Φ_n is an isomorphism. By the definition of Φ_n and Lemma 6.6, the collection $\{\Phi_n\}_n$ commutes with the structures of an $\mathcal{A}$–comodule and degeneracy maps. The assertion for the E_2–page immediately follows from the isomorphism of simplicial objects. □

Remark 6.12 The Euler number $\chi(M)$ can be recovered from the Poincaré algebra $\mathcal{H}^* = H^*(M)$. It is the image of $\Delta_{\mathcal{H}}$ by the composition

$$(\mathcal{H}^{\otimes 2})^{*=d} \xrightarrow{\text{multiplication}} \mathcal{H}^d \xrightarrow{\epsilon} \mathsf{k}.$$

So under the assumptions of Theorem 6.11, the E_2–page of the Čech s.s. is determined by the cohomology algebra $H^*(M)$. (Different orientations give apparently different presentations, but they are isomorphic.)

7 Examples

In this section, we compute some of the E_2–page of Čech s.s. for the spheres and products of two spheres $S^k \times S^l$ with $(k,l) = (\text{odd}, \text{even})$ or $(\text{even}, \text{even})$, and deduce some results on cohomology groups for the products of spheres. We also prove Corollary 1.3. Our computation is restricted to low degrees and consists of only elementary linear algebra on differentials and degree argument based on Theorem 6.11. We briefly state the results for the cases of spheres since, in these cases, the Čech s.s. only gives less information than the combination of Vassiliev's (or Sinha's) spectral sequence for long knots and the Serre spectral sequence for a fibration $\mathrm{Emb}(S^1, S^d) \to STS^d$ (see the proof of Proposition 7.2), at least in the degrees where we have computed. We give concrete descriptions of the differentials in the case of $M = S^k \times S^l$ with k odd and l even. In the rest of this section, we set $\mathcal{H} = H^*(M)$ for a fixed orientation.

7.1 The case of $M = S^d$ with d odd

In this case $A^{\star *}_{\bullet}(\mathcal{H})$ is described as

$$A^{\star *}_n(\mathcal{H}) = \bigwedge \{x_i, y_i, g_{ij} \mid 0 \le i, j \le n\}/\mathcal{I},$$

where $|x_i| = (0,d)$, $|y_i| = (0, d-1)$, $|g_{ij}| = (-1, d)$ and $\mathcal{I}$ is the ideal generated by

$$(x_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij} + g_{ji}, (x_i - x_j) g_{ij} \text{ and the 3–term relation for } g_{ij}.$$

The diagonal class is given by $\Delta_{\mathcal{H}} = x_0 - x_1 \in \mathcal{H} \otimes \mathcal{H}$.

Proposition 7.1 *Consider the Čech s.s. $\check{\mathbb{E}}^{pq}_r$ for the sphere S^d with odd $d \ge 5$. We abbreviate $\check{\mathbb{E}}^{pq}_2$ as (p,q). The following equalities hold*:

$$(-3, d) = \mathsf{k}\langle g_{12}\rangle, \quad (-1, d-1) = \mathsf{k}\langle y_1\rangle, \quad (0, d-1) = \mathsf{k}\langle y_0\rangle, \quad (0,d) = \mathsf{k}\langle x_0\rangle,$$

$$(-6,2d)=\mathsf{k}\langle g_{13}g_{24},-g_{12}g_{34}+g_{14}g_{23}\rangle,\quad (-4,2d-1)=\mathsf{k}\langle y_1g_{23}-y_2g_{13}+y_3g_{12}\rangle,$$
$$(-5,2d)=\mathsf{k}\langle g_{01}g_{23}+g_{02}g_{13}+g_{13}g_{23}\rangle,\quad (-3,2d-1)=\mathsf{k}\langle y_0g_{12}\rangle,$$
$$(-3,2d)=\mathsf{k}\langle x_0g_{12}\rangle,\quad (-1,2d-1)=\mathsf{k}\langle x_0y_1,x_1y_0,x_1y_1\rangle,\quad (0,2d-1)=\mathsf{k}\langle x_0y_0\rangle.$$

For other (p,q) with $p+q\le 2d-1$, we have $(p,q)=0$. □

Proposition 7.2 *Let d be an odd number with $d\ge 5$.*

(1) $\mathrm{Emb}(S^1,S^d)$ *is $(d-2)$–connected.*

(2) *The Čech s.s. for S^d does not collapse at the E_2–page in any coefficient ring.*

Proof For (1), consider the fiber sequence

$$\mathrm{Emb}_c(\mathbb{R},\mathbb{R}^d)\to\mathrm{Emb}(S^1,S^d)\to STS^d,$$

where STS^d is the tangent sphere bundle of S^d, the left map is given by taking the tangent vector at a fixed point, and the right space is the space of long knots. As is well known, STS^d is $(d-2)$–connected and $\mathrm{Emb}_c(\mathbb{R},\mathbb{R}^d)$ is $(2d-7)$–connected. As $d\ge 5$, we have the claim. Part (2) follows from (1) and Proposition 7.1. (There are nonzero elements in the total degrees $d-3$ and $d-2$.) □

Remark 7.3 The reader may find inconsistency between [8, Proposition 3.9(3)] and Proposition 7.2(1). This is just a typo; $n-j-2$ should be replaced with $n-j-1$ (and $n-j-1$ with $n-j$) in the former proposition (see its proof).

7.2 The case of $M=S^d$ with d even

In this subsection, we assume $2\in\mathsf{k}^\times$. $B_\bullet^{\star\star}(\mathcal{H})$ is described as

$$B_n^{\star\star}(\mathcal{H})=\bigwedge\{z_i,g_{ij},h_{ij}\mid 0\le i,j\le n\}/\mathcal{J},$$

where $|z_i|=(0,2d-1)$, $|g_{ij}|=(-1,d)$, $|h_{ij}|=(-1,2d-1)$ and $\mathcal{J}$ is the ideal generated by

$$(z_i)^2,(g_{ij})^2,(h_{ij})^2,g_{ii},h_{ii},g_{ij}-g_{ji},h_{ij}+h_{ji},(z_i-z_j)g_{ij},(z_i-z_j)h_{ij},(h_{ij}+h_{ki})g_{jk}-(h_{ij}+h_{jk})g_{ki},$$

and the 3–term relation for g_{ij} and h_{ij}. The diagonal classes are given by $\Delta_{\mathcal{H}}=0\in S\mathcal{H}\otimes S\mathcal{H}$ and $\Delta_{S\mathcal{H}}=z_0-z_1\in S\mathcal{H}\otimes S\mathcal{H}$.

Proposition 7.4 *Suppose $2\in\mathsf{k}^\times$. Consider the Čech s.s. $\check{\mathbb{E}}_r^{pq}$ for S^d with even $d\ge 4$. We abbreviate $\check{\mathbb{E}}_2^{pq}$ as (p,q). The following equalities hold:*

$$(-6,2d)=\mathsf{k}\langle g_{13}g_{24}\rangle,\qquad (-5,2d)=\mathsf{k}\langle g_{01}g_{23}+3g_{02}g_{13}+g_{03}g_{12}\rangle,$$
$$(-3,2d-1)=\mathsf{k}\langle h_{12}\rangle,\qquad (0,2d-1)=\mathsf{k}\langle z_0\rangle.$$

For other (p,q) with $p+q\le 2d-1$, we have $(p,q)=0$. □

For the case of $\mathsf{k}=\mathbb{F}_2$, the same statement as in Proposition 7.1 holds, except that "odd $d\ge 5$" is replaced with "even $d\ge 4$".

7.3 The case of $M = S^k \times S^l$ with k odd and l even

We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\wedge\{a, b\}$. We fix an orientation ϵ on $\mathcal{H}$ by $\epsilon(ab) = 1$. We write a_i for $e_i(a)$ and b_i for $e_i(b)$, and $A_n(\mathcal{H})$ is presented as

$$A_n(\mathcal{H}) = \bigwedge\{a_i, b_i, y_i, g_{ij} \mid 0 \le i, j \le n\}/\mathcal{I},$$

where $|y_i| = (0, k+l-1)$, $|g_{ij}| = (-1, k+l)$ and $\mathcal{I}$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij} + g_{ji}, (a_i - a_j)g_{ij}, (b_i - b_j)g_{ij} \text{ and the 3–term relation for } g_{ij}.$$

The diagonal class is given by $\Delta_{\mathcal{H}} = a_0b_0 - a_1b_0 + a_0b_1 - a_1b_1 \in \mathcal{H} \otimes \mathcal{H}$. The module $NA_n(\mathcal{H})$ is generated by the monomials of the form $a_{p_1} \cdots a_{p_s} b_{q_1} \cdots b_{q_t} g_{i_1 j_1} \cdots g_{i_r j_r}$ such that the set of subscripts $\{p_1, \dots, p_s, q_1, \dots, q_t, i_1, \dots, i_r, j_1, \dots, j_r\}$ contains the set $\{1, \dots, n\}$.

We shall present the total differential $\tilde{d}$ on

$$\check{\mathbb{E}}_1^{pq} = \bigoplus_{\star - \bullet = p} NA^{\star, q}_{\bullet}(\mathcal{H})$$

up to $p + q \le \max\{2k + l, k + 2l\}$. For $(p, q) = (-1, k)$, $(-1, l)$, $(-1, k+l-1)$, $(-1, k+l)$, $(-1, 2k)$, $(-1, 2l)$, $(-1, 2k+l)$, $(-1, k+2l)$, $(-1, 2k+l-1)$, $(-1, k+2l-1)$, $(-2, 2k)$, $(-2, 2l)$, $(-2, 3k)$ or $(-2, 3l)$, $\tilde{d}$ is zero.

For $(p, q) = (-3, k+l)$, $\tilde{d}$ is presented by the following matrix

	g_{12}
g_{01}	0
a_1b_2	1
a_2b_1	-1

This is read as $\tilde{d}(g_{12}) = a_1b_2 - a_2b_1$. For $(p, q) = (-2, k+l)$,

	g_{01}	a_1b_2	a_2b_1
a_0b_1	1	1	1
a_1b_0	-1	1	1
a_1b_1	-1	-1	-1

For $(p, q) = (-4, 2k+l)$,

	a_1g_{23}	a_2g_{13}	a_3g_{12}
a_0g_{12}	1	0	-1
a_1g_{02}	1	1	0
a_1g_{12}	-1	0	1
a_2g_{01}	0	1	1
$a_1a_2b_3$	-1	1	0
$a_1a_3b_2$	1	0	1
$a_2a_3b_1$	0	1	-1

For $(p,q)=(-3,2k+l)$,

	a_0g_{12}	a_1g_{02}	a_1g_{12}	a_2g_{01}	$a_1a_2b_3$	$a_1a_3b_2$	$a_2a_3b_1$
a_0g_{01}	0	0	0	0	0	0	0
$a_0a_1b_2$	−1	1	0	0	−1	−1	0
$a_0a_2b_1$	1	0	0	1	0	−1	−1
$a_1a_2b_0$	0	1	0	−1	1	0	−1
$a_1a_2b_1$	0	0	1	−1	0	1	1
$a_1a_2b_2$	0	1	1	0	−1	−1	0

For $(p,q)=(-2,2k+l)$,

	a_0g_{01}	$a_0a_1b_2$	$a_0a_2b_1$	$a_1a_2b_0$	$a_1a_2b_1$	$a_1a_2b_2$
$a_0a_1b_0$	1	−1	−1	0	−1	1
$a_0a_1b_1$	1	1	1	0	1	−1

For $(p,q)=(-2,2k+l-1)$,

	a_1y_2	a_2y_1
a_0y_1	1	1
a_1y_0	1	1
a_1y_1	−1	−1

For $(p,q)=(-4,k+2l)$,

	b_1g_{23}	b_2g_{13}	b_3g_{12}
b_0g_{12}	−1	0	1
b_1g_{02}	−1	−1	0
b_1g_{12}	1	0	−1
b_2g_{01}	0	−1	−1
$a_1b_2b_3$	0	1	1
$a_2b_1b_3$	1	0	−1
$a_3b_1b_2$	−1	−1	0

For $(p,q)=(-3,k+2l)$,

	b_0g_{12}	b_1g_{02}	b_1g_{12}	b_2g_{01}	$a_1b_2b_3$	$a_2b_1b_3$	$a_3b_1b_2$
b_0g_{01}	0	0	0	0	0	0	0
$a_0b_1b_2$	0	1	0	1	1	0	−1
$a_1b_0b_2$	1	0	0	−1	−1	1	0
$a_1b_1b_2$	0	0	1	−1	−1	−1	0
$a_2b_0b_1$	−1	−1	0	0	0	−1	1
$a_2b_1b_2$	0	−1	−1	0	0	1	1

For $(p,q)=(-2,k+2l)$,

	b_0g_{01}	$a_0b_1b_2$	$a_1b_0b_2$	$a_1b_1b_2$	$a_2b_0b_1$	$a_2b_1b_2$
$a_0b_0b_1$	1	2	1	1	1	1
$a_1b_0b_1$	−1	0	−1	1	−1	1

For $(p,q)=(-2,k+2l-1)$,

	b_1y_2	b_2y_1
b_0y_1	-1	-1
b_1y_0	1	1
b_1y_1	1	1

By direct computation based on the above presentation we obtain the following result. Let k_2 (resp. k^2) denote the module $\mathsf{k}/2\mathsf{k}$ (resp. $\mathsf{k}\oplus\mathsf{k}$).

Proposition 7.5 *Suppose k is either of $\mathbb{Z}$ or $\mathbb{F}_\mathfrak{p}$ where $\mathfrak{p}$ is prime. Let k be an odd number and l be an even numbers with $k+5\le l\le 2k-4$ and $|3k-2l|\ge 2$, or $l+7\le k\le 2l-7$ and $|3l-2k|\ge 2$. We abbreviate $\check{\mathbb{E}}_2^{pq}$ for $S^k\times S^l$ as (p,q). We have the following isomorphisms:*

$$(0,k)=(-1,k)=(0,l)=(-1,l)=(-1,2k)=(-2,2k)=(-1,2l)=(-2,2l)=\mathsf{k}$$

$$(-2,3k)=(-3,3k)=(-2,3l)=(-3,3l)=(0,k+l-1)=(-1,k+l-1)=\mathsf{k},$$

$$(0,k+l)=\mathsf{k},\qquad (-1,k+l)=\mathsf{k}\oplus\mathsf{k}_2 \text{ or } \mathsf{k}^2,\qquad (-2,k+l)=0 \text{ or } \mathsf{k},$$

$$(0,2k+l-1)=\mathsf{k},\qquad (-1,2k+l-1)=\mathsf{k}^2,\qquad (-2,2k+l-1)=\mathsf{k},$$

$$(-1,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k},\qquad (-2,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k}^2,\qquad (-3,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k}^2,$$

$$(-4,2k+l)=0 \text{ or } \mathsf{k},\qquad (0,k+2l-1)=\mathsf{k},\qquad (-1,k+2l-1)=\mathsf{k}^2,$$

$$(-2,k+2l-1)=\mathsf{k},\qquad (-1,k+2l)=\mathsf{k}_2 \text{ or } \mathsf{k},\qquad (-2,k+2l)=\mathsf{k} \text{ or } \mathsf{k}^2,$$

$$(-3,k+2l)=\mathsf{k}^2,\qquad (-4,k+2l)=\mathsf{k}.$$

Here "$(p,q)=A$ or B" means $(p,q)=A$ if $\mathsf{k}=\mathbb{Z}$ or $\mathbb{F}_\mathfrak{p}$ with $\mathfrak{p}\neq 2$ and $(p,q)=B$ if $\mathsf{k}=\mathbb{F}_2$. For other (p,q) with $p+q\le\max\{k+2l,2k+l\}$ we have $(p,q)=0$. □

The isomorphisms of Proposition 7.5 hold under milder conditions on k and l. It suffices to ensure the bidegrees presented above are pairwise distinct. By degree argument, we obtain the following corollary:

Corollary 7.6 *Suppose k is either $\mathbb{Z}$ or $\mathbb{F}_\mathfrak{p}$ where $\mathfrak{p}$ is a prime. Let k be an odd number and l be an even number with $k+5\le l\le 2k-4$ and $|3k-2l|\ge 2$, or $l+7\le k\le 2l-7$ and $|3l-2k|\ge 2$. We set $H^*=H^*(\mathrm{Emb}(S^1,S^k\times S^l))$.*

(1) *We have isomorphisms*

$$H^i=\mathsf{k}\quad\text{for } i=k-1,k,2k-2,2k-1,l-1,l,2l-2,2l-1,k+l.$$

(2) *If $\mathsf{k}=\mathbb{F}_\mathfrak{p}$ with $\mathfrak{p}\neq 2$, we have isomorphisms*

$$H^i=\begin{cases}\mathsf{k} & \text{if } i=k+l-2,2k+l-3,2k+l-1,\\ \mathsf{k}^2 & \text{if } i=k+l-1,2k+l-2,\\ 0 & \text{if } i=2k+l-4.\end{cases}$$

(3) *If* $\mathsf{k} = \mathbb{Z}$, *we have isomorphisms*

$$H^i = \begin{cases} \mathsf{k} & \text{if } i = k+l-2, \\ \mathsf{k}^2 \oplus \mathsf{k}_2 & \text{if } i = k+l-1, 2k+l-2, \\ \mathsf{k} \oplus \mathsf{k}_2 & \text{if } i = 2k+l-3, \\ 0 & \text{if } i = 2k+l-4. \end{cases}$$

(4) *We have* $H^i = 0$ *for an integer* i *that satisfies* $i \le \max\{k+2l, 2k+l\}$ *and is different from any of the following integers*:

$k-1, k, l-1, l, 2k-2, 2k-1, 2l-2, 2l-1, 3k-3, 3k-2, 3l-3, 3l-2, k+l-2, k+l-1,$
$k+l, 2k+l-4, 2k+l-3, 2k+l-2, 2k+l-1, k+2l-4, k+2l-3, k+2l-2, k+2l-1.$

Proof By an argument similar to the proof of Theorem 5.17, $\check{\mathbb{E}}_2^{-p,q} = 0$ if $q/p < \frac{1}{3}(k+l)$. We shall show that any differential $d_r : \check{\mathbb{E}}_r^{(-p-r,q+r-1)} \to \check{\mathbb{E}}_r^{-p,q}$ going into the term contained in the cohomology of the claim is zero. It is enough to show this for the case of $(-p,q) = (0, 2k+l-1)$ and $q+r-1 \ge k+2l-1$ since other cases are obvious, or follow from this case. We see

$$\frac{q+r-1}{p+r} = \frac{q-1}{r} + 1 \le \frac{2k+l-2}{l-k+1} + 1 = \frac{k+2l-1}{l-k+1} < \tfrac{1}{3}(k+l).$$

So $\mathbb{E}_r^{(-p-r,q+r-1)} = 0$ and $d_r = 0$. □

7.4 The case of $M = S^k \times S^l$ with k, l even

We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\wedge\{a,b\}$. We fix an orientation ϵ on $\mathcal{H}$ by $\epsilon(ab) = 1$. We set $c = \bar{a} \in S\mathcal{H}$ and $d = \bar{b} \in S\mathcal{H}$. We write a_i for $e_i(a)$, b_i for $e_i(b)$, etc, and $B_n(\mathcal{H})$ is presented as

$$B_n(\mathcal{H}) = \bigwedge\{a_i, b_i, c_i, d_i, g_{ij}, h_{ij} \mid 0 \le i, j \le n\}/\mathcal{J}$$

where $|g_{ij}| = (-1, k+l)$, $|h_{ij}| = (-1, 2(k+l)-1)$ and $\mathcal{J}$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (c_i)^2, (d_i)^2, a_ib_i, a_ic_i, b_id_i, c_id_i, a_id_i - b_ic_i (g_{ij})^2, (h_{ij})^2, g_{ii}, h_{ii}, g_{ij} - g_{ji}, h_{ij} + h_{ji},$$
$$(a_i - a_j)g_{ij}, (b_i - b_j)g_{ij}, (c_i - c_j)g_{ij} - a_ih_{ij}, (d_i - d_j)g_{ij} - b_ih_{ij}, (a_i - a_j)h_{ij}, (b_i - b_j)h_{ij}, (c_i - c_j)h_{ij},$$
$$(d_i - d_j)h_{ij}, (h_{ij} + h_{ik})g_{jk} - (h_{ij} + h_{jk})g_{ki} \text{ and the 3–term relations for } g_{ij} \text{ and } h_{ij}.$$

The diagonal classes are given by

$$\Delta_{\mathcal{H}} = a_0b_1 + a_1b_0 \in S\mathcal{H} \otimes S\mathcal{H} \quad \text{and} \quad \Delta_{S\mathcal{H}} = a_0d_0 + a_1d_0 + b_1c_0 - b_0c_1 - a_0d_1 - a_1d_1.$$

By an argument similar to the proof of Corollary 7.6, we obtain the following corollary:

Corollary 7.7 *Suppose* $2 \in \mathsf{k}^\times$. *Let* k *and* l *be two even numbers with* $k+2 \le l \le 2k-2$ *and* $|3k-2l| \ge 2$. *We set* $H^* = H^*(\mathrm{Emb}(S^1, S^k \times S^l))$. *We have isomorphisms*

$$H^i = \mathsf{k} \quad \text{for } i = k-1, k, l-1, l, k+l-3, k+l-2, k+l-1, 3k.$$

For any other degree $i \le 2k+l$, *we have* $H^i = 0$. □

7.5 The case of 4–dimensional manifolds

In this subsection, we prove Corollary 1.3. We assume that M is a simply connected 4–dimensional manifold. So, as is easily observed, $\mathcal{H}$ is a free k–module for any k.

Definition 7.8 Set $\chi = \chi(M)$. We define a map $\alpha\colon (\mathcal{H}^2)^{\otimes 2} \oplus \mathsf{k}g_{01} \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi\mathcal{H}^4$ by

$$\alpha(a \otimes b) = (-a \otimes b - b \otimes a) + ab, \quad \alpha(g_{01}) = \mathrm{pr}_1(\Delta_{\mathcal{H}}).$$

Here g_{01} is a formal free generator (which will correspond to the element of the same symbol in $\check{\mathbb{E}}_1^{-2,4}$) and pr_1 is the projection

$$(\mathcal{H}^{\otimes 2})^{*=4} \to (\mathcal{H}^2)^{\otimes 2} \oplus (1 \otimes \mathcal{H}^4) \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi\mathcal{H}^4.$$

The next proposition follows from direct computation and degree argument based on Theorem 6.11.

Lemma 7.9 *We use the notation in Definition 7.8. Suppose* k *is a field and* $\mathcal{H}^2$ *is not zero.*

(1) *When* $p+q=1$, $\check{\mathbb{E}}_r^{p,q}$ *is stationary after* E_2. *In particular,* $\check{\mathbb{E}}_2^{p,q} \cong \check{\mathbb{E}}_\infty^{p,q}$. *We have isomorphisms*

$$\check{\mathbb{E}}_2^{p,q} \cong \begin{cases} \mathcal{H}^2 & \text{if } (p,q) = (-1,2), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *There exists an isomorphism*

$$\check{\mathbb{E}}_2^{-2,4} \cong \mathrm{Ker}(\alpha)/\mathsf{k}(\mathrm{pr}_2(\Delta_{\mathcal{H}}) + 2g_{01}).$$

Here pr_2 *is the projection* $(\mathcal{H}^{\otimes 2})^{*=4} \to (\mathcal{H}^2)^{\otimes 2}$. *The differential* d_r *coming into this term is zero for* $r \geq 2$. □

Remark 7.10 Actually, Lemma 7.9 holds even when k is a not a field since torsion in the Künneth theorem does not affect the range.

Proof of Corollary 1.3 In this proof, we suppose k is a field. Set $H_2^{\mathbb{Z}} = H_2(M;\mathbb{Z})$. As is well known, there is a weak homotopy equivalence between $\mathrm{Imm}(S^1, M)$ and the free loop space LSM, and there is an isomorphism $\pi_1(LSM) \cong \pi_1(SM) \oplus \pi_2(SM)$. As M is simply connected, we have $\pi_1 \mathrm{Imm}(S^1, M) \cong \pi_2(SM) \cong \pi_2(M) \cong H_2^{\mathbb{Z}}$.

By the Goodwillie–Weiss convergence theorem, connectivity of the standard projection $\mathrm{holim}_\Delta \mathcal{C}^\bullet(M) \to \mathrm{holim}_{\Delta_n} \mathcal{C}^\bullet(M)$ increases as n increases. Since Δ_n is a compact category in the sense of [13] and $\mathcal{C}^n(M)$ is simply connected for any n, by [13, Theorem 2.2] we see that $\mathrm{Emb}(S^1, M)$ is $\mathbb{Z}$–complete. In particular, $\pi_1(\mathrm{Emb}(S^1, M))$ is a pro-nilpotent group. So, by a theorem of Stallings [38], we only have to prove that the composition

$$\mathrm{Emb}(S^1, M) \xrightarrow{i_M} \mathrm{Imm}(S^1, M) \xrightarrow{\simeq} LSM \xrightarrow{\mathrm{cl}_1} K(H_2^{\mathbb{Z}}, 1)$$

induces an isomorphism on $H_1(-;\mathbb{Z})$ and a surjection on $H_2(-;\mathbb{Z})$. Here the rightmost map cl_1 is the classifying map; see [15].

Consider the spectral sequence $E_r^{p,q}$ associated to the Hochschild complex of $C_*(\widetilde{\mathcal{T}}_M)$. This spectral sequence is isomorphic to the Bousfield–Kan type cohomology spectral sequence associated to the well-known cosimplicial model for LSM given by $[n] \mapsto SM^{n+1}$. The quotient map $\widetilde{\mathcal{T}}_M \to \mathcal{T}_M$ induces a map $f_r\colon E_r^{pq} \to \check{\mathbb{E}}_r^{pq}$ of spectral sequences. For $r = \infty$, this map is identified with the map on the associated graded induced by the inclusion i_M. For $p+q=1$, by Lemma 7.9 (and similar computation for E_r^{pq}), f_2 is an isomorphism for any field k. Since $\pi_1(\mathrm{Emb}(S^1, M))$ is the same as π_1 of a finite stage of Taylor tower which is the finite homotopy limit of a simply connected finite cell complex, it is finitely generated, and so is H_1. By the universal coefficient theorem, i_M induces an isomorphism on $H_1(-;\mathbb{Z})$. For the part of $p+q=2$, we see $E_2^{pq}=0$ for $p<-2$ and $E^{-2,4} \cong \mathrm{Ker}(\alpha) \cap (\mathcal{H}^2)^{\otimes 2}$. Consider the zigzag

$$LSM \xrightarrow{L(\mathrm{cl}_2)} LK(H_2^{\mathbb{Z}}, 2) \xleftarrow{i_K} \Omega K(H_2^{\mathbb{Z}}, 2),$$

where the left map is induced by the classifying map $\mathrm{cl}_2\colon SM \to K(H_2^{\mathbb{Z}}, 2)$ and the right one is the inclusion from the based loop space. Clearly the composition $\mathrm{cl}_1 \circ i_K\colon \Omega K(H_2^{\mathbb{Z}}, 2) \to K(H_2^{\mathbb{Z}}, 1)$ is a weak homotopy equivalence. Observe spectral sequences associated to the standard cosimplicial models of the above three spaces. Since the maps $L(\mathrm{cl}_2)$ and i_K are induced by cosimplicial maps, they induce maps on spectral sequences. In the part of total degree 2, we see that the filtration level F^{-2} for each of the three spectral sequences is the entire cohomology group, and the filtration level F^{-1} for the one for $\Omega K(H_2^{\mathbb{Z}}, 2)$ is zero. With these observations, we see that the image of $H^2(K(H_2^{\mathbb{Z}}, 1))$ in $H^2(LSM)$ by the map cl_1 is sent to a subspace V of $F^{-2}/F^{-1} \cong E_\infty^{-2,4} \subset E_2^{-2,4}$ isomorphically, and a basis of V is given by $\{a_i \otimes a_j - a_j \otimes a_i \mid i<j\}$ as elements of $E_2^{-2,4}$, where $\{a_i\}_i$ denotes a basis of $\mathcal{H}^2$. (We also see that these elements must be stationary.) If $\mathsf{k} \neq \mathbb{F}_2$, or if $\mathsf{k} = \mathbb{F}_2$ and the inverse of the intersection matrix has at least one nonzero diagonal component, the restriction of f_2 to V is a monomorphism by Lemmas 6.2 and 7.9. (Otherwise, the elements of the basis of V have the relation $\mathrm{pr}_0(\Delta_{\mathcal{H}}) = 0$.) This implies i_M induces a surjection on H_2 for any field k under the assumption of the theorem. By the universal coefficient theorem, we obtain the desired assertion on $H_2(-;\mathbb{Z})$. □

Remark 7.11 If all of the diagonal components of the inverse of the intersection matrix on $H_2(M;\mathbb{F}_2)$ are zero, the map $f_2\colon V \to \check{\mathbb{E}}_2^{-2,4}$ in the proof is not a monomorphism for $\mathsf{k} = \mathbb{F}_2$, but this does not necessarily imply the original (nonassociated graded) map is not a monomorphism. So in this case, it is still unclear whether i_M is an isomorphism on π_1.

8 Precise statement and proof of Theorem 1.5

Definition 8.1 • Fix a coordinate plane with coordinates (x, y). A *planar rooted n–tree* $(T, \mathfrak{e})$ consists of a 1–dimensional finite cell complex T and a continuous monomorphism $\mathfrak{e}$ from its realization $|T|$ to the half plane $y \geq 0$ such that:

– T is connected and $\pi_1(T)$ is trivial.

- The intersection of the image of $\mathfrak{e}$ and the x–axis consists of the image of n univalent vertices called *leaves*. These vertices are labeled by $1, \dots, n$ in the manner consistent with the standard order on the axis.
- T has a unique distinguished vertex, called the *root*, which is at least bivalent.
- Any vertex except for the leaves and root is at least trivalent.

An *isotopy* between n–trees $(T_1, \mathfrak{e}_1) \to (T_2, \mathfrak{e}_2)$ is an isotopy of the half plane onto itself which maps $\mathfrak{e}_1(|T_1|)$ onto $\mathfrak{e}_2(|T_2|)$ and the root to the root. (So an isotopy preserves the leaves, including the labels.) We will denote an isotopy class of planar rooted n–trees simply by T. The root vertex of a tree is usually denoted by v_r. For a vertex v of a tree, $|v|$ denotes the number which is the valence minus 1 if $v \neq v_r$, and equal to the valence if $v = v_r$ ($|v|$ is the number of the "out-going edges").

• Let Ψ_n be a category defined as follows. An object of Ψ_n is an isotopy class of planar rooted n–trees. There is a unique morphism $T \to T'$ if T' is obtained from T by successive contractions of internal edges (ie edges not adjacent to leaves).

• Let *Cat* be the category of small categories and functors. Let $i_n\colon \Psi_n \to \Psi_{n+1}$ be a functor which sends T to the tree made from T by attaching two edges to the n^{th} leaf of T and labeling the new leaves with n and $n+1$. We define a category Ψ as the colimit of the sequence $\Psi_1 \xrightarrow{i_1} \Psi_2 \xrightarrow{i_2} \cdots$ taken in *Cat*. $\mathcal{F}_n\colon \Psi_{n+1} \to \mathrm{P}_n$ denotes the functor given in [37, Definition 4.14], which sends a tree $T \in \Psi_{n+1}$ to the set of numbers i such that the shortest paths from i and $i+1$ to the root in T intersect only at the root. For the functor $\mathcal{G}_n\colon \mathrm{P}_{n+1} \to \Delta_n$, see Section 2.1. The square

$$\begin{tikzcd}
\Psi_{n+2} \arrow[r, "\mathcal{G}_{n+1}\circ\mathcal{F}_n"] \arrow[d, "i_n"] & \Delta_n \arrow[d, "i_n"] \\
\Psi_{n+3} \arrow[r, "\mathcal{G}_{n+2}\circ\mathcal{F}_{n+1}"] & \Delta_{n+1}
\end{tikzcd}$$

is clearly commutative, where the right vertical arrow is the natural inclusion, so we have the induced functor $\mathcal{G} \circ \mathcal{F}\colon \Psi \to \Delta$.

• Henceforth, for a symmetric sequence X and a vertex v of a tree in Ψ, we denote $X(|v|)$, $X(|v|-1)$ and $\underline{|v|-1}$ by $X(v)$, $X(v-1)$ and $\underline{v-1}$, respectively.

• For a $\mathcal{K}$–comodule X in $\mathcal{SP}$, we shall define a functor $\mathsf{F}^n X : \Psi_{n+2}^{\mathrm{op}} \to \mathcal{SP}$. The definition is similar to (a dual of) the construction of $\mathcal{D}_n[M]$ in [37, Definition 5.6]. For a tree $T \in \Psi_{n+2}$, define a space $\mathcal{K}_T^{\mathrm{nr}}$ by

$$\mathcal{K}_T^{\mathrm{nr}} = \prod_v \mathcal{K}(v).$$

Here v runs through all the nonroot and nonleaf vertices of T. This is denoted by K_T^{nr} in [37]. We set

$$\mathsf{F}^n X(T) = \mathrm{Map}(\mathcal{K}_T^{\mathrm{nr}}, X(v_r - 1)).$$

For a morphism $T \to T'$ given by the contraction of a nonroot edge e (an edge not adjacent to the root), the map $e^*\colon \mathsf{F}^n X(T') \to \mathsf{F}^n X(T)$ is the pullback by the inclusion $\mathcal{K}_T^{\mathrm{nr}} \to \mathcal{K}_{T'}^{\mathrm{nr}}$ to a face corresponding to the edge contraction (see [37, Definition 4.26]). For the i^{th} root edge e, the corresponding map is given by the following composition:

$$\begin{aligned}\operatorname{Map}\Big(\prod_{v\in T'}\mathcal{K}(v),X(v_r'-1)\Big)&=\operatorname{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v),X(v_r'-1)\Big)\to\operatorname{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v),\operatorname{Map}(\mathcal{K}(v_t),X(v_r-1))\Big)\\ &\cong\operatorname{Map}\Big(\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v)\Big)\times\mathcal{K}(v_t),X(v_r-1)\Big)=\operatorname{Map}\Big(\prod_{v\in T}\mathcal{K}(v),X(v_r-1)\Big).\end{aligned}$$

Here v_t is the vertex of e which is not the root. For $1\le i\le |v_r|-1$, the arrow is the pushforward by the adjoint of the partial composition $(-\circ_i-)\colon \mathcal{K}(v_t)\mathbin{\hat{\otimes}}X(v_r'-1)\to X(v_r-1)$, and for $i=|v_r|$ it is the pushforward by the adjoint of the composition

$$\mathcal{K}(v_t)\mathbin{\hat{\otimes}}X(v_r'-1)\xrightarrow{\mathrm{id}\otimes(-)^{\sigma}}\mathcal{K}(v_t)\mathbin{\hat{\otimes}}X(v_r'-1)\xrightarrow{(-\circ_1-)}X(v_r-1),$$

where σ is the transposition of the first $|v_r'|-|v_t|$ and last $|v_t|-1$ letters. The functors $\{\mathsf{F}^n\}_n$ are compatible with the inclusions $i_n\colon\Psi_{n+2}\to\Psi_{n+3}$. Precisely speaking, there exists an obviously defined natural isomorphism $j_n\colon \mathsf{F}^nX\cong\mathsf{F}^{n+1}X|_{\Psi_{n+2}}$ because the inclusion does not change $|v_r|$. We define a functor $\mathsf{F}X\colon\Psi\to\mathcal{SP}$ by $\mathsf{F}X(T)$ being the colimit of the sequence $\mathsf{F}^nX(T)\xrightarrow{\cong}\mathsf{F}^{n+1}X(T)\xrightarrow{\cong}\mathsf{F}^{n+2}X(T)\xrightarrow{\cong}\cdots$.

- We define a category $\mathsf{G}(n)^+$ for an integer $n\ge 1$ as follows. Its objects are a symbol $*$ and the graphs G with set of vertices $V(G)=\underline{n}$ and set of edges $E(G)\subset\{(i,j)\mid i,j\in\underline{n}\text{ with } i\le j\}$. There is a unique morphism $G\to H$ if and only if either both of G and H are graphs and $E(G)\subset E(H)$, or $G=*$ and $H\neq\varnothing$, where $\varnothing$ denotes the graph with no edges. As in the definition, we allow graphs in $\mathsf{G}(n)^+$ to have *loops*, ie edges of the form (i,i) for $i\in\underline{n}$.

- We define a functor $\omega\colon\Psi_{n+2}^{\mathrm{op}}\to\mathit{Cat}$ by $\omega(T)=\mathsf{G}(|v_r|-1)^+$. For the contraction $T\to T'$ of an edge e, we define a map $e^*\colon\underline{v_r'-1}\to\underline{v_r-1}$ as follows. If e is a nonroot edge, e^* is the identity. If e is the i^{th} root edge for $1\le i\le|v_r|-1$, e^* is the order-preserving surjection with $e^*(j)=i$ for $i\le j\le i+|v_t|-1$. For $i=|v_r|$, e^* is the composition

$$\underline{v_r'-1}\xrightarrow{(-)^{\sigma}}\underline{v_r'-1}\xrightarrow{(e')^*}\underline{v_r-1},\quad\text{where }(e')^*(j)=\begin{cases}1 & \text{if } 1\le j\le|v_t|,\\ j-|v_t|+1 & \text{if } |v_t|+1\le j\le|v_r'|-1,\end{cases}$$

and σ is the permutation given in the previous item. For $G\in\mathsf{G}(|v_r'|-1)^+$, we define an object $e^*(G)\in\mathsf{G}(|v_r|-1)^+$ by

$$e^*(G)=\begin{cases}* & \text{if } G=*,\\ \text{the graph with the edge set }\{(e^*(s),e^*(t))\mid(s,t)\in E(G)\} & \text{otherwise.}\end{cases}$$

- We define a category $\widetilde{\Psi}_{n+2}$ as the Grothendieck construction for the (nonlax) functor ω

$$\widetilde{\Psi}_{n+2}=\int_{\Psi_{n+2}}\omega.$$

An object of $\widetilde{\Psi}_{n+2}$ is a pair (T,G) with $T \in \Psi_{n+2}$ and $G \in \omega(T)$. A map $(T,G) \to (T',G')$ is a pair of maps $e\colon T \to T' \in \Psi_{n+2}$ and $G \to e^*(G') \in \omega(T)$. The functor $i_n\colon \Psi_{n+2} \to \Psi_{n+3}$ and the identity $\omega(T) = \omega(i_n(T))$ naturally induce a functor $i_n\colon \widetilde{\Psi}_{n+2} \to \widetilde{\Psi}_{n+3}$. We denote by $\widetilde{\Psi}$ the colimit of the sequence $\{\widetilde{\Psi}_{n+2}; i_n\}$.

- We fix a map $\mathcal{K} \to \mathcal{D}$ of operads and regard $\widetilde{\mathcal{T}}_M$ as a $\mathcal{K}$–comodule via this map.
- We shall define a functor $\mathsf{T}^n_M\colon \widetilde{\Psi}^{\mathrm{op}}_{n+2} \to \mathcal{SP}$. We set

$$\mathsf{T}^n_M(T,G) = \begin{cases} * & \text{if G has at least one loop or } G = *, \\ \mathrm{Map}(\mathcal{K}^{\mathrm{nr}}_T, \mathcal{T}_G) & \text{otherwise.} \end{cases}$$

For a map $(T,G) \to (T',G')$, we set

$$\begin{aligned} \mathrm{Map}\Big(\prod_{v\in T'} \mathcal{K}(v), \mathcal{T}_{G'}\Big) \to \mathrm{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}} \mathcal{K}(v), \mathrm{Map}(\mathcal{K}(v_t), \mathcal{T}_G)\Big) &\cong \mathrm{Map}\Big(\Big(\prod_{\substack{v\in T\\ v\neq v_t}} \mathcal{K}(v)\Big) \times \mathcal{K}(v_t), \mathcal{T}_G\Big) \\ &= \mathrm{Map}\Big(\prod_{v\in T} \mathcal{K}(v), \mathcal{T}_G\Big). \end{aligned}$$

Here the arrow is the adjoint of the map $\mathcal{K}(v_t) \mathbin{\hat{\otimes}} \mathcal{T}_{G'} \to \mathcal{T}_G$ which is the composition of the map $\mathcal{K}(v_t) \mathbin{\hat{\otimes}} \mathcal{T}_{G'} \to \mathcal{T}_{e^*(G')}$ defined in view of Lemma 3.11 and the inclusion $\mathcal{T}_{e^*G'} \subset \mathcal{T}_G$ coming from $G \subset e^*(G')$. The collection $\{\mathsf{T}^n_M\}_n$ naturally induces a functor $\mathsf{T}_M\colon \widetilde{\Psi} \to \mathcal{SP}$ with natural isomorphism $\mathsf{T}_M|_{\widetilde{\Psi}_{n+2}} \cong \mathsf{T}^n_M$.

- Let $\mathcal{M}$ be a model category. Let $\eta\colon \widetilde{\Psi} \to \Psi$ be the functor given by the projection $\eta(T,G) = T$. Let $\eta_!\colon \mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M}) \to \mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{M})$ be the left Kan extension along η, ie

$$(\eta_! X)(T) = \operatorname*{colim}_{\omega(T)} X_T$$

for $X \in \mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M})$. Here abusing notation, for $T \in \Psi$ we denote by $\omega(T)$ the full subcategory $\{(T,G) \mid G \in \omega(T)\}$ of $\widetilde{\Psi}$, and by X_T the restriction of X to $\omega(T)$. Let $\eta^*\colon \mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{M}) \to \mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M})$ be the pullback, ie $\eta^*(Y) = Y \circ \eta$.

Remark 8.2 The category Ψ_n is equivalent to the category Ψ^o_n given in [37, Definition 4.12].

Notation Henceforth we omit $(-)^{\mathrm{op}}$ under (ho)colim. For example, $\mathrm{hocolim}_{\Psi}$ denotes $\mathrm{hocolim}_{\Psi^{\mathrm{op}}}$.

In the rest of this section, as before, all functor categories are endowed with the projective model structure (see Section 2.1).

Lemma 8.3 *Let $\mathcal{M}$ be a cofibrantly generated model category.*

(1) *The pair $(\eta_!, \eta^*)$ is a Quillen adjoint pair.*

(2) *The restriction*

$$\mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M}) \to \mathcal{F}un(\omega(T)^{\mathrm{op}}, \mathcal{M}), \quad X \mapsto X_T,$$

preserves weak equivalences and cofibrations. In particular, the natural map $\operatorname{hocolim}_{\omega(T)} X_T \to \mathbb{L}\eta_! X(T) \in \mathbf{Ho}(\mathcal{M})$ *is an isomorphism.*

(3) *For any functor* $X \in \mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M})$*, there is a natural isomorphism in* $\mathbf{Ho}(\mathcal{M})$

$$\operatorname*{hocolim}_{\Psi} \mathbb{L}\eta_! X \cong \operatorname*{hocolim}_{\widetilde{\Psi}} X.$$

Proof Part (1) is straightforward. We shall prove (2). Let I be a set of generating cofibrations of $\mathcal{M}$. Let C be a category. For objects $a \in C$ and $A \in \mathcal{M}$, the functor sending $b \in C$ to the coproduct of copies of A labeled by morphisms from b to a is denoted by $\boldsymbol{F}^a_A \in \mathcal{F}un(C^{\mathrm{op}}, \mathcal{M})$. A set of generating cofibrations of $\mathcal{F}un(C, \mathcal{M})$ is given by

$$I_C = \{\boldsymbol{F}^a_f : \boldsymbol{F}^a_A \to \boldsymbol{F}^a_B \mid a \in C \text{ and } f : A \to B \in I\}.$$

See [20, Theorem 11.6.1] for details. Since $\omega(T)$ is a full subcategory of $\widetilde{\Psi}$, the restriction functor sends $I_{\widetilde{\Psi}}$ into $I_{\omega(T)}$. Since the restriction preserves colimits, it preserves relative cell objects with respect to these generating sets. As any cofibration is a retract of a relative cell object, we have proved (2). Part (3) follows from (2) and a standard property of colimits. ◻

Theorem 8.4 (1) *There exists an isomorphism in* $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$

$$(\mathcal{G} \circ \mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee) \cong \mathbb{L}\eta_! \mathsf{T}_M.$$

(2) *If* M *is simply connected and of dimension* ≥ 4*, there exists an isomorphism in* $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$C^*(\operatorname{Emb}(S^1, M)) \cong \operatorname*{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M.$$

Proof Let $T \in \Psi$ be an object and set $m = |v_r| - 1$, where v_r is the root vertex of T. By definition $\mathcal{T}_M(m) = \operatorname{colim}_{G \in \omega(T)} \mathcal{T}_G$. We shall show that the natural map

$$\operatorname*{hocolim}_{G \in \omega(T)} \mathcal{T}_G \to \operatorname*{colim}_{G \in \omega(T)} \mathcal{T}_G = \mathcal{T}_M(m) \in \mathbf{Ho}(\mathcal{SP})$$

is an isomorphism. Put $N_1 = \#\{(i,j) \mid i, j \in \underline{m} \text{ with } i \leq j\}$. By abuse of notation, we denote by P_{N_1} the subcategory of $\omega(T)$ consisting of nonempty graphs, which is actually isomorphic to P_{N_1}. The functor $\mathrm{P}^{\mathrm{op}}_{N_1} \ni G \mapsto \mathcal{T}_G \in \mathcal{SP}$ satisfies the assumption of Lemma 2.2(2), so the natural map $\operatorname{hocolim}_{\mathrm{P}_{N_1}} \mathcal{T}_G \to \operatorname{colim}_{\mathrm{P}_{N_1}} \mathcal{T}_G$ is an isomorphism. More precisely, for each k, $\mathrm{P}^{\mathrm{op}}_{N_1} \ni G \mapsto (\mathcal{T}_G)_k \in \mathcal{CG}_*$ satisfies the assumption for $\mathcal{M} = \mathcal{CG}_*$. Since a trivial fibration in $\mathcal{SP}$ is a level equivalence and a finite homotopy colimit is obtained by successive applications of a homotopy pushout, the finite homotopy colimit of a diagram of semistable connective spectra is π_*–isomorphic to the levelwise homotopy colimit. As $\mathcal{T}_M(m)$ is a cofiber of the natural map $\operatorname{colim}_{\mathrm{P}_{N_1}} \mathcal{T}_G \to \widetilde{\mathcal{T}}_M$, which is also a (levelwise) homotopy cofiber, we have the assertion. We define a natural transformation $\mathsf{T}_M \to \eta^* \circ \mathsf{F}(\mathcal{T}_M)$ by the pushforward by the constant map $\mathcal{T}_G \to \{*\} \subset \mathcal{T}_M(m)$ for $G \neq \varnothing \in \omega(T)$, and by the quotient map $\mathcal{T}_\varnothing \to \mathcal{T}_M(m)$ for $G = \varnothing$. By the assertion and Lemma 8.3(2), the derived adjoint of the natural transformation $\mathbb{L}\eta_!\mathsf{T}_M \to \mathsf{F}\mathcal{T}_M$ is an isomorphism in $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$. It is clear that F preserves weak equivalences, so by Theorem 4.4 we have isomorphisms in $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$

$$\mathsf{F}(\mathcal{C}_M^\vee) \cong \mathsf{F}\mathcal{T}_M \cong \mathbb{L}\eta_!\mathsf{T}_M.$$

We define a natural transformation $(\mathcal{G}\circ\mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee) \to \mathsf{F}(\mathcal{C}_M^\vee)$ by the inclusion $\mathcal{C}^{m-1}(M) = \mathcal{C}_M(m) \subset \mathrm{Map}(\mathcal{K}_T^{\mathrm{nr}}, \mathcal{C}_M(m))$ onto constant maps. This is clearly a weak equivalence, so we have proved (1).

For (2), since the functor $C_*: \mathcal{SP} \to \mathcal{CH}_{\mathsf{k}}$ preserves homotopy colimits (of semistable spectra), by (1), Lemma 8.3(3) and Lemma 5.3, we have isomorphisms in $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$\operatorname*{hocolim}_{\Psi}(\mathcal{G}\circ\mathcal{F})^* C_*(\mathcal{C}^\bullet(M)^\vee) \cong \operatorname*{hocolim}_{\Psi} \mathbb{L}\eta_! C_* \circ \mathsf{T}_M \cong \operatorname*{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M.$$

By Lemma 5.3, Theorem 5.17 and the fact that $\mathcal{G}\circ\mathcal{F}\colon \Psi^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is (homotopy) right cofinal (see Proposition 4.15 and Theorem 6.7 of [37]), we have isomorphisms in $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$C^*(\mathrm{Emb}(S^1, M)) \cong \operatorname*{hocolim}_{\Delta} C^*(\mathcal{C}^\bullet(M)) \cong \operatorname*{hocolim}_{\Delta} C_*(\mathcal{C}^\bullet(M)^\vee) \cong \operatorname*{hocolim}_{\Psi}(\mathcal{G}\circ\mathcal{F})^* C_*(\mathcal{C}^\bullet(M)^\vee).$$

Thus, we have an isomorphism $C^*(\mathrm{Emb}(S^1, M)) \cong \operatorname{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M$. □

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Automorphismes du groupe des automorphismes d'un groupe de Coxeter universel

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À l'aide de l'outre-espace de Guirardel et Levitt d'un produit libre, nous démontrons que le groupe des automorphismes extérieurs du groupe des automorphismes extérieurs du groupe de Coxeter universel de rang $n \geq 5$ est trivial, et qu'il s'agit d'un groupe cyclique d'ordre 2 si $n = 4$. Nous démontrons aussi que le groupe des automorphismes extérieurs du groupe des automorphismes du groupe de Coxeter universel de rang $n \geq 4$ est trivial.

Using the Guirardel–Levitt outer space of a free product, we prove that the outer automorphism group of the outer automorphism group of the universal Coxeter group of rank $n \geq 5$ is trivial, and that it is a cyclic group of order 2 if $n = 4$. In addition we prove that the outer automorphism group of the automorphism group of the universal Coxeter group of rank $n \geq 4$ is trivial.

20E08, 20E36, 20F28, 20F55

1 Introduction

Soit n un entier plus grand que 2. On note $F = \mathbb{Z}/2\mathbb{Z}$ un groupe cyclique d'ordre 2 et $W_n = *_n F$ un groupe de Coxeter universel de rang n, produit libre de n copies de F. Si G est un groupe, on note $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ son groupe d'automorphismes extérieurs. Nous démontrons dans cet article les résultats suivants.

Théorème 1.1 *Si $n \geq 5$, alors $\mathrm{Out}(\mathrm{Out}(W_n)) = \{1\}$. Si $n = 4$, alors $\mathrm{Out}(\mathrm{Out}(W_n))$ est isomorphe à $\mathbb{Z}/2\mathbb{Z}$.*

Théorème 1.2 *Si $n \geq 4$, alors $\mathrm{Out}(\mathrm{Aut}(W_n)) = \{1\}$.*

De tels résultats sont déjà connus dans le cas où $n = 2$ (voir par exemple [Thomas 2020, Lemmas 1.4.2 and 1.4.3]) où tous les automorphismes de $\mathrm{Out}(W_2)$ sont intérieurs et où le groupe $\mathrm{Out}(\mathrm{Aut}(W_2))$ est un groupe cyclique d'ordre 2. Dans le cas où $n = 3$, les groupes $\mathrm{Aut}(W_3)$ et $\mathrm{Out}(W_3)$ sont isomorphes à $\mathrm{Aut}(\mathbb{F}_2)$ et $\mathbb{P}\mathrm{GL}(2, \mathbb{Z})$ respectivement, avec $\mathbb{F}_2$ un groupe libre de rang 2 (voir par exemple [Varghese 2021, Lemma 2.3]). Nous obtenons donc une description de $\mathrm{Out}(\mathrm{Out}(W_n))$ pour tout entier n.

De telles questions de rigidité algébrique ont déjà été résolues dans des cas similaires. En effet, Mostow [1973] a démontré que le groupe des automorphismes extérieurs de réseaux irréductibles uniformes de groupes de Lie réels, connexes, semi-simples et non localement isomorphes à $\mathrm{SL}_2(\mathbb{R})$ est fini. De même, Ivanov [1997, Theorem 2] a démontré un résultat similaire dans le cas du groupe modulaire d'une surface compacte, connexe, orientable de genre $g \geq 2$. Enfin, Bridson et Vogtmann [2000] ont démontré que tout automorphisme du groupe des automorphismes extérieurs d'un groupe libre de rang N (avec $N \geq 3$) est une conjugaison. Ce dernier cas a motivé l'étude de la rigidité algébrique de $\mathrm{Out}(W_n)$, d'une part à cause de la propriété d'universalité pour les groupes engendrés par des éléments d'ordre 2 de W_n, d'autre part car, si $n \geq 3$, le groupe $\mathrm{Aut}(W_n)$ s'injecte dans $\mathrm{Aut}(F_{n-1})$ (voir par exemple [Mühlherr 1997, Theorem A]). Néanmoins, cette injection ne donne aucune piste de déduction des théorèmes 1.1 et 1.2 à partir du théorème de Bridson et Vogtmann. Peu de choses sont connues pour $\mathrm{Aut}(W_n)$. Une partie génératrice finie a été construite par Mühlherr [1997, Theorem B], et une présentation a été donnée par Gilbert [1987] dans le contexte plus général du groupe des automorphismes d'un produit libre, suivant [Fouxe-Rabinovitch 1941]. Plus récemment, Varghese [2021, Theorem C] a démontré que le groupe $\mathrm{Aut}(W_n)$ n'a pas la propriété (T) de Kazhdan. Dans tous les cas, les techniques d'étude de $\mathrm{Aut}(W_n)$ sont principalement combinatoires et géométriques. Mentionnons enfin que l'étude du groupe $\mathrm{Aut}(W_n)$ est en lien avec l'étude du groupe des automorphismes symétriques d'un produit libre étudié par McCullough et Miller [1996] (mais celle-ci ne donne aucune piste de déduction des théorèmes 1.1 et 1.2) et qu'il serait intéressant de connaître des résultats similaires dans ce contexte plus général.

Pour démontrer les théorèmes 1.1 et 1.2, nous étudions l'action de $\mathrm{Out}(W_n)$ sur un complexe simplicial de drapeaux introduit par Guirardel et Levitt. Plus précisément, nous cherchons à comprendre les stabilisateurs de certains sommets de ce complexe. En effet, les stabilisateurs de ces sommets formant une partie génératrice de $\mathrm{Aut}(W_n)$ et $\mathrm{Out}(W_n)$, comprendre l'image de ces stabilisateurs par des automorphismes de $\mathrm{Aut}(W_n)$ et $\mathrm{Out}(W_n)$ nous permettra de faciliter l'étude de ces derniers. L'étude de l'action de $\mathrm{Out}(W_n)$ sur un complexe simplicial se justifie également par la démonstration des théorèmes similaires dans les cas des réseaux des groupes de Lie semi-simples, du groupe modulaire d'une surface de type fini et du groupe des automorphismes d'un groupe libre qui passait également par l'étude de l'action du groupe étudié sur un espace géométrique adapté. En particulier, dans le cas du groupe des automorphismes extérieurs d'un groupe libre de rang N, cet objet géométrique était *l'outre-espace de Culler et Vogtmann* CV_N [1986].

Dans le cas de W_n, Guirardel et Levitt [2007b] ont introduit un espace topologique analogue à l'outre-espace de Culler et Vogtmann, appelé *l'outre-espace d'un produit libre*. Dans le cas d'un produit libre de copies de F, cet espace sera noté $\mathbb{P}\mathbb{O}(W_n)$. Ce dernier est défini comme un ensemble de classes d'homothétie de graphes métriques marqués de groupes, dont le groupe fondamental est isomorphe à W_n. Muni de la topologie dite *faible*, l'espace $\mathbb{P}\mathbb{O}(W_n)$ se rétracte par déformation forte sur un complexe simplicial de drapeaux, appelé *l'épine de* $\mathbb{P}\mathbb{O}(W_n)$. Le groupe $\mathrm{Out}(W_n)$ agit naturellement sur $\mathbb{P}\mathbb{O}(W_n)$ et sur son épine par précomposition du marquage. Le groupe $\mathrm{Aut}(W_n)$ agit quant à lui sur *l'autre-espace de* W_n, noté $\mathbb{P}\mathcal{A}(W_n)$. Nous renvoyons à la section 2 pour des précisions.

La démonstration du théorème 1.1 est inspirée de celle de [Bridson et Vogtmann 2000] dans le cas d'un groupe libre, mais des complications structurelles apparaissent, nécessitant de nouvelles idées et méthodes. Nous présentons la démonstration dans le cas de $\mathrm{Out}(W_n)$, le cas de $\mathrm{Aut}(W_n)$ étant similaire et présenté

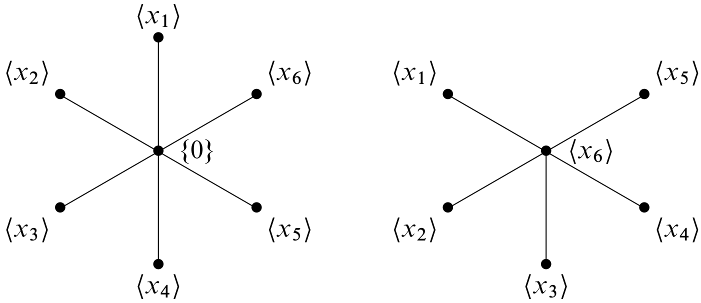

FIGURE 1: Exemples de graphes de groupes dont les classes d'équivalence sont respectivement une $\{0\}$–étoile et une F–étoile (cas $n=6$). Les arêtes ont des groupes associés triviaux. L'ensemble $\{x_1,\dots,x_6\}$ est une partie génératrice standard de W_6.

dans des remarques suivant les démonstrations dans le cas de $\mathrm{Out}(W_n)$. Son plan, très simplifié, est le suivant. L'épine de l'outre-espace $\mathbb{P}\mathbb{O}(W_n)$ contient deux types de sommets distingués, à savoir les $\{0\}$*–étoiles* et les F*–étoiles*, voir la section 2 et la figure 1. Ceci diffère de l'épine de l'outre-espace de Culler et Vogtmann pour lequel il existe un unique type de sommets distingués, à savoir les roses.

Nous étudions tout d'abord les stabilisateurs des $\{0\}$–étoiles et des F–étoiles sous l'action de $\mathrm{Out}(W_n)$. Le stabilisateur d'une $\{0\}$–étoile est isomorphe à $\mathfrak{S}_n$. Il correspond à la permutation des feuilles de la $\{0\}$–étoile. Le stabilisateur d'une F–étoile est isomorphe à $F^{n-2}\rtimes\mathfrak{S}_{n-1}$. Il correspond à la permutation des feuilles de la F–étoile ainsi qu'à l'application de conjugaisons partielles dont le conjuguant est contenu dans le groupe associé au centre de la F–étoile. Nous montrons dans la section 3 les caractérisations suivantes des stabilisateurs de $\{0\}$–étoiles et de F–étoiles (voir les propositions 3.1, 3.6 et 3.9).

Proposition 1.3 *Soit $n\geq 5$.*

(1) *Tout sous-groupe de* $\mathrm{Out}(W_n)$ *isomorphe à* $\mathfrak{S}_n$ *fixe un unique point de l'épine de* $\mathbb{P}\mathbb{O}(W_n)$; *et ce point est une* $\{0\}$*–étoile.*

(2) *Tout sous-groupe de* $\mathrm{Out}(W_n)$ *isomorphe à* $F^{n-2}\rtimes\mathfrak{S}_{n-1}$ *fixe un unique point de l'épine de* $\mathbb{P}\mathbb{O}(W_n)$; *et ce point est une* F*–étoile.*

(3) *Les stabilisateurs des* F*–étoiles sont les sous-groupes finis d'ordre maximaux de* $\mathrm{Out}(W_n)$.

La proposition 1.3 caractérise de ce fait les stabilisateurs de $\{0\}$–étoiles et de F–étoiles, qui sont les sous-groupes de $\mathrm{Out}(W_n)$ isomorphes à $\mathfrak{S}_n$ et $F^{n-2}\rtimes\mathfrak{S}_{n-1}$, respectivement. Ces caractérisations représentent une situation nouvelle en comparaison de la preuve de [Bridson et Vogtmann 2000] dans le cas d'un groupe libre. Notons par ailleurs que le stabilisateur d'une $\{0\}$–étoile correspond au groupe des permutations d'une partie génératrice de W_n et que ce groupe remarquable intervenait déjà dans l'étude faite en [Varghese 2021] de l'action de $\mathrm{Aut}(W_n)$ sur des espaces CAT(0).

La proposition 1.3 implique alors que tout automorphisme α de $\mathrm{Out}(W_n)$ préserve l'ensemble des stabilisateurs de $\{0\}$–étoiles et l'ensemble des stabilisateurs de F–étoiles. Fixons $\alpha\in\mathrm{Aut}(\mathrm{Out}(W_n))$.

Le groupe $\mathrm{Out}(W_n)$ agissant transitivement sur l'ensemble des $\{0\}$–étoiles, nous pouvons supposer que α induit un automorphisme du stabilisateur d'une $\{0\}$–étoile $\mathscr{X}$. Les stabilisateurs de $\{0\}$–étoiles étant isomorphes à $\mathfrak{S}_n$, si $n \geq 5$ et $n \neq 6$, nous pouvons supposer que la restriction de α au stabilisateur de $\mathscr{X}$ est égale à l'identité. Nous montrons alors qu'un tel α préserve le stabilisateur d'une F–étoile $\mathscr{Y}$ adjacente à $\mathscr{X}$, et que la restriction de α au stabilisateur de $\mathscr{Y}$ est en fait l'identité. Le groupe $\mathrm{Out}(W_n)$ étant engendré par l'union des stabilisateurs d'une $\{0\}$–étoile et d'une F–étoile adjacente, ceci conclut la démonstration si $n \geq 5$. Le cas $n = 4$, qui présente un automorphisme extérieur exceptionnel, est traité dans la section 4.

Remarquons enfin qu'une extension du théorème 1.1 à des produits libres de copies d'un groupe fini n'est pas immédiate, ni même aux produits libres de copies d'un groupe fini cyclique. En effet, même si la proposition 1.3 s'étend aux produits libres de groupes finis cycliques, il convient dès lors d'étudier le groupe des automorphismes du stabilisateur d'une $\{0\}$–étoile dans ce contexte, qui n'est cette fois plus trivial, ce qui nous semble nécessiter une nouvelle approche pour toute extension des théorèmes 1.1 et 1.2.

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2 Préliminaires

Nous rappelons tout d'abord la définition de l'outre-espace $\mathbb{P}\mathbb{O}(W_n)$ introduit dans [Guirardel et Levitt 2007b]. Un point de $\mathbb{P}\mathbb{O}(W_n)$ est une classe d'homothétie de graphes métriques X de groupes, dont le groupe fondamental est W_n et qui sont munis d'un isomorphisme de groupes appelé *marquage* $\rho\colon W_n \to \pi_1(X)$ (pour un choix indifférent de point base) vérifiant :

(1) Le graphe sous-jacent à X est un arbre fini, nous le noterons $\overline{X}$ tout au long de l'article.

(2) Tous les groupes d'arêtes sont triviaux.

(3) Il y a exactement n sommets de groupes associés isomorphes à F.

(4) Tous les autres sommets ont un groupe associé trivial.

(5) Toute feuille de l'arbre sous-jacent a un groupe associé non trivial.

(6) Si v est un sommet de groupe associé trivial, alors $\deg(v) \geq 3$.

Deux graphes métriques marqués (X, ρ) et (X', ρ') sont dans la même classe d'homothétie s'il existe une homothétie $f\colon X \to X'$ (ie un homéomorphisme multipliant toutes les longueurs des arêtes par un même scalaire strictement positif) telle que $f_* \circ \rho = \rho'$. On note $[X, \rho]$ la classe d'homothétie d'un tel graphe de groupes métrique marqué (X, ρ). Si le marquage est sous-entendu, on notera $\mathscr{X}$ la classe d'homothétie. Le groupe $\mathrm{Aut}(W_n)$ agit par précomposition du marquage. Par ailleurs, pour tout $\alpha \in \mathrm{Int}(W_n)$, et pour tout $\mathscr{X} \in \mathbb{P}\mathbb{O}(W_n)$, nous avons $\alpha(\mathscr{X}) = \mathscr{X}$. En effet, un automorphisme intérieur α de W_n agit par translation sur l'arbre de Bass–Serre associé à un graphe de groupes marqué X, ce qui implique que α préserve la classe d'équivalence de X. De ce fait, l'action de $\mathrm{Aut}(W_n)$ sur $\mathbb{PO}(W_n)$ induit une action de $\mathrm{Out}(W_n)$ sur $\mathbb{PO}(W_n)$.

La définition de *l'autre-espace de* W_n, noté $\mathbb{P}\mathcal{A}(W_n)$, est identique à celle de $\mathbb{PO}(W_n)$ à ceci près que chaque graphe de groupes métrique considéré est muni d'un point base v. Le *marquage* est alors un isomorphisme de groupes $\rho\colon W_n \to \pi_1(X, v)$. Les homothéties considérées préservent les points bases. Le groupe $\mathrm{Aut}(W_n)$ agit par précomposition du marquage.

Les ensembles $\mathbb{PO}(W_n)$ et $\mathbb{P}\mathcal{A}(W_n)$ sont munis d'une topologie. Pour tout élément $[X,\rho] \in \mathbb{PO}(W_n)$, soit (X,ρ) un représentant de cette classe d'équivalence tel que la somme des longueurs des arêtes du graphe $\overline{X}$ soit égale à 1. Le graphe de groupes (X,ρ) définit alors un simplexe ouvert obtenu en faisant varier les longueurs des arêtes du graphe $\overline{X}$, de manière à ce que la somme des longueurs des arêtes soit toujours égale à 1. Une classe d'équivalence $[X',\rho'] \in \mathbb{PO}(W_n)$ définit une face de codimension 1 du simplexe associé à (X,ρ) si l'on peut obtenir (X',ρ') à partir de (X,ρ) en écrasant une arête de $\overline{X}$. La *topologie faible* sur $\mathbb{PO}(W_n)$ est alors définie de la manière suivante : un ensemble est ouvert si et seulement si son intersection avec chaque simplexe ouvert est ouverte.

Nous rappelons à présent la définition d'un rétract par déformation forte $\mathrm{Out}(W_n)$–équivariant de $\mathbb{PO}(W_n)$, appelé *l'épine de l'outre-espace*. Rappelons qu'un complexe simplicial $\mathscr{C}$ est *de drapeaux* si, pour tout entier k, tout ensemble de k sommets deux à deux distincts et deux à deux adjacents de $\mathscr{C}$ forme l'ensemble des sommets d'un simplexe de $\mathscr{C}$ de dimension $k-1$. L'épine de $\mathbb{PO}(W_n)$ est le complexe simplicial de drapeaux dont les sommets sont les simplexes ouverts associés à chaque classe d'équivalence $[X,\rho]$, et où deux sommets correspondant à des classes d'équivalence de graphes de groupes marqués $[X,\rho]$ et $[X',\rho']$ sont reliés par une arête si $[X,\rho]$ définit une face du simplexe associé à $[X',\rho']$ ou réciproquement. L'épine de $\mathbb{P}\mathcal{A}(W_n)$ est définie de manière similaire. Il existe un plongement de l'épine de $\mathbb{PO}(W_n)$ dans $\mathbb{PO}(W_n)$ ayant pour image l'épine barycentrique de $\mathbb{PO}(W_n)$. Par la suite, nous identifierons l'épine de $\mathbb{PO}(W_n)$ avec son image par ce plongement. De même, il existe un plongement de l'épine de $\mathbb{P}\mathcal{A}(W_n)$ dans $\mathbb{P}\mathcal{A}(W_n)$ ayant pour image l'épine barycentrique de $\mathbb{P}\mathcal{A}(W_n)$.

Si X est un graphe de groupes, on note $\mathrm{Aut}_{\mathrm{gr}}(X)$ le groupe des automorphismes du graphe $\overline{X}$. Si X est un graphe de groupes pointé, la notation $\mathrm{Aut}_{\mathrm{gr}}(X)$ désigne le groupe des automorphismes du graphe pointé sous-jacent à X. Nous appellerons $\{0\}$*–étoile* la classe d'équivalence dans $\mathbb{PO}(W_n)$ d'un graphe de groupes marqué dont le graphe sous-jacent est un arbre ayant n feuilles et $n+1$ sommets et de longueur d'arêtes constante. Nous appellerons F*–étoile* la classe d'équivalence dans $\mathbb{PO}(W_n)$ d'un graphe de groupes marqué dont le graphe sous-jacent est un arbre ayant $n-1$ feuilles et n sommets et de longueur d'arêtes constante. Les sommets correspondants dans l'épine de $\mathbb{PO}(W_n)$ sont encore appelés $\{0\}$*–étoiles* et F*–étoiles*. Dans le cas de $\mathbb{P}\mathcal{A}(W_n)$, les définitions des $\{0\}$*–étoiles* et des F*–étoiles* sont identiques à ceci près que l'on suppose également que le point base est le centre (l'unique sommet qui n'est pas une feuille) du graphe.

On fixe désormais une partie génératrice standard $\{x_1,\ldots,x_n\}$ de W_n.

Le groupe $\mathrm{Aut}(W_n)$ (et donc $\mathrm{Out}(W_n)$) est de type fini. Nous décrivons maintenant une partie génératrice finie. Pour tout $i \in \{1, \dots, n-1\}$, on note $\tau_i \colon W_n \to W_n$ l'automorphisme envoyant x_i sur x_{i+1}, x_{i+1} sur x_i et qui fixe tous les autres générateurs. Pour tous les $i, j \in \{1, \dots, n\}$ tels que $i \neq j$, on note $\sigma_{i,j} \colon W_n \to W_n$ l'automorphisme qui envoie x_i sur $x_j x_i x_j$ et qui fixe tous les autres générateurs. La proposition suivante est due à Mühlherr (voir également [Fouxe-Rabinovitch 1941; Gilbert 1987]).

Proposition 2.1 [Mühlherr 1997, Theorem B] *Le groupe* $\mathrm{Aut}(W_n)$ *est engendré par* $\tau_1, \dots, \tau_{n-1}$ *et par* $\sigma_{1,2}$.

Si α est un élément de $\mathrm{Aut}(W_n)$, sa classe d'automorphismes extérieurs sera notée $[\alpha]$. Soit $p \colon \mathrm{Aut}(W_n) \to \mathrm{Out}(W_n)$ la projection canonique. On note $\widetilde{A}_n = \langle \tau_1, \dots, \tau_{n-1} \rangle$ et $A_n = p(\widetilde{A}_n)$. Les groupes $\widetilde{A}_n$ et A_n sont isomorphes au groupe symétrique $\mathfrak{S}_n$. On note $\widetilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n} \rangle$ et $U_n = p(\widetilde{U}_n)$. On voit que $\widetilde{U}_n$ est isomorphe au produit semi-direct $F^{n-1} \rtimes \mathfrak{S}_{n-1}$, alors que U_n est isomorphe au produit semi-direct $F^{n-2} \rtimes \mathfrak{S}_{n-1}$, où $\mathfrak{S}_{n-1}$ agit dans les deux cas par permutation des facteurs, en considérant F^{n-2} comme le quotient de F^{n-1} par le sous-groupe F diagonal. Soient $\widetilde{B}_n = \langle \tau_1, \dots, \tau_{n-2} \rangle$ et $B_n = p(\widetilde{B}_n)$. Les groupes $\widetilde{B}_n$ et B_n sont isomorphes à $\mathfrak{S}_{n-1}$.

Nous traitons à présent le cas où $n = 3$. Soit $\epsilon \colon W_3 \to \mathbb{Z}/2\mathbb{Z}$ le morphisme envoyant, pour tout $i \in \{1, 2, 3\}$, l'élément x_i sur 1. Mühlherr [1997, Theorem A] a montré que $\ker(\epsilon)$ est un sous-groupe caractéristique de W_3. De plus, $\ker(\epsilon)$ est un groupe libre à deux générateurs, librement engendré par $x_1 x_2$ et $x_2 x_3$. Ceci induit un morphisme $\rho \colon \mathrm{Aut}(W_3) \to \mathrm{Aut}(\mathbb{F}_2)$, qui est en fait un isomorphisme (cf [Varghese 2021, Lemma 2.3]).

Proposition 2.2 *Le morphisme* $\rho \colon \mathrm{Aut}(W_3) \to \mathrm{Aut}(\mathbb{F}_2)$ *induit un isomorphisme entre* $\mathrm{Out}(W_3)$ *et* $\mathbb{P}\mathrm{GL}(2, \mathbb{Z})$.

Démonstration Soient a et b les générateurs de $\mathbb{F}_2$. On remarque tout d'abord que $\mathrm{Int}(\mathbb{F}_2) \subseteq \rho(\mathrm{Int}(W_3))$. Donc le noyau du morphisme surjectif $\mathrm{Aut}(W_3) \to \mathrm{Out}(\mathbb{F}_2)$ est inclus dans $\mathrm{Int}(W_3)$. Pour tout $i \in \{1, 2, 3\}$, soit $\mathrm{ad}_{x_i} \in \mathrm{Aut}(W_3)$ la conjugaison globale par x_i. Un calcul immédiat montre que, pour tout $i \in \{1, 2, 3\}$, $\rho(\mathrm{ad}_{x_i})$ est dans la classe d'automorphisme extérieur du morphisme $\iota \colon \mathbb{F}_2 \to \mathbb{F}_2$ envoyant a sur a^{-1} et b sur b^{-1}. De ce fait, puisque le sous-groupe $\langle [\iota] \rangle$ est distingué dans $\mathrm{Out}(\mathbb{F}_2)$, le morphisme ρ induit un isomorphisme entre $\mathrm{Out}(W_3)$ et $\mathrm{Out}(\mathbb{F}_2)/\langle [\iota] \rangle$. Comme ι est envoyé par le morphisme d'abélianisation sur $-\mathrm{Id} \in \mathrm{GL}(2, \mathbb{Z})$, on voit que $\mathrm{Out}(W_3)$ est isomorphe à $\mathbb{P}\mathrm{GL}(2, \mathbb{Z})$. □

Nous allons démontrer les théorèmes 1.1 et 1.2 en étudiant les stabilisateurs des $\{0\}$–étoiles et des F–étoiles sous l'action de $\mathrm{Out}(W_n)$ et $\mathrm{Aut}(W_n)$. Pour cela, nous utiliserons les résultats suivants, dus respectivement à Hensel et Kielak et à Guirardel et Levitt, qui donnent des informations sur les points fixes de sous-groupes de $\mathrm{Out}(W_n)$.

Proposition 2.3 [Hensel et Kielak 2018, Corollary 6.1] *Soient* $n \geq 1$ *un entier et* H *un sous-groupe fini de* $\mathrm{Out}(W_n)$. *Alors* H *fixe un point de* $\mathbb{P}\mathbb{O}(W_n)$.

Corollaire 2.4 *Soient $n \geq 1$ un entier et H un sous-groupe fini de* $\mathrm{Aut}(W_n)$. *Alors H fixe un point de* $\mathbb{P}\mathcal{A}(W_n)$.

Démonstration Soit $p\colon \mathrm{Aut}(W_n) \to \mathrm{Out}(W_n)$ la projection canonique. Alors $p(H)$ est un sous-groupe fini de $\mathrm{Out}(W_n)$, donc par la proposition 2.3, $p(H)$ fixe un point $\mathscr{X}$ de l'outre-espace. Soit X un représentant de $\mathscr{X}$. Comme tout automorphisme intérieur agit sur X, et que $p(H)$ agit également sur X, on en déduit que H agit sur X. Étant donné que H est fini et que $\overline{X}$ est un arbre, on voit que H fixe un point v de $\overline{X}$. Donc la classe d'homothétie du graphe de groupes métrique marqué pointé (X, v) est fixée par H. □

Proposition 2.5 [Guirardel et Levitt 2007a, Theorem 8.3] *Soit $n \geq 2$ un entier. Si H est un sous-groupe de type fini de* $\mathrm{Out}(W_n)$ *(resp.* $\mathrm{Aut}(W_n)$*) fixant un point de* $\mathbb{P}\mathcal{O}(W_n)$ *(resp.* $\mathbb{P}\mathcal{A}(W_n)$*), alors l'ensemble des points fixes de H est contractile pour la topologie faible.* □

On note $\mathrm{Fix}_{\mathbb{P}\mathcal{O}(W_n)}(G)$ l'ensemble des points fixes d'un sous-groupe G de $\mathrm{Out}(W_n)$ dans $\mathbb{P}\mathcal{O}(W_n)$ (ou $\mathrm{Fix}(G)$ s'il n'y a pas d'ambiguïté). On note de plus $\mathrm{Fix}_{K_n}(G)$ l'ensemble des points fixes de G contenus dans l'épine de $\mathbb{P}\mathcal{O}(W_n)$. Puisque l'épine de $\mathbb{P}\mathcal{O}(W_n)$ est un rétract par déformation forte $\mathrm{Out}(W_n)$–équivariant de $\mathbb{P}\mathcal{O}(W_n)$, nous déduisons de la proposition 2.5 le résultat suivant.

Corollaire 2.6 *Soit $n \geq 2$ un entier. Si H est un sous-groupe de type fini de* $\mathrm{Out}(W_n)$ *fixant un point de l'épine de* $\mathbb{P}\mathcal{O}(W_n)$, *alors l'ensemble* $\mathrm{Fix}(H)$ *des points fixes de H dans l'épine de* $\mathbb{P}\mathcal{O}(W_n)$ *est connexe pour la topologie faible.*

Soit $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathcal{O}(W_n)$. On note X un représentant de $\mathscr{X}$ et T l'arbre de Bass–Serre associé à X. Nous définissons à présent un morphisme de groupes

$$\Phi\colon \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X).$$

Soient $[\alpha] \in \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})$ et $\alpha \in \mathrm{Aut}(W_n)$ un représentant de $[\alpha]$. Il existe un automorphisme $\widetilde{H}_\alpha \in \mathrm{Aut}(T)$ tel que $\alpha(g)\widetilde{H}_\alpha(x) = \widetilde{H}_\alpha(gx)$ pour tout $x \in T$ et pour tout $g \in W_n$. L'automorphisme $\widetilde{H}_\alpha$ induit un automorphisme $H_\alpha \in \mathrm{Aut}_{\mathrm{gr}}(X)$, et l'application $\alpha \mapsto H_\alpha$ passe au quotient pour donner un morphisme

$$\Phi\colon \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X).$$

Nous pouvons à présent démontrer un résultat identique au corollaire 2.6 dans le cas de $\mathbb{P}\mathcal{A}(W_n)$.

Corollaire 2.7 *Soit $n \geq 2$ un entier. Si H est un sous-groupe fini de* $\mathrm{Aut}(W_n)$ *fixant un point de l'épine de* $\mathbb{P}\mathcal{A}(W_n)$, *alors l'ensemble* $\mathrm{Fix}(H)$ *des points fixes de H dans l'épine de* $\mathbb{P}\mathcal{A}(W_n)$ *est connexe pour la topologie faible.*

Démonstration Soient $\mathscr{X}$ et $\mathscr{Y}$ deux points de l'épine de $\mathbb{P}\mathcal{A}(W_n)$ fixés par H. Soit $p_1\colon \mathbb{P}\mathcal{A}(W_n) \to \mathbb{P}\mathcal{O}(W_n)$ le morphisme canonique d'oubli du point base. On rappelle que $p\colon \mathrm{Aut}(W_n) \to \mathrm{Out}(W_n)$ est la projection canonique. Alors $p(H)$ fixe $p_1(\mathscr{X})$ et $p_1(\mathscr{Y})$, donc par le corollaire 2.6 il existe dans $\mathrm{Fix}_{K_n}(p(H))$ un chemin continu P de $p_1(\mathscr{X})$ vers $p_1(\mathscr{Y})$. Soient $\mathscr{X}_1, \dots, \mathscr{X}_n$ les sommets de K_n consécutifs dans P (on suppose $p_1(\mathscr{X}) = \mathscr{X}_1$ et $\mathscr{X}_n = p_1(\mathscr{Y})$) tels que, pour tout $i \in \{1, \dots, n-1\}$, X_i et X_{i+1}

sont reliés par une arête dans K_n. Soit X_1 un représentant de $\mathscr{X}_1$ et pour tout $i \in \{2, \dots, n\}$, soit X_i un représentant de $\mathscr{X}_i$ obtenu en écrasant ou en éclatant une forêt de X_{i-1}. Pour tout $i \in \{1, \dots, n\}$, comme tout automorphisme intérieur agit trivialement sur X_i, et puisque $p(H)$ agit également sur X_i, on en déduit que H agit sur X_i. De plus, étant donné que H est fini et que le graphe $\overline{X_i}$ est un arbre, on voit que H fixe un point v_i de $\overline{X_i}$. Pour tout i, soit $\widetilde{\mathscr{X}}_i$ la classe d'équivalence du graphe métrique marqué pointé (X_i, v_i) (on suppose que $\widetilde{\mathscr{X}}_1 = \mathscr{X}$ et $\widetilde{\mathscr{X}}_n = \mathscr{Y}$). Alors $\widetilde{\mathscr{X}}_i$ est fixé par H.

Nous construisons à présent pour tout $i \in \{1, \dots, n-1\}$ un chemin continu inclus dans l'ensemble des points fixes de H dans l'épine de $\mathbb{P}\mathscr{A}(W_n)$ entre $\widetilde{\mathscr{X}}_i$ et $\widetilde{\mathscr{X}}_{i+1}$, ce qui conclura. La construction étant symétrique, nous pouvons supposer, quitte à changer les représentants X_i et X_{i+1}, que X_{i+1} est obtenu à partir de X_i en écrasant une forêt $\mathscr{F}$. Soient Δ le simplexe ouvert dans $\mathbb{P}\mathscr{A}(W_n)$ associé à (X_i, v_i) et e l'arête de l'épine barycentrique de $\mathbb{P}\mathscr{A}(W_n)$ reliant $\widetilde{\mathscr{X}}_i$ et $\widetilde{\mathscr{X}}_{i+1}$. Pour toute arête f de $\mathscr{F}$, soit ℓ_f la longueur de f. Pour tout $t \in [0,1]$, soient X_i^t le graphe de groupes métrique obtenu à partir de X_i en donnant à toute arête $f \in \mathscr{F}$ la longueur $(1-t)\ell_f$, et $\mathrm{pr}_t \colon X_i \to X_i^t$ la projection canonique. On observe que $X_i^0 = X_i$ et que $X_i^1 = X_{i+1}$.

Puisque H stabilise X_i et X_{i+1}, on voit que H stabilise la forêt F. Donc, pour tout $t \in [0,1]$, le groupe H stabilise X_i^t. Puisque H fixe le sommet v_i de $\overline{X_i}$, il fixe également, pour tout $t \in [0,1]$, le sommet $\mathrm{pr}_t(x_i)$. Ceci induit un chemin continu de $\widetilde{\mathscr{X}}_i$ vers la classe d'équivalence dans K_n de $(X_{i+1}, \mathrm{pr}_1(v_i))$. Si $\mathrm{pr}_1(v_i) \neq v_{i+1}$, alors, puisque le graphe $\overline{X_{i+1}}$ est un arbre, H fixe l'unique arc dans $\overline{X_{i+1}}$ reliant $\mathrm{pr}_1(v_i)$ et v_{i+1}. Ceci induit alors un chemin continu contenu dans l'ensemble des points fixes de H dans l'épine de $\mathbb{P}\mathscr{A}(W_n)$ entre la classe d'équivalence dans K_n de $(X_{i+1}, \mathrm{pr}_1(v_i))$ et $\widetilde{\mathscr{X}}_{i+1}$, ce qui conclut. □

Soient $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ et X un représentant de $\mathscr{X}$. On note $\Phi \colon \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ le morphisme naturel. Donnons maintenant une description de $\ker(\Phi)$. Soit $[X, \rho]$ un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$. On note (X, ρ) un représentant de $[X, \rho]$. Soit e une arête de $\overline{X}$ reliant le sommet $v = o(e)$ au sommet $w = t(e)$. Soit $z \in G_v$ un élément du groupe associé au sommet v, et $\bar{z}$ son antécédent par ρ. Nous définissons à présent le *twist par z autour de e*. Soit G_u le groupe associé à un sommet u. Le twist par z autour de e, noté D_z, est l'automorphisme de W_n, bien défini modulo conjugaison, qui est égal à l'identité sur $\rho^{-1}(G_u)$ si u est dans la même composante connexe de $\overline{X}$ privé de l'intérieur de e que v, et qui à $x \in \rho^{-1}(G_u)$ associe $\bar{z}x\bar{z}^{-1}$ si u n'est pas dans la même composante connexe que v. Nous avons le résultat suivant, dû à Levitt.

Proposition 2.8 [Levitt 2005, Propositions 2.2 and 3.1] *Soit $n \geq 2$ un entier. Soient $\mathscr{X}$ un point de l'épine de l'outre-espace $\mathbb{P}\mathbb{O}(W_n)$ et X un représentant de $\mathscr{X}$. Soient $v_1, \dots, v_n$ les sommets du graphe $\overline{X}$ de groupe associé isomorphe à F et soit n_i le degré de v_i pour $i = 1, \dots, n$. Le noyau du morphisme* $\Phi \colon \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ (noté $\mathrm{Out}_0(W_n)$ dans [Levitt 2005]) *est isomorphe à $\prod_{i=1}^n F^{n_i - 1}$, et il est engendré par les twists autour des arêtes dont l'origine appartient à $\{v_1, \dots, v_n\}$ et n'est pas une feuille.*

Remarque 2.9 Dans le cas où $\mathscr{X} \in \mathbb{P}\mathscr{A}(W_n)$, le noyau est engendré par les twists autour des arêtes e dont l'origine $o(e)$ appartient à $\{v_1, \dots, v_n\}$ et n'est pas une feuille, et telles que, si $o(e)$ est distinct du point base v_*, ces arêtes ne soient pas contenues dans l'unique chemin reliant $o(e)$ à v_*. En particulier, si le groupe associé à v_* est trivial et si n_i est le degré de v_i pour $i = 1, \dots, n$, alors le noyau est isomorphe à $\prod_{i=1}^{n} F^{n_i - 1}$. Si le groupe associé à v_* est non trivial, et si on suppose $v_* = v_n$, alors le noyau est isomorphe à $\left(\prod_{i=1}^{n-1} F^{n_i-1}\right) \times F^{n_n}$.

3 Stabilisateurs des {0}–étoiles et des F–étoiles

Dans cette section nous donnons une caractérisation des stabilisateurs de $\{0\}$–étoiles et de F–étoiles. Nous présentons les démonstrations dans le cas de $\mathrm{Out}(W_n)$, les différences avec $\mathrm{Aut}(W_n)$ étant présentées dans des remarques suivant les démonstrations pour $\mathrm{Out}(W_n)$.

3.1 Stabilisateurs des {0}–étoiles

Nous étudions tout d'abord les stabilisateurs des $\{0\}$–étoiles. Nous démontrons dans cette section la proposition suivante.

Proposition 3.1 *Soient $n \geq 5$ un entier et G un sous-groupe de $\mathrm{Out}(W_n)$ isomorphe à $\mathfrak{S}_n$. Alors G est le stabilisateur dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$ d'une unique $\{0\}$–étoile.*

Afin de démontrer la proposition 3.1, nous avons besoin d'une étude des stabilisateurs de sommets de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ dont les graphes sous-jacents possèdent exactement n feuilles.

Lemme 3.2 *Soit $n \geq 4$ un entier. Soient G un sous-groupe fini de $\mathrm{Out}(W_n)$ isomorphe à $\mathfrak{S}_n$, et $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ fixé par G. On note X un représentant de $\mathscr{X}$. Si le nombre de feuilles de $\overline{X}$ est n, alors $\mathscr{X}$ est une $\{0\}$–étoile.*

Démonstration Soit v un sommet de $\overline{X}$ qui n'est pas une feuille et qui soit à distance maximale du centre[1] de $\overline{X}$.

Affirmation *Si $m = \deg(v)$, alors v est adjacent à au moins $m-1$ feuilles de $\overline{X}$.*

Démonstration L'hypothèse de maximalité sur v implique qu'il y a au plus un sommet adjacent à v qui n'est pas une feuille, car sinon nous pourrions trouver un sommet w adjacent à v qui ne serait pas une feuille et qui serait à distance strictement plus grande du centre que v. ◁

Maintenant, le groupe associé à v est trivial car $\overline{X}$ possède exactement n sommets de groupes associés non triviaux, et ces sommets sont tous des feuilles car $\overline{X}$ possède n feuilles. De ce fait, $\deg(v) \geq 3$ et v est adjacent à au moins deux feuilles, notées v_1 et v_2.

[1] Rappelons que le centre d'un arbre métrique compact non vide est l'unique milieu d'un segment de longueur maximale.

Soient L l'ensemble des feuilles de $\overline{X}$ et w une feuille de $\overline{X}$ distincte de v_1 et v_2. Puisque les seuls sommets de $\overline{X}$ dont les groupes associés sont non triviaux sont des feuilles, la proposition 2.8 montre que le morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est injectif. Ainsi, étant donné que le groupe G est isomorphe à $\mathfrak{S}_n$, et que $\overline{X}$ possède n feuilles, le morphisme naturel $\mathrm{Aut}_{\mathrm{gr}}(X) \hookrightarrow \mathrm{Bij}(L)$ est un isomorphisme. Il existe donc un automorphisme de $\overline{X}$ envoyant v_1 sur w et fixant v_2. De ce fait, w est adjacent à v. Ainsi, v est adjacent à toutes les feuilles de $\overline{X}$. Puisque le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathrm{Bij}(L)$, toutes les arêtes de $\overline{X}$ ont même longueur. On conclut que $\mathscr{X}$ est une $\{0\}$–étoile. □

Remarque 3.3 Le résultat est identique dans le cas de $\mathbb{P}\mathscr{A}(W_n)$. En effet, soit G un sous-groupe fini de $\mathrm{Aut}(W_n)$ isomorphe à $\mathfrak{S}_n$, et $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathscr{A}(W_n)$ fixé par G. On note X un représentant de $\mathscr{X}$. Supposons que $\overline{X}$ possède n feuilles. Alors la remarque 2.9 donne que le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est un sous-groupe distingué de G d'ordre au plus 2. Comme G est isomorphe à $\mathfrak{S}_n$ et que $n \geq 4$, le morphisme est injectif. La même démonstration que le lemme 3.2 montre alors que X possède n feuilles et $n+1$ sommets. Il reste à montrer que le point base est le centre de $\overline{X}$. Mais ceci provient du fait que le groupe G est isomorphe à $\mathrm{Aut}_{\mathrm{gr}}(X)$ qui lui-même est isomorphe à $\mathrm{Bij}(L)$. Ainsi, nécessairement, le point base est le centre de $\overline{X}$. Donc $\mathscr{X}$ est une $\{0\}$–étoile.

Démonstration de la proposition 3.1 Puisque G est fini, d'après la proposition 2.3, il existe un point $\mathscr{X}$ de l'épine de l'outre-espace qui est fixé par G. Soit X un représentant de $\mathscr{X}$. D'après la proposition 2.8, il existe un entier k tel que le noyau de l'application naturelle $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à $F^k \cap G$.

Or $F^k \cap G$ est un 2–sous-groupe distingué de $G \simeq \mathfrak{S}_n$. Donc, comme $n \geq 5$, un tel sous-groupe est trivial. De ce fait, G s'injecte dans $\mathrm{Aut}_{\mathrm{gr}}(X)$. Or tout automorphisme d'un arbre est entièrement déterminé par la permutation qu'il induit sur l'ensemble des feuilles. Ainsi, si L est l'ensemble des feuilles de $\overline{X}$,

$$G \hookrightarrow \mathrm{Aut}_{\mathrm{gr}}(X) \hookrightarrow \mathrm{Bij}(L).$$

Or, les représentants des éléments de $\mathbb{P}\mathbb{O}(W_n)$ possèdent au plus n sommets de groupes non triviaux et toutes les feuilles possèdent des groupes associés non triviaux. Donc $|L| \leq n$. Donc, comme G s'injecte dans $\mathrm{Bij}(L)$ et que G est isomorphe à $\mathfrak{S}_n$, on voit que G est isomorphe à $\mathrm{Aut}_{\mathrm{gr}}(X)$ et que $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathrm{Bij}(L)$. De ce fait, $\overline{X}$ possède n feuilles. Par le lemme 3.2, $\mathscr{X}$ est une $\{0\}$–étoile.

Montrons maintenant l'unicité. Puisque l'ensemble des $\{0\}$–étoiles est discret dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$, par le corollaire 2.6, on conclut que G fixe une unique $\{0\}$–étoile dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$. □

Remarque 3.4 Dans le cas de $\mathbb{P}\mathscr{A}(W_n)$, le résultat de la proposition 3.1 est vrai pour $n \geq 4$. En effet, dans le cas où $n \geq 5$, la démonstration est identique à celle de la proposition 3.1 en utilisant cette fois la remarque 3.3.

Dans le cas où $n = 4$, soit $\mathscr{X} \in \mathbb{P}\mathscr{A}(W_4)$ un point fixé par un sous-groupe G de $\mathrm{Aut}(W_n)$ isomorphe à $\mathfrak{S}_n$. On note X un représentant de $\mathscr{X}$ et v_* le point base de $\overline{X}$. Soit H le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$.

Supposons par l'absurde que H ne soit pas trivial. Alors, par la remarque 2.9, le groupe H est un 2–groupe. Comme le seul 2–sous-groupe distingué de $\mathfrak{S}_4$ est le groupe de Klein, le groupe H est isomorphe à F^2. Nous distinguons différents cas, selon le fait que le groupe associé à v_* soit trivial ou non et selon le nombre de sommets qui ne sont pas des feuilles et qui ont un groupe associé non trivial. On remarque immédiatement que, puisque tout arbre possède au moins deux feuilles, le nombre de sommets qui ne sont pas des feuilles et de groupes associés non triviaux est au plus 2.

Supposons que $\overline{X}$ contienne deux sommets qui ne soient pas des feuilles et dont les groupes associés sont isomorphes à F et que le groupe associé à v_* soit trivial.

L'hypothèse sur v_* implique que $\deg(v_*) \geq 3$. Comme chaque composante connexe de $\overline{X} - \{v_*\}$ contient au moins une feuille, $\overline{X}$ contiendrait cinq sommets de groupes associés non triviaux. Ceci contredit le fait qu'il y a exactement quatre sommets dans le graphe de groupes associés non triviaux.

Supposons que $\overline{X}$ contienne deux sommets qui ne sont pas des feuilles et dont les groupes associés sont isomorphes à F et que le groupe associé à v_* ne soit pas trivial.

Alors la description du noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ donné dans la remarque 2.9 donne que le cardinal du noyau est au moins 8, ce qui contredit le fait que H est de cardinal 4.

Supposons que $\overline{X}$ contienne un seul sommet, noté w, de groupe associé non trivial et qui ne soit pas une feuille et que le groupe associé à v_* soit trivial. Alors $\deg(v_*) \geq 3$. Comme chaque composante connexe de $\overline{X} - \{v_*\}$ contient au moins une feuille, et qu'il existe un sommet de groupe associé non trivial et qui ne soit pas une feuille, $\deg(v_*) = 3$. De plus, puisqu'il y a exactement quatre sommets dans le graphe de groupes associés non triviaux, chaque composante connexe de $\overline{X} - \{v_*\}$ contient exactement une feuille. Donc v_* est relié à exactement deux feuilles et w est relié à une seule feuille et à v_*. Or le cardinal du groupe des automorphismes d'un tel graphe est égal à 2. Comme le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal 4, ceci contredit le fait que G est isomorphe à $\mathfrak{S}_4$.

Supposons que $\overline{X}$ contienne un seul sommet, noté w, de groupe associé non trivial et qui ne soit pas une feuille. Si v_* est une feuille, alors le graphe possède exactement trois feuilles, dont l'une est le point base. De ce fait, comme tout automorphisme de $\overline{X}$ est induit par son action sur les feuilles, le groupe des automorphismes d'un tel graphe pointé est de cardinal 2. Comme le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal 4, ceci contredit le fait que G est isomorphe à $\mathfrak{S}_4$.

Supposons alors que le point base v_* ne soit pas une feuille. Par les cas précédents, $v_* = w$. Comme le nombre de sommets de groupe associé non trivial est exactement 4, et que tout sommet de groupe associé trivial est de degré au moins 3, le graphe $\overline{X}$ contient au plus un sommet de groupe associé trivial. Le cas où le nombre de sommets de groupe associé trivial est égal à 1 n'est pas possible car alors le cardinal du groupe des automorphismes d'un tel graphe est égal à 2, contredisant le fait que le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal 4 et que G est isomorphe à $\mathfrak{S}_4$.

Dans le cas où le nombre de sommets de groupe associé trivial est nul, on voit que $\mathscr{X}$ est une F–étoile. Or, par la remarque 2.9, le cardinal du noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est égal à 8, d'où une contradiction.

En conclusion, le morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est également injectif dans le cas où $\mathscr{X}$ appartient à $\mathbb{P}\mathscr{A}(W_4)$ et $n = 4$. La suite de la démonstration est alors identique à la proposition 3.1.

Remarque 3.5 La proposition 3.1 reste vraie si l'on remplace F par un groupe fini, dès lors que le groupe des automorphismes de ce groupe fini ne contient pas de sous-groupe isomorphe au groupe alterné.

3.2 Stabilisateurs des F–étoiles

Nous démontrons à présent une caractérisation des stabilisateurs de F–étoiles.

Proposition 3.6 *Soit $n \geq 5$. Soit G un sous-groupe de $\mathrm{Out}(W_n)$ isomorphe à $F^{n-2} \rtimes \mathfrak{S}_{n-1}$. Alors G est le stabilisateur d'une unique F–étoile.*

Le démonstration de la proposition 3.6 passe par l'étude des sous-groupes d'ordre maximaux de $\mathrm{Out}(W_n)$. Pour cela, nous avons besoin du lemme suivant.

Lemme 3.7 *Soient $n \geq 4$ un entier et $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$. On note X un représentant de $\mathscr{X}$. Soit k l'entier tel que le noyau du morphisme naturel*

$$\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$$

soit isomorphe à F^k. Alors $k \leq n-2$. Par ailleurs, $k = n-2$ si et seulement si l'ensemble $V\overline{X}$ des sommets de $\overline{X}$ est de cardinal n.

Démonstration Supposons que $|V\overline{X}| > n$. Soient v un sommet de groupe associé trivial et e une arête de X reliant v à un sommet w. Une telle arête existe car $\overline{X}$ est connexe et le nombre de sommets de $\overline{X}$ de groupe non trivial est égal à n.

Affirmation *Soient Y le graphe de groupes marqué obtenu à partir de X en contractant l'arête e et $\mathscr{Y}$ sa classe d'équivalence dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$. Alors le noyau du morphisme naturel $\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{Y}) \to \mathrm{Aut}_{\mathrm{gr}}(Y)$ est isomorphe à F^l, avec $l = k$ si le groupe associé à w est trivial, et $l \geq k+1$ sinon.*

Démonstration Si le groupe associé à w est trivial, alors contracter l'arête e ne modifie pas le degré des sommets dont le groupe associé est non trivial. Donc, dans ce cas, $k = l$. Supposons maintenant que le groupe associé à w ne soit pas trivial. Notons $\overline{vw}$ le sommet obtenu en contractant e. Le groupe associé à $\overline{vw}$ est non trivial. Alors, puisque, par hypothèse, $\deg(v) \geq 3$, nous avons

$$\deg(\overline{vw}) = \deg(v) + \deg(w) - 2 \geq \deg(w) + 1.$$

Ainsi, dans ce cas, $l \geq k+1$. $\triangleleft$

De ce fait, si $|V\overline{X}| > n$, il existe une arête reliant un sommet de groupe associé trivial et un sommet de groupe associé non trivial. Par l'affirmation précédente, l'entier k associé au morphisme $\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ n'est pas maximal.

Ainsi, pour calculer la borne maximale de k, nous pouvons supposer que $\overline{X}$ possède n sommets, tous de groupe associé non trivial. Donc,

$$\sum_{v \in V\overline{X}} \deg(v) = 2|E\overline{X}| = 2n-2,$$

la dernière égalité provenant du fait que $\overline{X}$ soit un arbre. Ainsi,

$$k = \sum_{v \in V\overline{X}} (\deg(v)-1) = \sum_{v \in V\overline{X}} \deg(v) - n = 2n-2-n = n-2.$$

Donc, $k \leq n-2$, et si $|V\overline{X}| = n$, alors $k = n-2$.

Supposons maintenant que $k = n-2$. Par l'affirmation précédente, la procédure de contraction présentée fait croître strictement k lorsque l'on contracte une arête reliant un sommet de groupe associé trivial et un sommet de groupe associé non trivial. Donc $\overline{X}$ ne peut pas contenir de sommets ayant un groupe associé trivial. On conclut que le cardinal de $V\overline{X}$ est égal à n. □

Remarque 3.8 Dans le cas de $\mathbb{P}\mathcal{A}(W_n)$, soit $\mathcal{X}$ un point de l'épine de $\mathbb{P}\mathcal{A}(W_n)$. On note X un représentant de $\mathcal{X}$. Soit k l'entier tel que le noyau du morphisme naturel $\mathrm{Stab}_{\mathrm{Aut}(W_n)}(\mathcal{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à F^k. Alors une démonstration identique au lemme 3.7 montre que $k \leq n-1$ avec égalité si et seulement si $|V\overline{X}| = n$.

Nous pouvons maintenant montrer le résultat suivant concernant les stabilisateurs de F–étoiles dans $\mathrm{Out}(W_n)$.

Proposition 3.9 (1) *Soit $n \geq 4$ un entier. Le cardinal maximal d'un sous-groupe fini de* $\mathrm{Out}(W_n)$ *est* $2^{n-2}(n-1)!$.

(2) *Supposons $n \geq 5$. Soient G un sous-groupe de* $\mathrm{Out}(W_n)$, *et $\mathcal{X}$ un point de l'épine de $\mathbb{P}\mathcal{O}(W_n)$ fixé par G. On note X un représentant de $\mathcal{X}$. Si $\overline{X}$ possède n feuilles, alors $|G| < 2^{n-2}(n-1)!$.*

(3) *Supposons $n \geq 4$. Soient G un sous-groupe de* $\mathrm{Out}(W_n)$ *isomorphe à $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ et $\mathcal{X}$ un point de l'épine de $\mathbb{P}\mathcal{O}(W_n)$ fixé par G. On note X un représentant de $\mathcal{X}$. Si le nombre de feuilles de $\overline{X}$ est $n-1$, alors $\mathcal{X}$ est une F–étoile.*

Démonstration Si $\mathcal{X}$ est un élément de l'épine de $\mathbb{P}\mathcal{O}(W_n)$, nous noterons X un représentant de $\mathcal{X}$. Nous noterons également L l'ensemble des feuilles de $\overline{X}$. Puisque $\overline{X}$ est un arbre, tout automorphisme de $\overline{X}$ est entièrement déterminé par son action sur les feuilles. Donc le morphisme de restriction de $\mathrm{Aut}_{\mathrm{gr}}(X)$ dans $\mathrm{Bij}(L)$ est injectif.

Montrons l'assertion (1). Puisque tout sous-groupe fini de $\mathrm{Out}(W_n)$ fixe un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ par la proposition 2.3, il suffit de montrer que, pour $\mathscr{X}$ un point de l'épine de l'outre-espace, $|\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})| \leq 2^{n-2}(n-1)!$. D'après la proposition 2.8, il existe un entier k tel que le noyau du morphisme naturel $\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X}) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à F^k. De ce fait, $|\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})| \leq 2^k |\mathrm{Aut}_{\mathrm{gr}}(X)|$.

Nous distinguons deux cas, selon le cardinal de L.

- Supposons que $|L| \leq n-1$. Alors $\mathrm{Aut}_{\mathrm{gr}}(X)$, qui s'injecte dans $\mathrm{Bij}(L)$, s'injecte dans $\mathfrak{S}_{n-1}$. Ainsi,
$$|\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})| \leq 2^k |\mathrm{Aut}_{\mathrm{gr}}(X)| \leq 2^k (n-1)! \leq 2^{n-2}(n-1)!,$$
où la dernière inégalité découle du lemme 3.7.
- Supposons que $|L| = n$. Alors tous les sommets ayant des groupes associés non triviaux sont des feuilles. Ainsi, $k = 0$ par la proposition 2.8. Puisque $\mathrm{Bij}(L)$ est isomorphe à $\mathfrak{S}_n$, nous avons
$$|\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})| \leq |\mathrm{Aut}_{\mathrm{gr}}(X)| \leq n!.$$
Or puisque $n \geq 4$, nous avons $n \leq 2^{n-2}$, donc $n! \leq 2^{n-2}(n-1)!$, ce qui conclut.

Donc, pour tout sous-groupe fini G de $\mathrm{Out}(W_n)$, l'ordre de G est au plus $2^{n-2}(n-1)!$. Cette borne est atteinte par le groupe $U_n = \langle [\tau_1], \dots, [\tau_{n-2}], [\sigma_{1,n}] \rangle$, qui est isomorphe au produit semi-direct $F^{n-2} \rtimes \mathfrak{S}_{n-1}$.

Soient $n \geq 5$ et G, $\mathscr{X}$ et X comme dans l'énoncé de l'assertion (2). Par la proposition 2.8, il existe un entier k tel que le noyau du morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à $F^k \cap G$. Puisque $\overline{X}$ possède n feuilles, par la proposition 2.8, l'entier k est nul. De ce fait, le groupe G s'injecte dans $\mathrm{Aut}_{\mathrm{gr}}(X)$, qui s'injecte dans $\mathrm{Bij}(L)$. Donc $|G| \leq n!$. Or $2^{n-2}(n-1)! \leq n!$ implique que $n \leq 4$. D'où $|G| < 2^{n-2}(n-1)!$.

Soient $n \geq 4$ et G, $\mathscr{X}$ et X comme dans l'énoncé de (3). Comme G est de cardinal maximal parmi les sous-groupes finis de $\mathrm{Out}(W_n)$, nous avons $G = \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})$. Donc, par la proposition 2.8, il existe un entier k tel que le noyau du morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à F^k. Ainsi, puisque $\mathrm{Aut}_{\mathrm{gr}}(X)$ s'injecte dans $\mathrm{Bij}(L)$ et que ce dernier est isomorphe à $\mathfrak{S}_{n-1}$, on voit que $|G| \leq 2^k(n-1)!$. Comme $k \leq n-2$ par le lemme 3.7, et puisque $|G| = 2^{n-2}(n-1)!$, on a nécessairement $k = n-2$. Le lemme 3.7 donne alors que $\overline{X}$ possède exactement n sommets. De ce fait, $\overline{X}$ possède $n-1$ feuilles et n sommets. Par ailleurs, on voit également que $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathrm{Bij}(L)$. Ainsi, toutes les arêtes de $\overline{X}$ ont la même longueur. Donc $\mathscr{X}$ est une F–étoile. □

Démonstration de la proposition 3.6 Supposons que $n \geq 5$ et que G soit un sous-groupe de $\mathrm{Out}(W_n)$ isomorphe à $F^{n-2} \rtimes \mathfrak{S}_{n-1}$. Par la proposition 2.3, le groupe G fixe un point $\mathscr{X}$ de l'épine de l'outre-espace. Comme G est de cardinal maximal parmi les sous-groupes finis de $\mathrm{Out}(W_n)$, nous avons $G = \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathscr{X})$. Donc, par la proposition 2.8, il existe un entier k tel que le noyau du morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à F^k.

Affirmation *L'arbre $\overline{X}$ possède exactement $n-1$ feuilles.*

Démonstration L'assertion (2) de la proposition 3.9 dit que $\overline{X}$ possède au plus $n-1$ feuilles. Nous avons

$$|G| = 2^{n-2}(n-1)! \leq 2^k |\mathrm{Aut}_{\mathrm{gr}}(X)| \leq 2^{n-2}|\mathrm{Aut}_{\mathrm{gr}}(X)|$$

où la dernière égalité provient du lemme 3.7. Donc $|\mathrm{Aut}_{\mathrm{gr}}(X)| \geq (n-1)!$. Ainsi, puisque $\overline{X}$ possède au plus $n-1$ feuilles, le groupe $\mathrm{Bij}(L)$ dans lequel s'injecte $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathfrak{S}_{n-1}$. Donc le cardinal de L est $n-1$. $\triangleleft$

De ce fait, $\overline{X}$ possède $n-1$ feuilles. Par l'assertion (3) de la proposition 3.9, $\mathscr{X}$ est une F–étoile dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$. Par le corollaire 2.6, l'ensemble des points fixes de G est connexe. Puisque l'ensemble des F–étoiles est discret dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$, on conclut que G fixe une unique F–étoile dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$. $\square$

Remarque 3.10 Dans le cas de $\mathrm{Aut}(W_n)$, soient G un sous groupe fini de $\mathrm{Aut}(W_n)$ et $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathscr{A}(W_n)$ fixé par G. On note X un représentant de $\mathscr{X}$.

(1) *Si $n \geq 4$, le cardinal de G est plus petit que $2^{n-1}(n-1)!$.*

La démonstration pour le cas où le nombre de feuilles de $\overline{X}$ est plus petit que $n-1$ est identique à celle de la proposition 3.9(1) en utilisant cette fois la remarque 3.8. Dans le cas où le nombre de feuilles est égal à n, le noyau du morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal plus petit que 2 par la remarque 2.9, donc $|G| \leq 2n! \leq 2^{n-1}(n-1)!$ car $n \geq 4$.

(2) *Si $n \geq 5$ et si $\overline{X}$ possède n feuilles, alors $|G| < 2^{n-1}(n-1)!$.*

En effet, par la remarque 2.9, le cardinal du noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est plus petit que 2, donc $|G| \leq 2n! < 2^{n-1}(n-1)!$ car $n \geq 5$.

(3) *Si $n \geq 4$, si G est isomorphe à $F^{n-1} \rtimes \mathfrak{S}_{n-1}$ et si $\overline{X}$ possède au plus $n-1$ feuilles, alors $\mathscr{X}$ est une F–étoile.*

En effet, une démonstration identique à celle de la proposition 3.9(3) montre que $\overline{X}$ possède $n-1$ feuilles et n sommets. Montrons alors que le point base est le centre de X. Ceci découle du fait que le groupe des automorphismes de $\overline{X}$ est isomorphe à $\mathfrak{S}_{n-1}$, car le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à F^{n-1} et que G est isomorphe à $F^{n-1} \rtimes \mathfrak{S}_{n-1}$.

(4) *Si $n \geq 4$ et si G est isomorphe à $F^{n-1} \rtimes \mathfrak{S}_{n-1}$, tout point de l'épine de $\mathbb{P}\mathscr{A}(W_n)$ fixé par G est une F–étoile.*

En effet, l'existence d'une F–étoile fixée par G lorsque $n \geq 5$ se déduit des faits précédents.

Dans le cas où $n = 4$, soit $\mathscr{X}$ un point de l'épine de l'outre-espace fixé par G, et soit X un représentant de $\mathscr{X}$. On note L l'ensemble des feuilles de $\overline{X}$. Si $\overline{X}$ possède au plus $n-1$ feuilles, alors, par le fait précédent, $\mathscr{X}$ est une F–étoile. Supposons que $\overline{X}$ possède exactement n feuilles. Alors la remarque 2.9

montre que le noyau du morphisme naturel $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal au plus 2. Il ne peut pas être injectif, car le cardinal de G est égal à 48 alors que le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$ s'injecte dans $\mathrm{Bij}(L)$ de cardinal égal à 24. Donc le noyau du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal égal à 2. Ainsi, le point base de $\overline{X}$ est une feuille. Or, puisque $\mathrm{Aut}_{\mathrm{gr}}(X)$ s'injecte dans $\mathrm{Bij}(L)$ et que l'image du morphisme $G \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est de cardinal égal à 24, on voit que $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathrm{Bij}(L)$. Ceci contredit le fait que le point base de $\overline{X}$ est une feuille. En conclusion, $\overline{X}$ possède au plus $n-1$ feuilles. Donc $\mathcal{X}$ est une F–étoile. La démonstration de l'unicité de la F–étoile fixée par G est alors identique à celle de la démonstration de la proposition 3.9(4).

Remarque 3.11 La proposition 3.6 reste vraie si l'on remplace F par un groupe cyclique fini.

3.3 Intersection des stabilisateurs d'une $\{0\}$–étoile et d'une F–étoile

Nous étudions dans cette section l'intersection d'un stabilisateur d'une $\{0\}$–étoile et d'un stabilisateur d'une F–étoile. Le résultat central de cette section (proposition 3.13) montre que l'intersection des stabilisateurs d'une $\{0\}$–étoile et d'une F–étoile fixe une unique classe d'équivalence de $\{0\}$–étoiles et une unique classe d'équivalence de F–étoiles.

Lemme 3.12 *Soit n un entier.*

(1) *Supposons que $n \geq 5$. Soit G un sous-groupe de $\mathfrak{S}_n$ isomorphe à $\mathfrak{S}_{n-1}$. Il existe un automorphisme de $\mathfrak{S}_n$ envoyant G sur $\{f \in \mathrm{Bij}(\{1,\dots,n\}) : f(n)=n\}$.*

(2) *Si $n \geq 4$ et $n \neq 6$ et si G est un sous-groupe de $\mathrm{Bij}(\{1,\dots,n\})$ isomorphe à $\mathfrak{S}_{n-1}$, alors il existe un entier $i \in \{1,\dots,n\}$ tel que $G = \{f \in \mathrm{Bij}(\{1,\dots,n\}) : f(i)=i\}$.*

Démonstration (1) L'action de $\mathfrak{S}_n$ sur $\mathfrak{S}_n/G$ par multiplication à gauche est un morphisme de groupes $\phi\colon \mathfrak{S}_n \to \mathrm{Bij}(\mathfrak{S}_n/G)$. Le noyau de ce morphisme est un sous-groupe distingué de $\mathfrak{S}_n$ inclus dans G. Or, G est d'indice n. Donc, étant donné que $n \geq 5$, le noyau de ce morphisme est trivial. Donc, puisque les groupes $\mathfrak{S}_n$ et $\mathrm{Bij}(\mathfrak{S}_n/G)$ ont même cardinal fini, le morphisme ϕ est un isomorphisme. Soit $\widetilde{\psi}\colon \mathfrak{S}_n/G \to \{1,\dots,n\}$ une bijection envoyant $\{G\}$ sur n, et $\psi\colon \mathrm{Bij}(\mathfrak{S}_n/G) \to \mathfrak{S}_n$ l'isomorphisme induit par $\widetilde{\psi}$. Alors $\psi \circ \phi$ est un automorphisme de $\mathfrak{S}_n$ envoyant G sur le sous-groupe de $\mathfrak{S}_n$ fixant n.

(2) Nous commençons par traiter le cas où $n=4$. Il découle d'une inspection des sous-groupes de $\mathfrak{S}_4$ isomorphes à $\mathfrak{S}_3$. En effet, $\mathfrak{S}_4$ possède exactement quatre sous-groupes isomorphes à $\mathfrak{S}_3$. Donc, il existe un entier $i \in \{1,2,3,4\}$ tel que $G = \{f \in \mathrm{Bij}(\{1,\dots,n\}) : f(i)=i\}$.

Supposons maintenant que $n \geq 5$ et que $n \neq 6$. Par le premier point du lemme, il existe un automorphisme ϕ de $\mathfrak{S}_n$ envoyant G sur $\{f \in \mathrm{Bij}(\{1,\dots,n\}) : f(n)=n\}$. Or, si $n \neq 6$, tout automorphisme de $\mathfrak{S}_n$ est intérieur. Comme les automorphismes intérieurs préservent le fait d'être le stabilisateur d'un entier, il existe un entier $i \in \{1,\dots,n\}$ tel que $G = \{f \in \mathrm{Bij}(\{1,\dots,n\}) : f(i)=i\}$. □

Étudions les points fixes du groupe B_n dans l'épine de l'outre-espace de W_n.

Proposition 3.13 *Soient $n \geq 4$ et $B_n = \langle [\tau_1], \dots, [\tau_{n-2}] \rangle$.*

(1) *Les seuls sommets fixés par B_n dans l'épine de l'outre-espace de W_n sont des $\{0\}$–étoiles et des F–étoiles.*

(2) *Le groupe B_n fixe une unique F–étoile et une unique $\{0\}$–étoile.*

Remarque La proposition 3.13 diffère des propositions 3.1 et 3.6 car elle porte uniquement sur un sous-groupe *particulier* de $\mathrm{Out}(W_n)$. Nous ne savons pas si le résultat reste vrai pour un sous-groupe de $\mathrm{Out}(W_n)$ isomorphe à $\mathfrak{S}_{n-1}$ quelconque.

Démonstration (1) Soient $\mathscr{X}$ un sommet de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ fixé par B_n et X un représentant de $\mathscr{X}$. Soient L l'ensemble des feuilles de $\overline{X}$ et $v_1, \dots, v_n$ les sommets de $\overline{X}$ dont les groupes associés sont non triviaux. Par la proposition 2.8, il existe un entier k tel que le noyau du morphisme naturel $B_n \to \mathrm{Aut}_{\mathrm{gr}}(X)$ soit isomorphe à $F^k \cap B_n$. Or, ce noyau est un sous-groupe de F^k, et ce dernier est engendré par des twists. Pour tout $i \in \{1, \dots, n\}$, soit y_i l'antécédent par le marquage de X du générateur du groupe associé à v_i. Les compositions de twists contenues dans $F^k \cap B_n$ préservent la classe de conjugaison dans W_n de y_i alors que les permutations du groupe engendré par $\{[\tau_1], \dots, [\tau_{n-2}]\}$ ne préservent pas ces dernières. De ce fait, nous avons $F^k \cap B_n = \{1\}$.

Le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$ s'injecte dans $\mathrm{Bij}(L)$. Par ailleurs, étant donné que le morphisme $\phi\colon B_n \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est injectif, et que B_n est isomorphe à $\mathfrak{S}_{n-1}$, nous avons $|L| \geq n-1$. De plus, chaque feuille ayant un groupe associé non trivial, nous avons $|L| \leq n$. Donc $|L| \in \{n-1, n\}$. Examinons les deux cas possibles.

Si $|L| = n-1$, alors $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathrm{Bij}(L)$. Montrons que $\mathscr{X}$ est une F–étoile. Soit v un sommet qui n'est pas une feuille à distance maximale du centre de $\overline{X}$. L'hypothèse de maximalité sur v implique qu'il y a au plus un sommet adjacent à v qui n'est pas une feuille, car sinon nous pourrions trouver un sommet w adjacent à v qui ne serait pas une feuille et qui serait à distance strictement plus grande du centre que v. De ce fait, v est adjacent à au moins $\deg(v) - 1$ feuilles.

Si le groupe associé à v est non trivial, alors v est fixé par B_n car c'est le seul sommet de $\overline{X}$ qui soit de groupe associé non trivial et qui ne soit pas une feuille. Donc, puisque B_n est isomorphe à $\mathrm{Aut}_{\mathrm{gr}}(X)$, le sommet v est fixé par $\mathrm{Aut}_{\mathrm{gr}}(X)$. Enfin, puisque tout élément de $\mathrm{Bij}(L)$ est induit par un élément de $\mathrm{Aut}_{\mathrm{gr}}(X)$, le sommet v est adjacent à toutes les feuilles et $\mathscr{X}$ est une F–étoile.

Si v est un sommet de groupe trivial, alors, par hypothèse, $\deg(v) \geq 3$. De ce fait, v est adjacent à au moins deux feuilles, notées v_1 et v_2. Soit w une feuille de $\overline{X}$ distincte de v_1 et v_2. Puisqu'il existe un automorphisme de $\overline{X}$ envoyant v_1 sur w et fixant v_2, alors w est nécessairement adjacent à v. Donc v est adjacent à toutes les feuilles. Ceci n'est pas possible, car alors X contiendrait uniquement $n-1$ sommets de groupe associé non trivial. Donc v est nécessairement un sommet de groupe associé non trivial et $\mathscr{X}$ est une F–étoile.

Supposons que $|L| = n$. Montrons alors que $\mathscr{X}$ est une $\{0\}$–étoile. Le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$ s'injecte dans $\mathrm{Bij}(L)$ qui est isomorphe à $\mathfrak{S}_n$. Par ailleurs, puisque B_n s'injecte dans $\mathrm{Aut}_{\mathrm{gr}}(X)$, l'image de $\mathrm{Aut}_{\mathrm{gr}}(X)$ dans $\mathrm{Bij}(L)$ contient un sous-groupe de $\mathrm{Bij}(L)$ isomorphe à $\mathfrak{S}_{n-1}$.

Soit H l'image de B_n dans $\mathrm{Aut}_{\mathrm{gr}}(X)$. Par le lemme 3.12(2), si $n \neq 6$, il existe une feuille v_1 de $\overline{X}$ telle que l'image de H dans $\mathrm{Bij}(L)$ soit égale à $\mathrm{Stab}_{\mathrm{Bij}(L)}(v_1)$. Soit v le sommet adjacent à v_1. Puisque v n'est pas une feuille, $\deg(v) \geq 3$. Ou bien v est adjacent à une autre feuille distincte de v_1, ou bien v est adjacent à une unique feuille.

Si v est adjacent à une unique feuille, il existe dans $\overline{X}$ des feuilles de $L - \{v_1\}$ à distance au moins 4. Soient w_1 et w_2 deux telles feuilles distinctes de v_1, telles que w_1 soit à distance maximale du centre et que w_2 soit une feuille distincte de v_1 à distance maximale de w_1. Puisque la valence de tout sommet de groupe associé trivial est au moins 3, il existe une feuille w_3 à distance 2 de w_2. Or l'image de H dans $\mathrm{Bij}(L)$ est égale à $\mathrm{Stab}_{\mathrm{Bij}(L)}(v_1)$. Il existe donc un automorphisme de $\overline{X}$ fixant w_3 et envoyant w_2 sur w_1, ce qui n'est pas possible par hypothèse sur w_1 et w_2.

Donc v est adjacent à une feuille distincte de v_1, que l'on note v_2. Soit w une feuille de $\overline{X}$ distincte de v_1 et v_2. Étant donne qu'il existe un automorphisme de $\overline{X}$ envoyant v_2 sur w et fixant v_1, le sommet w est à distance 2 de v_2. En particulier, $\mathscr{X}$ est une $\{0\}$–étoile.

Traitons maintenant le cas où $n = 6$. On numérote de 1 à 6 les feuilles. Une construction explicite d'un représentant de l'unique automorphisme extérieur non trivial de $\mathfrak{S}_6$ (voir par exemple [Miller 1958]) donne que l'unique (à conjugaison près) sous-groupe de $\mathrm{Bij}(L)$ isomorphe à $\mathfrak{S}_5$ et qui ne soit pas un stabilisateur de feuille est le groupe

$$H = \langle (1\,2)(3\,4)(5\,6), (1\,6)(2\,4)(3\,5), (1\,4)(2\,3)(5\,6), (1\,6)(2\,5)(3\,4) \rangle.$$

Supposons alors que H soit inclus dans l'image de $\mathrm{Aut}_{\mathrm{gr}}(X)$ dans $\mathrm{Bij}(L)$. Le groupe H agit transitivement sur les feuilles de $\overline{X}$. De ce fait, tous les sommets reliés à des feuilles sont adjacents à un même nombre k de feuilles. Les seules valeurs possibles pour k sont $k \in \{1, 2, 3, 6\}$. Le cas où $k = 1$ n'est pas possible car tout sommet qui n'est pas une feuille est de degré au moins 3 (tous les sommets dont les groupes associés sont non triviaux sont des feuilles). De plus, $k \neq 3$ car le groupe des automorphismes d'un tel graphe ne pourrait contenir simultanément les permutations $(1\,2)(3\,4)(5\,6)$, $(1\,6)(2\,4)(3\,5)$ et $(1\,4)(2\,3)(5\,6)$. Enfin, $k \neq 2$ car alors $\overline{X}$ posséderait trois sommets adjacents à deux feuilles. Cependant le groupe des automorphismes d'un tel graphe ne pourrait contenir simultanément les permutations $(1\,2)(3\,4)(5\,6)$, $(1\,6)(2\,4)(3\,5)$ et $(1\,6)(2\,5)(3\,4)$. Donc $k = 6$ et X est une $\{0\}$–étoile.

Ainsi, B_n fixe uniquement des $\{0\}$–étoiles et des F–étoiles.

(2) Montrons maintenant que B_n fixe une unique F–étoile. Soit X le graphe de groupes marqué pour lequel $\overline{X}$ possède n sommets, notés $v_1, \dots, v_n$, dont les feuilles sont $v_1, \dots, v_{n-1}$, et tel que pour tout $i \in \{1, \dots, n\}$, l'image réciproque par le marquage du générateur du groupe associé à v_i soit x_i. Soit $\mathscr{X}$ la

classe d'équivalence de X. Alors $\mathscr{X}$ est une F–étoile et le stabilisateur de $\mathscr{X}$ est U_n. Puisque $B_n \subseteq U_n$, ceci montre l'existence.

Montrons maintenant l'unicité. Soit $\mathscr{Y}$ une autre F–étoile fixée par B_n. On note Y un représentant de $\mathscr{Y}$. Par le corollaire 2.6, il existe dans $\mathrm{Fix}_{K_n}(B_n)$ un chemin continu de $\mathscr{X}$ vers $\mathscr{Y}$. Puisque deux F–étoiles distinctes ne sont pas reliées par une arête dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$, et puisque tout sommet de $\mathrm{Fix}_{K_n}(B_n)$ est une $\{0\}$–étoile ou une F–étoile, ce chemin passe par une $\{0\}$–étoile adjacente à $\mathscr{X}$.

Affirmation *Soient $\mathscr{Z}$ une $\{0\}$–étoile adjacente à $\mathscr{X}$ et Z un représentant de $\mathscr{Z}$. On note $v_1, \dots, v_n$ les sommets de $\overline{Z}$ dont les groupes associés sont non triviaux. Alors l'image réciproque par le marquage de Z des générateurs des groupes associés aux sommets $v_1, \dots, v_n$ est, à conjugaison près,*

$$\{x_n^{\alpha_1} x_1 x_n^{\alpha_1}, \dots, x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}}, x_n\},$$

avec $\alpha_i \in \{0, 1\}$ pour tout $i \in \{1, \dots, n-1\}$.

Démonstration Pour tout $i \in \{1, \dots, n\}$, soit y_i le générateur du groupe associé à v_i. Puisque $\mathscr{Z}$ est adjacente à $\mathscr{X}$, il existe une arête e de $\overline{Z}$ telle que le graphe de groupes marqué Z' dont le graphe $\overline{Z}'$ est obtenu à partir de $\overline{Z}$ en contractant e soit dans la classe $\mathscr{X}$. Quitte à renuméroter, on peut supposer que l'un des sommets de e est v_n. Soient T_X et $T_{Z'}$ les arbres de Bass–Serre associés à X et Z'. Les graphes de groupes X et Z' étant équivalents, il existe un homéomorphisme W_n–équivariant $f\colon T_X \to T_{Z'}$. Soit v le sommet de T_X de stabilisateur $\langle x_n \rangle$. Alors $f(v)$ a pour stabilisateur $\langle x_n \rangle$. Par ailleurs, étant donné que les sommets adjacents à v ont pour stabilisateurs $\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle, \langle x_n x_1 x_n \rangle, \dots, \langle x_n x_{n-1} x_n \rangle$, les sommets adjacents à $f(v)$ ont pour stabilisateurs $\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle, \langle x_n x_1 x_n \rangle, \dots, \langle x_n x_{n-1} x_n \rangle$. Donc, tout sous-graphe fini et connexe de $T_{Z'}$ ayant n sommets et $n-1$ feuilles et de centre $f(v)$ est tel que les stabilisateurs des feuilles sont

$$\langle x_n^{\alpha_1} x_1 x_n^{\alpha_1} \rangle, \dots, \langle x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}} \rangle,$$

avec $\alpha_i \in \{0, 1\}$ pour tout $i \in \{1, \dots, n-1\}$. Ainsi, l'image réciproque par le marquage de Z des générateurs des groupes associés aux sommets $v_1, \dots, v_n$ est, à conjugaison près,

$$\langle x_n^{\alpha_1} x_1 x_n^{\alpha_1} \rangle, \dots, \langle x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}} \rangle,$$

avec $\alpha_i \in \{0, 1\}$ pour tout $i \in \{1, \dots, n-1\}$. $\triangleleft$

Ainsi, au vu de la description des $\{0\}$–étoiles adjacentes à $\mathscr{X}$, le groupe B_n fixe une unique $\{0\}$–étoile adjacente à $\mathscr{X}$: la $\{0\}$–étoile Z telle que les antécédents par le marquage des générateurs des groupes de sommets non triviaux soient, à conjugaison près, $x_1, \dots, x_n$. On note $\mathscr{Z}$ la classe d'équivalence de Z.

Soit $\mathscr{Y}'$ une F–étoile adjacente à $\mathscr{Z}$. Notons Y' un représentant de $\mathscr{Y}'$. Il existe une arête e de $\overline{Z}$ telle que le graphe de groupes Z' obtenu en contractant e soit dans $\mathscr{Y}'$. Les antécédents par le marquage de Y' des générateurs des groupes de sommets sont donc, à conjugaison près, $x_1, \dots, x_n$.

Ainsi, puisque B_n permute les sommets de tout point de l'épine de $\mathbb{P}\mathcal{O}(W_n)$ dont l'image réciproque par le marquage des groupes associés sont $\langle x_1\rangle, \dots, \langle x_{n-1}\rangle$, on voit que l'unique F–étoile adjacente à $\mathcal{Z}$ fixée par B_n est $\mathcal{X}$. Donc, B_n fixe une unique F–étoile dans l'épine de $\mathbb{P}\mathcal{O}(W_n)$.

Montrons enfin que B_n fixe une unique $\{0\}$–étoile. Soit Z le graphe de groupes marqué dont le graphe sous-jacent possède $n+1$ sommets, n feuilles, notées $w_1, \dots, w_n$, et tel que pour tout $i \in \{1, \dots, n\}$, l'image réciproque par le marquage du générateur du groupe associé à w_i soit x_i. Soit $\mathcal{Z}$ la classe d'équivalence de Z. Alors $\mathcal{Z}$ est une $\{0\}$–étoile et le stabilisateur de $\mathcal{Z}$ est A_n. Puisque $B_n \subseteq A_n$, ceci montre l'existence.

Montrons l'unicité. Soit $\mathcal{Y}$ une autre $\{0\}$–étoile fixée par B_n. Par le corollaire 2.6, il existe un chemin continu dans $\mathrm{Fix}_{K_n}(B_n)$ de $\mathcal{Z}$ vers $\mathcal{Y}$. Au vu de l'assertion (1) de la proposition, ce chemin passe uniquement par des $\{0\}$–étoiles et des F–étoiles. Or, B_n fixe une unique F–étoile $\mathcal{X}$, et par la dernière affirmation, l'unique $\{0\}$–étoile adjacente à $\mathcal{X}$ et fixée par B_n est $\mathcal{Z}$. Donc B_n fixe une unique $\{0\}$–étoile dans l'épine de $\mathbb{P}\mathcal{O}(W_n)$. $\square$

Remarque 3.14 Soit $n \geq 4$. Dans le cas de $\mathrm{Aut}(W_n)$, soit $\widetilde{B}_n = \langle \tau_1, \dots, \tau_{n-2}\rangle$, qui est encore isomorphe à $\mathfrak{S}_{n-1}$. Soit $\mathcal{X}$ un point de l'épine de $\mathbb{P}\mathcal{A}(W_n)$ fixé par $\widetilde{B}_n$. On note X un représentant de $\mathcal{X}$.

(1) *Soit $\mathcal{X}$ est une F–étoile, soit $\overline{X}$ possède n feuilles et $n+1$ sommets.*

En effet, une démonstration identique à celle de la proposition 3.13(1) montre que le morphisme $\widetilde{B}_n \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est injectif, et que le nombre de feuilles de $\overline{X}$ est soit égal à $n-1$, soit égal à n. S'il est égal à $n-1$, une démonstration identique à celle de la proposition 3.13(1) montre que $\overline{X}$ possède n sommets et $n-1$ feuilles. Comme le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$ contient un sous-groupe isomorphe à $\mathfrak{S}_{n-1}$ et que $\overline{X}$ possède $n-1$ feuilles, on voit que, nécessairement, le point base de X est son centre. Donc $\mathcal{X}$ est une F–étoile. Si le nombre de feuilles de $\overline{X}$ est égal à n, une démonstration identique à celle de la proposition 3.13(1) montre que $\overline{X}$ possède $n+1$ sommets et n feuilles.

(2) *Le groupe $\widetilde{B}_n$ fixe une unique F–étoile.*

En effet, il fixe une F–étoile car $\widetilde{B}_n$ est un sous-groupe de $\widetilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n}\rangle$ et ce dernier est isomorphe à $F^{n-1} \rtimes \mathfrak{S}_{n-1}$. De ce fait, la remarque 3.10(4) permet de conclure. Nous appellerons $\mathcal{X}$ l'unique F–étoile fixée par $\widetilde{U}_n$.

Pour l'unicité, soit $\mathcal{Y}$ une autre F–étoile fixée par $\widetilde{B}_n$. Puisque l'ensemble des F–étoiles dans l'épine de $\mathbb{P}\mathcal{A}(W_n)$ n'est pas connexe, tout chemin continu entre $\mathcal{X}$ et $\mathcal{Y}$ et contenu dans l'ensemble des points fixes de $\widetilde{B}_n$ pour l'action de $\mathrm{Aut}(W_n)$ sur l'épine de $\mathbb{P}\mathcal{A}(W_n)$ passe par un point $\mathcal{Z}$ ayant un représentant Z de graphe sous-jacent possédant n feuilles et $n+1$ sommets. Soient $v_1, \dots, v_n$ les feuilles de $\overline{Z}$. Une démonstration identique à celle de la première affirmation de la démonstration de la proposition 3.13(2) montre que l'image réciproque par le marquage de Z des générateurs des groupes associés aux sommets $v_1, \dots, v_n$ est respectivement ou bien $x_1, \dots, x_{n-1}, x_n$ ou bien $x_n x_1 x_n, \dots, x_n x_{n-1} x_n, x_n$. De plus, la description de $\widetilde{B}_n$ montre que le point base de Z est contenu dans l'arête reliant le centre de $\overline{Z}$ et v_n.

Soit maintenant $\mathscr{X}'$ un sommet de l'épine de $\mathbb{P}\mathscr{A}(W_n)$ fixé par $\widetilde{B}_n$, adjacent à $\mathscr{X}$ et qui n'est pas une F–étoile. Puisque Z' possède n feuilles et $n+1$ sommets par le premier point de la remarque, un représentant Z' de $\mathscr{X}'$ est obtenu à partir de Z en déplaçant le point base dans l'arête reliant le centre de $\overline{Z}$ et v_n. De ce fait, les images réciproques par le marquage des générateurs des groupes associés aux feuilles de $\overline{Z}'$ sont les mêmes que pour $\mathscr{X}$.

Donc, pour conclure sur l'unicité de la F–étoile fixée par $\widetilde{B}_n$, il suffit d'étudier les F–étoiles fixées par $\widetilde{B}_n$ qui sont adjacentes à $\mathscr{X}$. Soit $\mathscr{Y}'$ une F–étoile adjacente à $\mathscr{X}$. Notons Y' un représentant de $\mathscr{Y}'$. Il existe une arête e de $\overline{Z}$ telle que le graphe de groupes Z' obtenu en contractant e soit dans $\mathscr{Y}'$. Les antécédents par le marquage de Y' des générateurs des groupes de sommets sont donc, à conjugaison près, $x_1,\dots,x_n$. Ainsi, puisque $\widetilde{B}_n$ permute transitivement les sommets de tout point de l'épine de $\mathbb{P}\mathscr{A}(W_n)$ dont les images réciproques par le marquage des groupes associés sont $\langle x_1\rangle,\dots,\langle x_{n-1}\rangle$, on voit que l'unique F–étoile adjacente à $\mathscr{X}$ fixée par $\widetilde{B}_n$ est $\mathscr{X}$. Donc, $\widetilde{B}_n$ fixe une unique F–étoile dans l'épine de $\mathbb{P}\mathscr{A}(W_n)$.

Remarque 3.15 La proposition 3.13 reste vraie lorsqu'on remplace F par un groupe cyclique fini.

4 Rigidité des automorphismes extérieurs d'un groupe de Coxeter universel

Le but de cette section est de démontrer le théorème 1.1. Nous distinguons différents cas, selon la valeur de n. Soit $\alpha\in\mathrm{Aut}(\mathrm{Out}(W_n))$.

4.1 Démonstration dans le cas $n\geq 5$ et $n\neq 6$

Soit $\mathscr{X}_1$ la $\{0\}$–étoile fixée par le sous-groupe fini A_n de $\mathrm{Out}(W_n)$ (l'unicité provient de la proposition 3.1). Alors, d'après la proposition 3.1, $\alpha(A_n)$ est le stabilisateur d'une unique $\{0\}$–étoile $\mathscr{X}_2$. Or $\mathrm{Out}(W_n)$ agit transitivement sur l'ensemble des $\{0\}$–étoiles, donc il existe $\psi\in\mathrm{Out}(W_n)$ tel que $\psi(\mathscr{X}_1)=\mathscr{X}_2$. Posons $\alpha_0=\mathrm{ad}(\psi)\circ\alpha$, alors $\alpha_0(A_n)=\mathrm{ad}(\psi)\circ\alpha(A_n)=A_n$.

Puisque $\alpha_0|_{A_n}$ est un automorphisme de A_n, que A_n est isomorphe à $\mathfrak{S}_n$ et que, pour $n\neq 6$, le groupe $\mathrm{Out}(\mathfrak{S}_n)$ est trivial, quitte à changer α_0 dans sa classe d'automorphismes extérieurs, on peut supposer que $\alpha_0|_{A_n}=\mathrm{id}_{A_n}$.

Maintenant, étant donné que $B_n\subseteq U_n$, nous avons $\alpha_0(B_n)=B_n\subseteq\alpha_0(U_n)$. Or, par la proposition 3.13(2), B_n fixe une unique F–étoile. Par ailleurs, le stabilisateur de cette F–étoile est U_n. Donc, puisque $\alpha_0(U_n)$ est également le stabilisateur d'une unique F–étoile par la proposition 3.6, on obtient que $\alpha_0(U_n)=U_n$.

Or, U_n est isomorphe au produit semi-direct $F^{n-2}\rtimes B_n$, et B_n agit sur F^{n-2} (vu comme le quotient de F^{n-1} par son sous-groupe diagonal F) par permutation des facteurs. Soit $\sigma\in B_n$. On note $\mathrm{fix}(\sigma)$ l'ensemble des points fixes de σ agissant par conjugaison dans F^{n-2}. Puisque $\alpha_0(\sigma)=\sigma$ pour tout $\sigma\in B_n$, on voit que $\alpha_0(\sigma g\sigma^{-1}) = \sigma\alpha_0(g)\sigma^{-1}$ pour tout $\sigma \in \{0\} \rtimes B_n$ et pour tout $g \in F^{n-2} \rtimes \{1\}$; en particulier, si $g \in \mathrm{fix}(\sigma)$, alors $\alpha_0(g) \in \mathrm{fix}(\sigma)$.

Soit maintenant $\sigma = (2 \dots n-1) \in B_n$. Alors $\mathrm{fix}(\sigma) = \{0, [\sigma_{1,n}]\}$. Donc, puisque $\alpha_0([\sigma_{1,n}]) \in \mathrm{fix}(\sigma)$, on a $\alpha_0([\sigma_{1,n}]) = [\sigma_{1,n}]$. De même, pour tout $i \in \{1, \dots, n-1\}$, $\alpha_0([\sigma_{i,n}]) = [\sigma_{i,n}]$. Ainsi, $\alpha_0|_{F^{n-2}} = \mathrm{id}_{F^{n-2}}$. Puisque, par ailleurs, α_0 est l'identité sur B_n, on voit que $\alpha_0|_{U_n} = \mathrm{id}_{U_n}$. De ce fait, étant donné que $\alpha_0|_{A_n} = \mathrm{id}_{A_n}$ et que A_n et U_n engendrent $\mathrm{Out}(W_n)$ par la proposition 2.1, on voit que $\alpha_0 = \mathrm{id}$ et le résultat s'en déduit.

4.2 Démonstration dans le cas $n = 6$

Dans le cas où $n = 6$, la proposition 3.1 s'appliquant encore, soit α_0 un représentant de la classe d'automorphismes extérieurs de α tel que $\alpha_0(A_n) = A_n$. Supposons que la classe d'automorphismes extérieurs de $\alpha_0|_{A_n}$ soit non triviale. Alors une description explicite d'un automorphisme engendrant l'unique classe d'automorphismes extérieurs de $\mathfrak{S}_6$ (cf [Miller 1958]) donne, en identifiant A_n et $\mathfrak{S}_6$ par l'unique isomorphisme envoyant τ_i sur la permutation $(i\ i+1)$ pour $1 \leq i \leq 5$, que

$$\alpha_0(B_n) = \langle [(1\,2)(3\,4)(5\,6)], [(1\,6)(2\,4)(3\,5)], [(1\,4)(2\,3)(5\,6)], [(1\,6)(2\,5)(3\,4)] \rangle.$$

Ainsi, $\alpha_0(B_n)$ agit transitivement sur les classes de conjugaison de $\{x_1, \dots, x_n\}$. Alors, puisque $\alpha_0(B_n) \subseteq \alpha_0(U_n)$, par la proposition 3.6, $\alpha_0(B_n)$ fixe une F–étoile $\mathcal{X}$. Soit X un représentant de $\mathcal{X}$. Par la proposition 2.8, le noyau du morphisme $\alpha_0(B_n) \to \mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $F^{n-2} \cap \alpha_0(B_n)$.

Or $F^{n-2} \cap \alpha_0(B_n)$ est un 2–sous-groupe distingué de $\alpha_0(B_n)$. Comme $\alpha_0(B_n)$ est isomorphe à $\mathfrak{S}_{n-1}$ et que $n = 6$, nous avons $F^{n-2} \cap \alpha_0(B_n) = \{1\}$. Donc $\alpha_0(B_n)$ est isomorphe à $\mathrm{Aut}_{\mathrm{gr}}(X)$ car $\mathrm{Aut}_{\mathrm{gr}}(X)$ est isomorphe à $\mathfrak{S}_{n-1}$. Soient maintenant $v_1, \dots, v_{n-1}$ les feuilles de $\overline{X}$ et v_n le centre de $\overline{X}$. Pour $j \in \{1, \dots, n\}$, soit $\langle y_j \rangle$ l'image réciproque par le marquage du groupe associé à v_j. Le groupe $\mathrm{Aut}_{\mathrm{gr}}(X)$, et donc $\alpha_0(B_n)$, s'identifie à l'ensemble des bijections de $\{v_1, \dots, v_n\}$ fixant v_n. Or, par la proposition 2.1, il existe $\pi \in \mathrm{Bij}(\{x_1, \dots, x_n\})$ telle que pour tout $i \in \{1, \dots, n\}$, il existe $z_i \in W_n$ vérifiant

$$y_i = z_i x_{\pi(i)} z_i^{-1}.$$

Ceci contredit le fait que $\alpha_0(B_n)$ s'identifie à l'ensemble des bijections de $\{v_1, \dots, v_n\}$ fixant v_n, car le groupe $\alpha_0(B_n)$ agit transitivement sur l'ensemble des classes de conjugaison de $\{x_1, \dots, x_n\}$. Donc la classe d'automorphismes extérieurs de $\alpha_0|_{A_n}$ est triviale et on conclut comme dans la section 4.1.

4.3 Démonstration dans le cas $n = 4$

Dans le cas où $n = 4$, les propositions 3.1 et 3.6 ne sont plus valables, car alors tout sous-groupe de $\mathrm{Out}(W_n)$ isomorphe à $\mathfrak{S}_4$ est isomorphe au produit semi-direct $V \rtimes \mathfrak{S}_3$, où V est le groupe de Klein. Nous avons cependant la proposition suivante.

Proposition 4.1 *Soient $n = 4$ et G un sous-groupe de $\mathrm{Out}(W_n)$ isomorphe au produit semi-direct $F^{n-2} \rtimes \mathfrak{S}_{n-1}$. Alors G est soit le stabilisateur d'une unique F–étoile, soit le stabilisateur d'une unique $\{0\}$–étoile. Les deux cas sont mutuellement exclusifs.*

Démonstration Soient $\mathscr{X}$ un point de l'épine de $\mathbb{P}\mathbb{O}(W_n)$ fixé par G (qui existe par la proposition 2.3), et X un représentant de $\mathscr{X}$. Soit L l'ensemble des feuilles de $\overline{X}$. La proposition 4.1 se démontre de manière identique à la proposition 3.9(3), à ceci près que l'on ne peut pas exclure le cas où $\overline{X}$ possède n feuilles. Il faut alors distinguer le cas où $|L| = n-1$ et $|L| = n$. Si $\overline{X}$ possède n feuilles, le lemme 3.2 donne que $\mathscr{X}$ est une $\{0\}$–étoile. Si $\overline{X}$ possède $n-1$ feuilles, alors la proposition 3.9(3) donne que $\mathscr{X}$ est une F–étoile.

Montrons maintenant que G ne peut fixer à la fois une $\{0\}$–étoile et une F–étoile. Par la proposition 3.9(1), G est le stabilisateur de tout point fixé par G.

Supposons que G soit le stabilisateur d'une $\{0\}$–étoile $\mathscr{X}$. Soit X un représentant de $\mathscr{X}$. Soient $v_1, \dots, v_n$ les sommets de $\overline{X}$ dont les groupes associés sont non triviaux et, pour tout $i \in \{1, \dots, n\}$, soit y_i l'image réciproque par le marquage du générateur du groupe associé à v_i. Alors le groupe G est le groupe engendré par les permutations de $\{y_1, \dots, y_n\}$.

Soit $\mathscr{Y}$ une F–étoile dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$ fixée par G. Par le corollaire 2.6, $\mathrm{Fix}_{K_n}(G)$ est connexe. Il existe donc un chemin continu dans $\mathrm{Fix}_{K_n}(G)$ de $\mathscr{X}$ vers $\mathscr{Y}$. Les sommets par lesquels passe ce chemin sont uniquement des $\{0\}$–étoiles et des F–étoiles au vu des points stabilisés par G. Or, le groupe engendré par les permutations de $\{y_1, \dots, y_n\}$ ne fixe aucune F–étoile adjacente à X. En effet, le groupe G contiendrait un élément permutant le centre de la F–étoile avec une feuille, ce qui n'est pas possible. Donc G ne fixe aucune F–étoile.

Enfin, l'unicité du point fixe provient du fait que l'ensemble des $\{0\}$–étoiles et l'ensemble des F–étoiles sont discrets dans l'épine de $\mathbb{P}\mathbb{O}(W_n)$ alors que l'ensemble des points fixes de G est connexe par le corollaire 2.6. □

Nous pouvons maintenant montrer le théorème 1.1 dans le cas $n = 4$.

Soit $\alpha \in \mathrm{Aut}(\mathrm{Out}(W_n))$. Soit $\mathscr{X}_1$ la $\{0\}$–étoile fixée par le sous-groupe fini $A_n \simeq \mathfrak{S}_4$ de $\mathrm{Out}(W_n)$. Par la proposition 4.1, $\alpha(A_n)$ fixe soit une $\{0\}$–étoile, soit une F–étoile.

Si $\alpha(A_n)$ fixe une $\{0\}$–étoile, alors la même démonstration que pour le cas où $n \neq 6$ dans la section 4.1 montre que quitte à changer α dans sa classe d'automorphismes extérieurs, nous avons $\alpha|_{A_n} = \mathrm{id}_{A_n}$. Par la proposition 4.1, le groupe $U_n \simeq F^2 \rtimes \mathfrak{S}_3$ fixe soit une $\{0\}$–étoile, soit une F–étoile. Étant donné que $B_n \subseteq U_n$ fixe une unique $\{0\}$–étoile ρ et une unique F–étoile ρ' et que $\alpha|_{B_n} = \mathrm{id}_{B_n}$, on voit que $\alpha(U_n)$ est soit le stabilisateur de ρ, soit le stabilisateur de ρ'. Cependant, puisque le stabilisateur de ρ est A_n et que $\alpha|_{A_n} = \mathrm{id}_{A_n}$, on voit que $\alpha(U_n)$ est le stabilisateur de ρ'. Donc $\alpha(U_n) = U_n$. Le reste de la démonstration est alors identique à celle du cas où $n \neq 6$ dans la section 4.1.

Supposons que $\alpha(A_n)$ fixe une unique F–étoile. Construisons à présent un représentant de la classe d'automorphismes extérieurs de α. Puisque $\mathrm{Out}(W_n)$ agit transitivement sur les F–étoiles, quitte à changer α dans sa classe d'automorphismes extérieurs, on peut supposer que $\alpha(A_n)=U_n$. Soit V le groupe de Klein contenu dans A_n. Alors $\alpha(V)$ est l'unique 2–sous-groupe distingué non trivial de U_n. Donc

$$\alpha(V)=\langle[\sigma_{1,4}],[\sigma_{2,4}],[\sigma_{3,4}]\rangle.$$

Ainsi, puisque $B_n\cap V=\{\mathrm{id}\}$, on voit que $\alpha(B_n)\cap\alpha(V)=\{\mathrm{id}\}$. Par ailleurs, $A_n=B_nV$, donc $U_n=\alpha(B_n)\alpha(V)$. De ce fait, $\alpha(B_n)$ est un sous-groupe de U_n d'ordre 6. Or, il existe une unique classe de conjugaison de sous-groupes d'ordre 6 dans U_n. Donc, quitte à changer α dans sa classe d'automorphismes extérieurs, on peut supposer que $\alpha(B_n)=B_n$. De même, puisque B_n est isomorphe à $\mathfrak{S}_3$, quitte à changer α dans sa classe d'automorphismes extérieurs, on peut supposer que $\alpha|_{B_n}=\mathrm{id}_{B_n}$.

Déterminons à présent l'image de $[\tau_3]$ et $[\sigma_{3,4}]$ par α. Puisque $[\tau_1][\tau_3]\in V$, on voit que $\alpha([\tau_1][\tau_3])\in\{[\sigma_{1,4}],[\sigma_{2,4}],[\sigma_{3,4}]\}$. Or, $[\tau_1]$ commute avec $[\tau_1][\tau_3]$, donc $\alpha([\tau_1][\tau_3])$ doit également commuter avec $[\tau_1]$. De ce fait, $\alpha([\tau_1][\tau_3])=[\sigma_{3,4}]$ et $\alpha([\tau_3])=[\tau_1][\sigma_{3,4}]$.

Déterminons l'image de $[\sigma_{3,4}]$ par α. Puisque $\alpha(B_n)=B_n$, le groupe $\alpha(U_n)$ est le stabilisateur d'un point fixe de B_n. Par la proposition 3.13, B_n fixe uniquement deux sommets de l'épine de $\mathbb{P}\mathcal{O}(W_n)$: la $\{0\}$–étoile stabilisée par A_n et la F–étoile stabilisée par U_n. Comme $\alpha(A_n)=U_n$, on a nécessairement $\alpha(U_n)=A_n$. Donc $\alpha([\sigma_{3,4}])\in V$. Puisque $[\sigma_{3,4}]$ commute avec $[\tau_1]$, on obtient que $\alpha([\sigma_{3,4}])=[\tau_1][\tau_3]$.

Donc α se restreint en l'identité sur B_n, envoie $[\tau_3]$ sur $[\tau_1][\sigma_{3,4}]$ et $[\sigma_{3,4}]$ sur $[\tau_1][\tau_3]$. Comme B_n, $[\tau_3]$ et $[\sigma_{3,4}]$ engendrent $\mathrm{Out}(W_4)$, ceci montre qu'un tel automorphisme α, s'il existe, est unique modulo automorphisme intérieur.

Réciproquement, montrons que l'application α de $B_n\cup\{[\tau_3],[\sigma_{3,4}]\}$ dans $\mathrm{Out}(W_4)$ définie par $\alpha|_{B_n}=\mathrm{id}_{B_n}$, $\alpha([\tau_3])=[\tau_1][\sigma_{3,4}]$ et $\alpha([\sigma_{3,4}])=[\tau_1][\tau_3]$ s'étend de manière unique en un morphisme de groupes de $\mathrm{Out}(W_4)$. Comme $[\tau_1]$ commute avec $[\tau_3]$ et $[\sigma_{3,4}]$, ceci montre que α est involutif, donc un automorphisme de $\mathrm{Out}(W_4)$. Sa classe dans $\mathrm{Out}(\mathrm{Out}(W_4))$ est non triviale (car son action sur l'épine de $\mathbb{P}\mathcal{O}(W_4)$ est non triviale), ce qui montre le théorème 1.1 lorsque $n=4$.

Pour simplifier les notations, nous notons $[i\ j]$ la classe d'automorphismes extérieurs de la transposition permutant x_i et x_j. Notons

$$S=\{[i\ j]:1\le i,j\le 4\}\cup\{[\sigma_{i,j}]:1\le i\neq j\le 4\},$$

qui est une partie génératrice de $\mathrm{Out}(W_4)$ par la proposition 2.1. Un petit calcul élémentaire montre que, si $i=1,2$, alors

$$[i\ 4]=[i\ 3][3\ 4][i\ 3],\qquad \alpha([i\ 3])\alpha([3\ 4])\alpha([i\ 3])=[j\ k][\sigma_{i,4}],$$
$$[\sigma_{i,4}]=[i\ 3][\sigma_{3,4}][i\ 3],\quad \alpha([i\ 3])\alpha([\sigma_{3,4}])\alpha([i\ 3])=[j\ k][i\ 4],$$

où $\{j,k\}=\{1,2,3\}-\{i\}$. Considérons l'application $\widetilde{\alpha}$ de S dans $\mathrm{Out}(W_4)$ étendant α sur $S\cap(B_n\cup\{[3\ 4],[\sigma_{3,4}]\})$ et telle que, si $i=1,2$,

$$\widetilde{\alpha}([i\ 4])=[j\ k][\sigma_{i,4}]\quad\text{et}\quad \widetilde{\alpha}([\sigma_{i,4}])=[j\ k][i\ 4],$$

où $\{j,k\} = \{1,2,3\} - \{i\}$. Des calculs élémentaires pour lesquels nous renvoyons à [Guerch 2022] montrent que cette application préserve, quand $n = 4$, la présentation de $\mathrm{Out}(W_n)$ donnée par Gilbert [1987, Theorem 2.20], ce qui conclut.

4.4 Démonstration de la rigidité de $\mathrm{Aut}(W_n)$

Nous démontrons à présent le théorème 1.2. Soient $n \geq 4$ et $\alpha \in \mathrm{Aut}(\mathrm{Aut}(W_n))$. Soient $\widetilde{A}_n = \langle \tau_1, \dots, \tau_{n-1} \rangle$, $\widetilde{B}_n = \langle \tau_1, \dots, \tau_{n-2} \rangle$ et $\widetilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n} \rangle$. En utilisant les remarques 3.4, 3.10(4) et 3.14(2), et en effectuant une démonstration identique à celle du théorème 1.1 dans les cas où $n \geq 5$, on voit que, quitte à changer α dans sa classe d'automorphismes extérieurs, $\alpha|_{\widetilde{A}_n} = \mathrm{id}_{\widetilde{A}_n}$ et $\alpha(\widetilde{U}_n) = \widetilde{U}_n$.

Or $\widetilde{U}_n$ est isomorphe à $F^{n-1} \rtimes \widetilde{B}_n$. Soit $\sigma \in \widetilde{B}_n$. On note $\mathrm{fix}(\sigma)$ l'ensemble des points fixes de σ agissant par conjugaison dans F^{n-1}. On voit que $\alpha(\sigma g \sigma^{-1}) = \sigma \alpha(g) \sigma^{-1}$ pour tout $\sigma \in \{0\} \rtimes \widetilde{B}_n$ et pour tout $g \in F^{n-1} \rtimes \{1\}$; en particulier, si $g \in \mathrm{fix}(\sigma)$, alors $\alpha(g) \in \mathrm{fix}(\sigma)$.

Soit maintenant $\sigma = (2 \dots n-1) \in B_n$. Alors $\mathrm{fix}(\sigma) = \bigl\{0, \sigma_{1,n}, \prod_{i \neq 1,n} \sigma_{i,n}, \prod_{i=1}^{n-1} \sigma_{i,n}\bigr\}$. Donc $\alpha(\sigma_{1,n}) \in \bigl\{\sigma_{1,n}, \prod_{i \neq 1,n} \sigma_{i,n}, \prod_{i=1}^{n-1} \sigma_{i,n}\bigr\}$. Comme $\prod_{i=1}^{n-1} \sigma_{i,n}$ est l'unique élément non trivial dans le centre de $\widetilde{U}_n$, on voit que $\alpha(\sigma_{1,n}) \neq \prod_{i=1}^{n-1} \sigma_{i,n}$.

Supposons par l'absurde que $\alpha(\sigma_{1,n}) = \prod_{i \neq 1,n} \sigma_{i,n}$. Pour $j \in \{1, \dots, n-1\}$, notons $(1\ j)$ la transposition de $\widetilde{B}_n$ permutant x_1 et x_j. Alors, on voit que $\alpha(\sigma_{j,n}) = \alpha((1\ j)\sigma_{1,n}(1\ j)) = \prod_{i \neq j,n} \sigma_{i,n}$ pour tout $j \in \{1, \dots, n-1\}$.

Un calcul immédiat montre alors que, pour tout $j \neq k, n$, et $k < n$,

$$\alpha(\sigma_{k,j}) = \alpha((j\ n)\sigma_{k,n}(j\ n)) = \prod_{i \neq j,k} \sigma_{i,j}.$$

Or $\sigma_{1,2}\sigma_{3,4} = \sigma_{3,4}\sigma_{1,2}$, alors que

$$\begin{aligned} \alpha(\sigma_{1,2})\alpha(\sigma_{3,4})(x_1) &= \prod_{i \neq 1,2} \sigma_{i,2} \prod_{i \neq 3,4} \sigma_{i,4}(x_1) = x_2 x_4 x_2 x_1 x_2 x_4 x_2, \\ \alpha(\sigma_{3,4})\alpha(\sigma_{1,2})(x_1) &= \prod_{i \neq 3,4} \sigma_{i,4} \prod_{i \neq 1,2} \sigma_{i,2}(x_1) = x_4 x_1 x_4. \end{aligned}$$

Donc $\alpha(\sigma_{1,2})\alpha(\sigma_{3,4}) \neq \alpha(\sigma_{3,4})\alpha(\sigma_{1,2})$. Ceci contredit le fait que α est un morphisme de groupes. Ainsi, $\alpha(\sigma_{1,n}) = \sigma_{1,n}$. Par la proposition 2.1, nous avons $\alpha = \mathrm{id}$. Ceci conclut la démonstration du théorème 1.2.

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The RO(C_4) cohomology of the infinite real projective space

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Following the Hu–Kriz method of computing the C_2 genuine dual Steenrod algebra $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{C_2}$, we calculate the C_4–equivariant Bredon cohomology of the classifying space $\mathbb{R}P^{\infty\rho} = B_{C_4}\Sigma_2$ as an RO(C_4) graded Green-functor. We prove that as a module over the homology of a point (which we also compute), this cohomology is not flat. As a result, it can't be used as a test module for obtaining generators in $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{C_4}$ as Hu and Kriz use it in the C_2 case. Their argument for the Borel equivariant dual Steenrod algebra does generalize, however, and we give a complete description of $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{hC_{2^n}}$ for any $n \geq 2$.

55N91, 55P91

1 Introduction

Historically, computations in stable equivariant homotopy theory have been much more difficult than their nonequivariant counterparts, even when the groups involved are as simple as possible (ie cyclic). In recent years, there has been a resurgence in such calculations for power-2 cyclic groups C_{2^n}, owing to the crucial involvement of C_8–equivariant homology in the solution of the Kervaire invariant problem; see Hill, Hopkins and Ravenel [8].

The case of $G = C_2$ is the simplest and most studied one, partially due to its connections to motivic homotopy theory over $\mathbb{R}$ by means of realization functors; see Heller and Ormsby [6]. It all starts with the $\mathrm{RO}(C_2)$ homology of a point, which was initially described in Lewis [12]. The types of modules over it that can arise as the equivariant homology of spaces were described in May [14], and this description was subsequently used in the computation of the $\mathrm{RO}(C_2)$ homology of C_2–surfaces in Hazel [5]. The C_2–equivariant dual Steenrod algebra (in characteristic 2) was computed in Hu and Kriz [10] and gives rise to a C_2–equivariant Adams spectral sequence that has been more recently leveraged in Isaksen, Wang and Xu [11]. Another application of the Hu–Kriz computation is the definition of equivariant Dyer–Lashof operations by Wilson [17] in the $\mathbb{F}_2$–homology of C_2–spectra with symmetric multiplication. Many of these results rely on the homology of certain spaces being free as modules over the homology of a point, and there is a robust theory of such free spectra, described in Hill [7].

The case of $G = C_4$ has been much less explored and is indeed considerably more complicated. This can already be seen in the homology of a point in integer coefficients (see Zeng [18] and Georgakopoulos [2]) and the case of $\mathbb{F}_2$ coefficients is not much better (compare Sections 3.1 and 3.2 for the C_2 and C_4 cases, respectively). The greater complexity in the ground ring (or to be more precise, ground Green functor), means that modules over it can also be more complicated and indeed, certain freeness results that are easy to obtain in the C_2 case no longer hold when generalized to C_4 (compare Section 4.1 with Sections 6–8).

The computation of the dual Steenrod algebra relies on the construction of Milnor generators. Nonequivariantly, the Milnor generators ξ_i of the mod 2 dual Steenrod algebra can be defined through the completed coaction of the dual Steenrod algebra on the cohomology of $B\Sigma_2 = \mathbb{R}P^\infty$: one has that $H^*(BC_{2+};\mathbb{F}_2) = \mathbb{F}_2[x]$ and the completed coaction $\mathbb{F}_2[x] \to (H\mathbb{F}_2)_*(H\mathbb{F}_2)[\![x]\!]$ is

$$x \mapsto \sum_i x^{2^i} \otimes \xi_i.$$

In the C_2–equivariant case, the space replacing $B\Sigma_2$ is the equivariant classifying space $B_{C_2}\Sigma_2$. This is still $\mathbb{R}P^\infty$ but now equipped with a nontrivial C_2 action (described in Section 4.1). Over the homology of a point, we no longer have a polynomial algebra on a single generator x, but rather a polynomial algebra on two generators c and b, modulo the relation

$$c^2 = a_\sigma c + u_\sigma b,$$

where a_σ and u_σ are the C_2–Euler and orientation classes respectively (defined in Section 2). As a module, this is still free over the homology of a point, and the completed coaction is

$$c \mapsto c \otimes 1 + \sum_i b^{2^i} \otimes \tau_i, \quad b \mapsto \sum_i b^{2^i} \otimes \xi_i.$$

The τ_i and ξ_i are the C_2–equivariant analogues of the Milnor generators, and Hu and Kriz show that they span the genuine dual Steenrod algebra.

For C_4, the cohomology of $B_{C_4}\Sigma_2$ is significantly more complicated (see Section 6) and most importantly is *not* a free module over the homology of a point. In fact, it's not even flat (Proposition 5.3), bringing into question whether we even have a coaction by the dual Steenrod algebra in this case.

There is another related reason to consider the space $B_{C_4}\Sigma_2$. In [17], Wilson describes a framework for equivariant total power operations over an $H\mathbb{F}_2$–module A equipped with a symmetric multiplication. The total power operation is induced from a map of spectra

$$A \to A^{t\Sigma_{[2]}},$$

where $(-)^{t\Sigma_{[2]}}$ is a variant Tate construction defined in [17].

In the nonequivariant case, $A \to A^{t\Sigma_{[2]}}$ induces a map $A_* \to A_*((x))$ and the Dyer–Lashof operations Q^i can be obtained as the components of this map:

$$Q(x) = \sum_i Q^i(x)x^i.$$

In the C_2–equivariant case, we have a map $A_\bigstar \to A_\bigstar[c, b^\pm]/(c^2 = a_\sigma c + u_\sigma b)$ and we get power operations

$$Q(x) = \sum_i Q^{i\rho}(x)b^i + \sum_i Q^{i\rho+\sigma}(x)cb^i.$$

When $A = H\mathbb{F}_2$, $A_\bigstar[c, b^\pm]/(c^2 = a_\sigma c + u_\sigma b)$ is the cohomology of $B_{C_2}\Sigma_2$ localized away from the class b.

For C_4 we would have to use the cohomology of $B_{C_4}\Sigma_2$ (localized at a certain class) but that is no longer free, meaning that the resulting power operations would have extra relations between them, further complicating the other arguments in [17].

The computation of $H^\bigstar(B_{C_4}\Sigma_{2+}; \mathbb{F}_2)$ also serves as a test case of RO(G) homology computations for equivariant classifying spaces where G is not of prime order. We refer the reader to Shulman [16], Chonoles [1], Wilson [17], and Sankar and Wilson [15] for such computations in the $G = C_p$ case.

As for the organization of this paper, Section 2 describes the conventions and notation that we shall be using throughout this document, as well as the Tate diagram for a group G and a G–equivariant spectrum.

Sections 3.1 and 3.3 describe the Tate diagram for C_2 and C_4, respectively, using coefficients in the constant Mackey functor $\mathbb{F}_2$.

In Section 4 we define equivariant classifying spaces $B_G H$ and briefly explain the elementary computation of the cohomology of $B_{C_2}\Sigma_2$ (this argument also appears in [17]).

In Section 5 we present the result of the computation of $H^\bigstar(B_{C_4}\Sigma_{2+}; \mathbb{F}_2)$ and prove that it's not flat as a Mackey functor module over $(H\mathbb{F}_2)_\bigstar$. Sections 6–8 contain the proofs of the computation of the cohomology of $B_{C_4}\Sigma_2$.

We have included three appendices in the end. Appendix A contains pictures of the spectral sequence converging to $H^\bigstar(B_{C_4}\Sigma_{2+};\mathbb{F}_2)$ while Appendix B contains a detailed description of $H^\bigstar(S^0;\mathbb{F}_2)$, which is the ground Green functor over which all our Mackey functors are modules.

Appendix C contains the description of the G–equivariant Borel dual Steenrod algebra $(H\mathbb{F}_2\wedge H\mathbb{F}_2)^{hG}_\bigstar$ where $G=C_{2^n}$ and $n\geq 1$. This is independent of the rest of the paper and is related to our work in the following way: In [10], the Borel dual Steenrod algebra is a key ingredient in the computation of the genuine dual Steenrod algebra over the group $G=C_2$; the other key ingredient is the computation of $H^\bigstar(B_{C_2}\Sigma_{2+};\mathbb{F}_2)$. For $G=C_{2^n}$, $n\geq 2$, the Borel equivariant description admits a straightforward generalization as we show in Appendix C.

To aid in the creation of the first two appendices, we extensively used the computer program of [2] available here. In fact, we have introduced new functionality in the software that computes the RO(G)–graded homology of spaces such as $B_{C_4}\Sigma_2$ given an explicit equivariant CW decomposition (such as we discuss in Section 6.1). This assisted in the discovery of a nontrivial d^2 differential in the spectral sequence of $B_{C_4}\Sigma_2$ (see Remark 7.8), although the provided proof is independent of the computer computation.

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2 Conventions and notations

We will use the letter k to denote the field $\mathbb{F}_2$, the constant Mackey functor $k=\underline{\mathbb{F}}_2$ and the corresponding Eilenberg–Mac Lane spectrum Hk. The meaning should always be clear from the context.

All our homology and cohomology will be in k coefficients.

The data of a C_4 Mackey functor M can be represented by a diagram displaying the values of M on orbits, its restriction and transfer maps and the actions of the Weyl groups. We shall refer to $M(C_4/C_4)$, $M(C_4/C_2)$ and $M(C_4/e)$ as the top, middle and bottom levels of the Mackey functor M, respectively. The Mackey functor diagram takes the form

$$M=\begin{tikzcd} M(C_4/C_4) \arrow[d, bend right, "\mathrm{Res}^4_2"'] \\ M(C_4/C_2) \arrow[u, bend right, "\mathrm{Tr}^4_2"'] \arrow[d, bend right, "\mathrm{Res}^2_1"'] \arrow[loop right, "C_4/C_2"] \\ M(C_4/e) \arrow[u, bend right, "\mathrm{Tr}^2_1"'] \arrow[loop right, "C_4"] \end{tikzcd}$$

If X is a G–spectrum then $X_\bigstar$ denotes the $\mathrm{RO}(G)$–graded G–Mackey functor, defined on orbits as

$$X_\bigstar(G/H) = X_\bigstar^H = \pi^H(S^{-\bigstar} \wedge X) \quad = [S^\bigstar, X]^H.$$

The index $\bigstar$ will always be an element of the real representation ring $\mathrm{RO}(G)$.

$\mathrm{RO}(C_4)$ is spanned by the irreducible representations 1, σ and λ, where σ is the 1–dimensional sign representation and λ is the 2–dimensional representation given by rotation by $\pi/2$.

For $V=\sigma$ or $V=\lambda$, denote by $a_V \in k_{-V}^{C_4}$ the Euler class induced by the inclusion of north and south poles $S^0 \hookrightarrow S^V$; also denote by $u_V \in k_{|V|-V}^{C_4}$ the orientation class generating the Mackey functor $k_{|V|-V}=k$. We will use the notation $\overline{a}_V, \overline{u}_V$ to denote the restrictions of a_V, u_V to middle level, and $\overline{\overline{u}}_V$ to denote the restriction of u_V to bottom level. This notation is consistent with its use in [9].

We also write $a_{\sigma_2} \in k_{-\sigma_2}^{C_2}$ and $u_{\sigma_2} \in k_{1-\sigma_2}^{C_2}$ for the C_2 Euler and orientation classes, where σ_2 is the sign representation of C_2.

The Gold relation [8] in k coefficients takes the form

$$a_\sigma^2 u_\lambda = 0.$$

Let EG be a contractible free G–space and $\widetilde{E}G$ be the cofiber of the collapse map $EG_+ \to S^0$. We use the notation

$$X_h = EG_+ \wedge X, \quad \widetilde{X} = \widetilde{E}G \wedge X, \quad X^h = F(EG_+, X) \quad \text{and} \quad X^t = \widetilde{X}^h.$$

The Tate diagram [4] then takes the form

$$\begin{tikzcd} X_h \arrow[r] \arrow[d, "\simeq"] & X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ X_h \arrow[r] & X^h \arrow[r] & X^t \end{tikzcd}$$

The square on the right is a homotopy pullback diagram and is called the Tate square.

Applying $\pi_\bigstar^G$ on the Tate diagram gives

$$\begin{tikzcd} X_{hG\bigstar} \arrow[r] \arrow[d, "\simeq"] & X_\bigstar^G \arrow[r] \arrow[d] & \widetilde{X}_\bigstar^G \arrow[d] \\ X_{hG\bigstar} \arrow[r] & X_\bigstar^{hG} \arrow[r] & X_\bigstar^{tG} \end{tikzcd}$$

3 The Tate diagram for C_2 and C_4

3.1 The Tate diagram for C_2

For $X=k$ and $G=C_2$ the corners of the Tate square are

$$k_\bigstar^{C_2} = k[a_{\sigma_2}, u_{\sigma_2}] \oplus k\left\{\frac{\theta_{\sigma_2}}{a_{\sigma_2}^i u_{\sigma_2}^j}\right\}_{i,j\ge 0}, \quad \widetilde{k}_\bigstar^{C_2} = k[a_{\sigma_2}^{\pm}, u_{\sigma_2}], \quad k_\bigstar^{hC_2} = k[a_{\sigma_2}, u_{\sigma_2}^{\pm}], \quad k_\bigstar^{tC_2} = k[a_{\sigma_2}^{\pm}, u_{\sigma_2}^{\pm}],$$

where $\theta_{\sigma_2}=\mathrm{Tr}_1^2(\overline{u}_{\sigma_2}^{-2})$. The map $k_h\to k$ in the Tate diagram induces

$$k_{hC_2\bigstar}=\Sigma^{-1}k_\bigstar^{tC_2}/k_\bigstar^{hC_2}\to k_\bigstar^{C_2},\quad a_{\sigma_2}^{-i}u_{\sigma_2}^{-j}\mapsto\frac{\theta_{\sigma_2}}{a_{\sigma_2}^i u_{\sigma_2}^{j-1}}.$$

3.2 The RO(C_4) homology of a point

The RO(C_4) homology of a point (in k coefficients) is significantly more complicated than the RO(C_2) one; see [2] for the integer coefficient case. Appendix B contains a very detailed description of it, and the goal in this subsection is to provide a more compact version. We have also included a summary table at the end of the subsection.

The top level is

(1) $$k_\bigstar^{C_4}\doteq k\left[a_\sigma,u_\sigma,a_\lambda,u_\lambda,\frac{u_\lambda}{u_\sigma^{1+i}},\frac{a_\sigma^2}{a_\lambda^{1+i}}\right]\oplus k[a_\lambda^{\pm}]\left\{\frac{\theta}{a_\sigma^i u_\sigma^j}\right\}\oplus k\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{u_\sigma^i a_\lambda^j}\right\}\oplus k[u_\sigma^{\pm}]\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{a_\lambda^j u_\lambda^{1+m}}\right\},$$

where the indices i,j,m range in $0,1,2,\dots$ and ϵ ranges in $0,1$.

The use of $\doteq$ as opposed to $=$ is meant to signify some subtlety present in (1) that needs to be clarified before the equality can be used. This subtlety has to do with how quotients are defined (cf [2]) and how elements multiply (the multiplicative relations). For example, the first summand in (1) is not actually a polynomial algebra, but rather a quotient of one, owing to the families of relations

$$\frac{u_\lambda}{u_\sigma^i}\cdot\frac{a_\sigma^2}{a_\lambda^j}=0,\quad u_\sigma\cdot\frac{u_\lambda}{u_\sigma^{1+i}}=\frac{u_\lambda}{u_\sigma^i},\quad a_\lambda\cdot\frac{a_\sigma^2}{a_\lambda^{1+i}}=\frac{a_\sigma^2}{a_\lambda^i},$$

where $i,j\geq 0$ (and $u_\lambda/u_\sigma^0=u_\lambda$ and $a_\sigma^2/a_\lambda^0=a_\sigma^2$).

We begin this process of carefully interpreting (1) by first noting that the middle level $\overline{u}_\sigma$ and bottom level $\overline{\overline{u}}_\sigma,\overline{\overline{u}}_\lambda$ are invertible. The element θ is then defined as

$$\theta=\mathrm{Tr}_2^4(\overline{u}_\sigma^{-2}).$$

We further introduce the elements

$$x_{n,m}=\mathrm{Tr}_1^4(\overline{\overline{u}}_\sigma^{-n}\overline{\overline{u}}_\lambda^{-m})\quad\text{for } n\geq 0, m\geq 1.$$

Observe that

$$x_{n,m}=\frac{x_{0,1}}{u_\sigma^n u_\lambda^{m-1}}.$$

The relation between $x_{0,1}=\mathrm{Tr}_1^4(\overline{\overline{u}}_\lambda^{-1})$ and θ is

$$x_{0,1}=a_\sigma^2\frac{\theta}{a_\lambda}=\theta\frac{a_\sigma^2}{a_\lambda}.$$

With this notation, the second curly bracket in (1) contains elements of the form

$$\frac{x_{n,1}}{a_\lambda^i}\quad\text{and}\quad\frac{x_{n,1}}{a_\sigma a_\lambda^i},$$

and the third contains
$$\frac{x_{n,m}}{a_\lambda^i} \quad \text{and} \quad \frac{x_{n,m}}{a_\sigma a_\lambda^i} \quad \text{for } m>1.$$

The behavior of the $x_{n,m}$ depends crucially on whether $m=1$ or not: $x_{n,1}u_\sigma=0$ but $x_{n,m}u_\sigma\neq 0$ for $m>1$; the $x_{n,1}$ are infinitely a_σ divisible, since
$$\frac{x_{n,1}}{a_\sigma^2}=\frac{\theta}{u_\sigma^n a_\lambda},$$
while the $x_{n,m}$ for $m>1$ can only be divided by a_σ once. That's why we separate them into two distinct summands in (1).

The third curly bracket in (1) for $\epsilon=0$ consists of quotients of
$$s:=\frac{(\theta/a_\lambda)a_\sigma}{u_\lambda}u_\sigma=\frac{x_{0,2}u_\sigma}{a_\sigma},$$
which is the mod 2 reduction of the element s from [2]. Note that $su_\lambda=sa_\lambda=0$.

The quotients in the RHS of (1) are all chosen coherently (cf [2]), that is, we always have the cancellation property
$$z\cdot\frac{y}{xz}=\frac{y}{x}.$$
We also have that
$$\frac{x}{y}\cdot\frac{z}{w}=\frac{xz}{yw}$$
as long as $xz\neq 0$ — this condition is necessary: $(\theta/a_\lambda)a_\sigma\neq 0$ is not $(\theta a_\sigma)/a_\lambda$, as $\theta a_\sigma=0$.

To compute any product of two elements in the RHS of (1) we follow the following procedure:

- If both elements involve θ then the product is automatically 0.
- If neither element involves θ, perform all possible cancellations and use the relation
$$\frac{u_\lambda}{u_\sigma^i}\cdot\frac{a_\sigma^2}{a_\lambda^j}=0,$$
where i,j range in $0,1,2,\ldots$.
- If only one element involves θ, perform all possible cancellations and use
$$\frac{x}{y}\cdot\frac{z}{w}=\frac{xz}{yw}$$
as long as xz appears in (1). If the resulting element appears in (1) then that's the product; if not, then the product is 0.

These are all the remarks needed to properly interpret the formula in (1) for the top level $k_\bigstar^{C_4}$.

The middle level is

(2) $$k_\bigstar^{C_2}\doteq k[\bar a_\lambda,\bar u_\lambda,\sqrt{\bar a_\lambda\bar u_\lambda},\bar u_\sigma^{\pm}]\oplus k[\bar u_\sigma^{\pm}]\left\{\frac{v}{\bar a_\lambda^i\bar u_\lambda^j\sqrt{\bar a_\lambda\bar u_\lambda}^{\,\epsilon}}\right\}.$$

element	also known as	degree in $k_\bigstar$	restriction	is transfer of
θ	-	$2\sigma-2$	0	$\overline{u}_\sigma^{-2}$
s	$(x_{0,2}u_\sigma)/a_\sigma$	$2\lambda-3$	$v/\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$	-
a_σ^2/a_λ	-	$\lambda-2\sigma$	$v\overline{u}_\sigma^2$	-
$x_{0,1}$	$(a_\sigma^2/a_\lambda)\theta$	$\lambda-2$	0	v
$x_{0,2}$	$x_{0,1}/u_\lambda$	$2\lambda-4$	0	$v/\overline{u}_\lambda$
$(a_\sigma u_\lambda)/u_\sigma$	-	$1-\lambda$	0	$\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$
v	θ_{σ_2}	$\lambda-2$	0	$\overline{\overline{u}}_\lambda^{-1}$

Table 1

Here, $\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$ is the (unique) element whose square is $\overline{a}_\lambda\overline{u}_\lambda$, and v is defined by $v=\mathrm{Tr}_1^2(\overline{\overline{u}}_\lambda^{-1})$. Further,

$$\mathrm{Tr}_2^4(\sqrt{\overline{a}_\lambda\overline{u}_\lambda})=\frac{a_\sigma u_\lambda}{u_\sigma},\quad \mathrm{Tr}_2^4(v)=x_{0,1},\quad \overline{s}:=\mathrm{Res}_2^4(s)=\frac{v}{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}},\quad \mathrm{Res}_2^4\left(\frac{a_\sigma^2}{a_\lambda}\right)=v\overline{u}_\sigma^2.$$

The interpretation of (2) is complete. We note that the restriction $\mathrm{Res}_2^4\colon k_\bigstar^{C_4}\to k_\bigstar^{C_2}$ makes $k_\bigstar^{C_2}$ into a $k_\bigstar^{C_4}$–module,

$$k_\bigstar^{C_2}=\frac{k_\bigstar^{C_4}[u_\sigma^{-1}]}{a_\sigma}\{1,\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\}.$$

In terms of the notation of the C_2 generators,

$$\overline{a}_\lambda=a_{\sigma_2}^2,\quad \overline{u}_\lambda=u_{\sigma_2}^2,\quad \sqrt{\overline{a}_\lambda\overline{u}_\lambda}=a_{\sigma_2}u_{\sigma_2},\quad v=\theta_{\sigma_2}.$$

Finally, the bottom level is very simple:

$$k_\bigstar^e=k[\overline{\overline{u}}_\lambda^\pm,\overline{\overline{u}}_\sigma^\pm].$$

We conclude with Table 1 giving the interesting/important elements of $k_\bigstar^{C_4}$ (outside of $a_\sigma,u_\sigma,a_\lambda,u_\lambda$). For more details, consult Appendix B.

3.3 The Tate diagram for C_4

Using the notation of the previous subsection, the corners of the Tate square are

$$k_\bigstar^{C_4}\doteq k\left[a_\sigma,u_\sigma,a_\lambda,u_\lambda,\frac{u_\lambda}{u_\sigma^{1+i}},\frac{a_\sigma^2}{a_\lambda^{1+i}}\right]\oplus k[a_\lambda^\pm]\left\{\frac{\theta}{a_\sigma^iu_\sigma^j}\right\}\oplus k\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{u_\sigma^ia_\lambda^j}\right\}\oplus k[u_\sigma^\pm]\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{a_\lambda^ju_\lambda^{1+m}}\right\},$$

$$\widetilde{k}_\bigstar^{C_4}=a_\lambda^{-1}k_\bigstar^{C_4}\doteq k\left[a_\sigma,u_\sigma,a_\lambda^\pm,u_\lambda,\frac{u_\lambda}{u_\sigma^{1+i}}\right]\oplus k[a_\lambda^\pm]\left\{\frac{\theta}{a_\sigma^iu_\sigma^j}\right\},$$

$$k_\bigstar^{hC_4}=k[a_\sigma,u_\sigma^\pm,a_\lambda,u_\lambda^\pm]/a_\sigma^2,\qquad k_\bigstar^{tC_4}=k[a_\sigma,u_\sigma^\pm,a_\lambda^\pm,u_\lambda^\pm]/a_\sigma^2.$$

The map $k_h\to k$ in the Tate diagram induces

$$k_{hC_4\bigstar}=\Sigma^{-1}k_\bigstar^{tC_4}/k_\bigstar^{hC_4}\to k_\bigstar^{C_4},\quad u_\sigma^{-i}a_\lambda^{-j}u_\lambda^{-m}\mapsto\frac{(\theta/a_\lambda)a_\sigma}{u_\sigma^{i-1}a_\lambda^{j-1}u_\lambda^m}.$$

An important distinction between the Tate diagram of C_4 and of C_2 is that, for the group C_2, the operation $\widetilde{E}C_2 \wedge -$ amounts to taking geometric fixed points: $\Phi^{C_2} k = \widetilde{E}C_2 \wedge k = \widetilde{k}$. This is not the case for C_4, and indeed

$$(\Phi^{C_4} k)_\bigstar = a_\sigma^{-1} \widetilde{k}_\bigstar^{C_4} = k[a_\sigma^\pm, u_\sigma, a_\lambda^\pm].$$

4 Equivariant classifying spaces

For groups G and K, denote by $E_G K$ any $G \times K$ space that is K–free and for which $(E_G K)^\Gamma$ is contractible for any subgroup $\Gamma \subseteq G \times K$ with $\Gamma \cap (\{1\} \times K) = \{1\}$ (a graph subgroup). The spaces $E_G \Sigma_n$ are those appearing in a G–E_∞–operad.

We define the equivariant classifying space

$$B_G K = E_G K / K.$$

4.1 The case of C_2

For $G = C_2$, the spaces $B_{C_2} \Sigma_2$ are used in the computation of the C_2 dual Steenrod algebra by Hu and Kriz [10] and for the construction of the total C_2–Dyer–Lashof operations in [17]. Both use the computation

$$k_{C_2}^\bigstar(B_{C_2}\Sigma_{2+}) = k_{C_2}^\bigstar[c, b]/(c^2 = a_{\sigma_2} c + u_{\sigma_2} b), \tag{3}$$

where c and b are classes in cohomological degrees σ_2 and $1 + \sigma_2$, respectively. Let us note here that $B_{C_2}\Sigma_2$ is $\mathbb{R}P^\infty$ with a nontrivial C_2 action; the restrictions of c, b are the generators of degree $1, 2$ of $k^*(\mathbb{R}P^\infty)$.

We shall now summarize this computation, since part of it will be needed for the analogous computation when $G = C_4$, which takes place in Sections 5–8.

Let σ, τ be the sign representations of C_2, Σ_2 respectively, and let $\rho = 1+\sigma$. Then $E_{C_2}\Sigma_2 = S(\infty(\rho \otimes \tau))$; here $S(V)$ denotes the unit G–sphere inside a G–representation V. The graph subgroups of $C_2 \times \Sigma_2$ are C_2 and Δ, and their orbits correspond to the cells

$$\frac{C_2 \times \Sigma_2}{C_2} = S(1 \otimes \tau) \quad \text{and} \quad \frac{C_2 \times \Sigma_2}{\Delta} = S(\sigma \otimes \tau).$$

Wilson [17] defines a filtration on $E_{C_2}\Sigma_2$ given by

$$S(1 \otimes \tau) \subseteq S(\rho \otimes \tau) \subseteq S((\rho + 1) \otimes \tau) \subseteq S(2\rho \otimes \tau) \subseteq \cdots,$$

whose quotients (after adjoining disjoint basepoints) are

$$\mathrm{gr}_{2j+1}\, E_{C_2}\Sigma_{2+} = \frac{C_2 \times \Sigma_2}{\Delta}_+ \wedge S^{(j+1)\rho_{C_2} - 1}, \quad \mathrm{gr}_{2j}\, E_{C_2}\Sigma_{2+} = \Sigma_{2+} \wedge S^{j\rho_{C_2}}.$$

Taking the quotient by Σ_2 gives a filtration for $B_{C_2}\Sigma_{2+}$ with

$$\mathrm{gr}_{2j+1}\, B_{C_2}\Sigma_{2+} = S^{(j+1)\rho_{C_2} - 1}, \quad \mathrm{gr}_{2j}\, B_{C_2}\Sigma_{2+} = S^{j\rho_{C_2}}.$$

Applying $k^\bigstar$ yields a spectral sequence

$$E^1 = k^\bigstar\{e^{j\rho}, e^{j\rho+\sigma}\} \Rightarrow k^\bigstar(B_{C_2}\Sigma_{2+})$$

of modules over the Green functor $k^\bigstar$. The fact that the differentials are module maps gives $E_1 = E_2$ for degree reasons. Furthermore, the vanishing of the RO(C_2) homology of a point in a certain range gives $E_2 = E_\infty$. The E_∞ page is free as a module over the Green functor $k^\bigstar$, hence there can't be any extension problems, and we get the module structure

$$k^\bigstar(B_{C_2}\Sigma_{2+}) = k^\bigstar\{e^{j\rho}, e^{j\rho+\sigma}\}.$$

It's easier to prove (using the homotopy fixed point spectral sequence) that

$$k^{h\bigstar}(B_{C_2}\Sigma_{2+}) = k^{h\bigstar}[w],$$

where w has cohomological degree 1. The map $k \to k^h$ from Section 2 induces

$$k^\bigstar(B_{C_2}\Sigma_{2+}) \to k^{h\bigstar}(B_{C_2}\Sigma_{2+}),$$

which is the localization which inverts u_{σ_2}. Thus we can see that $c = e^\sigma$ maps to $u_{\sigma_2}w$ (or $a_{\sigma_2} + u_{\sigma_2}w$), $b = e^\rho$ maps to $a_{\sigma_2}w + u_{\sigma_2}w^2$ and conclude that

$$k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+}) = k^\bigstar_{C_2}[c,b]/(c^2 = a_{\sigma_2}c + u_{\sigma_2}b).$$

$B_{C_2}\Sigma_2$ is a C_2–H–space so $k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+})$ is a Hopf algebra (since it is flat over $k^\bigstar_{C_2}$). For degree reasons, we can see that

$$\Delta(c) = c\otimes 1 + 1\otimes c, \quad \Delta(b) = b\otimes 1 + 1\otimes b, \quad \epsilon(c) = \epsilon(b) = 0.$$

(We can add a_{σ_2} to c to force $\epsilon(c) = 0$.) The primitive elements are spanned by c, b^{2^i}.

5 The cohomology of $B_{C_4}\Sigma_2$

In the next section we shall construct a cellular decomposition of $B_{C_4}\Sigma_2$ giving rise to a spectral sequence computing $k^\bigstar(B_{C_4}\Sigma_{2+})$. Here's the result of the computation, describing $k^\bigstar(B_{C_4}\Sigma_{2+})$ as a Green functor algebra over $k^\bigstar$.

Proposition 5.1 *There exist elements*

$$e^a \in k^{\sigma+\lambda}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^u \in k^{\sigma+\lambda-2}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^\lambda \in k^{\lambda}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^\rho \in k^{\rho}_{C_4}(B_{C_4}\Sigma_{2+})$$

such that

$$k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+}) = \frac{k^\bigstar_{C_4}\left[e^a, \dfrac{e^u}{u^i_\sigma}, \dfrac{e^\lambda}{u^i_\sigma}, e^\rho\right]_{i\geq 0}}{S}.$$

The relation set S consists of two types of relations (we use indices $i, j \geq 0$):

- ***Module relations*** $$\begin{cases} \dfrac{a_\sigma^2}{a_\lambda^j}\dfrac{e^u}{u_\sigma^i}=0, \\ \dfrac{(\theta/a_\lambda)a_\sigma}{u_\sigma^{i-2}a_\lambda^{j-1}}e^a+\dfrac{s}{u_\sigma^{i-1}a_\lambda^{j-2}}e^u=\dfrac{a_\sigma^2}{a_\lambda^j}\dfrac{e^\lambda}{u_\sigma^i}. \end{cases}$$

- ***Multiplicative relations*** $$\begin{cases} \dfrac{e^u}{u_\sigma^i}\dfrac{e^u}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j-2}}e^\lambda, \\ \dfrac{e^\lambda}{u_\sigma^i}\dfrac{e^u}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j}}e^a+a_\lambda\dfrac{e^u}{u_\sigma^{i+j}}, \\ e^a\dfrac{e^u}{u_\sigma^i}=\dfrac{u_\lambda}{u_\sigma^{i-1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^i}e^a, \\ \dfrac{e^\lambda}{u_\sigma^i}\dfrac{e^\lambda}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j+1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^{i+j+2}}e^a+a_\lambda\dfrac{e^\lambda}{u_\sigma^{i+j}}, \\ e^a\dfrac{e^\lambda}{u_\sigma^i}=\dfrac{e^u}{u_\sigma^{i+1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^{i+1}}e^\rho, \\ (e^a)^2=u_\sigma e^\lambda e^\rho+a_\sigma\dfrac{e^u}{u_\sigma}e^\rho+u_\sigma a_\lambda e^\rho+a_\sigma a_\lambda e^a. \end{cases}$$

The middle level of $k^\bigstar(B_{C_4}\Sigma_{2+})$ *is generated by the restrictions of* $e^a, e^u, e^\lambda, e^\rho$, *which we denote by* $\overline{e}^a, \overline{e}^u, \overline{e}^\lambda, \overline{e}^\rho$, *respectively, and two fractional elements*:

$$k_{C_2}^\bigstar(B_{C_4}\Sigma_{2+})=\frac{k_{C_2}^\bigstar\Big[\overline{e}^a,\overline{e}^u,\overline{e}^\lambda,\overline{e}^\rho,\dfrac{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^u}{\overline{u}_\lambda},\dfrac{\overline{a}_\lambda\overline{u}_\sigma^{-1}\overline{e}^u+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^\lambda}{\overline{u}_\lambda}\Big]}{\mathrm{Res}_2^4(S)}.$$

Here, $\mathrm{Res}_2^4(S)$ *denotes the relation set obtained by applying the ring homomorphism* Res_2^4 *on each relation of* S. *That is*, *we have the module relations*

$$\frac{v}{\overline{a}_\lambda^i}\overline{e}^u=\frac{v}{\overline{a}_\lambda^i}\overline{e}^\lambda=0\quad \textit{for any } i\geq 0,$$

and the multiplicative relations

$$\begin{aligned} &(\overline{e}^u)^2=\overline{u}_\sigma^2\overline{u}_\lambda\overline{e}^\lambda, && \overline{e}^\lambda\overline{e}^u=\overline{u}_\lambda\overline{e}^a+\overline{a}_\lambda\overline{e}^u, && \overline{e}^a\overline{e}^u=\overline{u}_\lambda\overline{u}_\sigma\overline{e}^\rho, \\ &(\overline{e}^\lambda)^2=\overline{u}_\lambda\overline{u}_\sigma^{-1}\overline{e}^\rho+\overline{a}_\lambda\overline{e}^\lambda, && \overline{e}^a\overline{e}^\lambda=\overline{u}_\sigma^{-1}\overline{e}^u\overline{e}^\rho, && (\overline{e}^a)^2=\overline{u}_\sigma\overline{e}^\lambda\overline{e}^\rho+\overline{u}_\sigma\overline{a}_\lambda\overline{e}^\rho. \end{aligned}$$

As for the Mackey functor structure, *the Weyl group* C_4/C_2 *action on the generators is trivial and we have*

- ***Mackey functor relations*** $$\begin{cases} \mathrm{Tr}_2^4\Big(\overline{u}_\sigma^{-i}\dfrac{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^u}{\overline{u}_\lambda}\Big)=a_\sigma\dfrac{e^u}{u_\sigma^{i+1}}, \\ \mathrm{Tr}_2^4\Big(\overline{u}_\sigma^{-i}\dfrac{\overline{a}_\lambda\overline{u}_\sigma^{-1}\overline{e}^u+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^\lambda}{\overline{u}_\lambda}\Big)=a_\sigma\dfrac{e^\lambda}{u_\sigma^{i+1}}. \end{cases}$$

Finally, the bottom level is

$$k_e^{\bigstar}(B_{C_4}\Sigma_{2+}) = k_e^{\bigstar}[\mathrm{Res}_1^4(e^u)]$$

with trivial Weyl group C_4 action and Mackey functor relations obtained by applying Res_1^4 to the multiplicative relations of S:

$$\mathrm{Res}_1^4 e^{\lambda} = \overline{\overline{u}}_{\sigma}^{-2}\overline{\overline{u}}_{\lambda}^{-1}\,\mathrm{Res}_1^4(e^u)^2, \quad \mathrm{Res}_1^4 e^{a} = \overline{\overline{u}}_{\sigma}^{-2}\overline{\overline{u}}_{\lambda}^{-2}\,\mathrm{Res}_1^4(e^u)^3, \quad \mathrm{Res}_1^4 e^{\rho} = \overline{\overline{u}}_{\lambda}^{-3}\overline{\overline{u}}_{\sigma}^{-3}\,\mathrm{Res}_1^4(e^u)^4.$$

Note: for every quotient y/x there is a defining relation $x\cdot(y/x) = y$. We have omitted these implicit module relations from the description above.

The best description of the middle level is in terms of the generators c, b of

$$k_{C_2}^{\star}(B_{C_2}\Sigma_{2+}) = \frac{k_{C_2}^{\star}[c,b]}{c^2 = a_{\sigma_2}c + u_{\sigma_2}b}.$$

Here, $\star$ ranges in $\mathrm{RO}(C_2)$, and to get $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ for $\bigstar$ in $\mathrm{RO}(C_4)$, we have to restrict to $\mathrm{RO}(C_2)$ representations of the form $n + 2m\sigma_2$. In this way,

$$k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+}) = k_{C_2}^{\star}(B_{C_2}\Sigma_{2+})[\overline{u}_{\sigma}^{\pm}],$$

where $\star$ needs to be restricted to oriented C_2 representations: $\bigstar = n + m\sigma + k\lambda$ in $\mathrm{RO}(C_4)$ corresponds to $\star = n + m + 2k\sigma_2$ in $\mathrm{RO}(C_2)$. The correspondence of generators is

$$\overline{e}^a = \overline{u}_{\sigma}(a_{\sigma_2}b + bc), \quad \overline{e}^u = \overline{u}_{\sigma}u_{\sigma_2}c, \quad \overline{e}^{\lambda} = c^2, \quad \overline{e}^{\rho} = \overline{u}_{\sigma}b^2.$$

We can also express the map to homotopy fixed points in terms of our generators:

Proposition 5.2 *There is a choice of the degree-1 element w in*

$$k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+}) = k^{hC_4\bigstar}[w]$$

such that the localization map $k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+}) \to k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+})$ induced by $k \to k^h$ and inverting u_{σ} and u_{λ} is

$$\begin{aligned} e^u &\mapsto u_{\sigma}u_{\lambda}w, & e^{\lambda} &\mapsto u_{\lambda}w^2,\\ e^a &\mapsto u_{\sigma}u_{\lambda}w^3 + u_{\sigma}a_{\lambda}w, & e^{\rho} &\mapsto u_{\sigma}u_{\lambda}w^4 + a_{\sigma}u_{\lambda}w^3 + u_{\sigma}a_{\lambda}w^2 + a_{\sigma}a_{\lambda}w. \end{aligned}$$

Proposition 5.3 *The module $k^{\bigstar}(B_{C_4}\Sigma_{2+})$ is not flat over $k^{\bigstar}$.*

Proof Let $R = k^{\bigstar}$ and $M = k^{\bigstar}(B_{C_4}\Sigma_{2+})$. Consider the map of $\mathrm{RO}(C_4)$–graded Mackey functors $f\colon R \to \Sigma^{2\sigma-\lambda}R$ given on top level by multiplication with a_{σ}^2/a_{λ}, and determined on the lower levels by restricting (so it's multiplication with $v\overline{u}_{\sigma}^2$ on the middle level and 0 on the bottom level). If M is a flat R–module then we have an exact sequence

$$0 \to M \boxtimes_R \mathrm{Ker}(f) \to M \xrightarrow{f} \Sigma^{2\sigma-\lambda}M.$$

Here, $\boxtimes$ is the box product of Mackey functors and $\boxtimes_R$ is the corresponding box product over R–modules.

The restriction functor Res^4_2 from R–modules to $\operatorname{Res}^4_2 R$–modules is exact and symmetric monoidal, so we replace M, R and $\operatorname{Ker}(f)$ by $\operatorname{Res}^4_2 M$, $\operatorname{Res}^4_2 R$ and $\operatorname{Res}^4_2 \operatorname{Ker}(f)$, respectively, and have an exact sequence of C_2 Mackey functors. Using the notation involving the C_2 generators c and b, and writing $a = a_{\sigma_2}$ and $u = u_{\sigma_2}$, we have

$$M = \bigoplus_{i\geq 0} R\{b^{2i}, cb^{2i+1}\} \oplus \bigoplus_{i\geq 0} R\{ab^{2i+1}, ub^{2i+1}, acb^{2i}, ucb^{2i}\}/\sim.$$

The map f maps each summand to itself, so we may replace M by $R\{c, ab, ub, acb, ucb\}/\sim$ and continue to have the same exact sequence as above. The top level then is

$$0 \to (M \boxtimes_R \operatorname{Ker}(f))(C_2/C_2) \to M(C_2/C_2) \xrightarrow{v\bar{u}_\sigma^2} M(C_2/C_2)$$

and v acts trivially on ab, ub, ac, uc, ie on $M(C_2/C_2)$, so we get

$$(M \boxtimes_R \operatorname{Ker}(f))(C_2/C_2) = M(C_2/C_2).$$

We compute from definition that $(M \boxtimes_R \operatorname{Ker}(f))(C_2/C_2)$ is isomorphic to $M(C_2/C_2) \otimes_{R(C_2/C_2)} I$, where $I := \operatorname{Ker}(R \xrightarrow{v} R)$. But $M(C_2/C_2) \otimes_{R(C_2/C_2)} I \to M(C_2/C_2)$ has image $IM(C_2/C_2)$ hence

$$IM(C_2/C_2) = M(C_2/C_2).$$

This contradicts the fact that $ab = e'^{\lambda+1}$ is not divisible by any element of the ideal I (since $e'^{\lambda+1}$ is only divisible by $\bar{u}_\sigma^{\pm i} \in R$, which are not in I). $\square$

This proof does not depend on the explicit computation of $k^{\bigstar}_{C_4}(B_{C_4}\Sigma_{2+})$ but rather on the fact that, while $k^{\star}_{C_2}(B_{C_2}\Sigma_{2+})$ is free over $k^{\star}_{C_2}$, where $\star \in \mathrm{RO}(C_2)$, this is no longer the case when restricting $\star$ to range over the image of $\mathrm{RO}(C_4) \to \mathrm{RO}(C_2)$.

6 A cellular decomposition of $B_{C_4}\Sigma_2$

We denote the generators of C_4 and Σ_2 by g and h, respectively; let also τ be the sign representation of Σ_2, and $\rho = 1 + \sigma + \lambda$ the regular representation of C_4.

The graph subgroups of $C_4 \times \Sigma_2$ are $C_4 = \langle g\rangle$, $C_2 = \langle g^2\rangle$, $\Delta = \langle gh\rangle$, $\Delta' = \langle g^2 h\rangle$ and e.

Since $\rho \otimes \tau$ contains a trivial representation when restricted to any of these graph subgroups, we have a model for the universal space

$$E_{C_4}\Sigma_2 = S(\infty(\rho \otimes \tau)),$$

and $B_{C_4}\Sigma_2$ is $\mathbb{R}P^\infty$ with nontrivial C_4 action

$$g(x_1, x_2, x_3, x_4, \dots) = (x_1, -x_2, -x_4, x_3, \dots).$$

$S(\infty(\rho \otimes \tau))$ is the space

$$S(\infty) = \left\{ (x_n) : \text{finitely supported and } \sum_i x_i^2 = 1 \right\}$$

with $C_4 \times \Sigma_2$ action

$$g(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, -x_2, -x_4, x_3, x_5, \dots), \quad h(x_1, x_2, \dots) = (-x_1, -x_2, \dots).$$

We shall use the notation $(x_1, \dots, x_n)$ for the point $(x_1, \dots, x_n, 0, 0, \dots) \in S(\infty)$. Moreover, the subspace of $S(\infty)_+$ where only $x_1, \dots, x_n$ are allowed to be nonzero shall be denoted by $\{(x_1, \dots, x_n)\}$.

We now describe a cellular decomposition of $E_{C_4}\Sigma_{2+}$ where the orbits are $C_4 \times \Sigma_{2+}/H \wedge S^V$, where V is a C_4 representation.

- Start with $\{(x_1)\}$ the union of two points $(1), (-1)$ and the basepoint. This is $C_4 \times \Sigma_2/C_{4+}$.
- $\{(x_1)\}$ includes in $\{(x_1, x_2)\} = S(1+\sigma)_+$. The cofiber is the wedge of two circles, corresponding to x_2 being positive or negative, and the action is

$$g(x_1, +) = (x_1, -), \quad h(x_1, +) = (-x_1, -).$$

After applying the self-equivalence given by $f(x_1, +) = (x_1, +)$ and $f(x_1, -) = (-x_1, -)$, the action becomes

$$g(x_1, +) = (-x_1, -), \quad h(x_1, +) = (x_1, -).$$

This is exactly $C_4 \times \Sigma_2/\Delta_+ \wedge S^\sigma$.

- $\{(x_1, x_2)\}$ includes in $\{(x_1, x_2, x_3, 0), (x_1, x_2, 0, x_4)\}$. The cofiber is the wedge of four spheres corresponding to the sign of the nonzero coordinate among the last two coordinates. If we number the spheres from 1 to 4 and use $(x, y)^i$ coordinates to denote them for $i = 1, 2, 3, 4$ then

$$g(x, y)^i = (x, -y)^{i+1}, \quad h(x, y)^i = (-x, -y)^{i+2}.$$

Applying the self-equivalence

$$f(x, y)^1 = (x, y)^1, \quad f(x, y)^2 = (-y, x)^2, \quad f(x, y)^3 = (-x, -y)^3, \quad f(x, y)^4 = (y, -x)^4,$$

the action becomes $g(x, y)^i = (-y, x)^{i+1}$ and $h(x, y)^i = (x, y)^{i+2}$, ie we have $C_4 \times \Sigma_2/\Delta'_+ \wedge S^\lambda$.

- $\{(x_1, x_2, x_3, 0), (x_1, x_2, 0, x_4)\}$ includes in $\{(x_1, x_2, x_3, x_4)\} = S(\rho \otimes \tau)$ and the cofiber is the wedge of four S^3's corresponding to the signs of x_3, x_4. Analogously to the item above, we get the space $C_4 \times \Sigma_2/\Delta'_+ \wedge S^{1+\lambda}$.
- The process now repeats: $\{(x_1, x_2, x_3, x_4)\}$ includes in $\{(x_1, x_2, x_3, x_4, x_5)\}$ and the cofiber is the wedge of two S^4's corresponding to the sign of x_5 and we get $C_4 \times \Sigma_2/C_{4+} \wedge S^{1+\sigma+\lambda}$. And so on.

We get the decomposition of $B_{C_4}\Sigma_{2+}$ where the associated graded is

$$\mathrm{gr}_{4j} = S^{j\rho}, \quad \mathrm{gr}_{4j+1} = S^{j\rho+\sigma}, \quad \mathrm{gr}_{4j+2} = \Sigma^{j\rho+\lambda} C_4/C_{2+}, \quad \mathrm{gr}_{4j+3} = \Sigma^{j\rho+1+\lambda} C_4/C_{2+}.$$

This filtration gives a spectral sequence of $k^\star$–modules converging to $k^\star(B_{C_4}\Sigma_{2+})$, which we shall analyze in the next section.

6.1 A decomposition using trivial spheres

The cellular decomposition of $B_{C_4}\Sigma_2$ we just established consists of one cell in every dimension, where by "cell" we mean a space of the form $(C_4/H)_+ \wedge S^V$ for H a subgroup of C_4 and V a real nonvirtual C_4–representation; let us call this a "type I" decomposition. It is also possible to obtain a decomposition using only "trivial spheres", namely with cells of the form $(C_4/H)_+ \wedge S^n$; we shall refer to this as a "type II" decomposition. A type I decomposition can be used to produce a type II decomposition by replacing each type I cell $(C_4/H)_+ \wedge S^V$ with its type II decomposition. This is useful for computer-based calculations, since type II decompositions lead to chain complexes as opposed to spectral sequences — $k_*((C_4/H)_+ \wedge S^V)$ is concentrated in a single degree if and only if V is trivial. Equipped with a type II decomposition, the computer program of [2] can calculate the additive structure of $k^\bigstar(B_{C_4}\Sigma_{2+})$ in a finite range (this can be helpful with our spectral sequence calculations: see Remark 7.8).

We note however that a minimal type I decomposition may expand to a nonminimal type II decomposition; this is the case for $B_{C_4}\Sigma_2$, where the minimal type II decomposition uses $2d+3$ cells in each dimension $d \geq 1$, while the one obtained by expanding the type I decomposition uses $3d+3$ cells in each dimension $d \geq 1$. It is the minimal decomposition that we have used as input for the computer program of [2].

7 The spectral sequence for $B_{C_4}\Sigma_2$

Applying $k^\bigstar$ on the filtration of $B_{C_4}\Sigma_{2+}$ gives a spectral sequence

$$E_1^{V,s} = k^V \operatorname{gr}_s \Rightarrow k^V B_{C_4}\Sigma_{2+}.$$

The differential d^r has (V,s) bidegree $(1,r)$ so it goes 1 unit to the right and r units up in (V,s) coordinates.

Before we can write down the E_1 page, we will need some notation. For a G–Mackey functor M and subgroup $H \subseteq G$, let $M_{G/H}$ denote the G–Mackey functor defined on orbits as $M_{G/H}(G/K) = M(G/H \times G/K)$; the restriction, transfer and Weyl group action in $M_{G/H}$ are induced from those in M. Equivalently, $M_{G/H}$ can be thought of as restricting M to an H Mackey functor and then inducing up to a G–Mackey functor.

For $G = C_4$ and $H = C_2$, the bottom level of M_{C_4/C_2} is

$$M_{C_4/C_2}(C_4/e) = M(C_4/e \times C_4/C_2) = M(C_4/e) \oplus M(C_4/e) = M(C_4/e)\{x,y\},$$

where x and y are used to distinguish the two copies of $M(C_4/e)$, ie so that any element of $M_{C_4/C_2}(C_4/e)$ can be uniquely written as $mx + m'y$ for $m, m' \in M(C_4/e)$. The Weyl group $W_{C_4}e = C_4$ acts as

$$g(mx + m'y) = (gm)(gx) + (gm')(gy) = (gm)y + (gm')x,$$

ie $y = gx$ for a fixed generator $g \in C_4$.

We can then describe M_{C_4/C_2} in terms of M and the computation of the restriction and transfer on x, which are shown in the diagram

$$M_{C_4/C_2} = \begin{tikzcd} M(C_4/C_2)\{x+gx\} \arrow[d, bend right, "m(x+gx)\mapsto m(x+gx)"'] \\ M(C_4/C_2)\{x,gx\} \arrow[u, bend right, "mx\mapsto m(x+gx)"'] \arrow[d, bend right, "mx\mapsto \mathrm{Res}(m)x"'] \arrow[loop right, "C_4/C_2"] \\ M(C_4/e)\{x,gx\} \arrow[u, bend right, "mx\mapsto \mathrm{Tr}(m)x"'] \arrow[loop right, "C_4"] \end{tikzcd}$$

where in each map, m is any element of the appropriate level $M(C_4/H)$, with $H \subseteq C_4$.

If $M = R$ is a Green functor, then R_{C_4/C_2} is an R–module. Its top level, namely $R(C_4/C_2)\{x+gx\}$, is an $R(C_4/C_4)$–module via extension of scalars along the restriction map $\mathrm{Res}^4_2\colon R(C_4/C_4) \to R(C_4/C_2)$.

7.1 The E_1 page

The rows in the E_1 page are

$$E_1^{\bigstar,4j} = k^{\bigstar-j\rho}, \qquad E_1^{\bigstar,4j+1} = k^{\bigstar-j\rho-\sigma},$$
$$E_1^{\bigstar,4j+2} = (k^{\bigstar-j\rho-\lambda})_{C_4/C_2}, \quad E_1^{\bigstar,4j+3} = (k^{\bigstar-j\rho-\lambda-1})_{C_4/C_2}.$$

We will write $e^{j\rho}$, $e^{j\rho+\sigma}$, $e^{j\rho+\lambda}$ and $e^{j\rho+\lambda+1}$ for the unit elements corresponding to the E_1 terms above, living in degrees $V = j\rho, j\rho+\sigma, j\rho+\lambda, j\rho+\lambda+1$ and filtrations $s = 4j, 4j+1, 4j+2, 4j+3$, respectively. We also write $\overline{e}^V, \overline{\overline{e}}^V$ for their restrictions to the middle and bottom levels respectively. In this way,

$$E_1^{\bigstar,*} = k^{\bigstar}\{e^{j\rho}, e^{j\rho+\sigma}\} \oplus (k^{\bigstar})_{C_4/C_2}\{e^{j\rho+\lambda}, e^{j\rho+\lambda+1}\},$$

and the three levels of the Mackey functor $E_1^{\bigstar,*}$, from top to bottom, are

$$k^{\bigstar}_{C_4}\{e^{j\rho}, e^{j\rho+\sigma}\} \oplus k^{\bigstar}_{C_2}\{e^{j\rho+\lambda}(x+gx), e^{j\rho+\lambda+1}(x+gx)\},$$
$$k^{\bigstar}_{C_2}\{\overline{e}^{j\rho}, \overline{e}^{j\rho+\sigma}\} \oplus k^{\bigstar}_{C_2}\{\overline{e}^{j\rho+\lambda}x, \overline{e}^{j\rho+\lambda}gx, \overline{e}^{j\rho+\lambda+1}x, \overline{e}^{j\rho+\lambda+1}gx\},$$
$$k^{\bigstar}_{e}\{\overline{\overline{e}}^{j\rho}, \overline{\overline{e}}^{j\rho+\sigma}\} \oplus k^{\bigstar}_{e}\{\overline{\overline{e}}^{j\rho+\lambda}x, \overline{\overline{e}}^{j\rho+\lambda}gx, \overline{\overline{e}}^{j\rho+\lambda+1}x, \overline{\overline{e}}^{j\rho+\lambda+1}gx\}.$$

For the top level, $k^{\bigstar}_{C_2}$ is a $k^{\bigstar}_{C_4}$–module through the restriction $\mathrm{Res}^4_2\colon k^{\bigstar}_{C_4} \to k^{\bigstar}_{C_2}$:

$$k^{\bigstar}_{C_2} = \frac{k^{\bigstar}_{C_4}[u_\sigma^{-1}]}{a_\sigma}\{1, \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\}.$$

It's important to note that this is *not* a cyclic $k^{C_4}_{\bigstar}$–module.

At this point, the reader may want to look over pictures of the E_1 page that we have included in Appendix A. We will reference them in the following subsections.

7.2 The d^1 differentials

In this subsection, we explain how the d^1 differentials on each level are computed. We shall need this crucial remark.

Remark 7.1 The restriction of the C_4 action on $B_{C_4}\Sigma_2$ to $C_2 \subseteq C_4$ results in a C_2 space equivalent to $B_{C_2}\Sigma_2$. The equivariant cohomology of this space is known from Section 4.1 and we shall use this result to compute the middle level spectral sequence for $B_{C_4}\Sigma_2$. Further restricting to the trivial group $e \subseteq C_4$, we get the nonequivariant space $\mathbb{R}P^\infty$ and this will be used to compute the bottom level spectral sequence.

Proposition 7.2 *The nontrivial d^1 differentials are generated by*

$$d^1(\overline{e}^{j\rho+\sigma}) = v\overline{u}_\sigma \overline{e}^{j\rho+\lambda}(x+gx),$$
$$d^1(e^{j\rho+\sigma}) = v\overline{u}_\sigma e^{j\rho+\lambda}(x+gx),$$
$$d^1(\overline{e}^{j\rho+\lambda}x) = \overline{e}^{j\rho+\lambda+1}(x+gx).$$

Proof First of all, the bottom level spectral sequence is concentrated on the diagonal and the nontrivial d^1 differentials are $k\{x,gx\} \to k\{x,gx\}$, $x \mapsto x+gx$, since $k^*(\mathbb{R}P^\infty)$ is k in every nonnegative degree. See Figures 1, 2, 3, 4 and 5.

The d^1 differentials on middle and top level are computed from the fact that they are $k^\star$–module maps, hence determined on

$$e^{j\rho}, \quad e^{j\rho+\sigma}, \quad \overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}^{\epsilon} e^{j\rho+\lambda+\epsilon'}(x+gx)$$

for the top level ($\epsilon, \epsilon' = 0, 1$), and on

$$\overline{e}^{j\rho}, \quad \overline{e}^{j\rho+\sigma}, \quad \overline{e}^{j\rho+\lambda}x, \quad \overline{e}^{j\rho+\lambda+1}x$$

for the middle level. We remark that because $k_{C_2}^\star$ is not a cyclic $k_{C_4}^\star$–module, it does not suffice to compute the top level d^1 on $e^{j\rho}, e^{j\rho+\sigma}, e^{j\rho+\lambda}, e^{j\rho+\lambda+1}$.

The d^1 differentials from row $4j$ to row $4j+1$ are all determined by $d^1\colon ke^{j\rho} \to k^{1-\sigma}e^{j\rho+\sigma}$. Note that $k^{1-\sigma}$ is generated by $0 \mid \overline{u}_\sigma^{-1} \mid \overline{\overline{u}}_\sigma^{-1}$ — this notation was defined in [2] and expresses the generators of all three levels from top to bottom, separated by vertical columns. The d^1 is trivial on bottom level, and using the fact that it commutes with restriction we can see that it's trivial in all levels. See Figure 1 and degrees $V = 0, 1$.

Similarly, the d^1 differentials from row $4j+1$ to row $4j+2$ are all determined by $d^1\colon ke^{j\rho+\sigma} \to (k^{\sigma-\lambda+1})_{C_4/C_2}e^{j\rho+\lambda}$. Note that $(k^{\sigma-\lambda+1})_{C_4/C_2}$ is generated by

$$v\overline{u}_\sigma(x+gx) \mid v\overline{u}_\sigma(x,gx) \mid \overline{\overline{u}}_\sigma\overline{\overline{u}}_\lambda^{-1}(x,gx).$$

The differential is trivial on bottom level, but on middle level the C_2 computation gives $k^{\sigma}_{C_2}(B_{C_4}\Sigma_{2+})=0$ forcing the differential to be nontrivial (the only other way to kill $E_1^{\sigma-\lambda+1,4j+2}(C_4/C_2)=k^2$ is for the d^1 differential from row $4j+2$ to $4j+3$ to be the identity $k^2\to k^2$ on middle level, which can't happen as we show in the next paragraph). Thus

$$d^1(\overline{e}^{j\rho+\sigma})=v\overline{u}_\sigma\overline{e}^{j\rho+\lambda}(x+gx) \quad \text{and} \quad d^1(e^{j\rho+\sigma})=v\overline{u}_\sigma e^{j\rho+\lambda}(x+gx).$$

See Figure 2 and degrees $V=\sigma,\sigma+1$.

The d^1 differentials from row $4j+2$ to row $4j+3$ are determined by

$$d^1\colon k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda}\to k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda+1},$$
$$d^1\colon k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}\to k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}.$$

On bottom level, these d^1's all are $x\mapsto x+gx$ and the commutation with restriction and transfer gives

$$d^1(\overline{e}^{j\rho+\lambda}x)=\overline{e}^{j\rho+\lambda+1}(x+gx),\quad d^1(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}(x+gx))=0,\quad d^1(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}(x+gx))=0.$$

See Figure 3 and degrees $V=\lambda,\lambda+1$.

Finally, the d^1 differentials from row $4j+3$ to row $4j+4$ are determined by

$$d^1\colon k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda+1}\to k^{1-\sigma}e^{j\rho+\rho},\quad d^1\colon k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}\to k^{2-\sigma+\lambda}e^{j\rho+\rho}.$$

These are trivial on the bottom level and by the commutation with restriction and transfer we can see that they are trivial on all levels. See Figure 4 and degrees $V=\rho-1,\rho$. □

This settles the E_1 page computation.

7.3 Bottom level computation

We can immediately conclude that the bottom level spectral sequence collapses in E_2, giving a single k in every $RO(C_4)$ degree. Thus there are no extension problems and the C_4 (Weyl group) action is trivial.

7.4 Middle level computation

By Remark 7.1 and comparing the description of the middle level E_2 with that of $k^{\bigstar}_{C_2}(B_{C_2}\Sigma_{2+})$ of Section 4.1 we can see that the middle level spectral sequence collapses on $E_2=E_\infty$.

To go from E_∞ to $k^{\bigstar}_{C_4}(B_{C_2}\Sigma_{2+})$ we need to be able to choose unique lifts for the permanent cycles when they are multiple candidates. To be consistent in our choices, we use the following rules.

7.4.1 Choosing unique lifts If we have a middle level element $\alpha\in E_\infty^{s,V}$ and the group $E_\infty^{t,V}$ vanishes for $t>s$, then α lifts uniquely to $k^{\bigstar}_{C_2}(B_{C_4}\Sigma_{2+})$. If on the other hand $E_\infty^{t,V}\neq 0$ for some $t>s$, then there are multiple lifts of α. In that case, we pick the lift for which there are no exotic restrictions (if possible). For example, if $\mathrm{Res}^2_1(\alpha)=0$ in E_∞ and there is a unique lift β of α such that $\mathrm{Res}^2_1(\beta)=0$, then we use β as our lift of α.

For the purposes of the following proposition let us temporarily write $a \rightsquigarrow b$ where b is the notation for unique lift for a.

Proposition 7.3 *There are unique lifts*

$$\overline{e}^{j\rho} \rightsquigarrow \overline{e}^{j\rho}, \qquad \overline{u}_\lambda \overline{e}^{j\rho+\sigma} \rightsquigarrow \overline{e}^{j,u}, \qquad \overline{a}_\lambda \overline{e}^{j\rho+\sigma} \rightsquigarrow \widetilde{e}^{j,a},$$
$$\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma} \rightsquigarrow e^{j,au}, \quad \overline{e}^{j\rho+\lambda}(x+gx) \rightsquigarrow \overline{e}^{j\rho+\lambda}, \quad \overline{e}^{j\rho+\lambda+1}x = \overline{e}^{j\rho+\lambda+1}gx \rightsquigarrow e'^{j\rho+\lambda+1}.$$

These lifts generate $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ as a $k^\bigstar_{C_2}$–module, and we have the relation

$$v\overline{e}^{j\rho+\lambda} = 0.$$

Proof We shall only explain a few of these, as most are immediate from Section 7.4.1 and the description of the E_2 page. The elements $\overline{u}^i_\sigma \overline{e}^{j\rho+\sigma}$ don't survive, but every other multiple of $\overline{e}^{j\rho+\sigma}$ does (since v annihilates them). These multiples are generated (as a $k^\bigstar_{C_2}$–module) by

$$\overline{a}_\lambda \overline{e}^{j\rho+\sigma}, \quad \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}, \quad \overline{u}_\lambda \overline{e}^{j\rho+\sigma}.$$

Note that we don't need to include elements involving $v\overline{e}^{j\rho+\sigma}$, since

$$\frac{v}{\overline{a}^i_\lambda \sqrt{\overline{a}_\lambda \overline{u}_\lambda}^{\,\epsilon} u^k_\lambda}\overline{e}^{j\rho+\sigma} = \frac{v}{\overline{a}^i_\lambda \sqrt{\overline{a}_\lambda \overline{u}_\lambda}^{\,\epsilon} u^{k+1}_\lambda}\overline{u}_\lambda \overline{e}^{j\rho+\sigma},$$

where $i,k \geq 0$ and $\epsilon = 0, 1$.

For each $j \geq 0$, the element $\overline{a}_\lambda \overline{e}^{j\rho+\sigma}$ has two distinct lifts. On E_∞ we have that $\mathrm{Res}^2_1(\overline{a}_\lambda \overline{e}^{j\rho+\sigma}) = 0$, and on $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ only one of the two lifts has trivial restriction.

Similarly, the elements $\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}$ have trivial restriction on E_∞ and unique lifts with trivial restriction on $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$. □

Remark 7.4 We should explain the notation used for the generators above. First, the elements $\overline{e}^{j,u}$ and $\overline{e}^{j\rho+\lambda}$ will turn out to be the restrictions of top level elements $e^{j,u}$ and $e^{j\rho+\lambda}$ respectively, both in E_∞ and in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$, hence their notation. Second, the elements $e^{j,au}$ and $e'^{j\rho+\lambda+1}$ are never restrictions, neither in E_∞ nor in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$, so their notation is rather ad hoc: the au in $e^{j,au}$ serves as a reminder of the $\sqrt{\overline{a}_\lambda \overline{u}_\lambda}$ in $e^{j,au} = \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}$, while the prime $'$ in $e'^{j\rho+\lambda+1}$ is used to distinguish them from the top level generators $e^{j\rho+\lambda+1}$ that the $e'^{j\rho+\lambda+1}$ transfer to. Finally, the elements $\widetilde{e}^{j,a}$ are restrictions of top level elements $e^{j,a}$ in E_∞, but not in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$ due to nontrivial Mackey functor extensions (exotic restrictions). That's why we denote them by $\widetilde{e}^{j,a}$ as opposed to $\overline{e}^{j,a}$; the $\overline{e}^{j,a}$ are reserved for $\mathrm{Res}^4_2(e^{j,a}) = \widetilde{e}^{j,a} + \overline{u}_\sigma e'^{j\rho+\lambda+1}$; see Lemma 8.4.

For convenience, when $j = 0$ we write $\widetilde{e}^a$, e^{au} and $\overline{e}^u$ in place of $\widetilde{e}^{0,a}$, $e^{0,au}$ and $\overline{e}^{0,u}$, respectively.

Now recall that $k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+})$ is freely generated over $k^\bigstar_{C_2}$ under the elements $e^{j\rho_2}$ and $e^{j\rho_2+\sigma_2}$; see Section 4.1. We shall write our middle level C_4 generators in terms of the C_2 generators.

Proposition 7.5 *We have*

$$\begin{aligned}
\overline{e}^{j\rho} &= \overline{u}_\sigma^j e^{2j\rho_2}, & \overline{e}^{j,u} &= \overline{u}_\sigma^{j+1} u_{\sigma_2} e^{2j\rho_2+\sigma_2},\\
e^{j,au} &= \overline{u}_\sigma^{j+1} a_{\sigma_2} e^{2j\rho_2+\sigma_2}, & \widetilde{e}^{j,a} &= \overline{u}_\sigma^{j+1} a_{\sigma_2} e^{(2j+1)\rho_2},\\
\overline{e}^{j\rho+\lambda} &= \overline{u}_\sigma^j a_{\sigma_2} e^{2j\rho_2+\sigma_2} + \overline{u}_\sigma^j u_{\sigma_2} e^{(2j+1)\rho_2}, & e'^{j\rho+\lambda+1} &= \overline{u}_\sigma^j e^{(2j+1)\rho_2+\sigma_2}.
\end{aligned}$$

Proof The map $f\colon E_{C_4}\Sigma_2 \to E_{C_2}\Sigma_2$, $f(x_1,x_2,x_3,x_4,\dots) = (x_1,x_3,x_2,x_4,\dots)$ is a $C_2{\times}\Sigma_2$–equivariant homeomorphism and induces a map on filtrations

$$\begin{tikzcd}
S(1) \arrow[r, hook] \arrow[d, leftrightarrow] & S(1+\sigma) \arrow[r, hook] \arrow[dr] & X \arrow[r, hook] \arrow[dr] & S(\rho) \arrow[r, hook] \arrow[d, leftrightarrow] & \cdots \\
S(1) \arrow[r, hook] & S(\rho_2) \arrow[r, hook] \arrow[ur] & S(1+\rho_2) \arrow[r, hook] \arrow[u] & S(2\rho_2) \arrow[r, hook] & \cdots
\end{tikzcd}$$

(The downwards arrows are f while the arrows in the opposite direction are f^{-1}.) To keep the notation tidy, we verify the correspondence of generators for $j=0$.

In the C_4 spectral sequence, we have $\widetilde{e}^a\overline{u}_\sigma^{-1}$ and $e'^{\lambda+1}$ in degree $\lambda+1$ and filtrations 1 and 3, respectively. In the C_2 spectral sequence, we have $a_{\sigma_2}e^{\sigma_2+1}$ and $e^{2\sigma_2+1}$ in the same degree and filtrations 2 and 3, respectively. The correspondence of filtrations gives

$$e'^{\lambda+1} = e^{2\sigma_2+1} \quad\text{and}\quad \widetilde{e}^a\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2+1} + \epsilon e^{2\sigma_2+1},$$

where $\epsilon = 0, 1$. Applying restriction on the second equation reveals that $\epsilon = 0$ and thus $\widetilde{e}^a\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2+1}$. The correspondence of filtrations in degrees $\lambda-1$ and λ gives

$$\overline{e}^u\overline{u}_\sigma^{-1} = \epsilon_1 a_{\sigma_2}u_{\sigma_2} + u_{\sigma_2}e^{\sigma_2}, \quad e^{au}\overline{u}_\sigma^{-1} = \epsilon_2 a_{\sigma_2}e^{\sigma_2} + \epsilon_3 u_{\sigma_2}e^{\sigma_2+1}, \quad \overline{e}^\lambda = \epsilon_4 a_{\sigma_2}e^{\sigma_2} + \epsilon_5 u_{\sigma_2}e^{\sigma_2+1},$$

where $\epsilon_i = 0, 1$. Applying restriction shows that

$$\epsilon_3 = 0 \quad\text{and}\quad \epsilon_5 = 1,$$

which further forces $\epsilon_2 = 1$. Looking at degree $2\lambda+\sigma-2$ in the C_4 spectral sequence, we see that we have a relation

$$\overline{a}_\lambda\overline{e}^u = \overline{u}_\lambda\widetilde{e}^a + \epsilon_6\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma\overline{e}^\lambda + \epsilon_7\overline{u}_\sigma\overline{u}_\lambda e'^{\lambda+1},$$

where again $\epsilon_i = 0, 1$. Combining the equations above we conclude that

$$\overline{a}_\lambda\overline{e}^u = \overline{u}_\lambda\widetilde{e}^a + \sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma\overline{e}^\lambda,$$

and

$$\overline{e}^u\overline{u}_\sigma^{-1} = u_{\sigma_2}e^{\sigma_2}, \quad e^{au}\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2}, \quad \overline{e}^\lambda = a_{\sigma_2}e^{\sigma_2} + u_{\sigma_2}e^{\sigma_2+1}.$$

To compute the ϵ_i we used the freeness of $k_{C_2}^\bigstar(B_{C_2}\Sigma_{2+})$ over $k_{C_2}^\bigstar$. □

As a corollary we obtain the relations

$$\begin{aligned}
\overline{u}_\lambda \widetilde{e}^{j,a} &= \overline{a}_\lambda \overline{e}^{j,u} + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma \overline{e}^{j\rho+\lambda}, \\
\overline{u}_\lambda e^{j,au} &= \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j,u}, \\
\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\, e^{j,au} &= \overline{a}_\lambda \overline{e}^{j,u}, \\
\overline{a}_\lambda e^{j,au} &= \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\widetilde{e}^{j,a} + \overline{a}_\lambda \overline{u}_\sigma \overline{e}^{j\rho+\lambda}, \\
\frac{v}{\overline{a}_\lambda^i} \overline{e}^{j\rho+\lambda} &= 0.
\end{aligned}$$

Thus, $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ is spanned as a $k^\bigstar_{C_2}$–module by $\overline{e}^{j\rho}$, $\widetilde{e}^{j,a}$, $e^{j,au}$, $\overline{e}^{j,u}$, $\overline{e}^{j\rho+\lambda}$ and $e'^{j\rho+\lambda+1}$ under the relations above. The bottom level $k^\bigstar_e(B_{C_4}\Sigma_{2+})$ is free on the restrictions of $\overline{e}^{j\rho}$, $\overline{e}^{j,u}$, $\overline{e}^{j\rho+\lambda}$ and $e'^{j\rho+\lambda+1}$.

The C_4/C_2 (Weyl group) action is trivial: the only extensions that may arise are $g\widetilde{e}^{j,a} = \widetilde{e}^{j,a} + \epsilon e'^{j\rho+\lambda+1}$ and $ge^{j,au} = e^{j,au} + \epsilon' \overline{e}^{j\rho+\lambda}$ where $\epsilon, \epsilon' = 0, 1$; applying restriction shows that $\epsilon = \epsilon' = 0$.

The cup product structure can be understood in terms of the C_2 generators c and b of Section 4.1. As an algebra, $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ is generated by $\widetilde{e}^a$, e^{au}, $\overline{e}^u$, $\overline{e}^\lambda$, $e'^{\lambda+1}$ and $\overline{e}^\rho$ under multiplicative relations that are implied by the correspondence of generators:

$$\begin{aligned}
&e^\rho = \overline{u}_\sigma b^2, &&\widetilde{e}^a = \overline{u}_\sigma a_{\sigma_2} b, &&e^{au} = \overline{u}_\sigma a_{\sigma_2} c, \\
&\overline{e}^u = \overline{u}_\sigma u_{\sigma_2} c, &&\overline{e}^\lambda = c^2 = a_{\sigma_2} c + u_{\sigma_2} b, &&e'^{\lambda+1} = cb.
\end{aligned}$$

Remark 7.6 The reader may notice that this description of the middle level $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ is rather different from the one given in Proposition 5.1. Let us now explain this discrepancy. First, the relation

$$\overline{u}_\lambda e^{au} = \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u$$

allows us to replace e^{au} by the quotient

$$\frac{\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u}{\overline{u}_\lambda},$$

which is why e^{au} does not appear in Proposition 5.1 but $(\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u)/\overline{u}_\lambda$ does. Second, in Lemma 8.4, we shall see that $\widetilde{e}^a + \overline{u}_\sigma e'^{\lambda+1}$ is the restriction of a top level generator e^a, which we denote by $\overline{e}^a$. We can replace the generator $\widetilde{e}^a$ by the element $\overline{e}^a$ and get the relation

$$\overline{u}_\sigma \overline{u}_\lambda e'^{\lambda+1} = \overline{u}_\lambda \overline{e}^a + \overline{a}_\lambda \overline{e}^u + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma \overline{e}^\lambda.$$

Thus we can replace the generator $e'^{\lambda+1}$ by the quotient

$$\frac{\overline{u}_\sigma^{-1} \overline{a}_\lambda \overline{e}^u + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^\lambda}{\overline{u}_\lambda},$$

which is what we do in the description of the middle level $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ found in Proposition 5.1. For our convenience, we shall continue to use the generators $\widetilde{e}^{j,a}$, $e^{j,au}$ and $e'^{j\rho+\lambda+1}$ in the following subsections, instead of their replacements.

7.5 Top level differentials

In this subsection, we compute the top level of the E_∞ page.

From Section 7.2, we know that (the top level of) the E_2 page is generated by

$$e^{j\rho}, \quad \alpha e^{j\rho+\sigma}, \quad \bar{u}_\sigma^{-i}\sqrt{\bar{a}_\lambda \bar{u}_\lambda}^{\epsilon} e^{j\rho+\lambda+\epsilon'},$$

where $i, j \geq 0$, $\epsilon, \epsilon' = 0, 1$ and $\alpha \in \mathrm{Ker}(k_\bigstar^{C_4} \xrightarrow{\mathrm{Res}_2^4} k_\bigstar^{C_2} \xrightarrow{v} k_\bigstar^{C_2})$. We also have the relation

$$ve^{j\rho+\lambda} = 0.$$

For degree reasons, the elements $e^{j\rho}$ survive the spectral sequence.

The elements $\bar{u}_\sigma^{-i} e^{j\rho+\lambda+1}$ and $\bar{u}_\sigma^{-i}\sqrt{\bar{a}_\lambda \bar{u}_\lambda} e^{j\rho+\lambda+1}$ are transfers, hence also survive (by the middle level computation of Section 7.4).

If $\alpha \in k_\bigstar^{C_4}$ is a transfer then so are the elements $\alpha e^{j\rho+\sigma}$ and thus they survive. The elements $\alpha \in k^\bigstar_{C_4} \setminus \{u_\sigma^m, m \geq 0\}$ that are not transfers can be broken into three categories:

- multiples of a_λ,
- multiples of u_λ / u_σ^i,
- $a_\sigma u_\sigma^i$.

Proposition 7.7 *The elements $a_\lambda e^{j\rho+\sigma}$ survive the spectral sequence, while the elements $a_\sigma u_\sigma^i e^{j\rho+\sigma}$ support nontrivial differentials*

$$d^2(a_\sigma u_\sigma^i e^{j\rho+\sigma}) = v\bar{u}_\sigma^{i+2} e^{j\rho+\lambda+1} \quad \text{for } i, j \geq 0.$$

Proof The elements $a_\lambda e^{j\rho+\sigma}$ can only support $d^3(a_\lambda e^{j\rho+\sigma}) = e^{(j+1)\rho}$ and applying restriction shows that this cannot happen.

Fix $j \geq 0$. For degree reasons, the only differential $a_\sigma e^{j\rho+\sigma}$ can support is $d^2(a_\sigma e^{j\rho+\sigma}) = v\bar{u}_\sigma^2 e^{j\rho+\lambda+1}$. If $a_\sigma e^{j\rho+\sigma}$ survives then it lifts to a unique element α of $k_{C_4}^{j\rho+2\sigma}(B_{C_4}\Sigma_{2+})$, while $a_\lambda e^{j\rho+\sigma}$ has two possible lifts to $k_{C_4}^{j\rho+2\sigma}(B_{C_4}\Sigma_{2+})$ that differ by $\mathrm{Tr}_2^4(u_\sigma e'^{j\rho+\lambda+1})$. Both lifts have the same restriction, which by Lemma 8.4 is computed to be $\tilde{e}^{j,a} + \bar{u}_\sigma e'^{j\rho+\lambda+1}$ (the proof of the lemma works regardless of the survival of $a_\sigma e^{j\rho+\sigma}$). Now one of those lifts, that we shall call β, satisfies

$$\frac{a_\sigma^2}{a_\lambda}\beta = a_\sigma \alpha$$

in $k_{C_4}^{j\rho+3\sigma}(B_{C_4}\Sigma_{2+})$. Applying Res_2^4 gives that

$$\mathrm{Res}_2^4\left(\frac{a_\sigma^2}{a_\lambda}\right)\mathrm{Res}_2^4(\beta) = 0 \Longrightarrow v\bar{u}_\sigma^2(\tilde{e}^{j,a} + \bar{u}_\sigma e'^{j\rho+\lambda+1}) = 0 \Longrightarrow v\bar{u}_\sigma^3 e'^{j\rho+\lambda+1} = 0,$$

which contradicts the computation of the module structure of the middle level. □

This differential is depicted by a dashed arrow in Figure 5, top. It does not appear in Figure 5 center and bottom.

Remark 7.8 The nonsurvival of $a_\sigma e^\sigma$ is consistent with the computation that $k_{C_4}^{2\sigma}$ has dimension 1 (spanned by $a_\sigma^2 e^0$) by the computer program of [2].

All the other elements of E_2 survive the spectral sequence:

Proposition 7.9 *The elements* $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$, $\overline{u}_\sigma^{-i}e^{j\rho+\lambda}$ *and* $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}$ *survive the spectral sequence for* $i, j \geq 0$.

Proof We work page by page. On E_2 we have

$$d^2(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}) = \epsilon_1\frac{\theta}{a_\sigma u_\sigma^{i-2}}e^{(j+1)\rho}, \quad d^2(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}) = \epsilon_2\frac{\theta}{a_\sigma^2 u_\sigma^{i-3}}a_\lambda e^{(j+1)\rho} + \epsilon_3\frac{u_\lambda}{u_\sigma^{i+1}}e^{(j+1)\rho},$$

where $\epsilon_i = 0, 1$. Multiplying by a_σ and using that $a_\sigma\overline{u}_\sigma^{-i}e^{j\rho+\lambda} = 0$ and that $a_\sigma\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda} = 0$ shows that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.

On E_3 we have

$$d^3(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}) = \epsilon_1\frac{\theta}{a_\sigma^2 u_\sigma^{i-2}}e^{(j+1)\rho+\sigma},$$
$$d^3(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}) = \epsilon_2\frac{\theta}{a_\sigma^3 u_\sigma^{i-3}}a_\lambda e^{(j+1)\rho+\sigma},$$
$$d^3\left(\frac{u_\lambda}{u_\sigma^i}e^{j\rho+\sigma}\right) = \epsilon_3\frac{\theta}{a_\sigma^2 u_\sigma^{i-4}}e^{(j+1)\rho},$$

where again $\epsilon_i = 0, 1$. We see that $\epsilon_1 = \epsilon_2 = 0$ by multiplication with a_σ, while $\epsilon_3 = 0$ can be seen by multiplying with a_σ^2.

The pattern of higher differentials is the same as in E_2 and E_3, and the same arguments show that there are no higher differentials. □

In conclusion:

Corollary 7.10 *The* E_∞ *page is generated as a* $k_{C_4}^\bigstar$*–module by*

$$e^{j\rho}, \quad a_\lambda e^{j\rho+\sigma}, \quad (u_\lambda/u_\sigma^i)e^{j\rho+\sigma}, \quad \overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}^{\,\epsilon}e^{j\rho+\lambda+\epsilon'},$$

where $i, j \geq 0$ *and* $\epsilon, \epsilon' = 0, 1$. *We have relations*

$$v\overline{u}_\sigma e^{j\rho+\lambda} = v\overline{u}_\sigma^2 e^{j\rho+\lambda+1} = 0.$$

8 Lifts and extension problems

8.1 Coherent lifts

If we have a top level element $\alpha \in E_\infty^{s,V}$ and $E_\infty^{t,V}$ vanishes for $t > s$, then α lifts uniquely to $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$. If on the other hand $E_\infty^{t,V}$ does not vanish for some $t > s$, then there are multiple choices of lifts of α.

When it comes to fractions y/x, we should make sure our choices of lifts are "coherent". Let us explain what that means with an example. The element $u_\lambda e^\sigma$ has a unique lift x_0, while $(u_\lambda/u_\sigma^i)e^\sigma$ has multiple distinct lifts if $i \geq 5$. If we choose x_i to lift $(u_\lambda/u_\sigma^i)e^\sigma$ then it will always be true that $u_\sigma^i x_i = x_0$; however, we shouldn't write $x_i = x_0/u_\sigma^i$ unless we can also guarantee that

$$u_\sigma x_i = x_{i-1}.$$

This expresses the coherence of fractions (also discussed in Section 3.2 and Appendix B) which is the cancellation property,

$$u_\sigma \frac{u_\lambda}{u_\sigma^i} e^\sigma = \frac{u_\lambda}{u_\sigma^{i-1}} e^\sigma.$$

This holds on E_∞, and we also want it to hold on $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$.

One more property enjoyed by the $(u_\lambda/u_\sigma^i)e^\sigma$ is that $a_\sigma^2(u_\lambda/u_\sigma^i)e^\sigma = 0$; it turns out that there are unique lifts x_i of $(u_\lambda/u_\sigma^i)e^\sigma$ such that $a_\sigma^2 x_i = 0$, and those lifts also satisfy the coherence property $u_\sigma x_i = x_{i-1}$:

Proposition 8.1 *For $i, j \geq 0$, there are unique lifts $e^{j,u}/u_\sigma^i$ and $e^{j\rho+\lambda}/u_\sigma^i$ of the elements $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ and $\bar{u}_\sigma^{-i} e^{j\rho+\lambda}$, respectively, that satisfy*

$$a_\sigma^2 \frac{e^{j,u}}{u_\sigma^i} = 0 \quad \textit{and} \quad a_\sigma^2 \frac{e^{j\rho+\lambda}}{u_\sigma^i} = 0.$$

These lifts are also coherent.

Proof Fix $i, j \geq 0$. We first deal with lifts of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$.

- **Existence** Fix $\bigstar$ to be the $\mathrm{RO}(C_4)$ degree of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ and write F^s for the decreasing filtration on $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ defining the spectral sequence, namely

$$E_\infty^{s,\bigstar} = F^s/F^{s+1}.$$

We start with any random lift $\alpha_0 \in F^{4j+1}$ of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$; if $a_\sigma^2\alpha_0 = 0$ then we are done. Otherwise take s_0 maximal with $a_\sigma^2\alpha_0 \in F^{s_0}$; since $a_\sigma^2(u_\lambda/u_\sigma^i)e^{j\rho+\sigma} = 0$ we have $s_0 > 4j+1$. In fact $s_0 > 4j+2$ since $E^{4j+2,\bigstar} = 0$.

We now prove that $s_0 > 4j+3$: $E^{4j+3,\bigstar}$ is spanned by $\bar{u}_\sigma^{3-i} e^{j\rho+\lambda+1}$ so we need to investigate the possibility $a_\sigma^2\alpha = \bar{u}_\sigma^{3-i} e^{j\rho+\lambda+1}$ on $E^{4j+3,\bigstar}$. Multiplying by u_σ^i reduces us to the case $i = 0$, where $u_\sigma^i\alpha$ is the unique lift of $u_\lambda e^{j\rho+\sigma}$. But we can see directly that $(a_\sigma^2/a_\lambda)u_\sigma^i\alpha = 0$ for degree reasons, hence $a_\sigma^2 u_\sigma^i \alpha = 0$ as well.

As $s_0 > 4j+3$, we can see directly that $F^{s_0}/F^{s_0+1} = E_\infty^{s_0,\bigstar}$ is generated by an element βe^V where $\beta \in k_{C_4}^{\bigstar-V}$ is divisible by a_σ^2. If $\alpha' \in F^{s_0}$ is a lift of $(\beta/a_\sigma^2)e^V$ then $\alpha_1 = \alpha_0 + \alpha'$ is a lift of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$. If $a_\sigma^2\alpha_1 = 0$ then we are done, otherwise $a_\sigma^2\alpha_1 \in F^{s_1}$ for $s_1 > s_0$ so we get α_2 by the same argument as above. Since $F^s = 0$ for large enough s, this inductive process will eventually end with the desired lift.

- **Uniqueness** If α and α' are two lifts of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$, then their difference is a finite sum p of elements $\beta' e^V$, where each $\beta' \in k_{C_4}^\bigstar$ is a fraction with a_σ^2 in its denominator. If $a_\sigma^2\alpha = a_\sigma^2\alpha' = 0$ then $a_\sigma^2 p = 0 \Longrightarrow a_\sigma^2\beta' = 0 \Longrightarrow \beta' = 0 \Longrightarrow p = 0$.

- **Coherence** Unfix i and let x_i be the lift of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ with $a_\sigma^2 x_i = 0$. Then $u_\sigma x_i$ is a lift of $(u_\lambda/u_\sigma^{i-1})e^{j\rho+\sigma}$ and $a_\sigma^2(u_\sigma x_i) = 0$, hence by uniqueness,
$$u_\sigma x_i = x_{i-1}.$$

The case of $\overline{u}_\sigma^{-i} e^{j\rho+\lambda}$ is near identical to what we did above for $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$. The changes are as follows. First, $s_0 > 4j+2$ (instead of $s_0 > 4j+1$). Next, we can see that $s_0 > 4j+4$ if $i > 1$, and multiplying by u_σ also proves the $i = 0, 1$ cases (this replaces the argument that showed $s_0 > 4j+3$). The rest of the arguments are identical. □

8.2 Top-level generators

The elements $e^{j\rho}$ have unique lifts to $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$, which we continue to denote by $e^{j\rho}$.

On the other hand, for each $j \geq 0$ there are two possible lifts of $a_\lambda e^{j\rho+\sigma}$. There is no good way to make a unique choice at this point, so we shall write $e^{j,a}$ for either.

In this subsection we shall prove:

Proposition 8.2 *The $k_{C_4}^\bigstar$–module $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ is generated by*
$$e^{j\rho}, \quad e^{j,a}, \quad \frac{e^{j,u}}{u_\sigma^i}, \quad \frac{e^{j\rho+\lambda}}{u_\sigma^i}, \quad \text{where } i, j \geq 0.$$

By Corollary 7.10 it suffices to prove that the $k_{C_4}^\bigstar$–module generated by the elements $e^{j\rho}$, $e^{j,a}$, $e^{j,u}/u_\sigma^i$ and $e^{j\rho+\lambda}/u_\sigma^i$ contains lifts of the elements
$$\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1}, \quad \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma^{-i} e^{j\rho+\lambda}, \quad \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1} \quad \in E_\infty.$$

Lemma 8.3 *The elements*
$$\frac{e^{j\rho+\lambda+1}}{u_\sigma^i} := \mathrm{Tr}_2^4(\overline{u}_\sigma^{-i} e'^{j\rho+\lambda+1})$$
are coherent lifts of $\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1} \in E_\infty$. Furthermore,
$$a_\sigma \frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{e^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.$$

Proof We see directly that $\mathrm{Tr}_2^4(\overline{u}_\sigma^{-i} e'^{j\rho+\lambda+1})$ lift $\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1}$ and coherence follows from the Frobenius relations.

Next, we see directly from the E_∞ page that $e^{j\rho+\lambda}$ is not in the image of the transfer Tr_2^4. Since $\mathrm{Ker}(a_\sigma)=\mathrm{Im}(\mathrm{Tr}_2^4)$ in $k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+})$, we must have a module extension of the form

$$a_\sigma e^{j\rho+\lambda}=u_\sigma e^{j\rho+\lambda+1}.$$

By Proposition 8.1, $a_\sigma^2 e^{j\rho+\lambda}/u_\sigma^i=0$, hence $a_\sigma e^{j\rho+\lambda}/u_\sigma^i$ is a transfer. The equation above shows that $a_\sigma e^{j\rho+\lambda}/u_\sigma^i\neq 0$, and the only way $a_\sigma e^{j\rho+\lambda}/u_\sigma^i$ can be a nonzero transfer is for $a_\sigma e^{j\rho+\lambda}/u_\sigma^i=\mathrm{Tr}_2^4(\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1})$. □

Before we can lift the rest of the E_∞ generators, we will need the following exotic restriction:

Lemma 8.4 *Both choices of $e^{j,a}$ have the same (exotic) restriction*

$$\mathrm{Res}_2^4(e^{j,a})=\tilde{e}^{j,a}+\overline{u}_\sigma e'^{j\rho+\lambda+1}.$$

Proof The two choices of $e^{j,a}$ differ by $u_\sigma e^{j\rho+\lambda+1}=\mathrm{Tr}_2^4(\overline{u}_\sigma e'^{j\rho+\lambda+1})$ hence have the same restriction. From the E_∞ page,

$$\mathrm{Res}_2^4(e^{j,a})=\tilde{e}^{j,a}+\epsilon\overline{u}_\sigma e'^{j\rho+\lambda+1},$$

where $\epsilon=0,1$. Transferring this gives

$$\mathrm{Tr}_2^4(\tilde{e}^{j,a})=\epsilon u_\sigma e^{j\rho+\lambda+1}.$$

Now transferring the middle level relation

$$\overline{u}_\lambda\tilde{e}^{j,a}=\overline{a}_\lambda\overline{e}^{j,u}+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma\overline{e}^{j\rho+\lambda}$$

shows that

$$u_\lambda\,\mathrm{Tr}_2^4(\tilde{e}^{j,a})=a_\sigma u_\lambda e^{j\rho+\lambda}$$

and thus $\mathrm{Tr}_2^4(\tilde{e}^{j,a})\neq 0$, which proves $\epsilon=1$. □

Lemma 8.5 *The elements*

$$\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i}:=\frac{u_\lambda}{u_\sigma^{i+1}}e^{j,a}+a_\lambda\frac{e^{j,u}}{u_\sigma^{i+1}}$$

are coherent lifts of $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}\in E_\infty$.

Proof Fix $i,j\geq 0$ and fix $\bigstar$ to be the degree of the element

$$\frac{u_\lambda}{u_\sigma^i}e^{j,a}+a_\lambda\frac{e^{j,u}}{u_\sigma^i}.$$

This element is by definition in filtration $4j+1$, however its projection to $E_\infty^{4j+1,\bigstar}$ is

$$\frac{u_\lambda}{u_\sigma^i}a_\lambda e^{j\rho+\sigma}+a_\lambda\frac{u_\lambda}{u_\sigma^i}e^{j\rho+\sigma}=0,$$

so it is actually in filtration $4j+2$. But observe that $E_\infty^{4j+2,\bigstar}$ is generated by $\overline{u}_\sigma^{-i+1}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}$, so it suffices to check that

$$\frac{u_\lambda}{u_\sigma^i}e^{j,a}+a_\lambda\frac{e^{j,u}}{u_\sigma^i}$$

is not 0 when projected to $E_\infty^{4j+2,\bigstar}$. Multiplying by u_σ^i reduces us to the case $i=0$, and then

$$\begin{aligned}\operatorname{Res}_2^4(u_\lambda e^{j,a}+a_\lambda e^{j,u}) &= \overline{u}_\lambda\tilde{e}^{j,a}+\overline{u}_\sigma\overline{u}_\lambda e'^{j\rho+\lambda+1}+\overline{a}_\lambda\overline{e}^{j,u}\\ &= \overline{u}_\sigma\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^{j\rho+\lambda}+\overline{u}_\sigma\overline{u}_\lambda e'^{j\rho+\lambda+1}\end{aligned}$$

using Lemma 8.4 and the middle level computation of Section 7.4. Projecting this restriction to $E_\infty^{4j+2,\bigstar}$ returns

$$\overline{u}_\sigma\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^{j\rho+\lambda}\neq 0,$$

as desired.

Coherence of $e_{\surd}^{j\rho+\lambda}/u_\sigma^i$ follows from the coherence of u_λ/u_σ^i and $e^{j,u}/u_\sigma^i$. □

Lemma 8.6 *The elements*

$$\frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^i}:=\operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e'^{j\rho+\lambda+1})$$

are coherent lifts of $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}\in E_\infty$. *Furthermore,*

$$a_\sigma\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i}=\frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.$$

Proof The fact that these transfers are lifts follows from the E_∞ page; coherence follows from the Frobenius relations. We check the equality directly:

$$\begin{aligned}a_\sigma\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i} &= \frac{a_\sigma u_\lambda}{u_\sigma^{i+1}}e^{j,a}+a_\lambda\frac{a_\sigma e^{j,u}}{u_\sigma^{i+1}}\\ &= \operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda})e^{j,a}+a_\lambda\operatorname{Tr}_2^4(e^{j,au}\overline{u}_\sigma^{-i})\\ &= \operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\tilde{e}^{j,a}+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1}+\overline{a}_\lambda e^{j,au}\overline{u}_\sigma^{-i})\\ &= \operatorname{Tr}_2^4(\overline{a}_\lambda\overline{u}_\sigma^{-i+1}\overline{e}^{j\rho+\lambda}+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1})\\ &= \operatorname{Tr}_2^4(\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1})\\ &= \frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.\end{aligned}$$

We used the middle level relation $\overline{a}_\lambda e^{j,au}=\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\tilde{e}^{j,a}+\overline{a}_\lambda\overline{u}_\sigma\overline{e}^{j\rho+\lambda}$ and the fact that $\overline{u}_\sigma^{-i}\overline{e}^{j\rho+\lambda}$ is the restriction of $e^{j\rho+\lambda}/u_\sigma^i$, which follows from the same fact on E_∞. □

Lemmas 8.3, 8.5 and 8.6 combined with Corollary 7.10 prove Proposition 8.2.

8.3 Mackey functor structure

Proposition 8.7 *The Mackey functor structure of $k^\bigstar(B_{C_4}\Sigma_{2+})$ is determined by*

$$\mathrm{Res}^4_2(e^{j\rho})=\overline{e}^{j\rho},\qquad \mathrm{Res}^4_2\left(\frac{e^{j,u}}{u^i_\sigma}\right)=\overline{e}^{j,u}\overline{u}_\sigma^{-i},\qquad \mathrm{Res}^4_2\left(\frac{e^{j\rho+\lambda}}{u^i_\sigma}\right)=\overline{e}^{j\rho+\lambda}\overline{u}_\sigma^{-i},$$

$$\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j,u}}{u^{i+1}_\sigma},\quad \mathrm{Tr}^4_2(e'^{j\rho+\lambda+1}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j\rho+\lambda}}{u^{i+1}_\sigma},\qquad \mathrm{Res}^4_2(e^{j,a})=\widetilde{e}^{j,a}+\overline{u}_\sigma e'^{j\rho+\lambda+1},$$

where $i,j\geq 0$.

Proof We can see directly that there are no Mackey functor extensions for $e^{j\rho}$, $e^{j,u}/u^i_\sigma$ and $e^{j\rho+\lambda}/u^i_\sigma$. The rest were established in the previous two subsections, apart from

$$\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j,u}}{u^{i+1}_\sigma}.$$

To see this, recall that $a^2_\sigma(e^{j,u}/u^i_\sigma)=0$, hence $a_\sigma(e^{j,u}/u^i_\sigma)$ is a transfer. Moreover, $a_\sigma(e^{j,u}/u^i_\sigma)\neq 0$, which is seen on the E_∞ page, and the only way that $a_\sigma(e^{j,u}/u^i_\sigma)$ can be a nonzero transfer is for $a_\sigma(e^{j,u}/u^i_\sigma)=\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})$. □

We did not list $\mathrm{Tr}^4_2(\widetilde{e}^{j,a})=a_\sigma e^{j\rho+\lambda}$ as this immediately follows by applying Tr^4_2 on

$$\mathrm{Res}^4_2(e^{j,a})=\widetilde{e}^{j,a}+\overline{u}_\sigma e'^{j\rho+\lambda+1}.$$

8.4 Top level module relations

With the exception of relations expressing coherence (ie $u_\sigma(e^{j,u}/u^i_\sigma)=e^{j,u}/u^{i-1}_\sigma$ and $u_\sigma(e^{j\rho+\lambda}/u^i_\sigma)=e^{j\rho+\lambda}/u^{i-1}_\sigma$), the rest of the module relations are given as follows.

Proposition 8.8 *The $k^\bigstar_{C_4}$–module $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$ is generated by*

$$e^{j\rho},\quad e^{j,a},\quad \frac{e^{j,u}}{u^i_\sigma},\quad \frac{e^{j\rho+\lambda}}{u^i_\sigma},$$

under the relations

$$\frac{a^2_\sigma}{a^m_\lambda}\frac{e^{j,u}}{u^i_\sigma}=0\quad\text{and}\quad \frac{(\theta/a_\lambda)a_\sigma}{u^{i-2}_\sigma a^{m-1}_\lambda}e^{j,a}+\frac{s}{u^{i-1}_\sigma a^{m-2}_\lambda}e^{j,u}=\frac{a^2_\sigma}{a^m_\lambda}\frac{e^{j\rho+\lambda}}{u^i_\sigma}$$

for $i,j,m\geq 0$.

Proof For $m>0$, we have the possible extensions

$$\frac{a^2_\sigma}{a^m_\lambda}\frac{e^{j,u}}{u^i_\sigma}=\sum_*\epsilon_*\frac{\theta}{a^*_\sigma u^*_\sigma a^*_\lambda}e^{*\rho+*},$$

where each $*$ denotes a nonnegative index (with different instances of $*$ being possibly different indices) and each $\epsilon_* = 0, 1$. Thus, multiplication by a_λ is an isomorphism for both sides — recall that a_λ acts invertibly on elements of the form $\theta/(a_\sigma^* u_\sigma^*)$ and a_σ^2 — which reduces us to $m = 1$. For $m = 1$ and $i > 0$ there are no extensions as there are no elements of the degree of $(a_\sigma^2/a_\lambda)(e^{j,u}/u_\sigma^i)$ in the right-hand side; in other words, $\epsilon_* = 0$ for all $*$. This establishes

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j,u}}{u_\sigma^i} = 0.$$

Similarly, if $m > 0$, we have the possible extensions

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{s}{u_\sigma^{i-2}a_\lambda^{m-1}}e_{\surd}^{j\rho+\lambda} + \sum_* \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*}e^{*\rho+*},$$

and multiplying with a_λ^m reduces us to

$$a_\sigma^2\frac{e^{j\rho+\lambda}}{u_\sigma^i} = \sum_* \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^{*-m}}e^{*\rho+*}.$$

But

$$a_\sigma^2\frac{e^{j\rho+\lambda}}{u_\sigma^i} = a_\sigma\frac{e^{j\rho+\lambda+1}}{u_\sigma^{i-1}} = a_\sigma \operatorname{Tr}_2^4(e'^{j\rho+\lambda+1}\overline{u}_\sigma^{-i+1}) = 0,$$

hence $\epsilon_* = 0$ for all $*$. Thus

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{s}{u_\sigma^{i-2}a_\lambda^{m-1}}e_{\surd}^{j\rho+\lambda},$$

and substituting

$$e_{\surd}^{j\rho+\lambda} = \frac{u_\lambda}{u_\sigma}e^{j,a} + a_\lambda\frac{e^{j,u}}{u_\sigma}$$

gives the desired relation. For $i = m = 1$ we get $(a_\sigma^2/a_\lambda)(e^{j\rho+\lambda+1}/u_\sigma) = 0$. □

As special cases, for $i, j, m \geq 0$ we get the relations

$$a_\sigma^2\frac{e^{j\rho+\lambda}}{u_\sigma^i} = 0, \quad \frac{a_\sigma^2}{a_\lambda}\frac{e^{j\rho+\lambda}}{u_\sigma} = 0, \quad \frac{\frac{\theta}{a_\lambda}a_\sigma}{u_\sigma^i}e^{j,a} = \frac{a_\sigma^2}{a_\lambda}\frac{e^{j\rho+\lambda}}{u_\sigma^{i+2}}, \quad \frac{s}{a_\lambda^m}e^{j,u} = \frac{a_\sigma^2}{a_\lambda^{m+2}}\frac{e^{j\rho+\lambda}}{u_\sigma}.$$

8.5 Top level cup products

Proposition 8.9 *As a $k_{C_4}^\bigstar$ algebra, $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ is generated by e^a, e^u/u_σ^i, e^λ/u_σ^i and e^ρ.*

Proof First of all, $e^{j\rho} = (e^\rho)^j$ since there are no extensions in degree $j\rho$ (to see that $(e^\rho)^j \neq 0$, apply restriction). Let A be the algebra span of e^a, e^u/u_σ^i, e^λ/u_σ^i and e^ρ. To see that $e^{j,a} \in A$ observe

$$e^{j\rho}e^a = \epsilon a_\sigma a_\lambda e^{j\rho} + e^{j,a} + \epsilon' u_\sigma e^{j\rho+\lambda+1},$$

and since

$$e^{j\rho+\lambda+1} = \operatorname{Tr}_2^4(e'^{j\rho+\lambda+1}) = \operatorname{Tr}_2^4(e^{j\rho}e'^{\lambda+1}) = e^{j\rho}e^{\lambda+1},$$

we get that $e^{j,a} \in A$ regardless of the status of ϵ and ϵ'.

Now suppose by induction that all elements in filtration $\le 4j$ are in A. We have that

$$e^{j\rho}\frac{e^u}{u_\sigma^i} = \cdots + \frac{e^{j,u}}{u_\sigma^i} + \sum \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*} e^{*\rho} + \sum \epsilon'_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*} e^{*,a},$$

where $\cdots$ are in filtration $< 4j+1$, hence in A. Since $e^{*\rho}, e^{*,a} \in A$ for any $* \ge 0$, we get $e^{j,u}/u_\sigma^i \in A$. This establishes that everything in filtration $\le 4j+1$ is in A.

Finally,

$$e^{j\rho}\frac{e^\lambda}{u_\sigma^i} = \cdots + \frac{e^{j\rho+\lambda}}{u_\sigma^i} + \sum \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*} e^{*\rho} + \sum \epsilon'_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*} e^{*,a},$$

where $\cdots$ are in filtration $< 4j+2$, so by the same argument $e^{j\rho+\lambda}/u_\sigma^i \in A$ as well. This completes the induction step. □

Inverting u_σ and u_λ gives

$$k^{hC_4\bigstar}[e^\rho, e^a, e^u, e^\lambda]$$

modulo relations, which is isomorphic to

$$k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+}) = k[a_\sigma, a_\lambda, u_\sigma^\pm, u_\lambda^\pm, w]/a_\sigma^2, \quad \text{where } |w| = 1.$$

There are two possible choices for w, differing by $a_\sigma u_\sigma^{-1}$, but both work equally well for the following arguments.

Proposition 8.10 *After potentially replacing the generators e^a, e^u/u_σ^i and e^λ/u_σ^i with algebra generators in the same degrees of $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ and satisfying the same already established relations, the localization map*

$$k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+}) \to k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+})$$

is given by

$$\begin{aligned} &e^u \mapsto u_\sigma u_\lambda w, && e^\lambda \mapsto u_\lambda w^2, \\ &e^a \mapsto u_\sigma u_\lambda w^3 + u_\sigma a_\lambda w, && e^\rho \mapsto u_\sigma u_\lambda w^4 + a_\sigma u_\lambda w^3 + u_\sigma a_\lambda w^2 + a_\sigma a_\lambda w. \end{aligned}$$

Proof Using the C_2 result (see Section 4.1), we have the correspondence on the middle level generators:

- $\overline{e}^u \mapsto \overline{u}_\sigma \overline{u}_\lambda w$,
- $e^\lambda \mapsto \overline{u}_\lambda w^2$,
- $\mathrm{Res}_2^4(e^a) \mapsto \overline{u}_\sigma(\overline{u}_\lambda w^3 + \overline{a}_\lambda w)$,
- $\overline{e}^\rho \mapsto \overline{u}_\sigma(\overline{a}_\lambda w^2 + \overline{u}_\lambda w^4)$,

from which we can deduce that the correspondence on top level is

- $e^u \mapsto u_\sigma u_\lambda w + \epsilon_1 a_\sigma u_\lambda$,
- $e^\lambda \mapsto u_\lambda w^2 + \epsilon_2 a_\sigma u_\sigma^{-1} u_\lambda w$,
- $e^a \mapsto u_\sigma u_\lambda w^3 + \epsilon_3 a_\sigma u_\lambda w^2 + u_\sigma a_\lambda w$,
- $e^\rho \mapsto u_\sigma u_\lambda w^4 + \epsilon_4 a_\sigma u_\lambda w^3 + u_\sigma a_\lambda w^2 + \epsilon_5 a_\sigma a_\lambda w$,

where the ϵ_i range in $0, 1$.

We may add $\epsilon_1 a_\sigma u_\lambda/u_\sigma^i$ to e^u/u_σ^i to force $\epsilon_1=0$; we may add $\epsilon_2 a_\sigma e^u/u_\sigma^{i+2}$ to e^λ/u_σ^i to force $\epsilon_2=0$, and we may add $\epsilon_3 a_\sigma e^\lambda$ to e^a to force $\epsilon_3=0$.

It remains to prove that $\epsilon_4=\epsilon_5=1$. This is a computation based on the Bockstein homomorphism $\beta\colon k_{C_4}^\bigstar(X)\to k_{C_4}^{\bigstar+1}(X)$. For $X=S^0$ we have

$$\beta(a_\sigma)=\beta(a_\lambda)=\beta(u_\lambda)=0 \quad\text{and}\quad \beta(u_\sigma)=a_\sigma.$$

For $X=B_{C_4}\Sigma_{2+}$, we see that $\beta(e^\rho)=0$ for degree reasons ($k_{C_4}^{\rho+1}(B_{C_4}\Sigma_{2+})=0$) and in the homotopy fixed points, $\beta(w)=w^2$ and $\beta(w^3)=w^4$. Thus, applying β on $e^\rho\mapsto u_\sigma u_\lambda w^4+\epsilon_4 a_\sigma u_\lambda w^3+u_\sigma a_\lambda w^2+\epsilon_5 a_\sigma a_\lambda w$ shows that $\epsilon_4=\epsilon_5=1$. $\square$

Proposition 8.11 *In $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ we have the multiplicative relations*

$$\frac{e^u}{u_\sigma^i}\frac{e^u}{u_\sigma^j}=\frac{u_\lambda}{u_\sigma^{i+j-2}}e^\lambda,\qquad \frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j}=\frac{u_\lambda}{u_\sigma^{i+j}}e^a+a_\lambda\frac{e^u}{u_\sigma^{i+j}},$$

$$e^a\frac{e^u}{u_\sigma^i}=\frac{u_\lambda}{u_\sigma^{i-1}}e^\rho+a_\sigma\frac{u_\lambda}{u_\sigma^i}e^a,\qquad \frac{e^\lambda}{u_\sigma^i}\frac{e^\lambda}{u_\sigma^j}=\frac{u_\lambda}{u_\sigma^{i+j+1}}e^\rho+a_\sigma\frac{u_\lambda}{u_\sigma^{i+j+2}}e^a+a_\lambda\frac{e^\lambda}{u_\sigma^{i+j}},$$

$$e^a\frac{e^\lambda}{u_\sigma^i}=\frac{e^u}{u_\sigma^{i+1}}e^\rho+a_\sigma\frac{u_\lambda}{u_\sigma^{i+1}}e^\rho,\qquad (e^a)^2=u_\sigma e^\lambda e^\rho+a_\sigma\frac{e^u}{u_\sigma}e^\rho+u_\sigma a_\lambda e^\rho+a_\sigma a_\lambda e^a.$$

Proof First,

$$\frac{e^u}{u_\sigma^i}\frac{e^u}{u_\sigma^j}=\epsilon_0 a_\lambda\frac{u_\lambda}{u_\sigma^{i+j-2}}+\epsilon_1 a_\sigma\frac{u_\lambda}{u_\sigma^{i+j}}e^u+\epsilon_2\frac{u_\lambda}{u_\sigma^{i+j-2}}e^\lambda+\cdots,$$

where $\epsilon_i=0,1$ and $\cdots$ is the sum of elements mapping to 0 in homotopy fixed points, but all having denominator a_σ^2. Mapping to homotopy fixed points shows $\epsilon_0=\epsilon_1=0$ and $\epsilon_2=1$, while multiplying by a_σ^2 trivializes the LHS (by $a_\sigma^2(e^u/u_\sigma^i)=0$) and thus shows that $\cdots=0$.

The same argument applied to

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j}=\epsilon_0\frac{\theta a_\lambda^2}{a_\sigma u_\sigma^{i+j-2}}+\epsilon_1 a_\sigma a_\lambda\frac{u_\lambda}{u_\sigma^{i+j}}+\epsilon_2 a_\lambda\frac{e^u}{u_\sigma^{i+j}}+\epsilon_3\frac{u_\lambda}{u_\sigma^{i+j}}e^a+\epsilon_4 a_\sigma\frac{u_\lambda}{u_\sigma^{i+j}}e^\lambda+\cdots$$

shows that

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j}=\epsilon_0\frac{\theta a_\lambda^2}{a_\sigma u_\sigma^{i+j-2}}+a_\lambda\frac{e^u}{u_\sigma^{i+j}}+\frac{u_\lambda}{u_\sigma^{i+j}}e^a.$$

There are two ways to show that $\epsilon_0=0$: the first is to multiply with $a_\sigma u_\sigma^{i+j-2}$ and compute $a_\sigma e^\lambda(e^u/u_\sigma^2)$ using $a_\sigma e^\lambda=\mathrm{Tr}_2^4(e'^{\lambda+1})$ together with the Frobenius relation and our knowledge of the multiplicative structure of the middle level from Section 7.4. The alternative is to observe that in the spectral sequence, if a and b live in filtrations $\geq n$ then so does ab. Before the modifications to the generators done in the proof of Proposition 8.10, $e^u/u_\sigma^i, e^a$ were in filtration ≥ 1 and e^λ/u_σ^i were in filtration ≥ 2. Thus, with the original generators, the extension for $e^\lambda e^u/u_\sigma^2$ does not involve the filtration 0 term $a_\lambda^2\theta/a_\sigma$. This is true even after performing the modifications prescribed in the proof of Proposition 8.10, since said modifications never involve terms with θ. Thus $\epsilon_0=0$.

Similarly we have

$$e^a\frac{e^u}{u_\sigma^i}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i-4}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-3}}e^a+\epsilon_2 a_\sigma a_\lambda\frac{e^u}{u_\sigma^i}+\epsilon_3 a_\sigma\frac{u_\lambda}{u_\sigma^i}e^a+\epsilon_4 a_\lambda\frac{e^\lambda}{u_\sigma^{i-2}}+\epsilon_5\frac{u_\lambda}{u_\sigma^{i-1}}e^\rho+\cdots$$

for $i\ge 3$, and mapping to homotopy fixed points and multiplying by a_σ^2 shows that

$$e^a\frac{e^u}{u_\sigma^i}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i-4}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-3}}e^a+a_\sigma\frac{u_\lambda}{u_\sigma^i}e^a+\frac{u_\lambda}{u_\sigma^{i-1}}e^\rho.$$

Multiplying by a_σ and using that $a_\sigma(e^u/u_\sigma^i)=\mathrm{Tr}_2^4(e^{au}\overline{u}_\sigma^{-i})$ shows that $\epsilon_1=0$. To show $\epsilon_0=0$, we use the filtration argument above.

These arguments also work with

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^\lambda}{u_\sigma^j}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i+j-2}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i+j-2}}e^a+\epsilon_2 a_\sigma a_\lambda\frac{e^u}{u_\sigma^{i+j-2}}+\epsilon_3 a_\sigma\frac{u_\lambda}{u_\sigma^{i+j+2}}e^a+\epsilon_4 a_\lambda\frac{e^\lambda}{u_\sigma^{i+j}}$$
$$+\epsilon_5\frac{u_\lambda}{u_\sigma^{i+j+1}}e^\rho+\cdots,$$
$$e^a\frac{e^\lambda}{u_\sigma^i}=\epsilon_6\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-2}}e^a+\epsilon_7 a_\sigma a_\lambda\frac{e^\lambda}{u_\sigma^{i-1}}+\epsilon_8\frac{a_\sigma u_\lambda}{u_\sigma^{i+1}}e^\rho+\epsilon_9\frac{\theta a}{a_\sigma a_\lambda u_\sigma^{i-3}}e^\rho+\epsilon_{10}\frac{u_\lambda}{u_\sigma^{i+1}}e^\rho e^u+\cdots,$$
$$(e^a)^2=\epsilon_{11}a_\sigma^2a_\lambda^2+\epsilon_{12}a_\sigma a_\lambda e^a+\epsilon_{13}u_\sigma a_\lambda e^\rho+\epsilon_{14}a_\sigma\frac{e^u}{u_\sigma}e^\rho+\epsilon_{16}u_\sigma e^\lambda e^\rho,$$

to complete the proof. □

We also have the nontrivial Bockstein

$$\beta(e^u/u_\sigma)=e^\lambda.$$

Appendix A Pictures of the spectral sequence

In this appendix, we have included 15 pictures of the E_1 page of the spectral sequence from Section 7. On each page, the three levels of the spectral sequence are drawn in three separate figures from top to bottom, using (V,s) coordinates. For notational simplicity and due to limited space, we suppress the e^V's and x, gx's from the generators. The e^V's can be recovered by looking at the filtration s (e.g. in filtration $s=4j$ we get $e^{j\rho}$) and to denote the presence of 2–dimensional vector spaces $k\{x,gx\}$ we write k^2 next to each generator. This k^2 is actually a $k[C_4/C_2]$ when considering the C_4 action; we only write k^2 in the diagrams as it is shorter, which helps with alignment.

For example, in Figure 1, top, there is an element $x_{0,1}/u_\sigma^2$ in coordinates $(5,5)$. This represents the fact that the top level of $E_1^{5,5}$ is generated by $(x_{0,1}/u_\sigma^2)e^{\rho+\sigma}$. In Figure 1, center, we have $v\overline{u}_\sigma^{-2}$ in the same

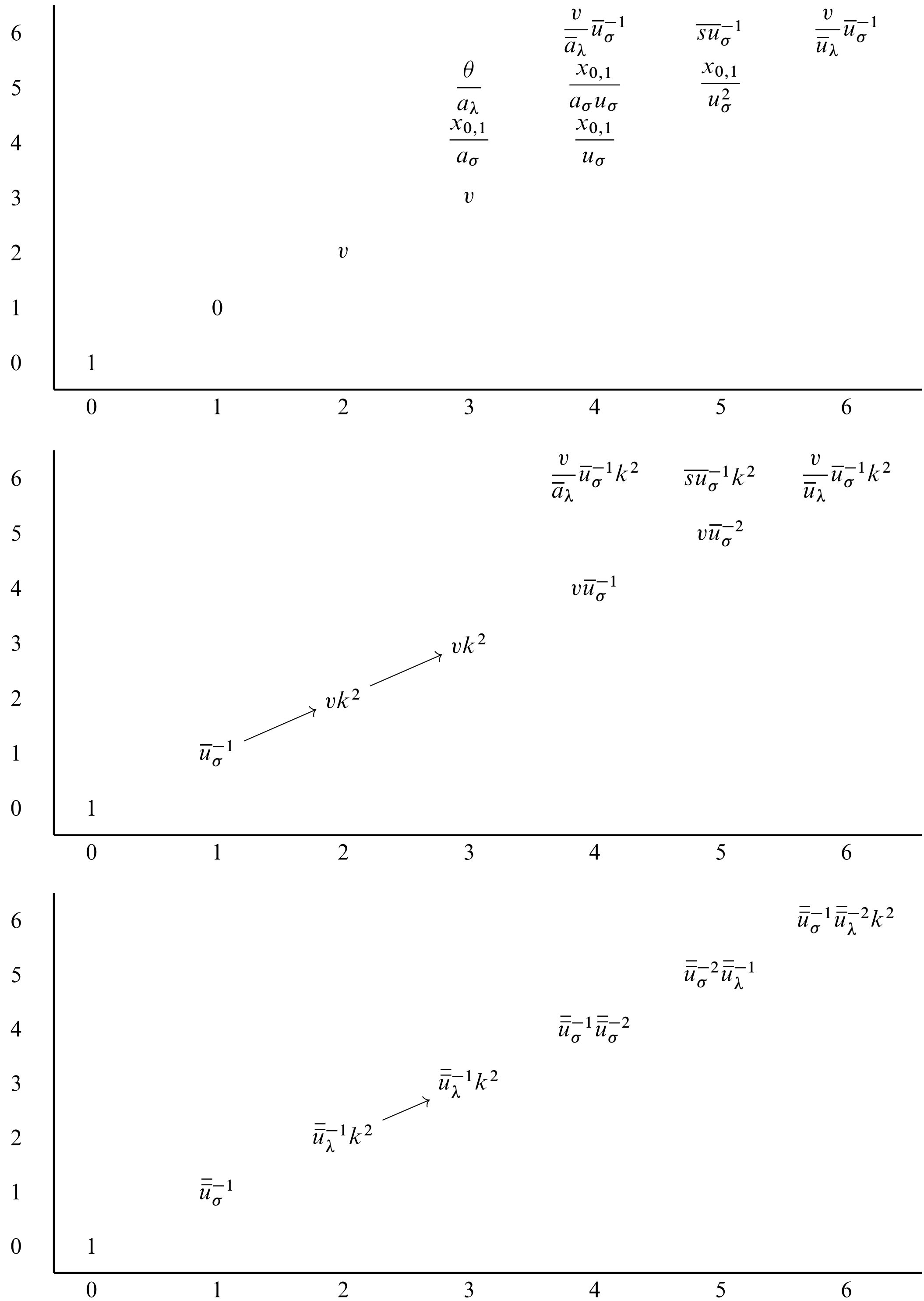

Figure 1: Top: $\underline{E}_1^{*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{*,s}(C_4/e)$, $*\in\mathbb{Z}$.

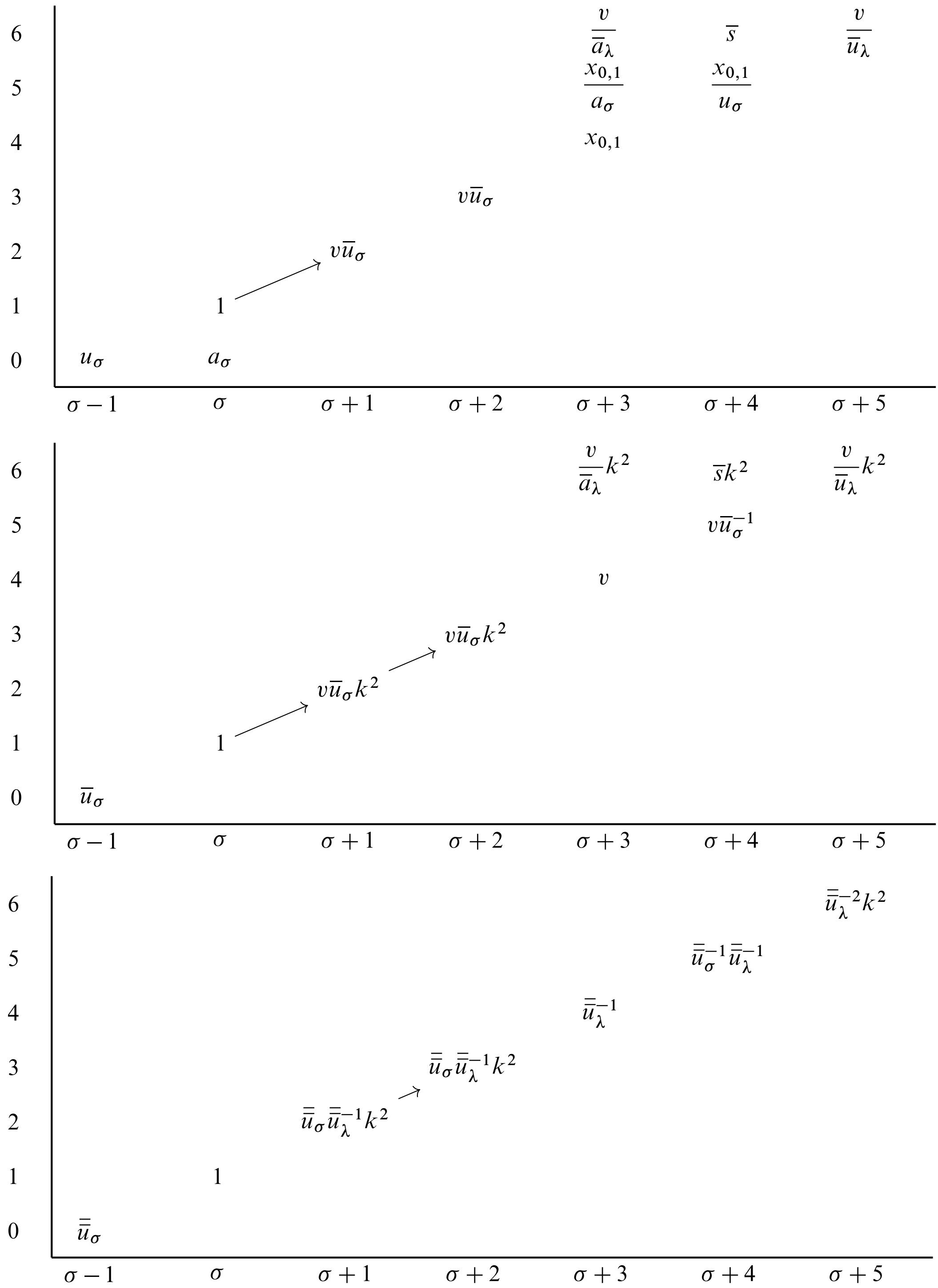

Figure 2: Top: $\underline{E}_1^{\sigma+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{\sigma+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{\sigma+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

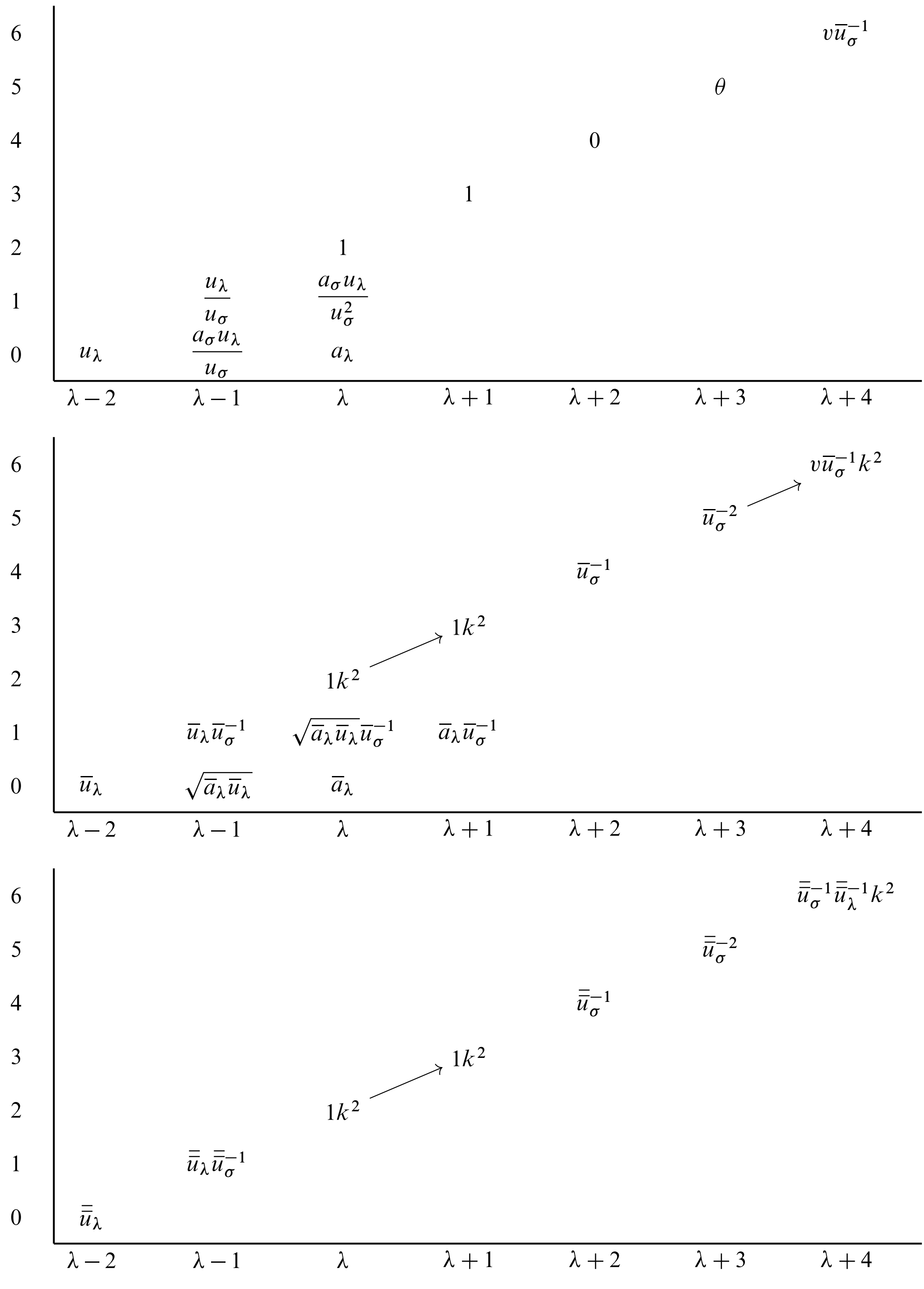

Figure 3: Top: $\underline{E}_1^{\lambda+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{\lambda+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{\lambda+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

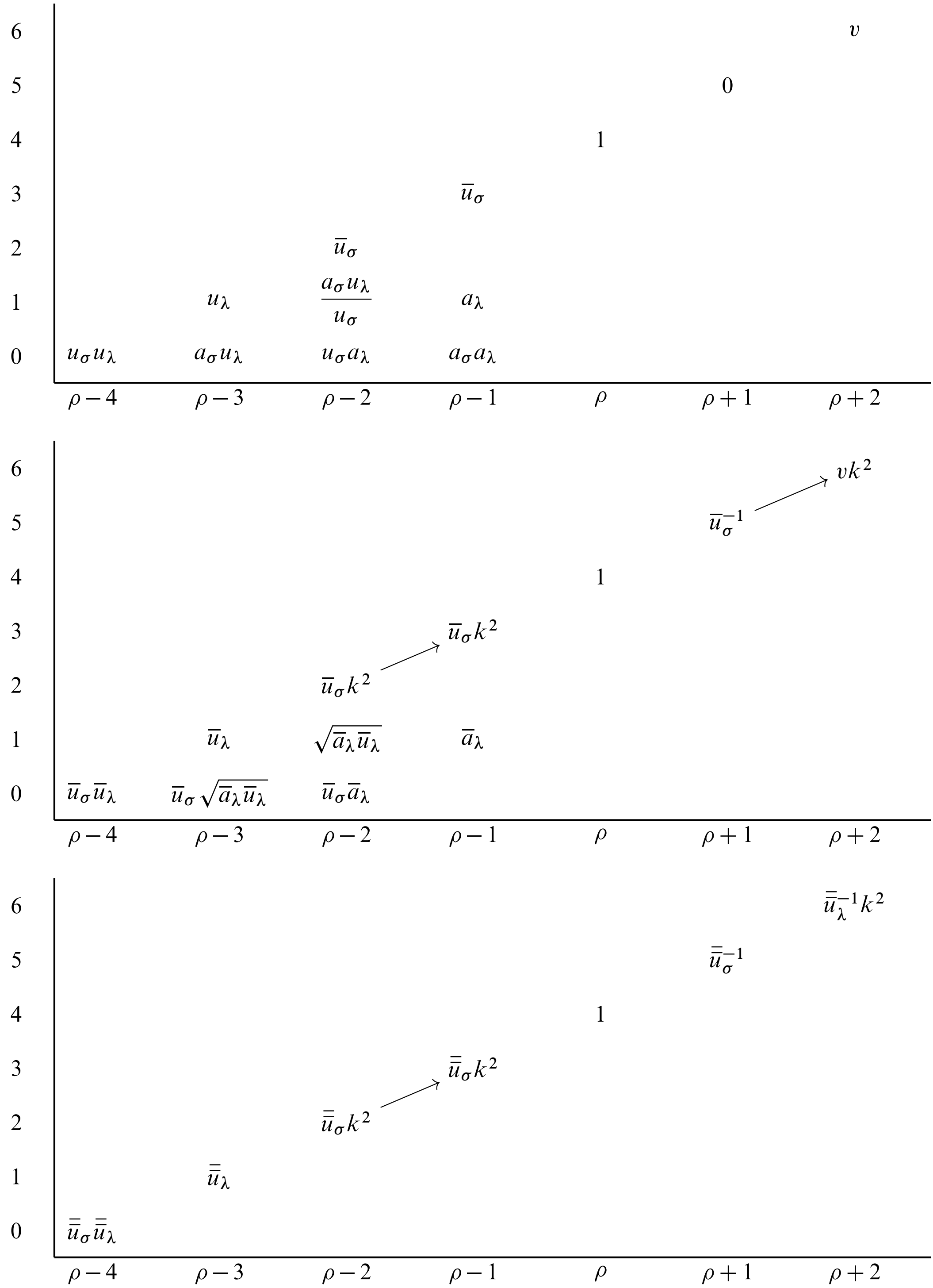

Figure 4: Top: $\underline{E}_1^{\rho+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{\rho+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{\rho+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

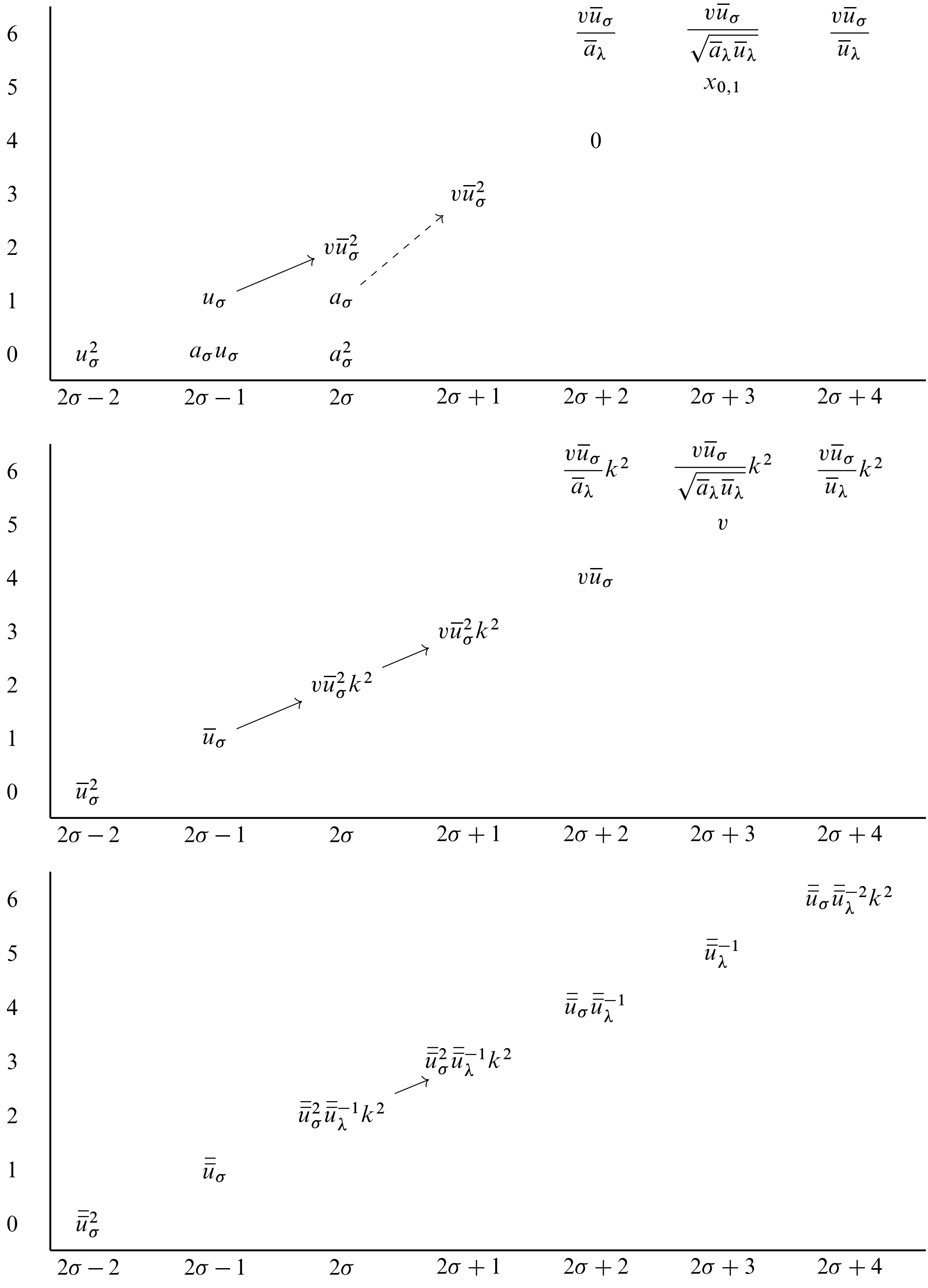

Figure 5: Top: $\underline{E}_1^{2\sigma+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{2\sigma+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{2\sigma+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

coordinates, meaning that the middle level of $E_1^{5,5}$ is generated by $(v\bar{u}_\sigma^{-2})\bar{e}^{\rho+\sigma}$. We have

$$\mathrm{Tr}_2^4(v\bar{u}_\sigma^{-2}\bar{e}^{\rho+\sigma}) = \frac{x_{0,1}}{u_\sigma^2}e^{\rho+\sigma}$$

In the top, middle and bottom graphs of Figure 1, if we look at coordinates $(2,2)$ we see v, vk^2 and $\bar{\bar{u}}_\lambda^{-1}k^2$, respectively. This represents the fact that the three levels of $E_1^{2,2}$ are generated by $ve^\lambda(x+gx)$ for the top, $v\bar{e}^\lambda x, v\bar{e}^\lambda gx$ for the middle, and $\bar{\bar{u}}_\lambda^{-1}\bar{\bar{e}}^\lambda x, \bar{\bar{u}}_\lambda^{-1}\bar{\bar{e}}^\lambda gx$ for the bottom level. We have:

$$\mathrm{Tr}_2^4(v\bar{e}^\lambda x) = ve^\lambda(x+gx)$$

These pictures are all obtained automatically by the computer program of [2], available on the author's GitHub page.

Appendix B The $\mathrm{RO}(C_4)$ homology of a point in $\mathbb{F}_2$ coefficients

In this appendix, we write down the detailed computation of $k_\bigstar$ for $\bigstar \in \mathrm{RO}(C_4)$. We use the following notation for Mackey functors (compare with [2]):

$$k = \begin{tikzcd} k \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \end{tikzcd} \qquad k_- = \begin{tikzcd} 0 \arrow[d, bend right] \\ k \arrow[u, bend right] \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \end{tikzcd} \qquad \langle k \rangle = \begin{tikzcd} k \arrow[d, bend right] \\ 0 \arrow[u, bend right] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \qquad \overline{\langle k \rangle} = \begin{tikzcd} 0 \arrow[d, bend right] \\ k \arrow[u, bend right] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

$$L = \begin{tikzcd} k \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \end{tikzcd} \qquad p^*L = \begin{tikzcd} k \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \end{tikzcd} \qquad Q = \begin{tikzcd} k \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \qquad Q^\sharp = \begin{tikzcd} k \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

$$L^\sharp = \begin{tikzcd} k \arrow[d, bend right, "1"'] \\ k \arrow[u, bend right, "0"'] \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \end{tikzcd} \qquad k_-^\flat = \begin{tikzcd} 0 \arrow[d, bend right] \\ 0k \arrow[u, bend right] \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \end{tikzcd}$$

Our notation corresponds to that used in [9] according to Table 2 below. Note that [9] is concerned with the integral homology of a point, while we are working over the characteristic 2. As a result, in this table, the Mackey functors of [9] are levelwise tensored with $k = \mathbb{F}_2$, an operation we denote by $-\otimes k$. We also note that L is dual to k and $k_-^\flat$ is dual to k_-.

Henceforth $n, m \geq 0$. We employ the notation $a \mid b \mid c$ to denote the generators of all three levels of a Mackey functor, from top to bottom, used in [2].

notation in [9]	notation in [2]
$\square \otimes k$	k
$\overline{\square} \otimes k$	k_-
$\bullet \otimes k$	$\langle k \rangle$
$\overline{\bullet} \otimes k$	$\langle \overline{k} \rangle$
◪ $\otimes k =$ $\dot{⬕}$ $\otimes k$	L
⬕ $\otimes k = \dot{\square} \otimes k$	p^*L
$\blacktriangledown \otimes k$	Q
$\circ \otimes k = \blacktriangle \otimes k$	$Q^\sharp$
⬒ $\otimes k$	$L^\sharp$
$\overline{⬕}$ $\otimes k$	$k_-^\flat$

Table 2

B.1 $k_* S^{n\sigma + m\lambda}$

We have

$$k_*(S^{n\sigma+m\lambda}) = \begin{cases} k & \text{if } * = n+2m, \\ Q^\sharp & \text{if } n \le * < n+2m \text{ and } *-n \text{ is even}, \\ Q & \text{if } n+1 \le * < n+2m \text{ and } *-n \text{ is odd}, \\ \langle k \rangle & \text{if } 0 \le * < n. \end{cases}$$

Moreover,

- $u_\sigma^n u_\lambda^m \mid \overline{u}_\sigma^n \overline{u}_\lambda^m \mid \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^m$ generates $k_{n+2m} = k$,
- $u_\sigma^n a_\lambda^{m-i} u_\lambda^i \mid \overline{u}_\sigma^n \overline{a}_\lambda^{m-i} \overline{u}_\lambda^i \mid 0$ generates $k_{n+2i} = Q^\sharp$ for $0 \le i < m$,
- $a_\sigma u_\sigma^{n-1} a_\lambda^{m-i} u_\lambda^i \mid \overline{u}_\sigma^n \overline{a}_\lambda^{m-i} \overline{u}_\lambda^{i-1} \sqrt{\overline{a}_\lambda \overline{u}_\lambda} \mid 0$ generates $k_{n+2i-1} = Q$ for $1 \le i \le m$, $n > 0$,
- $\dfrac{a_\sigma a_\lambda^{m-i} u_\lambda^i}{u_\sigma} \,\Big|\, \overline{a}_\lambda^{m-i} \overline{u}_\lambda^{i-1} \sqrt{\overline{a}_\lambda \overline{u}_\lambda} \mid 0$ generates $k_{2i-1} = Q$ for $1 \le i \le m$, $n = 0$,
- $a_\sigma^{n-i} u_\sigma^i a_\lambda^m \mid 0 \mid 0$ generates $k_i = \langle \mathbb{Z}/2 \rangle$ for $0 \le i < n$.

B.2 $k_* S^{-n\sigma - m\lambda}$

If n and m are not both 0:

$$k_*(S^{-n\sigma-m\lambda}) = \begin{cases} L & \text{if } * = -n-2m \text{ and } m \ne 0, \\ p^*L & \text{if } * = -n-2m \text{ and } n > 1, m = 0, \\ k_- & \text{if } * = -1 \text{ and } n = 1, m = 0, \\ Q^\sharp & \text{if } -n-2m < * < -n-1 \text{ and } *+n \text{ is odd}, \\ Q & \text{if } -n-2m < * < -n-1 \text{ and } *+n \text{ is even}, \\ \langle k \rangle & \text{if } -n-1 \le * < -1 \text{ and } m \ne 0, \\ \langle k \rangle & \text{if } -n+1 \le * < -1 \text{ and } m = 0. \end{cases}$$

Moreover,

- $\mathrm{Tr}_1^4\left(\dfrac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}\right)\Big|\,\mathrm{Tr}_1^2\left(\dfrac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}\right)\Big|\,\dfrac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}$ generates $k_{-n-2m}=L$ for $m\neq 0$,
- $\dfrac{\theta}{u_\sigma^{n-2}}\Big|\,\overline{u}_\sigma^{-n}\mid\overline{\overline{u}}_\sigma^{-n}$ generates $k_{-n}=p^*L$ for $m=0, n\geq 2$,
- $0\mid\overline{u}_\sigma^{-1}\mid\overline{\overline{u}}_\sigma^{-1}$ generates $k_{-1}=k_-$ for $n=1$, $m=0$,
- $\dfrac{s}{u_\sigma^n a_\lambda^{i-2}u_\lambda^{m-i}}\Big|\,\dfrac{\overline{s}}{\overline{u}_\sigma^n\overline{a}_\lambda^{i-2}\overline{u}_\lambda^{m-i}}\Big|\,0$ generates $k_{-n-2m+2i-3}=Q^\sharp$ for $2\leq i\leq m$,
- $\dfrac{x_{0,1}}{u_\sigma^n a_\lambda^{m-i}u_\lambda^{i-1}}\Big|\,\dfrac{v}{\overline{u}_\sigma^n\overline{a}_\lambda^{m-i}\overline{u}_\lambda^{i-1}}\Big|\,0$ generates $k_{-n-2i}=Q$ for $1\leq i<m$,
- $\dfrac{x_{0,1}}{a_\sigma^{n-i}u_\sigma^i a_\lambda^{m-1}}\Big|\,0\mid 0$ generates $k_{-i-2}=\langle k\rangle$ for $0\leq i\leq n-1$, $m\neq 0$,
- $\dfrac{\theta}{a_\sigma^{n-i}u_\sigma^{n-2}}\Big|\,0\mid 0$ generates $k_{-i}=\langle k\rangle$ for $2\leq i<n$, $m=0$.

B.3 $k_*S^{m\lambda-n\sigma}$

If n,m are both nonzero,

$$k_*(S^{m\lambda-n\sigma})=\begin{cases}\langle k\rangle & \text{if } 2m-n<*\leq -2,\\ k & \text{if } *=2m-n\geq -1,\\ \langle k\rangle\oplus k & \text{if } *=2m-n\leq -2,\\ Q & \text{if } -1\leq *<2m-n \text{ and } *+n \text{ is odd},\\ Q^\sharp & \text{if } -1\leq *<2m-n \text{ and } *+n \text{ is even},\\ \langle k\rangle\oplus Q & \text{if } -n+1\leq *<2m-n \text{ and } *+n \text{ is odd, and } *\leq -2,\\ \langle k\rangle\oplus Q^\sharp & \text{if } -n+2\leq *<2m-n \text{ and } *+n \text{ is even, and } *\leq -2,\\ Q & \text{if } *=-n \text{ and } n\geq 2,\\ \overline{\langle k\rangle} & \text{if } *=-1 \text{ and } n=1.\end{cases}$$

Moreover,

- $\dfrac{u_\lambda^m}{u_\sigma^n}\Big|\,\dfrac{\overline{u}_\lambda^m}{\overline{u}_\sigma^n}\Big|\,\dfrac{\overline{\overline{u}}_\lambda^m}{\overline{\overline{u}}_\sigma^n}$ generates the k in k_{2m-n},
- $\dfrac{a_\lambda^i u_\lambda^{m-i}}{u_\sigma^n}\Big|\,\dfrac{\overline{a}_\lambda^i\overline{u}_\lambda^{m-i}}{\overline{u}_\sigma^n}\Big|\,0$ generates the $Q^\sharp$ in $k_{2m-n-2i}$ for $0<i<m$,
- $\dfrac{a_\sigma a_\lambda^i u_\lambda^{m-i}}{u_\sigma^{n+1}}\Big|\,\dfrac{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{a}_\lambda^i\overline{u}_\lambda^{m-i-1}}{\overline{u}_\sigma^n}\Big|\,0$ generates the Q in $k_{2m-n-2i-1}$ for $0\leq i<m$,
- $\dfrac{\theta a_\lambda^m}{a_\sigma^{n-i}u_\sigma^{i-2}}\Big|\,0\mid 0$ generates the $\langle k\rangle$ in k_{-i} for $2\leq i<n$,

- $\dfrac{\theta a_\lambda^m}{u_\sigma^{n-2}} \,\Big|\, \overline{a}_\lambda^m \overline{u}_\sigma^{-n} \mid 0$ generates $k_{-n} = Q$ for $n \geq 2$,
- $0 \mid \overline{u}_\sigma^{-1} \overline{a}_\lambda^m \mid 0$ generates $k_{-1} = \overline{\langle k \rangle}$ for $n = 1$.

B.4 $k_* S^{n\sigma - m\lambda}$

If n, m are both nonzero,

$$k_*(S^{n\sigma-m\lambda}) = \begin{cases} Q^\sharp & \text{if } * = n-2 \text{ and } n, m \geq 2, \\ \overline{\langle k \rangle} & \text{if } * = -1 \text{ and } n = 1, m \geq 2, \\ \langle k \rangle \oplus Q & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is even, and } * \geq 0, \\ \langle k \rangle \oplus Q^\sharp & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is odd, and } * \geq 0, \\ Q & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is even, and } * < 0, \\ Q^\sharp & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is odd, and } * < 0, \\ L \oplus \langle k \rangle & \text{if } * = n-2m \text{ and } n-2m \geq 0 \text{ and } m \geq 2, \\ L & \text{if } * = n-2m \text{ and } n-2m < 0 \text{ and } m \geq 2, \\ L^\sharp & \text{if } * = n-2 \text{ and } n > 1 \text{ and } m = 1, \\ k_-^\flat & \text{if } * = -1 \text{ and } n = 1 \text{ and } m = 1, \\ \langle k \rangle & \text{if } 0 \leq * < n-2m. \end{cases}$$

Moreover,

- $\dfrac{a_\sigma^2 u_\sigma^{n-2}}{a_\lambda^m} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{a}_\lambda^{m-1}} \,\Big|\, 0$ generates $k_{n-2} = Q^\sharp$ for $n, m \geq 2$,
- $0 \,\Big|\, \dfrac{v\overline{u}_\sigma}{\overline{a}_\lambda^{m-1}} \,\Big|\, 0$ generates $k_{n-2} = \overline{\langle k \rangle}$ for $n = 1, m \geq 2$,
- $\dfrac{x_{0,2} u_\sigma^n}{a_\lambda^{i-1} u_\lambda^{m-i-1}} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{a}_\lambda^{i-1} \overline{u}_\lambda^{m-i}} \,\Big|\, 0$ generates the Q in $k_{n-2m+2i-2}$ for $2 \leq i \leq m-1$,
- $\dfrac{s u_\sigma^n}{a_\lambda^{i-2} u_\lambda^{m-i}} \,\Big|\, \dfrac{\overline{s}\overline{u}_\sigma^n}{\overline{a}_\lambda^{i-2} \overline{u}_\lambda^{m-i}} \,\Big|\, 0$ generates the $Q^\sharp$ in $k_{n-2m+2i-3}$ for $2 \leq i \leq m$,
- $\dfrac{x_{0,2} u_\sigma^n}{u_\lambda^{m-2}} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{u}_\lambda^{m-1}} \,\Big|\, \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^{-m}$ generates the L in k_{n-2m} for $m \geq 2$,
- $\dfrac{a_\sigma^2 u_\sigma^{n-2}}{a_\lambda} \,\Big|\, v\overline{u}_\sigma^n \mid \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^{-1}$ generates $k_{n-2} = L^\sharp$ for $n > 1, m = 1$,
- $0 \mid v\overline{u}_\sigma \mid \overline{\overline{u}}_\sigma \overline{\overline{u}}_\lambda^{-1}$ generates $k_{-1} = k_-^\flat$ for $n = m = 1$,
- $\dfrac{a_\sigma^i u_\sigma^{n-i}}{a_\lambda^m} \,\Big|\, 0 \mid 0$ generates the $\langle k \rangle$ in k_{n-i} for $2 < i \leq n$.

B.5 Subtleties about quotients

In this subsection, we investigate the subtleties regarding quotients y/x, similar to what we did in [2] for the integer coefficient case.

The crux of the matter is as follows: If we have $ax = y$ in $k_{\bigstar}^{C_4}$ then we can immediately conclude that $a = y/x$ as long as a is the *unique* element in its $\mathrm{RO}(C_4)$ degree satisfying $ax = y$. Unfortunately, as we can see from the detailed description of $k_{\bigstar}^{C_4}$, there are degrees $\bigstar$ for which $k_{\bigstar}^{C_4}$ is a two-dimensional vector space, generated by elements a and b both satisfying $ax = bx = y$; in this case a and b are both candidates for y/x and we need to distinguish them somehow. This is done by looking at the products of a and b with other Euler/orientation classes.

For a concrete example, take $k_{-2+4\sigma-\lambda}^{C_4}$, which is k^2 with generators a and b such that

$$u_\sigma a = u_\sigma b = \frac{u_\lambda}{u_\sigma^3},$$

so both a and b are candidates for u_λ/u_σ^4 (for degree reasons, there is a unique choice for u_λ/u_σ^3). To distinguish a and b, we use multiplication by a_σ^2: for one generator, say a, we have $a_\sigma^2 a = 0$, while for the other generator we get $a_\sigma^2 b = \theta a_\lambda$. So now

$$a_\sigma^2(a+b) = a_\sigma^2 b = \theta a_\lambda,$$

and both $a+b$ and b are candidates for $(\theta a_\lambda)/a_\sigma^2$. However, θ/a_σ^2 is defined uniquely and we insist

$$\frac{x}{z}\frac{y}{w} = \frac{xy}{zw}$$

whenever $xy \neq 0$, thus $(\theta a_\lambda)/a_\sigma^2$ is uniquely defined from

$$\frac{\theta a_\lambda}{a_\sigma^2} = \frac{\theta}{a_\sigma^2} a_\lambda.$$

Multiplying with u_σ returns 0 and as $u_\sigma b \neq 0$, we conclude that

$$a + b = \frac{\theta a_\lambda}{a_\sigma^2}.$$

Since $a_\sigma^2 a = 0$ and $a_\sigma^2 b = \theta a_\lambda$, we conclude that

$$a = \frac{u_\lambda}{u_\sigma^4} \quad \text{and} \quad b = \frac{u_\lambda}{u_\sigma^4} + \frac{\theta a_\lambda}{a_\sigma^2}.$$

More generally, we can use u_σ and a_σ multiplication to distinguish

$$\frac{a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*}, \quad \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*} \quad \text{and} \quad \frac{a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*} + \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*}.$$

Here, $* \geq 0$ is a generic index, ie the 12 total instances of $*$ can all be different; the important thing is that the $*$ indices are chosen so that these three elements are in the same $\mathrm{RO}(C_4)$–degree.

We can also distinguish between

$$\frac{a_\sigma a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*}, \quad \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*} \quad \text{and} \quad \frac{a_\sigma a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*} + \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*}$$

by u_σ and a_σ multiplication, although it's easier to use that only the first of the three elements is a transfer. We distinguish

$$\frac{a_\sigma^{*\geq 2} u_\sigma^*}{a_\lambda^*}, \quad \frac{x_{0,2} u_\sigma^*}{a_\lambda^* u_\lambda^*} \quad \text{and} \quad \frac{a_\sigma^{*\geq 2} u_\sigma^*}{a_\lambda^*} + \frac{x_{0,2} u_\sigma^*}{a_\lambda^* u_\lambda^*}$$

by a_λ^i multiplication (which for large enough i annihilates only the second term) and a_σ multiplication (which annihilates only the first term). We similarly distinguish

$$\frac{a_\sigma^{*\geq 2} u_\sigma^*}{a_\lambda^*}, \quad \frac{s u_\sigma^*}{a_\lambda^* u_\lambda^*} \quad \text{and} \quad \frac{a_\sigma^{*\geq 2} u_\sigma^*}{a_\lambda^*} + \frac{s u_\sigma^*}{a_\lambda^* u_\lambda^*}$$

by a_λ^i and a_σ^2 multiplication.

Appendix C The C_{2^n} Borel equivariant dual Steenrod algebra

In this appendix we compute the Borel dual Steenrod algebra $(k \wedge k)_\bigstar^{hG}$ for the groups $G = C_{2^n}$ as an RO(G)–graded Hopf algebroid over the Borel homology of a point $k_\bigstar^{hG}$, where $k = H\mathbb{F}_2$. We also compare our result with the description of the Borel Steenrod algebra given in [3], which is dual to our calculation. Henceforth, $G = C_{2^n}$, with $n \geq 1$.

C.1 The Borel homology of a point

The real representation ring RO(G) is spanned by the irreducible representations 1, σ and $\lambda_{s,k}$, where σ is the 1–dimensional sign representation and $\lambda_{s,m}$ is the 2–dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1 \leq m$ dividing 2^{n-2} and odd $1 \leq s < 2^n/m$. Note that 2–locally, $S^{\lambda_{s,m}} \simeq S^{\lambda_{1,m}}$ as C_{2^n}–equivariant spaces, by the s–power map. Therefore, to compute $k_\bigstar(X)$ for $\bigstar \in \mathrm{RO}(C_{2^n})$ it suffices to only consider $\bigstar$ in the span of 1, σ and $\lambda_k := \lambda_{1,2^k}$ for $0 \leq k \leq n-2$ ($\lambda_{n-1} = 2\sigma$ and $\lambda_n = 2$). For $V = \sigma$ or $V = \lambda_m$, denote by $a_V \in k_{-V}^{C_{2n}}$ the Euler class induced by the inclusion of north and south poles $S^0 \hookrightarrow S^V$; also denote by $u_V \in k_{|V|-V}^{C_{2n}}$ the orientation class generating the Mackey functor $k_{|V|-V} = k$; see [8]. The orientation classes $u_V \colon k \wedge S^{|V|} \to k \wedge S^V$ are nonequivariant equivalences hence they act invertibly on $k_{hG\bigstar}$, $k_\bigstar^{hG}$ and $k_\bigstar^{tG}$.

Proposition C.1 *For $G = C_{2^n}$ and $n > 1$,*

$$k_\bigstar^{hG} = k[a_\sigma, a_{\lambda_0}, u_\sigma^\pm, u_{\lambda_0}^\pm, \dots, u_{\lambda_{n-2}}^\pm]/a_\sigma^2, \quad k_\bigstar^{tG} = k[a_\sigma, a_{\lambda_0}^\pm, u_\sigma^\pm, u_{\lambda_0}^\pm, \dots, u_{\lambda_{n-2}}^\pm]/a_\sigma^2,$$

while for $n = 1$,

$$k_\bigstar^{hC_2} = k[a_\sigma, u_\sigma^\pm], \quad k_\bigstar^{tC_2} = k[a_\sigma^\pm, u_\sigma^\pm].$$

In all cases, $k_{hG\bigstar} = \Sigma^{-1} k_\bigstar^{tG} / k_\bigstar^{hG}$ (forgetting the ring structure) and the norm map $k_{hG\bigstar} \to k_\bigstar^{hG}$ is trivial.

Proof The homotopy fixed-point spectral sequence becomes

$$H^*(G;k)[u_\sigma^\pm, u_{\lambda_0}^\pm, \dots, u_{\lambda_{n-2}}^\pm] \Rightarrow k_\bigstar^{hG}.$$

We have $H^*(G;k) = k^*BG = k[a]/a^2 \otimes k[b]$, where $|a| = 1$ and $|b| = 2$. The spectral sequence collapses with no extensions and we can identify $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$. Finally, $\widetilde{E}G = S^{\infty\lambda_0} = \operatorname{colim}(S^{\lambda_0} \xrightarrow{a_{\lambda_0}} S^{\lambda_0} \xrightarrow{a_{\lambda_0}} \cdots)$ so to get $k_\bigstar^{tG}$ we are additionally inverting a_{λ_0}. □

C.2 The Borel dual Steenrod algebra

We now compute the G–Borel dual Steenrod algebra

$$(k \wedge k)_\bigstar^{hG}$$

as a Hopf algebroid over $k_\bigstar^{hG}$, for $G = C_{2^n}$.

We will implicitly be completing it at the ideal generated by a_σ for $G = C_2$, and at the ideal generated by a_{λ_0} for $G = C_{2^n}$ with $n > 1$; see [10, page 373] for more details in the case of $G = C_2$. With this convention, Hu and Kriz computed the C_2–Borel dual Steenrod algebra to be

$$(k \wedge k)_\bigstar^{hC_2} = k_\bigstar^{hC_2}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ (with $\xi_0 = 1$). The generators ξ_i restrict to the Milnor generators in the nonequivariant dual Steenrod algebra and

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k, \quad \epsilon(\xi_i) = 0 \quad \text{for } i \ge 1, \qquad \eta_R(a_\sigma) = a_\sigma, \qquad \eta_R(u_\sigma)^{-1} = \sum_{i=0}^{\infty} a_\sigma^{2^i-1} u_\sigma^{-2^i} \xi_i.$$

Proposition C.2 *For $G = C_{2^n}$, with $n > 1$,*

$$(k \wedge k)_\bigstar^{hG} = k_\bigstar^{hG}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ restricting to the $C_{2^{n-1}}$ generators ξ_i, with

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k, \quad \epsilon(\xi_i) = 0 \quad \text{for } i \ge 1,$$

$$\eta_R(a_\sigma) = a_\sigma, \quad \eta_R(a_{\lambda_0}) = a_{\lambda_0}, \quad \eta_R(u_\sigma) = u_\sigma + a_\sigma \xi_1, \quad \eta_R(u_{\lambda_m}) = u_{\lambda_m} \quad \text{for } m > 0,$$

$$\eta_R(u_{\lambda_0})^{-1} = \sum_i a_{\lambda_0}^{2^i-1} u_{\lambda_0}^{-2^i} \xi_i^2.$$

Proof The computation of $(k \wedge k)_*^{hG} = (k \wedge k)^*(BG)$ follows from the computation of $k_*^{hG} = k^*(BG) = k[a]/a^2 \otimes k[b]$ and the fact that nonequivariantly, $k \wedge k$ is a free k–module. To see that the homotopy fixed point spectral sequence for $k \wedge k$ converges strongly, let $F^i BG$ be the skeletal filtration on the Lens space $BG = S^\infty / C_{2^n}$; we can then compute directly that $\lim_i^1 (k \wedge k)^*(F^i BG) = \lim_i^1 (k[a]/a^2 \otimes k[b]/b^i) = 0$.

Thus we get $(k \wedge k)^{hG}_{\bigstar} = k^{hG}_{\bigstar}[\xi_i]$, and the diagonal Δ and augmentation ϵ are the same as in the nonequivariant case. The Euler classes a_σ and a_{λ_0} are maps of spheres so they are preserved under η_R. The action of η_R on u_σ and u_{λ_0} can be computed through the right coaction on $k^{hG}_{\bigstar}$: the (completed) coaction of the nonequivariant dual Steenrod algebra on $k^*(BG) = k[a]/a^2 \otimes k[b]$ is

$$a \mapsto a \otimes 1, \quad b \mapsto \sum_i b^{2^i} \otimes \xi_i^2.$$

To verify the formula for the coaction on b we need to check that $\mathrm{Sq}^1(b) = 0$ (the alternative is $\mathrm{Sq}^1(b) = ab$). From the long exact sequence associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, we can see that the vanishing of the Bockstein on b follows from $H^2(C_{2^n}; \mathbb{Z}/4) = \mathbb{Z}/4$ for $n > 1$.

After identifying $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$ we get the formula for $\eta_R(u_{\lambda_0})$, and also that

$$\eta_R(u_\sigma) = u_\sigma + \epsilon a_\sigma \xi_1,$$

where ϵ is either 0 or 1. This is equivalent to

$$\eta_R(u_\sigma^{-1}) = u_\sigma^{-1} + \epsilon a_\sigma u_\sigma^{-2} \xi_1,$$

and to see that $\epsilon = 1$ we use the map $k^{hC_2} = k^{h(C_{2^n}/C_{2^{n-1}})} \to k^{hC_{2^n}}$ that sends a_σ and u_σ to a_σ and u_σ, respectively. Finally, to compute $\eta_R(u_{\lambda_m})$ for $m > 0$ note that

$$k^{hC_{2^{n-m}}} = k^{hC_{2^n}/C_{2^m}} \to k^{hC_{2^n}}$$

sends a_{λ_0} and u_{λ_0} to $a_{\lambda_m} = 0$ and u_{λ_m}, respectively. □

C.3 Comparison with Greenlees's description

We now compare with the dual description given in [3].

In our notation, the G–spectrum b of [3] is $b = k^h$ and $b^V(X)$ corresponds to $(k^h)^{|V|}_G(X)$; to get $(k^h)^V_G(X)$ we need to multiply with the invertible element $u_V \in k^{hG}_{|V|-V}$. The Borel Steenrod algebra is $b^{\bigstar}_G b = (k^h)^{\bigstar}_G(k^h)$ and the Borel dual Steenrod algebra is $b^G_{\bigstar} b = (k^h)^G_{\bigstar}(k^h) = (k \wedge k)^{hG}_{\bigstar}$.

Greenlees proves that the Borel Steenrod algebra is given by the Massey–Peterson twisted tensor product (see [13]) of the nonequivariant Steenrod algebra k^*k and the Borel cohomology of a point $(k^h)^{\bigstar}_G = k^{hG}_{-\bigstar}$. The twisting has to do with the fact that the action of the Borel Steenrod algebra on $x \in (k^h)^{\bigstar}_G(X)$ is given by

$$(\theta \otimes a)(x) = \theta(ax),$$

where $\theta \in k^*k$ and $a \in k^{hG}_{\bigstar}$. The product of elements $\theta \otimes a$ and $\theta' \otimes a'$ in the Borel Steenrod algebra is not $\theta\theta' \otimes aa'$, since θ does not commute with cup products, but rather satisfies the Cartan formula

$$\theta(ab) = \sum_i \theta_i'(a)\theta_i''(b), \quad \Delta\theta = \sum_i \theta_i' \otimes \theta_i''.$$

Therefore,

$$(\theta \otimes a)(\theta' \otimes a')(x) = \theta(a\theta'(a'x)) = \sum_i \theta_i'(a)(\theta_i''\theta')(a'x),$$

so

$$(\theta \otimes a)(\theta' \otimes a') = \sum_i \theta_i'(a)(\theta_i''\theta' \otimes a') \tag{4}$$

(we have ignored signs as we are working in characteristic 2).

So the Borel Steenrod algebra is $k^*k \otimes k_\star^{hG}$ with twisted algebra structured defined by (4).

Moreover, Greenlees expresses the action of k^*k on $(k^h)_G^\star(X)$ in terms of the action of k^*k on the orientation classes u_V and the usual (nonequivariant) action of k^*k on $(k^h)_G^*(X) = k^*(X \wedge_G EG_+)$. This is done through the Cartan formula: if $x \in (k^h)_G^V(X)$, then $u_V^{-1}x \in (k^h)_G^{|V|}(X)$ and

$$\theta(x) = \theta(u_V u_V^{-1}x) = \sum_i \theta_i'(u_V)\theta_i''(u_V^{-1}x).$$

What remains to compute is $\theta_i'(u_V)$, namely the action of k^*k on orientation classes.

In our case, for $G = C_{2^n}$, we can see that:

Proposition C.3 *The action of k^*k on orientation classes is determined by*

$$\mathrm{Sq}^i(u_\sigma) = \begin{cases} u_\sigma & \text{if } i = 0, \\ a_\sigma & \text{if } i = 1, \\ 0 & \text{otherwise}; \end{cases} \qquad \mathrm{Sq}^i(u_{\lambda_m}) = \begin{cases} u_{\lambda_m} & \text{if } i = 0, \\ a_{\lambda_0} & \text{if } i = 2, m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Compare with the proof of Proposition C.2. □

The twisting in the case of the Borel dual Steenrod algebra corresponds to the fact that $(k \wedge k)_\star^{hG}$ is a Hopf algebroid and not a Hopf algebra; computationally this amounts to the formula for η_R of Proposition C.2.

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Annular Khovanov homology and augmented links

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Given an annular link L, there is a corresponding augmented link $\widetilde{L}$ in S^3 obtained by adding a meridian unknot component to L. We construct a spectral sequence with the second page isomorphic to the annular Khovanov homology of L that converges to the reduced Khovanov homology of $\widetilde{L}$. As an application, we classify all the links with the minimal rank of annular Khovanov homology. We also give a proof that annular Khovanov homology detects unlinks.

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1 Introduction

Khovanov [9] defined an invariant for links which assigns a bigraded abelian group $\mathrm{Kh}(L)$ to each link $L \subset S^3$. It is a categorification of the Jones polynomial in the sense that it replaces terms in the Jones polynomial by graded abelian groups. Since then, many related invariants have been studied, including Lee's deformation invariant [13] and Rasmussen's s–invariant [15], Khovanov's reduced version [10], the tangle invariant of Bar-Natan [5] and Khovanov–Rozansky homology [11].

Several spectral sequences that reveal the relationship between Khovanov homology theories and Floer theories have been established. The first one is due to Ozsváth and Szabó [14], which builds a connection between the reduced Khovanov homology of the mirror of a link L and the Heegaard Floer homology of the branched double cover of S^3 over L. Kronheimer and Mrowka [12] constructed a spectral sequence with the E_1 term isomorphic to Khovanov homology and converging to a version of singular instanton Floer homology.

Asaeda, Przytycki and Sikora [2] constructed Khovanov-type invariants for links in $\Sigma \times I$, where Σ is a surface. When $\Sigma = A$ is an annulus (sometimes it is convenient to view A as a punctured disk), the resulting invariant is called the *annular Khovanov homology*. Roberts [16] constructed a spectral sequence from annular Khovanov homology to Heegaard Floer homology. Grigsby, Licata and Wehrli [7] studied the analogue of Rasmussen's s–invariant in the annular setting. Xie [18] introduced annular instanton Floer homology for annular links as an analogue of the annular Khovanov homology, and they are also related by a spectral sequence, which can be used to distinguish braids from other tangles; see [18] and Xie and Zhang [19].

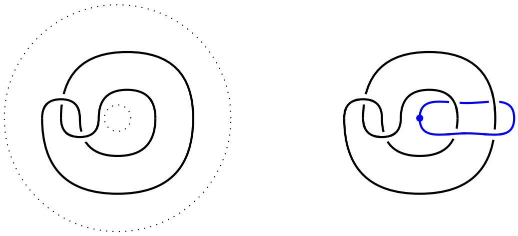

Figure 1: An annular link and its augmentation.

The relationship between annular Khovanov homology and the original Khovanov homology has been previously studied. There is a natural spectral sequence between them given by ignoring the punctured point [16, Lemma 2.3]. Stoffregen and Zhang [17] established a spectral sequence relating the (annular) Khovanov homologies of periodic knots and their quotients.

Considering the augmentation of links is an alternative approach to preserve the information about the punctured point.

Definition 1.1 Let $L \subset A \times I$ be an annular link. The *augmentation* of L is a pointed link $(\widetilde{L}, p) \subset \mathbb{R}^3$ obtained as follows. We view the thickened annulus $A \times I$ as a solid torus in $\mathbb{R}^3$, and $\widetilde{L}$ is given by the union of L and a meridian circle of $A \times I$ (sometimes we call it an *augmenting circle*). The basepoint p is chosen on the augmenting circle.

Under this convention, Xie [18, Section 4.3] showed that the annular instanton Floer homology $\mathrm{AHI}(L)$ is isomorphic to $\mathrm{I}^\natural(\widetilde{L})$, the reduced singular instanton Floer homology of the augmented link. We will prove the following theorem as an analogue of Xie's result on the Khovanov side. To avoid sign issues, all the coefficient rings will be $\mathbb{Z}/2\mathbb{Z}$ unless otherwise specified.

Theorem 1.2 *Let $L \subset A \times I$ be an annular link and let $(\widetilde{L}, p) \subset S^3$ be the corresponding augmented link of L. Then there is a spectral sequence with the E_2 term isomorphic to the annular Khovanov homology* $\mathrm{AKh}(L)$ *and it converges to the reduced Khovanov homology* $\mathrm{Khr}(\widetilde{L}, p)$.

We immediately obtain the following rank inequality:

Corollary 1.3 *Given an annular link L and its augmentation $\widetilde{L}$, we have*

$$\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) \geq \operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L}, p).$$

Question 1.4 *For what links L is* $\mathrm{AKh}(L)$ *isomorphic to* $\mathrm{Khr}(\widetilde{L}, p)$?

Theorem 1.2 provides an alternative way to prove some detection results by referring to the parallel consequences in reduced Khovanov homology. For a link L with n components, it is well known that $\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(L, p) \geq 2^{n-1}$. Hence, by the previous corollary, for an annular link L, we have

$$\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) \geq \operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L}, p) \geq 2^n.$$

On the other hand, links of minimal rank in $A \times I$ can be classified following Xie and Zhang [20]. Before stating the result, we first explain the notation. Recall that a *forest* is a graph (not necessarily connected) without cycles. Given a forest G, its corresponding link L_G is defined by assigning to each vertex of G an unknot component and linking two unknots in the way of Hopf links whenever their corresponding vertices are adjacent. For annular links, we need to assign which vertex corresponds to a nontrivial circle. We say such vertices are *annular* for convenience.

Theorem 1.5 *Let L be an n–component annular link. Then $\operatorname{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) = 2^n$ if and only if L is a forest of unknots such that each connected component of the corresponding graph of L contains at most one annular vertex.*

We say an annular link U is an *unlink* if it has a link diagram D without any crossing. Notice that our definition given here is slightly different to [18]. The following corollary is a generalization of Theorem 3.1 of Baldwin and Grigsby [3] and Corollary 1.4 of Xie and Zhang [19], where the unlinks are required to have all the components trivial or nontrivial:

Corollary 1.6 *Let L be an annular link with n components and let U be an annular unlink with n components (which might be trivial or nontrivial). Assume that*

$$\mathrm{AKh}(L) \cong \mathrm{AKh}(U)$$

as bigraded (by homological and annular gradings) abelian groups. Then L is isotopic to U.

The paper is organized as follows. In Section 2 we review the construction and properties of Khovanov homology. After some preparation in Section 3, we prove Theorem 1.2 in the last section and discuss its applications.

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2 Review of Khovanov homology theories

In this section, we review the construction and properties of the reduced version and the annular version of Khovanov homology.

2.1 Reduced Khovanov homology

The reduced version of Khovanov homology is defined in [10] as a categorification of the (normalized) Jones polynomial. We first recall the definition of the original Khovanov homology.

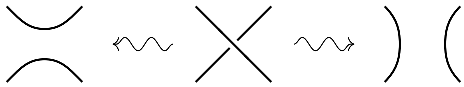

Figure 2: Two types of smoothings.

For a link diagram D with n crossings, denote the number of right-handed (resp. left-handed) crossings of D by n_+ (resp. n_-). For a crossing of D, we can use the 0–smoothing or 1–smoothing to resolve it, as shown in Figure 2. Fix an order of crossings; we can then use vectors $\boldsymbol{v} \in \{0,1\}^n$ to encode resolutions of D. Denote the resolution indicated by $\boldsymbol{v}$ by $D_{\boldsymbol{v}}$, and let $|\boldsymbol{v}|$ be the number of 1–smoothings in $D_{\boldsymbol{v}}$. Two resolutions that only differ on one smoothing of crossings are related by a cobordism. The resolutions of D are disjoint unions of circles, and the cobordisms are the merging or splitting of circles.

In original Khovanov homology, we apply a (1+1)D TQFT to the resolution cube to obtain a chain complex by assigning to each circle a graded free abelian group $V := \mathbb{Z}/2\mathbb{Z}\{v_+, v_-\}$. The resulting complex has two gradings: the homological one and the quantum one, and the latter is specified by $q\deg v_\pm = \pm 1$. Following [4], we denote the shifts on these two gradings by $[\bullet]$ and $\{\bullet\}$, respectively. We then take a shift on the quantum grading of chain groups by $|\boldsymbol{v}|$ to ensure the differential preserves the quantum grading and a global shift $[-n_-]\{n_+ - 2n_-\}$ to ensure invariance under Reidemeister moves. We finally take cohomology on the chain complex $(\mathrm{CKh}(L), d)$ to obtain $\mathrm{Kh}(L)$.

Remark 2.1 The gradings of Khovanov homology can be read as follows [8]. For a diagram D and a state S associated to a fixed resolution, let $w(D) = n_+ - n_-$ be the writhe number of D, $\sigma(S)$ be the difference between the numbers of 1–resolutions and 0–resolutions of S, and $\tau(S)$ be the difference between the numbers of v_+ and v_- appearing in S. Then the homological grading and the quantum grading of an element are given by

$$h\deg = \tfrac{1}{2}(\sigma(S) + w(D)), \quad q\deg = \tfrac{1}{2}(\sigma(S) + 2\tau(S) + 3w(D)),$$

respectively.

To define the reduced version of Khovanov homology, as in other reduced theories, we need to choose a basepoint p on the link L. Every resolution of L has exactly one circle containing p, and the generators that take v_- (with the q–grading omitted) on this pointed circle span a subcomplex $\mathrm{CKhr}(L,p) \subset \mathrm{CKh}(L)$. The *reduced Khovanov homology* $\mathrm{Khr}(L,p)$ is then defined by the cohomology of $\mathrm{CKhr}(L,p)$. The basepoint is sometime omitted in the notation if it is clear from the context (eg when we are considering an augmented link). As an example, for the Hopf link H with a positive linking number, we have

$$\mathrm{Khr}(H,p) = (\mathbb{Z}/2\mathbb{Z})^{(0,1)} \oplus (\mathbb{Z}/2\mathbb{Z})^{(2,5)}.$$

In general, the following proposition describes the effect on Khovanov homology of making a connected sum with a Hopf link. Here our statement is slightly different to the original description because of the different grading conventions. See [1, Remark 1.6] and Remark 2.1.

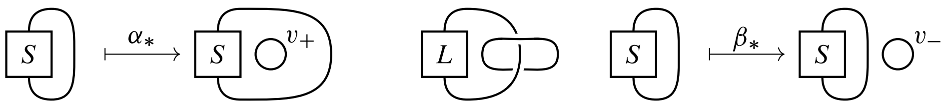

Figure 3: The maps α_* and β_*.

Proposition 2.2 [1, Theorem 6.1] *Let L be a pointed link and let H be the Hopf link with a positive linking number. Then we have a short exact sequence*

$$0 \to \mathrm{Khr}^{i-1,j-2}(L) \xrightarrow{\alpha_*} \mathrm{Khr}^{i+1,j+3}(L \# H) \xrightarrow{\beta_*} \mathrm{Khr}^{i+1,j+2}(L) \to 0.$$

Here α_* and β_* are given on a state S as in Figure 3.

2.2 Annular Khovanov homology

The annular version of Khovanov homology can be viewed as a special case of the link homology for links in thickened surfaces defined in [2]. Let A be an annulus. The annular Khovanov homology assigns a triply graded abelian group $\mathrm{AKh}(L)$ for each annular link $L \subset A \times I$. We follow the process and notation of [18].

Let D be a link diagram of L and define $n, n_\pm, \boldsymbol{v}, D_{\boldsymbol{v}}, V$ as in the previous subsection. In the annular case, there might be two types of circles in a resolution: circles that bound disks and circles with nontrivial homologies. We call the first type of circles *trivial* and the second ones *nontrivial*. To obtain the chain groups, we assign V to trivial circles and assign $W := \mathbb{Z}/2\mathbb{Z}\{w_+, w_-\}$ to nontrivial circles. The differentials are specified by the map corresponding to the merging or splitting of circles, as follows:

- Two trivial circles merge into a trivial circle, or one trivial circle splits into two trivial circles. In these cases, the maps are given the same as in Khovanov's original TQFT.
- One trivial circle and one nontrivial circle merge into a nontrivial circle. In this case, the maps are given by
$$v_+ \otimes w_\pm \mapsto w_\pm, \quad v_- \otimes w_\pm \mapsto 0.$$
- One nontrivial circle splits into a trivial circle and a nontrivial circle. In this case, the maps are given by
$$w_\pm \mapsto v_- \otimes w_\pm.$$
- Two nontrivial circles merge into a trivial circle. In this case, the maps are given by
$$w_\pm \otimes w_\pm \mapsto 0, \quad w_\pm \otimes w_\mp \mapsto v_-.$$
- One trivial circle splits into two nontrivial circles. In this case, the maps are given by
$$v_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+, \quad v_- \mapsto 0.$$

The homological and quantum grading are given the same as the original case with the additional request that $q \deg w_\pm = \pm 1$. After appropriate shifts, the differential is still filtered of degree $(1, 0)$.

There is the third grading on the chain complex, the so-called *annular grading* or f–grading, which is specified by $f\deg v_{\pm} = 0$ and $f\deg w_{\pm} = \pm 1$. The differential preserves the f–grading and hence it descends onto the cohomology groups $\mathrm{AKh}(L)$, the annular Khovanov homology.

Theorem 2.3 [2] *The annular Khovanov homology* $\mathrm{AKh}(L)$ *is an invariant of annular links in the sense that it is independent of the choice of link diagrams and the order of crossings.*

We conclude this section with some additional remarks.

Remark 2.4 Sometimes we write $\mathrm{AKh}(L, m)$ to indicate the f–degree m summand of $\mathrm{AKh}(L)$. If L is contained in a ball $B^3 \subset A \times I$, then $\mathrm{AKh}(L)$ is supported on $f = 0$ and $\mathrm{AKh}(L) \cong \mathrm{Kh}(L)$. Both the reduced Khovanov homology and the annular Khovanov homology are functorial [8]. That is, a link cobordism $\rho\colon L_1 \to L_2$ between links (resp. annular links) induces a (filtered) map between Khovanov homology groups

$$\mathrm{Khr}(\rho)\colon \mathrm{Khr}(L_1) \to \mathrm{Khr}(L_2) \quad (\text{resp. } \mathrm{AKh}(\rho)\colon \mathrm{AKh}(L_1) \to \mathrm{AKh}(L_2)).$$

3 The unlink case

In this section, we construct an isomorphism between the annular Khovanov homology of an annular unlink and the reduced Khovanov homology of its augmentation. We show that such an isomorphism is compatible with the group homomorphisms induced by the cobordism maps.

3.1 Homology groups

Denote the annular unlink with n nontrivial unknot components by U_n and let $\widetilde{U}_n$ be its augmentation, which corresponds to the graph shown in Figure 4 in the language of [20], as described before Theorem 1.5.

The obvious diagram of U_n contains n disjoint nontrivial circles. In this section, we will use this diagram to calculate homology groups. We assign the numbers 1 to n from the innermost nontrivial circle to the outermost one. Applying Proposition 2.2 inductively on unknot components, we can calculate the Poincaré polynomial of $\mathrm{Khr}(\widetilde{U}_n)$ as

$$P(\widetilde{U}_n) = (tq^3)^n (tq^2 + t^{-1}q^{-2})^n.$$

Here the homological and quantum gradings are indicated by t and q, respectively.

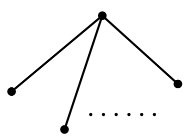

Figure 4: The tree corresponding to $\widetilde{U}_n$.

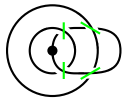

Figure 5: The symmetric resolution (10) of $\widetilde{U}_2$.

Each original component of $\widetilde{U}_n$ has two crossings with the meridian circle. There are 2^n resolutions such that every pair of crossings is resolved by the same smoothing. We say such resolutions are *symmetric* and encode them by 0–1 sequences of length n, as illustrated in Figure 5. Notice that a symmetric resolution always has n (unpointed) components. We denote the cobordism of changing one crossing (on the k^{th} strand) from 0–smoothing to 1–smoothing by $(\cdots\bullet\cdots)$ (here the mark $\bullet$ is on the k^{th} digit).

We can now describe the generators of $\mathrm{Khr}(\widetilde{U}_n)$ explicitly.

Proposition 3.1 *For each symmetric resolution $v \in \{0,1\}^n$, we can choose an element e_v lying in the chain group corresponding to this resolution. The collection of the e_v descends to a generating set of* $\mathrm{Khr}(\widetilde{U}_n)$.

Proof We prove the proposition by induction. There is nothing to say for $n=0$. For $n=1$, one can easily check that $e_{(1)} = v_+$ and $e_{(0)} = v_-$ gives a generating set of $\mathrm{Khr}(\widetilde{U}_1)$. In general, by applying Proposition 2.2 to $L = \widetilde{U}_{n-1}$ and $L \# H = \widetilde{U}_n$, we obtain a short exact sequence

$$0 \to \mathrm{Khr}^{i-1,j-2}(\widetilde{U}_{n-1}) \xrightarrow{\alpha_*} \mathrm{Khr}^{i+1,j+3}(\widetilde{U}_n) \xrightarrow{\beta_*} \mathrm{Khr}^{i+1,j+2}(\widetilde{U}_{n-1}) \to 0. \tag{3.2}$$

Here α_* and β_* come from the corresponding maps on the chain level.

Let $v = (v_1, v_2, \dots, v_n) \in \{0,1\}^n$ and let $v' = (v_1, \dots, v_{n-1})$. The sequence v' corresponds to a symmetric resolution $R'_{v'}$ of $\widetilde{U}_{n-1}$. If $v_n = 1$, we just need to take

$$e_v = \alpha_*(e_{v'}) = e_{v'} \otimes v_+.$$

If $v_n = 0$ and $e_{v'} = A \otimes v_+ + B \otimes v_-$, where $v_\pm$ are associated to the $(n-1)^{\text{st}}$ circle, we take

$$e_v = e_{v'} \otimes v_- + A \otimes v_- \otimes v_+.$$

It remains to show that e_v is a cycle. Assuming this, then we have $\beta_*(e_v) = e_{v'}$, and the conclusion follows from the short exact sequence (3.2) and the inductive hypothesis. Notice that the cobordism $(v', \bullet)$ always corresponds to a merging (rather than a splitting) of circles, and the construction ensures that $\mathrm{Khr}((v',\bullet))(e_v) = 0$. We show that other cobordisms also vanish by discussing the value of v_{n-1} (see Figure 6).

Notice that the cobordism map vanishes on A and B if the change is on the i^{th} strand $(1 \le i \le n-2)$. Hence, if $v_{n-1} = 1$, then there is no possibly nonvanishing cobordism map. Now assume that $v_{n-1} = 0$ and let $v'' = (v_1, \dots, v_{n-2})$,

$$e_{v''} = A_1 \otimes v_+ + B_1 \otimes v_-.$$

$v_{n-1} = 1$ $\qquad$ $v_{n-1} = 0$

Figure 6: Possible resolutions with $v_n = 0$.

Then

$$e_{v'} = (A_1 \otimes v_+ + B_1 \otimes v_-) \otimes v_- + A_1 \otimes v_- \otimes v_+,$$
$$e_v = ((A_1 \otimes v_+ + B_1 \otimes v_-) \otimes v_- + A_1 \otimes v_- \otimes v_+) \otimes v_- + A_1 \otimes v_- \otimes v_- \otimes v_+,$$

and hence $\mathrm{Khr}((v'', \bullet, 1))(e_v) = 0$. □

We now construct an explicit identification between $\mathrm{AKh}(U_n)$ and $\mathrm{Khr}(\widetilde{U}_n)$. On the level of homology, this is quite easy: The Poincaré polynomial of $\mathrm{AKh}(U_n)$ is given by

$$P(U_n) = (fq + f^{-1}q^{-1})^n.$$

Here the f–grading is indicated by f. The substitution $f \mapsto tq$ gives an isomorphism between $\mathrm{AKh}(U_n)$ and $\mathrm{Khr}(\widetilde{U}_n)$ (up to shifting). More concretely, the generator

$$w = w_\pm^{(1)} \otimes w_\pm^{(2)} \otimes \cdots \otimes w_\pm^{(n)} \in \mathrm{AKh}(U_n)$$

is identified with the generator corresponding to the symmetric resolution of label $(v_1, v_2, \ldots, v_n)$, where $v_i = 1$ if and only if $w_+^{(i)}$ appears in w ($i = 1, 2, \ldots, n$), as in Proposition 3.1.

The effect of adding a trivial unknot component to U_n is just taking two copies of the original homology groups with generators tensoring with $v_\pm$, respectively, by the Künneth formula.

Now we discuss the grading shifts. Let U be an annular unlink with m trivial unknot components and n nontrivial unknot components. Let

$$w = v_\pm^{(1)} \otimes \cdots \otimes v_\pm^{(m)} \otimes w_\pm^{(1)} \otimes w_\pm^{(2)} \otimes \cdots \otimes w_\pm^{(n)}$$

be a generator of $\mathrm{AKh}^{i,j,k}(U)$ and $\Phi_U(w) \in \mathrm{Khr}^{i',j'}(\widetilde{U})$ be the generator corresponding to w. Assume that w_+ (resp. w_-) appears t_+ (resp. t_-) times in w. Then $k = t_+ - t_-$ and $n = t_+ + t_-$. The homological grading i' increases by $2t_+ = k + n$, and the quantum grading j' increases by $2n + 2t_+ = k + 3n$.

We summarize the consequence of this subsection in the following form:

Theorem 3.3 *Let U be an annular unlink with n nontrivial unknot components, and let $\widetilde{U}$ be its augmentation. Then there is an isomorphism Φ_U between the annular Khovanov homology of U and the reduced Khovanov homology of $\widetilde{U}$. More precisely, we have an isomorphism*

$$\Phi_U : \mathrm{AKh}^{i,j,k}(U) \to \mathrm{Khr}^{i+k+n, j+k+3n}(\widetilde{U}).$$

The correspondence of generators is as given above. □

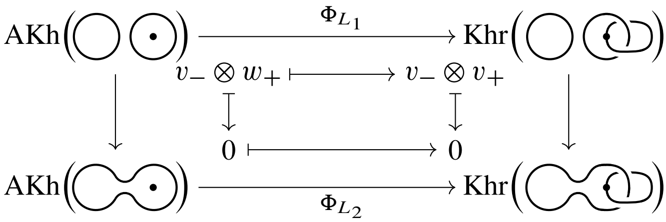

Figure 7: Case (a).

3.2 Functoriality

A cobordism between annular links naturally induces a cobordism between their augmentations. In this subsection, we show that the isomorphism Φ_L defined in Theorem 3.3 is compatible with cobordisms. For our purpose (see the next section), we don't need to deal with the Reidemeister moves on the diagram of L, and we concentrate on Morse moves, ie the merging and splitting of circles. We first verify the compatibility with only related circles and then consider the effect of adding other unlink components. There are four cases we need to discuss:

(a) one trivial circle and one nontrivial circle merge into a nontrivial circle;

(b) one nontrivial circle splits into a trivial circle and a nontrivial circle;

(c) two nontrivial circles merge into a trivial circle;

(d) one trivial circle splits into two nontrivial circles.

Since the homomorphisms induced by cobordisms are well defined [8], we may choose specific link diagrams to calculate them. Cases (a) and (b) are simple diagram chasing. Figure 7 illustrates this process.

In cases (c) and (d), we need to check the diagrams in Figure 8 commute.

Denote the upper and the lower links in the leftmost column of Figure 8 by L_3 and L_4, respectively. We have

$$\mathrm{Khr}(\widetilde{L}_3) \cong (\mathbb{Z}/2\mathbb{Z})^{(0,2)} \oplus ((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{(4,10)},$$
$$\mathrm{Khr}(\widetilde{L}_4) \cong (\mathbb{Z}/2\mathbb{Z})^{(0,1)} \oplus (\mathbb{Z}/2\mathbb{Z})^{(0,-1)}.$$

We first check case (c). Notice that $w(D)$, $\sigma(S)$ and $\tau(S)$ decrease by 4, 0 and 1 from $\widetilde{L}_3$ to $\widetilde{L}_4$, respectively. Hence, the cobordism map $\mathrm{Khr}(\widetilde{L}_3) \to \mathrm{Khr}(\widetilde{L}_4)$ is of degree $(-2,-7)$ by Remark 2.1, and

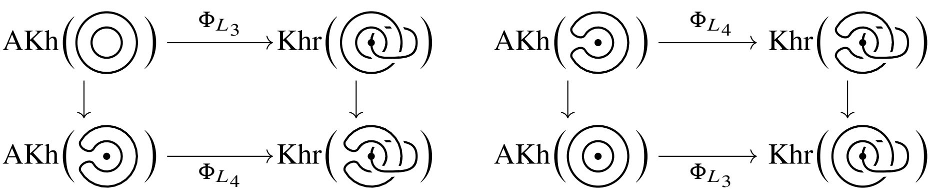

Figure 8: Cases (c) and (d).

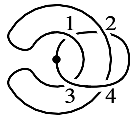

Figure 9: The labeling of crossings on $\widetilde{L}_4$.

the only possibly nontrivial map is

$$((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2} \to (\mathbb{Z}/2\mathbb{Z})^{(0,-1)},$$

which corresponds to the merging map in the leftmost column of Figure 8,

$$w_+ \otimes w_-, w_- \otimes w_+ \mapsto v_-.$$

By the algorithm given in Theorem 3.3, $w_- \otimes w_+$ and $w_+ \otimes w_-$ correspond to $v_- \otimes v_+$ (associated to the symmetric resolution (01)) and $v_+ \otimes v_- + v_- \otimes v_+$ (associated to the symmetric resolution (10)), respectively. Their images are $v_- \otimes v_+ \otimes v_-$ and v_-, respectively. It suffices to show they are nonvanishing and cohomologous.

To write down the differentials

$$d^{(-1,-1)}\colon \mathrm{CKhr}^{(-1,-1)}(\widetilde{L}_4) \to \mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4) \quad \text{and} \quad d^{(0,-1)}\colon \mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4) \to \mathrm{CKhr}^{(1,-1)}(\widetilde{L}_4)$$

in matrix form, we need to fix orders of bases of the chain groups as follows. Assign the crossing numbers 1 to 4 as in Figure 9. We first take the lexicographical order on the resolutions (ie take the states associated to the resolution (1100) first, then (1010), etc). Most resolutions correspond to exactly one state in these chain groups, except resolutions (0100), (0001) and (0101). For them, we give the order by where the unique v_+ appears (from top to bottom). Under this convention, we denote the bases of chain groups $\mathrm{CKhr}^{(-1,-1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(0,-1)}(\widetilde{L}_4)$ by e_i $(1 \le i \le 6)$ and f_j $(1 \le j \le 8)$, respectively. We have

$$d^{(-1,-1)} = \begin{pmatrix} 1&1&0&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&1&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&1 \end{pmatrix} \quad \text{and} \quad d^{(0,-1)} = \begin{pmatrix} 1&0&1&0&1&0&0&0\\ 0&0&0&1&0&0&1&1 \end{pmatrix}.$$

The elements v_- and $v_- \otimes v_+ \otimes v_-$ correspond to the vectors f_2 and f_6, respectively. It is easy to see that $f_2 - f_6 = d^{(-1,-1)}(e_1 + e_2 + e_5)$ and $f_2 \notin \operatorname{Im} d^{(-1,-1)}$. This finishes the verification in case (c).

The verification in case (d) is essentially the same. The only possibly nontrivial map in the rightmost column of Figure 8 is

$$(\mathbb{Z}/2\mathbb{Z})^{(0,1)} \to ((\mathbb{Z}/2\mathbb{Z})^{(2,6)})^{\oplus 2},$$

which corresponds to the splitting map

$$v_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+$$

in the third column of Figure 8. We give orders for the bases of $\mathrm{CKhr}^{(-1,1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(0,1)}(\widetilde{L}_4)$, $\mathrm{CKhr}^{(1,6)}(\widetilde{L}_3)$ and $\mathrm{CKhr}^{(2,6)}(\widetilde{L}_3)$ as in case (c). The only exception is that, for the resolution (0101) in $\mathrm{CKhr}^{(0,1)}(\widetilde{L}_4)$, we give the order according to the position of the unique v_- (from top to bottom). Under this convention, we have

$$d^{(0,1)} = \begin{pmatrix} 1&1&0&1&0&0&0&0\\ 0&1&1&0&0&0&0&1\\ 1&0&1&0&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&1&0&1&0&1\\ 0&0&0&0&1&0&0&1 \end{pmatrix} \quad \text{and} \quad d^{(-1,1)} = \begin{pmatrix} 1&0&0&1&0&1&1&0\\ 0&0&1&0&1&1&0&1 \end{pmatrix}^T.$$

The generator of $\mathrm{Khr}^{(0,1)}(\widetilde{L}_4)$ can be represented by $v^{(0,1)} = (1,1,0,0,1,1,1,1)^T$, and we have

$$\mathrm{Khr}(\widetilde{L}_4) \to \mathrm{Khr}(\widetilde{L}_3) \colon v^{(0,1)} \mapsto (1,0,1,0,0,1,0,1)^T.$$

The boundary subgroup of degree $(2,6)$ is spanned by the image of

$$d^{(1,6)} = \begin{pmatrix} 1&1&0&0\\ 1&0&1&0\\ 0&0&0&0\\ 1&0&0&1\\ 0&1&1&0\\ 0&1&0&1\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix}.$$

Therefore, under the map $\mathrm{Khr}(\widetilde{L}_4) \to \mathrm{Khr}(\widetilde{L}_3)$, we have

$$v^{(0,1)} \mapsto \Phi_{L_3}(w_+ \otimes w_- + w_- \otimes w_+) + d^{(1,6)}(1,0,0,1)^T.$$

This completes the verification in case (d).

It remains to consider the effect of adding a new unlink component to the cobordism. The case of adding a trivial unknot component is trivial and we assume that the additional unknot component is nontrivial. Let L_1 and L_2 be two annular unlinks and let $\rho \colon L_1 \to L_2$ be a cobordism obtained by a Morse move. We have

$$\mathrm{AKh}(\rho \amalg \mathrm{id}) = \mathrm{AKh}(\rho) \otimes \mathrm{id}_U;$$

here $U = U_1$ is the nontrivial annular unknot. Take $S \in \mathrm{AKh}(L_1)$ and let $T = \mathrm{AKh}(\rho)(S)$. By Proposition 2.2 and Theorem 3.3, the following diagram commutes:

$$\begin{tikzcd} S \otimes w_+ \arrow[r, mapsto, "\Phi_{L_1}"] \arrow[d, mapsto, "\mathrm{AKh}(\rho)"'] & \Phi_{L_1}(S) \otimes v_+ \arrow[d, mapsto, "\mathrm{Khr}(\rho')"] \\ T \otimes w_+ \arrow[r, mapsto, "\Phi_{L_2}"'] & \Phi_{L_2}(S) \otimes v_+ \end{tikzcd}$$

Assume that $\Phi_{L_1}(S) = A \otimes v_+ + B \otimes v_-$ and $\Phi_{L_2}(T) = C \otimes v_+ + D \otimes v_-$. By Proposition 2.2 and Theorem 3.3, the diagram

$$\begin{tikzcd} S \otimes w_- \arrow[r, mapsto, "\Phi_{L_1}"] \arrow[d, mapsto, "\mathrm{AKh}(\rho)"'] & \Phi_{L_1}(S) \otimes v_- + A \otimes v_- \otimes v_+ \arrow[d, mapsto, "\mathrm{Khr}(\widetilde{\rho})"] \\ T \otimes w_- \arrow[r, mapsto, "\Phi_{L_2}"'] & \Phi_{L_2}(T) \otimes v_- + C \otimes v_- \otimes v_+ \end{tikzcd}$$

commutes, which completes the proof.

In summary, we have shown the following theorem. Roughly speaking, it gives a natural isomorphism between two cohomology theories on annular unlinks.

Theorem 3.4 *Let L_1 and L_2 be two annular unlinks and let $\rho\colon L_1 \to L_2$ be a cobordism obtained by composition of Morse moves. The cobordism ρ induces a cobordism $\widetilde{\rho}$ between the augmentations $\widetilde{L}_1$ and $\widetilde{L}_2$. Let Φ_{L_1} and Φ_{L_2} be the isomorphisms given in Theorem 3.3. Then the following diagram commutes*:

$$\begin{tikzcd} \mathrm{AKh}(L_1) \arrow[r, "\Phi_{L_1}"] \arrow[d, "\mathrm{AKh}(\rho)"'] & \mathrm{Khr}(\widetilde{L}_1) \arrow[d, "\mathrm{Khr}(\widetilde{\rho})"] \\ \mathrm{AKh}(L_2) \arrow[r, "\Phi_{L_2}"'] & \mathrm{Khr}(\widetilde{L}_2) \end{tikzcd}$$

□

4 The spectral sequence

In this section, we prove Theorem 1.2 and discuss some examples and applications. To prove Theorem 1.2, we choose a link diagram as shown in Figure 10. For convenience, we call the strands appearing in the right of the left diagram the *annular strands*.

Proof of Theorem 1.2 Fix a link diagram D as in Figure 10. Crossings of $\widetilde{L}$ can be classified into two types: crossings of the augmenting circle and the annular strands, and the original crossings of L. We encode the resolutions of the first type of crossings by 0–1 sequences $\boldsymbol{w}_1$ and the second type by $\boldsymbol{w}_2$. Then the chain complex $\mathrm{CKhr}(\widetilde{L})$ can be encoded by the concatenation $\boldsymbol{v} = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. Every summand of the

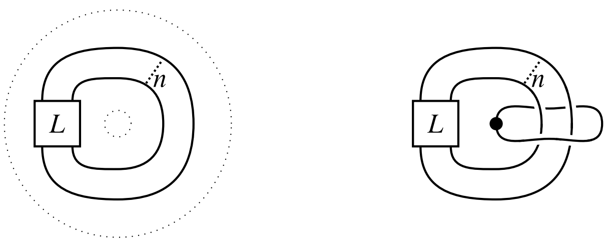

Figure 10: A standard link diagram and its augmentation.

differential comes from exactly one change of the smoothing, and the differential splits as $d = d_1 + d_2$, where d_i corresponds to the changes of smoothing on type i crossings. Denote the partial resolution of $\widetilde{L}$ on $\boldsymbol{w}_2$ by $\widetilde{L}_{\boldsymbol{w}_2}$, which is also the augmentation of the annular unlink $L_{\boldsymbol{w}_2}$ and hence there is no ambiguity.

The chain complex $\mathrm{CKhr}(\widetilde{L})$ is bigraded by $(|\boldsymbol{w}_1|, |\boldsymbol{w}_2|)$, and the differentials d_1 and d_2 have degrees $(1,0)$ and $(0,1)$, respectively. The spectral sequence of double complexes [6, Section III.7, Proposition 10] applies. The E_1 term is given by the cohomology of $(\mathrm{CKhr}(\widetilde{L}), d_1)$, which is a chain complex with chain groups $\mathrm{Khr}(\widetilde{L}_{\boldsymbol{w}_2})$ and differentials given by cobordisms. Since the link diagram is fixed, such cobordisms correspond to Morse moves. By Theorem 3.4, the E_1 term is isomorphic to the chain complex that calculates $\mathrm{AKh}(L)$, and hence the E_2 term is isomorphic to $\mathrm{AKh}(L)$. The spectral sequence converges to the cohomology of $(\mathrm{CKhr}(\widetilde{L}), d)$, ie $\mathrm{Khr}(\widetilde{L})$.

A Reidemeister move induces an isomorphism between the converging terms that is compatible with the filtration, and an isomorphism between the E_2 terms. The comparison theorem then applies and hence the spectral sequence is independent of the choice of the link diagram. □

Example 4.1 Consider the annular link L shown in Figure 1. The augmentation $\widetilde{L}$ is isotopic to the link L5a1 and $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}(L) = 8 = \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}(\widetilde{L})$. Hence, the spectral sequence collapses at the E_2 term. This illustrates that the spectral sequence can collapse for links not isotopic to braid closures.

We can derive a finer rank inequality from Theorem 1.2.

Corollary 4.2 *Let L and $\widetilde{L}$ be as in Theorem 1.2. Let n_0 be the number of annular strands and n'_- be the number of left-handed crossings on the augmenting circle. Then*

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}^n(\widetilde{L}) \leq \sum_{n_a+f_a+n_0-n'_-=n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}^{n_a}(L, f_a).$$

Proof Denote the gradings of $\mathrm{AKh}(L)$ by (n_a, q_a, f_a). Let $\tilde{n}_-$ be the number of left-handed crossings of $\widetilde{L}$. Then $\tilde{n}_- = n_- + n'_-$ and $n_a = |\boldsymbol{w}_2| - n_-$. Let n'_0 be the number of nontrivial unknot components of a specific partial resolution. Let $(n'_0)_+$ (resp. $(n'_0)_-$) be the number of 1–smoothings (resp. 0–smoothings). Then, by Theorem 3.4, we have

$$f_a = (n'_0)_+ - n'_0 = \tfrac{1}{2}((n'_0)_+ - (n'_0)_-) = |\boldsymbol{w}_1| - n_0$$

on the E_1 term. On the E_∞ term, we have $n = |\boldsymbol{w}_1| + |\boldsymbol{w}_2| - \tilde{n}_-$. Therefore, from Theorem 1.2, we obtain

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Khr}^n(\widetilde{L}) &= \sum_{|\boldsymbol{w}_1|+|\boldsymbol{w}_2|-\tilde{n}_-=n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} E_\infty^{|\boldsymbol{w}_1|,|\boldsymbol{w}_2|} \\ &\leq \sum_{|\boldsymbol{w}_1|+|\boldsymbol{w}_2|-\tilde{n}_-=n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} E_2^{|\boldsymbol{w}_1|,|\boldsymbol{w}_2|} \\ &= \sum_{n_a+f_a+n_0-n'_-=n} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}} \mathrm{AKh}^{n_a}(L, f_a).\end{aligned}$$ □

We now prove Theorem 1.5 and Corollary 1.6. The following simple observation is useful:

Lemma 4.3 *Let L be an annular link with a link diagram such that there is only one annular strand. View L as a link in S^3 and let p be a basepoint on this annular strand. Then $\mathrm{AKh}(L)$ is supported on $f=\pm 1$, and*

$$\mathrm{AKh}(L,\pm 1)\cong \mathrm{Khr}(L,p).$$

Proof There is exactly one nontrivial circle in each resolution of L, which is the circle containing p. Hence, the chain complex is supported on $f=\pm 1$. Furthermore, the subcomplexes of f–grading ± 1 are isomorphic to $\mathrm{CKhr}(L)$ by replacing the generators $w_\pm$ of the nontrivial circle by v_-, respectively. □

Proof of Theorem 1.5 Let G be a forest such that each connected component contains at most one annular vertex. Then L_G is a disjoint union of links with at most one annular strand. Then Lemma 4.3 applies and we have $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\mathrm{AKh}(L_G)=2^n$ by the Künneth formula.

Conversely, let L be an annular link with n components and

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\mathrm{AKh}(L)=2^n.$$

Then Corollary 1.3 gives $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\mathrm{Khr}(\widetilde{L})=2^n$. By [20, Theorem 1.2], $\widetilde{L}$ is a forest of unknots in S^3. Therefore, L is a forest of unknots in $A\times I$. Denote their corresponding forests by $\widetilde{G}$ and G, respectively. Notice that $\widetilde{G}$ is constructed from G by adding a vertex adjacent to all the annular vertices. Two annular vertices cannot lie in the same connected component of G since otherwise a cycle would occur in $\widetilde{G}$, which is absurd since $\widetilde{G}$ is a forest. □

Proof of Corollary 1.6 By Theorem 1.5, L is a forest of unknots in $A\times I$. Denote the corresponding forest by G. If G had an edge, then $\mathrm{AKh}(L)$ would not be supported on $t=0$ as $\mathrm{AKh}(U)$ is (see the discussion in Section 3.1), which is a contradiction. Hence, every vertex is an independent connected component of G, ie L is an annular unlink. The number of nontrivial unknot components in L can be read from the Poincaré polynomial of L. Therefore, L is isotopic to U. □

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Smith ideals of operadic algebras in monoidal model categories

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Building upon Hovey's work on Smith ideals for monoids, we develop a homotopy theory of Smith ideals for general operads in a symmetric monoidal category. For a sufficiently nice stable monoidal model category and an operad satisfying a cofibrancy condition, we show that there is a Quillen equivalence between a model structure on Smith ideals and a model structure on algebra morphisms induced by the cokernel and the kernel. For symmetric spectra, this applies to the commutative operad and all Σ–cofibrant operads. For chain complexes over a field of characteristic zero and the stable module category, this Quillen equivalence holds for all operads. We end with a comparison between the semi-model category approach and the ∞–category approach to encoding the homotopy theory of algebras over Σ–cofibrant operads that are not necessarily admissible.

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1 Introduction

A major part of stable homotopy theory is the study of structured ring spectra. These include strict ring spectra, commutative ring spectra, A_∞–ring spectra, E_∞–ring spectra, E_n–ring spectra, and so forth. Based on an unpublished talk by Jeff Smith, Hovey [2014] developed a homotopy theory of Smith ideals for ring spectra and monoids in more general symmetric monoidal model categories.

Let us briefly recall Hovey's work. For a symmetric monoidal closed category M, its arrow category $\vec{\mathsf{M}}$ is the category whose objects are morphisms in M and whose morphisms are commutative squares in M. It has two symmetric monoidal closed structures, namely the tensor product monoidal structure $\vec{\mathsf{M}}^{\otimes}$ and the pushout product monoidal structure $\vec{\mathsf{M}}^{\square}$. A monoid in $\vec{\mathsf{M}}^{\square}$ is a Smith ideal, and a monoid in $\vec{\mathsf{M}}^{\otimes}$ is a monoid morphism. If M is a model category, then $\vec{\mathsf{M}}^{\otimes}$ has the injective model structure $\vec{\mathsf{M}}^{\otimes}$, where weak equivalences and cofibrations are defined entrywise, and the category of monoid morphisms inherits a model structure from $\vec{\mathsf{M}}^{\otimes}$. Likewise, $\vec{\mathsf{M}}^{\square}$ has the projective model structure $\vec{\mathsf{M}}^{\square}$, where weak equivalences and fibrations are defined entrywise, and the category of Smith ideals inherits a model structure from $\vec{\mathsf{M}}^{\square}$. Surprisingly, when M is pointed (resp. stable), the cokernel and the kernel form a Quillen adjunction (resp. Quillen equivalence) between $\vec{\mathsf{M}}^{\square}$ and $\vec{\mathsf{M}}^{\otimes}$ and also between Smith ideals and monoid morphisms.

Since monoids are algebras over the associative operad, a natural question is whether there is a satisfactory theory of Smith ideals for algebras over other operads. For the commutative operad, White [2017] showed that commutative Smith ideals in symmetric spectra, equipped with either the positive flat (stable) or the positive (stable) model structure, inherit a model structure. The purpose of this paper is to generalize Hovey's work to Smith ideals for general operads in monoidal model categories. For an operad $\mathcal{O}$, we define a Smith $\mathcal{O}$–ideal as an algebra over an associated operad $\vec{\mathcal{O}}^{\square}$ in the arrow category $\vec{\mathsf{M}}^{\square}$. We will prove a precise version of the following result in Theorem 4.4.1:

Theorem A *Suppose* M *is a sufficiently nice stable monoidal model category, and* $\mathcal{O}$ *is a* $\mathfrak{C}$*–colored operad in* M *such that cofibrant Smith* $\mathcal{O}$*–ideals are also entrywise cofibrant in the arrow category of* M *with the projective model structure. Then there is a Quillen equivalence*

$$\{\text{Smith } \mathcal{O}\text{–ideals}\} \underset{\text{ker}}{\overset{\text{coker}}{\rightleftarrows}} \{\mathcal{O}\text{–algebra maps}\}$$

induced by the cokernel and the kernel.

For example, this theorem holds in the following situations:

1. (1) $\mathcal{O}$ is an arbitrary $\mathfrak{C}$–colored operad, and M is the category $\mathsf{Ch}(R)$ of bounded or unbounded chain complexes over a semisimple ring containing $\mathbb{Q}$ (Corollary 5.2.4); the stable module category of $k[G]$–modules for some field k and finite group G (Corollary 6.2.5); or the category of classical, equivariant or motivic symmetric spectra with the positive or positive flat stable model structure (Example 4.4.2).
2. (2) $\mathcal{O}$ is the commutative operad, and M is any of the examples above or equivariant orthogonal spectra, Hausmann's G–symmetric spectra [2017], or Schwede's global equivariant spectra [2018] with positive flat model structures (Section 5.1).
3. (3) $\mathcal{O}$ is $\Sigma_{\mathfrak{C}}$–cofibrant (eg the associative operad, A_∞–operads, E_∞–operads, and E_n–operads), and M is any of the examples above, or $\mathsf{Ch}(R)$ for a commutative ring R; StMod($k[G]$), where k is a principal ideal domain; an injective or projective model structure on spectra; S–modules [Elmendorf et al. 1997]; Mandell's equivariant symmetric spectra [2004]; or a Lydakis-style model structure on enriched functors (Corollary 5.2.3 and Examples 5.2.5 and 5.2.6).

The rest of this paper is organized as follows. In Section 2 we recall some basic facts about model categories and arrow categories. In Section 3 we define Smith ideals for an operad and prove that, when M is pointed, there is an adjunction between Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms given by the cokernel and the kernel. In Section 4 we define the model structures on Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms and prove the theorem above. We also include a discussion of what happens when there are only semi-model structures on Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms. In Section 5 we apply the theorem to the commutative operad and $\Sigma_{\mathfrak{C}}$–cofibrant operads. In Section 6 we apply the theorem to entrywise cofibrant operads. In Section 7 we include a comparison between various approaches to encoding the homotopy theory of operad algebras, including model categories, semi-model categories and ∞–categories. This discussion holds in general, beyond the situation of Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms.

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2 Model structures on the arrow category

In this section we recall a few facts about monoidal model categories and arrow categories. Our main references for model categories are [Hirschhorn 2003; Hovey 1999; Schwede and Shipley 2000]. In this paper, $(\mathsf{M}, \otimes, \mathbb{1}, \mathrm{Hom})$ will usually be a bicomplete symmetric monoidal closed category [Mac Lane 1998, VII.7] with monoidal unit $\mathbb{1}$, internal hom Hom, initial object $\varnothing$ and terminal object $*$. Since M is closed, $\varnothing \otimes X = \varnothing$ for any X.

2.1 Monoidal model categories

A model category is *cofibrantly generated* if there are sets I of cofibrations and J of trivial cofibrations (that is, morphisms that are both cofibrations and weak equivalences) that permit the small object argument (with respect to some cardinal κ), and a morphism is a fibration (resp. trivial fibration) if and only if it satisfies the right lifting property with respect to all morphisms in J (resp. I).

Let I–cell denote the class of transfinite compositions of pushouts of morphisms in I, and let I–cof denote retracts of such [Hovey 1999, 2.1.9]. In order to run the small object argument, we will assume the domains K of the morphisms in I (and J) are κ–small relative to I–cell (resp. J–cell). In other words, given a regular cardinal $\lambda \geq \kappa$ and any λ–sequence $X_0 \to X_1 \to \cdots$ formed of morphisms $X_\beta \to X_{\beta+1}$ in I–cell, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathsf{M}(K, X_\beta) \to \mathsf{M}\big(K, \operatorname*{colim}_{\beta<\lambda} X_\beta\big)$$

is a bijection. An object is *small* if there is some κ for which it is κ–small. We will say that a model category is *strongly cofibrantly generated* if the domains and codomains of I and J are small with respect to the entire category.

In Section 4, we will produce homotopy theories for operad algebras valued in arrow categories equipped with some model structure. Depending on the colored operad and properties of M, sometimes we will only have a semi-model structure on a category of algebras. However, as shown in Section 7, it still encodes the correct ∞–category. A semi-model category satisfies axioms similar to those of a model category, but one only knows that morphisms *with cofibrant domain* admit a factorization into a trivial cofibration followed by a fibration, and one only knows that trivial cofibrations *with cofibrant domain* lift against fibrations. To the authors' knowledge, every result about model categories has a corresponding result for semi-model categories, often obtained by first cofibrantly replacing everything in sight (see for example [Batanin and White 2024]).

Definition 2.1.1 [Batanin and White 2024, Definition 2.1] A *semi-model structure* on a category M consists of classes of weak equivalences W, fibrations F and cofibrations Q satisfying the following axioms:

(M1) Fibrations are closed under pullback.

(M2) The class W is closed under the two-out-of-three property.

(M3) W, F and Q are all closed under retracts.

(M4) (i) Cofibrations have the left lifting property with respect to trivial fibrations.
(ii) Trivial cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.

(M5) (i) Every morphism in M can be functorially factored into a cofibration followed by a trivial fibration.
(ii) Every morphism whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

If, in addition, M is bicomplete, then we call M a *semi-model category*. M is said to be *cofibrantly generated* if there are sets of morphisms I and J in M such that the class of fibrations (resp. trivial fibrations) is characterized by the right lifting property with respect to J (resp. I), the domains of I are small relative to I–cell, and the domains of J are small relative to morphisms in J–cell whose domain is cofibrant.

An adjunction with left adjoint L and right adjoint R is denoted by $L \dashv R$.

Definition 2.1.2 Suppose $L\colon \mathsf{M} \rightleftarrows \mathsf{N} : R$ is an adjunction between (semi-)model categories.

(1) We call $L \dashv R$ a *Quillen adjunction* if the right adjoint R preserves fibrations and trivial fibrations. In this case, we call L a *left Quillen functor* and R a *right Quillen functor*.

(2) We call a Quillen adjunction $L \dashv R$ a *Quillen equivalence* if, for each morphism $f : LX \to Y \in \mathsf{N}$ with X cofibrant in M and Y fibrant in N, f is a weak equivalence in N if and only if its adjoint $f^{\#}\colon X \to RY$ is a weak equivalence in M.

Definition 2.1.3 Suppose M is a category with pushouts and pullbacks.

(1) Given a solid-arrow commutative diagram

$$\begin{tikzcd}
A \arrow[dr, dotted, "f \boxslash g"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \times_D C \arrow[r] \arrow[d] & C \arrow[d, "g"] \\
& B \arrow[r, "f"] & D
\end{tikzcd}$$

in M in which the square is a pullback, the unique dotted induced morphism is denoted by $f \boxslash g$ and called the *pullback corner morphism* of f and g.

(2) Given a solid-arrow commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "g"] & \\
B \arrow[r] \arrow[drr, bend right, "f"'] & B \amalg_A C \arrow[dr, dotted, "f \circledast g"] & \\
& & D
\end{tikzcd}$$

in M in which the square is a pushout, the unique dotted induced morphism is denoted by $f \circledast g$ and called the *pushout corner morphism* of f and g.

In the next definition, we follow simplicial notation $0 \to 1$ so the reader can distinguish source and target at a glance.

Definition 2.1.4 Suppose $(\mathsf{M}, \otimes, \mathbb{1})$ is a monoidal category with pushouts. Suppose $f : X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ are morphisms in M. The pushout corner morphism

$$\begin{tikzcd}
X_0 \otimes Y_0 \arrow[r, "f\otimes 1"] \arrow[d, "1\otimes g"] & X_1 \otimes Y_0 \arrow[d] \arrow[ddr, bend left, "1\otimes g"] & \\
X_0 \otimes Y_1 \arrow[r] \arrow[drr, bend right, "f\otimes 1"] & (X_0 \otimes Y_1) \amalg_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \arrow[dr, "f \square g"] & \\
& & X_1 \otimes Y_1
\end{tikzcd}$$

of $f \otimes 1$ and $1 \otimes g$ is denoted by $f \square g$ and called the *pushout product* of f and g.

Definition 2.1.5 A symmetric monoidal closed category M equipped with a model structure is called a *monoidal model category* if it satisfies the following *pushout product axiom* [Schwede and Shipley 2000, Definition 3.1]:

- Given any cofibrations $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$, the pushout product morphism
$$(X_0 \otimes Y_1) \amalg_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \,\square\, g} X_1 \otimes Y_1$$
is a cofibration. If, in addition, either f or g is a weak equivalence, then $f \,\square\, g$ is a trivial cofibration.

Additionally, in order to guarantee that the unit $\mathbb{1}$ descends to the unit in the homotopy category, it is sometimes convenient to assume the *unit axiom* [Hovey 1999, 4.2.6]: if $Q\mathbb{1} \to \mathbb{1}$ is a cofibrant replacement, then, for any cofibrant object X, the induced morphism $Q\mathbb{1} \otimes X \to \mathbb{1} \otimes X \cong X$ is a weak equivalence. Since $(-) \otimes X$ is a left Quillen functor, if the unit axiom holds for one cofibrant replacement of $\mathbb{1}$, then it holds for any cofibrant replacement of $\mathbb{1}$.

2.2 Arrow categories

Definition 2.2.1 A *lax monoidal functor* $F\colon \mathsf{M} \to \mathsf{N}$ between two monoidal categories is a functor equipped with structure morphisms
$$FX \otimes FY \xrightarrow{F^2_{X,Y}} F(X \otimes Y), \quad \mathbb{1}^{\mathsf{N}} \xrightarrow{F^0} F\mathbb{1}^{\mathsf{M}}$$
for X and Y in M that are associative and unital in a suitable sense, as discussed in [Mac Lane 1998, XI.2], where this notion is referred to simply as a *monoidal functor*. If, furthermore, M and N are symmetric monoidal categories and F^2 is compatible with the symmetry isomorphisms, then F is called a *lax symmetric monoidal functor*. If the structure morphisms F^2 and F^0 are isomorphisms (resp. identity morphisms), then F is called a *strong monoidal functor* (resp. *strict monoidal functor*).

We now recall the two monoidal structures on the arrow category from [Hovey 2014].

Definition 2.2.2 Suppose $(\mathsf{M}, \otimes, \mathbb{1})$ is a symmetric monoidal category with pushouts.

(1) The *arrow category* $\vec{\mathsf{M}}$ is the category whose objects are morphisms in M, in which a morphism $\alpha\colon f \to g$ is a commutative square

(2.2.3)
$$\begin{tikzcd} X_0 \arrow[r, "\alpha_0"] \arrow[d, "f"'] & Y_0 \arrow[d, "g"] \\ X_1 \arrow[r, "\alpha_1"] & Y_1 \end{tikzcd}$$

in M. We will also write $\mathrm{Ev}_0\, f = X_0$, $\mathrm{Ev}_1\, f = X_1$, $\mathrm{Ev}_0\, \alpha = \alpha_0$ and $\mathrm{Ev}_1\, \alpha = \alpha_1$. The definition of $\vec{\mathsf{M}}$ does not require a monoidal structure on M.

(2) The *tensor product monoidal structure* on $\vec{\mathsf{M}}$ is given by the monoidal product
$$X_0 \otimes Y_0 \xrightarrow{f \otimes g} X_1 \otimes Y_1$$

for $f: X_0 \to X_1$ and $g: Y_0 \to Y_1$. The arrow category equipped with this monoidal structure is denoted by $\vec{\mathsf{M}}^{\otimes}$. The monoidal unit is $\mathrm{Id}: \mathbb{1} \to \mathbb{1}$.

(3) The *pushout product monoidal structure* on $\vec{\mathsf{M}}$ is given by the pushout product

$$(X_0 \otimes Y_1) \amalg_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

for $f: X_0 \to X_1$ and $g: Y_0 \to Y_1$. The arrow category equipped with this monoidal structure is denoted by $\vec{\mathsf{M}}^{\square}$. The monoidal unit is $\varnothing \to \mathbb{1}$.

(4) Defining $L_0(X) = (\mathrm{Id}: X \to X)$ and $L_1(X) = (\varnothing \to X)$ for $X \in \mathsf{M}$, there are adjunctions

(2.2.4)
$$\mathsf{M} \underset{\mathrm{Ev}_0}{\overset{L_0}{\rightleftarrows}} \vec{\mathsf{M}}^{\otimes}, \qquad \mathsf{M} \underset{\mathrm{Ev}_1}{\overset{L_1}{\rightleftarrows}} \vec{\mathsf{M}}^{\square}$$

with left adjoints on top and all functors strict symmetric monoidal.

2.3 Injective model structure

Theorem 2.3.1 [Hovey 2014, 2.1 and 2.2] *Suppose* M *is a model category.*

(1) *There is a model structure on* $\vec{\mathsf{M}}$, *called the* ***injective model structure***, *in which a morphism* $\alpha: f \to g$ *as in* (2.2.3) *is a weak equivalence (resp. cofibration) if and only if* α_0 *and* α_1 *are weak equivalences (resp. cofibrations) in* M. *A morphism* α *is a (trivial) fibration if and only if* α_1 *and the pullback corner morphism*

$$X_0 \xrightarrow{\alpha_1 \boxslash g} X_1 \times_{Y_1} Y_0$$

are (trivial) fibrations in M. *Note that this implies that* α_0 *is also a (trivial) fibration. The arrow category equipped with the injective model structure is denoted by* $\vec{\mathsf{M}}$.

(2) *If* M *is cofibrantly generated, then so is* $\vec{\mathsf{M}}$.

(3) *If* M *is a monoidal model category, then* $\vec{\mathsf{M}}^{\otimes}$ *equipped with the injective model structure is a monoidal model category, denoted by* $\vec{\mathsf{M}}^{\otimes}$.

(4) *If* M *satisfies the unit axiom, then so does* $\vec{\mathsf{M}}^{\otimes}$.

Proof This model structure is a special case of the injective model structure on a diagram category [Barwick 2010, 2.16]. Since the indexing category $\bullet \to \bullet$ is so simple, we can directly write down the generating (trivial) cofibrations and hence avoid the need to assume M is combinatorial, as in [White 2017, 5.5.1]. The generating cofibrations are of the form $L_1 i$ (where $i \in I$) and unit morphisms $\alpha_i: i \to U_1 \mathrm{Ev}_1 i$, where U_1 is the right adjoint of Ev_1 given by $U_1(X) = 1_X$. The generating trivial cofibrations are analogous, with $j \in J$ instead of $i \in I$. A morphism $\beta: f \to g$ has the right lifting property with respect to $L_1 i$ if and only if $\mathrm{Ev}_1 \beta$ has the right lifting property with respect to i, and β has the right lifting property with respect to α_i if and only if $\mathrm{Ev}_0 f \to \mathrm{Ev}_1 f \times_{\mathrm{Ev}_1 g} \mathrm{Ev}_0 g$ has the right lifting property with respect to i. Thus, these sets generate the injective model structure. The pushout product axiom and the unit axiom on $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ follows from the same on M [Barwick 2010, 4.51]. $\square$

2.4 Projective model structure

Theorem 2.4.1 [Hovey 2014, 3.1] *Suppose* M *is a model category.*

(1) *There is a model structure on* $\vec{\mathsf{M}}$, *called the* ***projective model structure***, *in which a morphism* $\alpha\colon f \to g$ *as in* (2.2.3) *is a weak equivalence* (*resp. fibration*) *if and only if* α_0 *and* α_1 *are weak equivalences* (*resp. fibrations*) *in* M. *A morphism* α *is a* (*trivial*) *cofibration if and only if* α_0 *and the pushout corner morphism*

$$X_1 \amalg_{X_0} Y_0 \xrightarrow{\alpha_1 \circledast g} Y_1$$

are (*trivial*) *cofibrations in* M. *Note that this implies that* α_1 *is also a* (*trivial*) *cofibration. The arrow category equipped with the projective model structure is denoted by* $\vec{\mathsf{M}}$.

(2) *If* M *is cofibrantly generated, then so is* $\vec{\mathsf{M}}$.

(3) *If* M *is a monoidal model category, then* $\vec{\mathsf{M}}^{\square}$ *equipped with the projective model structure is a monoidal model category, denoted by* $\vec{\mathsf{M}}^{\square}$.

(4) *If* M *satisfies the unit axiom, then so does* $\vec{\mathsf{M}}^{\square}$.

Proof (1) and (2) follow from [Hirschhorn 2003, 11.6.1]. For (3), Hovey [2014, 3.1] had the additional assumption that M be cofibrantly generated. However, White and Yau [2019a] proved that, if M is a monoidal model category, then so is $\vec{\mathsf{M}}^{\square}$. Lastly, for (4), note that a cofibrant replacement for the unit $\varnothing \to \mathbb{1}$ is $L_1(Q\mathbb{1})\colon \varnothing \to Q\mathbb{1}$. If f is cofibrant in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ (equivalently, a cofibration between cofibrant objects), then $L_1(Q\mathbb{1}) \mathbin{\square} f \to f$ is the same as $Q\mathbb{1} \otimes f \to f$. Thus, the unit axiom on $\vec{\mathsf{M}}^{\square}$ follows from the unit axiom on M. $\square$

For a category M with all small limits and colimits, recall from [Hovey 1999, Sections 1.1 and 6.1] that M is *pointed* if the unique morphism $\varnothing \to *$ is an isomorphism. In such a category, we define the *cokernel* of a morphism $f: X_0 \to X_1$ to be the morphism $\operatorname{coker} f: X_1 \to Z$ defined by the pushout:

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d] & X_1 \arrow[d, "\operatorname{coker} f"] \\ * \arrow[r] & Z \end{tikzcd}$$

Dually, the *kernel* of $f: X_0 \to X_1$ is the morphism $\ker f: A \to X_0$ defined by the pullback:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ker f"'] & * \arrow[d] \\ X_0 \arrow[r, "f"] & X_1 \end{tikzcd}$$

For the left adjoints L_0 and L_1 in (2.2.4), we note the equalities, for each object X,

(2.4.2)
$$\begin{aligned} \ker(L_0(X)) &= \ker(\mathrm{Id}\colon X \to X) = (\varnothing \to X) = L_1(X), \\ \operatorname{coker}(L_1(X)) &= \operatorname{coker}(\varnothing \to X) = (\mathrm{Id}\colon X \to X) = L_0(X). \end{aligned}$$

Most of the observations in Proposition 2.4.3 are from [Hovey 2014, 1.4, 4.1 and 4.3]. We provide proofs here for completeness.

Proposition 2.4.3 *Suppose M is a pointed symmetric monoidal category with all small limits and colimits.*

(1) *The cokernel is a strictly unital strong symmetric monoidal functor from $\vec{\mathsf{M}}^{\square}$ to $\vec{\mathsf{M}}^{\otimes}$ whose right adjoint is the kernel.*

(2) *The strong symmetric monoidality of the cokernel induces a strictly unital lax symmetric monoidal structure on the kernel such that the adjunction* (coker, ker) *is monoidal.*

(3) *If M is also a model category, then* (coker, ker) *is a Quillen adjunction.*

(4) *If M is a stable model category* [Hovey 1999, Chapter 7], *then* (coker, ker) *is a Quillen equivalence.*

Proof For (1), first note that coker preserves the units since the cokernel of $\varnothing \to \mathbb{1}$ is $\mathrm{Id}_{\mathbb{1}}$. Next, it is strong monoidal because, given $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ we can form the commutative diagram:

$$\begin{tikzcd}
X_1 \otimes Y_1 & X_0 \otimes Y_1 \arrow[l] \arrow[r] & * \\
X_1 \otimes Y_0 \arrow[u] \arrow[d, equal] & X_0 \otimes Y_0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u, equal] \arrow[d, equal] \\
X_1 \otimes Y_0 \arrow[r, equal] & X_1 \otimes Y_0 \arrow[r] & *
\end{tikzcd}$$

Vertical pushouts yield a span whose pushout is $\mathrm{coker}(f \mathbin{\square} g)$. Horizontal pushouts yield a span whose pushout is $\mathrm{coker}\, f \otimes \mathrm{coker}\, g$. Since pushouts commute, we obtain the natural isomorphism

$$(\mathrm{coker}\, f) \otimes (\mathrm{coker}\, g) \xrightarrow[\cong]{\mathrm{coker}^2_{f,g}} \mathrm{coker}(f \mathbin{\square} g). \tag{2.4.4}$$

We take this isomorphism as the (f, g) component of the monoidal constraint for coker. Using similar reasoning and the universal property of pushouts, one can show that the symmetric monoidal coherence diagrams commute.

For the statement that coker is left adjoint to ker, note that a morphism α from $\mathrm{coker}\, f$ to g is given by the diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d] & X_1 \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{coker}\, f"] & Y_0 \arrow[d, "g"] \\
* \arrow[r] & Z \arrow[r, "\alpha_1"'] & Y_1
\end{tikzcd}$$

These data are equivalent to a morphism from f to $\ker g$, since A is a pullback and Z is a pushout:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "\ker g"] & * \arrow[d] \\
X_1 \arrow[r, "\alpha_0"'] & Y_0 \arrow[r, "g"'] & Y_1
\end{tikzcd}$$

For (2), first note that $\ker\colon \vec{\mathsf{M}}^{\otimes} \to \vec{\mathsf{M}}^{\Box}$ preserves the monoidal units because the kernel of $\mathrm{Id}\colon \mathbb{1} \to \mathbb{1}$ is $\varnothing \to \mathbb{1}$. The monoidal constraint of the kernel at a pair of morphisms f and g,

$$\ker^2_{f,g}\colon (\ker f) \,\Box\, (\ker g) \to \ker(f \otimes g),$$

is adjoint to the following composite, with coker^2 the monoidal constraint in (2.4.4) and $\varepsilon\colon \mathrm{coker} \circ \ker \to \mathrm{Id}$ the counit of the adjunction:

(2.4.5)
$$\begin{tikzcd} \mathrm{coker}((\ker f) \,\Box\, (\ker g)) \arrow[d, "(\mathrm{coker}^2)^{-1}"', "\cong"] & \\ \mathrm{coker}(\ker f) \otimes \mathrm{coker}(\ker g) \arrow[r, "\varepsilon_f \otimes \varepsilon_g"] & f \otimes g \end{tikzcd}$$

The lax symmetric monoidal axioms for the kernel follow from those for the cokernel and the adjunction.

The assertion that the adjunction $(\mathrm{coker}, \ker)$ is monoidal means that its unit and counit are monoidal natural transformations [Mac Lane 1998, XI.2]. To prove this, first note that, by the above description of the adjunction, its unit and counit are the identity morphisms of the monoidal units in $\vec{\mathsf{M}}^{\Box}$ and $\vec{\mathsf{M}}^{\otimes}$, respectively.

To prove that the unit $\eta\colon \mathrm{Id} \to \ker \circ \mathrm{coker}$ is a monoidal natural transformation, it remains to show that the following diagram commutes for each pair of morphisms f and g:

$$\begin{tikzcd} f \,\Box\, g \arrow[r, "\eta_{f \Box g}"] \arrow[d, "\eta_f \Box \eta_g"'] & \ker(\mathrm{coker}(f \,\Box\, g)) \\ \ker(\mathrm{coker} f) \,\Box\, \ker(\mathrm{coker}\, g) \arrow[r, "\ker^2"] & \ker(\mathrm{coker} f \otimes \mathrm{coker}\, g) \arrow[u, "\ker(\cong)"'] \end{tikzcd}$$

This diagram commutes because the adjoint of each composite is the identity morphism of $\mathrm{coker}(f \,\Box\, g)$. For the long composite, this uses the naturality of $(\mathrm{coker}^2)^{-1}$ and one of the triangle identities for the adjunction $(\mathrm{coker}, \ker)$ [ibid., IV.1, Theorem 1].

To prove that the counit $\varepsilon\colon \mathrm{coker} \circ \ker \to \mathrm{Id}$ is a monoidal natural transformation, it remains to show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{coker}(\ker f) \otimes \mathrm{coker}(\ker g) \arrow[r, "\varepsilon_f \otimes \varepsilon_g"] \arrow[d, "\mathrm{coker}^2"', "\cong"] & f \otimes g \\ \mathrm{coker}(\ker f \,\Box\, \ker g) \arrow[r, "\mathrm{coker}(\ker^2_{f,g})"] & \mathrm{coker}(\ker(f \otimes g)) \arrow[u, "\varepsilon_{f \otimes g}"'] \end{tikzcd}$$

This diagram commutes because, starting from the lower left corner to $f \otimes g$, each composite is adjoint to $\ker^2_{f,g}$.

For (3), let α be a (trivial) cofibration and note that $\mathrm{coker}\,\alpha$ is the colimit of a morphism of pushout diagrams. That morphism of pushout diagrams is a Reedy (trivial) cofibration. The colimit functor is left Quillen as a functor from the Reedy model structure to the underlying category [Hovey 1999, Section 5.2]. Hence, $\mathrm{coker}\,\alpha$ is again a (trivial) cofibration, so coker is a left Quillen functor. See Lemma 6.1.8 for an analogous proof.

For (4), we must prove that, if f is cofibrant in $\vec{\mathsf{M}}^{\square}$ (so a cofibration of cofibrant objects) and g is fibrant in $\vec{\mathsf{M}}^{\otimes}$ (so a fibration of fibrant objects), then $\alpha\colon \operatorname{coker} f \to g$ is a weak equivalence if and only if its adjoint $\beta\colon f \to \ker g$ is a weak equivalence [ibid., 1.3.12]. We display both morphisms:

$$\begin{tikzcd} X_1 \arrow[r, "\operatorname{coker} f"] \arrow[d, "\alpha_0"'] & Z \arrow[d, "\alpha_1"] \\ Y_0 \arrow[r, "g"'] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "\beta_0"'] & X_1 \arrow[d, "\beta_1=\alpha_0"] \\ A \arrow[r, "\ker g"'] & Y_0 \end{tikzcd}$$

In the homotopy category, these data give rise to fiber and cofiber sequences. Since M is stable, every fiber sequence is canonically isomorphic to a cofiber sequence [ibid., Chapter 7]. We can extend to the right and realize α and β as giving a morphism of cofiber sequences in the homotopy category:

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "\beta_0"'] & X_1 \arrow[r, "\operatorname{coker} f"] \arrow[d, "\beta_1=\alpha_0"] & Z \arrow[r] \arrow[d, "\alpha_1"] & \Sigma X_0 \arrow[d, "\Sigma\beta_0"] \\ A \arrow[r, "\ker g"'] & Y_0 \arrow[r, "g"'] & Y_1 \arrow[r] & \Sigma A \end{tikzcd}$$

If either α or β is a weak equivalence, then so is the other, by the two-out-of-three property. Hence, coker and ker form a Quillen equivalence. $\square$

Proposition 2.4.6 *Suppose* M *is a cofibrantly generated model category in which the domains and the codomains of all the generating cofibrations and the generating trivial cofibrations are small in* M*. Then* $\vec{\mathsf{M}}_{\mathrm{inj}}$ *and* $\vec{\mathsf{M}}_{\mathrm{proj}}$ *are both strongly cofibrantly generated model categories.*

Proof The generating (trivial) cofibrations in $\vec{\mathsf{M}}_{\mathrm{inj}}$ are the morphisms $L_1 i$ and the morphisms

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$

for $i \in I$ (resp. $i \in J$) [Hovey 2014, 2.2]. The generating (trivial) cofibrations in $\vec{\mathsf{M}}_{\mathrm{proj}}$ are the morphisms $L_0 I \cup L_1 I$ (resp. $L_0 J \cup L_1 J$). So the smallness of the domains and codomains of the generating (trivial) cofibrations in $\vec{\mathsf{M}}_{\mathrm{inj}}$ and $\vec{\mathsf{M}}_{\mathrm{proj}}$ follows from our assumption on the domains and the codomains in I and J, since a morphism in the arrow category from f into a transfinite composition is determined by morphisms from $\mathrm{Ev}_0 f$ and $\mathrm{Ev}_1 f$ into transfinite compositions in M. $\square$

3 Smith ideals for operads

Suppose $(\mathsf{M}, \otimes, \mathbb{1})$ is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits on both sides, which is automatically true if M is a closed symmetric monoidal category.

In this section we define Smith ideals for an arbitrary colored operad $\mathcal{O}$ in M. When M is pointed, we observe in Theorem 3.4.2 that the cokernel and the kernel induce an adjunction between the categories of Smith $\mathcal{O}$–ideals and of $\mathcal{O}$–algebra morphisms. This will set the stage for the study of the homotopy theory of Smith $\mathcal{O}$–ideals in the next several sections.

3.1 Operads, algebras and bimodules

The following material on profiles and colored symmetric sequences is from [Yau and Johnson 2015]. For colored operads our references are [Yau 2016; White and Yau 2018a].

Definition 3.1.1 Suppose $\mathfrak{C}$ is a set, whose elements will be called *colors*.

(1) A $\mathfrak{C}$–*profile* is a finite, possibly empty sequence $\underline{c} = (c_1, \dots, c_n)$ with each $c_i \in \mathfrak{C}$.

(2) When permutations act on $\mathfrak{C}$–profiles from the left (resp. right), the resulting groupoid is denoted by $\Sigma_{\mathfrak{C}}$ (resp. $\Sigma_{\mathfrak{C}}^{\mathrm{op}}$).

(3) The category of $\mathfrak{C}$–*colored symmetric sequences* in M is the diagram category $\mathsf{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}}$. For a $\mathfrak{C}$–colored symmetric sequence X, we think of $\Sigma_{\mathfrak{C}}^{\mathrm{op}}$ (resp. $\mathfrak{C}$) as parametrizing the inputs (resp. outputs). For $(\underline{c}; d) \in \Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}$, the corresponding entry of a $\mathfrak{C}$–colored symmetric sequence X is denoted by $X\binom{d}{\underline{c}}$.

(4) A $\mathfrak{C}$–*colored operad* $(\mathcal{O}, \gamma, 1)$ in M consists of

- a $\mathfrak{C}$–colored symmetric sequence $\mathcal{O}$ in M;
- a structure morphism $\gamma \colon \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$, where $\circ$ is the circle product of $\mathcal{O}$ in [White and Yau 2018a, Definition 3.2.3], explicitly
$$\mathcal{O}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathcal{O}\binom{c_i}{\underline{b}_i} \xrightarrow{\gamma} \mathcal{O}\binom{d}{\underline{b}}$$
in M for all $d \in \mathfrak{C}$, $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$ and $\underline{b}_i \in \Sigma_{\mathfrak{C}}$ for $1 \le i \le n$, where $\underline{b} = (\underline{b}_1, \dots, \underline{b}_n)$ is the concatenation of the $\underline{b}_i$; and
- colored units $1_c \colon \mathbb{1} \to \mathcal{O}\binom{c}{c}$ for $c \in \mathfrak{C}$.

These data are required to satisfy the associativity, unity and equivariant conditions in [Yau 2016, Definition 11.2.1].

(5) For a $\mathfrak{C}$–colored operad $\mathcal{O}$ in M, an $\mathcal{O}$–*algebra* (A, λ) consists of

- objects $A_c \in \mathsf{M}$ for $c \in \mathfrak{C}$, and
- structure morphisms $\mathcal{O} \circ A \to A$, explicitly
$$\mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_1} \otimes \cdots \otimes A_{c_n} \xrightarrow{\lambda} A_d$$
in M for all $d \in \mathfrak{C}$ and $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$.

These data are required to satisfy the associativity, unity and equivariant conditions in [ibid., Definition 13.2.3]. Morphisms of $\mathbb{O}$–algebras are required to preserve the structure morphisms as in [ibid., Definition 13.2.8]. The category of $\mathbb{O}$–algebras in M is denoted by $\mathsf{Alg}(\mathbb{O};\mathsf{M})$. The forgetful functor is denoted by $U\colon \mathsf{Alg}(\mathbb{O};\mathsf{M}) \to \mathsf{M}^{\mathfrak{C}}$.

(6) Suppose (A,λ) is an $\mathbb{O}$–algebra for some $\mathfrak{C}$–colored operad $\mathbb{O}$ in M. An *A–bimodule* (X,θ) consists of

- objects $X_c \in \mathsf{M}$ for $c \in \mathfrak{C}$, and
- structure morphisms

$$\mathbb{O}\binom{d}{\underline{c}} \otimes A_{c_1} \otimes \cdots \otimes A_{c_{i-1}} \otimes X_{c_i} \otimes A_{c_{i+1}} \otimes \cdots \otimes A_{c_n} \xrightarrow{\theta} X_d$$

  in M for all $1 \le i \le n$ with $n \ge 1$, $d \in \mathfrak{C}$ and $\underline{c} = (c_1,\dots,c_n) \in \Sigma_{\mathfrak{C}}$.

These data are required to satisfy associativity, unity and equivariant conditions similar to those of an $\mathbb{O}$–algebra but with one input entry A and the output entry replaced by X. A morphism of A–bimodules is required to preserve the structure morphisms.

(7) For a $\mathfrak{C}$–colored operad $\mathbb{O}$ in M, we write

(3.1.2) $$\vec{\mathbb{O}}^{\otimes} = L_0\mathbb{O} \quad \text{and} \quad \vec{\mathbb{O}}^{\square} = L_1\mathbb{O}$$

for the $\mathfrak{C}$–colored operads in $\vec{\mathsf{M}}^{\otimes}$ and $\vec{\mathsf{M}}^{\square}$, respectively, where $L_0\colon \mathsf{M} \to \vec{\mathsf{M}}^{\otimes}$ and $L_1\colon \mathsf{M} \to \vec{\mathsf{M}}^{\square}$ are the strict monoidal functors in (2.2.4).

As a consequence of (2.4.2) and (3.1.2), we have

(3.1.3) $$\ker \vec{\mathbb{O}}^{\otimes} = \ker(L_0\mathbb{O}) = L_1\mathbb{O} = \vec{\mathbb{O}}^{\square}, \quad \operatorname{coker} \vec{\mathbb{O}}^{\square} = \operatorname{coker}(L_1\mathbb{O}) = L_0\mathbb{O} = \vec{\mathbb{O}}^{\otimes}.$$

Definition 3.1.4 Suppose, moreover, that M is a cofibrantly generated model category. We say that M is *operadically cofibrantly generated* if the domains and codomains of I (resp. J) are small with respect to a class of morphisms containing $U(\mathbb{O}\circ I)$–cell (resp. $U(\mathbb{O}\circ J)$–cell) for each $\mathfrak{C}$ and each $\mathfrak{C}$–colored operad $\mathbb{O}$. More explicitly, $\mathbb{O}\circ -\colon \mathsf{M}^{\mathfrak{C}} \to \mathsf{Alg}(\mathbb{O};\mathsf{M})$ is a left adjoint of the forgetful functor U [White and Yau 2018a, 4.1.11]. To form $\mathbb{O}\circ I$ and $\mathbb{O}\circ J$, we first embed M into the c–colored entry of $\mathsf{M}^{\mathfrak{C}}$ for some $c \in \mathfrak{C}$, with $1_{\varnothing}$ in all other entries, and then apply $\mathbb{O}\circ -$ to the images of I and J in $\mathsf{M}^{\mathfrak{C}}$. The condition for operadically cofibrantly generated is assumed to hold for each $c \in \mathfrak{C}$.

Example 3.1.5 Every strongly cofibrantly generated model category is operadically cofibrantly generated. The category of compactly generated topological spaces is *not* strongly cofibrantly generated. However, it is operadically cofibrantly generated. Indeed, the domains and codomains of $I \cup J$ are small relative to inclusions [Hovey 1999, 2.4.1], and the morphisms in $U(\mathbb{O}\circ I)$–cell and $U(\mathbb{O}\circ J)$–cell are inclusions [White and Yau 2020, 5.10].

3.2 Arrow category of operadic algebras

Definition 3.2.1 For each $\mathfrak{C}$–colored operad O in M, the arrow category, in the sense of Definition 2.2.2, of the category $\mathsf{Alg}(\mathsf{O};\mathsf{M})$ is denoted by $\overrightarrow{\mathsf{Alg}(\mathsf{O};\mathsf{M})}$.

Explicitly, an object in $\overrightarrow{\mathsf{Alg}(\mathsf{O};\mathsf{M})}$ is an O–algebra morphism. A morphism in $\overrightarrow{\mathsf{Alg}(\mathsf{O};\mathsf{M})}$ is a commutative square in $\mathsf{Alg}(\mathsf{O};\mathsf{M})$ as in (2.2.3), with each arrow an O–algebra morphism.

Proposition 3.2.2 *Suppose O is a $\mathfrak{C}$–colored operad in* M. *Then* $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$ *is canonically isomorphic to* $\overrightarrow{\mathsf{Alg}(\mathsf{O};\mathsf{M})}$.

Proof An $\vec{\mathsf{O}}^{\otimes}$–algebra $f=\{f_c\colon X_c\to Y_c\}$ consists of morphisms $f_c\in\mathsf{M}$ for $c\in\mathfrak{C}$ and structure morphisms

$$\vec{\mathsf{O}}^{\otimes}\binom{d}{\underline{c}}\otimes\bigotimes_{i=1}^{n} f_{c_i}\xrightarrow{\lambda} f_d$$

in $\vec{\mathsf{M}}^{\otimes}$ for all $d\in\mathfrak{C}$ and $\underline{c}=(c_1,\dots,c_n)\in\Sigma_{\mathfrak{C}}$. This structure morphism is equivalent to the commutative square

$$\begin{tikzcd}
\mathsf{O}\binom{d}{\underline{c}}\otimes\bigotimes_{i=1}^{n} X_{c_i} \arrow[r, "\lambda_0"] \arrow[d, "\mathrm{Id}\otimes\bigotimes f_{c_i}"'] & X_d \arrow[d, "f_d"] \\
\mathsf{O}\binom{d}{\underline{c}}\otimes\bigotimes_{i=1}^{n} Y_{c_i} \arrow[r, "\lambda_1"] & Y_d
\end{tikzcd}$$

in M. The associativity, unity, and equivariance of λ translate into those of λ_0 and λ_1, making (X,λ_0) and (Y,λ_1) into O–algebras in M. The commutativity of the previous square means that $f\colon(X,\lambda_0)\to(Y,\lambda_1)$ is a morphism of O–algebras. The identification of morphisms in $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$ and $\overrightarrow{\mathsf{Alg}(\mathsf{O};\mathsf{M})}$ is similar. □

Remark 3.2.3 For the associative operad As, whose algebras are monoids, the identification of $\vec{\mathsf{As}}^{\otimes}$–algebras (that is, monoids in $\vec{\mathsf{M}}^{\otimes}$) with monoid morphisms in M is [Hovey 2014, 1.5].

3.3 Operadic Smith ideals

Definition 3.3.1 Suppose O is a $\mathfrak{C}$–colored operad in M. The category of *Smith O–ideals* in M is defined as the category $\mathsf{Alg}(\vec{\mathsf{O}}^{\square};\vec{\mathsf{M}}^{\square})$.

Propositions 3.3.3 and 3.3.11 below unpack Definition 3.3.1. They should be compared with Proposition 3.2.2. For objects or morphisms $A_{c_s},\dots,A_{c_t}$ with $s\le t$, we use the abbreviation

(3.3.2)
$$A_{c_{s,t}}=\bigotimes_{k=s}^{t} A_{c_k}.$$

Proposition 3.3.3 *Suppose $\mathbb{O}$ is a $\mathfrak{C}$–colored operad in M. A Smith $\mathbb{O}$–ideal in M consists of precisely*

- *an $\mathbb{O}$–algebra (A, λ_1) in M,*
- *an A–bimodule (X, λ_0) in M, and*
- *an A–bimodule morphism $f\colon (X, \lambda_0) \to (A, \lambda_1)$*

such that, for $1 \le i < j \le n$, the diagram

(3.3.4)
$$\begin{tikzcd}
\mathbb{O}\binom{d}{\underline{c}} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,j-1}} \otimes X_{c_j} \otimes A_{c_{j+1,n}} \arrow[r, "\mathrm{Id}\otimes f_{c_j}\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes f_{c_i}\otimes\mathrm{Id}"'] & \mathbb{O}\binom{d}{\underline{c}} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,n}} \arrow[d, "\lambda_0^i"] \\
\mathbb{O}\binom{d}{\underline{c}} \otimes A_{c_{1,j-1}} \otimes X_{c_j} \otimes A_{c_{j+1,n}} \arrow[r, "\lambda_0^j"] & X_d
\end{tikzcd}$$

in M is commutative.

Proof An $\vec{\mathbb{O}}^{\square}$–algebra (f, λ) in $\vec{\mathsf{M}}^{\square}$ consists of

- morphisms $f_c\colon X_c \to A_c$ in M for $c \in \mathfrak{C}$, and
- structure morphisms
$$\vec{\mathbb{O}}^{\square}\binom{d}{\underline{c}} \mathbin{\square} f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n} \xrightarrow{\lambda} f_d$$
in $\vec{\mathsf{M}}^{\square}$ for all $d \in \mathfrak{C}$ and $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$

that are associative, unital and equivariant. Since $\vec{\mathbb{O}}^{\square}\binom{d}{\underline{c}}$ is the morphism $\varnothing \to \mathbb{O}\binom{d}{\underline{c}}$, when $n = 0$, the structure morphism λ is simply the morphism $\lambda_1\colon \mathbb{O}\binom{d}{\varnothing} \to A_d$ in M for $d \in \mathfrak{C}$. For $n \ge 1$, the structure morphism λ is equivalent to the commutative diagram

(3.3.5)
$$\begin{tikzcd}
\mathbb{O}\binom{d}{\underline{c}} \otimes \mathrm{dom}(f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n}) \arrow[r, "\lambda_0"] \arrow[d, "\mathrm{Id}\otimes f_*"'] & X_d \arrow[d, "f_d"] \\
\mathbb{O}\binom{d}{\underline{c}} \otimes A_{c_1} \otimes \cdots \otimes A_{c_n} \arrow[r, "\lambda_1"] & A_d
\end{tikzcd}$$

in M, where f_* is induced by the morphisms f_c. The bottom horizontal morphism λ_1 in (3.3.5) together with the morphisms $\lambda_1\colon \mathbb{O}\binom{d}{\varnothing} \to A_d$ for $d \in \mathfrak{C}$ give A the structure of an $\mathbb{O}$–algebra.

The domain of the iterated pushout product $f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n}$ is the colimit

(3.3.6)
$$\mathrm{dom}(f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n}) = \operatorname*{colim}_{(\epsilon_1,\dots,\epsilon_n)} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n}$$

in which $(\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \setminus \{(1, \dots, 1)\}$ and $f_{\epsilon_i} = X_{c_i}$ (resp. A_{c_i}) if $\epsilon_i = 0$ (resp. $\epsilon_i = 1$). The morphisms that define the colimit are given by the f_{c_i}. For each n–tuple of indices $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \setminus \{(1, \dots, 1)\}$, we denote by

(3.3.7)
$$f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n} \xrightarrow{\iota_\epsilon} \mathrm{dom}(f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n})$$

the morphism that comes with the colimit. For each $i \in \{1, \dots, n\}$, we denote by

$$\epsilon^i = (1, \dots, 0, \dots, 1) \in \{0,1\}^n$$

the n–tuple with 0 in the i^{th} entry and 1 in every other entry.

The top horizontal morphism λ_0 in (3.3.5) precomposed with $\mathrm{Id} \otimes \iota_{\epsilon^i}$, as in

(3.3.8)
$$\begin{tikzcd}
\mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,n}} \arrow[d, "\mathrm{Id}\otimes\iota_{\epsilon^i}"'] \arrow[dr, bend left, "\lambda_0^{\epsilon^i}"] & \\
\mathcal{O}\binom{d}{\underline{c}} \otimes \mathrm{dom}(f_{c_1} \,\square \cdots \square\, f_{c_n}) \arrow[r, "\lambda_0"] & X_d
\end{tikzcd}$$

for $1 \le i \le n$, gives X the structure of an A–bimodule. The commutative diagram (3.3.5), precomposed with $\mathrm{Id} \otimes \iota_{\epsilon^i}$ as in (3.3.8), implies that $f : (X, \lambda_0) \to (A, \lambda_1)$ is an A–bimodule morphism. The morphism λ_0^i in (3.3.4) is $\lambda_0^{\epsilon^i}$ in (3.3.8).

The diagram (3.3.4) is the boundary of the following diagram, where $D = \mathrm{dom}(f_{c_1} \,\square \cdots \square\, f_{c_n})$:

(3.3.9)
$$\begin{tikzcd}
\mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,j-1}} \otimes X_{c_j} \otimes A_{c_{j+1,n}} \arrow[r, "\mathrm{Id}\otimes f_{c_j}\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes f_{c_i}\otimes\mathrm{Id}"'] & \mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,n}} \arrow[d, "\mathrm{Id}\otimes\iota_{\epsilon^i}"'] \arrow[dd, bend left, "\lambda_0^{\epsilon^i}"] \\
 & \mathcal{O}\binom{d}{\underline{c}} \otimes D \arrow[d, "\lambda_0"] \\
\mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_{1,j-1}} \otimes X_{c_j} \otimes A_{c_{j+1,n}} \arrow[ur, "\mathrm{Id}\otimes\iota_{\epsilon^j}"] \arrow[r, "\lambda_0^{\epsilon^j}"] & X_d
\end{tikzcd}$$

The upper left quadrilateral is commutative because D is the colimit in (3.3.6). The other two triangles are commutative by the definition of $\lambda_0^{\epsilon^i}$ and $\lambda_0^{\epsilon^j}$ in (3.3.8).

The argument above can be reversed. In particular, to see that the commutative diagram (3.3.4), which is the boundary of (3.3.9), yields the top horizontal morphism λ_0 in (3.3.5), observe that the full subcategory of the punctured n–cube $\{0,1\}^n \setminus \{(1, \dots, 1)\}$ consisting of $(\epsilon_1, \dots, \epsilon_n)$ with at most two 0's is a final subcategory [Mac Lane 1998, IX.3]. Thus, the diagram (3.3.9) ensures that λ_0 exists. □

Remark 3.3.10 The special case of Proposition 3.3.3 for $\mathcal{O} = \mathrm{As}$ is [Hovey 2014, 1.7].

Proposition 3.3.11 *In the context of Proposition 3.3.3, a morphism of Smith $\mathcal{O}$–ideals*

$$((X, \lambda_0) \xrightarrow{f} (A, \lambda_1)) \xrightarrow{h} ((X', \lambda_0') \xrightarrow{f'} (A', \lambda_1'))$$

consists of precisely

- *a morphism $h^1 : A \to A'$ of $\mathcal{O}$–algebras, and*
- *a morphism $h^0 : X \to X'$ of A–bimodules, where X' becomes an A–bimodule via the restriction along h^1,*

such that the square

(3.3.12)
$$\begin{tikzcd} X_c \arrow[r, "h_c^0"] \arrow[d, "f_c"'] & X'_c \arrow[d, "f'_c"] \\ A_c \arrow[r, "h_c^1"] & A'_c \end{tikzcd}$$

is commutative for each $c \in \mathfrak{C}$.

Proof Following the proof of Proposition 3.3.3, we unravel the given morphism $h\colon (f,\lambda) \to (f',\lambda')$ of $\vec{\mathcal{O}}^{\square}$–algebras. The underlying datum of h is a morphism $f \to f'$ in $\vec{\mathsf{M}}^{\mathfrak{C}}$. Thus, h consists of, for each $c \in \mathfrak{C}$, morphisms

(3.3.13)
$$h_c^0\colon X_c \to X'_c \quad \text{and} \quad h_c^1\colon A_c \to A'_c$$

in M such that the square (3.3.12) commutes.

The compatibility of h with the $\vec{\mathcal{O}}^{\square}$–algebra structure means the following diagram commutes in $\vec{\mathsf{M}}$ for all $d, c_1, \dots, c_n \in \mathfrak{C}$:

(3.3.14)
$$\begin{tikzcd} \vec{\mathcal{O}}^{\square}\binom{d}{\underline{c}} \mathbin{\square} f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n} \arrow[r, "\lambda"] \arrow[d, "\mathrm{Id} \square h_{c_1} \square \cdots \square h_{c_n}"'] & f_d \arrow[d, "h_d"] \\ \vec{\mathcal{O}}^{\square}\binom{d}{\underline{c}} \mathbin{\square} f'_{c_1} \mathbin{\square} \cdots \mathbin{\square} f'_{c_n} \arrow[r, "\lambda'"] & f'_d \end{tikzcd}$$

If $n = 0$, then (3.3.14) is the commutative diagram:

(3.3.15)
$$\begin{tikzcd} \mathcal{O}\binom{d}{\varnothing} \arrow[r, "\lambda_1"] \arrow[d, equal] & A_d \arrow[d, "h_d^1"] \\ \mathcal{O}\binom{d}{\varnothing} \arrow[r, "\lambda'_1"] & A'_d \end{tikzcd}$$

For $n \geq 1$, using the abbreviation

$$D = \mathrm{dom}(f_{c_1} \mathbin{\square} \cdots \mathbin{\square} f_{c_n}) \quad \text{and} \quad D' = \mathrm{dom}(f'_{c_1} \mathbin{\square} \cdots \mathbin{\square} f'_{c_n}),$$

the diagram (3.3.14) becomes the commutative cube:

(3.3.16)
$$\begin{tikzcd} \mathcal{O}\binom{d}{\underline{c}} \otimes D \arrow[rr, "\lambda_0"] \arrow[dr, "\mathrm{Id} \otimes h_*"] \arrow[dd, "\mathrm{Id} \otimes f_*"'] & & X_d \arrow[dr, "h_d^0"] \arrow[dd, "f_d" near end] & \\ & \mathcal{O}\binom{d}{\underline{c}} \otimes D' \arrow[rr, "\lambda'_0" near start] \arrow[dd, "\mathrm{Id} \otimes f'_*" near start] & & X'_d \arrow[dd, "f'_d"] \\ \mathcal{O}\binom{d}{\underline{c}} \otimes A_{c_{1,n}} \arrow[rr, "\lambda_1" near start] \arrow[dr, "\mathrm{Id} \otimes h_*^1"'] & & A_d \arrow[dr, "h_d^1"] & \\ & \mathcal{O}\binom{d}{\underline{c}} \otimes A'_{c_{1,n}} \arrow[rr, "\lambda'_1"] & & A'_d \end{tikzcd}$$

The six commutative faces of (3.3.16) are as follows:

(1) The back face is (3.3.5) for (f,λ), expressing the $\vec{\mathcal{O}}^{\square}$–algebra structure λ on f.

(2) The front face is (3.3.5) for (f',λ'), expressing the $\vec{\mathcal{O}}^{\square}$–algebra structure λ' on f'.

(3) The right face is the square (3.3.12) for $d\in\mathfrak{C}$.

(4) The bottom face and the $n=0$ case (3.3.15) together express the fact that $h^1\colon (A,\lambda_1)\to(A',\lambda_1')$ is an $\mathcal{O}$–algebra morphism.

(5) The left face imposes no extra condition because D is the colimit in (3.3.6) and similarly for D'. In more detail, for each n–tuple $(\epsilon_1,\dots,\epsilon_n)\in\{0,1\}^n\setminus\{(1,\dots,1)\}$, the square

(3.3.17)
$$\begin{array}{ccc} f_{\epsilon_1}\otimes\cdots\otimes f_{\epsilon_n} & \xrightarrow{h_*} & f'_{\epsilon_1}\otimes\cdots\otimes f'_{\epsilon_n} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ A_{c_1}\otimes\cdots\otimes A_{c_n} & \xrightarrow{h^1_*} & A'_{c_1}\otimes\cdots\otimes A'_{c_n} \end{array}$$

is commutative because it is a tensor product of n commutative squares corresponding to the n tensor factors of the upper left corner.

- For a tensor factor with $\epsilon_i=0$, by definition $f_{\epsilon_i}=X_{c_i}$ and $f'_{\epsilon_i}=X'_{c_i}$. In this case, we have the commutative square (3.3.12) for $c_i\in\mathfrak{C}$.
- For a tensor factor with $\epsilon_i=1$, by definition $f_{\epsilon_i}=A_{c_i}$ and $f'_{\epsilon_i}=A'_{c_i}$. Both f_* and f'_* are given by the identity in the respective tensor factors, while both h_* and h^1_* are given by $h^1_{c_i}$.

Precomposing the top face of the commutative cube (3.3.16) with the morphism $\mathrm{Id}\otimes\iota_{\epsilon^i}$ in (3.3.8) yields the commutative diagram:

(3.3.18)
$$\begin{array}{ccc} \mathcal{O}\binom{d}{\underline{c}}\otimes A_{c_{1,i-1}}\otimes X_{c_i}\otimes A_{c_{i+1,n}} & \xrightarrow{\lambda_0^{\epsilon^i}} & X_d \\ \downarrow{\scriptstyle \mathrm{Id}\otimes h^0_{c_i}\otimes\mathrm{Id}} & & \\ \mathcal{O}\binom{d}{\underline{c}}\otimes A_{c_{1,i-1}}\otimes X'_{c_i}\otimes A_{c_{i+1,n}} & & \downarrow{\scriptstyle h^0_d} \\ \downarrow{\scriptstyle \mathrm{Id}\otimes h^1_{c_{1,i-1}}\otimes\mathrm{Id}\otimes h^1_{c_{i+1,n}}} & & \\ \mathcal{O}\binom{d}{\underline{c}}\otimes A'_{c_{1,i-1}}\otimes X'_{c_i}\otimes A'_{c_{i+1,n}} & \xrightarrow{(\lambda_0')^{\epsilon^i}} & X'_d \end{array}$$

This commutative diagram expresses the fact that $h^0\colon X\to X'$ is a morphism of A–bimodules, where X' becomes an A–bimodule via the restriction along h^1.

Finally, we observe that the top face of the cube (3.3.16) is actually equivalent to the commutative diagram (3.3.18). To see this, consider an n–tuple $\epsilon=(\epsilon_1,\dots,\epsilon_n)\in\{0,1\}^n$ with at least two entries equal to 0. Then the morphism ι_ϵ in (3.3.7) factors as follows for each index $i\in\{1,\dots,n\}$ with $\epsilon_i=0$, and similarly for f':

$$\begin{tikzcd} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n} \arrow[rr, "\iota_\epsilon"] \arrow[d, "f_*"'] & & D \\ A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,n}} \arrow[urr, "\iota_{\epsilon^i}"'] & & \end{tikzcd}$$

Thus, precomposing the top face of (3.3.16) with the morphism $\mathrm{Id} \otimes \iota_\epsilon$ yields a diagram that factors into two subdiagrams, one of which is (3.3.18). The other subdiagram commutes and imposes no extra condition by the same argument above for (3.3.17). □

The description of Smith $\mathcal{O}$–ideals and their morphisms in Propositions 3.3.3 and 3.3.11 imply the following result:

Proposition 3.3.19 *Suppose $\mathcal{O}$ is a $\mathfrak{C}$–colored operad in M. Then there exists a $(\mathfrak{C} \sqcup \mathfrak{C})$–colored operad $\mathcal{O}^s$ in M such that there is a canonical isomorphism of categories*

$$\mathsf{Alg}(\vec{\mathcal{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \cong \mathsf{Alg}(\mathcal{O}^s; \mathsf{M}).$$

Proof Denote the first and the second copies of $\mathfrak{C}$ in $\mathfrak{C} \sqcup \mathfrak{C}$ by $\mathfrak{C}^0$ and $\mathfrak{C}^1$, respectively. For an element $c \in \mathfrak{C}$, we write $c^\epsilon \in \mathfrak{C}^\epsilon$ for the same element for $\epsilon \in \{0, 1\}$. The entries of $\mathcal{O}^s$ are defined as, for $d, c_1, \ldots, c_n \in \mathfrak{C}$ and $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$,

$$\mathcal{O}^s\binom{d^1}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = \mathcal{O}\binom{d}{\underline{c}}, \qquad \mathcal{O}^s\binom{d^0}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = \begin{cases} \mathcal{O}\binom{d}{\underline{c}} & \text{if at least one } \epsilon_i = 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

The operad structure morphisms of $\mathcal{O}^s$ are either those of $\mathcal{O}$ or the unique morphism from the initial object $\varnothing$.

An $\mathcal{O}^s$–algebra in M consists of, first of all, a $(\mathfrak{C}^0 \sqcup \mathfrak{C}^1)$–colored object in M, that is, a $\mathfrak{C}^0$–colored object $X = \{X_c\}_{c \in \mathfrak{C}^0}$ and a $\mathfrak{C}^1$–colored object $A = \{A_c\}_{c \in \mathfrak{C}^1}$.

- The $\mathcal{O}^s$–algebra structure morphism

(3.3.20)
$$\mathcal{O}^s\binom{d^1}{c_1^1, \ldots, c_n^1} \otimes A_{c_1} \otimes \cdots \otimes A_{c_n} \xrightarrow{\lambda} A_d$$

  corresponds to the $\mathcal{O}$–algebra structure morphism λ_1 on A in (3.3.5).

- The $\mathcal{O}^s$–algebra structure morphism

(3.3.21)
$$\mathcal{O}^s\binom{d^0}{c_1^1, \ldots, c_{i-1}^1, c_i^0, c_{i+1}^1, \ldots, c_n^1} \otimes A_{c_{1,i-1}} \otimes X_{c_i} \otimes A_{c_{i+1,n}} \xrightarrow{\lambda} X_d$$

  corresponds to the A–bimodule structure morphism $\lambda_0^{\epsilon^i}$ on X in (3.3.8).

- The composite

$$\begin{tikzcd} X_d \arrow[d, "\cong"'] & & A_d \\ \mathbb{1}\otimes X_d \arrow[r, "1_d\otimes \mathrm{Id}"] & \mathbb{O}\binom{d}{d}\otimes X_d = \mathbb{O}^s\binom{d^1}{d^0}\otimes X_d \arrow[u, "\lambda"'] \end{tikzcd} \tag{3.3.22}$$

  corresponds to the morphism f_d in (3.3.5).

The identification of $\mathbb{O}^s$–algebra morphisms and Smith $\mathbb{O}$–ideal morphisms follows similarly from Proposition 3.3.11. More explicitly, a morphism h of $\mathbb{O}^s$–algebras consists of a $(\mathfrak{C}^0 \sqcup \mathfrak{C}^1)$–colored morphism in M. So h consists of component morphisms $h_c^0\colon X_c \to X_c'$ and $h_c^1\colon A_c \to A_c'$ as in (3.3.13). To see that these component morphisms make the diagram (3.3.12) commute, we use the fact that the components of f are the composites in (3.3.22) and similarly for f'. The desired diagram (3.3.12) is the boundary of the diagram:

$$\begin{tikzcd} X_c \arrow[r, "\cong"] \arrow[d, "h_c^0"'] & \mathbb{1}\otimes X_c \arrow[r, "1_c\otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes h_c^0"'] & \mathbb{O}^s\binom{c^1}{c^0}\otimes X_c \arrow[r, "\lambda"] \arrow[d, "\mathrm{Id}\otimes h_c^0"'] & A_c \arrow[d, "h_c^1"] \\ X_c' \arrow[r, "\cong"] & \mathbb{1}\otimes X_c' \arrow[r, "1_c\otimes \mathrm{Id}"] & \mathbb{O}^s\binom{c^1}{c^0}\otimes X_c' \arrow[r, "\lambda'"] & A_c' \end{tikzcd}$$

- The left square commutes by the naturality of the left unit isomorphism in the monoidal category M.
- The middle square commutes by the functoriality of $\otimes$.
- The right square commutes because h respects $\mathbb{O}^s$–algebra structures.

This shows that the diagram (3.3.12) is commutative.

The other two conditions in Proposition 3.3.11 are the following:

(i) $h^1\colon A \to A'$ is an $\mathbb{O}$–algebra morphism.

(ii) $h^0\colon X \to X'$ is an A–bimodule morphism.

Condition (i) consists of the $n = 0$ case (3.3.15) and the bottom face of the cube (3.3.16). These are obtained from the compatibility of h with the $\mathbb{O}^s$–algebra structure morphism (3.3.20). Condition (ii) is the diagram (3.3.18). This is obtained from the compatibility of h with the $\mathbb{O}^s$–algebra structure morphism (3.3.21). □

The colored operad $\mathbb{O}^s$ is somewhat similar to the two-colored operad for monoid morphisms in [Yau 2016, Section 14.3].

3.4 Operadic Smith ideals and morphisms of operadic algebras

In Proposition 2.4.3 we observe that, if M is a pointed symmetric monoidal category with all small limits and colimits, then there is an adjunction

$$\vec{\mathsf{M}}^{\square} \underset{\mathrm{ker}}{\overset{\mathrm{coker}}{\rightleftarrows}} \vec{\mathsf{M}}^{\otimes} \tag{3.4.1}$$

with cokernel as the left adjoint and kernel as the right adjoint. Since cokernel is a strictly unital strong symmetric monoidal functor, the kernel is a strictly unital lax symmetric monoidal functor, and the adjunction is monoidal. If M is a pointed model category, then $(\mathrm{coker}, \mathrm{ker})$ is a Quillen adjunction. If M is a stable model category, then $(\mathrm{coker}, \mathrm{ker})$ is a Quillen equivalence.

Theorem 3.4.2 *Suppose* M *is a complete and cocomplete symmetric monoidal pointed category in which the monoidal product commutes with colimits on both sides. Suppose* $\mathbb{O}$ *is a* $\mathfrak{C}$*–colored operad in* M*. Then the adjunction* (3.4.1) *induces an adjunction*

(3.4.3)
$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \underset{\mathrm{ker}}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$$

in which the left adjoint, the right adjoint, the unit and the counit are defined entrywise.

Proof To simplify the notation, in this proof we write $\mathsf{C} = \mathrm{coker}$ and $\mathsf{K} = \mathrm{ker}$. First we lift the functors C and K. Then we lift the unit and the counit for the adjunction.

Step 1: lifting the kernel and the cokernel to algebra categories The functors in (3.4.1) lifts entrywise to the functors in (3.4.3) for the following reasons:

- The functor
$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \xleftarrow{\mathsf{K}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$$
exists because $\mathsf{K}\colon \vec{\mathsf{M}}^{\otimes} \to \vec{\mathsf{M}}^{\square}$ is a lax symmetric monoidal functor and $\mathsf{K}\vec{\mathbb{O}}^{\otimes} = \vec{\mathbb{O}}^{\square}$ by (3.1.3).
- The functor
$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \xrightarrow{\mathsf{C}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$$
exists because $\mathsf{C}\colon \vec{\mathsf{M}}^{\square} \to \vec{\mathsf{M}}^{\otimes}$ is a strong symmetric monoidal functor and $\mathsf{C}\vec{\mathbb{O}}^{\square} = \vec{\mathbb{O}}^{\otimes}$ by (3.1.3).

More explicitly, suppose (f, λ) is an $\vec{\mathbb{O}}^{\square}$–algebra as in Proposition 3.3.3. Then $\mathsf{C}f$ becomes an $\vec{\mathbb{O}}^{\otimes}$–algebra with structure morphism $\lambda^{\#}$ given by the following composite for all $d, c_1, \dots, c_n \in \mathfrak{C}$, with $\mathsf{C}^2 = \mathrm{coker}^2$ the monoidal constraint of the cokernel in (2.4.4):

(3.4.4)
$$\begin{tikzcd}
\vec{\mathbb{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{C}f_{c_i} \arrow[r, "\lambda^{\#}"] \arrow[d, equal] & \mathsf{C}f_d \\
\mathsf{C}\vec{\mathbb{O}}^{\square}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{C}f_{c_i} \arrow[r, "\mathsf{C}^2"] & \mathsf{C}\big(\vec{\mathbb{O}}^{\square}\binom{d}{\underline{c}} \square f_{c_1} \square \cdots \square f_{c_n}\big) \arrow[u, "\mathsf{C}\lambda"']
\end{tikzcd}$$

The $\vec{\mathbb{O}}^{\otimes}$–algebra axioms for $(\mathsf{C}f, \lambda^{\#})$ follow from the $\vec{\mathbb{O}}^{\square}$–algebra axiom for (f, λ) and the symmetric monoidal axioms for the cokernel. The same reasoning also applies to the kernel.

Thus, there is a diagram of functors

(3.4.5)
$$\begin{tikzcd}
\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \arrow[r, shift left, "\mathsf{C}"] \arrow[d, "U"'] & \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes}) \arrow[l, shift left, "\mathsf{K}"] \arrow[d, "U"] \\
(\vec{\mathsf{M}}^{\square})^{\mathfrak{C}} \arrow[r, shift left, "\mathsf{C}"] & (\vec{\mathsf{M}}^{\otimes})^{\mathfrak{C}} \arrow[l, shift left, "\mathsf{K}"]
\end{tikzcd}$$

with both U forgetful functors and

$$U\mathsf{K} = \mathsf{K}U.$$

To see that this equality holds, suppose (f,λ) is an $\vec{\mathrm{O}}^{\otimes}$–algebra as in the proof of Proposition 3.2.2. As in (3.4.4), the $\vec{\mathrm{O}}^{\square}$–algebra $\mathsf{K}(f,\lambda)$ is given by $(\mathsf{K}f,\lambda')$, where the $\vec{\mathrm{O}}^{\square}$–algebra structure morphism λ' is constructed from the monoidal constraint K^2 and $\mathsf{K}\lambda$. Since each U forgets the operad algebra structure morphism, we obtain the equalities

$$U\mathsf{K}(f,\lambda) = U(\mathsf{K}f,\lambda') = \mathsf{K}f = \mathsf{K}U(f,\lambda).$$

The equality $U\mathsf{K} = \mathsf{K}U$ holds on $\vec{\mathrm{O}}^{\otimes}$–algebra morphisms because both K apply entrywise to morphisms, and both U do not change the morphisms.

Next we show that the unit and the counit,

$$\eta\colon \mathrm{Id} \to \mathsf{KC} \quad \text{and} \quad \varepsilon\colon \mathsf{CK} \to \mathrm{Id},$$

of the bottom adjunction $\mathsf{C} \dashv \mathsf{K}$ in (3.4.5) lift to the top between algebra categories.

Step 2: lifting the unit To show that η defines a natural transformation for the top functors in (3.4.5), first we need to show that, for each $\vec{\mathrm{O}}^{\square}$–algebra (f,λ), the unit component morphism $\eta_f\colon f \to \mathsf{KC}f$ in $\vec{\mathsf{M}}^{\mathfrak{C}}$ is an $\vec{\mathrm{O}}^{\square}$–algebra morphism. So we must show that the diagram

(3.4.6)
$$\begin{tikzcd}
\vec{\mathrm{O}}^{\square}\binom{d}{\underline{c}} \square \mathop{\square}_{i=1}^{n} f_{c_i} \arrow[r, "\lambda"] \arrow[d, "\mathrm{Id}\square\square_i \eta_i"'] & f_d \arrow[dd, "\eta_{f_d}"] \\
\mathsf{K}\vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}} \square \mathop{\square}_{i=1}^{n} \mathsf{KC}f_{c_i} \arrow[d, "\mathsf{K}^2"'] & \\
\mathsf{K}\Bigl(\vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{C}f_{c_i}\Bigr) \arrow[r, "\mathsf{K}\lambda^{\#}"] & \mathsf{KC}f_d
\end{tikzcd}$$

in $\vec{\mathsf{M}}$ is commutative for $d, c_1, \dots, c_n \in \mathfrak{C}$, with $\lambda^{\#}$ as in (3.4.4), $\eta_i = \eta_{f_{c_i}}$, $\vec{\mathrm{O}}^{\square} = \mathsf{K}\vec{\mathrm{O}}^{\otimes}$ by (3.1.3), and $\mathsf{K}^2 = \ker^2$ the monoidal constraint defined in (2.4.5).

To see that (3.4.6) is commutative, we consider the adjoint of each composite, which yields the boundary of the following diagram in $\vec{\mathsf{M}}$:

(3.4.7)
$$\begin{tikzcd}
\mathsf{C}\Bigl(\vec{\mathrm{O}}^{\square}\binom{d}{\underline{c}} \square \mathop{\square}_{i=1}^{n} f_{c_i}\Bigr) \arrow[rr, "\mathsf{C}\lambda"] \arrow[dr, "(\mathsf{C}^2)^{-1}"] \arrow[dd, "\mathsf{C}(\mathrm{Id}\square\square_i \eta_i)"'] & & \mathsf{C}(f_d) \\
& \mathsf{C}\vec{\mathrm{O}}^{\square}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{C}f_{c_i} \arrow[r, equal] \arrow[dr, "\mathrm{Id}\otimes\bigotimes_i \mathsf{C}\eta_i"'] & \vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{C}f_{c_i} \arrow[u, "\lambda^{\#}"'] \\
\mathsf{C}\Bigl(\mathsf{K}\vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}} \square \mathop{\square}_{i=1}^{n} \mathsf{KC}f_{c_i}\Bigr) \arrow[r, "(\mathsf{C}^2)^{-1}"'] & \mathsf{CK}\vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{CKC}f_{c_i} \arrow[ur, "\varepsilon_{\vec{\mathrm{O}}^{\otimes}\binom{d}{\underline{c}}}\otimes\bigotimes_i \varepsilon_{\mathsf{C}f_{c_i}}"'] &
\end{tikzcd}$$

The three subregions in (3.4.7) are commutative for the following reasons:

- The left triangle is commutative by naturality of the monoidal constraint $\mathsf{C}^2 = \mathrm{coker}^2$ of the cokernel.
- The upper right region is commutative by the definition of $\lambda^\#$ in (3.4.4).
- To see that the lower right triangle is commutative, first note that the counit component morphism

(3.4.8)
$$\varepsilon_{\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}}} \colon \mathsf{CK}\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}} \to \vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}}$$

is the identity, since
$$\mathsf{CK}\vec{\mathcal{O}}^\otimes = \mathsf{C}\vec{\mathcal{O}}^\square = \vec{\mathcal{O}}^\otimes,$$
by (3.1.3). For each of the other n tensor factors in the lower right triangle, the composite $\varepsilon_{\mathsf{C}f_{c_i}} \circ \mathsf{C}\eta_i$ is the identity morphism by one of the triangle identities for the adjunction $\mathsf{C} \dashv \mathsf{K}$ [Mac Lane 1998, IV.1, Theorem 1].

This proves that $\eta_f \colon f \to \mathsf{KC}f$ is an $\vec{\mathcal{O}}^\square$–algebra morphism. Moreover, η is natural with respect to $\vec{\mathcal{O}}^\square$–algebra morphisms because a diagram in $\mathsf{Alg}(\vec{\mathcal{O}}^\square; \vec{\mathsf{M}}^\square)$ is commutative if and only if its underlying diagram in $\vec{\mathsf{M}}^{\mathfrak{C}}$ is commutative. Thus, the unit $\eta \colon \mathrm{Id} \to \mathsf{KC}$ is a natural transformation for the top horizontal functors in (3.4.5) between algebra categories.

Step 3: lifting the counit Next we show that the counit $\varepsilon \colon \mathsf{CK} \to \mathrm{Id}$ of the bottom adjunction $\mathsf{C} \dashv \mathsf{K}$ in (3.4.5) lifts to the top between algebra categories. First we need to show that, for each $\vec{\mathcal{O}}^\otimes$–algebra (g, λ), the counit component morphism $\varepsilon_g \colon \mathsf{CK}g \to g$ in $\vec{\mathsf{M}}^{\mathfrak{C}}$ is an $\vec{\mathcal{O}}^\otimes$–algebra morphism. Denote by
$$(\mathsf{K}g, \bar{\lambda}) = \mathsf{K}(g, \lambda)$$
the $\vec{\mathcal{O}}^\square$–algebra obtained by applying the top functor K in (3.4.5). The $\vec{\mathcal{O}}^\square$–algebra structure morphism $\bar{\lambda}$ is the analogue of (3.4.4) for the kernel. In other words, it is the composite

(3.4.9)
$$\bar{\lambda} = (\mathsf{K}\lambda) \circ \mathsf{K}^2$$

with $\mathsf{K}^2 = \ker^2$ the monoidal constraint (2.4.5).

As noted in (3.4.8), each component $\varepsilon_{\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}}}$ is the identity. With $(-)^\#$ as in (3.4.4), ε_g is an $\vec{\mathcal{O}}^\otimes$–algebra morphism if and only if the boundary of the diagram

(3.4.10)
$$\begin{tikzcd}
\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^n \mathsf{CK}g_{c_i} \arrow[rrr, "\bar{\lambda}^\#"] \arrow[dr, "\mathsf{C}^2"] \arrow[ddd, "\mathrm{Id}\otimes\bigotimes_i \varepsilon_i"'] & & & \mathsf{CK}g_d \arrow[ddd, bend left, "\varepsilon_{g_d}"] \\
& \mathsf{C}\big(\vec{\mathcal{O}}^\square\binom{d}{\underline{c}} \square \square_{i=1}^n \mathsf{K}g_{c_i}\big) \arrow[urr, "\mathsf{C}\bar{\lambda}"] \arrow[dr, "\mathsf{C}(\mathsf{K}^2)"] & & \\
& & \mathsf{CK}\big(\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^n g_{c_i}\big) \arrow[uur, "\mathsf{CK}\lambda"] \arrow[dll, "\varepsilon"] & \\
\vec{\mathcal{O}}^\otimes\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^n g_{c_i} \arrow[rrr, "\lambda"] & & & g_d
\end{tikzcd}$$

in $\vec{\mathsf{M}}$ commutes for $d, c_1, \ldots, c_n \in \mathfrak{C}$, with $\varepsilon_i = \varepsilon_{g_{c_i}}$, $\vec{\mathcal{O}}^\otimes = \mathsf{C}\vec{\mathcal{O}}^\square$ and $\vec{\mathcal{O}}^\square = \mathsf{K}\vec{\mathcal{O}}^\otimes$ by (3.1.3).

The four subregions in (3.4.10) are commutative for the following reasons:

- The top triangle is commutative by the definition of $(-)^{\#}$ in (3.4.4).
- The triangle to its lower right is commutative by the definition of $\overline{\lambda}$ in (3.4.9) and the functoriality of C.
- The lower right quadrilateral is commutative by the naturality of the counit $\varepsilon\colon \mathsf{CK} \to \mathrm{Id}$.

Using the inverse of $\mathsf{C}^2 = \mathrm{coker}^2$, the left triangle in (3.4.10) is equivalent to the diagram:

(3.4.11)
$$\begin{tikzcd}
\mathsf{C}\big(\mathsf{K}\vec{\mathcal{O}}^{\otimes}\binom{d}{\underline{c}} \mathbin{\square} \square_{i=1}^{n} \mathsf{K}g_{c_i}\big) \arrow[r, "\mathsf{C}(\mathsf{K}^2)"] \arrow[d, "(\mathsf{C}^2)^{-1}"'] & \mathsf{CK}\big(\vec{\mathcal{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} g_{c_i}\big) \arrow[d, "\varepsilon"] \\
\mathsf{CK}\vec{\mathcal{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} \mathsf{CK}g_{c_i} \arrow[r, "\varepsilon\otimes\bigotimes_i \varepsilon_i"] & \vec{\mathcal{O}}^{\otimes}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{n} g_{c_i}
\end{tikzcd}$$

The diagram (3.4.11) is commutative because the adjoint of each composite is $\mathsf{K}^2 = \ker^2$ defined in (2.4.5). This shows that (3.4.10) is commutative, and ε_g is an $\vec{\mathcal{O}}^{\otimes}$–algebra morphism.

Moreover, ε is natural with respect to $\vec{\mathcal{O}}^{\otimes}$–algebra morphisms because a diagram in $\mathsf{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ is commutative if and only if its underlying diagram in $\vec{\mathsf{M}}^{\mathfrak{C}}$ is commutative. Thus, the counit $\varepsilon\colon \mathsf{CK} \to \mathrm{Id}$ is a natural transformation for the top horizontal functors in (3.4.5) between algebra categories.

Finally, the lifted natural transformations η and ε satisfy the triangle identities for an adjunction [Mac Lane 1998, IV.1, Theorem 1] because diagrams in $\mathsf{Alg}(\vec{\mathcal{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ and $\mathsf{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ are commutative if and only if their underlying diagrams in $\vec{\mathsf{M}}^{\mathfrak{C}}$ are commutative. This proves that the top horizontal functors (C, K) in (3.4.5) form an adjunction with the lifted unit and counit. □

4 Homotopy theory of Smith ideals for operads

In this section, we study the homotopy theory of Smith ideals for an operad $\mathcal{O}$. Under suitable conditions on the underlying monoidal model category M, in Definition 4.2.3 we define model structures on the categories of Smith $\mathcal{O}$–ideals and of $\mathcal{O}$–algebra morphisms. When M is pointed, the cokernel and the kernel yield a Quillen adjunction between these model categories. Furthermore, in Theorem 4.4.1 we show that if M is stable and if cofibrant Smith $\mathcal{O}$–ideals are entrywise cofibrant in $\vec{\mathsf{M}}^{\square}$, then the cokernel and the kernel yield a Quillen equivalence between the categories of Smith $\mathcal{O}$–ideals and of $\mathcal{O}$–algebra morphisms.

Definition 4.0.1 We say that a $\mathfrak{C}$–colored operad $\mathcal{O}$ is *admissible* if $\mathsf{Alg}(\mathcal{O}; \mathsf{M})$ admits a transferred model structure, with weak equivalences and fibrations defined entrywise in $\mathsf{M}^{\mathfrak{C}}$.

4.1 Admissibility of operads

Theorem 4.1.1 [White and Yau 2018a, 6.1.1 and 6.1.3] *Suppose* M *is an operadically cofibrantly generated* (Definition 3.1.4) *monoidal model category satisfying the following condition*:

(♤) *For each $n \geq 1$ and for each object $X \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$, the function*

$$X \otimes_{\Sigma_n} (-)^{\square n} : \mathsf{M} \to \mathsf{M}$$

takes trivial cofibrations into some subclass of weak equivalences that is closed under transfinite composition and pushout.

Then each $\mathfrak{C}$–colored operad $\mathbb{O}$ in M is admissible in the sense of Definition 4.0.1.

Example 4.1.2 Strongly cofibrantly generated monoidal model categories that satisfy (♤) include

(1) pointed or unpointed simplicial sets [Quillen 1967] and all of their left Bousfield localizations [Hirschhorn 2003];

(2) bounded or unbounded chain complexes over a commutative ring containing the rationals $\mathbb{Q}$ [Quillen 1967];

(3) symmetric spectra built on either simplicial sets or compactly generated topological spaces, motivic symmetric spectra, and G–equivariant symmetric spectra with either the positive stable model structure or the positive flat stable model structure [Pavlov and Scholbach 2018];

(4) the category of small categories with the folk model structure [Rezk 2000];

(5) simplicial modules over a field of characteristic zero [Quillen 1967];

(6) the stable module category of $k[G]$–modules [Hovey 1999, 2.2], where k is a field and G is a finite group (we recall that the homotopy category of this example is trivial unless the characteristic of k divides the order of Gs, the setting for *modular* representation theory).

The condition (♤) for (1)–(2) is proved in [White and Yau 2018a, Section 8], which also handles symmetric spectra built on simplicial sets, and (4)–(5) can be proved using similar arguments. The condition (♤) for the stable module category is proved by the argument in [White and Yau 2020, 12.2]. For symmetric spectra built on topological spaces, motivic symmetric spectra and equivariant symmetric spectra, we refer to [Pavlov and Scholbach 2018, Section 2], starting with $\mathscr{C} = \mathsf{Top}$, sSet^G, Top^G, and the $\mathbb{A}^1$–localization of simplicial presheaves with the injective model structure.

In each of these examples except those built from Top, the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. So Proposition 2.4.6 applies to show that, in each case, the arrow category with either the injective or the projective model structure is strongly cofibrantly generated. The category of (equivariant) symmetric spectra built on topological spaces is operadically cofibrantly generated by an argument analogous to that of Example 3.1.5, as are the arrow categories, by the remark below.

Remark 4.1.3 In [White and Yau 2018a, 6.1.1 and 6.1.3], M is assumed to be strongly cofibrantly generated, but actually operadically cofibrantly generated suffices for the proof. The smallness hypothesis is required in order to run the small object argument, and $\mathbb{O} \circ I$ (resp. $\mathbb{O} \circ J$) are the generating (trivial) cofibrations. We have previously pointed out that *operadically cofibrantly generated* is a sufficient smallness hypothesis in [White and Yau 2020, 5.7]. The proof of Proposition 2.4.6 also proves that, if M is operadically cofibrantly generated, then so are $\vec{\mathsf{M}}^{\square}$ and $\vec{\mathsf{M}}^{\otimes}$.

Even if ($\spadesuit$) is not satisfied, sometimes the classes of morphisms defined in Theorem 4.1.1 in $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ define a semi-model structure [White and Yau 2018a, 6.2.3 and 6.3.1]. We therefore phrase our arguments in this section to only rely on the semi-model category axioms in categories of algebras. In Section 7, we include a comparison to the ∞–categorical approach to encoding the homotopy theory of operad algebras.

4.2 Admissibility of operads in the arrow category

Recall the injective model structure on the arrow category, which is a monoidal model category if M is, by Theorem 2.3.1.

Theorem 4.2.1 *If M is a monoidal model category satisfying ($\spadesuit$), then so is $\vec{\mathsf{M}}^{\otimes}$. Therefore, if M is also cofibrantly generated in which the domains and the codomains of all the generating (trivial) cofibrations are small in M, then every $\mathfrak{C}$–colored operad on $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ is admissible.*

Proof Suppose M satisfies ($\spadesuit$) with respect to a subclass $\mathscr{C}$ of weak equivalences that is closed under transfinite composition and pushout. We write $\mathscr{C}'$ for the subclass of weak equivalences β in $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ such that $\beta_0, \beta_1 \in \mathscr{C}$. Then $\mathscr{C}'$ is closed under transfinite composition and pushout.

Suppose $f_X : X_0 \to X_1$ is an object in $(\vec{\mathsf{M}}^{\otimes})^{\Sigma_n^{\mathrm{op}}}$ and $\alpha : f_V \to f_W$,

$$\begin{tikzcd} V_0 \arrow[r, "\alpha_0"] \arrow[d, "f_V"'] & W_0 \arrow[d, "f_W"] \\ V_1 \arrow[r, "\alpha_1"] & W \end{tikzcd} \tag{4.2.2}$$

is a trivial cofibration in $\vec{\mathsf{M}}^{\otimes}$. We will show that $f_X \otimes_{\Sigma_n} \alpha^{\square n}$ belongs to $\mathscr{C}'$. The morphism $f_X \otimes_{\Sigma_n} \alpha^{\square n}$ in $\vec{\mathsf{M}}^{\otimes}$ is the commutative square

$$\begin{tikzcd} X_0 \otimes_{\Sigma_n} \mathrm{dom}(\alpha_0^{\square n}) \arrow[r, "X_0 \otimes_{\Sigma_n} \alpha_0^{\square n}"] \arrow[d, "f_X \otimes_{\Sigma_n} f_*"'] & X_0 \otimes_{\Sigma_n} W_0^{\otimes n} \arrow[d, "f_X \otimes_{\Sigma_n} f_W^{\otimes n}"] \\ X_1 \otimes_{\Sigma_n} \mathrm{dom}(\alpha_1^{\square n}) \arrow[r, "X_1 \otimes_{\Sigma_n} \alpha_1^{\square n}"] & X_1 \otimes_{\Sigma_n} W_1^{\otimes n} \end{tikzcd}$$

in M, where f_* is induced by f_V and f_W. Since α_0 and α_1 are trivial cofibrations in M and since $X_0, X_1 \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$, the condition ($\spadesuit$) in M implies that the two horizontal morphisms in the previous diagram are both in $\mathscr{C}$. This shows that $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ satisfies ($\spadesuit$) with respect to the subclass $\mathscr{C}'$ of weak equivalences.

The second assertion is now a consequence of Proposition 2.4.6, Example 3.1.5 and Theorem 4.1.1. $\square$

Definition 4.2.3 Suppose M is a cofibrantly generated monoidal model category satisfying (♤) in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose O is a $\mathfrak{C}$–colored operad in M.

(1) Equip the category of Smith O–ideals $\mathsf{Alg}(\vec{\mathsf{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ with the model structure given by Proposition 3.3.19 and Theorem 4.1.1. In other words, a morphism α of Smith O–ideals is a weak equivalence (resp. fibration) if and only if α_0 and α_1 are colorwise weak equivalences (resp. fibrations) in M.

(2) Equip the category $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ with the model structure given by Theorem 4.2.1. In other words, a morphism α in $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ is a weak equivalence (resp. fibration) if and only if α^c (= the c–colored entry of α) is a weak equivalence (resp. fibration) in $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ for each $c \in \mathfrak{C}$.

When (♤) is not satisfied but the classes of morphisms above still define semi-model structures (eg Remark 5.1.6, Corollary 5.2.3 and Theorem 6.2.1), we still denote those semi-model structures by $\mathsf{Alg}(\vec{\mathsf{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ and $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$.

Remark 4.2.4 Recall diagram (3.4.5). In Definition 4.2.3 the (semi-)model structure on Smith O–ideals is induced by the forgetful functor to $\mathsf{M}^{\mathfrak{C} \sqcup \mathfrak{C}}$, so its weak equivalences and fibrations are defined entrywise in M, or equivalently in $\vec{\mathsf{M}}^{\square}$. On the other hand, the model structure on O–algebra morphisms $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ is induced by the forgetful functor to $(\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$. The (trivial) fibrations in $\mathsf{Alg}(\vec{\mathsf{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ are, in particular, entrywise (trivial) fibrations in M. However, they are *not* defined entrywise in M, since (trivial) fibrations in $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ are not defined entrywise in M, as explained in Theorem 2.3.1.

Suppose $K \subseteq \mathsf{M}$ is a subclass of morphisms in a category M with a chosen initial object and $\mathfrak{C}$ is a set with $c \in \mathfrak{C}$. We denote by

$$K_c \subseteq \mathsf{M}^{\mathfrak{C}}$$

the subclass of morphisms in which the morphisms in K are concentrated in the c–entry with all other entries the initial object. The following observation will be used in the proof of Theorem 6.2.1 below:

Proposition 4.2.5 *In the context of Definition 4.2.3, the (semi-)model structure on Smith O–ideals is cofibrantly generated with generating cofibrations $\vec{\mathsf{O}}^{\square} \circ (L_0 I \cup L_1 I)_c$ and generating trivial cofibrations $\vec{\mathsf{O}}^{\square} \circ (L_0 J \cup L_1 J)_c$ for $c \in \mathfrak{C}$, where I and J are the sets of generating cofibrations and generating trivial cofibrations in M.*

Proof The category $\mathsf{Alg}(\vec{\mathsf{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ already has a (semi-)model structure, namely the one in Definition 4.2.3(1), with weak equivalences and fibrations defined via the forgetful functor U in the free–forgetful adjunction

$$(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}} \underset{U}{\overset{\vec{\mathsf{O}}^{\square} \circ -}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathsf{O}}^{\square}; \vec{\mathsf{M}}^{\square}),$$

since the weak equivalences and fibrations in $\vec{\mathsf{M}}_{\mathrm{proj}}$ are defined in M. To see that $\mathsf{Alg}(\vec{\mathsf{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ has a cofibrantly generated model structure with weak equivalences and fibrations defined entrywise in $\vec{\mathsf{M}}_{\mathrm{proj}}$

and with generating (trivial) cofibrations as stated above, we refer to the computations of [Johnson and Yau 2009, Lemma 3.3], which produces the sets I and J, proves the requisite smallness, and proves that fibrations and trivial fibrations are characterized by lifting with respect to I and J. Hence, this proof works just as well for semi-model categories. Since a (semi-)model structure is uniquely determined by the classes of weak equivalences and fibrations, this second model structure on $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ must be the same as the one in Definition 4.2.3(1). $\square$

4.3 Quillen adjunction between operadic Smith ideals and algebra morphisms

Proposition 4.3.1 *Suppose* M *is a pointed cofibrantly generated monoidal model category, in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose* $\mathbb{O}$ *is a* $\mathfrak{C}$*–colored operad in* M *such that* $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ *and* $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ *admit transferred semi-model structures as in Definition 4.2.3. Then the adjunction*

$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \underset{\ker}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes}) \tag{4.3.2}$$

in (3.4.3) is a Quillen adjunction.

Proof Suppose α is a (trivial) fibration in $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$. We must show that $\ker\alpha$ is a (trivial) fibration in $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$, that is, an entrywise (trivial) fibration in M. Since (trivial) fibrations in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ are defined entrywise in M, it suffices to show that $U\ker\alpha$ is a (trivial) fibration in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. Since there is an equality — see (3.4.5) —

$$U\ker\alpha = \ker U\alpha$$

and since $\ker\colon (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}} \to (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is a right Quillen functor by Proposition 2.4.3(3), we finish the proof by observing that $U\alpha \in (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$ is a (trivial) fibration. $\square$

Recall that a pointed (semi-)model category is *stable* if its homotopy category is a triangulated category [Hovey 1999, 7.1.1].

Proposition 4.3.3 *In the setting of Proposition 4.3.1, suppose* M *is also a stable (semi-)model category. Then the right Quillen functor* $\ker$ *in (4.3.2) reflects weak equivalences between fibrant objects.*

Proof Suppose α is a morphism in $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ between fibrant objects such that $\ker\alpha \in \mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ is a weak equivalence. So $\ker\alpha$ is entrywise a weak equivalence in M, or equivalently $U\ker\alpha \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is a weak equivalence. We must show that α is a weak equivalence, that is, that $U\alpha \in (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$ is a weak equivalence. The morphism $U\alpha$ is still a morphism between fibrant objects, and

$$\ker U\alpha = U\ker\alpha$$

is a weak equivalence in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. Since $\ker\colon (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}} \to (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is a right Quillen equivalence by Proposition 2.4.3(4), it reflects weak equivalences between fibrant objects by [Hovey 1999, 1.3.16]. So $U\alpha$ is a weak equivalence. $\square$

4.4 Quillen equivalence between operadic Smith ideals and algebra morphisms

The following result says that, under suitable conditions, Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms have equivalent homotopy theories:

Theorem 4.4.1 *Suppose* M *is a cofibrantly generated stable monoidal model category, and* $\mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square})$ *and* $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$ *admit transferred (semi-)model structures as in Definition 4.2.3. Suppose* $\mathbb{O}$ *is a* $\mathfrak{C}$*–colored operad in* M *such that cofibrant* $\vec{\mathbb{O}}^{\square}$*–algebras are also underlying cofibrant in* $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$*. Then the Quillen adjunction*

$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square}) \underset{\ker}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$$

is a Quillen equivalence.

Proof Using Proposition 4.3.3 and [Hovey 1999, 1.3.16] (or [White 2017, Remark 4.3] for the semi-model category case), it remains to show that for each cofibrant object $f_X \in \mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square})$, the derived unit

$$f_X \xrightarrow{\eta} \ker R_{\mathbb{O}} \operatorname{coker} f_X$$

is a weak equivalence in $\mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square})$, where $R_{\mathbb{O}}$ is a fibrant replacement functor in $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$. In other words, we must show that $U\eta$ is a weak equivalence in the model category $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$.

Suppose R is a fibrant replacement functor in $(\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$. Consider the solid-arrow commutative diagram

$$\begin{tikzcd}
U \operatorname{coker} f_X \arrow[r, "\sim"] \arrow[d, tail, "\sim"'] & UR_{\mathbb{O}} \operatorname{coker} f_X \arrow[d, two heads] \\
RU \operatorname{coker} f_X \arrow[r, two heads] \arrow[ur, dotted, "\alpha"] & 0
\end{tikzcd}$$

in $(\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$. Here the left vertical morphism is a trivial cofibration and is a fibrant replacement of $U \operatorname{coker} f_X$. The top horizontal morphism is a weak equivalence and is U applied to a fibrant replacement of $\operatorname{coker} f_X$. The other two morphisms are fibrations. So there is a dotted morphism α that makes the whole diagram commutative. By the two-out-of-three property, α is a weak equivalence between fibrant objects in $(\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$. Since $\ker\colon (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}} \to (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is a right Quillen functor, by Ken Brown's lemma [Hovey 1999, 1.1.12] $\ker \alpha$ is a weak equivalence in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$.

We now have a commutative diagram

$$\begin{tikzcd}
Uf_X \arrow[rr, "U\eta"] \arrow[d, "\varepsilon"'] & & U \ker R_{\mathbb{O}} \operatorname{coker} f_X \arrow[d, equal] \\
\ker R \operatorname{coker} Uf_X \arrow[r, equal] & \ker RU \operatorname{coker} f_X \arrow[r, "\ker\alpha", "\sim"'] & \ker UR_{\mathbb{O}} \operatorname{coker} f_X
\end{tikzcd}$$

in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$, where ε is the derived unit of Uf_X. To show that $U\eta$ is a weak equivalence, it suffices to show that ε is a weak equivalence. By assumption Uf_X is a cofibrant object in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. Since $(\operatorname{coker}, \ker)$

is a Quillen equivalence between $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ and $(\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{C}}$, the derived unit ε is a weak equivalence by [ibid., 1.3.16]. □

Example 4.4.2 Among the model categories in Example 4.1.2,

(1) the categories of bounded or unbounded chain complexes over a semisimple ring that contains the rational numbers,

(2) the stable module category of $k[G]$–modules,

(3) the categories of symmetric spectra, G–equivariant symmetric spectra built on simplicial sets for a finite group G and motivic symmetric spectra, with either the positive or the positive flat stable model structure

satisfy the conclusion of Theorem 4.4.1 for every operad O. Admissibility is proven in [White and Yau 2018a, 6.1.1; 2020, 5.15]. Stability is discussed in [Hovey 1999, Chapter 7; White and Yau 2018a, 8.3; Pavlov and Scholbach 2018, Section 2]. All are strongly cofibrantly generated because they are combinatorial model categories [White and Yau 2020, Sections 11 and 12; Pavlov and Scholbach 2018, Section 2]. So all satisfy the conditions of Theorem 4.4.1 except that the condition about cofibrant Smith O–ideals being colorwise cofibrant in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ is more subtle. We will consider this issue in the next two sections, proving this condition for (1) in Corollary 5.2.4 and for (2) in Corollary 6.2.5.

For classical, equivariant or motivic symmetric spectra, we must tweak the proof of Theorem 4.4.1. Let $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ refer to the projective model structure on the arrow category where M is the *injective* stable model structure on the relevant category of symmetric spectra. Since the weak equivalences of the injective stable model structure coincide with those of the positive (flat) stable model structure, in the last paragraph of the proof, it is enough to prove that ϵ is a weak equivalence with respect to the injective stable model structure on spectra. Hence, it suffices for Uf_X to be a cofibrant object in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$, which follows from the proof of [White and Yau 2018a, 8.3.3], using our filtrations and the fact that the cofibrations of the injective stable model structure are the monomorphisms.

We note that we cannot add the injective stable model structure on symmetric spectra to the list in Example 4.4.2 because it is not true that every operad is admissible. A famous obstruction due to Gaunce Lewis prevents the Com operad from being admissible, for example.

5 Smith ideals for commutative and Σ–cofibrant operads

In this section we apply Theorem 4.4.1 and consider Smith ideals for the commutative operad and $\Sigma_{\mathfrak{C}}$–cofibrant operads (Definition 5.2.1). In particular, in Corollary 5.2.3 we will show that Theorem 4.4.1 is applicable to all $\Sigma_{\mathfrak{C}}$–cofibrant operads. On the other hand, the commutative operad is usually not Σ–cofibrant. However, as we will see in Example 5.1.3, Theorem 4.4.1 is applicable to the commutative operad in symmetric spectra with the positive flat stable model structure.

5.1 Commutative Smith ideals

For the commutative operad, which is entrywise the monoidal unit and whose algebras are commutative monoids, we use the following definition from [White 2017, 3.4]. The notation $?/\Sigma_n$ means taking the Σ_n–coinvariants.

Definition 5.1.1 A monoidal model category M is said to satisfy the *strong commutative monoid axiom* if, whenever $f\colon K \to L$ is a (trivial) cofibration, then so is $f^{\square n}/\Sigma_n$, where $f^{\square n}$ is the n–fold pushout product (which can be viewed as the unique morphism from the colimit Q_n of a punctured n–dimensional cube to $L^{\otimes n}$), and the Σ_n–action is given by permuting the vertices of the cube.

The following result says that, under suitable conditions, commutative Smith ideals and commutative monoid morphisms have equivalent homotopy theories:

Corollary 5.1.2 *Suppose* M *is a cofibrantly generated stable monoidal model category that satisfies the strong commutative monoid axiom, the monoid axiom, and in which cofibrant* $\overrightarrow{\mathsf{Com}}^{\square}$*–algebras are also underlying cofibrant in* $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ (*this occurs, for example, if the monoidal unit is cofibrant*). *Then there is a Quillen equivalence*

$$\mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\square};\vec{\mathsf{M}}^{\square}) \underset{\ker}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$$

in which Com *is the commutative operad in* M.

Proof First, [White 2017, 5.12 and 5.14] ensures that $\vec{\mathsf{M}}^{\otimes}$ and $\vec{\mathsf{M}}^{\square}$ satisfy the strong commutative monoid axiom, and [Hovey 2014, 2.2 and 3.2] (also Theorems 2.3.1 and 2.4.1) ensures that they satisfy the monoid axiom. Hence, by [White 2017, 3.2], $\mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\square};\vec{\mathsf{M}}^{\square})$ and $\mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$ carry transferred model structures.

For the commutative operad, it is proved in [ibid., 3.6 and 5.14] that, with the strong commutative monoid axiom and a cofibrant monoidal unit, cofibrant $\overrightarrow{\mathsf{Com}}^{\square}$–algebras are also underlying cofibrant in $\vec{\mathsf{M}}^{\square}$. So Theorem 4.4.1 applies. □

Example 5.1.3 (commutative Smith ideals in symmetric spectra) Example 4.4.2 shows that the category of symmetric spectra with the positive flat stable model structure satisfies the hypotheses in Theorem 4.4.1. It also satisfies the strong commutative monoid axiom [ibid., 5.7] and the monoid axiom [Schwede and Shipley 2000]. While the monoidal unit is not cofibrant, nevertheless, White [2017, 5.15] shows that cofibrant commutative Smith ideals forget to cofibrant objects of $\vec{\mathsf{M}}^{\square}$. Therefore, Corollary 5.1.2 applies to the commutative operad Com in symmetric spectra with the positive flat stable model structure.

Example 5.1.4 (commutative Smith ideals in algebraic settings) Let R be a commutative ring containing the ring of rational numbers $\mathbb{Q}$. Corollary 5.2.4 shows that the category of (bounded or unbounded) chain complexes of R–modules satisfies the conditions of Theorem 4.4.1. They also satisfy the strong commutative monoid axiom and the monoid axiom [ibid., Lemma 5.1]. Hence, Corollary 5.1.2 applies, to give a homotopy theory of ideals of CDGAs. The same is true of the stable module category of $R = k[G]$, where k is a field and G is a finite group, using Corollary 6.2.5. The result is a homotopy theory of ideals of commutative R–algebras.

Example 5.1.5 (commutative Smith ideals in (equivariant) orthogonal/symmetric spectra) Let G be a compact Lie group. The positive flat stable model structure on G–equivariant orthogonal spectra satisfies the strong commutative monoid axiom [ibid., 5.10], the monoid axiom [White 2022, Section 5.8] and the property that cofibrant commutative Smith ideals forget to cofibrant objects of $\vec{\mathsf{M}}^{\square}$ [White 2017, 5.15]. The same is true for Hausmann's G–symmetric spectra built on either simplicial sets or topological spaces for a finite group G by [Hausmann 2017, 6.4, 6.16 and 6.22], and for Schwede's positive flat model structure for global equivariant homotopy theory (where commutative monoids are ultracommutative ring spectra) [Schwede 2018, 4.3.28, 5.4.1 and 5.4.3]. Hence, Corollary 5.1.2 applies in all three settings.

Of course, taking G trivial in Example 5.1.5, one obtains that Corollary 5.1.2 applies to orthogonal spectra with the positive flat stable model structure [White 2022, Section 5.8].

Remark 5.1.6 If, in Corollary 5.1.2, M fails to satisfy the monoid axiom, then we still have semi-model structures on $\mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\square}; \vec{\mathsf{M}}^{\square})$ and $\mathsf{Alg}(\overrightarrow{\mathsf{Com}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ by [White 2017, 3.8]. In this case, Theorem 4.4.1 still applies, as long as cofibrant $\overrightarrow{\mathsf{Com}}^{\square}$–algebras are also underlying cofibrant in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ (eg if the monoidal unit is cofibrant, by [ibid., 3.6]).

5.2 Smith ideals for Σ–cofibrant operads

For a cofibrantly generated model category M and a small category $\mathfrak{D}$, recall that the diagram category $\mathsf{M}^{\mathfrak{D}}$ inherits a *projective model structure* with weak equivalences and fibrations defined entrywise in M [Hirschhorn 2003, 11.6.1]. We use this below when $\mathfrak{D} = \Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}$ is the groupoid in Definition 3.1.1. In this case, the category $\mathsf{M}^{\mathfrak{D}}$ is the category of $\mathfrak{C}$–colored symmetric sequences.

Definition 5.2.1 For a cofibrantly generated model category M, a $\mathfrak{C}$–colored operad in M is said to be *$\Sigma_{\mathfrak{C}}$–cofibrant* if its underlying $\mathfrak{C}$–colored symmetric sequence is cofibrant. If $\mathfrak{C}$ is the one-point set, then we say *Σ–cofibrant* instead of $\Sigma_{\{*\}}$–cofibrant

Proposition 5.2.2 *Suppose M is a cofibrantly generated model category and $\mathfrak{D}$ is a small category. If $X \in \mathsf{M}^{\mathfrak{D}}$ is cofibrant, then $L_1 X \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{D}}$ and $L_0 X \in (\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}})^{\mathfrak{D}}$ are cofibrant. In particular, if $\mathcal{O}$ is a $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operad in M, then $\vec{\mathcal{O}}^{\square} = L_1 \mathcal{O}$ is a $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operad in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and $\vec{\mathcal{O}}^{\otimes}$ is a $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operad in $\vec{\mathsf{M}}^{\otimes}$.*

Proof The Quillen adjunction $L_1 \colon \mathsf{M} \rightleftarrows \vec{\mathsf{M}}^{\square}_{\mathrm{proj}} \colon \mathrm{Ev}_1$ lifts to a Quillen adjunction of $\mathfrak{D}$–diagram categories

$$\mathsf{M}^{\mathfrak{D}} \underset{\mathrm{Ev}_1}{\overset{L_1}{\rightleftarrows}} (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{D}}$$

by [ibid., 11.6.5(1)], and similarly for (L_0, Ev_0). If $X \in \mathsf{M}^{\mathfrak{D}}$ is cofibrant, then $L_1 X$ and $L_0 X$ are cofibrant since L_1 and L_0 are left Quillen functors. $\square$

The following result says that, under suitable conditions, for a $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operad $\mathbb{O}$, Smith $\mathbb{O}$–ideals and $\mathbb{O}$–algebra morphisms have equivalent homotopy theories:

Corollary 5.2.3 *Suppose* M *is as in Theorem 4.4.1 and* $\mathbb{O}$ *is a* $\Sigma_{\mathfrak{C}}$*–cofibrant* $\mathfrak{C}$*–colored operad in* M*. Then* $\mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square})$ *and* $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes})$ *have transferred semi-model structures, where cofibrant* $\vec{\mathbb{O}}^{\square}$*–algebras are also underlying cofibrant in* $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$*. Hence, there is a Quillen equivalence*

$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square};\vec{\mathsf{M}}^{\square}) \underset{\ker}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes};\vec{\mathsf{M}}^{\otimes}).$$

Proof The arrow categories $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ are cofibrantly generated monoidal model categories by Theorems 2.3.1 and 2.4.1. By Proposition 5.2.2, the $\mathfrak{C}$–colored operads $\vec{\mathbb{O}}^{\square}$ in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and $\vec{\mathbb{O}}^{\otimes}$ in $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ are $\Sigma_{\mathfrak{C}}$–cofibrant. Theorem 6.3.1 in [White and Yau 2018a], applied to $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and $\vec{\mathsf{M}}^{\otimes}$, now gives the transferred semi-model structures and says that every cofibrant $\vec{\mathbb{O}}^{\square}$–algebra is underlying cofibrant in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. So Theorem 4.4.1 applies. □

The following provides one source of applications of Corollary 5.2.3, and answers a question Pavel Safranov asked the first author. This result generalizes [White 2017, Lemma 5.1; White and Yau 2018a, 8.1], as it applies in particular to fields of characteristic zero.

Corollary 5.2.4 *Suppose* R *is a commutative ring with unit and* M *is the category of bounded or unbounded chain complexes of* R*–modules, with the projective model structure. The following are equivalent:*

1. *R is a semisimple ring containing the rational numbers $\mathbb{Q}$.*
2. *Every symmetric sequence is projectively cofibrant.*

In particular, for such rings R*, every* $\mathfrak{C}$*–colored operad in* M *is* $\Sigma_{\mathfrak{C}}$*–cofibrant, so Corollary 5.2.3 is applicable for all colored operads in* M*. If* R *contains* $\mathbb{Q}$ *(but is not necessarily semisimple), then every entrywise cofibrant* $\mathfrak{C}$*–colored operad in* M *is* $\Sigma_{\mathfrak{C}}$*–cofibrant and admissible.*

Proof Assume (1). Maschke's theorem [Polcino Milies and Sehgal 2002, 3.4.7] guarantees that each group ring $R[\Sigma_n]$ is semisimple (since $1/n!$ exists in R, making $n!$ invertible). This means every module M over $R[\Sigma_n]$ is projective. In particular, M is a direct summand of a module induced from the trivial subgroup, and has a free Σ_n–action. Hence, (2) follows.

Conversely, if (2) is true, then it implies that, for every n, every module in $R[\Sigma_n]$ is projective. This means each $R[\Sigma_n]$ is a semisimple ring. By [loc. cit.], this implies that R is semisimple and $n!$ is invertible in R for every n. It follows that $\mathbb{Q}$ is contained in R.

For such R, the projective model structure on (bounded or unbounded) chain complexes of R–modules has every object cofibrant (so, automatically, cofibrant operad algebras forget to cofibrant chain complexes). Hence, any $\mathfrak{C}$–colored operad is entrywise cofibrant and hence $\Sigma_{\mathfrak{C}}$–cofibrant. Furthermore, Theorem 4.1.1 implies that all operads are admissible, since every $X \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ is Σ_n–projectively cofibrant.

If R contains $\mathbb{Q}$ but is not semisimple, then there can be nonprojective R–modules, but the argument of [loc. cit.] shows that an $R[\Sigma_n]$–module that is projective as an R–module is projective as an $R[\Sigma_n]$–module. It follows that Corollary 5.2.3 holds for entrywise cofibrant operads, including the operad Com. Indeed, all operads are admissible thanks to Theorem 4.1.1, since, for any trivial cofibration f and any $X \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$, maps of the form $X \otimes_{\Sigma_n} f^{\square n}$ are trivial h–cofibrations and this class of morphisms is closed under pushout and transfinite composition [White 2022, Section 5.8]. □

Example 5.2.5 Suppose M is as in Theorem 4.4.1, that is, cofibrantly generated, stable, monoidal and with (co)domains of $I \cup J$ small. Many examples of such M are provided in Examples 4.4.2 and 5.2.6 and in [White 2017; 2022; White and Yau 2018a; 2018b; 2019a; 2019b; 2020; Gutiérrez and White 2018; Hovey and White 2020]. Here are some examples of Σ–cofibrant operads, for which Corollary 5.2.3 is applicable:

Smith ideals The associative operad As, which has $\mathsf{As}(n) = \coprod_{\Sigma_n} \mathbb{1}$ as the n^{th} entry and which has monoids as algebras, is Σ–cofibrant. In this case, Corollary 5.2.3 is [Hovey 2014, Corollary 4.4(1)].

Smith A_∞–ideals Any A_∞–operad, defined as a Σ–cofibrant resolution of As, is Σ–cofibrant. In this case, Corollary 5.2.3 says that Smith A_∞–ideals and A_∞–algebra morphisms have equivalent homotopy theories. For instance, one can take the standard differential graded A_∞–operad [Markl 1996] and, for symmetric spectra, the Stasheff associahedra operad [1963a; 1963b].

Smith E_∞–ideals Any E_∞–operad, defined as a Σ–cofibrant resolution of the commutative operad Com, is Σ–cofibrant. In this case, Corollary 5.2.3 says that Smith E_∞–ideals and E_∞–algebra morphisms have equivalent homotopy theories. For example, for symmetric spectra, one can take the Barratt–Eccles E_∞–operad $E\Sigma_*$ [1974]. An elementary discussion of the Barratt–Eccles operad is in [Johnson and Yau 2021, Section 11.4].

Smith E_n–ideals For each $n \geq 1$, the little n–cubes operad $\mathscr{C}_n$ [Boardman and Vogt 1973; May 1972] is Σ–cofibrant and is an E_n–operad by definition [Fresse 2017, 4.1.13]. In this case, with M being symmetric spectra with the positive (flat) stable model structure, Corollary 5.2.3 says that Smith $\mathscr{C}_n$–ideals and $\mathscr{C}_n$–algebra morphisms have equivalent homotopy theories. One may also use other Σ–cofibrant E_n–operads [Fiedorowicz 1998], such as the Fulton–MacPherson operad [Getzler and Jones 1994; Fresse 2017, 4.3], which is actually a cofibrant E_n–operad. An elementary discussion of a categorical E_n–operad is in [Johnson and Yau 2021, Chapter 13].

Example 5.2.6 The power of restricting attention to the class of $\Sigma_{\mathfrak{C}}$–cofibrant colored operads is that Theorem 4.4.1 holds for a larger class of model categories. In particular, the following model categories satisfy the conditions of Theorem 4.4.1 for the class of $\Sigma_{\mathfrak{C}}$–cofibrant colored operads, as do all examples listed in Section 5.1:

(1) S–modules with the model structure from [Elmendorf et al. 1997].

(2) The projective, injective, positive or positive flat stable model structures [White 2022, 5.59 and 5.61] on symmetric spectra, G–equivariant orthogonal spectra (for a compact Lie group G) and motivic symmetric spectra.

(3) Mandell's model structure on G–equivariant symmetric spectra built on simplicial sets or topological spaces, where G is a finite group in the former case and a compact Lie group in the latter case [Mandell 2004].

(4) Model structures for (equivariant) stable homotopy theory based on Lydakis's theory of enriched functors [Dundas et al. 2003]. For example, this includes the model category of G–enriched functors from finite G–simplicial sets to G–simplicial sets, where G is a finite group, from [ibid., Theorem 2].

(5) Any model structure M on symmetric spectra built on $(\mathscr{C}, G)$ where $\mathscr{C}$ is a model category and G is an endofunctor, as long as M is an operadically cofibrantly generated, monoidal, stable model structure. For example, taking $\mathscr{C}$ to be the canonical model structure on small categories, and using the suspension discussed in [White and Yau 2020, Section 13], one obtains by [Hovey 2001, 7.3] a combinatorial, stable, monoidal model structure on symmetric spectra of small categories with applications to Goodwillie calculus. Using [Pavlov and Scholbach 2018, Section 2], one may obtain positive and positive flat variants. Another example is taking $\mathscr{C}$ to be the I–spaces or J–spaces of Sagave and Schlichtkrull, and building projective, positive or positive flat spectra on them as in [loc. cit.].

(6) The projective model structure on bounded or unbounded chain complexes over a commutative ring R [White and Yau 2020, Section 11].

(7) The stable module category of $k[G]$, where G is a finite group and k is a principal ideal domain [ibid., Section 12].

All of these examples are stable monoidal model categories, so Corollary 5.2.3 applies, once the requisite smallness hypothesis for the generating (trivial) cofibrations is checked. Symmetric spectra, motivic symmetric spectra, examples (6) and (7), and Mandell's model (3) of G–equivariant symmetric spectra built on simplicial sets are all combinatorial, as is the model structure on enriched functors (4) in simplicial contexts. Symmetric spectra as in (5) are combinatorial if $\mathscr{C}$ is combinatorial. S–modules, G–equivariant orthogonal spectra, Mandell's model (3) in topological contexts, and symmetric spectra built on topological spaces (another example of (5)) are operadically cofibrantly generated just as in Example 3.1.5, since they are built from compactly generated spaces. We recall that spaces are small relative to inclusions, and the morphisms in $(\mathbb{O}\circ(I\cup J))$–cell are inclusions [ibid., 5.10].

6 Smith ideals for entrywise cofibrant operads

In this section we apply Theorem 4.4.1 to operads that are not necessarily $\Sigma_{\mathfrak{C}}$–cofibrant. To do that, we need to redistribute some of the cofibrancy assumptions — that cofibrant Smith $\mathbb{O}$–ideals are underlying cofibrant in the arrow category — from the colored operad to the underlying category. We will show in Theorem 6.2.1 that Theorem 4.4.1 is applicable to all entrywise cofibrant operads provided that M satisfies the cofibrancy condition ($\heartsuit$) below. This implies that, over the stable module category [Hovey 1999, 2.2], Theorem 4.4.1 is always applicable.

6.1 Cofibrancy assumptions

Definition 6.1.1 Suppose M is a cofibrantly generated monoidal model category. Define the following conditions in M:

($\heartsuit$) For each $n \geq 1$ and each morphism $f \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ that is an underlying cofibration between cofibrant objects in M, the function

$$f \,\square_{\Sigma_n} (-) \colon \mathsf{M}^{\Sigma_n} \to \mathsf{M}$$

takes each morphism in M^{Σ_n} that is an underlying cofibration in M to a cofibration in M. More explicitly, this condition asks that, for each morphism $g \in \mathsf{M}^{\Sigma_n}$ that is an underlying cofibration in M, the morphism

$$f \,\square_{\Sigma_n} g = (f \,\square\, g)/\Sigma_n$$

is a cofibration in M.

$(\clubsuit)_{\mathrm{cof}}$ For each $n \geq 1$ and each object $X \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ that is underlying cofibrant in M, the function

$$X \otimes_{\Sigma_n} (-)^{\square n} \colon \mathsf{M} \to \mathsf{M}$$

preserves cofibrations.

$(\clubsuit)_{\mathrm{t.cof}}$ For each $n \geq 1$ and each object $X \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ that is underlying cofibrant in M, the function

$$X \otimes_{\Sigma_n} (-)^{\square n} \colon \mathsf{M} \to \mathsf{M}$$

preserves trivial cofibrations.

Remark 6.1.2 The condition ($\heartsuit$) implies $(\clubsuit)_{\mathrm{cof}}$, since $(\varnothing \to X) \,\square\, (-) = X \otimes (-)$. The condition $(\clubsuit)_{\mathrm{cof}}$ was introduced in [White and Yau 2018a, 6.2.1], where the authors proved that, if M satisfies $(\clubsuit)_{\mathrm{cof}}$ and $(\clubsuit)_{\mathrm{t.cof}}$, then there exist transferred semi-model structures on algebras over entrywise cofibrant (but not necessarily $\Sigma_{\mathfrak{C}}$–cofibrant) colored operads. It is, therefore, no surprise that we consider $(\clubsuit)_{\mathrm{cof}}$ and its variant ($\heartsuit$) here in order to use Theorem 4.4.1 for operads that are not necessarily $\Sigma_{\mathfrak{C}}$–cofibrant. Of course, ($\spadesuit$) implies $(\clubsuit)_{\mathrm{t.cof}}$, so $(\clubsuit)_{\mathrm{t.cof}}$ holds in all the model categories in Example 4.1.2.

Proposition 6.1.3 *The condition* ($\heartsuit$) *holds in the categories of*

(1) *simplicial sets with either the Quillen model structure or the Joyal model structure* [Lurie 2009], *where cofibrations are the monomorphisms*;

(2) *bounded or unbounded chain complexes over a field k of characteristic zero*, *where cofibrations are degreewise monomorphisms* [Hovey 1999, 2.3.9] *since every monomorphism of k–modules splits and every chain complex is cofibrant* (see Corollary 5.2.4);

(3) *small categories with the folk model structure where cofibrations are injective on objects* [Rezk 2000];

(4) *the stable module category of* $k[G]$*–modules with the characteristic of* k *dividing the order of* G*, where cofibrations are injections* [Hovey 1999, 2.2.12]; *and*

(5) *the injective model structure on symmetric spectra*, G*–equivariant symmetric spectra and motivic symmetric spectra*, *where the cofibrations are the monomorphisms* [Hovey 2001].

Proof For simplicial sets with either model structure, a cofibration is precisely an injection, and the pushout product of two injections is again an injection. Dividing an injection by a Σ_n–action is still an injection. The other cases are proved similarly. $\square$

Proposition 6.1.4 *If* $(\heartsuit)$ *holds in* M, *then it also holds in any left Bousfield localization of* M.

Proof The condition $(\heartsuit)$ only refers to cofibrations, which remain the same in any left Bousfield localization. $\square$

The next observation is the key that connects the cofibrancy condition $(\heartsuit)$ in M to the arrow category.

Theorem 6.1.5 *Suppose* M *is a cofibrantly generated monoidal model category satisfying* $(\heartsuit)$. *Then the arrow category* $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ *satisfies* $(\clubsuit)_{\mathrm{cof}}$.

Proof Suppose $f_X : X_0 \to X_1$ is an object in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\Sigma_n^{\mathrm{op}}}$ that is underlying cofibrant in $\vec{\mathsf{M}}^{\square}$. This means that f_X is a morphism in $\mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ that is an underlying cofibration between cofibrant objects in M. The condition $(\clubsuit)_{\mathrm{cof}}$ for $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ asks that the function

$$f_X \,\square_{\Sigma_n}\, (-)^{\square_2 n} : \vec{\mathsf{M}}^{\square}_{\mathrm{proj}} \to \vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$$

preserve cofibrations, where $\square$ and $\square_2$ are the pushout products in M and $\vec{\mathsf{M}}^{\square}$, respectively. When $n = 1$ the condition $(\clubsuit)_{\mathrm{cof}}$ for $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ is a special case of the pushout product axiom in $\vec{\mathsf{M}}^{\square}$, which is true by [White and Yau 2019a, Theorem A].

Next suppose $n \geq 2$ and $\alpha : f_V \to f_W$ is a morphism in $\vec{\mathsf{M}}$ as in (4.2.2). The iterated pushout product $\alpha^{\square_2 n} \in (\vec{\mathsf{M}}^{\square})^{\Sigma_n}$ is the commutative square

(6.1.6)
$$\begin{tikzcd} Z \arrow[r, "\zeta_1"] \arrow[d, "\zeta_0"'] & Y_1 \arrow[d, "f_W^{\square n}"] \\ Y_0 \arrow[r, "\alpha_1^{\square n}"] & W_1^{\otimes n} \end{tikzcd}$$

in M^{Σ_n} for some object Z with $\zeta_1 = \mathrm{Ev}_0(\alpha^{\square_2 n})$. Note that ζ_1 is not an iterated pushout product because Ev_0 and $\square_2$ do not commute. Applying $f_X \,\square_{\Sigma_n}\, (-)$, the morphism $f_X \,\square_{\Sigma_n}\, \alpha^{\square_2 n}$ is the commutative square

$$
\begin{tikzcd}
{[(X_1 \otimes Z) \amalg_{X_0 \otimes Z} (X_0 \otimes Y_0)]_{\Sigma_n}} \arrow[r, "\varphi"] \arrow[d, "f_X \square_{\Sigma_n} \zeta_0"'] & {[(X_1 \otimes Y_1) \amalg_{X_0 \otimes Y_1} (X_0 \otimes W_1^{\otimes n})]_{\Sigma_n}} \arrow[d, "f_X \square_{\Sigma_n} f_W^{\square n}"] \\
(X_1 \otimes Y_0)_{\Sigma_n} \arrow[r, "(X_1 \otimes \alpha_1^{\square n})_{\Sigma_n}"] & (X_1 \otimes W_1^{\otimes n})_{\Sigma_n}
\end{tikzcd}
\tag{6.1.7}
$$

in M. Suppose α is a cofibration in $\vec{\mathsf{M}}^{\square}$. This means that the morphism $\alpha_0 \colon V_0 \to W_0$ and the pushout corner morphism $\alpha_1 \circledast f_W \colon V_1 \amalg_{V_0} W_0 \to W_1$ are cofibrations in M. We must show that $f_X \square_{\Sigma_n} \alpha^{\square_2 n}$ is a cofibration in $\vec{\mathsf{M}}^{\square}$. In other words, we must show that, in (6.1.7):

(1) $\varphi = \mathrm{Ev}_0(f_X \square_{\Sigma_n} \alpha^{\square_2 n})$ is a cofibration in M.

(2) The pushout corner morphism of $f_X \square_{\Sigma_n} \alpha^{\square_2 n}$ is a cofibration in M.

We will prove (1) and (2) in Lemmas 6.1.8 and 6.1.10, respectively. $\square$

Lemma 6.1.8 *The morphism φ in (6.1.7) is a cofibration in* M.

Proof Taking Σ_n–coinvariants and taking pushouts commute by the commutation of colimits. So φ is also the induced morphism from the pushout of the top row to the pushout of the bottom row in the commutative diagram

$$
\begin{tikzcd}
(X_1 \otimes Z)_{\Sigma_n} \arrow[d, "(X_1 \otimes \zeta_1)_{\Sigma_n}"'] & (X_0 \otimes Z)_{\Sigma_n} \arrow[l, "(f_X \otimes Z)_{\Sigma_n}"'] \arrow[r, "(X_0 \otimes \zeta_0)_{\Sigma_n}"] \arrow[d, "(X_0 \otimes \zeta_1)_{\Sigma_n}"'] & (X_0 \otimes Y_0)_{\Sigma_n} \arrow[d, "(X_0 \otimes \alpha_1^{\square n})_{\Sigma_n}"] \\
(X_1 \otimes Y_1)_{\Sigma_n} & (X_0 \otimes Y_1)_{\Sigma_n} \arrow[l, "(f_X \otimes Y_1)_{\Sigma_n}"] \arrow[r, "(X_0 \otimes f_W^{\square n})_{\Sigma_n}"'] & (X_0 \otimes W_1^{\otimes n})_{\Sigma_n}
\end{tikzcd}
\tag{6.1.9}
$$

in M. Here the left square is commutative by definition, and the right square is $X_0 \otimes_{\Sigma_n} (-)$ applied to $\alpha^{\square_2 n}$ in (6.1.6).

We consider the Reedy category D with three objects $\{-1, 0, 1\}$, a morphism $0 \to -1$ that lowers the degree, a morphism $0 \to 1$ that raises the degree, and no other nonidentity morphisms. Using the Quillen adjunction [Hovey 1999, proof of 5.2.6],

$$
\mathsf{M}^{\mathsf{D}} \underset{\text{constant}}{\overset{\text{colim}}{\rightleftarrows}} \mathsf{M},
$$

to show that φ is a cofibration in M, it is enough to show that (6.1.9) is a Reedy cofibration in M^{D}. So we must show that, in (6.1.9):

(1) The left and the middle vertical arrows are cofibrations in M.

(2) The pushout corner morphism of the right square is a cofibration in M.

The objects X_0 and X_1 in $\mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ are cofibrant in M. The morphism $\zeta_1 = \mathrm{Ev}_0(\alpha^{\square_2 n}) \in \mathsf{M}^{\Sigma_n}$ is an underlying cofibration in M. Indeed, since $\alpha \in \vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ is a cofibration, so is the iterated pushout product

$\alpha^{\square 2n}$ by the pushout product axiom [White and Yau 2019a]. In particular, $\mathrm{Ev}_0(\alpha^{\square 2n})$ is a cofibration in M. The condition $(\heartsuit)$ in M (for the morphism $\varnothing \to X_i$) now implies that the left and the middle vertical morphisms $X_i \otimes_{\Sigma_n} \zeta_1$ in (6.1.9) are cofibrations in M.

Finally, since $X_0 \in \mathsf{M}^{\Sigma_n^{\mathrm{op}}}$ is cofibrant in M and since the pushout corner morphism of $\alpha^{\square 2n} \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\Sigma_n}$ is a cofibration in M, the condition $(\heartsuit)$ in M again implies the pushout corner morphism of the right square $X_0 \otimes_{\Sigma_n} \alpha^{\square 2n}$ in (6.1.9) is a cofibration in M. $\square$

Lemma 6.1.10 *The pushout corner morphism of $f_X \square_{\Sigma_n} \alpha^{\square 2n}$ in (6.1.7) is a cofibration in M.*

Proof The pushout corner morphism of $f_X \square_{\Sigma_n} \alpha^{\square 2n}$ is the morphism $f_X \square_{\Sigma_n} (\alpha_1^{\square n} \circledast f_W^{\square n})$. This is taking the Σ_n–coinvariants of the pushout product in the diagram

$$\begin{tikzcd}
X_0 \otimes (Y_0 \amalg_Z Y_1) \arrow[r, "\mathrm{Id}\otimes(\alpha_1^{\square n} \circledast f_W^{\square n})"] \arrow[d, "f_X \otimes \mathrm{Id}"'] \arrow[dr, phantom, "\text{pushout}"] & X_0 \otimes W_1^{\otimes n} \arrow[d] \arrow[ddr, bend left, "f_X \otimes \mathrm{Id}"] & \\
X_1 \otimes (Y_0 \amalg_Z Y_1) \arrow[r] \arrow[drr, bend right, "\mathrm{Id}\otimes(\alpha_1^{\square n} \circledast f_W^{\square n})"'] & {[X_1 \otimes (Y_0 \amalg_Z Y_1)] \amalg_{[X_0 \otimes (Y_0 \amalg_Z Y_1)]} (X_0 \otimes W_1^{\otimes n})} \arrow[dr, "f_X \square (\alpha_1^{\square n} \circledast f_W^{\square n})"] & \\
& & X_1 \otimes W_1^{\otimes n}
\end{tikzcd}$$

in M^{Σ_n} with $\alpha_1^{\square n} \circledast f_W^{\square n}$ the pushout corner morphism of $\alpha^{\square 2n} \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\Sigma_n}$ in (6.1.6). Since $\alpha^{\square 2n}$ is a cofibration in $\vec{\mathsf{M}}^{\square}$, its pushout corner morphism $\alpha_1^{\square n} \circledast f_W^{\square n}$ is a cofibration in M. So the condition $(\heartsuit)$ in M implies that $f_X \square_{\Sigma_n} (\alpha_1^{\square n} \circledast f_W^{\square n})$ is a cofibration in M. $\square$

6.2 Underlying cofibrancy of cofibrant Smith ideals for entrywise cofibrant operads

Theorem 6.2.1 *Suppose M is a cofibrantly generated monoidal model category satisfying $(\heartsuit)$ and $(\clubsuit)_{\mathrm{t.cof}}$, in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose $\mathbb{O}$ is an entrywise cofibrant $\mathfrak{C}$–colored operad in M. Then $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ and $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ admit transferred semi-model structures, and cofibrant Smith $\mathbb{O}$–ideals are underlying cofibrant in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. In particular, if M is also stable, then there is a Quillen equivalence*

$$\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \underset{\text{ker}}{\overset{\text{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes}).$$

Proof If $\mathbb{O}$ is entrywise cofibrant in M, then $\vec{\mathbb{O}}^{\square} = L_1\mathbb{O}$ is entrywise cofibrant in $\vec{\mathsf{M}}^{\square}$, and $\vec{\mathbb{O}}^{\otimes} = L_0\mathbb{O}$ is entrywise cofibrant in $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ by Proposition 5.2.2. Furthermore, because M satisfies $(\clubsuit)_{\mathrm{t.cof}}$, so does $\vec{\mathsf{M}}^{\otimes}$, by the exact same proof as in Theorem 4.2.1 (but now X_0 and X_1 are cofibrant in M, and we appeal to $(\clubsuit)_{\mathrm{t.cof}}$ in M instead of $(\spadesuit)$). Thus, we have transferred semi-model structures

- $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ by [White and Yau 2018a, 6.2.3] applied to $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$, and
- $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ by [loc. cit.] applied to the colored operad $\mathbb{O}^{s}$ in M in Proposition 3.3.19.

Using Theorem 4.4.1, it is enough to prove the assertion that cofibrant Smith $\mathbb{O}$–ideals are underlying cofibrant in $(\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. Writing $\varnothing^{\vec{\mathbb{O}}^{\square}}$ for the initial $\vec{\mathbb{O}}^{\square}$–algebra, first we claim that $\varnothing^{\vec{\mathbb{O}}^{\square}}$ is underlying cofibrant in $(\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. Indeed, for each color $d \in \mathfrak{C}$, the d–colored entry of the initial $\vec{\mathbb{O}}^{\square}$–algebra is the object

$$\varnothing_d^{\vec{\mathbb{O}}^{\square}} = \vec{\mathbb{O}}^{\square}\binom{d}{\varnothing} = \left(\varnothing^{\mathsf{M}} \to \mathbb{O}\binom{d}{\varnothing}\right)$$

in $\overrightarrow{\mathsf{M}}^{\square}$, where $\varnothing^{\mathsf{M}}$ is the initial object in M and the symbol $\varnothing$ in $\binom{d}{\varnothing}$ is the empty $\mathfrak{C}$–profile. Since $\mathbb{O}$ is assumed entrywise cofibrant, it follows that each entry of the initial $\vec{\mathbb{O}}^{\square}$–algebra $\varnothing^{\vec{\mathbb{O}}^{\square}}$ is underlying cofibrant in $\overrightarrow{\mathsf{M}}^{\square}$. Indeed, the pushout corner morphism of

$$\begin{array}{ccc} \varnothing^{\mathsf{M}} & \longrightarrow & \varnothing^{\mathsf{M}} \\ \downarrow & & \downarrow \\ \varnothing^{\mathsf{M}} & \longrightarrow & \mathbb{O}\binom{d}{\varnothing} \end{array}$$

is the cofibration $\varnothing^{\mathsf{M}} \to \mathbb{O}\binom{d}{\varnothing}$ in M, so, by Theorem 2.4.1(1), $\varnothing_d^{\vec{\mathbb{O}}^{\square}}$ is cofibrant in $\overrightarrow{\mathsf{M}}^{\square}$.

By Proposition 4.2.5, the semi-model structure on $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \overrightarrow{\mathsf{M}}^{\square})$ is right-induced by the forgetful functor U to $(\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ and is cofibrantly generated by $\vec{\mathbb{O}}^{\square} \circ (L_0 I \cup L_1 I)_c$ and $\vec{\mathbb{O}}^{\square} \circ (L_0 J \cup L_1 J)_c$ for $c \in \mathfrak{C}$, where I and J are the generating (trivial) cofibrations in M. Suppose A is a cofibrant $\vec{\mathbb{O}}^{\square}$–algebra. We must show that A is underlying cofibrant in $(\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. By [Hirschhorn 2003, 11.2.2], the cofibrant $\vec{\mathbb{O}}^{\square}$–algebra A is the retract of the colimit of a transfinite composition, starting with $\varnothing^{\vec{\mathbb{O}}^{\square}}$, of pushouts of morphisms in $\vec{\mathbb{O}}^{\square} \circ (L_0 I \cup L_1 I)_c$ for $c \in \mathfrak{C}$. Since $\varnothing^{\vec{\mathbb{O}}^{\square}}$ is underlying cofibrant in $\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and since the class of cofibrations in a model category, such as $(\overrightarrow{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$, is closed under transfinite compositions [ibid., 10.3.4], the following lemma will finish the proof. □

The proof of Lemma 6.2.3 below uses the next definition, from [White and Yau 2018a, 4.3.5]:

Definition 6.2.2 ($\mathbb{O}_A$ for $\mathbb{O}$–algebras) For a $\mathfrak{C}$–colored operad $\mathbb{O}$ in M and $A \in \mathsf{Alg}(\mathbb{O}; \mathsf{M})$, define the $\mathfrak{C}$–colored symmetric sequence $\mathbb{O}_A$ as follows. For $d \in \mathfrak{C}$ and orbit $[\underline{c}] \in \Sigma_{\mathfrak{C}}$, define the component

$$\mathbb{O}_A\binom{d}{[\underline{c}]} \in \mathsf{M}^{\Sigma_{[\underline{c}]}^{\mathrm{op}} \times \{d\}}$$

as the reflexive coequalizer of the diagram, in $\mathsf{M}^{\Sigma_{[\underline{c}]}^{\mathrm{op}} \times \{d\}}$,

$$\begin{tikzcd} \displaystyle\coprod_{[\underline{a}] \in \Sigma_{\mathfrak{C}}} \mathbb{O}\binom{d}{[\underline{a}], [\underline{c}]} \otimes_{\Sigma_{[\underline{a}]}} (\mathbb{O} \circ A)_{[\underline{a}]} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \displaystyle\coprod_{[\underline{a}] \in \Sigma_{\mathfrak{C}}} \mathbb{O}\binom{d}{[\underline{a}], [\underline{c}]} \otimes_{\Sigma_{[\underline{a}]}} A_{[\underline{a}]}. \arrow[l, bend right, "s"'] \end{tikzcd}$$

The three arrows in this diagram are as follows:

- d_0 is induced by the composition of $\mathbb{O}$.
- d_1 is induced by the $\mathbb{O}$–algebra structure on A.
- The common section s is induced by the unit $A \to \mathbb{O} \circ A$.

Lemma 6.2.3 *Under the hypotheses of Theorem 6.2.1, suppose $\alpha\colon f \to g$ is a morphism in $(L_0 I \cup L_1 I)_c$ for some color $c \in \mathfrak{C}$, and*

$$\begin{tikzcd} \vec{\mathrm{O}}^{\square} \circ f \arrow[r] \arrow[d, "\vec{\mathrm{O}}^{\square} \circ \alpha"'] & B_0 \arrow[d, "j"] \\ \vec{\mathrm{O}}^{\square} \circ g \arrow[r] & B_\infty \end{tikzcd}$$

is a pushout in $\mathsf{Alg}(\vec{\mathrm{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ with B_0 cofibrant and $UB_0 \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ cofibrant. Then Uj is a cofibration in $(\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$. In particular, B_∞ is also cofibrant and $UB_\infty \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is cofibrant.

Proof By the filtration in [White and Yau 2018a, 4.3.16] and the fact that cofibrations are closed under pushouts, to show that $Uj \in (\vec{\mathsf{M}}^{\square}_{\mathrm{proj}})^{\mathfrak{C}}$ is a cofibration, it is enough to show that, for each $n \geq 1$ and each color $d \in \mathfrak{C}$, the morphism

(6.2.4) $$\vec{\mathrm{O}}^{\square}_{B_0}\binom{d}{nc} \square_{\Sigma_n} \alpha^{\square 2n}$$

in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ is a cofibration, where $nc = (c, \ldots, c)$ is the $\mathfrak{C}$–profile with n copies of the color c. The object $\vec{\mathrm{O}}^{\square}_{B_0}$ is as in Definition 6.2.2 for $\vec{\mathrm{O}}^{\square}$ and B_0, and $\alpha^{\square 2n}$ is the n–fold pushout product of α. Recall that $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ satisfies $(\clubsuit)_{\mathrm{cof}}$ by Theorem 6.1.5 and that $\vec{\mathrm{O}}^{\square}$ is entrywise cofibrant in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ because O is entrywise cofibrant in M. The cofibrancy of $B_0 \in \mathsf{Alg}(\vec{\mathrm{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ and [ibid., 6.2.4] applied to $\vec{\mathrm{O}}^{\square}$ now imply that $\vec{\mathrm{O}}^{\square}_{B_0}$ is entrywise cofibrant in $\vec{\mathsf{M}}^{\square}$. By the condition $(\clubsuit)_{\mathrm{cof}}$ in $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ once again, we can conclude that the morphism (6.2.4) is a cofibration because α is a cofibration in $\vec{\mathsf{M}}^{\square}$. □

Corollary 6.2.5 *Suppose M is the stable module category of $k[G]$–modules for some field k whose characteristic divides the order of G. Then, for each $\mathfrak{C}$–colored operad O in M, there is a Quillen equivalence*

$$\mathsf{Alg}(\vec{\mathrm{O}}^{\square}; \vec{\mathsf{M}}^{\square}) \underset{\mathrm{ker}}{\overset{\mathrm{coker}}{\rightleftarrows}} \mathsf{Alg}(\vec{\mathrm{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes}).$$

Proof The stable module category is a stable model category that satisfies the hypotheses of Theorem 6.2.1 in which every object is cofibrant [Hovey 1999, 2.2.12; White and Yau 2020, Section 12]. □

There are several more examples where Theorem 4.4.1 likely applies to all entrywise cofibrant operads, but where ($\heartsuit$) has not been checked. For example, the positive flat stable model structure on symmetric spectra built on compactly generated spaces have the property that, for any entrywise cofibrant colored operad O, cofibrant O–algebras forget to cofibrant spectra [Pavlov and Scholbach 2018, Section 2], but the authors do not know a reference proving the same for $\vec{\mathsf{M}}^{\square}$.

Conjecture 6.2.6 The positive flat stable model structure on symmetric spectra built on compactly generated spaces satisfies the conclusion of Theorem 6.2.1.

Similarly, by analogy with the positive flat model structure on symmetric spectra, one would expect the positive flat model structure on G–equivariant orthogonal spectra to satisfy this property.

Conjecture 6.2.7 If $\mathsf{M} = G\mathsf{Sp}_O$ is the positive flat stable model structure on G–equivariant orthogonal spectra, then it satisfies the property that, if $\mathbb{O}$ is an entrywise cofibrant $\mathfrak{C}$–colored operad and A is a cofibrant $\mathbb{O}$–algebra, then UA is cofibrant in $\mathsf{M}^{\mathfrak{C}}$. Furthermore, M satisfies the conclusion of Theorem 6.2.1 for any compact Lie group G.

Recent work of Hill, Hopkins and Ravenel has illustrated that the positive (flat) model structure on $G\mathsf{Sp}_O$ is not quite right. One also needs an equifibrancy condition, also known as completeness. There is a positive complete model structure on $G\mathsf{Sp}_O$, and it satisfies the commutative monoid axiom [Gutiérrez and White 2018, Section 5]. However, the authors do not know if a positive, complete, flat variant has been worked out.

Problem 6.2.8 Let G be a compact Lie group.

(1) Work out a positive complete flat stable model structure on $G\mathsf{Sp}_O$.

(2) Prove that it satisfies the condition that all colored operads are admissible.

(3) Prove that cofibrant operad algebras forget to cofibrant underlying objects.

(4) Prove that this model structure satisfies the conclusion of Theorem 6.2.1.

In a related vein, we have the following problem:

Problem 6.2.9 Let M_s (resp. M_s^+) denote Schwede's global positive (flat) model structure [2018] and let M_h (resp. M_h^+) denote Hausmann's positive (flat) model structure for G–symmetric spectra [2017].

(1) Prove that all colored operads are admissible in M_s, M_s^+, M_h and M_h^+.

(2) Prove that, if $\mathbb{O}$ is entrywise cofibrant, then cofibrant $\mathbb{O}$–algebras forget to underlying cofibrant objects in M_s^+ and M_h^+ in each color.

(3) Prove that M_s^+ and M_h^+ satisfy the conclusion of Theorem 6.2.1.

Lastly, injective model structures on various categories of spectra have the property that all objects are cofibrant, so the condition about the forgetful functor preserving cofibrancy is trivial. However, not all operads are admissible. A likely remedy is to develop *positive injective* model structures (by requiring cofibrations to be isomorphisms in level zero), which would automatically be Quillen equivalent to existing stable model structures on spectra, but the authors do not know a reference where this is done.

Problem 6.2.10 Let M denote the category of symmetric spectra.

(1) Prove that the positive injective stable model structure M_i^+ is a monoidal model category.

(2) Prove that all operads are admissible in M_i^+. If so, then automatically cofibrant $\mathbb{O}$–algebras forget to cofibrant underlying objects.

(3) Prove that M_i^+ satisfies the conclusion of Theorem 6.2.1.

(4) Do the same for symmetric spectra valued in a general base model category $\mathscr{C}$, where stabilization is with respect to an endofunctor G.

(5) Do the same for orthogonal spectra and equivariant orthogonal spectra, possibly restricting to Δ–generated spaces, as is done in [White 2022, Section 5.8].

(6) Produce a model structure on the category of S–modules, Quillen equivalent to the one in [Elmendorf et al. 1997], with the property that cofibrant commutative ring spectra are underlying cofibrant. Do the same for general entrywise cofibrant colored operads, and prove that the conclusion of Theorem 6.2.1 holds in this setting.

7 Semi-model categories and ∞–categories for operad algebras

In this paper, we often transferred model structures, using (♤), or semi-model structures, using Definition 6.1.1 or using $\Sigma_{\mathfrak{C}}$–cofibrant operads $\mathbb{O}$, to categories of $\mathbb{O}$–algebras. The language of ∞–categories could also be used to study the homotopy theory of $\mathbb{O}$–algebras. We work in the model of quasicategories, ie everywhere we write ∞–category we mean quasicategory. The main results of this section, Theorems 7.3.1 and 7.3.3, show that the two approaches — namely, semi-model categories and ∞–categories — are equivalent in a suitable sense for $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operads that are *not* necessarily admissible.

7.1 Preliminaries on ∞–operads

As detailed in [Lurie 2017, 4.5.4.7 and 4.5.4.12], the crucial property needed to compare a model structure on $\mathbb{O}$–algebras with the corresponding ∞–category structure is that the forgetful functor

$$U : \mathsf{Alg}(\mathbb{O}; \mathsf{M}) \to \mathsf{M}^{\mathfrak{C}}$$

preserves and reflects homotopy sifted colimits, that is, $N(\mathscr{C})$–indexed homotopy colimits, where $\mathscr{C}$ is a small category such that the nerve $N(\mathscr{C})$ is sifted [Lurie 2009, 5.5.8.1].

Lurie [2017, 4.5.4.12] proves this property for the Com–operad and a restrictive class of model categories M, namely combinatorial and freely powered (4.5.4.2) monoidal model categories. Lurie then deduces [ibid., 4.5.4.7] that the underlying ∞–category $N(\mathsf{CAlg}(\mathsf{M})^c)[W_{\mathsf{Com}}^{-1}]$ of the model category $\mathsf{CAlg}(\mathsf{M})$ — where $(-)^c$ refers to taking cofibrant objects and W_{Com} is the class of weak equivalences of Com–algebras — is equivalent as an ∞–category to $\mathsf{CAlg}(N(\mathsf{M}^c)[W^{-1}])$, obtained as the ∞–category of commutative monoids valued in the symmetric monoidal ∞–category $N(\mathsf{M}^c)[W^{-1}]$ associated to M. Here $N(\mathsf{M}^c)$ denotes the homotopy coherent nerve of the simplicial category M^c, and the notation $(-)[W^{-1}]$ refers to the ∞–categorical meaning of inverting the class W [ibid., 1.3.4.1]. To be precise, the ∞–category $N(\mathsf{M}^c)[W^{-1}]$ can be constructed via a fibrant replacement of the pair (M^c, W) in the category of marked simplicial sets [loc. cit.].

Following the model of Lurie's proof, it is possible to prove that, whenever M is a simplicial monoidal model category and $\mathbb{O}$ is an *admissible* $\Sigma_{\mathfrak{C}}$–cofibrant simplicial colored operad (Theorems 4.1.1 and 5.2.1), then the forgetful functor preserves and reflects homotopy sifted colimits, and the ∞–category obtained from the model category of $\mathbb{O}$–algebras is equivalent as an ∞–category to the ∞–category obtained from

$N^{\otimes}\mathbb{O}$–algebras in the ∞–category associated to M [Pavlov and Scholbach 2018, 7.9 and 7.11]. Here $N^{\otimes}\mathbb{O}$ is the operadic nerve of $\mathbb{O}$ [Lurie 2017, 2.1.1.23], ie Lurie's model for the ∞–operad associated to the simplicial colored operad $\mathbb{O}$. Consequently, for *admissible* $\Sigma_{\mathfrak{C}}$–cofibrant colored simplicial operads, the homotopy theory obtained via the model category route matches the homotopy theory obtained via the ∞–category route.

We extend this result in two ways. First, we will show that it holds when $\mathbb{O}$ is only *semiadmissible* instead of admissible (ie $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ has a transferred semi-model structure). Second, we will show the same thing for the setting of enriched ∞–operads. For the latter, we work in a monoidal model category M (not necessarily simplicial) and consider a colored operad $\mathbb{O}$ valued in M. Note that, if M is a $\mathscr{V}$–model category for some monoidal model category $\mathscr{V}$ and $\mathbb{O}$ is a colored operad valued in $\mathscr{V}$, then there is a colored operad $\mathbb{O}'$ valued in M with the same algebras (obtained by tensoring the levels of $\mathbb{O}$ with the unit of M), so we focus on the case when $\mathbb{O}$ is valued in M. In this case, there is an associated *enriched ∞–operad* [Chu and Haugseng 2020] as we now describe. First, we must restate [Haugseng 2019, 4.1]:

Definition 7.1.1 Let M be a monoidal model category. A *subcategory of flat objects* is a full symmetric monoidal subcategory $\mathsf{M}^{\flat}$ (which implies the unit is flat) that satisfies the following two conditions:

(1) All cofibrant objects are flat (that is, are in $\mathsf{M}^{\flat}$).

(2) If X is flat and f is a weak equivalence in $\mathsf{M}^{\flat}$, then $X \otimes f$ is a weak equivalence.

If the unit of M is cofibrant, then the subcategory of cofibrant objects is a subcategory of flat objects [Haugseng 2019, 4.2], by Ken Brown's lemma. We note that, if the unit of M is cofibrant, then the same is true for both $\vec{\mathsf{M}}^{\square}_{\mathrm{proj}}$ and $\vec{\mathsf{M}}^{\otimes}$. The purpose of the definition above is to avoid assuming the monoidal unit is cofibrant, as this would rule out positive (flat) model structures on spectra (which do admit a subcategory of flat objects, namely the cofibrant objects of the flat model structure, by [ibid., 4.11]). White [2017; 2022] gives many examples of model categories with a subcategory of flat objects (namely, the subcategory of cofibrant objects), including spaces, simplicial sets, chain complexes, diagram categories, simplicial presheaves and various categories of spectra.

With Definition 7.1.1 in hand, we are ready to describe the enriched ∞–operad associated to a colored operad $\mathbb{O}$ valued in M, following [Haugseng 2019, Section 4]. First, the inclusions $\mathsf{M}^{c} \hookrightarrow \mathsf{M}^{\flat} \hookrightarrow \mathsf{M}$ induce equivalences of localizations when all three are localized with respect to their subcategories of weak equivalences. Next, the symmetric monoidal localization $\mathsf{M}^{\flat} \to \mathsf{M}^{\flat}[W^{-1}] \simeq \mathsf{M}[W^{-1}]$ of [Lurie 2017, 4.1.7.4] gives a functor from ∞–operads enriched in $\mathsf{M}^{\flat}$ to ∞–operads enriched in $\mathsf{M}[W^{-1}]$. But, because $\mathsf{M}^{\flat}$ is a 1–category, the former are simply strict colored operads in $\mathsf{M}^{\flat}$. The following is a combination of [Chu and Haugseng 2020, 1.1.3; Haugseng 2019, 4.4]:

Proposition 7.1.2 *Let M be a symmetric monoidal model category and $\mathsf{M}^{\flat}$ a subcategory of flat objects. Then the ∞–category of ∞–operads enriched in $\mathsf{M}[W^{-1}]$ is equivalent to the ∞–category of enriched colored operads in $\mathsf{M}^{\flat}$, with the Dwyer–Kan equivalences inverted.*

With these preliminary results and definitions in hand, we are ready to prove the main results of the section.

7.2 Homotopy sifted colimits

Following the model of [Lurie 2017, 4.5.4.7 and 4.5.4.12], we must first prove that the forgetful functor

$$U : \mathsf{Alg}(\mathbb{O}; \mathsf{M}) \to \mathsf{M}^{\mathfrak{C}}$$

preserves and reflects homotopy sifted colimits, even when $\mathsf{Alg}(\mathbb{O}; \mathsf{M})$ is only a semi-model category. It suffices to prove this in the case where $\mathbb{O}$ is a colored operad in M, as the case where $\mathbb{O}$ is a simplicial colored operad and M is a simplicial model category follows from our discussion above regarding $\mathscr{V}$–model categories.

It is known that, for every cofibrantly generated monoidal model category M, every $\Sigma_{\mathfrak{C}}$–cofibrant colored operad $\mathbb{O}$ in M is semiadmissible. In other words, there is a transferred semi-model structure on $\mathbb{O}$–algebras [White and Yau 2018a, 6.3.1]. An alternative approach assumes M satisfies (♣) and appeals to [ibid., 6.2.3] for such a semi-model structure. It is also known that there are $\Sigma_{\mathfrak{C}}$–cofibrant colored operads $\mathbb{O}$ whose category of $\mathbb{O}$–algebras do *not* admit a full model structure [Batanin and White 2021, 2.9]. Hence, the results in this section really do apply to previously unknown examples, and complete the study of semi-model structures on operad algebras set out in [White and Yau 2018a; 2019b; 2020; 2023]. For completeness, we handle the case of both symmetric and nonsymmetric colored operads [Muro 2011], noting that, for the nonsymmetric case, being $\Sigma_{\mathfrak{C}}$–cofibrant is the same as being entrywise cofibrant.

Proposition 7.2.1 *Suppose M is a cofibrantly generated monoidal model category and $\mathbb{O}$ is a $\Sigma_{\mathfrak{C}}$–cofibrant (symmetric) $\mathfrak{C}$–colored operad valued in M. Then the forgetful functor $U : \mathsf{Alg}(\mathbb{O}; \mathsf{M}) \to \mathsf{M}^{\mathfrak{C}}$ preserves and reflects homotopy sifted colimits.*

Proof We follow the proof from [Pavlov and Scholbach 2018, 7.9], which is itself based on the proof of [Lurie 2017, 4.5.4.12]. First, as pointed out in [Lurie 2017], the reflection property is implied by the preservation property, and it is sufficient to prove that U preserves homotopy colimits indexed by a small category $\mathfrak{D}$ such that the nerve $N(\mathfrak{D})$ is homotopy sifted.

Consider the projective model structure $(\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}$, the projective semi-model structure $\mathsf{Alg}(\mathbb{O}; \mathsf{M})^{\mathfrak{D}}$ guaranteed by [Barwick 2010, 3.4] and the forgetful functor

$$U^{\mathfrak{D}} : \mathsf{Alg}(\mathbb{O}; \mathsf{M})^{\mathfrak{D}} \to (\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}.$$

Let

$$F : (\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}} \to \mathsf{M}^{\mathfrak{C}} \quad \text{and} \quad F_{\mathsf{Alg}(\mathbb{O})} : \mathsf{Alg}(\mathbb{O}; \mathsf{M})^{\mathfrak{D}} \to \mathsf{Alg}(\mathbb{O}; \mathsf{M})$$

denote the colimit functors with respect to $\mathfrak{D}$. The proof in [Lurie 2017, 4.5.4.12] reduces us to proving that the canonical isomorphism of functors

$$\alpha : F \circ U^{\mathfrak{D}} \cong U \circ F_{\mathsf{Alg}(\mathbb{O})} : \mathsf{Alg}(\mathbb{O}; \mathsf{M})^{\mathfrak{D}} \to \mathsf{M}^{\mathfrak{C}}$$

persists after everything is derived.

Let LF and $LF_{\mathsf{Alg}(\mathbb{O})}$ denote the left derived functors of F and $F_{\mathsf{Alg}(\mathbb{O})}$, obtained via cofibrant replacement in $(\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}$ and $\mathsf{Alg}(\mathbb{O};\mathsf{M})^{\mathfrak{D}}$, respectively. Since U and $U^{\mathfrak{D}}$ preserve weak equivalences, as in [loc. cit.], we are reduced to proving that the induced natural transformation $\bar{\alpha}\colon LF\circ U^{\mathfrak{D}}\to U\circ LF_{\mathsf{Alg}(\mathbb{O})}$ is an isomorphism in the homotopy category. This means that, for every cofibrant A in $\mathsf{Alg}(\mathbb{O};\mathsf{M})^{\mathfrak{D}}$, we must show that

$$\bar{\alpha}\colon LF(U^{\mathfrak{D}}A)\to U(LF_{\mathsf{Alg}(\mathbb{O})}(A))$$

is a weak equivalence.

The right-hand side is canonically weakly equivalent to $U(F_{\mathsf{Alg}(\mathbb{O})}(A))$ because A is projectively cofibrant, and this is weakly equivalent to $F(U^{\mathfrak{D}}A)$ via α. At this point, the proof in [loc. cit.] requires a detailed analysis of so-called "good" objects and morphisms in $(\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}$. However, when $\mathbb{O}$ is $\Sigma_{\mathfrak{C}}$–cofibrant, the situation is much simpler, because U takes cofibrant algebras to cofibrant objects of $\mathsf{M}^{\mathfrak{C}}$ [White and Yau 2018a, 6.3.1] (and [Muro 2011, 9.5] for the nonsymmetric case).

Furthermore, the $\mathfrak{D}$–constant operad $\mathbb{O}^{\mathfrak{D}}$, taking value $\mathbb{O}$ at every $a\in\mathfrak{D}$, is $\Sigma_{\mathfrak{C}}$–cofibrant in $\mathsf{Alg}(\mathbb{O};\mathsf{M})^{\mathfrak{D}}$. This can be seen directly, as $\Sigma_{\mathfrak{C}}$–cofibrancy for an operad P valued in $\mathsf{M}^{\mathfrak{D}}$ is the condition that, for each $a\in\mathfrak{D}$ and each $(\underline{c};d)\in\Sigma_{\mathfrak{C}}^{\mathrm{op}}\times\mathfrak{C}$, the object $P_a\binom{d}{\underline{c}}$ $\bigl(=\mathbb{O}\binom{d}{\underline{c}}$ in our case$\bigr)$ is projectively cofibrant in $\mathsf{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}}\times\mathfrak{C}}$. Hence, by [White and Yau 2018a, 6.3.1] (and [Muro 2011, 9.5] for the nonsymmetric case), the functor $U^{\mathfrak{D}}$ also preserves cofibrancy, since the projective semi-model structure transferred from the semi-model structure on $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ is the same as the transferred semi-model structure on $\mathbb{O}^{\mathfrak{D}}$–algebras in the projective model structure $(\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}$. Hence, $U^{\mathfrak{D}}A$ is cofibrant in $(\mathsf{M}^{\mathfrak{C}})^{\mathfrak{D}}$, and so $LF(U^{\mathfrak{D}}A)\simeq F(U^{\mathfrak{D}}A)$, as required. □

Remark 7.2.2 Following the model of [Lurie 2017] (or [Pavlov and Scholbach 2018]), after establishing Proposition 7.2.1, the next step should be to prove that the semi-model category $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ describes the ∞–category of $N^{\otimes}\mathbb{O}$–algebras in the ∞–category associated to M, as discussed above. However, when $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ is only a semi-model structure, an additional step is needed. We need to know that homotopy colimits (given by colimits of projectively cofibrant objects in $\mathsf{Alg}(\mathbb{O};\mathsf{M})^{\mathfrak{D}}$) agree with ∞–categorical colimits. In the case of full model structures, one knows that the projective model structure on $\mathsf{Alg}(\mathbb{O};\mathsf{M})^{\mathfrak{D}}$ describes the ∞–category of functors, and that a Quillen adjunction gives rise to an adjunction of ∞–categories. For the case of semi-model categories, we invoke [Monaco 2021, A.10] for the latter.

Remark 7.2.3 We conjecture that Proposition 7.2.1 remains true for entrywise cofibrant colored operads $\mathbb{O}$ if M satisfies (♧) and we replace appeals to [White and Yau 2018a, 6.3.1] above by appeals to [ibid., 6.2.3]. However, the proof of this would require a detailed analysis of "good" objects and would take us too far afield.

7.3 Semi-model categories and ∞–categories of operad algebras

With the previous proposition in hand, we are ready for the main result of this section. The slogan for this result is that, for any $\Sigma_{\mathfrak{C}}$–free (symmetric) colored operad $\mathbb{O}$ and any reasonable monoidal model category M, the semi-model category of $\mathbb{O}$–algebras in M describes the corresponding ∞–category of $\mathbb{O}$–algebras in the symmetric monoidal ∞–category described by M. This is true of both

(1) the unenriched case, where M is a simplicial monoidal model category, $\mathbb{O}$ is a simplicial colored operad and the ∞–operad associated to $\mathbb{O}$ is the operadic nerve $N^{\otimes}\mathbb{O}$ of $\mathbb{O}$ [Lurie 2017, 2.1.1.23];

(2) the enriched case, where M is a monoidal model category, $\mathbb{O}$ is a colored operad valued in M and we use the theory of enriched ∞–operads to define the ∞–category of $\mathbb{O}$–algebras (as recalled in Section 7.1 and spelled out in [Chu and Haugseng 2020; Haugseng 2019]).

For both cases, we handle the cases where $\mathbb{O}$ is a symmetric colored operad and where $\mathbb{O}$ is a nonsymmetric colored operad simultaneously. We handle the enriched case first.

Theorem 7.3.1 *Suppose* M *is a cofibrantly generated monoidal model category that admits a subcategory of flat objects* $\mathsf{M}^{\flat}$ *and* $\mathbb{O}$ *is a* $\Sigma_{\mathfrak{C}}$*–cofibrant (symmetric)* $\mathfrak{C}$*–colored operad valued in* $\mathsf{M}^{\flat}$*.*

- *Let* $\mathsf{Alg}(\mathbb{O};\mathsf{M})^c[W_{\mathbb{O}}^{-1}]$ *be the* ∞*–category obtained from the semi-model category* $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ *by first passing to the subcategory of cofibrant objects, and then inverting the weak equivalences between* $\mathbb{O}$*–algebras.*
- *Let* $\mathsf{Alg}(\mathbb{O};\mathsf{M}[W^{-1}])$ *be the* ∞*–category obtained by first passing from* M *to the (symmetric) monoidal category* $\mathsf{M}[W^{-1}]$ *and then passing to* $\mathbb{O}$*–algebras, where* $\mathbb{O}$ *is viewed as a colored operad in* $\mathsf{M}[W^{-1}] \simeq \mathsf{M}^{\flat}[W^{-1}]$*.*

Then the natural comparison functor

$$\mathsf{Alg}(\mathbb{O};\mathsf{M})^c[W_{\mathbb{O}}^{-1}] \to \mathsf{Alg}(\mathbb{O};\mathsf{M}[W^{-1}])$$

is an equivalence of ∞*–categories.*

Proof The proof of [Haugseng 2019, 4.10] goes through directly by replacing the appeal to [Pavlov and Scholbach 2018, 7.8] with an appeal to Proposition 7.2.1. That is, we consider the forgetful functors from both categories to the ∞–category associated to $\mathsf{M}^{\mathfrak{C}}$, and appeal to the Barr–Beck theorem for ∞–categories [Lurie 2017, 4.7.3.16] to see that these forgetful functors are monadic right adjoints (this is where Proposition 7.2.1 is needed). We appeal to [Haugseng 2019, 3.8], which occurs entirely on the ∞–category level, for the usual formula for free $\mathbb{O}$–algebras and the observation that the two associated monads on $\mathsf{M}^{\mathfrak{C}}$ have equivalent underlying endofunctors. This proof works for both symmetric and nonsymmetric colored operads $\mathbb{O}$, as both are known to inherit transferred semi-model structures from $\mathsf{M}^{\mathfrak{C}}$, and as Proposition 7.2.1 applies in both settings. $\square$

Remark 7.3.2 The proof of [ibid., 4.10] relies on the observation that a Quillen adjunction $F:\mathsf{M} \rightleftarrows \mathsf{N}:G$ induces an adjunction between the underlying ∞–categories. We appeal to [Monaco 2021, A.10] for the semi-model category analogue of this fact.

We turn now to the unenriched case.

Theorem 7.3.3 *Suppose* M *is a cofibrantly generated simplicial monoidal model category and* $\mathbb{O}$ *is a* $\Sigma_{\mathfrak{C}}$*–cofibrant (symmetric) simplicial* $\mathfrak{C}$*–colored operad.*

- *Let* $N(\mathsf{Alg}(\mathbb{O};\mathsf{M})^c)[W^{-1}_{\mathsf{Alg}(\mathbb{O})}]$ *be the* ∞*–category obtained from the semi-model category* $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ *by first passing to the subcategory of cofibrant objects, then taking the nerve and then inverting the weak equivalences.*
- *Let* $\mathsf{Alg}(N^{\otimes}\mathbb{O}; N(\mathsf{M}^c)[W^{-1}])$ *be the* ∞*–category of* $N^{\otimes}\mathbb{O}$*–algebras valued in the* ∞*–category* $N(\mathsf{M}^c)[W^{-1}]$ *associated to* M*.*

Then the natural comparison functor

$$N(\mathsf{Alg}(\mathbb{O};\mathsf{M})^c)[W^{-1}_{\mathsf{Alg}(\mathbb{O})}] \to \mathsf{Alg}(N^{\otimes}\mathbb{O}; N(\mathsf{M}^c)[W^{-1}])$$

is an equivalence of ∞*–categories.*

Proof We deliberately phrased the proof of Theorem 7.3.1 so that word for word it proves this result as well (again with the critical step hinging on an appeal to Proposition 7.2.1). We only stated the two theorems separately to highlight the difference between enriched and unenriched ∞–operads, and the connection to where the colored operad $\mathbb{O}$ is valued. □

Remark 7.3.4 One can show that Theorems 7.3.1 and 7.3.3 are false in general in the symmetric case if the $\Sigma_{\mathfrak{C}}$–cofibrancy of $\mathbb{O}$ is dropped. Well-known counterexamples include the operad Com and $\mathsf{M} = \mathsf{Ch}(\mathbb{F}_p)$. However, every $\mathfrak{C}$–colored operad $\mathbb{O}$ admits a $\Sigma_{\mathfrak{C}}$–cofibrant replacement $Q\mathbb{O}$. If $\mathbb{O}$ is semiadmissible and admits rectification with $Q\mathbb{O}$ (meaning there is a Quillen equivalence of semi-model categories between $\mathsf{Alg}(\mathbb{O};\mathsf{M})$ and $\mathsf{Alg}(Q\mathbb{O};\mathsf{M})$), then Theorems 7.3.1 and 7.3.3 do apply to $\mathbb{O}$, since the weak equivalence $Q\mathbb{O} \to \mathbb{O}$ induces an equivalence $N^{\otimes}Q\mathbb{O} \to N^{\otimes}\mathbb{O}$, and hence we can use the two-out-of-three property to deduce the statement for $\mathbb{O}$ from the statement for $Q\mathbb{O}$. Conditions on M under which rectification hold are provided in [White 2017] (for Com rectifying to E_∞) and [White and Yau 2019b] (for general colored operads), among other places.

Remark 7.3.5 Theorem 7.3.1 answers positively the question raised in [Haugseng 2019, 4.13] about extending [ibid., 4.10] to Σ–cofibrant operads and semi-model category structure on $\mathsf{Alg}(\mathbb{O};\mathsf{M})$. As pointed out by Haugseng, the assumptions on M and $\mathbb{O}$ are much weaker than those required to get a full model structure on $\mathbb{O}$–algebras. In particular, Theorem 7.3.1 applies not only to the examples listed by Haugseng — namely, spaces, simplicial sets, chain complexes and symmetric spectra — but also to equivariant spaces, equivariant orthogonal spectra, motivic symmetric spectra, the stable module category, chain complexes over a field of nonzero characteristic, simplicial presheaves, the projective model structure on small functors [Chorny and White 2018], the folk model structure on the category of small categories (or groupoids), various abelian model structures arising from the theory of cotorsion pairs, and left Bousfield localizations of these categories.

These examples are detailed in [White 2017; 2022; White and Yau 2018a; 2020]. In several of these examples (eg chain complexes over a field of nonzero characteristic, examples arising from cotorsion pairs, and algebras over left Bousfield localizations $L_{\mathscr{C}}\mathsf{M}$), categories of algebras are known to have transferred semi-model structures but are not known to have transferred model structures. For chain complexes over $\mathbb{F}_2$, there is even an explicit example of a category of $\mathbb{O}$–algebras that has a transferred semi-model structure that is not a model structure [Batanin and White 2021, 2.9]. For algebras over a left Bousfield localization $L_{\mathscr{C}}\mathsf{M}$, many examples are discussed in [White and Batanin 2015; Batanin and White 2022; 2024; White 2021].

In most of the examples listed above, the unit is cofibrant and cofibrant objects are flat, so the category of cofibrant objects is our $\mathsf{M}^\flat$ (note that left Bousfield localization does not change the class of cofibrant objects). For the positive (flat) model structure on equivariant orthogonal spectra (resp. motivic symmetric spectra), one can use the cofibrant objects of the flat model structure, just as Haugseng [2019] does for symmetric spectra, as discussed in [Hovey and White 2020] (resp. [Pavlov and Scholbach 2018], building on work of Hornbostel).

We conclude with a specialization of Theorem 7.3.1 to the main examples of interest in the present paper.

Lemma 7.3.6 *Suppose* M *is a monoidal model category that admits a subcategory of flat objects*, $\mathsf{M}^\flat$. *Then* $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ *also admits a subcategory of flat objects.*

Proof In $\vec{\mathsf{M}}^{\otimes}$, we take the full subcategory consisting of arrows $f: X_1 \to X_2$, where X_1 and X_2 are in $\mathsf{M}^\flat$. This is a symmetric monoidal subcategory of $\vec{\mathsf{M}}^{\otimes}$, as the monoidal unit $\mathrm{Id}: \mathbb{1} \to \mathbb{1}$ is flat and the tensor product of two flat arrows is flat. Condition (1) of Definition 7.1.1 holds because cofibrations are entrywise, and (2) holds because the tensor product and weak equivalences are entrywise. $\square$

Corollary 7.3.7 *Suppose* M *is a cofibrantly generated monoidal model category that admits a subcategory of flat objects* $\mathsf{M}^\flat$. *Suppose* $\mathbb{O}$ *is a* $\Sigma_{\mathfrak{C}}$*–cofibrant* $\mathfrak{C}$*–colored operad valued in* $\mathsf{M}^\flat$. *Then the transferred semi-model structures of Corollary 5.2.3 on* $\mathsf{Alg}(\vec{\mathbb{O}}^{\otimes}; \vec{\mathsf{M}}^{\otimes})$ *and* $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$ *describe the corresponding* ∞*–categories*, *in the sense of Theorem 7.3.1. If*, *in addition*, M *is stable*, *then the Quillen equivalence of Corollary 5.2.3 yields an equivalence of* ∞*–categories.*

Proof This follows from Theorem 7.3.1, applied to

- $\vec{\mathsf{M}}^{\otimes}_{\mathrm{inj}}$ and the colored operad $\vec{\mathbb{O}}^{\otimes}$, appealing to Lemma 7.3.6 for the subcategory of flat objects and to Proposition 5.2.2 for the $\Sigma_{\mathfrak{C}}$–cofibrancy; and
- M and the colored operad $\mathbb{O}^s$, with the assumed subcategory of flat objects on M — as Proposition 3.3.19 shows, $\mathbb{O}^s$ is $\Sigma_{\mathfrak{C}\sqcup\mathfrak{C}}$–cofibrant, and the transferred semi-model structure on $\mathbb{O}^s$–algebras coincides with the transferred semi-model structure on $\mathsf{Alg}(\vec{\mathbb{O}}^{\square}; \vec{\mathsf{M}}^{\square})$.

The statement about Quillen equivalences follows from [Monaco 2021, A.11]. $\square$

We note that, in the examples mentioned after Definition 7.1.1, we could take $\mathsf{M}^\flat$ to be the subcategory of cofibrant objects of M. In these examples, every $\Sigma_{\mathfrak{C}}$–cofibrant $\mathfrak{C}$–colored operad is already entrywise cofibrant. Hence, it is no loss of generality to assume $\mathbb{O}$ is valued in $\mathsf{M}^\flat$ instead of in M for these examples.

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The persistent topology of optimal transport based metric thickenings

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A metric thickening of a given metric space X is any metric space admitting an isometric embedding of X. Thickenings have found use in applications of topology to data analysis, where one may approximate the shape of a dataset via the persistent homology of an increasing sequence of spaces. We introduce two new families of metric thickenings, the p–Vietoris–Rips and p–Čech metric thickenings for all $1 \le p \le \infty$, which include all probability measures on X whose p–diameter or p–radius is bounded from above, equipped with an optimal transport metric. The p–diameter (resp. p–radius) of a measure is a certain ℓ_p relaxation of the usual notion of diameter (resp. radius) of a subset of a metric space. These families recover the previously studied Vietoris–Rips and Čech metric thickenings when $p = \infty$. As our main contribution, we prove a stability theorem for the persistent homology of p–Vietoris–Rips and p–Čech metric thickenings, which is novel even in the case $p = \infty$. In the specific case $p = 2$, we prove a Hausmann-type theorem for thickenings of manifolds, and we derive the complete list of homotopy types of the 2–Vietoris–Rips thickenings of the n–sphere as the scale increases.

55N31; 51F99, 53C23

1 Introduction

Geometric simplicial complexes, such as Vietoris–Rips or Čech complexes, are one of the cornerstones of topological data analysis. One can approximate the shape of a dataset X by building a growing sequence of Vietoris–Rips complexes with X as the underlying set, and then computing persistent homology. The shape of the data, as measured by persistence, is reflective of important patterns within; see Carlsson [25].

The popularity of Vietoris–Rips complexes relies on at least three facts. First, Vietoris–Rips filtrations and their persistent homology signatures are computable; see Bauer [12]. Second, Vietoris–Rips persistent homology is stable (see Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [27] and Chazal, de Silva and Oudot [30]), meaning that the topological data analysis pipeline is robust to certain types of noise. Third, Vietoris–Rips complexes are topologically faithful at low scale parameters: one can use them to recover the homotopy types (see Latschev [50]) or homology groups (see Chazal and Oudot [28]) of an unknown underlying space, when given only a finite noisy sampling.

At higher scale parameters, we mostly do not know how Vietoris–Rips complexes behave. This is despite the fact that one of the key insights of persistent homology is to allow the scale parameter to vary from small to large, tracking the lifetimes of features as the scale increases. Our practice is ahead of our theory in this regard: data science practitioners are building Vietoris–Rips complexes with scale parameters larger than those for which the reconstruction results of [28; 50] apply.

Stability implies that, as more and more data points are sampled from some "true" underlying space M, the Vietoris–Rips persistent homology of the dataset X converges to the Vietoris–Rips persistent homology of M. The simplest possible case is when the dataset X is sampled from a manifold M, and so we cannot fully understand the Vietoris–Rips persistent homology of data without also understanding the Vietoris–Rips persistent homology of manifolds. However, not much is known about Vietoris–Rips complexes of manifolds, except at small scales; see Hausmann [42]. Even the Vietoris–Rips persistent homology of the n–sphere $\mathbb{S}^n$ is almost entirely unknown.

Two potential obstacles for understanding the homotopy types of Vietoris–Rips complexes of a manifold M are that

(i) the natural inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous, and

(ii) we do not yet have a full Morse theory for Vietoris–Rips complexes of manifolds.

There are by now several strategies for handling these two obstacles. One strategy is to remain in the setting of Vietoris–Rips simplicial complexes. Obstacle (i) is then unavoidable. Regarding obstacle (ii), Bestvina–Brady Morse theory has only been successfully applied at low scale parameters, allowing Zaremsky [75] to prove that $\mathrm{VR}(\mathbb{S}^n;r)$ recovers the homotopy type of $\mathbb{S}^n$ for r small enough, but not to derive new homotopy types that appear as r increases. Simplicial techniques have been considered for a long time, but even successes such as an understanding of the homotopy types of the Vietoris–Rips complexes of the circle at all scales (See Adamaszek and Adams [1]) are not accompanied by a broader Morse theory (though some techniques feel Morse-theoretic).

A second strategy is very recent. Lim, Mémoli and Okutan [52] and Okutan [62] show that the Vietoris–Rips simplicial complex filtration is equivalent to thickenings of the Kuratowski embedding into $L^\infty(M)$ or any other *injective* metric space. In particular, the two filtrations have the same persistent homology. This overcomes obstacle (i): the inclusion of a metric space into a thickening of its Kuratowski embedding is continuous, and indeed an isometry onto its image. This connection has created new opportunities, such as the Morse theoretic techniques employed by Katz [45; 46; 47; 48]. This Morse theory allows one to prove the first new homotopy types that occur for Vietoris–Rips simplicial complexes of the circle and 2–sphere, but have not yet inspired progress for larger scales, or for spheres above dimension two.

A third strategy is to consider Vietoris–Rips metric thickenings, which rely on optimal transport and Wasserstein distances; see Adamaszek, Adams and Frick [2]. We refer to these spaces as the ∞–metric thickenings, for reasons that will become clear in the following paragraph. Such thickenings were invented in order to enable Morse-theoretic proofs of the homotopy types of Vietoris–Rips type spaces. The first new homotopy type of the ∞–Vietoris–Rips metric thickening of the n–sphere is known [2], but only for a single (nonpersistent) scale parameter. It was previously only conjectured that the ∞–Vietoris–Rips metric thickenings have the same persistent homology as the more classical Vietoris–Rips simplicial complexes [2, Conjecture 6.12]; one of our contributions is to answer this conjecture in the affirmative. Mirth [61] considers a Morse theory in Wasserstein space, which is inspired in part by applications to ∞–Vietoris–Rips metric thickenings, but which does not apply as-is to these thickenings as the ∞–diameter functional is not "λ–convex"; see Santambrogio [69].

We introduce a generalization: the p–Vietoris–Rips metric thickening for any $1 \le p \le \infty$. Let X be an arbitrary metric space. For $1 \le p \le \infty$, the p–Vietoris–Rips metric thickening at scale parameter $r > 0$ contains all probability measures on X whose p–diameter is less than r. The p–Vietoris–Rips metric thickening will be equipped with the topology induced from the weak topology on $\mathcal{P}_X$. When X is bounded, the weak topology is generated by an optimal transport based metric; see Corollary A.2. For p finite, the p–diameter of a probability measure α on the metric space X is defined as

$$\operatorname{diam}_p(\alpha) := \left(\iint_{X\times X} d_X^p(x,x')\,\alpha(dx)\,\alpha(dx') \right)^{\frac{1}{p}},$$

and $\operatorname{diam}_\infty(\alpha)$ is defined to be the diameter of the support of α.

The p–Vietoris–Rips metric thickenings at scale r form a metric bifiltration of X that is covariant in r and contravariant in p. Indeed, we have an inclusion map $\mathrm{VR}_p(X;r) \hookrightarrow \mathrm{VR}_{p'}(X;r')$ for $r \le r'$ and $p \ge p'$; see Table 1:

$$\begin{tikzcd} \mathrm{VR}_{p'}(X;r) \arrow[r, hook] & \mathrm{VR}_{p'}(X;r') \\ \mathrm{VR}_p(X;r) \arrow[r, hook] \arrow[u, hook] & \mathrm{VR}_p(X;r') \arrow[u, hook] \end{tikzcd}$$

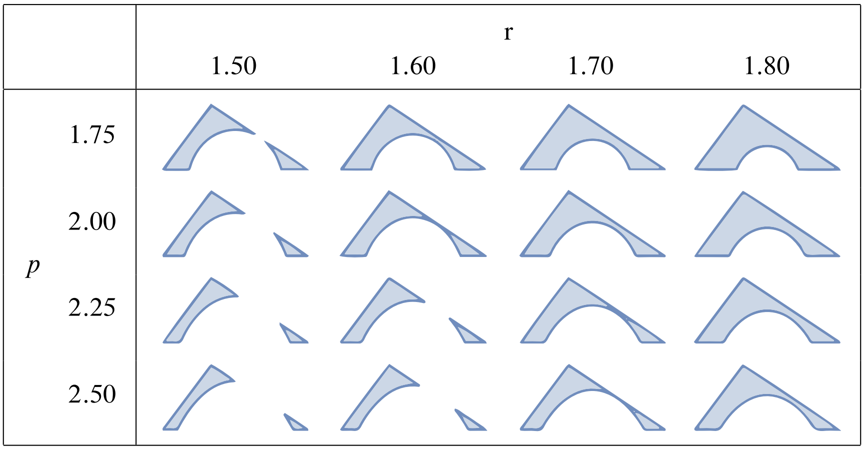

Table 1: The p–Vietoris–Rips bifiltration $\mathrm{VR}_p(X;r)$ for X a metric space of three points in $\mathbb{R}^2$, with $\mathcal{P}_X$ visualized as the convex hull of X in $\mathbb{R}^2$. Note that $\mathrm{VR}_p(X;r) \subseteq \mathrm{VR}_{p'}(X;r')$ for $r \le r'$ and $p \ge p'$.

As one of our main contributions, we prove that the p–Vietoris–Rips metric thickening is stable. This means that if two totally bounded metric spaces X and Y are close in the Gromov–Hausdorff distance, then their filtrations $\mathrm{VR}_p(X;\cdot)$ and $\mathrm{VR}_p(Y;\cdot)$ are close in the homotopy type distance. This was previously unknown even in the case $p=\infty$ (see [2, Conjecture 6.14]); we prove stability for all $1 \le p \le \infty$. As a consequence, it follows that the (undecorated) persistent homology diagrams for the Vietoris–Rips simplicial complexes $\mathrm{VR}(X;r)$ and for the $p=\infty$ Vietoris–Rips metric thickenings $\mathrm{VR}_\infty(X;\cdot)$ are identical. In other words, the persistent homology barcodes for $\mathrm{VR}(X,\cdot)$ and $\mathrm{VR}_\infty(X;\cdot)$ are identical, up to replacing closed interval endpoints with open endpoints, or vice versa. This answers [2, Conjecture 6.12] in the affirmative. Another consequence of stability is that the p–metric thickenings give the same persistence diagrams whether one considers all Radon probability measures, or instead the restricted setting of only measures with finite support.

The proof of stability for metric thickenings is more intricate than the proof of stability for the corresponding simplicial complexes (see for instance Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [27], Chazal, de Silva and Oudot [30] and Mémoli [57]). Whereas simplicial complexes can be compared via simplicial maps, the direct analogues of simplicial maps on metric thickenings are not necessarily continuous. Thus, new techniques are required to construct continuous maps between metric thickenings. Our technique, relying on partitions of unity, allows us to continuously approximate measures on one metric space X by measures on another metric space Y (where this approximation depends on the Gromov–Hausdorff distance between X and Y), thereby allowing us to construct the desired interleavings.

We furthermore introduce the p–Čech metric thickenings, again for $1 \le p \le \infty$, and prove analogues of all of the above results. The p–Čech metric thickening at scale parameter $r > 0$ contains all probability measures supported on X whose p–radius is less than r; see Table 2.

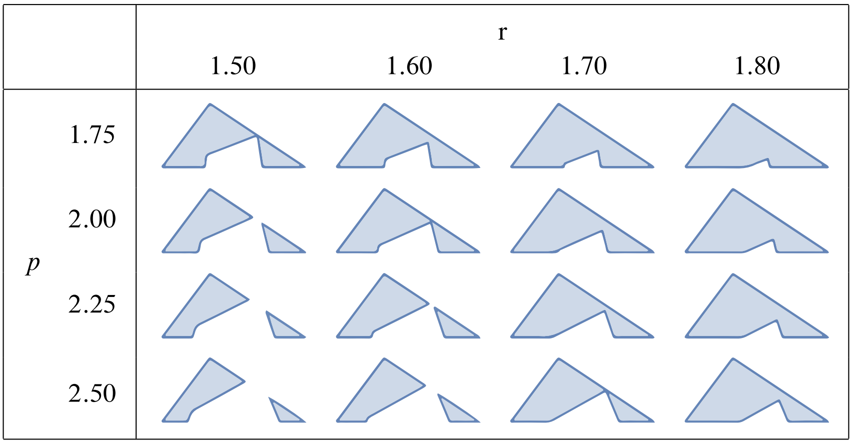

Table 2: The p–Čech bifiltration $\check{C}_p(X;r)$ for X a metric space of three points in $\mathbb{R}^2$, with $\mathcal{P}_X$ visualized as the convex hull of X in $\mathbb{R}^2$. Note that $\check{C}_p(X;r) \subseteq \check{C}_{p'}(X;r')$ for $r \le r'$ and $p \ge p'$.

We also deduce the complete spectrum of homotopy types of 2–Vietoris–Rips and 2–Čech metric thickenings of the n–sphere, equipped with the Euclidean metric ℓ_2: $\mathrm{VR}_2((\mathbb{S}^n, \ell_2); r)$ first attains the homotopy type of $\mathbb{S}^n$ for scales $r \le \sqrt{2}$, and then for all scales $r > \sqrt{2}$ the space is contractible. By contrast, the Vietoris–Rips simplicial complexes or the ∞–Vietoris–Rips metric thickenings of the n–sphere (with either the Euclidean or the geodesic metric) are only known for a bounded range of scales, including only a single change in homotopy type (see Adamaszek, Adams and Frick [2, Section 5]), even though infinitely many changes in homotopy type are conjectured (see Adams, Bush and Frick [4, Question 8.1]). See however Lim, Mémoli and Okutan [52, Corollary 7.18] for results for round spheres with the ℓ_∞ metric.

One of our main motivations for introducing the p–Vietoris–Rips and p–Čech metric thickenings is to enable effective Morse theories on these types of spaces. The p–variance for Čech metric thickenings is a minimum of linear functionals, and therefore fits in the framework of Morse theory for min-type functions; see Baryshnikov, Bubenik and Kahle [8], Bryzgalova [21], Gershkovich and Rubinstein [38] and Matov [54]. On the Vietoris–Rips side, we remark that gradient flows of functionals on Wasserstein space can be defined when the functional is "λ–convex"; see Mirth [61] and Santambrogio [69]. Though the ∞–diameter is not λ–convex, we hope that the p–diameter functional for $p < \infty$ may be λ–convex in certain settings.

Organization In Section 2 we describe background material and set notation. We define the p–Vietoris–Rips and p–Čech metric thickenings in Section 3, and consider their basic properties in Section 4. In Section 5 we prove stability. We consider Hausmann-type theorems in Section 6, and deduce the 2–Vietoris–Rips metric thickenings of Euclidean spheres in Section 7. In Section 8 we bound the length of intervals in p–Vietoris–Rips and p–Čech metric thickenings using a generalization of the spread of a metric space, called the p–spread.

We conclude the paper by providing some discussion in Section 9.

In Section A.1 we explain how the q–Wasserstein distance metrizes the weak topology for $1 \le q < \infty$. We describe connections to min-type Morse theories in Section A.2. In Section A.3 we show that p–Čech thickenings of finite metric spaces are homotopy equivalent to simplicial complexes. We prove the persistent homology diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of a family of discrete metric spaces in Section A.4, and we describe the 0–dimensional persistent homology of the p–Vietoris–Rips and p–Čech metric thickenings of an arbitrary metric space in Section A.5. We consider crushings in Section A.6. In Section A.7, we show that the main properties we prove for the (intrinsic) p–Čech metric thickening also hold for the ambient p–Čech metric thickening.

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2 Background

This section introduces background material and notation.

Metric spaces and the Gromov–Hausdorff distance Let (X, d_X) be a metric space. For any $x \in X$, we let $B(x; r) := \{y \in X \mid d_X(x, y) < r\}$ denote the open ball centered at x of radius r.

Given a metric space (X, d_X), the *diameter* of a nonempty subset $A \subset X$ is $\operatorname{diam}(A) := \sup_{a,a' \in A} d_X(a, a')$, whereas its *radius* is $\operatorname{rad}(A) := \inf_{x \in X} \sup_{a \in A} d_X(x, a)$. Note that, in general,

$$\tfrac{1}{2}\operatorname{diam}(A) \le \operatorname{rad}(A) \le \operatorname{diam}(A).$$

Definition 2.1 (uniform discrete metric space) For any natural number n, we use Z_n to denote the metric space consisting of n points with all interpoint distances equal to 1.

Definition 2.2 (ε–net) Let X be a metric space. A subset $U \subset X$ is called an *ε–net* of X if, for any point $x \in X$, there is a point $u \in U$ with $d_X(x, u) < \varepsilon$.

Let X and Y be metric spaces. The *distortion* of an arbitrary function $\varphi\colon X \to Y$ is

$$\operatorname{dis}(\varphi) := \sup_{x,x' \in X} |d_X(x, x') - d_Y(\varphi(x), \varphi(x'))|.$$

The *codistortion* of a pair of arbitrary functions $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ is

$$\operatorname{codis}(\varphi, \psi) := \sup_{x \in X, y \in Y} |d_X(x, \psi(y)) - d_Y(\varphi(x), y)|.$$

We will use the following expression for the Gromov–Hausdorff distance between X and Y [43]:

$$d_{\mathrm{GH}}(X,Y) = \tfrac{1}{2}\inf_{\varphi,\psi} \max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi,\psi)). \tag{1}$$

Remark 2.3 If $\mathrm{codis}(\varphi,\psi) < \eta$, then for all $(x,y) \in X \times Y$,

$$|d_X(x,\psi(y)) - d_Y(\varphi(x),y)| < \eta.$$

In particular, by letting $y = \varphi(x)$ above, this means that

$$d_X(x,\psi\circ\varphi(x)) < \eta \quad \text{for all } x \in X.$$

See [23, Chapter 7] for more details about the Gromov–Hausdorff distance.

Simplicial complexes Two of the most commonly used methods for producing filtrations in applied topology are the Vietoris–Rips and Čech simplicial complexes, defined as follows. Let X be a metric space, and let $r \geq 0$. The *Vietoris–Rips simplicial complex* $\mathrm{VR}(X;r)$ has X as its vertex set, and contains a finite subset $[x_0,\dots,x_k]$ as a simplex when $\mathrm{diam}([x_0,\dots,x_k]) := \max_{i,j} d_X(x_i,x_j) < r$. The *Čech simplicial complex* $\check{\mathrm{C}}(X;r)$ has X as its vertex set, and contains a finite subset $[x_0,\dots,x_k]$ as a simplex when $\bigcap_{i=0}^k B(x_i;r) \neq \varnothing$.

Probability measures and Wasserstein distances Our main reference for measure theory elements is [15] (albeit we use slightly different notation). Given a metric space (X,d_X), by $\mathcal{P}_X$ we denote the set of all *Radon* probability measures on X. We equip $\mathcal{P}_X$ with the weak topology: a sequence $\alpha_1,\alpha_2,\alpha_3,\ldots \in \mathcal{P}_X$ is said to converge weakly to $\alpha \in \mathcal{P}_X$ if, for all bounded, continuous functions $\varphi\colon X \to \mathbb{R}$, we have $\lim_{n\to\infty}\int_X \varphi(x)\,\alpha_n(dx) = \int_X \varphi(x)\,\alpha(dx)$. The *support* $\mathrm{supp}(\alpha)$ of a probability measure $\alpha \in \mathcal{P}_X$ is the largest closed set C such that every open set which has nonempty intersection with C has positive measure. If $\mathrm{supp}(\alpha)$ consists of a finite set of points, then α is called *finitely supported* and can be written as $\alpha = \sum_{i\in I} a_i\,\delta_{x_i}$, where I is finite, $a_i \geq 0$ for all i, $\sum_{i\in I} a_i = 1$, and each δ_{x_i} is a Dirac delta measure at x_i. Let $\mathcal{P}_X^{\mathrm{fin}}$ denote the set of all finitely supported Radon probability measures on X.

Given another metric space Y and a measurable map $f\colon X \to Y$, the *pushforward* map $f_\sharp\colon \mathcal{P}_X \to \mathcal{P}_Y$ induced by f is defined by $f_\sharp(\alpha)(B) = \alpha(f^{-1}(B))$ for every Borel set $B \subset Y$. In the case of finitely supported probability measures, we have the explicit formula $f_\sharp\big(\sum_{i\in I} a_i\,\delta_{x_i}\big) = \sum_{i\in I} a_i\,\delta_{f(x_i)}$, so $f_\sharp$ restricts to a function $\mathcal{P}_X^{\mathrm{fin}} \to \mathcal{P}_Y^{\mathrm{fin}}$. If f is a continuous map, then $f_\sharp$ is a continuous map between $\mathcal{P}_X$ and $\mathcal{P}_Y$ in the weak topology; see Chapter 5 in [15]. In the finitely supported case, the restriction of $f_\sharp$ is a continuous map from $\mathcal{P}_X^{\mathrm{fin}}$ to $\mathcal{P}_Y^{\mathrm{fin}}$.

Given $\alpha,\beta \in \mathcal{P}_X$, a *coupling* between them is any probability measure μ on $X \times X$ with marginals α and β, meaning that $(\pi_1)_\sharp\mu = \alpha$ and $(\pi_2)_\sharp\mu = \beta$, where $\pi_i\colon X \times X \to X$ is the projection map defined by $\pi_i(x_1,x_2) = x_i$ for $i = 1,2$. By $\mathrm{Cpl}(\alpha,\beta)$ we denote the set of all couplings between α and β. Notice that $\mathrm{Cpl}(\alpha,\beta)$ is always nonempty as the product measure $\alpha\otimes\beta$ is in $\mathrm{Cpl}(\alpha,\beta)$.

Given $q \in [1, \infty)$, let $\mathcal{P}_{q,X}$ be the subset of $\mathcal{P}_X$ consisting of Radon probability measures with finite moments of order q, that is, measures α with $\int_X d_X^q(x, x_0)\, \alpha(dx) < \infty$ for some, and thus any, $x_0 \in X$. Note that when X is a bounded metric space, $\mathcal{P}_{q,X} = \mathcal{P}_X$. We can equip $\mathcal{P}_{q,X}$ with the *q–Wasserstein distance* (or *Kantorovich q–metric*). In this setting, the q–Wasserstein distance is given by

$$d_{\mathrm{W},q}^X(\alpha, \beta) := \inf_{\mu \in \mathrm{Cpl}(\alpha,\beta)} \left(\iint_{X \times X} d_X^q(x, x')\, \mu(dx \times dx') \right)^{\frac{1}{q}}, \tag{2}$$

and it can be shown that $d_{\mathrm{W},q}^X$ defines a metric on $\mathcal{P}_{q,X}$; see Chapter 3.3 in [15]. If $q = \infty$, then again for any metric space (X, d_X) we define the *∞–Wasserstein distance*

$$d_{\mathrm{W},\infty}^X(\alpha, \beta) := \inf_{\mu \in \mathrm{Cpl}(\alpha,\beta)} \sup_{(x,x') \in \mathrm{supp}(\mu)} d_X(x, x') \tag{3}$$

for $\alpha, \beta \in \mathcal{P}_{\infty,X}$, the set of measures with bounded support. The ∞–Wasserstein distance $d_{\mathrm{W},\infty}^X$ is clearly symmetric, and the triangle inequality comes from a similar gluing trick as in Chapter 3.3 of [15]. Thus $d_{\mathrm{W},\infty}^X$ is a pseudometric that is bounded below by a metric — for example $d_{\mathrm{W},1}^X$. Therefore $d_{\mathrm{W},\infty}^X$ is also a metric. For general $q \le q'$, it follows from Hölder's inequality that $d_{\mathrm{W},q}^X \le d_{\mathrm{W},q'}^X$.

It is known that on a bounded metric space, for any $q \in [1, \infty)$, the q–Wasserstein metric generates the weak topology. For a summary of the result, see Section A.1. On the other hand, the ∞–Wasserstein metric $d_{\mathrm{W},\infty}^X$ generates a finer topology than the weak topology in general.

Let $f, g \colon X \to \mathbb{R}$ be measurable functions over the metric space X, and let $p \in [1, \infty)$. We will frequently use the Minkowski inequality, which states that

$$\left(\int_X |f(x) + g(x)|^p\, \alpha(dx) \right)^{\frac{1}{p}} \le \left(\int_X |f|^p(x)\, \alpha(dx) \right)^{\frac{1}{p}} + \left(\int_X |g|^p(x)\, \alpha(dx) \right)^{\frac{1}{p}}.$$

The following proposition shows that we may construct linear homotopies in spaces of probability measures, analogous to linear homotopies in Euclidean spaces. Linear homotopies will play an important role in Section 5.

Proposition 2.4 *Let Z be a metric space (or more generally a first-countable space), let X be a metric space, and let $f, g \colon Z \to \mathcal{P}_X$ be continuous. Then the linear homotopy $H \colon Z \times [0, 1] \to \mathcal{P}_X$ given by $H(z, t) = (1 - t) f(z) + t g(z)$ is continuous.*

Proof It suffices to show sequential continuity, because $Z \times I$ is metrizable (or more generally first-countable) and the weak topology on the set $\mathcal{P}_X$ is also metrizable; see [15, Theorem 3.1.4] and also Section A.1. If $\{(z_n, t_n)\}_n$ converges to (z, t) in $Z \times I$, then to show weak convergence of the image in $\mathcal{P}_X$, let $\gamma \colon X \to \mathbb{R}$ be any bounded, continuous function. We have

$$\begin{aligned}
\lim_{n\to\infty} \int_X \gamma(x) H(z_n, t_n)(dx) &= \lim_{n\to\infty} \left((1 - t_n) \int_X \gamma(x) f(z_n)(dx) + t_n \int_X \gamma(x) g(z_n)(dx) \right) \\
&= (1 - t) \int_X \gamma(x) f(z)(dx) + t \int_X \gamma(x) g(z)(dx) = \int_X \gamma(x) H(z, t)(dx),
\end{aligned}$$

where the second equality uses the fact that $f(z_n)$ and $g(z_n)$ are weakly convergent, since f and g are continuous. Therefore $H(z_n, t_n)$ converges weakly to $H(z, t)$, so H is continuous. □

Fréchet means For each $p \in [1, \infty]$, let the *p–Fréchet function of* $\alpha \in \mathcal{P}_X$, namely $F_{\alpha,p}\colon X \to \mathbb{R} \cup \{\infty\}$, be defined by

$$F_{\alpha,p}(x) := \begin{cases} \left(\int_X d_X^p(z,x)\,\alpha(dz)\right)^{1/p} & \text{for } p < \infty, \\ \sup_{z \in \mathrm{supp}(\alpha)} d_X(x,z) & \text{for } p = \infty. \end{cases}$$

Note that $F_{\alpha,p}(x) = d_{\mathrm{W},p}^X(\delta_x, \alpha)$. A point $x \in X$ that minimizes $F_{\alpha,p}(x)$ is called a *Fréchet mean* of α; in general, Fréchet means need not be unique. See [44] for some of the basic properties of Fréchet means.

Metric thickenings with $p = \infty$ Let X be a bounded metric space. The Vietoris–Rips and Čech metric thickenings were introduced in [2] with the notation $\mathrm{VR}^m(X; r)$ and $\check{\mathrm{C}}^m(X; r)$, where the superscript m denoted "metric." We instead denote these spaces by $\mathrm{VR}_\infty^{\mathrm{fin}}(X; r)$ and $\check{\mathrm{C}}_\infty^{\mathrm{fin}}(X; r)$, since one of our main contributions will be to introduce the generalizations $\mathrm{VR}_p(X; r)$ and $\check{\mathrm{C}}_p(X; r)$ — and their finitely supported variants $\mathrm{VR}_p^{\mathrm{fin}}(X; r)$ and $\check{\mathrm{C}}_p^{\mathrm{fin}}(X; r)$ — for any $1 \le p \le \infty$.

The *Vietoris–Rips metric thickening* $\mathrm{VR}_\infty^{\mathrm{fin}}(X; r)$ is the space of all finitely supported probability measures of the form $\sum_{i=0}^k a_i \delta_{x_i}$ such that $\mathrm{diam}(\{x_0, \dots, x_k\}) < r$, equipped with the q–Wasserstein metric for some $1 \le q < \infty$. The choice of $q \in [1, \infty)$ does not affect the homeomorphism type by Corollary A.2. The *Čech metric thickening* $\check{\mathrm{C}}_\infty^{\mathrm{fin}}(X; r)$ is the space of all finite probability measures of the form $\sum_{i=0}^k a_i \delta_{x_i}$ such that $\bigcap_{i=0}^k B(x_i; r) \neq \varnothing$, equipped with the q–Wasserstein metric for some $1 \le q < \infty$.

Comparisons We give a brief survey of the various advantages and disadvantages of using simplicial complexes and $p = \infty$ metric thickenings.

The Vietoris–Rips and Čech simplicial complexes were developed first, and they enjoy the benefits of simplicial and combinatorial techniques. For this reason, these complexes (and related complexes) have been used in (co)homology theories for metric spaces, and in discrete versions of homotopy theories; see [7; 13; 19; 20; 26; 32; 33; 59; 64; 66; 67; 71]. The persistent homology of Vietoris–Rips complexes on top of finite metric spaces can be efficiently computed [12]. Even when built on top of infinite metric spaces, much is known about the theory of Vietoris–Rips simplicial complexes using Hausmann's theorem [42], Latschev's theorem [50], the stability of Vietoris–Rips persistent homology [27; 30; 31], Bestvina–Brady Morse theory [75] and the Vietoris–Rips complexes of the circle [1].

Despite this rich array of simplicial techniques, there are some key disadvantages to Vietoris–Rips simplicial complexes. If X is not a discrete metric space, then the inclusion from X into $\mathrm{VR}(X; r)$ for any $r \ge 0$ is not continuous, since the vertex set of a simplicial complex is equipped with the discrete topology. Another disadvantage is that even though we start with a metric space X, the Vietoris–Rips simplicial complex $\mathrm{VR}(X; r)$ may not be metrizable. Indeed, a simplicial complex is metrizable if and only if it is locally finite [68, Proposition 4.2.16(2)], that is, if and only if each vertex is contained in only a finite number of simplices. So when X is infinite, $\mathrm{VR}(X;r)$ is often not metrizable, ie its topology cannot be induced by any metric. Though Vietoris–Rips complexes accept metric spaces as input, they do not remain in this same category, and may produce as output topological spaces that cannot be equipped with any metric structure.

By contrast, the metric thickening $\mathrm{VR}^{\mathrm{fin}}_\infty(X;r)$ is always a metric space that admits an isometric embedding $X \hookrightarrow \mathrm{VR}^{\mathrm{fin}}_\infty(X;r)$ [2]. Furthermore, metric thickenings allow for nicer proofs of Hausmann's theorem. For M a Riemannian manifold and $r>0$ sufficiently small depending on the curvature of M, Hausmann produces a map $T\colon \mathrm{VR}(M;r)\to M$ from the simplicial complex to the manifold that is not canonical in the sense that it depends on a total order of all points in the manifold. Also, the inclusion map $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous, and therefore cannot be a homotopy inverse for T. Nevertheless, Hausmann is able to prove T is a homotopy equivalence without constructing an explicit inverse. By contrast, in the context of metric thickenings, one can produce a canonical map $\mathrm{VR}^{\mathrm{fin}}_\infty(M;r)\to M$ by mapping a measure to its Fréchet mean (whenever r is small enough that measures of diameter less than r have unique Fréchet means). The (now continuous) inclusion $M \hookrightarrow \mathrm{VR}^{\mathrm{fin}}_\infty(M;r)$ can be shown to be a homotopy inverse via linear homotopies [2, Theorem 4.2].

One of our main contributions is showing how these various spaces relate to each other, especially when it comes to persistent homology. This allows one to work either simplicially, geometrically, or with measures — whichever perspective is most convenient for the task at hand.

Homology and persistent homology For each integer $k\geq 0$, let H_k denote the singular homology functor from the category Top of topological spaces to the category Vec of vector spaces and linear transformations. We use coefficients in a fixed field, so that homology groups are furthermore vector spaces. For background on persistent homology, we refer the reader to [34; 35; 76].

In applications of topology, such as topological data analysis [25], one often models a dataset not as a single space X, but instead as an increasing sequence of spaces. We refer to an increasing sequence of spaces, that is, a functor from the poset $(\mathbb{R},\leq)$ to Top, as a *filtration*. If X is a metric space, then a common filtration is the Vietoris–Rips simplicial complex filtration $\mathrm{VR}(X,\cdot)$. We will introduce relaxed versions, the p–Vietoris–Rips metric thickening filtrations $\mathrm{VR}_p(X;\cdot)$; see Section 3.

By applying homology (with coefficients in a field) to a filtration, we obtain a functor from the poset $(\mathbb{R},\leq)$ to Vec, ie a *persistence module*. We will use symbols like V, W and so on to denote persistence modules. Following [29], a persistence module is *Q–tame* if, for any $s<t$, the structure map $V(s)\to V(t)$ is of finite rank. In [29], it is shown that a Q–tame persistence module V can be converted into a *persistence diagram*,[1] $\mathrm{dgm}(V)$, which is a multiset in the extended open half-plane consisting of pairs $p=(b,d)$ for $-\infty\leq b<d\leq+\infty$. The persistence diagrams can be compared via the bottleneck distance d_{B}, which is defined as follows. Given persistence diagrams D_1 and D_2, a subset $M\subset D_1\times D_2$ is said to be a *partial matching* between D_1 and D_2 if:

[1] Specifically, we consider *undecorated* persistence diagrams, as described in [29].

- For every point p in D_1, there is at most one point q in D_2 such that $(p,q) \in M$. (If there is no such q, we then say p is unmatched.)
- For every point q in D_2, there is at most one point p in D_1 such that $(p,q) \in M$. (If there is no such p, we then say q is unmatched.)

The *bottleneck distance* d_{B} between two persistence diagrams D_1 and D_2 is

$$d_{\mathrm{B}}(D_1, D_2) := \inf_M \max\left\{ \sup_{(p,q)\in M} \|p-q\|_\infty, \sup_{s\in D_1 \sqcup D_2 \text{ unmatched}} \left|\tfrac{1}{2}(s_b - s_d)\right| \right\},$$

where s is an element in D_1 or D_2, s_b is the birth time of s, s_d is the death time of s, and M varies among all possible partial matchings.

Interleavings Let $\mathcal{C}$ be a category. We call any functor from the poset $(\mathbb{R}, \leq)$ to $\mathcal{C}$ an $\mathbb{R}$*–space*. Such a functor gives a *structure map* $X(s) \to X(t)$ for any $s \leq t$. For X an $\mathbb{R}$–space, the δ*–shift* of X is the functor $X^\delta : \mathbb{R} \to C$ with $X^\delta(t) = X(t+\delta)$ for all $t \in \mathbb{R}$. We have a natural transformation id_X^δ from X to X^δ which maps $X(t)$ to $X^\delta(t)$ using the structure maps from X. For two $\mathbb{R}$–spaces X and Y, we say they are δ*–interleaved* if there are natural transformations $F : X \to Y^\delta$ and $G : Y \to X^\delta$ such that $F \circ G = \mathrm{id}_X^{2\delta}$ and $G \circ F = \mathrm{id}_Y^{2\delta}$. Now we define a pseudodistance between $\mathbb{R}$–spaces, which we call the *interleaving distance* $d_{\mathrm{I}}^{\mathcal{C}}$, as follows:

$$d_{\mathrm{I}}^{\mathcal{C}}(X, Y) := \inf\{\delta \mid X \text{ and } Y \text{ are } \delta\text{–interleaved}\}.$$

We note that $d_{\mathrm{I}}^{\mathrm{Vec}}$ is the interleaving distance for persistence modules. The isometry theorem of [29; 51] states that $d_{\mathrm{I}}^{\mathrm{Vec}}(V, W) = d_{\mathrm{B}}(\mathrm{dgm}(V), \mathrm{dgm}(W))$ for Q–tame persistence modules V and W.

We will use the following lemma in our proofs in Section 5 on stability:

Lemma 2.5 *If a persistence module P can be approximated arbitrarily well in the interleaving distance by Q–tame persistence modules, then P is also Q–tame.*

Proof For any $s < t$, let $\varepsilon = t - s$, so there is a Q–tame persistence module P_ε such that $d_{\mathrm{I}}(P, P_\varepsilon) < \frac{1}{3}\varepsilon$. Then using the maps of an $\frac{1}{3}\varepsilon$–interleaving between P and P_ε, the structure map $P(s) \to P(t)$ can be factored as follows:

$$\begin{tikzcd}
P(s) \arrow[rrr] \arrow[dr] & & & P(t) \\
& P_\varepsilon\left(s+\frac{1}{3}\varepsilon\right) \arrow[r] & P_\varepsilon\left(s+\frac{2}{3}\varepsilon\right) \arrow[ur] &
\end{tikzcd}$$

As P_ε is a Q–tame module, the rank of $P_\varepsilon\left(s+\frac{1}{3}\varepsilon\right) \to P_\varepsilon\left(s+\frac{2}{3}\varepsilon\right)$ is finite, and hence so is the rank of $P(s) \to P(t)$. Since s and t are arbitrary, we get the Q–tameness of P. □

The homotopy type distance We next recall the definition of the homotopy type distance d_{HT} from [37]. The following is a small generalization of [37, Definition 2.2] in that we do not require the maps φ_X and φ_Y to be continuous.

Definition 2.6 Let (X,φ_X) and (Y,φ_Y) be two topological spaces with real-valued functions on them. For any $\delta \geq 0$, a *δ–map* between (X,φ_X) and (Y,φ_Y) is a continuous map $\Phi\colon X \to Y$ such that $\varphi_Y \circ \Phi(x) \leq \varphi_X(x)+\delta$ for any $x \in X$. For any two δ–maps $\Phi_0\colon X \to Y$ and $\Phi_1\colon X \to Y$, a *δ–homotopy* between Φ_0 and Φ_1 with respect to the pair (φ_X,φ_Y) is a continuous map $H\colon X \times [0,1] \to Y$ such that

(i) $\Phi_0 \equiv H(\cdot\,,0)$,

(ii) $\Phi_1 \equiv H(\cdot\,,1)$,

(iii) $H(\cdot\,,t)$ is a δ–map with respect to the pair (φ_X,φ_Y) for every $t \in [0,1]$.

Definition 2.7 [37, Definitions 2.5 and 2.6] For every $\delta \geq 0$ and for any two pairs (X,φ_X) and (Y,φ_Y), we say (X,φ_X) and (Y,φ_Y) are *δ–homotopy equivalent* if there exist δ–maps $\Phi\colon X \to Y$ and $\Psi\colon Y \to X$, with respect to (φ_X,φ_Y) and (φ_Y,φ_X), respectively, such that

- the map $\Psi \circ \Phi\colon X \to X$ is 2δ–homotopic to id_X with respect to (φ_X,φ_X), and
- the map $\Phi \circ \Psi\colon Y \to Y$ is 2δ–homotopic to id_Y with respect to (φ_Y,φ_Y).

The *d_{HT}–distance* between (X,φ_X) and (Y,φ_Y) is

$$d_{\mathrm{HT}}((X,\varphi_X),(Y,\varphi_Y)) := \inf\{\delta \geq 0 \mid (X,\varphi_X) \text{ and } (Y,\varphi_Y) \text{ are } \delta\text{–homotopy equivalent}\}.$$

If (X,φ_X) and (Y,φ_Y) are not δ–homotopy equivalent for any δ, then we declare

$$d_{\mathrm{HT}}((X,\varphi_X),(Y,\varphi_Y)) = \infty.$$

Proposition 2.8 [37, Proposition 2.10] *The d_{HT} distance is an extended pseudometric on the set of pairs of topological spaces and real-valued functions.*

A pair (X,φ_X), where X is a topological space and $\varphi_X\colon X \to \mathbb{R}$ is a real-valued function, induces an $\mathbb{R}$–space $[X,\varphi_X;\cdot\,]$ given by the sublevel set filtration, and defined by

$$[X,\varphi_X;r] := \varphi_X^{-1}((-\infty,r)) \quad \text{for } r \in \mathbb{R}$$

and

$$[X,\varphi_X;\cdot\,] := \{\varphi_X^{-1}((-\infty,r)) \subseteq \varphi_X^{-1}((-\infty,r'))\}_{r \leq r'}.$$

The following theorem shows that the interleaving distance of the persistent homology of the sublevel set filtrations is bounded by the d_{HT} distance of the respective pairs. The theorem is a slight generalization of [37, Lemma 3.1] in which we do not require the continuity of φ_X and φ_Y; we omit its (identical) proof.

Lemma 2.9 [37, Lemma 3.1] *Let (X,φ_X) and (Y,φ_Y) be two pairs. Then for any integer $k \geq 0$,*

$$d_{\mathrm{I}}^{\mathrm{Vec}}(H_k \circ [X,\varphi_X;\cdot\,], H_k \circ [Y,\varphi_Y;\cdot\,]) \leq d_{\mathrm{HT}}((X,\varphi_X),(Y,\varphi_Y)).$$

3 The p–relaxation of metric thickenings

We now describe the construction of the p–relaxations of metric thickenings.

3.1 The relaxed diameter and radius functionals

Throughout this section, (X, d_X) will denote a bounded metric space. Recall that $\mathcal{P}_X$ is the set of all Radon probability measures on X, equipped with the weak topology. We now introduce, for each $p \in [1, \infty]$, the *p–diameter* of a measure α in $\mathcal{P}_X$, where the ∞–diameter of α is precisely the diameter of its support $\mathrm{supp}\,(\alpha)$ (as a subset of X). Consider, for each $p \in [1, \infty]$, the *p–diameter* map $\mathrm{diam}_p\colon \mathcal{P}_X \to \mathbb{R}_{\geq 0}$ given by

$$\mathrm{diam}_p(\alpha) = \begin{cases} \left(\iint_{X\times X} d_X^p(x,x')\,\alpha(dx)\,\alpha(dx')\right)^{1/p} & \text{for } p < \infty, \\ \mathrm{diam}(\mathrm{supp}(\alpha)) & \text{for } p = \infty. \end{cases}$$

We remark that the p–diameter has been studied and critiqued as a measure of diversity in population biology [63; 65], and it has also been considered in relation to the Gromov–Wasserstein distance [55]. Similarly, define the *p–radius* map $\mathrm{rad}_p\colon \mathcal{P}_X \to \mathbb{R}_{\geq 0}$ via

$$\mathrm{rad}_p(\alpha) = \begin{cases} \inf_{x\in X}\left(\int_X d_X^p(x,x')\,\alpha(dx')\right)^{1/p} & \text{for } p < \infty, \\ \mathrm{rad}(\mathrm{supp}(\alpha)) & \text{for } p = \infty. \end{cases}$$

Note that the p–radius of a measure is simply the p^{th} root of its p–variance.

We observe that

$$\mathrm{diam}_p(\alpha) = \left(\int_X F_{\alpha,p}^p(x)\,\alpha(dx)\right)^{\frac{1}{p}} = \left(\int_X (d_{\mathrm{W},p}^X(\alpha,\delta_x))^p\,\alpha(dx)\right)^{\frac{1}{p}}$$

and

$$\mathrm{rad}_p(\alpha) = \inf_{x\in X} F_{\alpha,p}(x) = \inf_{x\in X} d_{\mathrm{W},p}^X(\delta_x,\alpha).$$

Proposition 3.1 *The functions* $\mathrm{diam}_p, \mathrm{rad}_p\colon \mathcal{P}_X \to \mathbb{R}$ *satisfy*

$$\mathrm{rad}_p(\alpha) \leq \mathrm{diam}_p(\alpha) \leq 2\,\mathrm{rad}_p(\alpha).$$

Proof To see that $\mathrm{rad}_p(\alpha) \leq \mathrm{diam}_p(\alpha)$, note that

$$\begin{aligned}\mathrm{diam}_p(\alpha) &= \left(\iint_{X\times X} (d_X(x,x'))^p\,\alpha(dx)\,\alpha(dx')\right)^{\frac{1}{p}} = \left(\int_X\left(\int_X d_X^p(x,x')\,\alpha(dx)\right)\alpha(dx')\right)^{\frac{1}{p}} \\ &\geq \left(\inf_{x'\in X}\left(\int_X d_X^p(x,x')\,\alpha(dx)\right)\right)^{\frac{1}{p}} = \inf_{x'\in X}\left(\int_X d_X^p(x,x')\,\alpha(dx)\right)^{\frac{1}{p}} = \mathrm{rad}_p(\alpha).\end{aligned}$$

To see that $\mathrm{diam}_p(\alpha) \leq 2\,\mathrm{rad}_p(\alpha)$, for any $\varepsilon > 0$ there is a point $z \in X$ with

$$\left(\int_X d_X^p(x,z)\,\alpha(dx)\right)^{\frac{1}{p}} \leq \mathrm{rad}_p(\alpha) + \varepsilon.$$

Then we have

$$\begin{aligned}\operatorname{diam}_p(\alpha) &= \Big(\iint_{X\times X}(d_X(x,x'))^p\,\alpha(dx)\,\alpha(dx')\Big)^{\frac{1}{p}} \le \Big(\iint_{X\times X}(d_X(x,z)+d_X(z,x'))^p\,\alpha(dx)\,\alpha(dx')\Big)^{\frac{1}{p}}\\ &\le \Big(\iint_{X\times X}(d_X(x,z))^p\,\alpha(dx)\,\alpha(dx')\Big)^{\frac{1}{p}} + \Big(\iint_{X\times X}(d_X(z,x'))^p\,\alpha(dx)\,\alpha(dx')\Big)^{\frac{1}{p}}\\ &\le 2\operatorname{rad}_p(\alpha)+2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $\operatorname{diam}_p(\alpha) \le 2\operatorname{rad}_p(\alpha)$. □

Remark 3.2 The tightness of the bound $\operatorname{rad}_p(\alpha) \le \operatorname{diam}_p(\alpha)$ can be seen from the calculation for the uniform measure on Z_n, the metric space on a set of size n with all interpoint distances equal to 1. To see the (asymptotic) tightness of $\operatorname{diam}_p(\alpha) \le 2\operatorname{rad}_p(\alpha)$, we consider the metric space $Z_n \cup \{O\}$, where the newly introduced "center" O has distance $\frac{1}{2}$ to every other point. Then for the measure $\alpha := \sum_{z\in Z_n}(1/n)\delta_z$, we have $\operatorname{rad}_p(\alpha) = \frac{1}{2}$ and $\operatorname{diam}_p(\alpha) = ((n-1)/n)^{1/p}$. We obtain asymptotic tightness by letting n go to infinity.

3.2 The relaxed Vietoris–Rips and Čech metric thickenings

Definition 3.3 (p–Vietoris–Rips filtration) For each $r > 0$ and $p \in [1,\infty]$, let the *p–Vietoris–Rips metric thickening at scale r* be

$$\operatorname{VR}_p(X;r) := \{\alpha \in \mathcal{P}_X \mid \operatorname{diam}_p(\alpha) < r\}.$$

We regard $\operatorname{VR}_p(X;r)$ as a topological space by endowing it with the subspace topology from $\mathcal{P}_X$ (ie the weak topology). By convention, when $r \le 0$, we will let $\operatorname{VR}_p(X;r) = \varnothing$. By

$$\operatorname{VR}_p(X;\cdot) := \{\operatorname{VR}_p(X;r) \xhookrightarrow{\nu^X_{r,r'}} \operatorname{VR}_p(X;r')\}_{r\le r'}$$

we will denote the filtration thus induced.

We use $\operatorname{VR}^{\mathrm{fin}}_p(X;r)$ and $\operatorname{VR}^{\mathrm{fin}}_p(X;\cdot)$ to denote the finitely supported variants obtained by replacing $\mathcal{P}_X$ in the above definition with $\mathcal{P}^{\mathrm{fin}}_X$.

Note that in the specific case $p = \infty$, the ∞–diameter of a measure is simply the diameter of its support. For a bounded metric space X, the weak topology is generated by the 1–Wasserstein metric, so the definition of $\operatorname{VR}^{\mathrm{fin}}_p(X;r)$ generalizes [2, Definition 3.1], which is the specific case $p = \infty$.

Remark 3.4 ($\operatorname{VR}_p(X;\cdot)$ as a softening of $\operatorname{VR}_\infty(X,\cdot)$) The definition of $\operatorname{VR}_p(X;\cdot)$ can be extended to the whole range $p \in [0,\infty]$ as follows. First note that $\operatorname{diam}_p(\alpha)$ can still be defined as above for $p \in (0,1)$, and by $\operatorname{diam}_0(\alpha) = 0$ when $p = 0$. Furthermore, if $\alpha = \sum_{i\in I} a_i\delta_{x_i}$, then

$$\lim_{p\downarrow 0}\operatorname{diam}_p(\alpha) = \prod_{i,j\in I}(d_X(x_i,x_j))^{a_i a_j},$$

which equals 0 since the product contains terms with $i = j$.

At any rate, by the standard generalized means inequality [22], we have $\mathrm{diam}_{p'}(\alpha) \le \mathrm{diam}_p(\alpha)$ for any $p' \le p$ in the range $[0,\infty]$. So, for fixed $r > 0$, not only does $\mathrm{VR}_p(X;r)$ become larger and larger as p decreases, but also for $p = 0$, $\mathrm{VR}_0(X;r)$ contains *all* Radon probability measures on X and thus has trivial reduced homology.

Definition 3.5 (p–Čech filtration) For each $r > 0$ and $p \in [1,\infty]$, let the *p–Čech metric thickening at scale r* be

$$\check{\mathrm{C}}_p(X;r) := \{\alpha \in \mathcal{P}_X \mid \mathrm{rad}_p(\alpha) < r\}.$$

We regard $\check{\mathrm{C}}_p(r;X)$ as a topological space by endowing it with the subspace topology from $\mathcal{P}_X$ (ie the weak topology). By convention, when $r \le 0$, we will let $\check{\mathrm{C}}_p(X;r) = \varnothing$. By

$$\check{\mathrm{C}}_p(X;\cdot) := \{\check{\mathrm{C}}_p(X;r) \xhookrightarrow{\nu^X_{r,r'}} \check{\mathrm{C}}_p(X;r')\}_{r \le r'}$$

we will denote the filtration thus induced.[2]

Remark 3.6 Let α be a measure in $\mathcal{P}_X$. As $\mathrm{rad}_p(\alpha) = \inf_{x\in X} d^X_{\mathrm{W},p}(\delta_x,\alpha)$, we have $\mathrm{rad}_p(\alpha) < r$ if and only if there is some $x \in X$ such that $d^X_{\mathrm{W},p}(\delta_x,\alpha) < r$. Therefore $\check{\mathrm{C}}_p(X;r)$ is exactly the union over all $x \in X$ of the balls $B(\delta_x;r)$, with respect to the p–Wasserstein metric, centered at points δ_x in the isometric image of X in $\mathcal{P}_X$.

We use $\check{\mathrm{C}}^{\mathrm{fin}}_p(X;r)$ and $\check{\mathrm{C}}^{\mathrm{fin}}_p(X;\cdot)$ to denote the finitely supported variants, obtained by replacing $\mathcal{P}_X$ in the above definition with $\mathcal{P}^{\mathrm{fin}}_X$.

Note that in the specific case $p = \infty$, the ∞–radius of a measure is simply the radius of its support. For a bounded metric space X, the weak topology is generated by the 1–Wasserstein metric, so the definition of $\check{\mathrm{C}}^{\mathrm{fin}}_p(X;r)$ generalizes [2], which considers the specific case $p = \infty$.

Though the above definitions are given with the $<$ convention, we remark that they instead could have been given with the $\le$ convention, namely $\mathrm{diam}_p(\alpha) \le r$ or $\mathrm{rad}_p(\alpha) \le r$. We restrict attention to the $<$ convention, even though many of the statements we give are also true with the $\le$ convention.

The next proposition shows that p–Vietoris–Rips and p–Čech metric thickenings are nested in the same way that Vietoris–Rips and Čech simplicial complexes are.

Proposition 3.7 *Let X be a bounded metric space. Then, for any $r > 0$,*

$$\mathrm{VR}_p(X;r) \subseteq \check{\mathrm{C}}_p(X;r) \subseteq \mathrm{VR}_p(X;2r).$$

Proof This follows immediately from Proposition 3.1, which implies that, for any $\alpha \in \mathcal{P}_X$,

$$\mathrm{diam}_p(\alpha) \ge \mathrm{rad}_p(\alpha) \ge \tfrac{1}{2}\,\mathrm{diam}_p(\alpha). \qquad \square$$

[2] We will use $\nu^X_{r,r'}$ to denote the inclusion maps in both Čech and Vietoris–Rips thickenings; this will not lead to confusion in this paper.

If X and Z are metric spaces with $X \subseteq Z$, and if the metric on Z is an extension of that on X, then following Gromov [41, Section 1.B] we say that Z is an *r–metric thickening of X* if, for all $z \in Z$, there is some $x \in X$ with $d_Z(x, z) \le r$.

Proposition 3.8 *Let X be a bounded metric space. When equipped with the q–Wasserstein metric for $1 \le q \le p$, we have that $\check{\mathrm{C}}_p(X; r)$ and $\mathrm{VR}_p(X; r)$ are each r–metric thickenings of X for all $r > 0$.*

Proof We use the isometric embedding $X \to \check{\mathrm{C}}_p(X; r)$ given by $x \mapsto \delta_x$. Let $\alpha \in \check{\mathrm{C}}_p(X; r)$. Hence there exists some $x \in X$ with $r > d^X_{\mathrm{W},p}(\delta_x, \alpha) \ge d^X_{\mathrm{W},q}(\delta_x, \alpha)$, which shows that $\check{\mathrm{C}}_p(X; r)$, equipped with the q–Wasserstein metric, is an r–metric thickening of X.

The Vietoris–Rips case follows immediately since $\mathrm{VR}_p(X; r) \subseteq \check{\mathrm{C}}_p(X; r)$ by Proposition 3.7. □

The next remark follows the perspective introduced in [52], which shows that the filling radius is related to the persistent homology of the Vietoris–Rips simplicial complex filtration $\mathrm{VR}(X, \cdot)$.

Remark 3.9 Let X be a closed connected n–dimensional manifold so that the *fundamental class* of X is well defined. Let Z be an r–metric thickening of X for some $r > 0$. It is shown in [40, Page 8, Another Corollary] that the map

$$H_n(X) \hookrightarrow H_n(Z)$$

induced from the inclusion $\iota\colon X \to Z$ is an injection whenever r is less than a scalar geometric invariant $\mathrm{FillRad}(X)$, called the *filling radius* of X. Hence Proposition 3.8 implies that, for all $p \in [1, \infty]$, the persistence diagram in dimension n of either the filtration $\mathrm{VR}_p(X; \cdot)$ or $\check{\mathrm{C}}_p(X; \cdot)$ contains an interval with left endpoint equal to zero and length at least $\mathrm{FillRad}(X)$. As proved in [45], the filling radius of the n–sphere (with its geodesic metric) is $\mathrm{FillRad}(\mathbb{S}^n) = \frac{1}{2}\arccos(-1/(n+1))$. Therefore, when $X = \mathbb{S}^n$, the n–dimensional persistence diagram of either $\mathrm{VR}_p(\mathbb{S}^n; \cdot)$ or $\check{\mathrm{C}}_p(\mathbb{S}^n; \cdot)$ contains an interval starting at zero which is no shorter than $\frac{1}{2}\arccos(-1/(n+1))$.

Lemma 3.10 *For all $r > 0$ and all $p, p' \in [1, \infty]$ with $p \ge p'$, one has*

$$\mathrm{VR}_p(X; r) \subseteq \mathrm{VR}_{p'}(X; r) \quad \text{and} \quad \check{\mathrm{C}}_p(X; r) \subseteq \check{\mathrm{C}}_{p'}(X; r).$$

Proof This comes from applying the Hölder inequality on diam_p and rad_p. □

Example 3.11 We recall that Z_{n+1} is the metric space consisting of $n+1$ points with all interpoint distances equal to 1. For any natural number n, the Vietoris–Rips or Čech simplicial complex filtrations of Z_{n+1} do not produce any nondiagonal point in their persistence diagram, except in homological dimension zero. However, for any $1 < p < \infty$, both the p–Vietoris–Rips and p–Čech filtrations of Z_{n+1} will contain nondiagonal points in their persistence diagrams for homological degrees from zero to $n-1$. More specifically, both the p–Vietoris–Rips and p–Čech filtrations are homotopy equivalent to a filtration

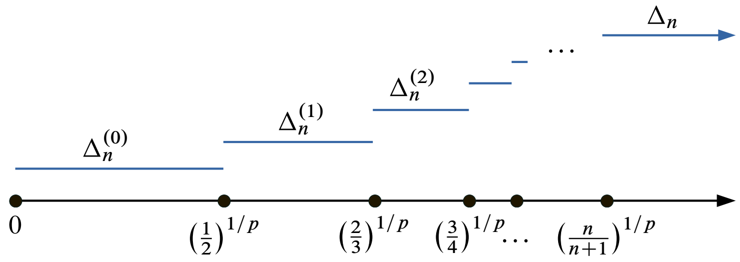

Figure 1: Homotopy types of both $\mathrm{VR}_p(Z_{n+1};\cdot)$ and $\check{\mathrm{C}}_p(Z_{n+1};\cdot)$.

of an n–simplex Δ_n by its k–skeleta, $\Delta_n^{(k)}$. We prove the following result in Section A.4. For r in the interval $((k/(k+1))^{1/p}, ((k+1)/(k+2))^{1/p}]$ with $0 \le k \le n-1$, we have

$$\mathrm{VR}_p(Z_{n+1};r) \simeq \check{\mathrm{C}}_p(Z_{n+1};r) \simeq \Delta_n^{(k)},$$

and when $r > (n/(n+1))^{1/p}$, both $\mathrm{VR}_p(Z_{n+1};r)$ and $\check{\mathrm{C}}_p(Z_{n+1};r)$ become the n–simplex Δ_n, which is contractible.

From this we get the persistence diagrams

$$\mathrm{dgm}_{k,p}^{\mathrm{VR}}(Z_{n+1}) = \mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(Z_{n+1}) = \begin{cases} \left(0, \left(\frac{1}{2}\right)^{1/p}\right)^{\otimes n} \oplus (0,\infty) & \text{if } k=0, \\ ((k/(k+1))^{1/p}, ((k+1)/(k+2))^{1/p})^{\otimes \binom{n}{k+1}} & \text{if } 1 \le k \le n-1, \\ \varnothing & \text{if } k \ge n. \end{cases}$$

Note that, from the definition of $\mathrm{VR}_\infty(Z_{n+1};\cdot)$, we have $\mathrm{dgm}_{k,\infty}^{\mathrm{VR}} = \varnothing$ for $k \ge 1$, and furthermore that $\lim_{p\uparrow\infty} \mathrm{dgm}_{k,p}^{\mathrm{VR}}(Z_{n+1}) = \mathrm{dgm}_{k,\infty}^{\mathrm{VR}}(Z_{n+1})$ for each $k \ge 0$. An analogous result holds for $\check{\mathrm{C}}_p(Z_{n+1};\cdot)$.

3.3 A more general setting: controlled invariants

We now generalize the relaxed p–Vietoris–Rips and p–Čech filtrations.

Definition 3.12 (controlled invariants on $\mathcal{P}_X$) Let i be a functional that associates to each bounded metric space X a function $\mathrm{i}^X : \mathcal{P}_X \to \mathbb{R}$. We say, for some $L > 0$, that i is an *L–controlled* invariant if the following conditions are satisfied:

(i) **Stability under pushforward** For any map f between finite metric spaces X and Y and any α in $\mathcal{P}_X$,

$$\mathrm{i}^Y(f_\sharp(\alpha)) \le \mathrm{i}^X(\alpha) + L\,\mathrm{dis}(f).$$

(ii) **Stability with respect to $d_{\mathrm{W},\infty}^X$** For any bounded metric space X and any α and β in $\mathcal{P}_X$,

$$|\mathrm{i}^X(\alpha) - \mathrm{i}^X(\beta)| \le 2L\, d_{\mathrm{W},\infty}^X(\alpha,\beta).$$

In the next section, we will prove that both diam_p and rad_p induce controlled invariants.

For any controlled invariant $\mathfrak{i}$, we will use the notation $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ to denote the sublevel set filtration induced by the pair $(\mathcal{P}_X, \mathfrak{i}^X)$. That is,

$$[\mathcal{P}_X, \mathfrak{i}^X; r] = (\mathfrak{i}^X)^{-1}((-\infty, r)) \text{ for } r \in \mathbb{R}$$

and

$$[\mathcal{P}_X, \mathfrak{i}^X; \cdot] = \{(\mathfrak{i}^X)^{-1}((-\infty, r)) \subseteq (\mathfrak{i}^X)^{-1}((-\infty, r'))\}_{r \le r'}.$$

This construction generalizes both the definition of $\mathrm{VR}_p(X; \cdot)$ and that of $\check{\mathrm{C}}_p(X; \cdot)$.

Similarly, any controlled invariant $\mathfrak{i} \colon \mathcal{P}_X^{\mathrm{fin}} \to \mathbb{R}$ induces an analogous sublevel set filtration $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ of $\mathcal{P}_X^{\mathrm{fin}}$. This construction generalizes both the definition of $\mathrm{VR}_p^{\mathrm{fin}}(X; \cdot) = [\mathcal{P}_X, \mathrm{diam}_p^X; \cdot]$ and that of $\check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot) = [\mathcal{P}_X, \mathrm{rad}_p^X; \cdot]$.

We'll see later on in Corollary 5.3 that the filtrations $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ and $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ yield persistence modules which are 0–interleaved.

4 Basic properties

We prove some basic properties. These include the convexity of the Wasserstein distance, the fact that diam_p and rad_p are 1–controlled and the fact that, for X finite, the $p = \infty$ metric thickenings $\mathrm{VR}_\infty(X; r)$ and $\check{\mathrm{C}}_\infty(X; r)$ are homeomorphic to the simplicial complexes $\mathrm{VR}(X; r)$ and $\check{\mathrm{C}}(X; r)$, respectively. We begin with the property that nearby measures have nearby integrals. Throughout this section, X is a bounded metric space.

Lemma 4.1 *Let φ be an L–Lipschitz function on X. Then, for any α and β in $\mathcal{P}_X$ and any $q \in [1, \infty]$,*

$$\left| \left(\int_X |\varphi(x)|^q \, \alpha(dx) \right)^{\frac{1}{q}} - \left(\int_X |\varphi(x)|^q \, \beta(dx) \right)^{\frac{1}{q}} \right| \le L \, d_{\mathrm{W},q}^X(\alpha, \beta).$$

Proof Let $\mu \in \mathrm{Cpl}(\alpha, \beta)$ be any coupling. Then

$$\begin{aligned}
&\left| \left(\int_X |\varphi(x)|^q \, \alpha(dx) \right)^{\frac{1}{q}} - \left(\int_X |\varphi(x)|^q \, \beta(dx) \right)^{\frac{1}{q}} \right| \\
&\qquad = \left| \left(\iint_{X \times X} |\varphi(x)|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} - \left(\iint_{X \times X} |\varphi(x')|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} \right| \\
&\qquad \le \left(\iint_{X \times X} |\varphi(x) - \varphi(x')|^q \, \mu(dx \times dx') \right)^{\frac{1}{q}} \le L \left(\iint_{X \times X} d_X^q(x, x') \, \mu(dx \times dx') \right)^{\frac{1}{q}}.
\end{aligned}$$

We obtain the claim by taking the infimum of the right-hand side with respect to the coupling μ. □

We will often use the following convexity result of the Wasserstein distance; for a more general result, see [72, Theorem 4.8].

Lemma 4.2 *Suppose that* $c_1, \dots, c_n$ *are nonnegative real numbers satisfying* $\sum_{k=1}^n c_k = 1$ *and that* $\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n \in \mathcal{P}_X$. *Then for all* $q \in [1, \infty)$, *we have*

$$d_{\mathrm{W},q}^X\left(\sum_{k=1}^n c_k\alpha_k, \sum_{k=1}^n c_k\alpha'_k\right) \le \left(\sum_{k=1}^n c_k\left(d_{\mathrm{W},q}^X(\alpha_k, \alpha'_k)\right)^q\right)^{\frac{1}{q}}$$

and

$$d_{\mathrm{W},\infty}^X\left(\sum_{k=1}^n c_k\alpha_k, \sum_{k=1}^n c_k\alpha'_k\right) \le \max_k d_{\mathrm{W},\infty}^X(\alpha_k, \alpha'_k).$$

Proof Let $\varepsilon > 0$ and $q \in [1, \infty)$. For each k, suppose μ_k is a coupling between α_k and α'_k such that

$$\iint_{X\times X} d_X^q(x, x')\,\mu_k(dx \times dx') < (d_{\mathrm{W},q}^X(\alpha_k, \alpha'_k) + \varepsilon)^q.$$

Then it can be checked that $\sum_{k=1}^n c_k\mu_k$ is a coupling between $\sum_{k=1}^n c_k\alpha_k$ and $\sum_{k=1}^n c_k\alpha'_k$. We have

$$\begin{aligned} d_{\mathrm{W},q}^X\left(\sum_{k=1}^n c_k\alpha_k, \sum_{k=1}^n c_k\alpha'_k\right) &\le \left(\iint_{X\times X} d_X^q(x, x')\sum_{k=1}^n c_k\,\mu_k(dx \times dx')\right)^{\frac{1}{q}} \\ &< \left(\sum_{k=1}^n c_k(d_{\mathrm{W},q}^X(\alpha_k, \alpha'_k) + \varepsilon)^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^n c_k(d_{\mathrm{W},q}^X(\alpha_k, \alpha'_k))^q\right)^{\frac{1}{q}} + \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, the claimed inequality holds. The case for $q = \infty$ can be checked separately using the same matching. $\square$

We define an invariant generalizing diam_p, and then prove that it is 1–controlled.

Definition 4.3 ($\mathrm{i}_{q,p}^X$ invariant) Let X be a bounded metric space. For any $p, q \in [1, \infty]$, we define the $\mathrm{i}_{q,p}$ *invariant*, which associates to each bounded metric space X a function $\mathrm{i}_{q,p}^X : \mathcal{P}_X \to \mathbb{R}$, by

$$\mathrm{i}_{q,p}^X(\alpha) = \left(\int_X (d_{\mathrm{W},q}^X(\alpha, \delta_x))^p\,\alpha(dx)\right)^{\frac{1}{p}}.$$

Note that $\mathrm{i}_{q,p}^X$ recovers diam_p^X when $p = q$.

Remark 4.4 In the construction of $\mathrm{i}_{q,p}^X$, one can get other interesting filtration functions by replacing the Wasserstein distance with other distances between measures. These include, for example, the Lévy–Prokhorov metric, the Fortet–Mourier metric and variants of the Kantorovich–Rubinstein metric. For definitions of these metrics, see [15, Section 3].

Lemma 4.5 *Let* X *and* Y *be bounded metric spaces, and let* $f : X \to Y$ *be a map. Then, for any element* $\alpha \in \mathcal{P}_X$ *and* $p \in [1, \infty]$,

$$\mathrm{i}_{q,p}^Y(f_\sharp(\alpha)) \le \mathrm{i}_{q,p}^X(\alpha) + \mathrm{dis}(f).$$

Proof We compute

$$\begin{aligned}
\mathrm{i}^{Y}_{q,p}(f_\sharp(\alpha)) &= \Big(\int_Y \big(d^{Y}_{\mathrm{W},q}(f_\sharp(\alpha),\delta_y)\big)^p\, f_\sharp(\alpha)(dy)\Big)^{\frac{1}{p}} = \Big(\int_X \big(d^{Y}_{\mathrm{W},q}(f_\sharp(\alpha),\delta_{f(x)})\big)^p\, \alpha(dx)\Big)^{\frac{1}{p}} \\
&= \Big(\int_X \Big(\int_Y d^q_Y(y,f(x))\, f_\sharp(\alpha)(dy)\Big)^{\frac{p}{q}}\, \alpha(dx)\Big)^{\frac{1}{p}} \\
&= \Big(\int_X \Big(\int_X d^q_Y(f(x'),f(x))\, \alpha(dx')\Big)^{\frac{p}{q}}\, \alpha(dx)\Big)^{\frac{1}{p}} \\
&\le \Big(\int_X \Big(\int_X (d_X(x',x)+\mathrm{dis}(f))^q\, \alpha(dx')\Big)^{\frac{p}{q}}\, \alpha(dx)\Big)^{\frac{1}{p}} \le \mathrm{i}^{X}_{q,p}(\alpha)+\mathrm{dis}(f). \qquad \square
\end{aligned}$$

Lemma 4.6 *Let X be a bounded metric space. Then, for any α,β in $\mathcal{P}_X$,*

$$|\mathrm{i}^{X}_{q,p}(\alpha)-\mathrm{i}^{X}_{q,p}(\beta)| \le d^{X}_{\mathrm{W},q}(\alpha,\beta)+d^{X}_{\mathrm{W},p}(\alpha,\beta).$$

In particular, $|\mathrm{i}^{X}_{q,p}(\alpha)-\mathrm{i}^{X}_{q,p}(\beta)| \le 2d^{X}_{\mathrm{W},\infty}(\alpha,\beta)$.

Proof We first note that

$$\begin{aligned}
\mathrm{i}^{X}_{q,p}(\alpha) &= \Big(\int_X (d^{X}_{\mathrm{W},q}(\alpha,\delta_x))^p\, \alpha(dx)\Big)^{\frac{1}{p}} \le \Big(\int_X (d^{X}_{\mathrm{W},q}(\beta,\delta_x)+d^{X}_{\mathrm{W},q}(\alpha,\beta))^p\, \alpha(dx)\Big)^{\frac{1}{p}} \\
&\le \Big(\int_X (d^{X}_{\mathrm{W},q}(\beta,\delta_x))^p\, \alpha(dx)\Big)^{\frac{1}{p}} + d^{X}_{\mathrm{W},q}(\alpha,\beta).
\end{aligned}$$

We also have

$$\begin{aligned}
&\Big|\Big(\int_X (d^{X}_{\mathrm{W},q}(\beta,\delta_x))^p\, \alpha(dx)\Big)^{\frac{1}{p}} - \mathrm{i}_{q,p}(\beta)\Big| \\
&\qquad = \Big|\Big(\int_X (d^{X}_{\mathrm{W},q}(\beta,\delta_x))^p\, \alpha(dx)\Big)^{\frac{1}{p}} - \Big(\int_X (d^{X}_{\mathrm{W},q}(\beta,\delta_x))^p\, \beta(dx)\Big)^{\frac{1}{p}}\Big| \\
&\qquad \le d^{X}_{\mathrm{W},p}(\alpha,\beta),
\end{aligned}$$

where the last line follows from Lemma 4.1 since $d^{X}_{\mathrm{W},q}(\beta,\delta_x)$ is a 1–Lipschitz function in x. Therefore,

$$\mathrm{i}_{q,p}(\alpha) \le \mathrm{i}_{q,p}(\beta)+d^{X}_{\mathrm{W},q}(\alpha,\beta)+d^{X}_{\mathrm{W},p}(\alpha,\beta).$$

We then get the result by swapping the roles of α and β. $\square$

Remark 4.7 This establishes the continuity of $\mathrm{i}_{q,p}$ for $q,p\in[1,\infty)$. If p or q equals infinity, then $\mathrm{i}_{q,p}$ is not necessarily continuous in the weak topology since $d^{X}_{\mathrm{W},\infty}$ does not necessarily induce the same topology on $\mathcal{P}_X$ as $d^{X}_{\mathrm{W},p}$ does for $p<\infty$. Indeed, if $X=[0,1]$ is the unit interval, then $((n-1)/n)\delta_0+(1/n)\delta_1\to\delta_0$ in $\mathcal{P}_X$ as $n\to\infty$, even though $\mathrm{diam}_\infty(((n-1)/n)\delta_0+(1/n)\delta_1)$ is equal to 1 for all $n\ge 1$, and hence does not converge to $0=\mathrm{diam}(\delta_0)$.

The above two lemmas imply that $\mathrm{i}_{q,p}$ (and hence $\mathrm{diam}_p=\mathrm{i}_{p.p}$) is a 1–controlled invariant. We next consider the analogous properties for rad_p.

Lemma 4.8 (rad$_p$ under a map with bounded distortion) *Let X and Y be bounded metric spaces, and let $f : X \to Y$ be a map. Then, for any $\alpha \in \mathcal{P}_X$ and $p \in [1, \infty]$,*

$$\mathrm{rad}_p(f_\sharp(\alpha)) \le \mathrm{rad}_p(\alpha) + \mathrm{dis}(f).$$

Proof We only give the proof for the case $p < \infty$; the case $p = \infty$ is similar. We have

$$\begin{aligned}
\mathrm{rad}_p(f_\sharp(\alpha)) &= \inf_{y \in Y} \Big(\int_Y d_Y^p(y, y') \, f_\sharp(\alpha)(dy') \Big)^{\frac{1}{p}} \le \inf_{y \in f(X)} \Big(\int_Y d_Y^p(y, y') \, f_\sharp(\alpha)(dy') \Big)^{\frac{1}{p}} \\
&= \inf_{x \in X} \Big(\int_Y d_Y^p(f(x), y') \, f_\sharp(\alpha)(dy') \Big)^{\frac{1}{p}} = \inf_{x \in X} \Big(\int_X d_Y^p(f(x), f(x')) \, \alpha(dx') \Big)^{\frac{1}{p}} \\
&\le \inf_{x \in X} \Big(\int_X (d_X(x, x') + \mathrm{dis}(f))^p \, \alpha(dx') \Big)^{\frac{1}{p}} \le \inf_{x \in X} \Big(\int_X d_X^p(x, x') \, \alpha(dx') \Big)^{\frac{1}{p}} + \mathrm{dis}(f) \\
&= \mathrm{rad}_p(\alpha) + \mathrm{dis}(f).
\end{aligned}$$

□

Lemma 4.9 (stability of rad$_p$) *Let X be a bounded metric space. For any two probability measures $\alpha, \beta \in \mathcal{P}_X$ and for every $p \in [1, \infty]$,*

$$|\mathrm{rad}_p(\alpha) - \mathrm{rad}_p(\beta)| \le d^X_{\mathrm{W},p}(\alpha, \beta).$$

Proof We compute

$$\begin{aligned}
\mathrm{rad}_p(\alpha) = \inf_{x \in X} d^X_{\mathrm{W},p}(\alpha, \delta_x) &\le \inf_{x \in X} (d^X_{\mathrm{W},p}(\beta, \delta_x) + d^X_{\mathrm{W},p}(\beta, \alpha)) = \inf_{x \in X} d^X_{\mathrm{W},p}(\beta, \delta_x) + d^X_{\mathrm{W},p}(\beta, \alpha) \\
&= \mathrm{rad}_p(\beta) + d^X_{\mathrm{W},p}(\beta, \alpha).
\end{aligned}$$

□

Remark 4.10 Lemma 4.9 establishes the continuity of rad$_p$ for $p \in [1, \infty)$, as these functions are 1–Lipschitz. Similarly to diam$_\infty$, we note that rad$_\infty$ is not necessarily continuous because the metric topology given by $d^X_{\mathrm{W},\infty}$ is not necessarily equal to the weak topology.

The above two lemmas imply that rad$_p$ is a 1–controlled invariant.

We end this section of basic properties by showing that $\mathcal{P}_X$ is homeomorphic to a simplex when X is finite. Hence for X finite, the $p = \infty$ metric thickenings $\mathrm{VR}_\infty(X; r)$ and $\check{\mathrm{C}}_\infty(X; r)$ are homeomorphic to the simplicial complexes $\mathrm{VR}(X; r)$ and $\check{\mathrm{C}}(X; r)$, respectively; see also [2, Corollary 6.4].

Lemma 4.11 *If X is a finite metric space with n points, then $\mathcal{P}_X$ is homeomorphic to the standard $(n-1)$–simplex.*

Proof Let $X = \{x_1, \dots, x_n\}$. The space $\mathcal{P}_X$ of probability measures is in bijection with the standard $(n-1)$–simplex $\Delta_{n-1} = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n y_i = 1 \text{ with } y_i \ge 0 \text{ for all } i\}$ via the function $f : \mathcal{P}_X \to \Delta_{n-1}$ defined by $f\big(\sum_{i=1}^n a_i \delta_{x_i}\big) = (a_1, \dots, a_n)$. Suppose we have a sequence $\{\alpha_k\}$ in $\mathcal{P}_X$ given by $\alpha_k = \sum_{i=1}^n a_{k,i} \delta_{x_i}$. By the definition of weak convergence, $\{\alpha_k\}$ converges to $\alpha = \sum_{i=1}^n a_i \delta_{x_i}$ in $\mathcal{P}_X$ if and only if $\int_X \phi(x) \, \alpha_k(dx)$ converges to $\int_X \phi(x) \, \alpha(dx)$ for all bounded and continuous functions $\phi \colon X \to \mathbb{R}$. These integrals are equal to $\sum_{i=1}^n a_{k,i} \phi(x_i)$ and $\sum_{i=1}^n a_i \phi(x_i)$, respectively, so $\{\alpha_k\}$

converges to α if and only if $\lim_{k\to\infty} a_{k,i} = a_i$ for each i. Therefore $\{\alpha_k\}$ converges to α in $\mathcal{P}_X$ if and only if $\{f(\alpha_k)\}$ converges to $f(\alpha)$ in Δ_{n-1}. □

In general, the $p=\infty$ metric thickenings $\mathrm{VR}_\infty(X;r)$ and $\check{\mathrm{C}}_\infty(X;r)$ of a finite metric space X are in bijection, as sets, with the geometric realizations of the usual Vietoris–Rips and Čech simplicial complexes on X, via the natural bijection $\sum_{i=1}^n a_i\delta_{x_i} \mapsto \sum_{i=1}^n a_i x_i$, where $\sum_{i=1}^n a_i x_i$ denotes the formal sum in the geometric realization. Therefore Lemma 4.11 implies the following; see also [2, Proposition 6.6].

Lemma 4.12 *For a finite metric space X and any $r>0$, we have homeomorphisms* $\mathrm{VR}_\infty(X;r) \cong \mathrm{VR}(X;r)$ *and* $\check{\mathrm{C}}_\infty(X;r) \cong \check{\mathrm{C}}(X;r)$.

5 Stability

In this section we establish the stability of all the filtrations we have introduced, under reasonable assumptions on the metric spaces. The proof requires new techniques in order to construct maps that adequately compare two filtrations of metric thickenings. In the proof of stability for simplicial complexes [30], simplicial maps are used to construct maps between filtrations of simplicial complexes, and the shift in parameter can be bounded using the Gromov–Hausdorff distance. A naive attempt to apply this technique to filtrations of metric thickenings fails because there is no analogue of simplicial maps for metric thickenings. Indeed, if X and Y are metric spaces that are close in the Gromov–Hausdorff distance, then we get a map $f: X\to Y$ that need not be continuous, but which is of bounded distortion. This induces a *continuous* map $\mathrm{VR}(X;r)\to\mathrm{VR}(Y;r')$ between simplicial complexes, as long as r' is chosen to be large enough, and from there one can obtain interleavings. However, the analogous map $\mathrm{VR}_p(X;r)\to\mathrm{VR}_p(Y;r')$ between metric thickenings cannot be continuous if $f: X\to Y$ is not continuous, since there are natural isometric embeddings of X and Y into $\mathrm{VR}_p(X;r)$ and $\mathrm{VR}_p(Y;r')$; see Proposition 3.8. In essence, the fact that metric thickenings have a more well-behaved topology means that it is more difficult to construct interleavings between them.

To overcome this difficulty, we instead construct continuous functions between metric thickenings by distributing mass according to finite partitions of unity, subordinate to open coverings by δ–balls. This ensures that these maps distort distances in a controlled way, allowing for a controlled change in the invariants defining the filtration and thus providing an interleaving. Creating these maps and checking their properties will require many of the ideas from the previous sections.

In Section 5.1 we state our main results: Theorem A and its immediate consequence, Theorem B. We give two applications of stability. First, Section 5.2 applies the stability theorem in order to show that various persistence modules of interest are Q–tame, and therefore have persistence diagrams. Next, in Section 5.3 we apply stability to show the close relationship between ∞–metric thickenings and simplicial complexes, generalizing and answering in the affirmative [2, Conjecture 6.12], which states that these filtrations have identical persistence diagrams. We give the proof of the stability theorem, Theorem A, in Section 5.4.

5.1 Statement of the stability theorem

Let X and Y be totally bounded metric spaces, let $p \in [1,\infty]$ and let $k \geq 0$. We will see, for example, that we have the following stability bound:

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(X;\cdot), H_k \circ \mathrm{VR}_p(Y;\cdot)) \leq 2d_{\mathrm{GH}}(X,Y). \tag{4}$$

This implies that if two metric spaces are close in the Gromov–Hausdorff distance, then their resulting filtrations and persistence modules are also close.

Example 5.1 Consider the metric space Z_{n+1} from Example 3.11, which has $n+1$ points all at interpoint distance 1 apart. Let $\zeta_{p,k}$ be the interleaving distance between persistent homology modules

$$\zeta_{p,k} := d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(Z_{n+1};\cdot), H_k \circ \mathrm{VR}_p(Z_{m+1};\cdot)) \quad \text{for } n \neq m.$$

Notice that, from Example 3.11, when $n \neq m$ and p is finite,

$$\zeta_{p,0} = \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{1/p} \quad \text{and} \quad \zeta_{p,1} = \tfrac{1}{2}\left(\left(\tfrac{2}{3}\right)^{1/p} - \left(\tfrac{1}{2}\right)^{1/p}\right).$$

However, when $p = \infty$, we have $\zeta_{\infty,0} = \frac{1}{2}$ and $\zeta_{\infty,k} = 0$ for all $k \geq 1$.

From these calculations we can make the following observations:

(i) For p infinite, the only value of k for which $\zeta_{\infty,k} \neq 0$ is $k = 0$.

(ii) For p finite, $\zeta_{p,1} > 0$.

(iii) For p finite, $\sup_k \zeta_{\infty,k} = \frac{1}{2} > \frac{1}{2}\left(\frac{1}{2}\right)^{1/p} = \sup_k \zeta_{p,k}$.

From items (i) and (ii) we can see that, whereas persistence diagrams for $k \geq 1$ of $p = \infty$ thickenings do not contain discriminative information for the Z_n spaces, in contrast, the analogous quantities for p finite do absorb useful information.

Since it is known (see [56, Example 4.1]) that $d_{\mathrm{GH}}(Z_{n+1}, Z_{m+1}) = \frac{1}{2}$ whenever $n \neq m$, item (iii) suggests that the lower bound for Gromov–Hausdorff given by (4) may not be tight for p finite. This phenomenon can actually be explained by the more general theorem below, which identifies a certain pseudometric between filtrations which mediates between the two terms appearing in (4).

Theorem A *Let $\mathfrak{i}$ be an L–controlled invariant. Then, for any two totally bounded metric spaces X and Y and any integer $k \geq 0$,*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X;\cdot], H_k \circ [\mathcal{P}_Y, \mathfrak{i}^Y;\cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_Y, \mathfrak{i}^Y)) \leq 2L\, d_{\mathrm{GH}}(X,Y),$$
$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X;\cdot], H_k \circ [\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}^Y;\cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X), (\mathcal{P}_Y^{\mathrm{fin}}, \mathfrak{i}^Y)) \leq 2L\, d_{\mathrm{GH}}(X,Y).$$

Note that there are instances when the quantity in the middle vanishes, yet the spaces X and Y are nonisometric; see Section A.6 for results about this in terms of the notion of *crushing* considered by Hausmann [42] and [2; 58].

Corollary 5.2 *For any two totally bounded metric spaces X and Y, for any $q, p \in [1,\infty]$, and for any integer $k \geq 0$, we have*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X_{q,p}; \cdot], H_k \circ [\mathcal{P}_Y, \mathfrak{i}^Y_{q,p}; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X_{q,p}), (\mathcal{P}_Y, \mathfrak{i}^Y_{q,p})) \leq 2 d_{\mathrm{GH}}(X,Y),$$
$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X_{q,p}; \cdot], H_k \circ [\mathcal{P}^{\mathrm{fin}}_Y, \mathfrak{i}^Y_{q,p}; \cdot]) \leq d_{\mathrm{HT}}((\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X_{q,p}), (\mathcal{P}^{\mathrm{fin}}_Y, \mathfrak{i}^Y_{q,p})) \leq 2 d_{\mathrm{GH}}(X,Y).$$

Proof From Lemmas 4.5 and 4.6 we know $\mathfrak{i}_{q,p}$ is a 1–controlled invariant; we then apply Theorem A. □

Theorem B *Let X and Y be totally bounded metric spaces, let $p \in [1,\infty]$ and let $k \geq 0$ be an integer. Then the k^{th} persistent homology of $\mathrm{VR}_p(X;\cdot)$ and $\mathrm{VR}_p(Y;\cdot)$ are ε–interleaved for any $\varepsilon \geq d_{\mathrm{GH}}(X,Y)$, and similarly for $\check{\mathrm{C}}_p(X;\cdot)$ and $\check{\mathrm{C}}_p(Y;\cdot)$:*

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p(X;\cdot), H_k \circ \mathrm{VR}_p(Y;\cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{diam}^X_p), (\mathcal{P}_Y, \mathrm{diam}^Y_p)) \leq 2 d_{\mathrm{GH}}(X,Y),$$
$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}^{\mathrm{fin}}_p(X;\cdot), H_k \circ \mathrm{VR}^{\mathrm{fin}}_p(Y;\cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}^{\mathrm{fin}}_X, \mathrm{diam}^X_p), (\mathcal{P}^{\mathrm{fin}}_Y, \mathrm{diam}^Y_p)) \leq 2 d_{\mathrm{GH}}(X,Y),$$
$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_p(X;\cdot), H_k \circ \check{\mathrm{C}}_p(Y;\cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{rad}^X_p), (\mathcal{P}_Y, \mathrm{rad}^Y_p)) \leq 2 d_{\mathrm{GH}}(X,Y),$$
$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}^{\mathrm{fin}}_p(X;\cdot), H_k \circ \check{\mathrm{C}}^{\mathrm{fin}}_p(Y;\cdot)) \leq d_{\mathrm{HT}}((\mathcal{P}^{\mathrm{fin}}_X, \mathrm{rad}^X_p), (\mathcal{P}^{\mathrm{fin}}_Y, \mathrm{rad}^Y_p)) \leq 2 d_{\mathrm{GH}}(X,Y).$$

Proof The p–Vietoris–Rips case follows from Corollary 5.2 by letting $q = p$, in which case $\mathfrak{i}^X_{p,p} = \mathrm{diam}^X_p$ (see Definition 4.3). The p–Čech case follows from Theorem A since rad_p is a 1–controlled invariant by Lemmas 4.8 and 4.9. □

In the corollary below, we use Theorem A to show that a controlled invariant on all Radon probability measures produces a filtration at interleaving distance zero from that same invariant restricted to only finitely supported measures.

Corollary 5.3 *Let X be a totally bounded metric space, let $k \geq 0$ be an integer and let $\mathfrak{i}$ be an L–controlled invariant. Then*

$$d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X; \cdot]) = d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X)) = 0.$$

Proof For any $\varepsilon > 0$, let U be an ε–net in X. Then the stability theorem, Theorem A, shows

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_U, \mathfrak{i}^U)) \leq 2L\, d_{\mathrm{GH}}(X,U) \leq 2L\varepsilon, \quad d_{\mathrm{HT}}((\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X), (\mathcal{P}^{\mathrm{fin}}_U, \mathfrak{i}^U)) \leq 2L\, d_{\mathrm{GH}}(X,U) \leq 2L\varepsilon.$$

Note, $(\mathcal{P}_U, \mathfrak{i}^U) = (\mathcal{P}^{\mathrm{fin}}_U, \mathfrak{i}^U)$ since U is finite. Since d_{HT} satisfies the triangle inequality (Proposition 2.8),

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X)) \leq 4L\varepsilon.$$

By letting ε go to zero, we see $d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X)) = 0$. It then follows from Lemma 2.9 that $d_{\mathrm{I}}(H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot], H_k \circ [\mathcal{P}^{\mathrm{fin}}_X, \mathfrak{i}^X; \cdot]) = 0$. □

5.2 Consequence of stability: tameness

We next determine conditions under which a persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ associated with a controlled invariant $\mathfrak{i}$ will be Q–tame. In particular, we apply the results to the Vietoris–Rips and Čech metric thickenings.

Corollary 5.4 *Let $\mathfrak{i}$ be a controlled invariant and let $k \geq 0$ be an integer. Suppose the persistence module $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ (resp. $H_k \circ [\mathcal{P}_V^{\mathrm{fin}}, \mathfrak{i}^V; \cdot]$) is Q–tame for any finite metric space V. Then, for any totally bounded metric space X, the persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ (resp. $H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$) is Q–tame.*

Proof Since X is totally bounded, Theorem A implies that the persistence module $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ can be approximated arbitrarily well in the interleaving distance by the Q–tame persistence modules on finite ε–nets V_ε as ε goes to zero. Then the result follows from Lemma 2.5. □

Now, we give a sufficient condition for the Q–tameness of the persistence module $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ over a finite metric space V.

Corollary 5.5 *Suppose $\mathfrak{i}$ is a controlled invariant such that, for any finite metric space V, $\mathfrak{i}^V$ is a continuous function on $\mathcal{P}_V$. Then, for any totally bounded metric space X, the persistence modules $H_k \circ [\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ and $H_k \circ [\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; \cdot]$ are Q–tame for any integer $k \geq 0$.*

Proof According to Corollary 5.4, it suffices to check Q–tameness over finite metric spaces. If V is a finite metric space, we may identify $\mathcal{P}_V$ with a simplex by Lemma 4.11. Then $[\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ is a sublevel set filtration on a simplex, and since $\mathfrak{i}^V$ is continuous, [29, Theorem 2.22] shows $H_k \circ [\mathcal{P}_V, \mathfrak{i}^V; \cdot]$ is Q–tame. □

Here we summarize the Q–tameness results related to relaxed Vietoris–Rips and Čech metric thickenings.

Corollary 5.6 *Let X be a totally bounded metric space, let $p \in [1, \infty]$ and let $k \geq 0$ be an integer. The persistence modules $H_k \circ \mathrm{VR}_p(X; \cdot)$, $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot)$ are Q–tame.*

Proof For $H_k \circ \mathrm{VR}_p(X; \cdot)$, when p is finite, Lemma 4.6 implies diam_p is a continuous function over any $\mathcal{P}_X$ where X is a bounded metric space. Then we get the result by applying Corollary 5.5. When $p = \infty$, Lemma 4.12 shows the persistent homology associated with diam_∞ is Q–tame over finite metric spaces. We then get the result by applying Corollary 5.4.

We obtain the case of $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$ from $H_k \circ \mathrm{VR}_p(X; \cdot)$ by using the interleaving distance result in Corollary 5.3 along with the Q–tame approximation result in Lemma 2.5.

For $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$, when p is finite, Lemma 4.9 implies rad_p is a continuous function over any $\mathcal{P}_X$ where X is a bounded metric space. The rest is similar to the Vietoris–Rips case. □

Remark 5.7 The proof that $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ is Q–tame can be made more direct at one step. For U finite, to see that $H_k \circ \check{\mathrm{C}}_p(U; \cdot)$ is Q–tame, one could appeal to Theorem F in Section A.3 instead of Theorem 2.22 of [29].

The Q–tame persistence modules given by this theorem allow us to discuss persistence diagrams, using the results of [29].

Corollary 5.8 *Let X be a totally bounded metric space, let $p \in [1, \infty]$ and let $k \geq 0$ be an integer. Then $H_k \circ \mathrm{VR}_p(X; \cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X; \cdot)$ have the same persistence diagram, $\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X)$. Similarly, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X; \cdot)$ have the same persistence diagram, $\mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(X)$.*

Proof Persistence diagrams are well defined for Q–tame persistence modules, so for any totally bounded metric space X, any $p \in [1,\infty]$ and any $k \geq 0$, by Corollary 5.6 we have persistence diagrams associated to $H_k \circ \mathrm{VR}_p(X;\cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X;\cdot)$. From Corollary 5.3 we know that the interleaving distance between $H_k \circ \mathrm{VR}_p(X;\cdot)$ and $H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X;\cdot)$ is zero, and so the isometry theorem [29, Theorem 4.11] implies that these persistence modules have the same (undecorated) persistence diagram, denoted by $\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X)$. The same proof also works for Čech metric thickenings. □

Combining the isometry theorem [29, Theorem 4.11] with Theorem B and Corollary 5.6, we obtain the following:

Corollary 5.9 *If X and Y are totally bounded metric spaces, then for any $p \in [1,\infty]$ and integer $k \geq 0$,*

$$d_{\mathrm{B}}(\mathrm{dgm}_{k,p}^{\mathrm{VR}}(X), \mathrm{dgm}_{k,p}^{\mathrm{VR}}(Y)) \leq 2d_{\mathrm{GH}}(X,Y) \quad \text{and} \quad d_{\mathrm{B}}(\mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(X), \mathrm{dgm}_{k,p}^{\check{\mathrm{C}}}(Y)) \leq 2d_{\mathrm{GH}}(X,Y).$$

5.3 Consequence of stability: connecting ∞–metric thickenings and simplicial complexes

We show how the ∞–Vietoris–Rips and ∞–Čech metric thickenings recover the persistent homology of the Vietoris–Rips and Čech simplicial complexes. Our Corollary 5.10 answers [2, Conjecture 6.12] in the affirmative.

We recall that $\mathrm{VR}_p(X;r)$ denotes the p–Vietoris–Rips metric thickening, that $\mathrm{VR}_p^{\mathrm{fin}}(X;r)$ denotes the p–Vietoris–Rips metric thickening for measures of finite support, and that $\mathrm{VR}(X;r)$ denotes the Vietoris–Rips simplicial complex. Theorem B shows, for any $p \in [1,\infty]$, any $\delta > 0$ and any finite δ–net U_δ of a totally bounded metric space X, that

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(X;\cdot), H_k \circ \mathrm{VR}_p^{\mathrm{fin}}(U_\delta;\cdot)) \leq 2\delta.$$

For $p=\infty$, by Lemma 4.12, $H_k \circ \mathrm{VR}_\infty(U_\delta;\cdot) \cong H_k \circ \mathrm{VR}(U_\delta,\cdot)$, so from the above we have

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X;\cdot), H_k \circ \mathrm{VR}(U_\delta,\cdot)) \leq 2\delta.$$

Now, by the triangle inequality for the interleaving distance, by the inequality above and by the Gromov–Hausdorff stability of $X \mapsto H_k \circ \mathrm{VR}(X,\cdot)$ [30; 27],

$$\begin{aligned} &d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X;\cdot), H_k \circ \mathrm{VR}(X,\cdot)) \\ &\qquad \leq d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X;\cdot), H_k \circ \mathrm{VR}(U_\delta,\cdot)) + d_{\mathrm{I}}(H_k \circ \mathrm{VR}(U_\delta,\cdot), H_k \circ \mathrm{VR}(X,\cdot)) \\ &\qquad \leq 4\delta. \end{aligned}$$

Since this holds for any $\delta > 0$, we find $d_{\mathrm{I}}(H_k \circ \mathrm{VR}_\infty^{\mathrm{fin}}(X;\cdot), H_k \circ \mathrm{VR}(X,\cdot)) = 0$. This implies that the bottleneck distance between persistence diagrams is 0, and the (undecorated) diagrams are in fact equal (see for instance [29, Theorem 4.20]). We can apply Corollary 5.3 to get the same result for $\mathrm{VR}_\infty(X;\cdot)$ (measures with infinite support), and the same proof works in the Čech case. We state this as the following theorem:

Corollary 5.10 *For any totally bounded metric space X and any integer $k \geq 0$,*

$$d_{\mathrm{I}}(H_k \circ \mathrm{VR}^{\mathrm{fin}}_{\infty}(X;\cdot), H_k \circ \mathrm{VR}(X,\cdot)) = 0, \quad d_{\mathrm{I}}(H_k \circ \mathrm{VR}_{\infty}(X;\cdot), H_k \circ \mathrm{VR}(X,\cdot)) = 0,$$
$$\mathrm{dgm}^{\mathrm{VR}}_{k,\infty}(X) = \mathrm{dgm}^{\mathrm{VR}}_{k}(X),$$
$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}^{\mathrm{fin}}_{\infty}(X;\cdot), H_k \circ \check{\mathrm{C}}(X;\cdot)) = 0, \quad d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_{\infty}(X;\cdot), H_k \circ \check{\mathrm{C}}(X;\cdot)) = 0,$$
$$\mathrm{dgm}^{\check{\mathrm{C}}}_{k,\infty}(X) = \mathrm{dgm}^{\check{\mathrm{C}}}_{k}(X).$$

Remark 5.11 Whereas the ∞–metric thickenings $\mathrm{VR}_{\infty}(X;\cdot)$ and the simplicial complexes $\mathrm{VR}(X,\cdot)$ yield persistence modules with an interleaving distance of 0, we are in the interesting landscape where the metric thickenings $\mathrm{VR}_p(X;\cdot)$ may yield something new and different for $p < \infty$. For example, in Section 7 we explore the new persistence modules that arise for p–metric thickenings of Euclidean spheres $(\mathbb{S}^n, \ell_2)$ in the case $p = 2$.

Example 5.12 Let $X = \mathbb{S}^1$ be the circle. The persistent homology diagrams of the simplicial complex filtrations $\mathrm{VR}(\mathbb{S}^1,\cdot)$ and $\check{\mathrm{C}}(\mathbb{S}^1;\cdot)$ are known from [1], and therefore Corollary 5.10 gives the persistent homology diagrams of $\mathrm{VR}_{\infty}(\mathbb{S}^1;\cdot)$ and $\check{\mathrm{C}}_{\infty}(\mathbb{S}^1;\cdot)$. However, although $\mathrm{VR}(\mathbb{S}^1;r)$ and $\check{\mathrm{C}}(\mathbb{S}^1;r)$ are known to obtain the homotopy types of all odd-dimensional spheres as r increases, the homotopy types of $\mathrm{VR}_{\infty}(\mathbb{S}^1;r)$ and $\check{\mathrm{C}}_{\infty}(\mathbb{S}^1;r)$ are still not proven.

Question 5.13 Is the simplicial complex $\mathrm{VR}(X;r)$ always homotopy equivalent to $\mathrm{VR}^{\mathrm{fin}}_{\infty}(X;r)$ or $\mathrm{VR}_{\infty}(X;r)$? Compare with [2, Remark 3.3].

Question 5.14 Theorem 5.2 of [52] states that, for X compact, bars in the persistent homology of the simplicial complex filtration $\mathrm{VR}(X,\cdot)$ are of the form $(a,b]$ or (a,∞). Is the same true for $\mathrm{VR}_{\infty}(X;\cdot)$? Note that we are using the $<$ convention, ie a simplex is included in $\mathrm{VR}(X;r)$ when its diameter is strictly less than r, and a measure is included in $\mathrm{VR}_{\infty}(X;r)$ when the diameter of its support is strictly less than r.

5.4 The proof of stability

As mentioned above, the construction of interleaving maps between filtrations of metric thickenings is more intricate than in the case of simplicial complexes. A main idea behind the proof of our stability result, Theorem A, is to approximate the metric space X by a finite subspace (a net) U. The advantage is that it is easier to construct a continuous map with domain $\mathcal{P}_U$ than one with domain $\mathcal{P}_X$. The crucial next step is the construction of a continuous map from $\mathcal{P}_X$ to $\mathcal{P}_U$ by using a partition of unity subordinate to an open covering by δ–balls. This map distorts distances by a controlled amount, allowing us to approximate measures of $\mathcal{P}_X$ in $\mathcal{P}_U$.

We begin with some necessary lemmas.

Lemma 5.15 *Let U be a finite δ–net of a bounded metric space X, and let $\{\zeta^U_u\}_{u\in U}$ be a continuous partition of unity subordinate to the open covering $\bigcup_{u\in U} B(u;\delta)$ of X.*

We have the continuous map $\Phi_U : \mathcal{P}_X \to \mathcal{P}_U$ defined by

$$\alpha \mapsto \sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_u.$$

For any $q \in [1,\infty]$, we have $d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) < \delta$.

Proof The map Φ_U is well-defined, as

$$\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx) = \int_X \sum_{u\in U} \zeta_u^U(x)\,\alpha(dx) = \int_X \alpha(dx) = 1.$$

For the continuity, since the weak topology on $\mathcal{P}_X$ and $\mathcal{P}_U$ can be metrized, it suffices to show, for a weakly convergent sequence $\alpha_n \in \mathcal{P}_X$, that the image $\Phi_U(\alpha_n)$ is also weakly convergent. As U is a finite metric space, it suffices to show that, for any fixed $u_0 \in U$, the sequence of real numbers $(\Phi_U(\alpha_n)(\{u_0\}))_n$ is itself convergent. Note that

$$\Phi_U(\alpha_n)(\{u_0\}) = \int_X \zeta_{u_0}^U(x)\,\alpha_n(dx).$$

Since $\zeta_{u_0}^U$ is a bounded continuous function on X, we obtain the desired convergence through the weak convergence of α_n.

Lastly, we must show that $d_{\mathrm{W},q}^X(\alpha, \Phi_U(\alpha)) < \delta$. For any $u \in U$, we use the notation $w_u^\alpha := \int_X \zeta_u^U(x)\,\alpha(dx)$, so that $\Phi_U(\alpha) = \sum_{u\in U} w_u^\alpha\,\delta_u$ and

$$\sum_{u\in U} w_u^\alpha = \sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx) = \int_X \sum_u \zeta_u^U(x)\,\alpha(dx) = \int_X \alpha(dx) = 1.$$

Let $\zeta_u^U\alpha$ denote the measure such that $\zeta_u^U\alpha(B) = \int_B \zeta_u^U(x)\,\alpha(dx)$ for any measurable set $B \subseteq X$. We have

$$\alpha = \sum_{u\in U} \zeta_u^U\alpha = \sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}} w_u^\alpha\Big(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big).$$

From this we get

$$d_{\mathrm{W},q}^X(\alpha,\Phi_U(\alpha)) = d_{\mathrm{W},q}^X\Bigg(\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}} w_u^\alpha\Big(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big), \sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}} w_u^\alpha\,\delta_u\Bigg) \le \Bigg(\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}} w_u^\alpha\Big(d_{\mathrm{W},q}^X\Big(\frac{1}{w_u^\alpha}\zeta_u^U\alpha,\delta_u\Big)\Big)^q\Bigg)^{\frac{1}{q}} < \delta.$$

The first inequality is by Lemma 4.2. The last one comes from the fact that each $(1/w_u^\alpha)\zeta_u^U\alpha$ is a probability measure supported in $B(u;\delta)$, and thus $d_{\mathrm{W},q}^X((1/w_u^\alpha)\zeta_u^U\alpha, \delta_u) < \delta$. □

For any two totally bounded metric spaces X and Y, through partitions of unity we can build continuous maps between $\mathcal{P}_X$ and $\mathcal{P}_Y$, as follows. Let X and Y be two totally bounded metric spaces, let $\eta > 2d_{\mathrm{GH}}(X,Y)$ and let $\delta > 0$. We fix finite δ–nets $U \subseteq X$ of X and $V \subseteq Y$ of Y. By the triangle inequality, $d_{\mathrm{GH}}(U,V) < \frac{1}{2}\eta + 2\delta$. So there exist maps $\varphi : U \to V$ and $\psi : V \to U$ with

$$\max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi,\psi)) \le \eta + 4\delta.$$

We then define the maps $\widehat{\Phi}\colon \mathcal{P}_X \to \mathcal{P}_Y$ and $\widehat{\Psi}\colon \mathcal{P}_Y \to \mathcal{P}_X$ to be

$$\widehat{\Phi} := \iota_V \circ \varphi_\sharp \circ \Phi_U \quad \text{and} \quad \widehat{\Psi} := \iota_U \circ \psi_\sharp \circ \Phi_V,$$

where $\iota_U\colon \mathcal{P}_U \hookrightarrow \mathcal{P}_X$ and $\iota_V\colon \mathcal{P}_V \hookrightarrow \mathcal{P}_Y$ are inclusions and Φ_U and Φ_V are the maps defined in Lemma 5.15:

$$\begin{tikzcd} \mathcal{P}_X \arrow[r, dotted, "\widehat{\Phi}"] \arrow[d, "\Phi_U"'] & \mathcal{P}_Y \\ \mathcal{P}_U \arrow[r, "\varphi_\sharp"] & \mathcal{P}_V \arrow[u, hook, "\iota_V"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_Y \arrow[r, dotted, "\widehat{\Psi}"] \arrow[d, "\Phi_V"'] & \mathcal{P}_X \\ \mathcal{P}_V \arrow[r, "\psi_\sharp"] & \mathcal{P}_U \arrow[u, hook, "\iota_U"'] \end{tikzcd}$$

Then both $\widehat{\Phi}$ and $\widehat{\Psi}$ are continuous maps by Lemma 5.15 and by the continuity of pushforwards of continuous maps.

The use of the finite δ–nets U and V is important, because they can be compared by continuous maps φ and ψ, which yield continuous maps $\varphi_\sharp$ and $\psi_\sharp$. The following lemma shows that $\widehat{\Phi}$ and $\widehat{\Psi}$ are homotopy equivalences.

Lemma 5.16 *With the above notation, we have homotopy equivalences $\widehat{\Psi} \circ \widehat{\Phi} \simeq \mathrm{id}_{\mathcal{P}_X}$ and $\widehat{\Phi} \circ \widehat{\Psi} \simeq \mathrm{id}_{\mathcal{P}_Y}$ via the linear families $H_t^X : \mathcal{P}_X \times [0,1] \to \mathcal{P}_X$ and $H_t^Y : \mathcal{P}_Y \times [0,1] \to \mathcal{P}_Y$ given by*

$$H_t^X(\alpha) := (1-t)\widehat{\Psi} \circ \widehat{\Phi}(\alpha) + t\alpha \quad \text{and} \quad H_t^Y(\beta) := (1-t)\widehat{\Phi} \circ \widehat{\Psi}(\beta) + t\beta.$$

Moreover, $d_{\mathrm{W},q}^X(H_t^X(\alpha), \alpha) < \eta + 6\delta$ and $d_{\mathrm{W},q}^Y(H_t^Y(\beta), \beta) < \eta + 6\delta$ for any $q \in [1,\infty]$ and any $t \in [0,1]$.

Proof The homotopies H_t^X and H_t^Y are continuous by Proposition 2.4 since $\widehat{\Phi}$ and $\widehat{\Psi}$ are continuous. We will only present the estimate $d_{\mathrm{W},q}^X(H_t^X(\alpha), \alpha) < \eta + 6\delta$, as the other inequality can be proved in a similar way. We first calculate the expression for $\widehat{\Psi} \circ \widehat{\Phi}(\alpha)$, obtaining

$$\begin{aligned}
\widehat{\Psi} \circ \widehat{\Phi}(\alpha) &= \psi_\sharp(\Phi_V(\varphi_\sharp(\Phi_U(\alpha)))) = \psi_\sharp\Big(\Phi_V\Big(\varphi_\sharp\Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_u\Big)\Big)\Big) \\
&= \psi_\sharp\Big(\Phi_V\Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_{\varphi(u)}\Big)\Big) \\
&= \psi_\sharp\Big(\sum_{v\in V} \int_Y \zeta_v^V(y)\Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\,\delta_{\varphi(u)}\Big)(dy)\,\delta_v\Big) \\
&= \psi_\sharp\Big(\sum_{v\in V} \Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx) \int_Y \zeta_v^V(y)\,\delta_{\varphi(u)}(dy)\Big)\,\delta_v\Big) \\
&= \psi_\sharp\Big(\sum_{v\in V} \Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\zeta_v^V(\varphi(u))\Big)\,\delta_v\Big) \\
&= \sum_{v\in V} \Big(\sum_{u\in U} \int_X \zeta_u^U(x)\,\alpha(dx)\zeta_v^V(\varphi(u))\Big)\,\delta_{\psi(v)} = \sum_{v\in V} \Big(\sum_{u\in U} w_u^\alpha \zeta_v^V(\varphi(u))\Big)\,\delta_{\psi(v)},
\end{aligned}$$

where we again use the notation $w_u^\alpha := \int_X \zeta_u^U(x)\,\alpha(dx)$. We have

$$H_t^X(\alpha) = (1-t) \sum_{v\in V} \sum_{u\in U} \big(w_u^\alpha \zeta_v^V(\varphi(u))\big)\,\delta_{\psi(v)} + t\alpha.$$

Now we show that $H_t^X(\alpha)$ is a convex combination of probability measures in $\mathcal{P}_X$ and hence itself lies in $\mathcal{P}_X$. That is, the sum of coefficients in front of all $\delta_{\psi(v)}$ and α in the above formula for H_t^X is 1:

$$(1-t)\sum_{v\in V}\sum_{u\in U}\big(w_u^\alpha \zeta_v^V(\varphi(u))\big)+t=(1-t)\sum_{u\in U} w_u^\alpha+t=1.$$

Next, we will bound the q–Wasserstein distance $d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)$. For this, we first rewrite α as

$$\begin{aligned}\alpha&=(1-t)\sum_{u\in U}\zeta_u^U\alpha+t\alpha=(1-t)\sum_{v\in V}\sum_{u\in U}\zeta_v^V(\varphi(u))(\zeta_u^U\alpha)+t\alpha\\&=(1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}}\big(w_u^\alpha\zeta_v^V(\varphi(u))\big)\Big(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big)+t\alpha.\end{aligned}$$

Based on these observations, we then apply the inequality from Lemma 4.2 to obtain

$$\begin{aligned}&d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)\\&\quad=d_{\mathrm{W},q}^X\Big((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}}\big(w_u^\alpha\zeta_v^V(\varphi(u))\big)\,\delta_{\psi(v)}+t\alpha,\,(1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}}\big(w_u^\alpha\zeta_v^V(\varphi(u))\big)\Big(\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big)+t\alpha\Big)\\&\quad\le\Big((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}}\big(w_u^\alpha\zeta_v^V(\varphi(u))\big)\Big(d_{\mathrm{W},q}^X\Big(\delta_{\psi(v)},\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big)\Big)^p\Big)^{\frac{1}{p}}\\&\quad\le\Big((1-t)\sum_{v\in V}\sum_{\substack{u\in U,\\ w_u^\alpha\neq 0}}\big(w_u^\alpha\zeta_v^V(\varphi(u))\big)\Big(d_{\mathrm{W},q}^X(\delta_{\psi(v)},\delta_u)+d_{\mathrm{W},q}^X\Big(\delta_u,\frac{1}{w_u^\alpha}\zeta_u^U\alpha\Big)\Big)^p\Big)^{\frac{1}{p}}.\end{aligned}$$

For each nonzero term in the summand of the last expression we have $d_Y(v,\varphi(u))<\delta$, which implies $d_X(\psi(v),u)<5\delta+\eta$ via the codistortion assumption. Therefore $d_{\mathrm{W},q}^X(\delta_{\psi(v)},\delta_u)<5\delta+\eta$ as well. Moreover, since $(1/w_u^\alpha)\zeta_u^U\alpha$ is a probability measure supported in the δ–ball centered at the point u, we have $d_{\mathrm{W},q}^X(\delta_u,(1/w_u^\alpha)\zeta_u^U\alpha)<\delta$. Therefore $d_{\mathrm{W},q}^X(H_t^X(\alpha),\alpha)<6\delta+\eta$. □

Remark 5.17 The above lemma still holds if we replace $\mathcal{P}_X$ by $\mathcal{P}_X^{\mathrm{fin}}$ and $\mathcal{P}_Y$ by $\mathcal{P}_Y^{\mathrm{fin}}$, as all related maps naturally restrict to the set of finitely supported measures, $\mathcal{P}_X^{\mathrm{fin}}$.

We are now ready to prove our main result, Theorem A.

Proof of Theorem A For i an L–controlled invariant, for totally bounded metric spaces X and Y, and for any integer $k\ge 0$, we must show

$$\begin{aligned}d_{\mathrm{I}}(H_k\circ[\mathcal{P}_X,\mathrm{i}^X;\cdot],H_k\circ[\mathcal{P}_Y,\mathrm{i}^Y;\cdot])&\le d_{\mathrm{HT}}((\mathcal{P}_X,\mathrm{i}^X),(\mathcal{P}_Y,\mathrm{i}^Y))\le 2L\,d_{\mathrm{GH}}(X,Y),\\ d_{\mathrm{I}}(H_k\circ[\mathcal{P}_X^{\mathrm{fin}},\mathrm{i}^X;\cdot],H_k\circ[\mathcal{P}_Y^{\mathrm{fin}},\mathrm{i}^Y;\cdot])&\le d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}},\mathrm{i}^X),(\mathcal{P}_Y^{\mathrm{fin}},\mathrm{i}^Y))\le 2L\,d_{\mathrm{GH}}(X,Y).\end{aligned}$$

We prove only the first line above, as the finitely supported case in the second line has a nearly identical proof. The inequality involving d_{I} and d_{HT} follows from Lemma 2.9. Hence it suffices to prove the inequality $d_{\mathrm{HT}}((\mathcal{P}_X,\mathrm{i}^X),(\mathcal{P}_Y,\mathrm{i}^Y))\le 2L\,d_{\mathrm{GH}}(X,Y)$ involving d_{HT} and d_{GH}.

Following the construction in Lemma 5.16, let $\eta > 2\,d_{\mathrm{GH}}(X,Y)$ and $\delta > 0$. We fix finite δ–nets $U \subset X$ of X and $V \subset Y$ of Y. By the triangle inequality, $d_{\mathrm{GH}}(U,V) < \frac{1}{2}\eta + 2\delta$, and so there exist maps $\varphi\colon U \to V$ and $\psi\colon V \to U$ with

$$\max(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi,\psi)) \le \eta + 4\delta.$$

For any $\delta > 0$, we will show that $(\mathcal{P}_X, \mathfrak{i}^X)$ and $(\mathcal{P}_Y, \mathfrak{i}^Y)$ are $(\eta+6\delta)L$–homotopy equivalent. By the definition of a δ–homotopy (Definition 2.6), it suffices to show that

- $\hat{\Phi}$ is an $(\eta+6\delta)L$–map from $(\mathcal{P}_X, \mathfrak{i}^X)$ to $(\mathcal{P}_Y, \mathfrak{i}^Y)$,
- $\hat{\Psi}$ is an $(\eta+6\delta)L$–map from $(\mathcal{P}_Y, \mathfrak{i}^Y)$ to $(\mathcal{P}_X, \mathfrak{i}^X)$,
- $\hat{\Psi} \circ \hat{\Phi}\colon \mathcal{P}_X \to \mathcal{P}_X$ is $(2\eta+12\delta)L$–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathfrak{i}^X, \mathfrak{i}^X)$,
- $\hat{\Phi} \circ \hat{\Psi}\colon \mathcal{P}_X \to \mathcal{P}_X$ is $(2\eta+12\delta)L$–homotopic to $\mathrm{id}_{\mathcal{P}_Y}$ with respect to $(\mathfrak{i}^Y, \mathfrak{i}^Y)$.

We will only present the proof for the first and third items; the other two can be proved similarly. We have

$$\mathfrak{i}^Y(\hat{\Phi}(\alpha)) = \mathfrak{i}^Y(\varphi_\sharp \circ \Phi_U(\alpha)) \le \mathfrak{i}^X(\Phi_U(\alpha)) + (\eta+4\delta)L \le \mathfrak{i}^X(\alpha) + (\eta+6\delta)L,$$

where the first inequality is from the stability of the invariant $\mathfrak{i}$ under pushforward and from the bound on the distortion of φ, and where the second inequality is from the stability of the invariant with respect to Wasserstein distance and from the bound $d^X_{\mathrm{W},q}(\alpha, \Phi_U(\alpha)) < \delta$ in Lemma 5.15. This proves the first item.

For the third item, we use the homotopy H^X_t in Lemma 5.16. Then it suffices to show, for any fixed $t \in [0,1]$, that the map H^X_t is a $(2\eta+12\delta)L$–map from $(\mathcal{P}_X, \mathfrak{i}^X)$ to $(\mathcal{P}_X, \mathfrak{i}^X)$, that is,

$$\mathfrak{i}^X(H^X_t(\alpha)) \le \mathfrak{i}^X(\alpha) + (2\eta+12\delta)L.$$

This comes from the inequality $d^X_{\mathrm{W},\infty}(H^X_t(\alpha), \alpha) < \eta + 6\delta$ from Lemma 5.16, and from the stability assumption of $\mathfrak{i}$ with respect to Wasserstein distance.

Now, since $(\mathcal{P}_X, \mathfrak{i}^X)$ and $(\mathcal{P}_Y, \mathfrak{i}^Y)$ are $(\eta+6\delta)L$–homotopy equivalent for any $\eta > 2\,d_{\mathrm{GH}}(X,Y)$ and $\delta > 0$, it follows from the definition of the d_{HT} distance (Definition 2.7) that $d_{\mathrm{HT}}((\mathcal{P}_X, \mathfrak{i}^X), (\mathcal{P}_Y, \mathfrak{i}^Y)) \le 2L\,d_{\mathrm{GH}}(X,Y)$. This completes the proof. □

6 A Hausmann type theorem for 2–Vietoris–Rips and 2–Čech thickenings of Euclidean submanifolds

Hausmann's theorem [42] states that if X is a Riemannian manifold and if the scale $r > 0$ is sufficiently small (depending on the curvature X), then the Vietoris–Rips simplicial complex $\mathrm{VR}(X;r)$ is homotopy equivalent to X. A short proof using an advanced version of the nerve lemma is given in [73]. Versions of Hausmann's theorem have been proven for ∞–metric thickenings in [2], where X is equipped with the Riemannian metric, and in [6], where X is equipped with a Euclidean metric. In this section we prove a Hausmann type theorem for the 2–Vietoris–Rips and 2–Čech metric thickenings. Our result is closest to that in [6] (now with $p = 2$ instead of $p = \infty$): we work with any Euclidean subset X of positive reach. This includes (for example) any embedded C^k submanifold of $\mathbb{R}^n$ for $k \ge 2$, with or without boundary [70].

We begin with a definition and some lemmas that will be needed in the proof.

Let α be a measure in $\mathcal{P}_1(\mathbb{R}^n)$, the collection of all Radon measures with finite first moment. For any coordinate function x_i, where $1 \le i \le n$, we have

$$\int_{\mathbb{R}^n} |x_i| \, \alpha(dx) \le \int_{\mathbb{R}^n} \|x\| \, \alpha(dx) < \infty.$$

Therefore the *Euclidean mean* map $m \colon \mathcal{P}_1(\mathbb{R}^n) \to \mathbb{R}^n$ given by the following vector-valued integral is well-defined:

$$m(\alpha) := \int_{\mathbb{R}^n} x \, \alpha(dx).$$

Lemma 6.1 *Let X be a metric space and let $f : X \to \mathbb{R}^n$ be a bounded continuous function. Then the induced map $m \circ f_\sharp \colon \mathcal{P}_X \to \mathbb{R}^n$ is continuous.*

Proof For a sequence α_n that weakly converges to α, we have the vector-valued integral

$$m \circ f_\sharp(\alpha_n) = \int_{\mathbb{R}^n} x f_\sharp(\alpha_n)(dx) = \int_X f(x) \, \alpha_n(dx).$$

As f is bounded and continuous, the above limit converges to $m \circ f_\sharp(\alpha)$ as α_n converges to α. Therefore the map $m \circ f_\sharp$ is continuous. □

Lemma 6.2 *Let α be a probability measure in $\mathcal{P}_1(\mathbb{R}^n)$. Then, for $p \in [1, \infty]$, there is some $z \in \operatorname{supp}(\alpha)$ with*

$$\|m(\alpha) - z\| \le \operatorname{diam}_p(\alpha).$$

Proof As $\operatorname{diam}_1(\alpha) \le \operatorname{diam}_p(\alpha)$ for all $p \in [1, \infty]$, it suffices to show $\|m(\alpha) - z\| \le \operatorname{diam}_1(\alpha)$. We consider the formula

$$\int_{\mathbb{R}^n} \|m(\alpha) - z\| \, \alpha(dz) = \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} (x - z) \, \alpha(dx) \right\| \alpha(dz) \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|(x - z)\| \, \alpha(dx) \, \alpha(dz) = \operatorname{diam}_1(\alpha).$$

So there must be some $z \in \operatorname{supp}(\alpha)$ with $\|m(\alpha) - z\| \le \operatorname{diam}_1(\alpha)$. □

Lemma 6.3 *For any probability measure $\alpha \in \mathcal{P}_1(\mathbb{R}^n)$ and for any $x \in \mathbb{R}^n$, we can write the associated squared 2–Fréchet function $F^2_{\alpha,2}(x)$ as*

$$F^2_{\alpha,2}(x) = \|x - m(\alpha)\|^2 + F^2_{\alpha,2}(m(\alpha)).$$

Proof By the linearity of the inner product,

$$\begin{aligned}
F^2_{\alpha,2}(x) &:= \int_{\mathbb{R}^n} \|x - y\|^2 \, \alpha(dy) = \int_{\mathbb{R}^n} (\|x\|^2 - 2\langle x, y\rangle + \|y\|^2) \, \alpha(dy) \\
&= \int_{\mathbb{R}^n} (\|x\|^2 - 2\langle x, y\rangle + \|m(\alpha)\|^2 - \|m(\alpha)\|^2 + \|y\|^2) \, \alpha(dy) \\
&= \|x\|^2 - 2\langle x, m(\alpha)\rangle + \|m(\alpha)\|^2 + \int_{\mathbb{R}^n} (-\|m(\alpha)\|^2 + \|y\|^2) \, \alpha(dy) \\
&= \|x - m(\alpha)\|^2 + \int_{\mathbb{R}^n} (\|m(\alpha)\|^2 - 2\langle m(\alpha), y\rangle + \|y\|^2) \, \alpha(dy) \\
&= \|x - m(\alpha)\|^2 + \int_{\mathbb{R}^n} \|m(\alpha) - y\| \, \alpha(dy) = \|x - m(\alpha)\|^2 + F^2_{\alpha,2}(m(\alpha)).
\end{aligned}$$

□

Recall the *medial axis* of X is defined as the closure

$$\mathrm{med}(X) = \overline{\{y \in \mathbb{R}^n \mid \text{there exist } x_1 \neq x_2 \in X \text{ with } \|y - x_1\| = \|y - x_2\| = \inf_{x \in X} \|y - x\|\}}.$$

The *reach* τ of X is the closest distance $\tau := \inf_{x\in X, y\in \mathrm{med}(X)} \|x - y\|$ between points in X and $\mathrm{med}(X)$. Let $U_\tau(X)$ be the τ–neighborhood of X in $\mathbb{R}^n$, that is, the set of points $y \in \mathbb{R}^n$ such that there is some $x \in X$ with $\|y - x\| < \tau$. The definition of reach implies that, for any point y in the open neighborhood $U_\tau(X)$, there is a unique closest point $x \in X$. The associated nearest projection map $\pi : U_\tau(X) \to X$ is continuous; see [36, Theorem 4.8(8)] or [6, Lemma 3.7].

Lemma 6.4 *Let X be a bounded subset of $\mathbb{R}^n$ with reach $\tau(X) > 0$. Let $\alpha \in \mathcal{P}_X$ have its Euclidean mean $m(\alpha)$ in the neighborhood $U_\tau(X)$. Then along the linear interpolation family*

$$\alpha_t := (1-t)\alpha + t\delta_{\phi(\alpha)},$$

where ϕ is the composition $\pi \circ m : \mathcal{P}_X \to X$, both diam_2 and rad_2 obtain their maximum at $t = 0$.

Proof According to Lemma 6.3, for any $x \in X$,

$$F^2_{\alpha,2}(x) = \|x - m(\alpha)\|^2 + F^2_{\alpha,2}(m(\alpha)).$$

As $m(\alpha)$ is inside $U_\tau(X)$, the first term $\|x - m(\alpha)\|$ is minimized over $x \in X$ at $x = \phi(\alpha)$, and hence so is $F_{\alpha,2}(x)$. Therefore $\mathrm{rad}_2(\alpha) = \inf_{x\in X} F_{\alpha,2}(x) = F_{\alpha,2}(\phi(\alpha))$. So for $\mathrm{rad}_2(\alpha_t)$,

$$\begin{aligned}(\mathrm{rad}_2(\alpha_t))^2 \le F^2_{\alpha_t,2}(\phi(\alpha)) &= \int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \,\alpha_t(dx)\\ &= (1-t)\int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \,\alpha(dx) + t\int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \delta_{\phi(\alpha)}(dx)\\ &= (1-t)\int_{\mathbb{R}^n} \|x - \phi(\alpha)\|^2 \,\alpha(dx) = (1-t)F^2_{\alpha,2}(\phi(\alpha)) \le (\mathrm{rad}_2(\alpha))^2.\end{aligned}$$

We now consider diam_2. Recall that $F_{\alpha,2}(\phi(\alpha)) \le F_{\alpha,2}(x)$ for all $x \in X$. From this, we have $(\mathrm{diam}_2(\alpha))^2 = \int_X F^2_{\alpha,2}(x)\,\alpha(dx) \ge F^2_{\alpha,2}(\phi(\alpha))$. Therefore

$$\begin{aligned}(\mathrm{diam}_2(\alpha_t))^2 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 \,\alpha_t(dx)\,\alpha_t(dy)\\ &= (1-t)^2 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 \,\alpha(dx)\,\alpha(dy) + 2t(1-t)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 \,\alpha(dx)\,\delta_{\phi(\alpha)}(dy)\\ &\qquad + t^2 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 \,\delta_{\phi(\alpha)}(dx)\,\delta_{\phi(\alpha)}(dy)\\ &= (1-t)^2(\mathrm{diam}_2(\alpha))^2 + 2t(1-t)F^2_{\alpha,2}(\phi(\alpha)) \le (1-t)^2(\mathrm{diam}_2(\alpha))^2 + 2t(1-t)(\mathrm{diam}_2(\alpha))^2\\ &= (1-t^2)(\mathrm{diam}_2(\alpha))^2 \le (\mathrm{diam}_2(\alpha))^2.\end{aligned}$$

□

Theorem C *Let X be a bounded subset of $\mathbb{R}^n$ with positive reach τ. Then for all $0 < r \le \tau$, the isometric embeddings $X \hookrightarrow \mathrm{VR}_2(X;r)$, $X \hookrightarrow \mathrm{VR}^{\mathrm{fin}}_2(X;r)$, $X \hookrightarrow \check{\mathrm{C}}_2(X;r)$ and $X \hookrightarrow \check{\mathrm{C}}^{\mathrm{fin}}_2(X;r)$ are homotopy equivalences.*

Proof We begin with the 2–Vietoris–Rips metric thickening. Let α be a measure in $\mathrm{VR}_2(X;r)$. Lemma 6.2 shows that there exists some $x \in X$ with $\|m(\alpha)-x\| \le \mathrm{diam}_2(\alpha) < r \le \tau$. Hence $m(\alpha)$ is in the neighborhood $U_\tau(X)$, and $\phi := \pi \circ m$ is well defined on $\mathrm{VR}_2(X;r)$. As both π and m are continuous, so is their composition ϕ. Let i be the isometric embedding of X into $\mathrm{VR}_2(X;r)$. Clearly we have $\phi \circ i = \mathrm{id}_X$. Then it suffices to show the homotopy equivalence $i \circ \phi \simeq \mathrm{id}_{\mathrm{VR}_2(X;r)}$.

For this, we use the linear family $(\alpha_t)_{t\in[0,1]}$, where $\alpha_t := (1-t)\alpha + t\delta_{\phi(\alpha)}$. According to Lemma 6.4, we know this family is inside $\mathrm{VR}_2(X;r)$, and therefore α_t is well defined. By Proposition 2.4, this is a continuous family as the map ϕ is continuous. Therefore the linear family indeed provides a homotopy $i \circ \phi \simeq \mathrm{id}_{\mathrm{VR}_2(X;r)}$.

If α is a finitely supported measure, then the above construction resides inside finitely supported measures as well and therefore we get the result for $\mathrm{VR}_2^{\mathrm{fin}}(X;r)$.

For the Čech case, we again map $\check{\mathrm{C}}_2(X;r)$ onto X using the map $\alpha \mapsto \phi(\alpha) := \pi(m(\alpha))$. To see this is well defined, note that as $\alpha \in \check{\mathrm{C}}_2(X;r)$, there is some $z \in X$ with $F_\alpha(z) = \|z-m(\alpha)\|^2 + F_\alpha(m(\alpha)) < r^2$. This implies $\|z-m(\alpha)\| < r \le \tau$ and so $m(\alpha)$ lies inside $U_\tau(X)$. We then get the result by the same homotopy construction as before. The finitely supported Čech case holds similarly. □

7 The 2–Vietoris–Rips and 2–Čech thickenings of spheres with Euclidean metric

Let $\mathbb{S}^n$ be the n–dimensional sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, and let $(\mathbb{S}^n, \ell_2)$ denote the unit sphere equipped with the Euclidean metric. In this section we determine the successive homotopy types of the 2–Vietoris–Rips and 2–Čech metric thickening filtrations $\mathrm{VR}_2((\mathbb{S}^n,\ell_2);\cdot)$ and $\check{\mathrm{C}}_2((\mathbb{S}^n,\ell_2);\cdot)$. We begin with a lemma.

Lemma 7.1 *For any measure $\alpha \in \mathcal{P}_{(\mathbb{S}^n,\ell_2)}$, we have*

$$\mathrm{diam}_2(\alpha) = (2-2\|m(\alpha)\|^2)^{1/2} \le \sqrt{2} \quad \textit{and} \quad \mathrm{rad}_2(\alpha) = (2-2\|m(\alpha)\|)^{1/2} \le \sqrt{2}.$$

Note that since $\|m(\alpha)\| \le 1$, we have that indeed $\mathrm{rad}_2(\alpha) \le \mathrm{diam}_2(\alpha)$, and by the same token $\mathrm{diam}_2(\alpha) \le 2\,\mathrm{rad}_2(\alpha)$, in agreement with Proposition 3.1.

Proof of Lemma 7.1 We can calculate $\mathrm{diam}_2(\alpha)$ as follows:

$$\begin{aligned}\mathrm{diam}_2(\alpha) &= \Big(\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \|x-y\|^2\,\alpha(dx)\,\alpha(dy)\Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} 2-2\langle x,y\rangle\,\alpha(dx)\,\alpha(dy)\Big)^{\frac{1}{2}} \\ &= \Big(2-2\Big\langle \int_{\mathbb{S}^n} x\,\alpha(dx), \int_{\mathbb{S}^n} y\,\alpha(dy)\Big\rangle\Big)^{\frac{1}{2}} = (2-2\|m(\alpha)\|^2)^{\frac{1}{2}} \le \sqrt{2}.\end{aligned}$$

The calculation for $\mathrm{rad}_2(\alpha)$ gives

$$\begin{aligned}\mathrm{rad}_2(\alpha) &= \inf_{x\in\mathbb{S}^n}\left(\int_{\mathbb{S}^n}\|x-y\|^2\,\alpha(dy)\right)^{\frac{1}{2}} = \inf_{x\in\mathbb{S}^n}\left(2-2\int_{\mathbb{S}^n}\langle x,y\rangle\,\alpha(dy)\right)^{\frac{1}{2}}\\ &= \inf_{x\in\mathbb{S}^n}(2-2\langle x,m(\alpha)\rangle)^{1/2} = \left(2-2\left\langle\frac{m(\alpha)}{\|m(\alpha)\|},m(\alpha)\right\rangle\right)^{\frac{1}{2}} = (2-2\|m(\alpha)\|)^{1/2}\le\sqrt{2}. \qquad\square\end{aligned}$$

We now see that, for spheres with euclidean metric, both the 2–Vietoris–Rips and the 2–Čech metric thickenings attain *only two* homotopy types:

Theorem D *The isometric embeddings $\mathbb{S}^n\hookrightarrow \mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)$ and $\mathbb{S}^n\hookrightarrow \check{\mathrm{C}}_2((\mathbb{S}^n,\ell_2);r)$ are homotopy equivalences for any $r\le\sqrt{2}$. When $r>\sqrt{2}$, the spaces $\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)$ and $\check{\mathrm{C}}_2((\mathbb{S}^n,\ell_2);r)$ are contractible. By restricting to finitely supported measures, we get the same result for $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n,\ell_2);r)$ and $\check{\mathrm{C}}_2^{\mathrm{fin}}((\mathbb{S}^n,\ell_2);r)$.*

Remark 7.2 A similar phenomenon is found for the Vietoris–Rips filtration of $(\mathbb{S}^n,\ell_\infty)$, where $\mathrm{VR}((\mathbb{S}^n,\ell_\infty);r)$ has the homotopy type of $\mathbb{S}^n$ when $r\le 2/\sqrt{n+1}$ and becomes contractible when $r>2/\sqrt{n+1}$; see [52, Section 7.3]. In contrast, the homotopy types of the Vietoris–Rips and Čech filtrations of geodesic circles include all possible odd-dimensional spheres [1]. The first new homotopy type of the ∞–Vietoris–Rips metric thickening of the n–sphere (with either the geodesic or the Euclidean metric) is known [2, Theorem 5.4], but so far only for a single (nonpersistent) scale parameter, and only when using the convention $\mathrm{diam}_\infty(\alpha)\le r$ instead of $\mathrm{diam}_\infty(\alpha)<r$.

We note that Theorem C only implies that $\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)\simeq\mathbb{S}^n$ for $r\le\tau(\mathbb{S}^n)=1$, whereas Theorem D extends this range to $r\le\sqrt{2}$.

Proof of Theorem D From Lemma 7.1 we know that $\sqrt{2}$ is the maximal possible 2–diameter on $\mathcal{P}_{(\mathbb{S}^n,\ell_2)}$. Therefore, when $r>\sqrt{2}$, the spaces $\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)=\mathcal{P}_{(\mathbb{S}^n,\ell_2)}$ and $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n,\ell_2);r)=\mathcal{P}^{\mathrm{fin}}_{(\mathbb{S}^n,\ell_2)}$ are both convex and hence contractible.

When $0<r\le\sqrt{2}$, by Lemma 7.1 we get that $m(\alpha)$ is not the origin for any $\alpha\in\mathcal{P}_{(\mathbb{S}^n,\ell_2)}$ (since otherwise we would have $\mathrm{diam}_2(\alpha)=\sqrt{2}$, which is too large). Therefore $m(\alpha)$ is inside the tubular neighborhood $U_1((\mathbb{S}^n,\ell_2))$. Let i be the inclusion map $(\mathbb{S}^n,\ell_2)\hookrightarrow\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)$ mapping points to delta measures, and let ϕ be the composition $\pi\circ m\colon \mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)\to(\mathbb{S}^n,\ell_2)$, which is well defined as $m(\alpha)$ is not the origin. Then we naturally have $\phi\circ i=\mathrm{id}_{(\mathbb{S}^n,\ell_2)}$. By applying Lemma 6.4, the linear family $\alpha_t=(1-t)\alpha+t\,\delta_{\phi(\alpha)}$ lies inside $\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)$ and therefore provides the desired homotopy between $i\circ\phi$ and $\mathrm{id}_{\mathrm{VR}_2((\mathbb{S}^n,\ell_2);r)}$. By restricting to the set of finitely supported measures, we get the result for $\mathrm{VR}_2^{\mathrm{fin}}((\mathbb{S}^n,\ell_2);r)$.

The proof is identical for the Čech case. $\square$

Remark 7.3 The above calculation gives, for any two positive integers $n \neq m$, that

$$d_{\mathrm{B}}(\mathrm{dgm}^{\mathrm{VR}}_{n,2}(\mathbb{S}^n,\ell_2),\mathrm{dgm}^{\mathrm{VR}}_{n,2}(\mathbb{S}^m,\ell_2)) = \tfrac{\sqrt{2}}{2} \quad \text{and} \quad d_{\mathrm{B}}(\mathrm{dgm}^{\check{\mathrm{C}}}_{n,2}(\mathbb{S}^n,\ell_2),\mathrm{dgm}^{\check{\mathrm{C}}}_{n,2}(\mathbb{S}^m,\ell_2)) = \tfrac{\sqrt{2}}{2}.$$

The stability theorem, Theorem B, gives

$$d_{\mathrm{GH}}((\mathbb{S}^n,\ell_2),(\mathbb{S}^m,\ell_2)) \geq \tfrac{\sqrt{2}}{4}. \tag{5}$$

Remark 7.4 In a more detailed analysis, Proposition 9.13 of [53] provides the lower bound $\frac{1}{2}$ for $d_{\mathrm{GH}}((\mathbb{S}^n,\ell_2),(\mathbb{S}^m,\ell_2))$ when $n \neq m$, which is larger than the lower bound $\frac{\sqrt{2}}{4}$ given in (5). As for the geodesic distance case, one can use [53, Corollary 9.8(1)] to obtain $d_{\mathrm{GH}}(\mathbb{S}^n,\mathbb{S}^m) \geq \arcsin\bigl(\frac{\sqrt{2}}{4}\bigr)$ from (5), where $d_{\mathrm{GH}}(\mathbb{S}^n,\mathbb{S}^m)$ is the Gromov–Hausdorff distance between spheres endowed with their geodesic distances for any $0 < m < n$. A better lower bound is found in [53]:

$$d_{\mathrm{GH}}(\mathbb{S}^n,\mathbb{S}^m) \geq \tfrac{1}{2}\arccos\left(\frac{-1}{m+1}\right) \geq \tfrac{\pi}{4},$$

which arises from considerations other than stability. The lower bound given by the quantity in the middle is shown to coincide with the exact Gromov–Hausdorff distance between $\mathbb{S}^1$ and $\mathbb{S}^2$, between $\mathbb{S}^1$ and $\mathbb{S}^3$, and also between $\mathbb{S}^2$ and $\mathbb{S}^3$.

8 Bounding barcode length via spread

The *spread* of a metric space is defined by Katz [45], and used in [52, Section 9] to bound the length of intervals in Vietoris–Rips simplicial complex persistence diagrams. In this section we identify a measure theoretic version of the notion of spread, and we use it to bound the length of intervals in p–Vietoris–Rips and p–Čech metric thickening persistence diagrams.

Definition 8.1 The p–spread of a bounded metric space X is defined as

$$\mathrm{spread}_p(X) := \inf_{\alpha_* \in \mathcal{P}^{\mathrm{fin}}_X} \sup_{\alpha \in \mathcal{P}_X} d^X_{\mathrm{W},p}(\alpha_*,\alpha).$$

Note that, for any $1 \leq p \leq q \leq \infty$,

$$\mathrm{spread}_1(X) \leq \mathrm{spread}_p(X) \leq \mathrm{spread}_q(X) \leq \mathrm{spread}_\infty(X) \leq \mathrm{rad}(X).$$

Remark 8.2 When $p \in [1,\infty)$, the space $\mathcal{P}^{\mathrm{fin}}_X$ is dense in $\mathcal{P}_{p,X}$, the set of Radon measures with finite moment of order p; see [15, Corollary 3.3.5]. Therefore, for a bounded metric space, the p–spread is just the radius of the metric space $(\mathcal{P}_X, d^X_{\mathrm{W},p})$.

Proposition 8.3 *Let X be a bounded metric space and let $\mathfrak{i}$ be an L–controlled invariant. For any $r > 0$ and any $\xi > \mathrm{spread}_\infty(X)$, the space $[\mathcal{P}_X, \mathfrak{i}^X; r]$ can be contracted inside of $[\mathcal{P}_X, \mathfrak{i}^X; r+2L\xi]$. In particular, any homology class of $[\mathcal{P}_X, \mathfrak{i}^X; r]$ will vanish in $[\mathcal{P}_X, \mathfrak{i}^X; r+2L\xi]$.*

Proof As $\xi > \mathrm{spread}_\infty(X)$, there is some $\alpha_\xi \in \mathcal{P}_X^{\mathrm{fin}}$ such that, for all $\alpha \in \mathcal{P}_X$, we have $d_{\mathrm{W},\infty}^X(\alpha_\xi, \alpha) < \xi$. Consider the linear homotopy

$$h_t\colon [0,1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$$

defined by

$$(t,\alpha) \mapsto (1-t)\alpha + t\alpha_\xi.$$

This gives a homotopy between the inclusion from $[\mathcal{P}_X, \mathfrak{i}^X; r]$ to $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$ and a constant map, and therefore implies our conclusion so long as the homotopy is well defined. It then suffices to show $\mathfrak{i}^X(h_t(\alpha)) < r + 2L\xi$. This comes from the stability condition of the invariant with respect to Wasserstein distances and from Lemma 4.2:

$$\mathfrak{i}^X(h_t(\alpha)) \le \mathfrak{i}^X(\alpha) + 2L\, d_{\mathrm{W},\infty}^X(h_t(\alpha), \alpha) \le \mathfrak{i}^X(\alpha) + 2L\, d_{\mathrm{W},\infty}^X(\alpha_\xi, \alpha) < r + 2L\xi. \qquad \square$$

Remark 8.4 It is not difficult to see that for the $\mathfrak{i}_{q,p}$ invariants we can improve the bound $\xi > \mathrm{spread}_\infty(X)$ to $\xi > \max\{\mathrm{spread}_p(X), \mathrm{spread}_q(X)\}$.

We also have the following stronger contractibility conclusion for $\mathrm{VR}_p(X;\cdot)$ and $\check{\mathrm{C}}_p(X;\cdot)$:

Theorem E *Let X be a bounded metric space. For any $p \in [1,\infty]$ and any $r > \mathrm{spread}_p(X)$, both $\mathrm{VR}_p(X;r)$ and $\check{\mathrm{C}}_p(X;r)$ are contractible.*

Proof As $r > \mathrm{spread}_p(X)$, there is some $\alpha_r \in \mathcal{P}_X^{\mathrm{fin}}$ such that, for all $\alpha \in \mathcal{P}_X$, we have $d_{\mathrm{W},p}^X(\alpha_r, \alpha) < r$. In particular, this implies $\mathrm{diam}_p(\alpha_r) = (\int_X F_{\alpha_r,p}^p(x)\, \alpha_r(dx))^{1/p} < r$ and $\mathrm{rad}_p(\alpha_r) = \inf_{x \in X} F_{\alpha_r,p}(x) < r$. Now let $\mathfrak{i}$ be either diam_p or rad_p. Consider the linear homotopy

$$h_t\colon [0,1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r]$$

defined by

$$(t,\alpha) \mapsto (1-t)\alpha + t\alpha_r.$$

It then suffices to show, for all $t \in [0,1]$, that $\mathrm{diam}_p(h_t(\alpha)) < r$ and $\mathrm{rad}_p(h_t(\alpha)) < r$, so that this linear homotopy from the identity map on $[\mathcal{P}_X, \mathfrak{i}^X; r]$ to the constant map to α_r is well-defined.

In the diam_p case, we have

$$\begin{aligned}
\mathrm{diam}_p^p(h_t(\alpha)) &= (1-t)^2 \iint_{X\times X} d_X^p(x,x')\,\alpha(dx)\,\alpha(dx') + 2t(1-t)\iint_{X\times X} d_X^p(x,x')\,\alpha_r(dx)\,\alpha(dx') \\
&\qquad + t^2 \iint_{X\times X} d_X^p(x,x')\,\alpha_r(dx)\,\alpha_r(dx') \\
&= (1-t)^2\,\mathrm{diam}_p^p(\alpha) + 2t(1-t)\int_X (d_{\mathrm{W},p}^X(\alpha_r, \delta_{x'}))^p\,\alpha(dx') + t^2\int_X (d_{\mathrm{W},p}^X(\alpha_r, \delta_{x'}))^p\,\alpha_r(dx') \\
&< (1-t)^2 r^p + 2t(1-t) r^p + t^2 r^p = r^p.
\end{aligned}$$

In the rad_p case, as $\mathrm{rad}_p(\alpha) < r$, there exists a point $y \in X$ such that $d_{\mathrm{W},p}^X(\delta_y, \alpha) < r$. Thus

$$\begin{aligned}
\mathrm{rad}_p^p(h_t(\alpha)) &= \big(\inf_{x\in X} d_{\mathrm{W},p}^X(\delta_x, h_t(\alpha))\big)^p \le \big(d_{\mathrm{W},p}^X(\delta_y, h_t(\alpha))\big)^p \\
&= (1-t)(d_{\mathrm{W},p}^X(\delta_y, \alpha))^p + t(d_{\mathrm{W},p}^X(\delta_y, \alpha_r))^p < (1-t)r^p + tr^p = r^p. \qquad \square
\end{aligned}$$

The upper bound for the lifetime of the persistent homology features in $[\mathcal{P}_X, \mathfrak{i}^X; \cdot]$ for L–controlled invariants is also related to the metric spread defined in [45].

Definition 8.5 The *metric spread* of a metric space X is defined to be

$$\operatorname{spread}(X) := \inf_{U \subset X, |U| < \infty} \max(d_{\mathrm{H}}(U, X), \operatorname{diam}(U)),$$

where d_{H} is the Hausdorff distance.

Let $\mathfrak{i}$ be an L–controlled invariant and let $\max(\mathfrak{i}^X)$ be the maximum of the function $\mathfrak{i}^X$ on $\mathcal{P}_X$, for X a bounded metric space. For any $r > \max(\mathfrak{i}^X)$ we have $[\mathcal{P}_X, \mathfrak{i}^X; r] = \mathcal{P}_X$, and therefore $[\mathcal{P}_X, \mathfrak{i}^X; r]$ is contractible. Inspired by the definition of the spread, we have the following result:

Lemma 8.6 *Let X be a metric space, let $\mathfrak{i}$ be an L–controlled invariant and let U be a finite subset of X. Then, for any $\xi > \max(d_{\mathrm{H}}(U, X), (\max(\mathfrak{i}^U) - r)/(2L))$, the space $[\mathcal{P}_X, \mathfrak{i}^X; r]$ is contractible in $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$. In particular, any homology class of $[\mathcal{P}_X, \mathfrak{i}^X; r]$ will vanish in $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$.*

Proof As $\xi > d_{\mathrm{H}}(U, X)$, the balls $\{B(u; \xi)\}_{u \in U}$ form a open covering of X. We choose a partition of unity subordinate to the covering and build the map $\Phi_U : [\mathcal{P}_X, \mathfrak{i}^X; r] \to \mathcal{P}_U$ as in Lemma 5.15. Since $r + 2L\xi > \max(\mathfrak{i}^U)$, we know $\mathcal{P}_U = [\mathcal{P}_U, \mathfrak{i}^U; r + 2L\xi]$. Since the inclusion map $\iota_U : U \to X$ is of zero distortion, $\mathfrak{i}^X((\iota_U)_\sharp(\beta)) \le \mathfrak{i}^U(\beta) \le \max(\mathfrak{i}^U)$ for all $\beta \in \mathcal{P}_U$. This implies that the contractible set $\mathcal{P}_U$ is inside $[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$, and so we have the following diagram:

$$\begin{tikzcd}
{[\mathcal{P}_X, \mathfrak{i}^X; r]} \arrow[r, "\nu^X_{r,r+2L\xi}"] \arrow[dr, "\Phi_U"'] & {[\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]} \\
 & \mathcal{P}_U \arrow[u, hook, "\iota_U"']
\end{tikzcd}$$

As the image of $\iota_U \circ \Phi_U$ maps $[\mathcal{P}_X, \mathfrak{i}^X; r]$ into a contractible subset $\mathcal{P}_U \subset [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$, it is homotopy equivalent to a constant map. To obtain this conclusion, it thus suffices to show that $\iota_U \circ \Phi_U$ is homotopy equivalent to the inclusion given by the structure map $\nu^X_{r,r+2L\xi}$. Consider the linear homotopy

$$h_t : [0, 1] \times [\mathcal{P}_X, \mathfrak{i}^X; r] \to [\mathcal{P}_X, \mathfrak{i}^X; r + 2L\xi]$$

defined by

$$(t, \alpha) \mapsto (1 - t)\alpha + t\,\Phi_U(\alpha).$$

From the stability property of $\mathfrak{i}$, Lemma 4.2 and the estimate in Lemma 5.15, we have

$$\mathfrak{i}^X(h_t(\alpha)) \le \mathfrak{i}^X(\alpha) + 2L\, d^X_{\mathrm{W},\infty}(\alpha, h_t(\alpha)) \le \mathfrak{i}^X(\alpha) + 2L\, d^X_{\mathrm{W},\infty}(\alpha, \Phi_U(\alpha)) \le \mathfrak{i}^X(\alpha) + 2L\xi < r + 2L\xi.$$

This shows that the homotopy h_t from $\iota_U \circ \Phi_U$ to $\nu^X_{r,r+2L\xi}$ is well defined. □

Remark 8.7 For the 1–controlled $\mathfrak{i}_{q,p}$ invariant on a bounded metric space X, the maximum of $\mathfrak{i}^X_{q,p}$ is bounded by $\operatorname{diam}(X)$. Therefore, for any finite subset U,

$$\max(d_{\mathrm{H}}(U, X), \operatorname{diam}(U)) \ge \max\bigl(d_{\mathrm{H}}(U, X), \tfrac{1}{2}(\max(\mathfrak{i}^U_{q,p}) - r)\bigr).$$

Hence, the previous lemma implies that the lifetime of features in $[\mathcal{P}_X, \mathrm{i}^X_{q,p}; \cdot]$ is bounded by $2\,\mathrm{spread}(X)$. In the diam_p case the factor 2 can be removed, as we show next, and therefore matches [52, Proposition 9.6] for Vietoris–Rips simplicial complexes.

Proposition 8.8 *For any $r > 0$, $p \in [1,\infty]$ and $\xi > \mathrm{spread}(X)$, the space $\mathrm{VR}_p(X;r)$ is contractible in $\mathrm{VR}_p(X;r+\xi)$. In particular, any homology class on $\mathrm{VR}_p(X;r)$ will vanish in $\mathrm{VR}_p(X;r+\xi)$.*

Proof As $\xi > \mathrm{spread}(X)$, there exists some $U \subset X$ such that $\mathrm{diam}(U) < \xi$ and $d_{\mathrm{H}}(U,X) < \xi$. As the maximum of diam^U_p on $\mathcal{P}_U$ is bounded by $\mathrm{diam}(U)$, the contractible subset $\mathcal{P}_U$ lies inside $\mathrm{VR}_p(X;r+\xi)$, and therefore $\iota_U \circ \Phi_U$ is homotopy equivalent to a constant map. Let h_t be the linear homotopy used in Lemma 8.6. What is left to get a homotopy equivalence between $\iota_U \circ \Phi_U$ and $\nu^X_{r,r+\xi}$ is to show $\mathrm{diam}_p(h_t(\alpha)) < r+\xi$. This comes from the following calculation:

$$\begin{aligned}
\mathrm{diam}^p_p(h_t(\alpha)) &= (1-t)^2 \iint_{X\times X} d^p(x,x')\,\alpha(dx)\,\alpha(dx') + 2t(1-t)\iint_{X\times X} d^p(x,x')\,\alpha(dx)\,\Phi_U(\alpha)(dx') \\
&\qquad + t^2 \iint_{X\times X} d^p(x,x')\,\Phi_U(\alpha)(dx)\,\Phi_U(\alpha)(dx') \\
&= (1-t)^2\,\mathrm{diam}^p_p(\alpha) + 2t(1-t)\int_X (d^X_{\mathrm{W},p}(\alpha,\delta_{x'}))^p\,\Phi_U(\alpha)(dx') + t^2\,\mathrm{diam}^p_p(\Phi_U(\alpha)) \\
&\le (1-t)^2\,\mathrm{diam}^p_p(\alpha) + 2t(1-t)\int_X \big(d^X_{\mathrm{W},p}(\alpha,\Phi_U(\alpha)) + d^X_{\mathrm{W},p}(\Phi_U(\alpha),\delta_{x'})\big)^p\,\Phi_U(\alpha)(dx') \\
&\qquad + t^2\,\mathrm{diam}^p_p(\Phi_U(\alpha)) \\
&\le (1-t)^2\,\mathrm{diam}^p_p(\alpha) + 2t(1-t)\big(\xi + \mathrm{diam}_p(\Phi_U(\alpha))\big)^p + t^2\,\mathrm{diam}^p_p(\Phi_U(\alpha)) \\
&< (1-t)^2 r^p + 2t(1-t)(r+\xi)^p + t^2\xi^p < (r+\xi)^p. \qquad \square
\end{aligned}$$

The following result shows the lifetime of features of $\check{\mathrm{C}}_p(X;\cdot)$ is also bounded by the metric spread of X, $\mathrm{spread}(X)$.

Proposition 8.9 *For any $r > 0$, $p \in [1,\infty]$ and $\xi > \mathrm{spread}(X)$, the space $\check{\mathrm{C}}_p(X;r)$ is contractible in $\check{\mathrm{C}}_p(X;r+\xi)$. In particular, any homology class on $\check{\mathrm{C}}_p(X;r)$ will vanish in $\check{\mathrm{C}}_p(X;r+\xi)$.*

Proof As $\xi > \mathrm{spread}(X)$, there exists some $U \subset X$ such that $\mathrm{diam}(U) < \xi$ and $d_{\mathrm{H}}(U,X) < \xi$. The maximum of rad^U_p on $\mathcal{P}_U$ is bounded by $\mathrm{diam}(U)$, and the contractible subset $\mathcal{P}_U$ lies inside $\check{\mathrm{C}}_p(X;r+\xi)$. This shows $\iota_U \circ \Phi_U$ is homotopy equivalent to a constant map. Let $h_t = (1-t)\alpha + t\,\Phi_U(\alpha)$ be the linear homotopy used in Lemma 8.6. What is left is to show is $\mathrm{rad}_p(h_t(\alpha)) < r+\xi$. By Lemmas 4.2 and 4.9,

$$\mathrm{rad}_p(h_t(\alpha)) \le \mathrm{rad}_p(\alpha) + d^X_{\mathrm{W},p}(\alpha, h_t(\alpha)) \le r + t^{1/p} d^X_{\mathrm{W},p}(\alpha,\Phi_U(\alpha)) \le r+\xi. \qquad \square$$

9 Conclusion

Filtrations, ie increasing sequences of spaces, play a foundational role in applied and computational topology, as they are the input to persistent homology. To produce a filtration from a metric space X, one often considers a Vietoris–Rips or Čech simplicial complex, with X as its vertex set, as the scale parameter increases. Since a point in the geometric realization of a simplicial complex is a convex combination of the vertices of the simplex $[x_0, x_1, \dots, x_k]$ in which it lies, each such point can alternatively be identified with a probability measure: a convex combination of Dirac delta masses δ_{x_0}, δ_{x_1}, …, δ_{x_k}. We can therefore reinterpret the Vietoris–Rips and Čech simplicial complex filtrations instead as filtrations in the space of probability measures, which are referred to as the Vietoris–Rips and Čech *metric thickenings*. In [2] it is argued that the metric thickenings have nicer properties for some purposes. For example, the inclusion from metric space X into the metric thickening is always an isometry onto its image, whereas an inclusion from metric space X into the vertex set of a simplicial complex is not even continuous unless X is discrete. We prove that these two perspectives are compatible: the ∞–Vietoris–Rips (resp. Čech) metric thickening filtration has the same persistent homology as the Vietoris–Rips (resp. Čech) simplicial complex filtration when X is totally bounded. Therefore, when analyzing these filtrations, one can choose to apply either simplicial techniques (simplicial homology, simplicial collapses, discrete Morse theory) or measure-theoretic techniques (optimal transport, Karcher or Fréchet means), whichever is more convenient for the task at hand.

The measure-theoretic perspective motivates new filtrations to build on top of a metric space X. Though the Vietoris–Rips simplicial complex filtration is closely related (at interleaving distance zero) to the metric thickening filtration obtained by looking at sublevel sets in the space of probability measures of the ∞–diameter functional, one can instead consider sublevel sets of the p–diameter functional for any $1 \le p \le \infty$. The same is true upon replacing Vietoris–Rips with Čech and replacing p–diameter with p–radius. These relaxed p–Vietoris–Rips and p–Čech metric thickenings enjoy the same stability results underlying the use of persistent homology: nearby metric spaces produce nearby persistence modules. The generalization to $p < \infty$ is a useful one: though determining the homotopy types of ∞–Vietoris–Rips thickenings of n–spheres is a hard open problem, we give a complete description of the homotopy types of 2–Vietoris–Rips thickenings of n–spheres for all n. We also prove a Hausmann-type theorem in the case $p = 2$, and ask if the $p < \infty$ metric thickenings may be amenable to study using tools from Morse theory.

More generally, one can consider sublevel sets of any L–controlled function on the space of probability measures on X. We prove stability in this much more general context. This allows one to consider metric thickenings that are tuned to a particular task, perhaps incorporating other geometric notions besides just proximity, such as curvature, centrality, eccentricity, etc. One can design an L–controlled functional to highlight specific features that may be useful for a particular data science task.

We hope these contributions inspire more work on metric thickenings and their relaxations. We end with some open questions.

(i) For X totally bounded, is the $p = \infty$ metric thickening $\mathrm{VR}_\infty(X; r)$ homotopy equivalent to the simplicial complex $\mathrm{VR}(X; r)$, and similarly is $\check{\mathrm{C}}_\infty(X; r)$ homotopy equivalent to $\check{\mathrm{C}}(X; r)$? Note, we are using the $<$ convention.

(ii) For X totally bounded, is $\mathrm{VR}_p(X;r)$ homotopy equivalent to $\mathrm{VR}_p^{\mathrm{fin}}(X;r)$, is $\check{\mathrm{C}}_p(X;r)$ homotopy equivalent to $\check{\mathrm{C}}_p^{\mathrm{fin}}(X;r)$ and, for $\mathfrak{i}$ a controlled invariant, is $[\mathcal{P}_X, \mathfrak{i}^X; r]$ homotopy equivalent to $[\mathcal{P}_X^{\mathrm{fin}}, \mathfrak{i}^X; r]$?

(iii) Is there an analogue to the Hausmann-type Theorem C which holds for $p \in (2,\infty)$? The case $p=\infty$ was tackled in [6]. In a similar spirit, it seems interesting to explore whether analogous theorems hold when the ambient space is a more general Hadamard space instead of $\mathbb{R}^d$.

(iv) Can one prove Latschev-type theorems [50] for p–metric thickenings?

(v) For $p \neq 2$, what are the homotopy types of p–Vietoris–Rips and p–Čech thickenings of spheres at all scales? Is the homotopy connectivity a nondecreasing function of the scale, and if so, how quickly does the homotopy connectivity increase?

(vi) What are the homotopy types of p–Vietoris–Rips and p–Čech metric thickenings of other manifolds, such as ellipses (see [3]), ellipsoids, tori and projective spaces [5; 45; 49]?

(vii) What versions of Morse theory [60] can be developed in order to analyze the homotopy types of p–metric thickenings of manifolds as the scale increases? See Section A.2 for some initial ideas in the case of p–Čech thickenings. For homogeneous spaces such as spheres, versions of Morse–Bott theory [16; 17; 18] may be needed.

(viii) For X finite, is $\mathrm{VR}_p(X;r)$ always homotopy equivalent to a subcomplex of the complete simplex on the vertex set X? See Section A.3 for a proof of the Čech case.

(ix) For X finite with $n+1$ points, the space $\mathcal{P}_X$ is an n–simplex in $\mathbb{R}^{n+1}$ where coordinates are the weights of a measure at each point. In this case, diam_p is a quadratic polynomial on $\mathbb{R}^{n+1}$ and rad_p is the minimum of $n+1$ linear equations. Therefore, both $\mathrm{VR}_p(X;r)$ and $\check{\mathrm{C}}_p(X;r)$ are semialgebraic sets in $\mathbb{R}^{n+1}$. Can one use linear programming along with the results of Section A.3 to calculate the homology groups of $\check{\mathrm{C}}(X;r)$, and the work on quadratic semialgebraic sets [9; 11; 24] to calculate the homology groups of $\mathrm{VR}_p(X;r)$? See also the recent paper [10], which provides a singly exponential complexity algorithm for computing the sublevel set persistent homology induced by a polynomial on a semialgebraic set up to some fixed homological dimension.

Appendix

The appendix contains results which are related to but not central to the main thread of the paper. In Section A.1 we explain how the q–Wasserstein distance metrizes the weak topology for $1 \le q < \infty$. We describe connections to min-type Morse theories in Section A.2, and ask what can be gained from these connections. In Section A.3 we show that p–Čech thickenings of finite metric spaces are homotopy equivalent to simplicial complexes with one vertex for each point in the metric space. We derive the persistent homology diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of a family of discrete metric spaces in Section A.4, and we describe the 0–dimensional persistent homology of the p–Vietoris–Rips and p–Čech metric thickenings of an arbitrary metric space in Section A.5. We consider crushings in Section A.6. In Section A.7, we show that the main properties we prove for the (intrinsic) p–Čech metric thickening also hold for the ambient p–Čech metric thickening.

A.1 Metrization of the weak topology

For $1 \le q < \infty$ the q–Wasserstein distance metrizes the weak topology, as we explain here for completeness.

Definition A.1 Let X be a metric space. The Lévy–Prokhorov metric on $\mathcal{P}_X$ is given by the formula

$$d_{\mathrm{LP}}(\alpha,\beta) := \inf\{\varepsilon > 0 \mid \alpha(E) \le \beta(E^{\varepsilon}) + \varepsilon, \beta(E) \le \alpha(E^{\varepsilon}) + \varepsilon \text{ for all } E \in \mathfrak{B}(X)\}.$$

Here $E^{\varepsilon} = \bigcup_{x\in E} B_{\varepsilon}(x)$ is the open ε–neighborhood of E in $\mathcal{P}_X$, and $\mathfrak{B}(X)$ is the Borel σ–algebra.

We state two theorems that we will use:[3]

Theorem [15, Theorem 3.1.4] *The weak topology on the set $\mathcal{P}_X$ is generated by the Lévy–Prokhorov metric.*[4]

Theorem [39, Theorem 2] *On a metric space X, for any α and β in $\mathcal{P}_X$, one has*

$$(d_{\mathrm{LP}})^2 \le d^X_{\mathrm{W},1} \le (\mathrm{diam}(X)+1)d_{\mathrm{LP}}.$$

Corollary A.2 *On a bounded metric space X, for any $q \in [1,\infty)$, the q–Wasserstein metric generates the weak topology on $\mathcal{P}_X$.*

Proof From [39, Theorem 2; 15, Theorem 3.1.4], we know $d^X_{\mathrm{W},1}$ generates the weak topology. For other values of q, notice that, for any coupling μ between α and β,

$$\left(\int_X d_X^q(x,x')\,\mu(dx\times dx')\right)^{\frac{1}{q}} \le (\mathrm{diam}(X))^{(q-1)/q}\left(\int_X d_X(x,x')\,\mu(dx\times dx')\right)^{\frac{1}{q}}.$$

This implies $d^X_{\mathrm{W},q} \le (\mathrm{diam}(X))^{(q-1)/q}(d^X_{\mathrm{W},1})^{1/q}$. Along with $d^X_{\mathrm{W},1} \le d^X_{\mathrm{W},q}$, this implies that, on a bounded metric space, all q–Wasserstein metrics with q finite are equivalent and generate the weak topology on $\mathcal{P}_X$. □

A.2 Min-type Morse theory

The paper [8] by Baryshnikov, Bubenik, and Kahle studies a Morse theory for min-type functions; see also [21; 38; 54]. The following notation is from [8, Section 3.1]. Let X be a compact metric space (called the *parameter space*), let M be a compact smooth manifold perhaps with boundary, and let $f\colon X\times M \to \mathbb{R}$ be a continuous function. For each $x \in X$, define $f_x\colon M \to \mathbb{R}$ by $f_x(m) = f(x,m)$. Let $\nabla f_x\colon M \to \mathbb{R}$ be the gradient of f_x with respect to m. We furthermore assume that the function

[3]The first statement is from [15]. We emphasize that we use $\mathcal{P}_X$ to denote the set of all Radon probability measures on X whereas the author instead uses $\mathcal{P}_r(X)$.

[4]Theorem 3.1.4 in [15] proves a stronger result, namely that the claim is true within the set of τ–additive probability measures. The restricted version we use follows from the fact that every Radon measure is τ–additive; see [14, Proposition 7.2.2(i)].

$X \times M \to \mathbb{R}$ defined by $(x, m) \mapsto \nabla f_x(m)$ is continuous. There is then an analogue of Morse theory, called "min-type Morse theory", for the min-type function $\tau\colon M \to \mathbb{R}$ defined by $\tau(m) = \min_{x\in X} f_x(m)$.

Whereas [8] uses min-type Morse theory to study configurations of hard spheres, we instead propose the use of min-type Morse theory to study p–Čech metric thickenings, as follows. Let X be a *finite* metric space with $n+1$ points. Define $M = \Delta_n$ to be the n–simplex on $n+1$ vertices; a point $m \in \Delta_n$ is given in barycentric coordinates as $m = (m_0, \dots, m_n)$ with $m_i \geq 0$ and $\sum_i m_i = 1$. For $x \in X$, let $f_x\colon \Delta_n \to \mathbb{R}$ be defined by $f_x(m) = \sum_i m_i d_X^p(x, x_i)$; note that this is equal to the p^{th} power of the p–Fréchet function, namely to $F^p_{\alpha,p}(x) = \sum_i m_i d_X^p(x, x_i)$, when α is the measure $\alpha = \sum_i m_i \delta_{x_i}$. So $f\colon X \times \Delta_n \to \mathbb{R}$ is defined by $f(x, m) = f_x(m)$. Note that each gradient $\nabla f_x\colon \Delta_n \to \mathbb{R}$ is linear, and hence continuous, and so the joint function $X \times \Delta_n \to \mathbb{R}$ given by $(x, m) \mapsto \nabla f_x(m)$ is continuous since X is discrete. The function $\tau\colon \Delta_n \to \mathbb{R}$ is then defined by $\tau(m) = \min_{x\in X} f_x(m)$; note that this is equal to $\mathrm{rad}_p^p(\alpha) = \inf_{x\in X} F^p_{\alpha,p}(x)$ for $\alpha = \sum_i m_i \delta_{x_i}$. By Lemma 4.11 and its proof, we have not only a homeomorphism $\mathcal{P}_X \cong \Delta_n$, but also a homeomorphism

$$\check{\mathrm{C}}_p(X; r) = \mathrm{rad}_p^{-1}((-\infty, r)) \cong \tau^{-1}((-\infty, r^p)),$$

meaning that the p–Čech metric thickenings are homeomorphic to the sublevel sets of the min-type function τ. In this setting, we have the additional convenience that each function f_x is affine.

Question A.3 Can the machinery from [8] be used to prove new results about p–Čech metric thickenings, such as homotopy types? A first step in this direction might be to use their balanced criterion (which derives from Farkas' lemma) to help identify which points are topological regular points or critical points of τ.

Question A.4 We have restricted to X finite (with $n+1$ points) so that $M = \Delta_n$ will be a manifold with boundary. Can one build up towards letting X be a manifold, such as a circle or n–sphere?

A.3 Finite p–Čech metric thickenings are homotopy equivalent to simplicial complexes

The p–Vietoris–Rips and p–Čech metric thickenings we consider are based on the corresponding simplicial complexes and are closely related to them. The most direct relationship is given by Lemma 4.12 in the case of finite metric spaces and $p = \infty$. Further similarities in the case of totally bounded metric spaces and $p = \infty$ are observed in the persistence diagrams, as shown in Corollary 5.10. For $p < \infty$, the metric thickenings are not as directly related to the corresponding simplicial complexes. However, in Section A.3, we establish that all p–Čech metric thickenings on finite metric spaces are homotopy equivalent to simplicial complexes (although generally not the corresponding Čech simplicial complexes).

Theorem F *Let (X, d_X) be a finite metric space with $n+1$ points. For any $p \in [1, \infty]$ and any $r > 0$, $\check{\mathrm{C}}_p(X; r)$ is homotopy equivalent to a simplicial complex on $n+1$ vertices, consisting of the simplices contained in the homeomorphic image of $\check{\mathrm{C}}_p(X; r)$ in the standard n–simplex.*

Proof The result holds for the case $p = \infty$ by Lemma 4.12, so we will suppose $p \in [1, \infty)$. We begin with some background notation and observations. Let σ be a simplex in a Euclidean space, let $z_0 \in \sigma$ and let σ_{z_0} be the union of the closed faces of σ not containing z_0 (we always have $\sigma_{z_0} \subseteq \partial\sigma$, and $\sigma_{z_0} = \partial\sigma$ if z_0 is in the interior of σ). Then there is a continuous function $P : \sigma \setminus \{z_0\} \to \sigma_{z_0}$ defined by projecting radially from z_0. Furthermore, if $C \subseteq \sigma$ is convex and contains z_0, then for any $z \in \sigma \setminus C$ the line segment connecting z and $P(z)$ is contained in $\sigma \setminus C$, since this lies in the line segment connecting z_0 and $P(z)$. Therefore a linear homotopy shows that $P|_{\sigma \setminus C} : \sigma \setminus C \to \sigma_{z_0} \setminus C$ is a strong deformation retraction. We will apply such retractions successively to a simplex and its faces as described below.

Let $X = \{x_0, \dots, x_n\}$. If $\alpha = \sum_i a_i \delta_{x_i}$, then using the notation for the Fréchet function from Section 2, $F^p_{\alpha,p}(x_j) = \sum_i a_i d^p_X(x_i, x_j)$. Thus $\alpha \in \check{\mathrm{C}}_p(X; r)$ if and only if $\sum_i a_i d^p_X(x_i, x_j) < r^p$ for some j. Let $\Delta = \{(y_0, \dots, y_n) \in \mathbb{R}^{n+1} \mid \sum_i y_i = 1 \text{ with } y_i \geq 0 \text{ for all } i\}$ be the standard n–simplex in $\mathbb{R}^{n+1}$. By Lemma 4.11, $\check{\mathrm{C}}_p(X; r)$ is homeomorphic to

$$Y = \left\{ (y_0, \dots, y_n) \in \Delta \;\middle|\; \sum_i y_i d^p_X(x_i, x_j) < r^p \text{ for some } j \right\}.$$

Equivalently, Y is a sublevel set of the function $\rho \colon \Delta \to \mathbb{R}$ given by

$$\rho(y_0, \dots, y_n) = \min_j \left\{ \sum_i y_i d^p_X(x_i, x_j) \right\}.$$

We note that Y contains the vertices of Δ by the assumption that $r > 0$. If $Y = \Delta$, then $\check{\mathrm{C}}_p(X; r) \cong \Delta$, and Δ is homeomorphic to a simplex with vertex set X, as required. If not, then $\Delta \setminus Y$ is convex as it is the intersection of half-spaces and Δ, so we may project radially from any point $z_0 \in \Delta \setminus Y$, as above. This shows $Y \simeq Y \cap \Delta_{z_0}$, where Δ_{z_0} is the union of the closed faces of Δ not containing z_0.

Since $Y \cap \Delta_{z_0}$ is contained in the boundary of Δ, we will next verify that we can define retractions within the $(n-1)$–dimensional faces of Δ contained in Δ_{z_0} that contain a point not in Y. More generally, we will repeat, for successively lower dimensional faces, inductively obtaining a sequence of strong deformation retractions $Y_n \to Y_{n-1} \to \cdots \to Y_0$, where $Y_n = Y$. Each Y_k will consist of all closed faces of Δ contained in Y, along with some subset of the remaining k–dimensional closed faces intersected with Y. Thus Y_0 will be a simplicial complex consisting of the closed faces of Δ contained in Y and will be homotopy equivalent to $\check{\mathrm{C}}_p(X; r)$, as required.

We will use induction, so suppose Y_k meets the description above. Let $\sigma_1, \dots, \sigma_m$ be those remaining k–dimensional faces of Δ whose intersections with Y are contained in Y_k and that contain a point in their

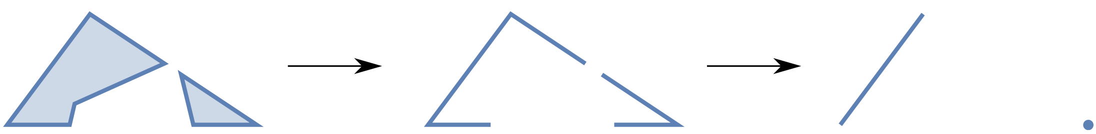

Figure 2: The sequence of deformation retractions used in the proof of Theorem F collapses a subset of a simplex to a simplicial complex on its vertices. In this example $n = 2$, and the sequence of deformation retractions is $Y_2 \to Y_1 \to Y_0$.

interior that is not in Y_k. Let $\tau_1, \dots, \tau_{m'}$ be the remaining k–dimensional faces whose intersections with Y are contained in Y_k and that have only some boundary point not in Y_k. On each σ_l, we may choose an interior point not in Y_k and project radially from this point as above. The homotopy is constant on the boundary, so we may define a homotopy on Y_k that simultaneously retracts each $\sigma_l \cap Y_k$ to $\partial\sigma_l \cap Y_k$ and is constant on all points not in any σ_l. Thus Y_k deformation retracts onto a subset Y'_k consisting of faces of Δ that are contained in Y, along with $\tau_l \cap Y_k$ for all l.

Next, choose any point $y \in \partial\tau_1 \setminus Y'_k$. Note that projecting radially from y as above affects any simplex in $\partial\tau_1$ that contains y, so we must extend this to a homotopy H on Y'_k in a way that is consistent on the simplices containing y. For any τ_l containing y, let H be defined on $\tau_l \cap Y'_k$ by the radial projection from y, as above. For any two τ_l and $\tau_{l'}$ containing y, the two definitions of H on $\tau_l \cap \tau_{l'} \cap Y$ are consistent, as they are both linear homotopies. We also let H be constant on any point in a face of Δ not containing y. This includes all faces of Δ contained in Y and all $\tau_l \cap Y$ such that $y \notin \tau_l$. The definitions are consistent on their overlap, since if τ_l contains y, then radial projection from y is constant on the faces of τ_l that do not contain y. Therefore H is a well-defined homotopy, which shows that Y'_k deformation retracts onto the subset that excludes the interiors of all τ_l containing y. We can repeat this for the remaining set of τ_l until all have been retracted. Composing these deformation retractions, we have shown that Y_k deformation retracts onto a subset Y_{k-1} consisting of faces of Δ contained in Y along with the intersections of Y with a subset of the remaining $(k-1)$–dimensional faces of Δ (subsets of the boundaries of those k–dimensional simplices that were collapsed). This completes the inductive step, so we obtain the sequence $Y_n \to Y_{n-1} \to \cdots \to Y_0$ of strong deformation retractions, as required. □

Corollary A.5 *Let (X, d_X) be a finite metric space, let $p \in [1, \infty]$ and, for any $r > 0$, let $\mathrm{S}(X; r)$ be the simplicial complex from Theorem F that is homotopy equivalent to $\check{\mathrm{C}}_p(X; r)$. These simplicial complexes form a filtration $\mathrm{S}(X; \cdot)$, and for any integer $k \geq 0$, $H_k \circ \check{\mathrm{C}}_p(X; \cdot)$ and $H_k \circ \mathrm{S}(X; \cdot)$ are isomorphic persistence modules.*

Proof The $p = \infty$ case again holds by Lemma 4.12, so let $p \in [1, \infty)$. We will identify $\check{\mathrm{C}}_p(X; r)$ with its homeomorphic image in the standard simplex. By Theorem F, $\mathrm{S}(X; r)$ consists of the simplices contained in $\check{\mathrm{C}}_p(X; r)$. So if $r_1 \leq r_2$, the fact that $\check{\mathrm{C}}_p(X; r_1) \subseteq \check{\mathrm{C}}_p(X; r_2)$ implies $\mathrm{S}(X; r_1) \subseteq \mathrm{S}(X; r_2)$, and thus $\mathrm{S}(X; \cdot)$ is a filtration of simplicial complexes.

Since in the proof of the theorem we constructed a deformation retraction $\check{\mathrm{C}}_p(X; r) \to \mathrm{S}(X; r)$, the inclusion $\mathrm{S}(X; r) \hookrightarrow \check{\mathrm{C}}_p(X; r)$ is a homotopy equivalence for each r. Therefore the induced maps on homology give isomorphisms $H_k(\mathrm{S}(X; r)) \cong H_k(\check{\mathrm{C}}_p(X; r))$ for each r. Furthermore, the following diagram commutes for all $r_1 < r_2$, since all maps are inclusions:

$$\begin{tikzcd}
\check{\mathrm{C}}_p(X; r_1) \arrow[r, hook] & \check{\mathrm{C}}_p(X; r_2) \\
\mathrm{S}(X; r_1) \arrow[r, hook] \arrow[u, hook] & \mathrm{S}(X; r_2) \arrow[u, hook]
\end{tikzcd}$$

This shows that the induced maps on homology commute, giving a morphism of persistence modules. Since each vertical map is an isomorphism, this is an isomorphism of persistence modules. □

These results show that p–Čech persistent homology can be computed, at least in principle. The complex $\mathrm{S}(X;r)$ in Corollary A.5 consists of all simplices contained in the r^p–sublevel set of the function ρ in the proof of Theorem F. So finding the filtration $\mathrm{S}(X;\cdot)$ would require finding the maximum of ρ on each face. Calculating persistent homology would involve finding the maximum of ρ on each face of the necessary dimensions.

A.4 Persistence diagrams of $\mathrm{VR}_p(Z_{n+1};\cdot)$ and $\check{\mathrm{C}}_p(Z_{n+1};\cdot)$

In this section, we will calculate the persistence diagrams of the p–Vietoris–Rips and p–Čech metric thickenings of the metric space Z_{n+1}, consisting of $n+1$ points with all interpoint distances equal to 1. Let Δ_n be the n–dimensional simplex on $n+1$ points, and let $\Delta_n^{(k)}$ denote its k–skeleton.

Proposition A.6 *Let Z_{n+1} be the metric space consisting of $n+1$ points with all interpoint distances equal to 1. For $(k/(k+1))^{1/p} < r \le ((k+1)/(k+2))^{1/p}$ with $0 \le k \le n-1$, we have*

$$\mathrm{VR}_p(Z_{n+1};r) \simeq \check{\mathrm{C}}_p(Z_{n+1};r) \simeq \Delta_n^{(k)},$$

and when $r > (n/(n+1))^{1/p}$, both $\mathrm{VR}_p(Z_{n+1};r)$ and $\check{\mathrm{C}}_p(Z_{n+1};r)$ become the n–simplex $\mathcal{P}_{Z_{n+1}}$, which is contractible.

Proof We will use the following observation, for which we omit the proof. Let α be a measure in $\mathcal{P}_{Z_{m+1}}$, where $m+1$ is any positive integer. Then both diam_p and rad_p will obtain the maximum $(m/(m+1))^{1/p}$ only at the uniform measure supported on the $m+1$ points of the metric space Z_{m+1}, ie the barycenter of the m–simplex $\mathcal{P}_{Z_{m+1}}$.

Let k be an integer where $0 \le k \le n-1$. For any r in $((k/(k+1))^{1/p}, ((k+1)/(k+2))^{1/p}]$, we will show both $\mathrm{VR}_p(Z_{n+1};r)$ and $\check{\mathrm{C}}_p(Z_{n+1};r)$ can be deformation retracted to the k–skeleton of the n–simplex $\mathcal{P}_{Z_{n+1}}$, denoted by $\Delta_n^{(k)}$. Since $(k/(k+1))^{1/p} < r \le ((k+1)/(k+2))^{1/p}$, we know both $\mathrm{VR}_p(Z_{n+1};r)$ and $\check{\mathrm{C}}_p(Z_{n+1};r)$ contain the k–skeleton of Δ_n, but not any higher-dimensional skeleta. We can then use the radial projection as in Corollary A.5 to work out the retraction. From the above observation, the barycenter of any simplex in Δ_n with dimension higher than k is not in $\mathrm{VR}_p(Z_{n+1};r)$ and $\check{\mathrm{C}}_p(Z_{n+1};r)$. We can use these barycenters as the basepoint for radial projection. We start from Δ_n: if the radial projection based at its barycenter can be restricted to $\mathrm{VR}_p(Z_{n+1};r)$ or $\check{\mathrm{C}}_p(Z_{n+1};r)$, then it will retract them onto their intersections with the $(n-1)$–skeleton $(\Delta_n)^{(n-1)}$. From the proof of Corollary A.5, we know the restriction is well defined for $\check{\mathrm{C}}_p(Z_{n+1};r)$. Here, it is also well defined for $\mathrm{VR}_p(Z_{n+1};r)$, because our basepoint for the radial projection is an interior maximum of the quadratic function diam_p^p, and diam_p^p will be concave along any line that passes through the basepoint. This in turn shows that the line segment connecting two points outside of $\mathrm{VR}_p(Z_{n+1};r)$ is also outside of $\mathrm{VR}_p(Z_{n+1};r)$. We can continue the retraction process inductively on $\mathrm{VR}_p(Z_{n+1};r)\cap\Delta_n^{(l)}$ and $\check{\mathrm{C}}_p(Z_{n+1};r)\cap\Delta_n^{(l)}$, where $k<l\le n-1$, with radial projections based at barycenters of l–simplices of Δ_n. This results in a retraction onto $\mathrm{VR}_p(Z_{n+1};r)\cap\Delta_n^{(l-1)}$ and $\check{\mathrm{C}}_p(Z_{n+1};r)\cap\Delta_n^{(l-1)}$, respectively. Eventually, we will deformation retract onto $\mathrm{VR}_p(Z_{n+1};r)\cap\Delta_n^{(k)}$ and $\check{\mathrm{C}}_p(Z_{n+1};r)\cap\Delta_n^{(k)}$, which are both equal to $\Delta_n^{(k)}$. □

Corollary A.7 *Let n be a positive integer and let Z_{n+1} be the metric space consisting of $n+1$ points with interpoint distance equal to 1. The persistence diagrams of $\mathrm{VR}_p(Z_{n+1};\cdot)$ and $\check{\mathrm{C}}_p(Z_{n+1};\cdot)$ are the same and are of the form*

$$\mathrm{dgm}^{\mathrm{VR}}_{k,p}(Z_{n+1})=\mathrm{dgm}^{\check{\mathrm{C}}}_{k,p}(Z_{n+1})=\begin{cases}\left(0,\left(\frac{1}{2}\right)^{1/p}\right)^{\otimes n}\oplus(0,\infty) & \text{if } k=0,\\ \left(\left(k/(k+1)\right)^{1/p},\left((k+1)/(k+2)\right)^{1/p}\right)^{\otimes\binom{n}{k+1}} & \text{if } 0<k\le n-1,\\ \varnothing & \text{if } k>n-1.\end{cases}$$

The superscripts denote the multiplicity of a point in the persistence diagram.

Proof For any integer k with $0<k<n-1$, we know the homology of the k–skeleton of an n–simplex Δ_n is given by

$$H_l(\Delta_n^{(k)},\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } l=0,\\ \mathbb{Z}^{\binom{n}{k+1}} & \text{if } l=k,\\ \varnothing & \text{otherwise.}\end{cases}$$

We get the result by combining this with the previous result on the homotopy types of $\mathrm{VR}_p(Z_{n+1};\cdot)$ and $\check{\mathrm{C}}_p(Z_{n+1};\cdot)$. □

A.5 Zero-dimensional persistent homology of $\mathrm{VR}_p(X;\cdot)$ and $\check{\mathrm{C}}_p(X;\cdot)$

For a finite metric space X, we will show that the 0–dimensional persistent homology of $\mathrm{VR}_p(X;\cdot)$ and $\check{\mathrm{C}}_p(X;\cdot)$ are the same, and that they both recover the single-linkage clustering up to a constant factor related to p.

Lemma A.8 *For X a finite metric space, the birth time for all intervals in the 0–dimensional barcodes of $\mathrm{VR}_p(X;r)$ and $\check{\mathrm{C}}_p(X;r)$ is zero.*

Proof Since all delta measures δ_x have diam_p and rad_p equal to zero, it suffices to show, for any measure α in $\mathrm{VR}_p(X;r)$ (or $\check{\mathrm{C}}_p(X;r)$), that there is a path in $\mathrm{VR}_p(X;r)$ (or $\check{\mathrm{C}}_p(X;r)$) that connects α with some delta measure.

For α a measure in $\mathrm{VR}_p(X;r)$,

$$\int_X (d^X_{\mathrm{W},p}(\alpha,\delta_x))^p\,\alpha(dx)=(\mathrm{diam}_p(\alpha))^p<r^p.$$

So there is some $x_0 \in \mathrm{supp}(\alpha)$ such that $d^X_{\mathrm{W},p}(\alpha, \delta_{x_0}) < r^p$. We then pick the path $\alpha_t = (1-t)\alpha + t\delta_{x_0}$ for $t \in [0,1]$. Now consider

$$\begin{aligned}\mathrm{diam}_p(\alpha_t) &= \Big(\int_X (d^X_{\mathrm{W},p}(\alpha_t, \delta_x))^p\, \alpha_t(dx)\Big)^{\frac{1}{p}} = \Big(\int_X \big((1-t)d^X_{\mathrm{W},p}(\alpha,\delta_x) + t\, d^X_{\mathrm{W},p}(\delta_{x_0},\delta_x)\big)^p\, \alpha_t(dx)\Big)^{\frac{1}{p}}\\ &\le (1-t)\Big(\int_X (d^X_{\mathrm{W},p}(\alpha,\delta_x))^p\, \alpha_t(dx)\Big)^{\frac{1}{p}} + t\Big(\int_X (d^X_{\mathrm{W},p}(\delta_{x_0},\delta_x))^p\, \alpha_t(dx)\Big)^{\frac{1}{p}}\\ &\le (1-t)\big((1-t)\,\mathrm{diam}_p^p(\alpha) + t(d^X_{\mathrm{W},p}(\alpha,\delta_{x_0}))^p\big)^{1/p} + t\big((1-t)(d^X_{\mathrm{W},p}(\alpha,\delta_{x_0}))^p\big)^{1/p}\\ &< (1-t)r + t(1-t)^{1/p} r < r.\end{aligned}$$

This shows $\alpha_t \in \mathrm{VR}_p(X;r)$, and the path is continuous by Proposition 2.4.

For α a measure in $\check{\mathrm{C}}_p(X;r)$, there is some x_0 with $d^X_{\mathrm{W},p}(\alpha,\delta_{x_0}) < r$. Then $d^X_{\mathrm{W},p}((1-t)\alpha + t\delta_{x_0}, \delta_{x_0}) = (1-t)d^X_{\mathrm{W},p}(\alpha,\delta_{x_0}) < r$. This shows α_t is a continuous path in $\check{\mathrm{C}}_p(X;r)$. □

Proposition A.9 *Let (X, d_X) be a finite metric space. Then the 0–dimensional persistence modules of $\mathrm{VR}_p(X;\cdot)$ and $\check{\mathrm{C}}_p(X;\cdot)$ are both equal to the 0–dimensional persistence module of the Vietoris–Rips simplicial complex filtration of the rescaled metric space $\big(X, \big(\frac{1}{2}\big)^p d_X\big)$.*

Proof From Lemma A.8, we know that all the bars for the 0–dimensional persistence module are born at 0. Let x be a point in (X,d), and let x' be a closest point to x in the finite metric space X. Then it suffices to show that δ_x and $\delta_{x'}$ will only be in the same connected component of $\mathrm{VR}_p((X,d);r)$ or $\check{\mathrm{C}}_p((X,d);r)$ for any $r > \big(\frac{1}{2}\big)^p d_X(x,x')$.

Since the path $\gamma_t = (1-t)\delta_x + t\delta_{x'}$ has maximal diam_p and rad_p given by $\big(\frac{1}{2}\big)^p d_X(x,x')$, we know δ_x and $\delta_{x'}$ will be in the same connected component of $\mathrm{VR}_p((X,d);r)$ or $\check{\mathrm{C}}_p((X,d);r)$ for any $r > \big(\frac{1}{2}\big)^p d_X(x,x')$.

On the other hand, let (Y, d_Y) be the metric space $(\{x,x'\}, d_X|_{\{x,x'\}})$. Then the map $g_{x,x'}\colon (X,d) \to (Y,d_Y)$ sending x to x and all other points to x' is a 1–Lipschitz map. Also, $g_{x,x'}$ induces a continuous map $G_{x,x'}\colon \mathrm{VR}_p(X;r) \to \mathrm{VR}_p(Y;r)$ for any $r > 0$ via pushforward. Note that, for any $r \le \big(\frac{1}{2}\big)^p d_X(x,x')$, the images of δ_x and $\delta_{x'}$ are not in the same connected component of $\mathrm{VR}_p(Y;r)$. By the continuity of $G_{x,x'}$, the delta masses δ_x and $\delta_{x'}$ cannot be in the same connected component of $\mathrm{VR}_p(X;r)$ either. A similar argument works for $\check{\mathrm{C}}_p(X;r)$. □

A.6 Crushings and the homotopy type distance

A crushing is a particular type of deformation retraction that doesn't increase distances. In this section we consider the effects of a crushing applied to the metric space underlying a metric thickening.

Let X be a metric space and $A \subseteq X$ a subspace. Following [42], a *crushing* from X to A is defined as a distance-nonincreasing strong deformation retraction from X to A, that is, a continuous map

$H : X \times [0,1] \to X$ satisfying

(i) $H(x,1) = x$, $H(x,0) \in A$ and $H(a,t) = a$ if $a \in A$,

(ii) $d_X(H(x,t'), H(y,t')) \le d_X(H(x,t), H(y,t))$ whenever $t' \le t$.

When this happens, we say that X *can be crushed onto* A.

In [42, Proposition 2.2], Hausmann proves that, if X can be crushed onto A, then inclusion $\mathrm{VR}(A;r) \hookrightarrow \mathrm{VR}(X;r)$ of Vietoris–Rips simplicial complexes is a homotopy equivalence. A similar result is proven for Vietoris–Rips and Čech metric thickenings with $p = \infty$ in [2, Appendix B]. We study how more general metric thickenings behave with respect to crushings. We begin with some preliminaries:

Lemma A.10 *Let X be a complete separable metric space and let ϕ be an L–Lipschitz function that is absolutely bounded by a constant $C > 0$. Then, for any $\alpha, \beta \in \mathcal{P}_X$,*
$$\left| \int_X \phi(x)\,\alpha(dx) - \int_X \phi(x)\,\beta(dx) \right| \le (L + 2C) d_{\mathrm{LP}}(\alpha, \beta).$$

Proof Let $\varepsilon = d_{\mathrm{LP}}(\alpha,\beta)$. Then by the expression of the Lévy–Prokhorov metric in [15, Theorem 3.1.5], there is a coupling between α and β by $\mu \in \mathcal{P}_{X \times X}$ such that
$$\mu(\{(x,x') \in X \times X \mid d_X(x,x') > \varepsilon\}) \le \varepsilon.$$
Let E be the set $\{(x,x') \in X \times X \mid d_X(x,x') > \varepsilon\}$. Then
$$\begin{aligned}\left| \int_X \phi(x)\,\alpha(dx) - \int_X \phi(x)\,\beta(dx) \right| &= \left| \iint_{X\times X} \phi(x) - \phi(x')\,\mu(dx \times dx') \right| \\ &\le \iint_E |\phi(x) - \phi(x')|\,\mu(dx\times dx') + \iint_{E^c} |\phi(x)-\phi(x')|\,\mu(dx \times dx') \\ &\le 2C\varepsilon + L \iint_{E^c} d_X(x,x')\,\mu(dx\times dx') = (L+2C) d_{\mathrm{LP}}(\alpha,\beta). \qquad \square\end{aligned}$$

Proposition A.11 *Let X be a complete separable metric space such that there is a crushing from X onto a subset $A \subset X$. Then there is an induced deformation retraction from $\mathcal{P}_X$ onto $\mathcal{P}_A$.*

Proof Let H be a crushing from X to A, so H is a continuous map from $X \times [0,1]$ to X. We use the notation $f_t(x)$ to denote the map $H(x,t) : X \to X$ for any fixed t in $[0,1]$. Then we can define a map $\widetilde{H} : \mathcal{P}_X \times [0,1] \to \mathcal{P}_X$ via
$$\widetilde{H}(\alpha,t) = (f_t)_\sharp(\alpha).$$
For continuity, let (α_n, t_n) be a sequence that converges to $(\alpha_\infty, t_\infty)$. Then, for any bounded continuous function $\gamma(x)$ on X, we have
$$\int_X \gamma(x) (f_{t_n})_\sharp(\alpha_n)(dx) = \int_X \gamma \circ f_{t_n}(x)\,\alpha_n(dx).$$
Without loss of generality, we may assume γ is 1–Lipschitz and absolutely bounded by $C > 0$. Therefore, every $\gamma \circ f_{t_n}(x)$ is 1–Lipschitz and bounded by C for any n. We use the notation $I_{i,j}$ to denote $\int_X \gamma \circ f_{t_i}(x)\,\alpha_j(dx)$. Then $|I_{i,j}|$ is uniformly bounded by C. Lemma A.10 then shows that, for any i,
$$|I_{i,j} - I_{i,\infty}| \le (1 + 2C) d_{\mathrm{LP}}(\alpha_j, \alpha_\infty).$$

The uniform bound on $|I_{i,\infty}|$ implies that $I_{i,\infty}$ converges to $I_{\infty,\infty}$. For any $\varepsilon > 0$, we can find an N such that, for any $n > N$, $|I_{n,\infty} - I_{\infty,\infty}| \le \frac{1}{2}\varepsilon$ and

$$d_{\mathrm{LP}}(\alpha_n, \alpha_\infty) \le \frac{\varepsilon}{2(1+2C)}.$$

Then

$$|I_{n,n} - I_{\infty,\infty}| \le |I_{n,n} - I_{n,\infty}| + |I_{n,\infty} - I_{\infty,\infty}| \le \varepsilon.$$

This shows that $I_{i,i}$ converges to $I_{\infty,\infty}$, and therefore $\widetilde{H}$ is continuous. As H satisfies $H(x,1) = x$, $H(x,0) = f_0(x) \in A$ and $H(a,t) = f_t(a) = a$ if $a \in A$, we get $\widetilde{H}(\alpha,1) = (\mathrm{id}_{\mathcal{P}_X})_\sharp(\alpha) = \alpha$, $\widetilde{H}(\alpha,0) = (f_0)_\sharp(\alpha) \in \mathcal{P}_A$ for any $\alpha \in \mathcal{P}_X$, and $\widetilde{H}(\beta,t) = \beta$ for $\beta \in \mathcal{P}_A$. Therefore $\widetilde{H}$ is a indeed a strong deformation retraction from $\mathcal{P}_X$ to $\mathcal{P}_A$. □

In the spirit of [2, Lemma B.1], we can apply the above deformation retraction to the sublevel set filtrations of a set of invariants that includes $\mathrm{i}_{q,p}$ and rad_p.

Theorem G *Let* i *be an invariant such that, for any metric spaces* X *and* Y *and any* 1*–Lipschitz map* $f\colon X \to Y$, *the induced map on* $\mathcal{P}_X$ *does not increase the values of* i. *More precisely, for any* $\alpha \in \mathcal{P}_X$, *we require* $\mathrm{i}^Y(f_\sharp(\alpha)) \le \mathrm{i}^X(\alpha)$. *Then, for any complete separable metric space* X *and any subset* A *such that* X *can crushed onto* A, *we have*

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{i}^X), (\mathcal{P}_A, \mathrm{i}^A)) = 0.$$

Proof Let H be the crushing from X onto A and let f be $H(x,0)$. Then as both f and the inclusion $\iota\colon A \to X$ are 1–Lipschitz maps, the above condition on the invariant i implies

- $f_\sharp$ is a 0–map from (X, i^X) to (A, i^A), and
- $\iota_\sharp$ is a 0–map from (A, i^A) to (X, i^X).

Note that $f_\sharp \circ \iota_\sharp$ is the identify map on $\mathcal{P}_A$ and $\iota_\sharp \circ f_\sharp$ is 0–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathrm{i}^X, \mathrm{i}^X)$. Therefore $d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{i}^X), (\mathcal{P}_A, \mathrm{i}^A)) = 0$. □

Remark A.12 The crushing result could be leveraged to analyze the persistent homology of a space using the persistent homology of embedded submanifolds; see [74].

A.7 Ambient filtrations from Lipschitz invariants

We now show that the main properties for intrinsic p–Čech metric thickenings also hold for ambient p–Čech metric thickenings.

Let M be a metric space and let X be a subset of M. Then any function i^M naturally restricts to $\mathcal{P}_X$ and induces a filtration. We have the following stability result, given that i^M is C–Lipschitz with respect to $d^M_{\mathrm{W},\infty}$ for some $C > 0$.

Theorem H *Let* M *be a metric space and let* i^M *be a* C*–Lipschitz function on* $\mathcal{P}_X$ *with respect to* $d^M_{\mathrm{W},\infty}$. *Then, for any two totally bounded subsets* X *and* Y *in* M,

$$d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{i}^M), (\mathcal{P}_Y, {}^M)) \le C d^M_{\mathrm{H}}(X,Y) \quad \textit{and} \quad d_{\mathrm{HT}}((\mathcal{P}^{\mathrm{fin}}_X, \mathrm{i}^M), (\mathcal{P}^{\mathrm{fin}}_Y, {}^M)) \le C d^M_{\mathrm{H}}(X,Y).$$

Here d_{H}^M is the Hausdorff distance in M.

Proof Overall, the proof follows a construction similar to that in Lemma 5.16 in the setting of Hausdorff distance. For $\eta > 2\, d_{\mathrm{H}}^M(X,Y)$ and $\delta > 0$, we fix finite δ–nets $U \subset X$ of X and $V \subset Y$ of Y. By the triangle inequality, $d_{\mathrm{H}}(U,V) < \frac{1}{2}\eta + 2\delta$. For any point $u \in U$ there is a point v in V with $d_M(u,v) < \frac{1}{2}\eta + 2\delta$. Through this construction, there are maps $\varphi\colon U \to V$ and $\psi\colon V \to U$ with

- $d_M(u,\varphi(u)) < \frac{1}{2}\eta + 2\delta$ for any $u \in U$,
- $d_M(v,\psi(v)) < \frac{1}{2}\eta + 2\delta$ for any $v \in V$,
- $\max(\operatorname{dis}(\varphi), \operatorname{dis}(\psi), \operatorname{codis}(\varphi,\psi)) \le \eta + 4\delta$.

We use the notations $\widehat{\Phi}$, $\widehat{\Psi}$, H_t^X and H_t^Y as in Lemma 5.16. Let α be a measure in $\mathcal{P}_X$ and β a measure in $\mathcal{P}_Y$. The last item implies that the following bound from Lemma 5.16 still holds here: we have $d_{\mathrm{W},\infty}^M(H_t^X(\alpha),\alpha) < \eta + 6\delta$ and $d_{\mathrm{W},\infty}^M(H_t^Y(\beta),\beta) < \eta + 6\delta$. Similar to the proof of Theorem A, it suffices to show that

- $\widehat{\Phi}\colon \mathcal{P}_X \to \mathcal{P}_Y$ is a $\left(\frac{1}{2}\eta+3\delta\right)C$–map from $(\mathcal{P}_X, \mathrm{i}^M)$ to $(\mathcal{P}_Y, \mathrm{i}^M)$,
- $\widehat{\Psi}\colon \mathcal{P}_Y \to \mathcal{P}_X$ is a $\left(\frac{1}{2}\eta+3\delta\right)C$–map from $(\mathcal{P}_Y, \mathrm{i}^M)$ to $(\mathcal{P}_X, \mathrm{i}^M)$,
- $\widehat{\Psi}\circ\widehat{\Phi}\colon \mathcal{P}_X \to \mathcal{P}_X$ is $(\eta+6\delta)C$–homotopic to $\mathrm{id}_{\mathcal{P}_X}$ with respect to $(\mathrm{i}^M, \mathrm{i}^M)$,
- $\widehat{\Phi}\circ\widehat{\Psi}\colon \mathcal{P}_X \to \mathcal{P}_X$ is $(\eta+6\delta)C$–homotopic to $\mathrm{id}_{\mathcal{P}_Y}$ with respect to $(\mathrm{i}^M, \mathrm{i}^M)$.

We will only show the first and the third items; the rest can be proved similarly. For the first item, using the fact that i^M is C–Lipschitz with respect to $d_{\mathrm{W},\infty}^M$ and the estimate in Lemma 5.15, we get

$$\begin{aligned}
\mathrm{i}^M(\widehat{\Phi})(\alpha) = \mathrm{i}^M(\varphi_\sharp(\Phi_U(\alpha))) &\le \mathrm{i}^M(\Phi_U(\alpha)) + d_{\mathrm{W},\infty}^M\big(\varphi_\sharp(\Phi_U(\alpha)), \Phi_U(\alpha)\big)C \\
&\le \mathrm{i}^M(\Phi_U(\alpha)) + \left(\tfrac{1}{2}\eta + 2\delta\right)C \\
&\le \mathrm{i}^M(\alpha) + d_{\mathrm{W},\infty}^M(\alpha, \Phi_U(\alpha))C + \left(\tfrac{1}{2}\eta + 2\delta\right)C \\
&\le \mathrm{i}^M(\alpha) + \left(\tfrac{1}{2}\eta + 3\delta\right)C.
\end{aligned}$$

For the third item, by the inequality $d_{\mathrm{W},\infty}^M(H_t^X(\alpha),\alpha) < \eta + 6\delta$, we have

$$\mathrm{i}^M(H_t^X(\alpha)) \le \mathrm{i}^M(\alpha) + (\eta + 6\delta)C. \qquad \square$$

An interesting case is the ambient p–radius, which leads to p–ambient Čech filtrations. Let X be a bounded subset in a metric space M. For any $\alpha \in \mathcal{P}_X$, we define the *p–ambient radius* of α to be

$$\mathrm{rad}_p^M(\alpha) := \inf_{m \in M} F_{\alpha,p}(m) = \inf_{m \in M} d_{\mathrm{W},p}^X(\delta_m, \alpha).$$

Definition A.13 (p–ambient Čech filtration) Let $X \subseteq M$ be metric spaces. For each $r > 0$ and $p \in [1,\infty]$, let the *p–ambient Čech metric thickening at scale r* be

$$\check{\mathrm{C}}_p(X, M; r) := \{\alpha \in \mathcal{P}_X \mid \mathrm{rad}_p^M(\alpha) < r\}.$$

Similarly, the p–ambient Čech metric thickening at scale r with finite support is defined as

$$\check{\mathrm{C}}_p^{\mathrm{fin}}(X, M; r) := \{\alpha \in \mathcal{P}_X^{\mathrm{fin}} \mid \mathrm{rad}_p^M(\alpha) < r\}.$$

Therefore, as a special instance of Theorem H:

Theorem I *Let X and Y be two totally bounded spaces sitting inside a metric space M. Then*

$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_p(X, M; \cdot), H_k \circ \check{\mathrm{C}}_p(Y, M; \cdot)) \le d_{\mathrm{HT}}((\mathcal{P}_X, \mathrm{rad}_p^M), (\mathcal{P}_Y, {}^M)) \le d_{\mathrm{H}}^M(X, Y),$$
$$d_{\mathrm{I}}(H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(X, M; \cdot), H_k \circ \check{\mathrm{C}}_p^{\mathrm{fin}}(Y, M; \cdot)) \le d_{\mathrm{HT}}((\mathcal{P}_X^{\mathrm{fin}}, \mathrm{rad}_p^M), (\mathcal{P}_Y^{\mathrm{fin}}, {}^M)) \le d_{\mathrm{H}}^M(X, Y).$$

Proof According to Lemma 4.9, rad_p^M is 1–Lipschitz with respect to $d_{\mathrm{W},\infty}^M$. We apply Theorem H to get the inequality on the right. The inequality on the left follows from Lemma 2.9. □

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A generalization of moment-angle manifolds with noncontractible orbit spaces

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We generalize the notion of moment-angle manifold over a simple convex polytope to an arbitrary nice manifold with corners. For a nice manifold with corners Q, we first compute the stable decomposition of the moment-angle manifold $\mathcal{Z}_Q$ via a construction called rim-cubicalization of Q. From this, we derive a formula to compute the integral cohomology group of $\mathcal{Z}_Q$ via the strata of Q. This generalizes the Hochster's formula for the moment-angle manifold over a simple convex polytope. Moreover, we obtain a description of the integral cohomology ring of $\mathcal{Z}_Q$ using the idea of partial diagonal maps. In addition, we define the notion of polyhedral product of a sequence of based CW–complexes over Q and obtain similar results for these spaces as we do for $\mathcal{Z}_Q$. Using this general construction, we can compute the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to its canonical torus action from the Davis–Januszkiewicz space of Q. The result leads to the definition of a new notion called the topological face ring of Q, which generalizes the notion of face ring of a simple polytope. Moreover, the topological face ring of Q computes the equivariant cohomology of all locally standard torus actions with Q as the orbit space when the corresponding principal torus bundle over Q is trivial. Meanwhile, we obtain some parallel results for the real moment-angle manifold $\mathbb{R}\mathcal{Z}_Q$ over Q as well.

57S12; 57N65, 57S17, 57S25

1 Introduction

The construction of a moment-angle manifold over a simple polytope is first introduced by Davis and Januszkiewicz in [17]. Suppose P is a simple (convex) polytope with m *facets* (codimension-one faces). A convex polytope in a Euclidean space is called *simple* if every codimension-k face is the intersection of exactly k facets of the polytope. The moment-angle manifold $\mathcal{Z}_P$ over P is a closed connected manifold with an effective action by the compact torus $T^m = (S^1)^m$ whose orbit space is P. It is shown in [17] that many important topological invariants of $\mathcal{Z}_P$ can be computed easily from the combinatorial structure of P. These manifolds play an important role in the research of toric topology. The reader is referred to Buchstaber and Panov [9; 10] for more discussions on the topological and geometrical aspects of moment-angle manifolds.

The notion of moment-angle manifold over a simple convex polytope has been generalized in many different ways. For example, Davis and Januszkiewicz [17] define a class of topological spaces now called *moment-angle complexes* — named by Buchstaber and Panov in [8] — where the simple polytope is replaced by a simple polyhedral complex. Later, Lü and Panov [26] defined the notion of moment-angle complex of a *simplicial poset*. In addition, Ayzenberg and Buchstaber [1] defined the notion of moment-angle spaces over arbitrary convex polytopes (not necessarily simple). Note that in all these generalizations, the orbit spaces of the canonical torus actions are all contractible. Yet an even wider class of spaces called *generalized moment-angle complexes* or *polyhedral products* over simplicial complexes were introduced by Bahri, Bendersky, Cohen and Gitler in [3], which has become the major subject in the homotopy-theoretic study of toric topology.

In this paper, we generalize the construction of moment-angle manifolds by replacing the simple polytope P by a nice manifold with corners Q which is not necessarily contractible. Such a generalization has been considered by Poddar and Sarkar [29] for polytopes with simple holes.

A motive for the study of this generalized construction is to compute the equivariant cohomology ring of locally standard torus actions. Recall that an action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally modeled on the standard representation of T^n on $\mathbb{C}^n$. Then the orbit space $Q = M/T^n$ is a manifold with corners. Conversely, every manifold with a locally standard T^n–action and with Q as the orbit space is equivariantly homeomorphic to the quotient construction $Y/{\sim}$, where Y is a principal T^n–bundle over Q and $\sim$ is an equivalence relation determined by the characteristic function on Q; see Yoshida [35]. Generally speaking, it is difficult to compute the equivariant cohomology ring of M from the corresponding principal bundle Y and the characteristic function on Q. But we will see in Corollary 5.5 that when Y is the trivial T^n–bundle over Q, the equivariant cohomology ring of M can be computed from the strata of Q directly. Examples of such kind include many toric origami manifolds — see Ayzenberg, Masuda, Park and Zeng [2], Cannas da Silva, Guillemin and Pires [12] and Holm and Pires [22] — with coorientable folding hypersurface where the faces of the orbit spaces may be nonacyclic.

Recall that an *n–dimensional manifold with corners* Q is a Hausdorff space with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the transitional functions are homeomorphisms which preserve the codimension of each point. Here the *codimension* $c(x)$ of a point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n_{\geq 0}$ is the number of x_i which are 0. So we have a well-defined map $c\colon Q \to \mathbb{Z}_{\geq 0}$, where $c(q)$ is the codimension of a point $q \in Q$. In particular, the interior Q° of Q consists of points of codimension 0, ie $Q^\circ = c^{-1}(0)$.

Suppose Q is an n–dimensional manifold with corners with $\partial Q \neq \varnothing$. An *open face* of Q of codimension k is a connected component of $c^{-1}(k)$. A (closed) *face* is the closure of an open face. A face of codimension one is called a *facet* of Q. Note that a face of codimension zero in Q is just a connected component of Q.

A manifold with corners Q is said to be *nice* if either its boundary ∂Q is empty or ∂Q is nonempty and any codimension-k face of Q is a component of the intersection of k different facets in Q.

Let Q be a nice n–manifold with corners. Let $\mathcal{F}(Q) = \{F_1, \dots, F_m\}$ be the set of facets of Q. For any subset $J \subseteq [m] = \{1, \dots, m\}$, let

$$F_J = \bigcup_{j \in J} F_j, \quad F_\varnothing = \varnothing, \quad F_{\cap J} = \bigcap_{j \in J} F_j, \quad F_{\cap \varnothing} = Q.$$

It is clear that, for all $J \subseteq J' \subseteq [m]$,

$$F_J \subseteq F_{J'}, \quad F_{\cap J'} \subseteq F_{\cap J}, \quad F_{\cap J} \subseteq F_J.$$

Let $\lambda \colon \mathcal{F}(Q) \to \mathbb{Z}^m$ be a map such that $\{\lambda(F_1), \dots, \lambda(F_m)\}$ is a unimodular basis of $\mathbb{Z}^m \subset \mathbb{R}^m$. Since $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$, we can identify the m–torus $(S^1)^m = \mathbb{R}^m / \mathbb{Z}^m$. The *moment-angle manifold over* Q is defined by

$$\mathcal{Z}_Q = Q \times (S^1)^m / {\sim}, \tag{1}$$

where $(x, g) \sim (x', g')$ if and only if $x = x'$ and $g^{-1} g' \in \mathbb{T}_x^\lambda$ where $\mathbb{T}_x^\lambda$ is the subtorus of $(S^1)^m$ determined by the linear subspace of $\mathbb{R}^m$ spanned by the set $\{\lambda(F_j) \mid x \in F_j\}$. There is a canonical action of $(S^1)^m$ on $\mathcal{Z}_Q$ defined by

$$g' \cdot [(x, g)] = [(x, g' g)] \quad \text{for } x \in Q \text{ and } g, g' \in (S^1)^m. \tag{2}$$

Since the manifold with corners Q is nice and λ is unimodular, it is easy to see from the above definition that $\mathcal{Z}_Q$ is a manifold.

Convention In the rest of this paper, we assume that any nice manifold with corners Q can be equipped with a CW–complex structure such that every face of Q is a subcomplex. In addition, we assume that Q has only finitely many faces. Note that a compact smooth nice manifold with corners always satisfies these two conditions since it is triangulable; see Johnson [25]. But in general we do not require Q to be compact or smooth. We do not assume Q to be connected either.

Similarly to the stable decomposition of (generalized) moment-angle complexes obtained in [3], we have the following stable decomposition of $\mathcal{Z}_Q$.

Theorem 1.1 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. There is a homotopy equivalence*

$$\mathbf{\Sigma}(\mathcal{Z}_Q) \simeq \bigvee_{J \subseteq [m]} \mathbf{\Sigma}^{|J|+1}(Q / F_J), \tag{3}$$

where $\bigvee$ denotes the wedge sum and $\mathbf{\Sigma}$ denotes the reduced suspension.

Here we will not explicitly write down the basepoints for our spaces unless it is necessary to do so.

Corollary 1.2 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. The integral* (*reduced*) *cohomology group of $\mathcal{Z}_Q$ is given by*

$$H^p(\mathcal{Z}_Q) \cong \bigoplus_{J \subseteq [m]} H^{p-|J|}(Q, F_J), \quad \widetilde{H}^p(\mathcal{Z}_Q) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^{p-|J|}(Q / F_J) \quad \text{for all } p \in \mathbb{Z}. \tag{4}$$

Note that when $J = \varnothing$, $H^*(Q, F_\varnothing) = H^*(Q, \varnothing) = H^*(Q) \cong \widetilde{H}^*(Q) \oplus \mathbb{Z}$.

The term "cohomology" of a space X, denoted by $H^*(X)$, in this paper always means singular cohomology with integral coefficients if not specified otherwise.

When Q is acyclic (ie $\widetilde{H}^*(Q) = 0$), we have $H^p(Q, F_J) \cong \widetilde{H}^{p-1}(F_J)$ by a cohomology long exact sequence for the pair (Q, F_J). So, in this case,

$$H^p(\mathcal{Z}_Q) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^{p-|J|-1}(F_J) \quad \text{for all } p \in \mathbb{Z}.$$

This recovers Hochster's formula for the moment-angle manifold over a simple polytope in [10, Theorem 3.2.9]; see also [10, Proposition 3.2.11].

Remark 1.3 There is an analogue of $\mathcal{Z}_Q$ by replacing the group $(S^1)^m$ by $(\mathbb{Z}_2)^m$. The counterpart in the $(\mathbb{Z}_2)^m$ construction, denoted by $\mathbb{R}\mathcal{Z}_Q$, is a special case of the basic construction of Davis [16, Chapter 5] for a mirror space along with a Coxeter system. A formula parallel to Corollary 1.2 for computing the integral cohomology group of $\mathbb{R}\mathcal{Z}_Q$ is contained in Davis [15, Theorem A]; see also [16, Chapter 8]. We call $\mathbb{R}\mathcal{Z}_Q$ the *real moment-angle manifold over Q*.

Given a nice manifold with corners Q with facets $F_1, \dots, F_m$, define

$$\mathcal{R}_Q^* := \bigoplus_{J \subseteq [m]} H^*(Q, F_J). \tag{5}$$

There is a graded ring structure $\Cup$ on $\mathcal{R}_Q^*$ defined as follows:

- If $J \cap J' \neq \varnothing$, then $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\Cup} H^*(Q, F_{J \cup J'})$ is trivial.
- If $J \cap J' = \varnothing$, then $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\Cup} H^*(Q, F_{J \cup J'})$ is the relative cup product $\cup$; see Hatcher [20, page 209].

We can prove the following theorem via the above stable decomposition of $\mathcal{Z}_Q$.

Theorem 1.4 *Let Q be a nice manifold with corners with m facets $F_1, \dots, F_m$. There exists a ring isomorphism* (*up to a sign*) *from $(\mathcal{R}_Q^*, \Cup)$ to the integral cohomology ring of $\mathcal{Z}_Q$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_J)$ up by $|J|$ for every $J \subseteq [m]$.*

It is indicated in [10, Exercise 3.2.14] that Theorem 1.4 holds for any simple polytope. Moreover, we can generalize Theorem 1.4 to describe the cohomology ring of the polyhedral product of any $(\mathbb{D}, \mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j)\}_{j=1}^m$ over Q (see Theorem 4.8). In particular, we have the following result for $\mathbb{R}\mathcal{Z}_Q$.

Theorem 1.5 (Corollary 4.10) *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Then*

$$\Sigma(\mathbb{R}\mathcal{Z}_Q) \simeq \bigvee_{J \subseteq [m]} \Sigma(Q/F_J), \quad H^p(\mathbb{R}\mathcal{Z}_Q) \cong \bigoplus_{J \subseteq [m]} H^p(Q, F_J), \quad \textit{for all } p \in \mathbb{Z}.$$

Moreover, the integral cohomology ring of $\mathbb{R}\mathcal{Z}_Q$ is isomorphic as a graded ring to the ring $(\mathcal{R}^*_Q, \cup)$ where $\cup$ is the relative cup product

$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\cup} H^*(Q, F_{J\cup J'}) \quad \textit{for all } J, J' \subseteq [m].$$

We can describe the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to the canonical action of $(S^1)^m$ as follows.

Let $\boldsymbol{k}$ denote a commutative ring with a unit. For any $J \subseteq [m]$, let $R^J_{\boldsymbol{k}}$ be the subring of the polynomial ring $\boldsymbol{k}[x_1, \dots, x_m]$ defined by

$$R^J_{\boldsymbol{k}} := \begin{cases} \mathrm{span}_{\boldsymbol{k}}\{x^{n_1}_{j_1} \cdots x^{n_s}_{j_s} \mid n_1 > 0, \dots, n_s > 0\} & \text{if } J = \{j_1, \dots, j_s\} \neq \varnothing, \\ \boldsymbol{k} & \text{if } J = \varnothing. \end{cases} \tag{6}$$

We can multiply $f(x) \in R^J_{\boldsymbol{k}}$ and $f'(x) \in R^{J'}_{\boldsymbol{k}}$ in $\boldsymbol{k}[x_1, \dots, x_m]$ and obtain an element $f(x)f'(x) \in R^{J\cup J'}_{\boldsymbol{k}}$.

Definition 1.6 (topological face ring) Let Q be a nice manifold with corners with m facets $F_1, \dots, F_m$. For any coefficient ring $\boldsymbol{k}$, the *topological face ring of Q* over $\boldsymbol{k}$ is defined to be

$$\boldsymbol{k}\langle Q\rangle := \bigoplus_{J\subseteq[m]} H^*(F_{\cap J}; \boldsymbol{k}) \otimes R^J_{\boldsymbol{k}}. \tag{7}$$

Here if $F_{\cap J} = \varnothing$, we use the convention $H^*(\varnothing; \boldsymbol{k}) = \{0\}$.

For any $J, J' \subseteq [m]$, the product $\star$ on $\boldsymbol{k}\langle Q\rangle$,

$$(H^*(F_{\cap J}; \boldsymbol{k}) \otimes R^J_{\boldsymbol{k}}) \otimes (H^*(F_{\cap J'}; \boldsymbol{k}) \otimes R^{J'}_{\boldsymbol{k}}) \xrightarrow{\star} (H^*(F_{\cap(J\cup J')}; \boldsymbol{k}) \otimes R^{J\cup J'}_{\boldsymbol{k}})$$

is defined, for $\phi \in H^*(F_{\cap J}; \boldsymbol{k})$, $f(x) \in R^J_{\boldsymbol{k}}$, $\phi' \in H^*(F_{\cap J'}; \boldsymbol{k})$ and $f'(x) \in R^{J'}_{\boldsymbol{k}}$, by

$$(\phi \otimes f(x)) \star (\phi' \otimes f'(x)) := (\kappa^*_{J\cup J', J}(\phi) \cup \kappa^*_{J\cup J', J'}(\phi')) \otimes f(x)f'(x), \tag{8}$$

where $\kappa_{I',I} : F_{\cap I'} \to F_{\cap I}$ is the inclusion map for any $I \subseteq I' \subseteq [m]$ and $\kappa^*_{I',I} : H^*(F_{\cap I}; \boldsymbol{k}) \to H^*(F_{\cap I'}; \boldsymbol{k})$ is the induced homomorphism on cohomology.

In addition, we can consider $\boldsymbol{k}\langle Q\rangle$ as a graded ring if we choose a degree for every indeterminate x_j in $\boldsymbol{k}[x_1, \dots, x_m]$ and define

$$\deg(\phi \otimes (x^{n_1}_{j_1} \cdots x^{n_s}_{j_s})) = \deg(\phi) + n_1 \deg(x_{j_1}) + \cdots + n_s \deg(x_{j_s}).$$

Theorem 1.7 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Then the equivariant cohomology ring of $\mathcal{Z}_Q$ (or $\mathbb{R}\mathcal{Z}_Q$) with $\mathbb{Z}$–coefficients (or $\mathbb{Z}_2$–coefficients) with respect to the canonical $(S^1)^m$–action (or $(\mathbb{Z}_2)^m$–action) is isomorphic as a graded ring to the topological face ring $\mathbb{Z}\langle Q\rangle$ (or $\mathbb{Z}_2\langle Q\rangle$) of Q by choosing $\deg(x_j) = 2$ (or $\deg(x_j) = 1$) for all $1 \leq j \leq m$.*

Moreover, the natural $H^*(BT^m)$–module structure on the integral equivariant cohomology ring $H^*_{T^m}(\mathcal{Z}_Q)$ is described in (52) where $T^m = (S^1)^m$.

Remark 1.8 A calculation of the equivariant cohomology group of $\mathscr{Z}_Q$ with $\mathbb{Z}$–coefficients was announced earlier by T Januszkiewicz in a talk in 2020 [24]. The formula given in Januszkiewicz's talk is equivalent to our $\mathbb{Z}\langle Q\rangle$. But the ring structure of the equivariant cohomology of $\mathscr{Z}_Q$ was not described in [24].

For a nice manifold with corners Q, there are two other notions which reflect the stratification of Q. One is the *face poset* of Q which is the set of all faces of Q ordered by inclusion, denoted by $\mathscr{S}_Q$ (note that each connected component of Q is also a face). The other one is the *nerve simplicial complex* of the covering of ∂Q by its facets, denoted by K_Q. The *face ring* (or *Stanley–Reisner ring*) of a simplicial complex is an important tool to study combinatorial objects in algebraic combinatorics and combinatorial commutative algebra; see [28; 30].

When Q is a simple polytope, all faces of Q, including Q itself, and all their intersections are acyclic. Then it is easy to see that the topological face ring of Q is isomorphic to the face ring of K_Q (see Example 5.2). But in general, the topological face ring of Q encodes more topological information of Q than the face ring of K_Q.

There is another way to think of the topological face ring $\boldsymbol{k}\langle Q\rangle$. Let

$$\mathscr{R}^*_{\cap Q,\boldsymbol{k}} := \bigoplus_{J\subseteq[m]} H^*(F_{\cap J};\boldsymbol{k}),$$

where product $*$ on $\mathscr{R}^*_{\cap Q,\boldsymbol{k}}$ is defined, for any $\phi\in H^*(F_{\cap J};\boldsymbol{k})$ and $\phi'\in H^*(F_{\cap J'};\boldsymbol{k})$, by

$$\phi*\phi' := \kappa^*_{J\cup J',J}(\phi)\cup\kappa^*_{J\cup J',J'}(\phi')\in H^*(F_{\cap(J\cup J')};\boldsymbol{k}).$$

Moreover,

$$\boldsymbol{k}[x_1,\dots,x_m] = \bigoplus_{J\subseteq[m]} R^J_{\boldsymbol{k}},$$

so we can think of both $\mathscr{R}^*_{\cap Q,\boldsymbol{k}}$ and $\boldsymbol{k}[x_1,\dots,x_m]$ as $2^{[m]}$–graded rings where $2^{[m]}=\{J\subseteq[m]\}$ is the power set of $[m]$. Then the topological face ring $\boldsymbol{k}\langle Q\rangle$ can be thought of as the Segre product of $\mathscr{R}^*_{\cap Q,\boldsymbol{k}}$ and $\boldsymbol{k}[x_1,\dots,x_m]$ with respect to their $2^{[m]}$–gradings. By definition, the *Segre product* of two rings R and S graded by a common semigroup $\mathscr{A}$ — using the notation of Hoa [21] — is

$$R\underline{\otimes} S = \bigoplus_{\boldsymbol{a}\in\mathscr{A}} R_{\boldsymbol{a}}\otimes S_{\boldsymbol{a}}.$$

So $R\underline{\otimes} S$ is a subring of the tensor product of R and S (as graded rings). The Segre product of two graded rings (or modules) is studied in algebraic geometry and commutative algebra; see Chow [13] and Fröberg and Hoa [19; 21], for example.

Here we can think of $2^{[m]}$ as a semigroup where the product of two subsets of $[m]$ is just their union. Then by this notation, we can write

$$\boldsymbol{k}\langle Q\rangle = \mathscr{R}^*_{\cap Q,\boldsymbol{k}}\,\underline{\otimes}\,\boldsymbol{k}[x_1,\dots,x_m].$$

From this form, we see that $\boldsymbol{k}\langle Q\rangle$ is essentially determined by $\mathscr{R}^*_{\cap Q,\boldsymbol{k}}$.

The paper is organized as follows. In Section 2, we first construct an embedding of Q into $Q \times [0,1]^m$ which is analogous to the embedding of a simple polytope into a cube. This induces an embedding of $\mathcal{Z}_Q$ into $Q \times (D^2)^m$, from which we can do the stable decomposition of $\mathcal{Z}_Q$ and give a proof of Theorem 1.1. Our argument proceeds along the same line as the argument given in [3, Section 6] but with some extra ingredients. In fact, we will not do the stable decomposition of $\mathcal{Z}_Q$ directly, but the stable decomposition of the disjoint union of $\mathcal{Z}_Q$ with a point. In Section 3, we obtain a description of the product structure of the cohomology of $\mathcal{Z}_Q$ using the stable decomposition of $\mathcal{Z}_Q$ and the partial diagonal map introduced in [4]. From this we give a proof of Theorem 1.4. In Section 4, we define the notion of polyhedral product of a sequence of based CW–complexes over a nice manifold with corners Q and obtain some results parallel to $\mathcal{Z}_Q$ for these spaces. In particular, we obtain a description of the integral cohomology ring of real moment-angle manifold $\mathbb{R}\mathcal{Z}_Q$ (see Corollary 4.10). In Section 5, we compute the equivariant cohomology ring of $\mathcal{Z}_Q$ and prove Theorem 1.7. In Section 6, we discuss more generalizations of the construction of $\mathcal{Z}_Q$ and extend our main theorems to some wider settings.

2 Stable decomposition of $\mathcal{Z}_Q$

Let Q be a nice manifold with corners with m facets. To obtain the stable decomposition of $\mathcal{Z}_Q$, we first construct a special embedding of Q into $Q \times [0,1]^m$, called the rim-cubicalization of Q. This construction can be thought of as a generalization of the embedding of a simple polytope with m facets into $[0,1]^m$ defined in [9, Chapter 4].

2.1 Rim-cubicalization of Q in $Q \times [0,1]^m$

Let $F_1, \dots, F_m$ be all the facets of Q. For a face f of Q, let I_f be the subset of $[m]$ called the *strata index* of f,

$$I_f = \{i \in [m] \mid f \subseteq F_i\} \subseteq [m].$$

Then we define a subset $\hat{f}$ of $Q \times [0,1]^m$ associated to f as follows. We write

$$[0,1]^m = \prod_{j \in [m]} [0,1]_{(j)}$$

and define

$$\hat{f} = f \times \prod_{j \in I_f} [0,1]_{(j)} \times \prod_{j \in [m]\setminus I_f} 1_{(j)}.$$

In particular,

$$\widehat{F}_i = F_i \times [0,1]_{(i)} \times \prod_{j \in [m]\setminus\{i\}} 1_{(j)}, \quad 1 \le i \le m.$$

Let $\mathcal{S}_Q$ be the face poset of Q and define

$$\widehat{Q} = \bigcup_{f \in \mathcal{S}_Q} \hat{f} \subseteq Q \times [0,1]^m. \tag{9}$$

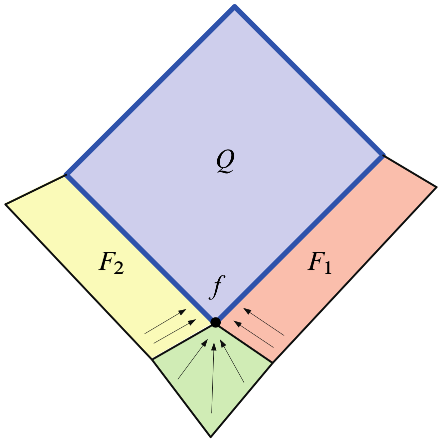

Figure 1: Rim-cubicalization of Q in $Q \times [0,1]^m$.

It is easy to see that $\widehat{Q}$ is a nice manifold with corners whose facets are $\widehat{F}_1, \dots, \widehat{F}_m$. If we identify Q with the subspace $Q \times \prod_{j \in [m]} 1_{(j)} \subseteq \widehat{Q}$, then we can think of $\widehat{Q}$ as inductively gluing the product of all codimension-k strata of Q with a k–cube to ∂Q (see Figure 1),

$$Q \xleftarrow{\text{glue}} F_j \times [0,1] \xleftarrow{\text{glue}} \cdots \xleftarrow{\text{glue}} f \times [0,1]^{|I_f|} \xleftarrow{\text{glue}} \cdots .$$

Due to this viewpoint, we call $\widehat{Q}$ the *rim-cubicalization* of Q in $Q \times [0,1]^m$.

Lemma 2.1 *$\widehat{Q}$ is homeomorphic to Q as a manifold with corners.*

Proof For any face f of Q and $0 \le t \le 1$, let

$$\hat{f}(t) = f \times \prod_{j \in I_f} [t,1]_{(j)} \times \prod_{j \in [m] \setminus I_f} 1_{(j)}, \quad \widehat{Q}(t) = \bigcup_{f \in \mathcal{S}_Q} \hat{f}(t) \subseteq Q \times [t,1]^m.$$

Then $\widehat{Q}(t)$ determines an isotopy (see Figure 1) from $\widehat{Q}(0) = \widehat{Q}$ to

$$\widehat{Q}(1) = \bigcup_{f \in \mathcal{S}_Q} \Bigl(f \times \prod_{j \in [m]} 1_{(j)} \Bigr) = Q \times \prod_{j \in [m]} 1_{(j)} \cong Q.$$

Around a codimension-k stratum of Q, the isotopy $\widehat{Q}(t)$ is locally equivalent to the standard isotopy from $C_k^n(-1)$ to $C_k^n(0)$ defined in Example 2.2.

Clearly, the isotopy $\widehat{Q}(t)$ sends each face $\hat{f}$ of $\widehat{Q}$ to $f \times \prod_{j \in [m]} 1_{(j)}$. So, under the identification of $Q \times \prod_{j \in [m]} 1_{(j)}$ with Q, $\widehat{Q}$ is homeomorphic to Q as a manifold with corners. □

Example 2.2 Let $C_k^n(0)$ and $C_k^n(-1)$ be two subspaces of $\mathbb{R}^n$ defined by

$$C_k^n(0) := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_1, \dots, x_k < 1, -1 < x_{k+1}, \dots, x_n < 1\},$$
$$C_k^n(-1) := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid -1 \le x_1, \dots, x_k < 1, -1 < x_{k+1}, \dots, x_n < 1\}.$$

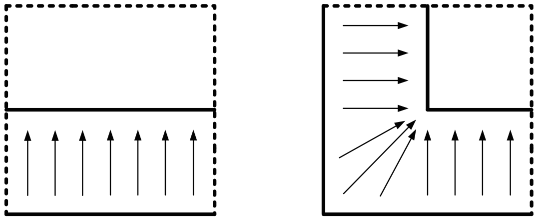

Figure 2: Isotopy from $C^n_k(-1)$ to $C^n_k(0)$.

There is a strong deformation retraction from $C^n_k(-1)$ to $C^n_k(0)$ defined by

$$H(x_1,\dots,x_n,t) = (\delta_{x_1}(t)\cdot x_1,\dots,\delta_{x_k}(t)\cdot x_k, x_{k+1},\dots,x_n)$$

for any $(x_1,\dots,x_n) \in C^n_k(-1)$ and $t \in [0,1]$, where

$$\delta_x(t) = \begin{cases} 1-t & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

It is easy to see that for any $t \in [0,1]$, the image of $H(\cdot\,,t)$ is

$$C^n_k(t-1) = \{(x_1,\dots,x_n) \mid t-1 \leq x_1,\dots,x_k < 1, -1 < x_{k+1},\dots,x_n < 1\}.$$

So H actually defines an isotopy from $C^n_k(-1)$ to $C^n_k(0)$ (see Figure 2).

2.2 Embedding $\mathcal{Z}_Q$ into $Q \times (D^2)^m$

Using the above rim-cubicalization of Q in $Q\times[0,1]^m$, we can embed the manifold $\mathcal{Z}_Q$ into $Q\times(D^2)^m$, where $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$ is the unit disk.

In the following, we consider $[0,1]$ as a subset of D^2 and the cube $[0,1]^m$ as a subset of $(D^2)^m \subset \mathbb{C}^m$. For any $j \in [m]$, let $S^1_{(j)}$ and $D^2_{(j)}$ denote the corresponding spaces indexed by j.

There is a canonical action of $(S^1)^m$ on $Q\times(D^2)^m$ defined by

$$(g_1,\dots,g_m)\cdot(x,z_1,\dots,z_m) = (x, g_1z_1,\dots,g_mz_m),$$

where $x \in Q$, $g_j \in S^1_{(j)}$ and $z_j \in D^2_{(j)}$ for $1 \leq j \leq m$. The orbit space of this action can be identified with $Q\times[0,1]^m$. We denote the quotient map by

$$p\colon Q\times(D^2)^m \to Q\times[0,1]^m.$$

For any face f of Q, we define

$$(D^2,S^1)^f := p^{-1}(\hat{f}) = f \times \prod_{j\in I_f} D^2_{(j)} \times \prod_{j\in[m]\setminus I_f} S^1_{(j)} \subseteq Q\times(D^2)^m, \tag{10}$$

$$(D^2,S^1)^Q := \bigcup_{f\in\mathcal{S}_Q} (D^2,S^1)^f = \bigcup_{f\in\mathcal{S}_Q} \Bigl(f\times\prod_{j\in I_f} D^2_{(j)} \times \prod_{j\in[m]\setminus I_f} S^1_{(j)} \Bigr). \tag{11}$$

There is a canonical action of $(S^1)^m$ on $(D^2,S^1)^Q$ induced by the canonical action of $(S^1)^m$ on $Q\times(D^2)^m$.

Lemma 2.3 $(D^2,S^1)^Q$ *is equivariantly homeomorphic to* $\mathcal{X}_Q$.

Proof By Lemma 2.1, it is equivalent to show that $(D^2,S^1)^Q$ is equivariantly homeomorphic to $\mathcal{X}_{\widehat{Q}}$. We consider $[0,1)^m$ as a nice manifold with corners whose facets are $F_1^{\square},\dots,F_m^{\square}$, where

$$F_i^{\square}=0_{(i)}\times\prod_{j\in[m]\backslash\{i\}}[0,1)_{(j)},\quad 1\le i\le m.$$

It is clear that $\mathcal{X}_{[0,1)^m}=[0,1)^m\times(S^1)^m/{\sim}$ is homeomorphic to $(D^2\backslash S^1)^m$. The quotient map $[0,1)^m\times(S^1)^m\to\mathcal{X}_{[0,1)^m}=(D^2\backslash S^1)^m$ extends to a map $\pi\colon[0,1]^m\times(S^1)^m\to(D^2)^m$ which can be written explicitly as

$$\pi\colon[0,1]^m\times(S^1)^m\to(D^2)^m,\quad((t_1,\dots,t_m),(g_1,\dots g_m))\mapsto(g_1t_1,\dots,g_mt_m).\tag{12}$$

Define

$$\pi_Q=\mathrm{id}_Q\times\pi\colon Q\times[0,1]^m\times(S^1)^m\to Q\times(D^2)^m.$$

Notice that the facets of $\widehat{Q}$ are the intersections of $\widehat{Q}$ with $Q\times F_1^{\square},\dots,Q\times F_m^{\square}$,

$$\widehat{F}_i=\widehat{Q}\cap(Q\times F_i^{\square}),\quad 1\le i\le m.$$

We can easily check that the restriction of π_Q to $\widehat{Q}\times(S^1)^m$ gives exactly $\mathcal{X}_{\widehat{Q}}$, ie

$$\mathcal{X}_{\widehat{Q}}=\pi_Q(\widehat{Q}\times(S^1)^m).$$

Moreover, for any face f of Q,

$$\begin{aligned}\pi_Q(\hat{f}\times(S^1)^m)&=\pi_Q\Bigl(f\times\prod_{j\in I_f}([0,1]_{(j)}\times S^1_{(j)})\times\prod_{j\in[m]\backslash I_f}S^1_{(j)}\Bigr)\\&=f\times\prod_{j\in I_f}D^2_{(j)}\times\prod_{j\in[m]\backslash I_f}S^1_{(j)}=(D^2,S^1)^f.\end{aligned}$$

So we have a homeomorphism

$$\mathcal{X}_{\widehat{Q}}\cong\pi_Q(\widehat{Q}\times(S^1)^m)=\bigcup_{f\in\mathcal{S}_Q}\pi_Q(\hat{f}\times(S^1)^m)=\bigcup_{f\in\mathcal{S}_Q}(D^2,S^1)^f=(D^2,S^1)^Q.$$

Clearly, the above homeomorphism is equivariant with respect to the canonical actions of $(S^1)^m$ on $\mathcal{X}_{\widehat{Q}}$ and $(D^2,S^1)^Q$. □

2.3 Viewing $\mathcal{X}_Q$ as a colimit of CW–complexes

By Lemma 2.3, studying the stable decomposition of $\mathcal{X}_Q$ is equivalent to studying that for $(D^2,S^1)^Q$. To do the stable decomposition as in [3], we want to first think of $(D^2,S^1)^Q$ as the colimit of a diagram of CW–complexes over a finite *poset* (partially ordered set). The following are some basic definitions; see [38].

- Let CW be the category of CW–complexes and continuous maps.

- Let CW_* be the category of based CW–complexes and based continuous maps.
- A *diagram* $\mathcal{D}$ of CW–complexes or based CW–complexes over a finite poset $\mathcal{P}$ is a functor
$$\mathcal{D}\colon \mathcal{P} \to \mathrm{CW} \text{ or } \mathrm{CW}_*$$
such that for every $p \le p'$ in $\mathcal{P}$, there is a map $d_{pp'}\colon \mathcal{D}(p') \to \mathcal{D}(p)$ with
$$d_{pp} = \mathrm{id}_{\mathcal{D}(p)}, \quad d_{pp'}d_{p'p''} = d_{pp''} \quad \text{for all } p \le p' \le p''.$$
- The *colimit* of $\mathcal{D}$ is the space
$$\mathrm{colim}(\mathcal{D}) := \Big(\coprod_{p\in\mathcal{P}} \mathcal{D}(p)\Big)\Big/\sim$$
where $\sim$ denotes the equivalence relation generated by requiring that for each $x \in \mathcal{D}(p')$, $x \sim d_{pp'}(x)$ for every $p < p'$.

To think of $(D^2,S^1)^Q$ as a colimit of CW–complexes, we need to introduce a finer decomposition of $(D^2,S^1)^Q$ as follows. By the notations in Section 2.1, for any face f of Q and any subset $L \subseteq I_f \subseteq [m]$, let

$$(D^2,S^1)^{(f,L)} := f \times \prod_{j\in I_f\backslash L} D^2_{(j)} \times \prod_{j\in[m]\backslash(I_f\backslash L)} S^1_{(j)}. \tag{13}$$

Clearly, $(D^2,S^1)^{(f,L)} \subseteq (D^2,S^1)^{(f,L')}$ if and only if $L \supseteq L'$. So we have

$$(D^2,S^1)^f = (D^2,S^1)^{(f,\varnothing)} = \bigcup_{L\subseteq I_f} (D^2,S^1)^{(f,L)},$$
$$(D^2,S^1)^Q = \bigcup_{f\in\mathcal{S}_Q} (D^2,S^1)^f = \bigcup_{f\in\mathcal{S}_Q}\bigcup_{L\subseteq I_f} (D^2,S^1)^{(f,L)}. \tag{14}$$

Corresponding to this decomposition, we define a poset associated to Q by

$$\mathcal{P}_Q = \{(f,L) \mid f \in \mathcal{S}_Q, L \subseteq I_f \subseteq [m]\}, \tag{15}$$

where $(f,L) \le (f',L')$ if and only if $f \supseteq f'$ and $I_f\backslash L \supseteq I_{f'}\backslash L'$. It follows from the definition (13) that

$$(f,L) \le (f',L') \iff (D^2,S^1)^{(f',L')} \subseteq (D^2,S^1)^{(f,L)}.$$

Note that $\mathcal{P}_Q$ is a finite poset since by our convention Q only has finitely many faces.

Definition 2.4 Let $\boldsymbol{D}\colon \mathcal{P}_Q \to \mathrm{CW}$ be a diagram of CW–complexes where

$$\boldsymbol{D}((f,L)) = (D^2,S^1)^{(f,L)} \quad \text{for all } (f,L) \in \mathcal{P}_Q.$$

For any $(f,L) \le (f',L') \in \mathcal{P}_Q$, $d_{(f,L),(f',L')}\colon \boldsymbol{D}((f',L')) \to \boldsymbol{D}((f,L))$ is the natural inclusion.

Clearly, $(D^2,S^1)^Q$ is the colimit of the diagram $\boldsymbol{D}$. So we have

$$\mathcal{Z}_Q \cong (D^2,S^1)^Q = \mathrm{colim}(\boldsymbol{D}) = \bigcup_{(f,L)\in\mathcal{P}_Q} (D^2,S^1)^{(f,L)}. \tag{16}$$

Remark 2.5 Here we do not write $(D^2,S^1)^Q$ as the colimit of a diagram of based CW–complexes. This is because in general it is not possible to choose a basepoint in each $(D^2,S^1)^{(f,L)}$ to adapt to the colimit construction of a diagram in CW_*.

2.4 Stable decomposition of $\mathcal{Z}_Q$

First of all, let us recall a well-known theorem — see [23; 34] — which allows us to decompose the Cartesian product of a collection of based CW–complexes into a wedge of spaces after doing a suspension.

Let (X_i,x_i) for $1\le i\le m$ be based CW–complexes. For $I=\{i_1,\dots,i_k\}\subseteq[m]$ with $1\le i_1<\cdots<i_k\le m$, define

$$\hat{X}^I = X_{i_1}\wedge\cdots\wedge X_{i_k},$$

which is the quotient space of $X^I=X_{i_1}\times\cdots\times X_{i_k}$ by the subspace given by

$$FW(X^I)=\{(y_{i_1},\dots,y_{i_k})\in X^I \mid y_{i_j} \text{ is the basepoint } x_{i_j}\in X_{i_j} \text{ for at least one } i_j\}.$$

Theorem 2.6 *Let (X_i,x_i) for $1\le i\le m$ be based connected CW–complexes. There is a based, natural homotopy equivalence*

$$h\colon \mathbf{\Sigma}(X_1\times\cdots\times X_m)\to\mathbf{\Sigma}\Bigl(\bigvee_{\varnothing\neq I\subseteq[m]}\hat{X}^I\Bigr),$$

where I runs over all the nonempty subsets of $[m]$. Furthermore, the map h commutes with colimits.

In our proof later, we need a slightly generalized version of Theorem 2.6. Before that, let us first prove three simple lemmas.

Lemma 2.7 *If (X,x_0) and (Y,y_0) are based CW–complexes with X contractible, then $X\wedge Y$ is also contractible.*

Proof The deformation retraction from X to x_0 naturally induces a deformation retraction from

$$X\wedge Y = X\times Y/(\{x_0\}\times Y)\cup(X\times\{y_0\})$$

to its canonical basepoint $[(x_0,y_0)]=[(\{x_0\}\times Y)\cup(X\times\{y_0\})]$. □

Lemma 2.8 *Suppose a CW–complex X has N connected components $X_1,\dots,X_N$. Then there is a homotopy equivalence*

$$\mathbf{\Sigma}(X)\simeq\mathbf{\Sigma}(X_1)\vee\cdots\vee\mathbf{\Sigma}(X_N)\vee\bigvee_{N-1}S^1.$$

where $\bigvee_{N-1}S^1$ is the wedge sum of $N-1$ copies of S^1.

Proof This follows easily from the definition of reduced suspension. □

Lemma 2.9 *Let $X_1 = X_1' \cup \{x_1\}$ where X_1' is a connected based CW–complex and $x_1 \notin X_1'$ is the basepoint of X_1.*

(a) *For any connected based CW–complex X_2, there is a homotopy equivalence*

$$\mathbf{\Sigma}(X_1 \wedge X_2) \simeq \mathbf{\Sigma}(X_2) \vee \mathbf{\Sigma}(X_1' \wedge X_2).$$

(b) *If $X_2 = X_2' \cup \{x_2\}$, where X_2' is a connected based CW–complex and $x_2 \notin X_2'$ is the basepoint of X_2, then $X_1 \wedge X_2$ is the disjoint union of $X_1' \wedge X_2'$ and a point represented by $\{x_1\} \times \{x_2\}$.*

Proof (a) By the definition of smash product, we have a homeomorphism

$$X_1 \wedge X_2 = (X_1' \cup \{x_1\}) \times X_2/(\{x_1\} \times X_2 \cup X_1' \times \{x_2\}) \cong X_1' \times X_2/X_1' \times \{x_2\}.$$

Then we have

$$\begin{aligned} \mathbf{\Sigma}(X_1 \wedge X_2) &= \mathbf{\Sigma}(X_1' \times X_2/X_1' \times \{x_2\}) \\ &\simeq \mathbf{\Sigma}(X_1' \times X_2)/\mathbf{\Sigma}(X_1' \times \{x_2\}) \\ &\simeq \big(\mathbf{\Sigma}(X_1') \vee \mathbf{\Sigma}(X_2) \vee \mathbf{\Sigma}(X_1' \wedge X_2)\big)/\mathbf{\Sigma}(X_1') \quad \text{(by Theorem 2.6)} \\ &\simeq \mathbf{\Sigma}(X_2) \vee \mathbf{\Sigma}(X_1' \wedge X_2). \end{aligned}$$

(b) This follows directly from the definition of smash product. □

We can generalize Theorem 2.6 to the following form.

Theorem 2.10 *Let (X_i, x_i) for $1 \le i \le m$, be based CW–complexes. Assume that for some $1 \le n \le m$,*

- *$X_i = Y_i \cup \{x_i\}$ for $1 \le i \le n$, where Y_i is a connected CW–complex and $x_i \notin Y_i$.*
- *X_i, for $n+1 \le i \le m$, is a connected CW–complex.*

There is a based, natural homotopy equivalence which commutes with colimits,

$$h\colon \mathbf{\Sigma}(X_1 \times \cdots \times X_m) \to \mathbf{\Sigma}\Big(\bigvee_{\varnothing \ne I \subseteq [m]} \hat{X}^I\Big).$$

Proof For brevity, let $[n_1, n_2] = \{n_1, \dots, n_2\}$ for any integer $n_1 \le n_2$. Let

$$x^I = \{x_{i_1}\} \times \cdots \times \{x_{i_k}\}, \quad Y^I = Y_{i_1} \times \cdots \times Y_{i_k}, \quad I = \{i_1, \dots, i_k\}, \quad i_1 < \cdots < i_k.$$

There are 2^n connected components in $X^{[m]} = X_1 \times \cdots \times X_m$, which are

$$\{x^{[n]\setminus I} \times Y^I \times X^{[n+1,m]} \mid I \subseteq [1,n]\}.$$

We choose a basepoint for each Y_i with $1 \le i \le n$. So, by Lemma 2.8,

$$\mathbf{\Sigma}(X_1 \times \cdots \times X_m) \simeq \bigvee_{I \subseteq [1,n]} \mathbf{\Sigma}(Y^I \times X^{[n+1,m]}) \vee \bigvee_{2^n - 1} S^1.$$

Since $Y_1, \dots, Y_n, X_{n+1}, \dots, X_m$ are all connected based CW–complexes, we can apply Theorem 2.6 to each $Y^I \times X^{[n+1,m]}$ and obtain

$$\boldsymbol{\Sigma}(X_1 \times \cdots \times X_m) \simeq \bigvee_{I \subseteq [1,n]} \bigvee_{\substack{L \cup J \neq \varnothing, L \subseteq I \\ J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J) \vee \bigvee_{2^n - 1} S^1. \tag{17}$$

On the other hand, for any $I = \{i_1, \dots, i_k\} \subseteq [1,n]$ and $J \subseteq [n+1,m]$,

$$\widehat{X}^I \wedge \widehat{X}^J = X_{i_1} \wedge \cdots \wedge X_{i_k} \wedge \widehat{X}^J = (Y_{i_1} \cup \{x_{i_1}\}) \wedge \cdots \wedge (Y_{i_k} \cup \{x_{i_k}\}) \wedge \widehat{X}^J.$$

If $J \neq \varnothing$, $\widehat{X}^I \wedge \widehat{X}^J$ is a connected CW–complex. Then by iteratively using Lemma 2.9(a), we obtain

$$\boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J) \simeq \bigvee_{L \subseteq I} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J).$$

Alternatively, if $J = \varnothing$ and $I \neq \varnothing$, by iteratively using Lemma 2.9(b), we can deduce that $\widehat{X}^I$ is the disjoint union of $\widehat{Y}^I$ and a point represented by x^I. So, by Lemma 2.8, $\boldsymbol{\Sigma}(\widehat{X}^I) \simeq \boldsymbol{\Sigma}(\widehat{Y}^I) \vee S^1$.

So we have

$$\begin{aligned}
\bigvee_{\substack{H \subseteq [m] \\ H \neq \varnothing}} \boldsymbol{\Sigma}(\widehat{X}^H) &= \bigvee_{\substack{I \cup J \neq \varnothing, I \subseteq [1,n] \\ J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J) \\
&= \Bigg(\bigvee_{\substack{I \subseteq [1,n] \\ \varnothing \neq J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{X}^I \wedge \widehat{X}^J) \Bigg) \vee \Bigg(\bigvee_{\varnothing \neq I \subseteq [1,n]} \boldsymbol{\Sigma}(\widehat{X}^I) \Bigg) \\
&\simeq \Bigg(\bigvee_{I \subseteq [1,n]} \bigvee_{\substack{L \subseteq I \\ \varnothing \neq J \subseteq [n+1,m]}} \boldsymbol{\Sigma}(\widehat{Y}^L \wedge \widehat{X}^J) \Bigg) \vee \Bigg(\bigvee_{\varnothing \neq I \subseteq [1,n]} (\boldsymbol{\Sigma}(\widehat{Y}^I) \vee S^1) \Bigg).
\end{aligned}$$

By comparing the above expression with (17), we prove the theorem. □

Remark 2.11 By Theorem 2.10, it is not hard to see that all the main theorems in [3] also hold for based CW–complex pairs $\{(X_i, A_i, a_i)\}_{i=1}^m$ where each of X_i and A_i is either connected or is a disjoint union of a connected CW–complex with its basepoint. In particular, [3, Corollary 2.24] also holds for (D^1, S^0).

Remark 2.12 It is possible to extend Theorem 2.10 further to deal with spaces which are a disjoint union of a connected CW–complex with finitely many points. But since Theorem 2.10 is already enough for our discussion in this paper, we leave the more generalized statement to the reader.

Definition 2.13 For any based CW–complexes (X, x_0) and (Y, y_0), let

$$X \rtimes Y := X \times Y / x_0 \times Y, \quad X \ltimes Y := X \times Y / X \times y_0.$$

If each of X and Y is either connected or is a disjoint union of a connected CW–complex with its basepoint, there is a homotopy equivalence by Theorem 2.10,

$$\Sigma(X \rtimes Y) \simeq \Sigma(X \times Y)/\Sigma(x_0 \times Y) \simeq \Sigma(X) \vee \Sigma(X \wedge Y), \tag{18}$$

$$\Sigma(X \ltimes Y) \simeq \Sigma(X \times Y)/\Sigma(X \times y_0) \simeq \Sigma(Y) \vee \Sigma(X \wedge Y). \tag{19}$$

We can further generalize Theorem 2.6 to the following form. We will use the following convention in the rest of the paper:

Convention For any based space Y, define $Y \wedge \hat{X}^I := Y$ when $I = \varnothing$.

Theorem 2.14 *Let (X_i, x_i) for $1 \le i \le m$ and (B, b_0) be a collection of based CW–complexes where each of X_i and B is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a based, natural homotopy equivalence which commutes with colimits,*

$$h\colon \Sigma(B \rtimes (X_1 \times \cdots \times X_m)) \to \Sigma\Big(\bigvee_{I \subseteq [m]} B \wedge \hat{X}^I\Big).$$

Proof By definition,

$$\begin{aligned}
\Sigma(B \wedge (X_1 \times \cdots \times X_m)) &= \Sigma(B \times (X_1 \times \cdots \times X_m)/B \vee (X_1 \times \cdots \times X_m)) \\
&\simeq \Sigma(B \times X_1 \times \cdots \times X_m)/\Sigma(B) \vee \Sigma(X_1 \times \cdots \times X_m) \\
&\simeq \bigvee_{\varnothing \ne I \subseteq [m]} \Sigma(B \wedge \hat{X}^I) \quad \text{(by Theorem 2.10)}.
\end{aligned}$$

Then, by (18),

$$\begin{aligned}
\Sigma(B \rtimes (X_1 \times \cdots \times X_m)) &= \Sigma(B) \vee \Sigma(B \wedge (X_1 \times \cdots \times X_m)) \\
&\simeq \Sigma(B) \vee \bigvee_{\varnothing \ne I \subseteq [m]} \Sigma(B \wedge \hat{X}^I) \\
&\simeq \Sigma\Big(\bigvee_{I \subseteq [m]} B \wedge \hat{X}^I\Big).
\end{aligned}$$

□

To apply the above stable decomposition lemmas to $(D^2, S^1)^Q$, we need to choose a basepoint for each $(D^2, S^1)^{(f,L)}$ in the first place. But by Remark 2.5, there is no good way to choose a basepoint inside each $(D^2, S^1)^{(f,L)}$ to adapt to the colimit construction of $(D^2, S^1)^Q$. So, in the following, we add an auxiliary point to all $(D^2, S^1)^{(f,L)}$ as their common basepoint:

- Let $1_{(j)}$ be the basepoint of $S^1_{(j)}$ and $D^2_{(j)}$ for every $j \in [m]$.
- Let $Q_+ = Q \cup q_0$ where $q_0 \notin Q$ is the basepoint of Q_+.
- For any face f of Q, let $f_+ = f \cup q_0$ with basepoint q_0.
- For any $(f, L) \in \mathcal{P}_Q$, define $(D^2, S^1)^{(f,L)}_+ := (D^2, S^1)^{(f,L)} \cup \hat{q}_0$, where $\hat{q}_0 = q_0 \times \prod_{j \in [m]} 1_{(j)}$ is the basepoint.
- Let $(D^2, S^1)^Q_+ = (D^2, S^1)^Q \cup \hat{q}_0$ with basepoint $\hat{q}_0$.

Let $\boldsymbol{D}_+\colon \mathcal{P}_Q \to \mathrm{CW}_*$ be a diagram of based CW–complexes, where

$$\boldsymbol{D}_+((f,L)) = (D^2,S^1)_+^{(f,L)} \quad \text{for all } (f,L)\in\mathcal{P}_Q,$$

and $(d_+)_{(f,L),(f',L')}\colon \boldsymbol{D}_+((f',L'))\to \boldsymbol{D}_+((f,L))$ is the natural inclusion for any $(f,L)\le(f',L')\in\mathcal{P}_Q$. Then it is clear that

$$(D^2,S^1)_+^Q = (D^2,S^1)^Q \cup \hat{q}_0 = \mathrm{colim}(\boldsymbol{D})\cup\hat{q}_0 = \mathrm{colim}(\boldsymbol{D}_+). \tag{20}$$

Next, we analyze the reduced suspension $\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+))$ from the colimit point of view. Since all the $(D^2,S^1)_+^{(f,L)}$ share the same basepoint $\hat{q}_0$,

$$\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+)) = \boldsymbol{\Sigma}\Bigl(\bigcup_{(f,L)\in\mathcal{P}_Q}(D^2,S^1)_+^{(f,L)}\Bigr) = \bigcup_{(f,L)\in\mathcal{P}_Q}\boldsymbol{\Sigma}((D^2,S^1)_+^{(f,L)}).$$

Lemma 2.15 *For any $(f,L)\in\mathcal{P}_Q$, there is a natural homeomorphism which commutes with taking the colimit,*

$$(D^2,S^1)_+^{(f,L)} \cong f_+ \rtimes \Bigl(\prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)}\Bigr).$$

Proof By our definitions,

$$\begin{aligned} f_+ \rtimes \prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)} \\ &= (f\cup q_0)\times \prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)} \Big/ q_0 \times \prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)} \\ &\cong \Bigl(f\times \prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)}\Bigr)\cup\hat{q}_0 = (D^2,S^1)_+^{(f,L)}. \end{aligned}$$

The above homeomorphism "$\cong$" is induced by the global homeomorphism

$$Q_+ \rtimes \prod_{j\in[m]} D^2_{(j)} \to \Bigl(Q\times\prod_{j\in[m]} D^2_{(j)}\Bigr)\cup\hat{q}_0$$

which identifies $q_0\times\prod_{j\in[m]} D^2_{(j)} \big/ q_0\times\prod_{j\in[m]} D^2_{(j)}$ with $\hat{q}_0$. □

Since we assume that each face f of Q is a CW–complex in our convention, we can deduce from Theorem 2.14 and Lemma 2.15 that

$$\begin{aligned} \boldsymbol{\Sigma}((D^2,S^1)_+^{(f,L)}) &\cong \boldsymbol{\Sigma}\Bigl(f_+\rtimes\Bigl(\prod_{j\in I_f\setminus L} D^2_{(j)} \times \prod_{j\in[m]\setminus(I_f\setminus L)} S^1_{(j)}\Bigr)\Bigr) \\ &\simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}\Bigl(f_+\wedge \bigwedge_{j\in J\cap(I_f\setminus L)} D^2_{(j)} \wedge \bigwedge_{j\in J\setminus(I_f\setminus L)} S^1_{(j)}\Bigr). \end{aligned} \tag{21}$$

According to (21), we define a family of diagrams of based CW–complexes

$$\widehat{\boldsymbol{D}}^J_+ : \mathcal{P}_Q \to \mathrm{CW}_*, \quad J \subseteq [m],$$

(22) $$\widehat{\boldsymbol{D}}^J_+((f,L)) := f_+ \wedge \bigwedge_{j\in J\cap(I_f\setminus L)} D^2_{(j)} \wedge \bigwedge_{j\in J\setminus(I_f\setminus L)} S^1_{(j)} \subseteq Q_+ \wedge \bigwedge_{j\in J} D^2_{(j)},$$

where $(\hat{d}^J_+)_{(f,L),(f',L')} : \widehat{\boldsymbol{D}}^J_+((f',L')) \to \widehat{\boldsymbol{D}}^J_+((f,L))$ is the natural inclusion for $(f,L)\le(f',L')\in\mathcal{P}_Q$. The basepoint of $\widehat{\boldsymbol{D}}^J_+((f,L))$ is

$$[\hat{q}^J_0] := \Big[q_0 \times \prod_{j\in J} 1_{(j)}\Big].$$

Since here the reduced suspension commutes with colimits up to homotopy equivalence [3, Theorem 4.3], we obtain a homotopy equivalence

(23) $$\boldsymbol{\Sigma}(\mathrm{colim}(\boldsymbol{D}_+)) \simeq \mathrm{colim}(\boldsymbol{\Sigma}(\boldsymbol{D}_+)) \simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+)).$$

The following theorem from [3] will be useful in our proof of Theorem 1.1. It is a modification of the "homotopy lemma" given in [31; 33; 38].

Theorem 2.16 [3, Corollary 4.5] *Let D and E be two diagrams over a finite poset $\mathcal{P}$ with values in CW_* for which the maps $\mathrm{colim}_{q>p} D(q) \hookrightarrow D(p)$ and $\mathrm{colim}_{q>p} E(q) \hookrightarrow E(p)$ are all closed cofibrations. If f is a map of diagrams over $\mathcal{P}$ such that for every $p\in\mathcal{P}$, $f_p : D(p)\to E(p)$ is a homotopy equivalence, then f induces a homotopy equivalence $f : \mathrm{colim}(D(P)) \to \mathrm{colim}(E(P))$.*

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1 By (20) and (23), we obtain a homotopy equivalence

(24) $$\boldsymbol{\Sigma}((D^2,S^1)^Q_+) \simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+)).$$

Notice that when $J\cap(I_f\setminus L)\neq\varnothing$, $\widehat{\boldsymbol{D}}^J_+((f,L))$ is contractible by Lemma 2.7. So, for any $J\subseteq[m]$, we define another diagram of based CW–complexes

$$\widehat{\boldsymbol{E}}^J_+ : \mathcal{P}_Q \to \mathrm{CW}_*,$$

(25) $$\widehat{\boldsymbol{E}}^J_+((f,L)) := \begin{cases} \widehat{\boldsymbol{D}}^J_+((f,L)) = f_+ \wedge \bigwedge_{j\in J} S^1_{(j)} & \text{if } J\cap(I_f\setminus L)=\varnothing, \\ [\hat{q}^J_0] & \text{if } J\cap(I_f\setminus L)\neq\varnothing. \end{cases}$$

For $(f,L)\le(f',L')\in\mathcal{P}_Q$, $(\hat{e}^J_+)_{(f,L),(f',L')} : \widehat{\boldsymbol{E}}^J_+((f',L')) \to \widehat{\boldsymbol{E}}^J_+((f,L))$ is either the natural inclusion or the constant map $\boldsymbol{c}_{[\hat{q}^J_0]}$ (mapping all points to $[\hat{q}^J_0]$). The basepoint of $\widehat{\boldsymbol{E}}^J_+((f,L))$ is $[\hat{q}^J_0]$.

Moreover, let $\Phi^J : \widehat{\boldsymbol{D}}^J_+ \to \widehat{\boldsymbol{E}}^J_+$ be a map of diagrams over $\mathcal{P}_Q$ defined by

$$\Phi^J_{(f,L)} : \widehat{\boldsymbol{D}}^J_+((f,L)) \to \widehat{\boldsymbol{E}}^J_+((f,L)), \qquad \Phi^J_{(f,L)} = \begin{cases} \mathrm{id}_{\widehat{\boldsymbol{D}}^J_+((f,L))} & \text{if } J\cap(I_f\setminus L)=\varnothing, \\ \boldsymbol{c}_{[\hat{q}^J_0]} & \text{if } J\cap(I_f\setminus L)\neq\varnothing. \end{cases}$$

Then by Theorem 2.16, there exists a homotopy equivalence

$$\operatorname{colim}(\widehat{\boldsymbol{D}}^J_+) \simeq \operatorname{colim}(\widehat{\boldsymbol{E}}^J_+).$$

Note that we always have $\widehat{\boldsymbol{D}}^J_+((f, I_f)) \subseteq \widehat{\boldsymbol{D}}^J_+((f, L))$ for any $L \subsetneq I_f \neq \varnothing$. So we can ignore the terms $\{\widehat{\boldsymbol{D}}^J_+((f, I_f)) \mid I_f \neq \varnothing, f \in \mathcal{S}_Q\}$ when computing $\operatorname{colim}(\widehat{\boldsymbol{D}}^J_+)$. If $I_f = \varnothing$, then $f \subseteq Q^\circ$ and $\widehat{\boldsymbol{D}}^J_+((f, L)) = \widehat{\boldsymbol{D}}^J_+((f, \varnothing)) = f_+ \wedge \bigwedge_{j \in J} S^1_{(j)}$.

To understand $\operatorname{colim}(\widehat{\boldsymbol{E}}^J_+)$, we need to figure out in (25) what are those faces f of Q with some $L \subsetneq I_f$ such that $J \cap (I_f \backslash L) \neq \varnothing$.

- There exists $L \subsetneq I_f$ with $J \cap (I_f \backslash L) \neq \varnothing$ if and only if $J \cap I_f \neq \varnothing$, which is equivalent to $f \subseteq F_J$. Conversely,

$$F_J = \bigcup_{f \in \mathcal{S}_Q} f \cap F_J = \bigcup_{f \in \mathcal{S}_Q} \bigcup_{j \in J} f \cap F_j = \bigcup_{f \in \mathcal{S}_Q} \bigcup_{j \in J \cap I_f} f \cap F_j \subseteq \bigcup_{f \in \mathcal{S}_Q} \bigcup_{J \cap I_f \neq \varnothing} f.$$

  This implies

$$\bigcup_{f \in \mathcal{S}_Q} \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \backslash L) \neq \varnothing}} f = F_J. \tag{26}$$

- There exists $L \subsetneq I_f$ with $J \cap (I_f \backslash L) = \varnothing$ if and only if $f \subseteq F_{[m] \backslash J}$. So

$$\bigcup_{f \in \mathcal{S}_Q} \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \backslash L) = \varnothing}} f = F_{[m] \backslash J}. \tag{27}$$

The above discussion implies

$$\bigcup_{f \in \mathcal{S}_Q} \bigcup_{\substack{\exists L, L' \subsetneq I_f \\ J \cap (I_f \backslash L) = \varnothing \\ J \cap (I_f \backslash L') \neq \varnothing}} f = F_{[m] \backslash J} \cap F_J. \tag{28}$$

By the definition of $\widehat{\boldsymbol{E}}^J_+$, if we have a face f of Q and two subsets $L, L' \subsetneq I_f$ such that $J \cap (I_f \backslash L) = \varnothing$ while $J \cap (I_f \backslash L') \neq \varnothing$, then $J \cap (I_f \backslash (L \cup L')) = \varnothing$ and $J \cap (I_f \backslash (L \cap L')) \neq \varnothing$. So, in this case,

- $(\hat{e}^J_+)_{(f, L'), (f, L \cup L')} \colon \widehat{\boldsymbol{E}}^J_+((f, L \cup L')) \to \widehat{\boldsymbol{E}}^J_+((f, L'))$ is the constant map $\boldsymbol{c}_{[\hat{q}^J_0]}$,

$$f_+ \wedge \bigwedge_{j \in J} S^1_{(j)} \mapsto [\hat{q}^J_0];$$

- $(\hat{e}^J_+)_{(f, L), (f, L \cup L')} \colon \widehat{\boldsymbol{E}}^J_+((f, L \cup L')) \to \widehat{\boldsymbol{E}}^J_+((f, L))$ is identity map,

$$f_+ \wedge \bigwedge_{j \in J} S^1_{(j)} \mapsto f_+ \wedge \bigwedge_{j \in J} S^1_{(j)}.$$

Then in $\operatorname{colim}(\widehat{\boldsymbol{E}}^J_+)$, the image of any of such $f_+ \wedge \bigwedge_{j \in J} S^1_{(j)}$ is equivalent to the point $[\hat{q}^J_0]$. So we can deduce that:

- For $J \neq \varnothing$,

$$\operatorname{colim}(\hat{\boldsymbol{E}}^J_+) \cong \Bigl(Q^\circ \cup \bigcup_{\substack{\exists L \subsetneq I_f \\ J \cap (I_f \setminus L) = \varnothing}} f_+\Bigr) \wedge \bigwedge_{j \in J} S^1_{(j)} \Big/ \Bigl(\bigcup_{\substack{\exists L, L' \subsetneq I_f \\ J \cap (I_f \setminus L) = \varnothing \\ J \cap (I_f \setminus L') \neq \varnothing}} f_+ \Bigr) \wedge \bigwedge_{j \in J} S^1_{(j)}$$

$$\cong \bigl(((Q^\circ \cup F_{[m]\setminus J}) \cup q_0) / ((F_{[m]\setminus J} \cap F_J) \cup q_0)\bigr) \wedge \bigwedge_{j \in J} S^1_{(j)} \quad \text{(by (27) and (28))}$$

$$\cong (Q/F_J) \wedge \bigwedge_{j \in J} S^1_{(j)} \cong \boldsymbol{\Sigma}^{|J|}(Q/F_J).$$

- For $J = \varnothing$, $\operatorname{colim}(\hat{\boldsymbol{E}}^{\varnothing}_+) \cong Q^\circ \cup F_{[m]} \cup q_0 = Q \cup q_0 = Q_+$.

Combining all the above arguments, we obtain homotopy equivalences

$$\begin{aligned}
\boldsymbol{\Sigma}((D^2, S^1)^Q_+) &\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\operatorname{colim}(\hat{\boldsymbol{D}}^J_+)) \\
&\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\operatorname{colim}(\hat{\boldsymbol{E}}^J_+)) \\
&\simeq \boldsymbol{\Sigma}(Q_+) \vee \bigvee_{\varnothing \neq J \subseteq [m]} \boldsymbol{\Sigma}^{|J|+1}(Q/F_J) \\
&\simeq S^1 \vee \boldsymbol{\Sigma}(Q) \vee \bigvee_{\varnothing \neq J \subseteq [m]} \boldsymbol{\Sigma}^{|J|+1}(Q/F_J) \\
&\simeq S^1 \vee \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}^{|J|+1}(Q/F_J).
\end{aligned}$$

On the other hand,

$$\boldsymbol{\Sigma}((D^2, S^1)^Q_+) = \boldsymbol{\Sigma}((D^2, S^1)^Q \cup \hat{q}_0) \simeq S^1 \vee \boldsymbol{\Sigma}((D^2, S^1)^Q) \cong S^1 \vee \boldsymbol{\Sigma}(\mathcal{Z}_Q). \qquad \square$$

3 Cohomology ring structure of $\mathcal{Z}_Q$

The cohomology ring of the moment-angle complex over a simplicial complex K was computed by Franz [18] and Baskakov, Buchstaber and Panov [7]. The cohomology rings of a much wider class of spaces called *generalized moment-angle complexes* or *polyhedral products* were computed by Bahri, Bendersky, Cohen and Gitler [4] via partial diagonal maps and by Bahri, Bendersky, Cohen and Gitler [5] by a spectral sequence under certain freeness conditions (coefficients in a field for example). The study in this direction is further extended in [6]. A computation using different methods was carried out by Wang and Zheng [32] and Zheng [37].

It was shown by Bahri, Bendersky, Cohen and Gitler [4] that the product structure on the cohomology of a polyhedral product over a simplicial complex can be formulated in terms of the stable decomposition and partial diagonal maps of the polyhedral product. For a nice manifold with corners Q, since we also have the stable decomposition of $\mathcal{Z}_Q$, we should be able to describe the cohomology ring of $\mathcal{Z}_Q$ in a similar way.

Let us first recall the definition of partial diagonal in product spaces from [4]. Let $X_1, \dots, X_m$ be a collection of based CW–complexes. Using the notations in Section 2.4, for any $I \subseteq [m]$, there are natural projections $X^{[m]} \to \hat{X}^I$ obtained as the composition

$$\hat{\Pi}_I : X^{[m]} \xrightarrow{\Pi_I} X^I \xrightarrow{\rho_I} \hat{X}^I,$$

where $\Pi_I : X^{[m]} \to X^I$ is the natural projection and ρ_I is the quotient map in the definition of the smash product $\hat{X}^I$. In addition, let

$$W_I^{J,J'} := \bigwedge_{|J|+|J'|} W_i, \quad J \cup J' = I,$$

where

$$W_i = \begin{cases} X_i & \text{if } i \in I \backslash (J \cap J'), \\ X_i \wedge X_i & \text{if } i \in J \cap J'. \end{cases}$$

Note that if $J \cup J' = I$ and $J \cap J' = \varnothing$, $W_I^{J,J'} = \hat{X}^I$.

Define $\psi_I^{J,J'} : \hat{X}^I \to W_I^{J,J'}$ as $\psi_I^{J,J'} = \bigwedge_{i \in I} \psi_i$, where $\psi_i : X_i \to W_i$ is defined by

$$\psi_i = \begin{cases} \mathrm{id} & \text{if } i \in I \backslash (J \cap J'), \\ \Delta_i : X_i \to X_i \wedge X_i & \text{if } i \in J \cap J', \end{cases}$$

where $\Delta_i : X_i \to X_i \times X_i \to X_i \wedge X_i$ is the *reduced diagonal* of X_i.

Note that the smash products $W_I^{J,J'}$ and $\hat{X}^J \wedge \hat{X}^{J'}$ have the same factors, but in a different order arising from the natural shuffles. Let

$$\Theta_I^{J,J'} : W_I^{J,J'} \to \hat{X}^J \wedge \hat{X}^{J'}, \quad J \cup J' = I, \tag{29}$$

be the natural homeomorphism given by a shuffle. Define the *partial diagonal*

$$\hat{\Delta}_I^{J,J'} : \hat{X}^I \xrightarrow{\psi_I^{J,J'}} W_I^{J,J'} \xrightarrow{\Theta_I^{J,J'}} \hat{X}^J \wedge \hat{X}^{J'} \tag{30}$$

be the composition of $\Theta_I^{J,J'}$ and $\psi_I^{J,J'}$. There is a commutative diagram

$$\begin{tikzcd} X^{[m]} \arrow[r, "\Delta^X_{[m]}"] \arrow[d, "\hat{\Pi}_I"'] & X^{[m]} \wedge X^{[m]} \arrow[d, "\hat{\Pi}_J \wedge \hat{\Pi}_{J'}"] \\ \hat{X}^I \arrow[r, "\hat{\Delta}_I^{J,J'}"] & \hat{X}^J \wedge \hat{X}^{J'} \end{tikzcd}$$

where $\Delta^X_{[m]}$ is the reduced diagonal map of $X^{[m]}$.

Let $\boldsymbol{k}$ denote a commutative ring with a unit. For any $J \subseteq [m]$, there is a homomorphism of rings given by the *reduced cross product* $\times$ (see [20, page 223]),

$$\bigotimes_{j \in J} \tilde{H}^*(X_j; \boldsymbol{k}) \xrightarrow{\times} \tilde{H}^*(\hat{X}^J; \boldsymbol{k}).$$

In particular, this ring homomorphism becomes a ring isomorphism if all (possibly except one) $\widetilde{H}^*(X_j;\boldsymbol{k})$ are free $\boldsymbol{k}$–modules; see [20, Theorem 3.21].

Lemma 3.1 *For any $\phi_j \in \widetilde{H}^*(X_j;\boldsymbol{k})$ with $j \in J$ and any $\phi'_j \in \widetilde{H}^*(X_j;\boldsymbol{k})$ with $j \in J'$,*

$$(\widehat{\Delta}_I^{J,J'})^* \colon H^*(\widehat{X}^J \wedge \widehat{X}^{J'};\boldsymbol{k}) \to H^*(\widehat{X}^I;\boldsymbol{k}), \quad I = J \cup J',$$
$$(\widehat{\Delta}_I^{J,J'})^*\Big(\Big(\bigtimes_{j\in J}\phi_j\Big)\times\Big(\bigtimes_{j\in J'}\phi'_j\Big)\Big) = \Big(\bigtimes_{j\in J\setminus J'}\phi_j\Big)\times\Big(\bigtimes_{j\in J'\setminus J}\phi'_j\Big)\times\Big(\bigtimes_{j\in J\cap J'}\Delta_j^*(\phi_j\times\phi'_j)\Big).$$

Proof The above formula follows easily from the definition of $\widehat{\Delta}_I^{J,J'}$. Note that the shuffle $\Theta_I^{J,J'}$ (see (29)) sorts all the cohomology classes $\{\phi_j\}_{j\in J}$ and $\{\phi'_j\}_{j\in J'}$ in order without introducing any $\pm$ sign. This is because for any space X and Y,

$$T \colon X \wedge Y \to Y \wedge X, \quad [(x,y)] \mapsto [(y,x)],$$

induces a group isomorphism $T^* \colon H^*(Y \wedge X;\boldsymbol{k}) \to H^*(X \wedge Y;\boldsymbol{k})$ such that

$$T^*(\phi_Y \times \phi_X) = \phi_X \times \phi_Y, \quad \phi_X \in H^*(X;\boldsymbol{k}), \phi_Y \in H^*(Y;\boldsymbol{k}).$$

So, when $\Theta_I^{J,J'}$ transposes the space factors, the cohomology classes in the reduced cross product are transposed accordingly. $\square$

The following lemma will be useful for our proof of Theorem 1.4 later.

Lemma 3.2 *Let X be a CW–complex and A and B be two subcomplexes of X. The relative cup product $H^*(X,A)\otimes H^*(X,B) \xrightarrow{\cup} H^*(X,A\cup B)$ induces a product*

$$\widetilde{H}^*(X/A)\otimes \widetilde{H}^*(X/B) \xrightarrow{\widetilde{\cup}} \widetilde{H}^*(X/(A\cup B)),$$

which can be factored as

$$\phi \mathbin{\widetilde{\cup}} \phi' = \Delta_X^*(\phi\times\phi'),\ \phi\in\widetilde{H}^*(X/A), \phi'\in\widetilde{H}^*(X/B)$$

where $\Delta_X \colon X \to X \times X$ is the diagonal map and $\phi\times\phi'$ is the reduced cross product of ϕ and ϕ'.

Proof This can be verified directly from the following diagram when A and B are nonempty:

$$\begin{tikzcd}
H^*(X,A)\otimes H^*(X,B) \arrow[r, "\times", "\text{relative cross product}"'] & H^*(X\times X,(A\times X)\cup(X\times B)) \arrow[r, "\Delta_X^*"] & H^*(X,A\cup B) \\
H^*(X/A,A/A)\otimes H^*(X/B,B/B) \arrow[u, "\cong"] \arrow[r, "\times"] \arrow[d, phantom, "\cong"] & H^*\big(X/A\times X/B,(A/A\times X/B)\cup(X/A\times B/B)\big) \arrow[u, "\cong"] \arrow[d, phantom, "\cong"] & H^*\big(X/(A\cup B),A\cup B/A\cup B\big) \arrow[u, "\cong"] \arrow[d, phantom, "\cong"] \\
\widetilde{H}^*(X/A)\otimes\widetilde{H}^*(X/B) & \widetilde{H}^*(X/A\wedge X/B) & \widetilde{H}^*(X/(A\cup B))
\end{tikzcd}$$

where the lower $\xrightarrow{\times}$ is the reduced cross product on $\widetilde{H}^*(X/A)\otimes\widetilde{H}^*(X/B)$.

If A or B is empty, we should replace $\widetilde{H}^*(X/A)$ or $\widetilde{H}^*(X/B)$ by $H^*(X)$ in the above diagram. Moreover, since $H^*(X)\cong\widetilde{H}^*(X)\oplus\mathbb{Z}$, the $\widetilde{\cup}$ on $\widetilde{H}^*(X)$ is just the restriction of $\cup$ from $H^*(X)$. $\square$

Another useful fact is when X_i is the suspension of some space, the reduced diagonal $\Delta_i\colon X_i \to X_i \wedge X_i$ is nullhomotopic; see [4]. So we have the following lemma.

Lemma 3.3 *If for some $j \in J \cap J'$, X_j is a suspension space, then the partial diagonal*
$$\widehat{\Delta}_I^{J,J'}\colon \widehat{X}^I \to \widehat{X}^J \wedge \widehat{X}^{J'}$$
is nullhomotopic, where $I = J \cup J'$.

Now we are ready to give a proof of Theorem 1.4. Our argument is parallel to the argument used in the proof of [4, Theorem 1.4].

Proof of Theorem 1.4 For brevity, we will use the following notation in the proof:
$$Q_+ \times (D^2)^{[m]} := Q_+ \times \prod_{j\in[m]} D^2_{(j)}, \quad Q_+ \wedge (\widehat{D}^2)^J := Q_+ \wedge \bigwedge_{j\in J} D^2_{(j)}.$$
Considering the partial diagonals (30) for $Q_+ \times (D^2)^{[m]}$, we obtain a map
$$\widehat{\Delta}^{J,J'}_{J\cup J',Q_+}\colon Q_+ \wedge (\widehat{D}^2)^{J\cup J'} \to (Q_+ \wedge (\widehat{D}^2)^J) \wedge (Q_+ \wedge (\widehat{D}^2)^{J'})$$
for any $J, J' \subseteq [m]$ and a commutative diagram

$$\begin{tikzcd} Q_+ \times (D^2)^{[m]} \arrow[r, "\Delta^{Q_+,D^2}_{[m]}"] \arrow[d, "\widehat{\Pi}_{J\cup J'}"] & (Q_+ \times (D^2)^{[m]}) \wedge (Q_+ \times (D^2)^{[m]}) \arrow[d, "\widehat{\Pi}_J \wedge \widehat{\Pi}_{J'}"] \\ Q_+ \wedge (\widehat{D}^2)^{J\cup J'} \arrow[r, "\widehat{\Delta}^{J,J'}_{J\cup J',Q_+}"] & (Q_+ \wedge (\widehat{D}^2)^J) \wedge (Q_+ \wedge (\widehat{D}^2)^{J'}) \end{tikzcd} \tag{31}$$

where $\Delta^{Q_+,D^2}_{[m]}$ is the reduced diagonal map of $Q_+ \times (D^2)^{[m]}$. By restricting the above diagram to $\mathrm{colim}(\boldsymbol{D}_+)$, we obtain a commutative diagram for all $J, J' \subseteq [m]$,

$$\begin{tikzcd} \mathrm{colim}(\boldsymbol{D}_+) \arrow[r, "\Delta^{Q_+,D^2}_{[m]}"] \arrow[d, "\widehat{\Pi}_{J\cup J'}"] & \mathrm{colim}(\boldsymbol{D}_+) \wedge \mathrm{colim}(\boldsymbol{D}_+) \arrow[d, "\widehat{\Pi}_J \wedge \widehat{\Pi}_{J'}"] \\ \mathrm{colim}(\widehat{\boldsymbol{D}}^{J\cup J'}_+) \arrow[r, "\widehat{\Delta}^{J,J'}_{J\cup J',Q_+}"] & \mathrm{colim}(\widehat{\boldsymbol{D}}^J_+) \wedge \mathrm{colim}(\widehat{\boldsymbol{D}}^{J'}_+) \end{tikzcd} \tag{32}$$

Given cohomology classes $u \in \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+))$ and $v \in \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^{J'}_+))$, let
$$u \circledast v = (\widehat{\Delta}^{J,J'}_{J\cup J',Q_+})^*(u \times v) \in \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^{J\cup J'}_+)), \tag{33}$$
where $u \times v \in \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+) \wedge \mathrm{colim}(\widehat{\boldsymbol{D}}^{J'}_+))$ is the reduced cross product of u and v. This defines a ring structure on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+))$.

The commutativity of diagram (32) implies
$$\widehat{\Pi}^*_{J\cup J'}(u \circledast v) = \widehat{\Pi}^*_J(u) \cup \widehat{\Pi}^*_{J'}(v),$$
where $\cup$ is the cup product for $\mathrm{colim}(\boldsymbol{D}_+)$.

By (23), the direct sum of $\widehat{\Pi}^*_J$ induces an additive isomorphism

$$(34)\qquad \bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J\colon \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+)) \to \widetilde{H}^*(\mathrm{colim}(\boldsymbol{D}_+)) = \widetilde{H}^*((D^2,S^1)^Q_+).$$

Then since $\widehat{\Pi}^*_J\colon \widetilde{H}^*(\mathrm{colim}(\widehat{\boldsymbol{D}}^J_+)) \to \widetilde{H}^*(\mathrm{colim}(\boldsymbol{D}_+))$ is a ring homomorphism for every $J \subseteq [m]$, we can assert that $\bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J$ is a ring isomorphism. Then by the proof of Theorem 1.1, this induces a ring isomorphism

$$(35)\qquad \bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J\colon \Big(\bigoplus_{J\subseteq[m]} \widetilde{H}^*(Q/F_J \wedge \bigwedge_{j\in J} S^1_{(j)}), \circledast\Big) \to \widetilde{H}^*((D^2,S^1)^Q).$$

Finally, let us define a ring isomorphism from $(\mathcal{R}^*_Q, \Cup)$ to the cohomology ring $H^*((D^2,S^1)^Q)$ via $\bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J$.

For any $1 \le j \le m$, let $\iota^1_{(j)}$ denote a generator of $\widetilde{H}^1(S^1_{(j)})$. Then for any subset $J=\{j_1,\dots,j_s\}\subseteq[m]$ with $j_1<\cdots<j_s$, we have a generator

$$\iota^J = \iota^1_{(j_1)} \times\cdots\times \iota^1_{(j_s)} \in \widetilde{H}^{|J|}\Big(\bigwedge_{j\in J} S^1_{(j)}\Big).$$

For each $J\subseteq[m]$, there is a canonical linear isomorphism (see [20, page 223]),

$$\widetilde{H}^*(Q/F_J) \xrightarrow{\cong} \widetilde{H}^*\Big(Q/F_J \wedge \bigwedge_{j\in J} S^1_{(j)}\Big) \cong \widetilde{H}^*(\boldsymbol{\Sigma}^{|J|}(Q/F_J)), \quad \phi \mapsto \phi\times\iota^J.$$

Let

$$\widetilde{\mathcal{R}}^*_Q := \bigoplus_{J\subseteq[m]} \widetilde{H}^*(Q/F_J).$$

Then $\mathcal{R}^*_Q = \widetilde{\mathcal{R}}^*_Q \oplus \mathbb{Z}$. By Lemma 3.2, there is natural ring structure on $\widetilde{\mathcal{R}}^*_Q$, denoted by $\widetilde{\Cup}$, that is induced from the product $\Cup$ on $\mathcal{R}^*_Q$ (see (5)). We have a commutative diagram

$$(36)\qquad \begin{array}{ccc} H^*(Q,F_J)\otimes H^*(Q,F_{J'}) & \xrightarrow{\Cup} & H^*(Q,F_{J\cup J'}) \\ \downarrow & & \downarrow \\ \widetilde{H}^*(Q/F_J)\otimes \widetilde{H}^*(Q/F_{J'}) & \xrightarrow{\widetilde{\Cup}} & \widetilde{H}^*(Q/F_{J\cup J'}) \end{array}$$

such that for any $J,J'\subseteq[m]$ with $J\cap J'\neq\varnothing$, $\widetilde{\Cup}$ is trivial; and for any $J,J'\subseteq[m]$ with $J\cap J=\varnothing$, $\widetilde{\Cup}=\widetilde{\cup}$ is induced from the relative cup product $\cup$ on $H^*(Q,F_J)\otimes H^*(Q,F_{J'})$ (see Lemma 3.2).

It is clear that $(\mathcal{R}^*_Q,\Cup)$ and $(\widetilde{\mathcal{R}}^*_Q,\widetilde{\Cup})$ determine each other.

- When $J\cap J'\neq\varnothing$, since $(D^2,S^1)\cong(\Sigma D^1,\Sigma S^0)$ is a pair of suspension spaces, Lemma 3.3 implies that
$$\widehat{\Delta}^{J,J'}_{J\cup J',Q_+}\colon \mathrm{colim}(\widehat{\boldsymbol{D}}^{J\cup J'}_+) \to \mathrm{colim}(\widehat{\boldsymbol{D}}^J_+)\wedge \mathrm{colim}(\widehat{\boldsymbol{D}}^{J'}_+)$$
is nullhomotopic. So, by (33), $\circledast$ is trivial in this case which corresponds to the definition of $\widetilde{\Cup}$ on $\widetilde{\mathcal{R}}^*_Q$.

- When $J \cap J' = \varnothing$, suppose in (35), we have elements
$$u = \phi \times \iota^J \in \widetilde{H}^*\Big(Q/F_J \wedge \bigwedge_{j \in J} S^1_{(j)}\Big) = \widetilde{H}^*(\boldsymbol{\Sigma}^{|J|}(Q/F_J)),$$
$$v = \phi' \times \iota^{J'} \in \widetilde{H}^*\Big(Q/F_{J'} \wedge \bigwedge_{j \in J'} S^1_{(j)}\Big) = \widetilde{H}^*(\boldsymbol{\Sigma}^{|J'|}(Q/F_{J'})).$$
Then Lemmas 3.1 and 3.2 imply that
$$u \circledast v = (\widehat{\Delta}^{J,J'}_{J\cup J',Q_+})^*((\phi \times \iota^J) \times (\phi' \times \iota^{J'})) = (-1)^{|J||\phi'|}(\phi \mathbin{\widetilde{\cup}} \phi') \times \iota^{J \cup J'}.$$
So we have a commutative diagram
$$\begin{tikzcd}
\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) \arrow[r, "\widetilde{\Cup}"] \arrow[d, "\times\iota^J \otimes \times\iota^{J'}"'] & \widetilde{H}^*(Q/F_{J\cup J'}) \arrow[d, "\times\iota^{J\cup J'}"] \\
\widetilde{H}^*(\boldsymbol{\Sigma}^{|J|}(Q/F_J)) \otimes \widetilde{H}^*(\boldsymbol{\Sigma}^{|J'|}(Q/F_{J'})) \arrow[r, "\circledast"] & \widetilde{H}^*(\boldsymbol{\Sigma}^{|J\cup J'|}(Q/F_{J\cup J'}))
\end{tikzcd} \tag{37}$$
which implies that the product $\widetilde{\Cup}$ on $\widetilde{\mathcal{R}}^*_Q$ corresponds to the product $\circledast$ in (35) in this case.

Combining the above arguments, we obtain isomorphisms of rings,
$$(\widetilde{\mathcal{R}}^*_Q, \widetilde{\Cup}) \xrightarrow{\cong} \Big(\bigoplus_{J \subseteq [m]} \widetilde{H}^*(Q/F_J \wedge \bigwedge_{j \in J} S^1_{(j)}), \circledast\Big) \xrightarrow{\bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J} \widetilde{H}^*((D^2,S^1)^Q) = \widetilde{H}^*(\mathcal{Z}_Q).$$
It follows that there is a ring isomorphism (up to a sign) from $(\mathcal{R}^*_Q, \Cup)$ to $H^*(\mathcal{Z}_Q)$.

Note that the above ring isomorphism is not degree-preserving. But by the diagram in (37), we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_J)$ up by $|J|$ for every $J \subseteq [m]$. □

4 Polyhedral product over a nice manifold with corners

Let Q be a nice manifold with corners whose facets are $F_1, \dots, F_m$. Let $(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j)\}_{j=1}^m$, where X_j and A_j are CW–complexes with a basepoint $a_j \in A_j \subseteq X_j$.

For any face f of Q, define
$$(\mathbb{X},\mathbb{A})^f := f \times \prod_{j \in I_f} X_j \times \prod_{j \in [m]\setminus I_f} A_j, \quad (\mathbb{X},\mathbb{A})^Q := \bigcup_{f \in \mathcal{S}_Q} (\mathbb{X},\mathbb{A})^f \subseteq Q \times \prod_{j\in[m]} X_j.$$
If $(\mathbb{X}, \mathbb{A}) = \{(X_j, A_j, a_j) = (X, A, a_0)\}_{j=1}^m$, we also denote $(\mathbb{X},\mathbb{A})^Q$ by $(X, A)^Q$.

We call $(\mathbb{X},\mathbb{A})^Q$ the *polyhedral product* of $(\mathbb{X},\mathbb{A})$ over Q. Note that in general, the homeomorphism type of $(\mathbb{X},\mathbb{A})^Q$ depends on the ordering of the facets of Q and the ordering of the X_j. We consider $(\mathbb{X},\mathbb{A})^Q$ as an analogue of polyhedral products over a simplicial complex; see [8].

In the rest of this section, we assume that each of X_j and A_j in $(\mathbb{X},\mathbb{A})$ is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then we can study the stable decomposition and cohomology ring of $(\mathbb{X},\mathbb{A})^Q$ in the same way as we do for $\mathcal{Z}_Q$.

- Let $Q_+ = Q \cup q_0$ where $q_0 \notin Q$ is the basepoint of Q_+.
- For any face f of Q, let $f_+ = f \cup q_0$ with basepoint q_0.
- For any $(f,L) \in \mathcal{P}_Q$, define

$$(\mathbb{X},\mathbb{A})^{(f,L)} := f \times \prod_{j\in I_f\setminus L} X_j \times \prod_{j\in[m]\setminus(I_f\setminus L)} A_j, \quad (\mathbb{X},\mathbb{A})_+^{(f,L)} := (\mathbb{X},\mathbb{A})^{(f,L)} \cup \hat{q}_0^{(\mathbb{X},\mathbb{A})}, \tag{38}$$

  where $\hat{q}_0^{(\mathbb{X},\mathbb{A})}$ is the basepoint defined by $\hat{q}_0^{(\mathbb{X},\mathbb{A})} := q_0 \times \prod_{j\in[m]} a_j$.
- Let $(\mathbb{X},\mathbb{A})_+^Q = (\mathbb{X},\mathbb{A})^Q \cup \hat{q}_0^{(\mathbb{X},\mathbb{A})}$ with basepoint $\hat{q}_0^{(\mathbb{X},\mathbb{A})}$.

Let $\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+} \colon \mathcal{P}_Q \to \mathrm{CW}_*$ be the diagram of based CW–complexes, where

$$\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}((f,L)) := (\mathbb{X},\mathbb{A})_+^{(f,L)} \quad \text{for all } (f,L) \in \mathcal{P}_Q,$$

and let $(d_{(\mathbb{X},\mathbb{A})+})_{(f,L),(f',L')} \colon \boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}((f',L')) \to \boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}((f,L))$ be the natural inclusion for any $(f,L) \le (f',L') \in \mathcal{P}_Q$. Then

$$(\mathbb{X},\mathbb{A})_+^Q = \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) = \bigcup_{(f,L)\in\mathcal{P}_Q} (\mathbb{X},\mathbb{A})_+^{(f,L)}. \tag{39}$$

By Theorem 2.10, we can prove the following lemma parallel to Lemma 2.15.

Lemma 4.1 *For any $(f,L) \in \mathcal{P}_Q$, there is a natural homeomorphism which commutes with taking the colimit*

$$(\mathbb{X},\mathbb{A})_+^{(f,L)} \cong f_+ \rtimes \Big(\prod_{j\in I_f\setminus L} X_j \times \prod_{j\in[m]\setminus(I_f\setminus L)} A_j \Big).$$

So, by Theorem 2.14,

$$\begin{aligned}\boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})_+^{(f,L)}) &\cong \boldsymbol{\Sigma}\Big(f_+ \rtimes \Big(\prod_{j\in I_f\setminus L} X_j \times \prod_{j\in[m]\setminus(I_f\setminus L)} A_j \Big)\Big) \\ &\simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}\Big(f_+ \wedge \bigwedge_{j\in J\cap(I_f\setminus L)} X_j \wedge \bigwedge_{j\in J\setminus(I_f\setminus L)} A_j \Big).\end{aligned} \tag{40}$$

Accordingly, we define a family of diagrams of based CW–complexes,

$$\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+} \colon \mathcal{P}_Q \to \mathrm{CW}_*, \quad J \subseteq [m],$$

$$\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L)) := f_+ \wedge \bigwedge_{j\in J\cap(I_f\setminus L)} X_j \wedge \bigwedge_{j\in J\setminus(I_f\setminus L)} A_j \quad \text{for all } (f,L) \in \mathcal{P}_Q.$$

Define $(\hat{d}^J_{(\mathbb{X},\mathbb{A})+})_{(f,L),(f',L')}\colon \hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f',L')) \to \hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ to be the natural inclusion for any $(f,L) \le (f',L') \in \mathcal{P}_Q$. The basepoint of $\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ is $[q_0 \times \prod_{j\in J} a_j]$. So we have the following theorem by [3, Theorem 4.3].

Theorem 4.2 *Let $(\mathbb{X},\mathbb{A}) = \{(X_j,A_j,a_j)\}_{j=1}^m$ where each X_j and A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there are homotopy equivalences*

$$S^1 \vee \boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q) \simeq \boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q_+) = \boldsymbol{\Sigma}(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \simeq \bigvee_{J\subseteq[m]} \boldsymbol{\Sigma}(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})).$$

This implies

$$H^*((\mathbb{X},\mathbb{A})^Q) \cong \widetilde{H}^*(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})).$$

Moreover, using the partial diagonal map for $Q_+ \times \prod_{j\in[m]} X_j$ as in the proof of Theorem 1.4, we have a diagram parallel to diagram (32) for any $J, J' \subseteq [m]$,

$$\begin{tikzcd}
\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) \arrow[r, "\Delta^{Q_+,\mathbb{X}}_{[m]}"] \arrow[d, "\hat{\Pi}_{J\cup J'}"] & \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) \wedge \operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+}) \arrow[d, "\hat{\Pi}_J \wedge \hat{\Pi}_{J'}"] \\
\operatorname{colim}(\hat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{X},\mathbb{A})+}) \arrow[r, "\hat{\Delta}^{J,J'}_{J\cup J',Q_+}"] & \operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}) \wedge \operatorname{colim}(\hat{\boldsymbol{D}}^{J'}_{(\mathbb{X},\mathbb{A})+})
\end{tikzcd} \tag{41}$$

Similarly, we can obtain the following theorem parallel to Theorem 1.4.

Theorem 4.3 *Let $(\mathbb{X},\mathbb{A}) = \{(X_j,A_j,a_j)\}_{j=1}^m$ where each X_j and A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a ring isomorphism*

$$\bigoplus_{J\subseteq[m]} \hat{\Pi}^*_J\colon \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})) \to \widetilde{H}^*(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) \cong H^*((\mathbb{X},\mathbb{A})^Q),$$

where the product $\circledast$ on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}))$ is defined by

$$\begin{gathered}
\widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})) \otimes \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^{J'}_{(\mathbb{X},\mathbb{A})+})) \xrightarrow{\circledast} \widetilde{H}^*(\operatorname{colim}(\hat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{X},\mathbb{A})+})), \\
u \circledast v := (\hat{\Delta}^{J,J'}_{J\cup J',Q_+})^*(u \times v).
\end{gathered} \tag{42}$$

In the following two subsections, we will study the stable decomposition and cohomology ring of $(\mathbb{X},\mathbb{A})^Q$ under some special conditions on $(\mathbb{X},\mathbb{A})$.

4.1 The case of $(\mathbb{X},\mathbb{A})^Q$ with each X_j contractible

Observe that in the proof of Theorem 1.1, the only properties of (D^2,S^1) that we actually use are that

(i) D^2 is contractible;

(ii) $X \wedge S^1$ is homeomorphic to $\boldsymbol{\Sigma}(X)$ for any based CW–complex X.

So, if we assume that every X_j in $(\mathbb{X},\mathbb{A})$ is contractible, we can obtain the following theorem parallel to Theorem 1.1.

Theorem 4.4 *Let Q be a nice manifold with corners with facets $F_1,\dots,F_m$. Let*

$$(\mathbb{X},\mathbb{A})=\{(X_j,A_j,a_j)\}_{j=1}^m,$$

where each X_j is contractible and each A_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a homotopy equivalence

$$\boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q)\simeq\bigvee_{J\subseteq[m]}\boldsymbol{\Sigma}\Big(Q/F_J\wedge\bigwedge_{j\in J}A_j\Big).$$

So the reduced cohomology group

$$\widetilde{H}^*((\mathbb{X},\mathbb{A})^Q)\cong\bigoplus_{J\subseteq[m]}\widetilde{H}^*\Big(Q/F_J\wedge\bigwedge_{j\in J}A_j\Big).$$

Proof We can easily extend the argument in the proof of Theorem 1.1 to show

$$\operatorname{colim}(\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})\simeq\begin{cases}Q/F_J\wedge\bigwedge_{j\in J}A_j & \text{if } J\neq\varnothing,\\ Q_+ & \text{if } J=\varnothing.\end{cases}$$

Then the statements of the theorem follow from Theorem 4.2 and the fact that $\boldsymbol{\Sigma}(Q_+)\simeq S^1\vee\boldsymbol{\Sigma}(Q)$. $\square$

Moreover, we have the following theorem which is parallel to [4, Theorem 1.4].

Theorem 4.5 *Under the condition in Theorem 4.4, there is a ring isomorphism*

$$\Big(\bigoplus_{J\subseteq[m]}\widetilde{H}^*\Big(Q/F_J\wedge\bigwedge_{j\in J}A_j\Big),\circledast\Big)\to\widetilde{H}^*((\mathbb{X},\mathbb{A})^Q)\quad\textit{induced by}\quad\bigoplus_{J\subseteq[m]}\widehat{\Pi}^*_J.$$

Remark 4.6 If any combination of Q/F_J and A_j's satisfies the *strong smash form of the Künneth formula* as defined in [3, page 1647] over a coefficient ring $\boldsymbol{k}$, ie the natural map

$$\widetilde{H}^*(Q/F_J;\boldsymbol{k})\otimes\bigotimes_{j\in I}\widetilde{H}^*(A_j;\boldsymbol{k})\to\widetilde{H}^*\Big(Q/F_J\wedge\bigwedge_{j\in I}A_j;\boldsymbol{k}\Big)$$

is an isomorphism for any $I,J\subseteq[m]$, then we can write the cohomology ring structure of $(\mathbb{X},\mathbb{A})^Q$ with $\boldsymbol{k}$–coefficients more explicitly via the formula in Lemma 3.1.

In the following, we demonstrate the product $\circledast$ for $(\mathbb{D},\mathbb{S})^Q$ where

$$(\mathbb{D},\mathbb{S})=\{(D^{n_j+1},S^{n_j},a_j)\}_{j=1}^m.$$

Here D^{n+1} is the unit ball in $\mathbb{R}^{n+1}$ and $S^n=\partial D^{n+1}$.

In particular, if $(\mathbb{D},\mathbb{S})=\{(D^{n_j+1},S^{n_j},a_j)=(D^{n+1},S^n,a_0)\}_{j=1}^m$, we also write

$$(\mathbb{D},\mathbb{S})^Q=(D^{n+1},S^n)^Q.$$

Example 4.7 $\mathscr{X}_Q \cong (D^2, S^1)^Q$ and $\mathbb{R}\mathscr{X}_Q \cong (D^1, S^0)^Q$ (see Remark 1.3).

We define a graded ring structure $\Cup^{(\mathbb{D},\mathbb{S})}$ on $\mathscr{R}^*_Q$ according to $(\mathbb{D},\mathbb{S})$ as follows.

- If $J \cap J' = \varnothing$ or $J \cap J' \neq \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$, then
$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\Cup^{(\mathbb{D},\mathbb{S})}} H^*(Q, F_{J\cup J'})$$
is the relative cup product $\cup$.
- If $J \cap J' \neq \varnothing$ and there exists $n_j \geq 1$ for some $j \in J \cap J'$, then
$$H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\Cup^{(\mathbb{D},\mathbb{S})}} H^*(Q, F_{J\cup J'})$$
is trivial.

By Lemma 3.2, the product $\Cup^{(\mathbb{D},\mathbb{S})}$ on $\mathscr{R}^*_Q$ induces a product $\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}$ on $\widetilde{\mathscr{R}}^*_Q$.

- If $J \cap J' = \varnothing$ or $J \cap J' \neq \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$, then
$$\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) \xrightarrow{\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}} \widetilde{H}^*(Q/F_{J\cup J'})$$
is the product $\widetilde{\cup}$ induced by the relative cup product $H^*(Q, F_J) \otimes H^*(Q, F_{J'}) \xrightarrow{\cup} H^*(Q, F_{J\cup J'})$.
- If $J \cap J' \neq \varnothing$ and there exists $n_j \geq 1$ for some $j \in J \cap J'$, then
$$\widetilde{H}^*(Q/F_J) \otimes \widetilde{H}^*(Q/F_{J'}) \xrightarrow{\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}} \widetilde{H}^*(Q/F_{J\cup J'})$$
is trivial.

We have the following theorem which generalizes Theorems 1.1 and 1.4.

Theorem 4.8 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Then for any*
$$(\mathbb{D},\mathbb{S}) = \{(D^{n_j+1}, S^{n_j}, a_j)\}_{j=1}^m,$$

(a) *there is a homotopy equivalence*
$$\Sigma((\mathbb{D},\mathbb{S})^Q) \simeq \bigvee_{J\subseteq[m]} \Sigma\Big(Q/F_J \wedge \bigwedge_{j\in J} S^{n_j}\Big) \cong \bigvee_{J\subseteq[m]} \Sigma^{1+\sum_{j\in J} n_j}(Q/F_J),$$
which implies
$$H^p((\mathbb{D},\mathbb{S})^Q) \cong \bigoplus_{J\subseteq[m]} H^{p-\sum_{j\in J} n_j}(Q, F_J) \quad \text{for all } p \in \mathbb{Z};$$

(b) *there is a ring isomorphism (up to a sign) from $(\mathscr{R}^*_Q, \Cup^{(\mathbb{D},\mathbb{S})})$ to the integral cohomology ring of $(\mathbb{D},\mathbb{S})^Q$; moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of the elements in $H^*(Q, F_J)$ for every $J \subseteq [m]$.*

Proof For brevity, we use the notation

$$N_J = \sum_{j \in J} n_j, \quad J \subseteq [m].$$

Statement (a) follows from Theorem 4.4 and the simple fact that

$$Q/F_J \wedge \bigwedge_{j \in J} S^{n_j} \cong Q/F_J \wedge S^{N_J} \cong \boldsymbol{\Sigma}^{N_J}(Q/F_J).$$

For statement (b), note that, by Theorem 4.5, we have a ring isomorphism

$$\bigoplus_{J \subseteq [m]} \widehat{\Pi}_J^* \colon \Bigl(\bigoplus_{J \subseteq [m]} \widetilde{H}^*\Bigl(Q/F_J \wedge \bigwedge_{j \in J} S^{n_j}\Bigr), \circledast \Bigr) \to \widetilde{H}^*((\mathbb{D},\mathbb{S})^Q). \tag{43}$$

For any $1 \le j \le m$, let ι^{n_j} denote a generator of $\widetilde{H}^{n_j}(S^{n_j})$. Let

$$\iota^J_{(\mathbb{D},\mathbb{S})} = \bigtimes_{j \in J} \iota^{n_j} \in \widetilde{H}^{N_J}\Bigl(\bigwedge_{j \in J} S^{n_j}\Bigr)$$

be a generator.

(i) Assume $J \cap J' \neq \varnothing$ and there exists $n_j \ge 1$ for some $j \in J \cap J'$. Then since S^{n_j} is a suspension space, the map $\widehat{\Delta}^{J,J'}_{J \cup J', Q_+}$ in (41) is nullhomotopic. This implies that the product $\circledast$ in (43) is trivial which corresponds to the definition of $\widetilde{\mathbb{U}}^{(\mathbb{D},\mathbb{S})}$ on $\widetilde{\mathcal{R}}^*_Q$ in this case.

(ii) Assume $J \cap J' \neq \varnothing$ but $n_j = 0$ for all $j \in J \cap J'$. Let

$$J_0 = \{ j \in [m] \mid n_j = 0 \} \subseteq [m].$$

So the condition on J and J' is equivalent to $J \cap J' \subseteq J_0$ which implies

$$(J \backslash J_0) \cap (J' \backslash J_0) = \varnothing. \tag{44}$$

Since $X \wedge S^0 \cong X$ for any based space X, we have for any $J \subseteq [m]$,

$$Q/F_J \wedge \bigwedge_{j \in J} S^{n_j} \cong Q/F_J \wedge \bigwedge_{j \in J \backslash J_0} S^{n_j} \cong \boldsymbol{\Sigma}^{N_{J \backslash J_0}}(Q/F_J).$$

By Lemmas 3.1 and 3.2, we can derive an explicit formula for the product $\circledast$ in (43) as follows. For any elements

$$u = \phi \times \iota^{J \backslash J_0}_{(\mathbb{D},\mathbb{S})} \in \widetilde{H}^*\Bigl(Q/F_J \wedge \bigwedge_{j \in J \backslash J_0} S^{n_j}\Bigr) = \widetilde{H}^*(\boldsymbol{\Sigma}^{N_{J \backslash J_0}}(Q/F_J)),$$

$$v = \phi' \times \iota^{J' \backslash J_0}_{(\mathbb{D},\mathbb{S})} \in \widetilde{H}^*\Bigl(Q/F_{J'} \wedge \bigwedge_{j \in J' \backslash J_0} S^{n_j}\Bigr) = \widetilde{H}^*(\boldsymbol{\Sigma}^{N_{J' \backslash J_0}}(Q/F_{J'})),$$

we have

$$u \circledast v = (\widehat{\Delta}^{J \backslash J_0, J' \backslash J_0}_{(J \cup J') \backslash J_0, Q_+})^* \bigl((\phi \times \iota^{J \backslash J_0}_{(\mathbb{D},\mathbb{S})}) \times (\phi' \times \iota^{J' \backslash J_0}_{(\mathbb{D},\mathbb{S})}) \bigr) = (-1)^{N_{J \backslash J_0} |\phi'|} (\phi \mathbin{\widetilde{\cup}} \phi') \times \iota^{(J \cup J') \backslash J_0}_{(\mathbb{D},\mathbb{S})}$$

by (44). So we have a commutative diagram parallel to diagram (37),

$$\begin{tikzcd}
\widetilde{H}^*(Q/F_J)\otimes\widetilde{H}^*(Q/F_{J'}) \arrow[r, "\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}"] \arrow[d, "\times\iota^{J\backslash J_0}_{(\mathbb{D},\mathbb{S})}\otimes\times\iota^{J'\backslash J_0}_{(\mathbb{D},\mathbb{S})}"'] & \widetilde{H}^*(Q/F_{J\cup J'}) \arrow[d, "\times\iota^{(J\cup J')\backslash J_0}_{(\mathbb{D},\mathbb{S})}"] \\
\widetilde{H}^*(\Sigma^{N_{J\backslash J_0}}(Q/F_J))\otimes\widetilde{H}^*(\Sigma^{N_{J'\backslash J_0}}(Q/F_{J'})) \arrow[r, "\circledast"] & \widetilde{H}^*(\Sigma^{N_{(J\cup J')\backslash J_0}}(Q/F_{J\cup J'}))
\end{tikzcd}$$

This implies that the product $\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}$ on $\widetilde{\mathcal{R}}^*_Q$ corresponds to the product $\circledast$ in (43) in this case.

(iii) When $J\cap J'=\varnothing$, the proof of the correspondence between the product $\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})}$ on $\widetilde{\mathcal{R}}^*_Q$ and the product $\circledast$ in (43) is the same as case (ii).

The above discussion implies that there is an isomorphism of rings

$$(\widetilde{\mathcal{R}}^*_Q,\widetilde{\Cup}^{(\mathbb{D},\mathbb{S})})\to\Big(\bigoplus_{J\subseteq[m]}\widetilde{H}^*(Q/F_J\wedge\bigwedge_{j\in J}S^{n_j}),\circledast\Big)\xrightarrow{\bigoplus_{J\subseteq[m]}\widehat{\Pi}^*_J}\widetilde{H}^*((\mathbb{D},\mathbb{S})^Q).$$

This implies that $(\mathcal{R}^*_Q,\Cup^{(\mathbb{D},\mathbb{S})})$ is isomorphic (up to a sign) to the integral cohomology ring $H^*((\mathbb{D},\mathbb{S})^Q)$. Moreover, according to the above diagram, we can make the ring isomorphism between $(\mathcal{R}^*_Q,\Cup^{(\mathbb{D},\mathbb{S})})$ and $H^*((\mathbb{D},\mathbb{S})^Q)$ degree-preserving by shifting the degrees of all the elements in $H^*(Q,F_J)$ up by $N_{J\backslash J_0}$ for every $J\subseteq[m]$. □

Remark 4.9 S^0 is not a suspension of any space and the reduced diagonal map

$$\Delta_{S^0}=\mathrm{id}_{S^0}\colon S^0\to S^0\wedge S^0\cong S^0$$

is not nullhomotopic. This is the essential reason why for a general $(\mathbb{D},\mathbb{S})$, the cohomology ring of $(\mathbb{D},\mathbb{S})^Q$ is more subtle than that of $\mathcal{Z}_Q$.

A very special case of Theorem 4.8 is $(D^1,S^0)^Q=\mathbb{R}\mathcal{Z}_Q$ where the product $\Cup^{(D^1,S^0)}$ on $\mathcal{R}^*_Q$ is exactly the relative cup product for all $J,J'\subseteq[m]$.

Corollary 4.10 *Let Q be a nice manifold with corners with facets $F_1,\dots,F_m$. Then*

$$\Sigma(\mathbb{R}\mathcal{Z}_Q)\simeq\bigvee_{J\subseteq[m]}\Sigma(Q/F_J),\quad H^p(\mathbb{R}\mathcal{Z}_Q)\cong\bigoplus_{J\subseteq[m]}H^p(Q,F_J)\quad\text{for all }p\in\mathbb{Z}.$$

Moreover, the integral cohomology ring of $\mathbb{R}\mathcal{Z}_Q$ is isomorphic as a graded ring to the ring $(\mathcal{R}^*_Q,\cup)$ where $\cup$ is the relative cup product

$$H^*(Q,F_J)\otimes H^*(Q,F_{J'})\xrightarrow{\cup}H^*(Q,F_{J\cup J'})\quad\text{for all }J,J'\subseteq[m].$$

Note that in the case of $\mathbb{R}\mathcal{Z}_Q$, the sign factor of the isomorphism between $(\mathcal{R}^*_Q,\cup)$ and $H^*(\mathbb{R}\mathcal{Z}_Q)$ is trivial because the degree of $\iota^J_{(D^1,S^0)}$ is always zero.

Remark 4.11 When Q is a simple polytope, the ring structure of the integral cohomology of $\mathbb{R}\mathcal{Z}_Q$ was studied in [11] via a different method.

4.2 The case of $(\mathbb{X},\mathbb{A})^Q$ with each A_j contractible

If in $(\mathbb{X},\mathbb{A})=\{(X_j,A_j,a_j)\}_{j=1}^m$, each A_j is contractible, we can derive the stable decomposition of $(\mathbb{X},\mathbb{A})^Q$ from Theorem 4.2 as follows.

Theorem 4.12 *Let Q be a nice manifold with corners with facets $F_1,\dots,F_m$. Let*

$$(\mathbb{X},\mathbb{A})=\{(X_j,A_j,a_j)\}_{j=1}^m,$$

where each A_j is contractible and each X_j is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then there is a homotopy equivalence

$$S^1\vee\mathbf{\Sigma}((\mathbb{X},\mathbb{A})^Q)\simeq\mathbf{\Sigma}((\mathbb{X},\mathbb{A})^Q_+)\simeq\bigvee_{J\subseteq[m]}\mathbf{\Sigma}\biggl((F_{\cap J}\cup q_0)\wedge\bigwedge_{j\in J}X_j\biggr).$$

So we have $\mathbf{\Sigma}((\mathbb{X},\mathbb{A})^Q)\simeq\bigvee_{J\subseteq[m]}\mathbf{\Sigma}\bigl(F_{\cap J}\ltimes\bigwedge_{j\in J}X_j\bigr)$ and

$$H^*((\mathbb{X},\mathbb{A})^Q)\cong\bigoplus_{J\subseteq[m]}\widetilde{H}^*\biggl((F_{\cap J}\cup q_0)\wedge\bigwedge_{j\in J}X_j\biggr).$$

Proof By Lemma 2.7 and our assumption on A_j, when $J\backslash(I_f\backslash L)\neq\varnothing$,

$$\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))=f_+\wedge\bigwedge_{j\in J\cap(I_f\backslash L)}X_j\wedge\bigwedge_{j\in J\backslash(I_f\backslash L)}A_j$$

is contractible. So, for any $J\subseteq[m]$, we define a diagram of based CW–complexes

$$\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}:\mathcal{P}_Q\to\mathrm{CW}_*$$

$$\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}((f,L)):=\begin{cases}\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))=f_+\wedge\bigwedge_{j\in J}X_j & \text{if } J\backslash(I_f\backslash L)=\varnothing,\\ [\hat{q}^J_0] & \text{if } J\backslash(I_f\backslash L)\neq\varnothing,\end{cases}\tag{45}$$

where $(\hat{g}^J_{(\mathbb{X},\mathbb{A})+})_{(f,L),(f',L')}:\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}((f',L'))\to\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ is either the natural inclusion or the constant map $\boldsymbol{c}_{[\hat{q}^J_0]}$ for any $(f,L)\le(f',L')\in\mathcal{P}_Q$. The basepoint of $\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))$ is $[\hat{q}^J_0]$.

Let $\Psi^J_{(\mathbb{X},\mathbb{A})+}:\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}\to\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}$ be a map of diagrams over $\mathcal{P}_Q$ defined by

$$(\Psi^J_{(\mathbb{X},\mathbb{A})+})_{(f,L)}:\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))\to\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}((f,L)),$$

$$(\Psi^J_{(\mathbb{X},\mathbb{A})+})_{(f,L)}=\begin{cases}\mathrm{id}_{\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+}((f,L))} & \text{if } J\backslash(I_f\backslash L)=\varnothing,\\ \boldsymbol{c}_{[\hat{q}^J_0]} & \text{if } J\backslash(I_f\backslash L)\neq\varnothing.\end{cases}$$

Then by Theorem 2.16, there exists a homotopy equivalence

$$\mathrm{colim}(\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})\simeq\mathrm{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}),\quad J\subseteq[m].$$

To understand $\mathrm{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+})$, we need to figure out in (45) what are those faces f of Q with some $L\subseteq I_f$ such that $J\backslash(I_f\backslash L)=\varnothing$.

- There exists $L \subseteq I_f$ such that $J \backslash (I_f \backslash L) = \varnothing$ if and only if $J \subseteq I_f$. So

$$\bigcup_{f \in \mathcal{S}_Q} \bigcup_{\substack{\exists L \subseteq I_f \\ J \backslash (I_f \backslash L) = \varnothing}} f = \bigcup_{f \in \mathcal{S}_Q} \bigcup_{J \subseteq I_f} f = F_{\cap J}. \tag{46}$$

- There exists $L \subseteq I_f$ such that $J \backslash (I_f \backslash L) \neq \varnothing$ if and only if $J \neq \varnothing$.

Then for any $\varnothing \neq J \subseteq [m]$,

$$\operatorname{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}) \cong \bigcup_{\substack{\exists L \subseteq I_f \\ J \backslash (I_f \backslash L) = \varnothing}} \Big(f_+ \wedge \bigwedge_{j \in J} X_j \Big) \Big/ \bigcup_{\substack{\exists L \subseteq I_f \\ J \backslash (I_f \backslash L) \neq \varnothing}} \Big(f_+ \wedge \bigwedge_{j \in I_f \backslash L} X_j \wedge \bigwedge_{j \in J \backslash (I_f \backslash L)} A_j \Big)$$

$$\simeq \Big(\Big(\bigcup_{J \subseteq I_f} f_+ \Big) \wedge \bigwedge_{j \in J} X_j \Big) \Big/ \bigcup_{\substack{f \in \mathcal{S}_Q, L \subseteq I_f \\ J \backslash (I_f \backslash L) \neq \varnothing}} \Big(f_+ \wedge \bigwedge_{j \in I_f \backslash L} X_j \wedge \bigwedge_{j \in J \backslash (I_f \backslash L)} a_j \Big) \tag{47}$$

$$\cong \Big(\bigcup_{J \subseteq I_f} f_+ \Big) \wedge \bigwedge_{j \in J} X_j \Big/ [\hat{q}^J_0] = (F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \quad \text{(by (46))}, \tag{48}$$

where the "$\simeq$" in (47) is because each A_j is contractible, so A_j deformation retracts to its basepoint a_j, and the "$\cong$" in (48) is because $f_+ \times \prod_{j \in I_f \backslash L} X_j \times \prod_{j \in J \backslash (I_f \backslash L)} a_j$ is equivalent to the basepoint $[\hat{q}^J_0]$ in $f_+ \wedge \bigwedge_{j \in J} X_j$ since a_j is the basepoint of X_j.

When $J = \varnothing$,

$$\operatorname{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+}) = \bigcup_{f \in \mathcal{S}_Q} f_+ = Q \cup q_0 = F_{\cap \varnothing} \cup q_0.$$

So, by Theorem 4.2, we have homotopy equivalences

$$\begin{aligned}\boldsymbol{\Sigma}((\mathbb{X},\mathbb{A})^Q_+) = \boldsymbol{\Sigma}(\operatorname{colim}(\boldsymbol{D}_{(\mathbb{X},\mathbb{A})+})) &\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\operatorname{colim}(\widehat{\boldsymbol{D}}^J_{(\mathbb{X},\mathbb{A})+})) \\ &\simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}(\operatorname{colim}(\widehat{\boldsymbol{G}}^J_{(\mathbb{X},\mathbb{A})+})) \simeq \bigvee_{J \subseteq [m]} \boldsymbol{\Sigma}\Big((F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \Big).\end{aligned}$$

By Definition 2.13,

$$(F_{\cap J} \cup q_0) \wedge \bigwedge_{j \in J} X_j \cong \begin{cases} F_{\cap J} \ltimes \bigwedge_{j \in J} X_j & \text{if } J \neq \varnothing, \\ Q \cup q_0 & \text{if } J = \varnothing. \end{cases}$$

Then since $\boldsymbol{\Sigma}(Q \cup q_0) \simeq S^1 \vee \boldsymbol{\Sigma}(Q)$, the theorem is proved. □

The cohomology ring structure of $(\mathbb{X},\mathbb{A})^Q$ can be computed by Theorem 4.3. In particular, if any combination of $F_{\cap J}$ and X_j's satisfies the strong smash form of the Künneth formula over a coefficient ring $\boldsymbol{k}$, we can give an explicit description of the cohomology ring of $(\mathbb{X},\mathbb{A})^Q$ with $\boldsymbol{k}$–coefficients. Indeed, by Theorems 4.3 and 4.12 we obtain an isomorphism of rings

$$(49)\qquad \bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J\colon \bigoplus_{J\subseteq[m]} \Bigl(H^*(F_{\cap J};\boldsymbol{k})\otimes \bigotimes_{j\in J} \widetilde{H}^*(X_j;\boldsymbol{k})\Bigr) \to H^*((\mathbb{X},\mathbb{A})^Q;\boldsymbol{k}),$$

where the product $\circledast$ on the left-hand side is defined by (42) via the partial diagonal maps. We will do some computation of this kind in the next section to describe the equivariant cohomology ring of the moment-angle manifold $\mathcal{Z}_Q$.

5 Equivariant cohomology ring of $\mathcal{Z}_Q$ and $\mathbb{R}\mathcal{Z}_Q$

Let Q be a nice manifold with corners whose facets are $F_1,\dots,F_m$. Since there is a canonical action of $(S^1)^m$ on $\mathcal{Z}_Q$ (see (2)), it is a natural problem to compute the equivariant cohomology ring of $\mathcal{Z}_Q$ with respect to this action.

For a simple polytope P, it is shown in Davis and Januszkiewicz [17] that the equivariant cohomology of $\mathcal{Z}_P$ with integral coefficients is isomorphic to the face ring (or Stanley–Reisner ring) $\mathbb{Z}[P]$ of P defined by

$$\mathbb{Z}[P]=\mathbb{Z}[x_1,\dots,x_m]/\mathcal{I}_P,$$

where $\mathcal{I}_P$ is the ideal generated by all square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_s}$ such that $F_{i_1}\cap\cdots\cap F_{i_s}=\varnothing$ in P. A liner basis of $\mathbb{Z}[P]$ is given by

$$(50)\qquad \{1\}\cup\{x_{i_1}^{n_1}\cdots x_{i_s}^{n_s}\mid F_{i_1}\cap\cdots\cap F_{i_s}\neq\varnothing,\ n_1>0,\dots,n_s>0\}.$$

We can also think of $\mathbb{Z}[P]$ as the face ring of ∂P^* where P^* is the dual simplicial polytope of P; see [9, Chapter 3].

For brevity, let $T^m=(S^1)^m$. By definition, the *equivariant cohomology* of $\mathcal{Z}_Q$, denoted by $H^*_{T^m}(\mathcal{Z}_Q)$, is the cohomology of the *Borel construction*

$$ET^m\times_{T^m}\mathcal{Z}_Q=ET^m\times\mathcal{Z}_Q/{\sim},$$

where $(e,x)\sim(eg,g^{-1}x)$ for any $e\in ET^m$, $x\in\mathcal{Z}_Q$ and $g\in T^m$. Here we let

$$ET^m=(ES^1)^m=(S^\infty)^m.$$

Associated to the Borel construction, there is a canonical fiber bundle

$$(51)\qquad \mathcal{Z}_Q\to ET^m\times_{T^m}\mathcal{Z}_Q\to BT^m,$$

where $BT^m=(BS^1)^m=(S^\infty/S^1)^m=(\mathbb{C}P^\infty)^m$ is the *classifying space* of T^m.

By Lemma 2.3, $\mathcal{Z}_Q$ is equivariantly homeomorphic to $(D^2,S^1)^Q$. So computing the equivariant cohomology of $\mathcal{Z}_Q$ is equivalent to computing that for $(D^2,S^1)^Q$.

By the colimit construction of $(D^2,S^1)^Q$ in (16) and our notation for polyhedral products (38), the Borel construction

$$\begin{aligned} ET^m \times_{T^m} (D^2,S^1)^Q &= \bigcup_{(f,L)\in\mathcal{P}_Q} ET^m \times_{T^m} (D^2,S^1)^{(f,L)} \\ &= \bigcup_{(f,L)\in\mathcal{P}_Q} (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^{(f,L)} \\ &= (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^Q. \end{aligned}$$

Then, by the homotopy equivalence of the pairs

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1) \to (\mathbb{C}P^\infty, *),$$

we can derive from Theorem 2.16 that there is a homotopy equivalence

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^Q \simeq (\mathbb{C}P^\infty, *)^Q.$$

We call $(\mathbb{C}P^\infty,*)^Q$ the *Davis–Januszkiewicz space* of Q, denoted by $\mathcal{D}\mathcal{J}(Q)$. So the equivariant cohomology ring of $\mathcal{Z}_Q$ is isomorphic to the ordinary cohomology ring of $\mathcal{D}\mathcal{J}(Q)$.

Similarly, we can prove that the Borel construction of $\mathbb{R}\mathcal{Z}_Q$ with respect to the canonical $(\mathbb{Z}_2)^m$–action is $(\mathbb{R}P^\infty,*)^Q$.

Proof of Theorem 1.7 By the proof of Theorem 4.12 and the fact that $H^*(\mathbb{C}P^\infty)$ is torsion free, we can deduce from (48) that

$$\widetilde{H}^*(\widehat{\boldsymbol{D}}^J_{(\mathbb{C}P^\infty,*)+}) \cong \widetilde{H}^*(\widehat{\boldsymbol{G}}^J_{(\mathbb{C}P^\infty,*)+}) \cong H^*(F_{\cap J}) \otimes \bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}) \quad \text{for all } J \subseteq [m],$$

where $(\mathbb{C}P^\infty)^m = \prod_{j\in[m]} \mathbb{C}P^\infty_{(j)}$. Then we obtain a ring isomorphism from (49),

$$\bigoplus_{J\subseteq[m]} \widehat{\Pi}^*_J \colon \bigoplus_{J\subseteq[m]} \Bigl(H^*(F_{\cap J}) \otimes \bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}) \Bigr) \to H^*((\mathbb{C}P^\infty,*)^Q) \cong H^*_{T^m}(\mathcal{Z}_Q),$$

where the product $\circledast$ on the left-hand side is defined by (42) via the partial diagonal maps

$$\operatorname{colim}(\widehat{\boldsymbol{D}}^{J\cup J'}_{(\mathbb{C}P^\infty,*)+}) \xrightarrow{\widehat{\Delta}^{J,J'}_{J\cup J',Q+}} \operatorname{colim}(\widehat{\boldsymbol{D}}^{J}_{(\mathbb{C}P^\infty,*)+}) \wedge \operatorname{colim}(\widehat{\boldsymbol{D}}^{J'}_{(\mathbb{C}P^\infty,*)+}).$$

Example 5.1 If $Q=[0,1)$, the moment-angle manifold $\mathcal{Z}_{[0,1)} = D^2 \backslash S^1$, whose Borel construction is homotopy equivalent to $\mathbb{C}P^\infty$. Then

$$H^*_{S^1}(\mathcal{Z}_{[0,1)}) \cong H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \ \deg(x)=2.$$

The above ring isomorphism implies that the homomorphism $\Delta^*_{\mathbb{C}P^\infty}$ induced by the reduced diagonal map $\Delta_{\mathbb{C}P^\infty} \colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty$ on the integral cohomology is given by

$$\Delta^*_{\mathbb{C}P^\infty} \colon \widetilde{H}^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \cong \widetilde{H}^*(\mathbb{C}P^\infty) \otimes \widetilde{H}^*(\mathbb{C}P^\infty) \to \widetilde{H}^*(\mathbb{C}P^\infty), \quad \theta\otimes\theta' \to \theta\cup\theta'.$$

Then, by Lemma 3.1 and the above example, for any elements

$$u = \phi \otimes \bigotimes_{j \in J} \theta_j, \qquad \phi \in H^*(F_{\cap J}),\ \theta_j \in \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}),$$
$$v = \phi' \otimes \bigotimes_{j \in J'} \theta'_j, \quad \phi' \in H^*(F_{\cap J'}),\ \theta'_j \in \widetilde{H}^*(\mathbb{C}P^\infty_{(j)}),$$

we have

$$u \circledast v = (\kappa^*_{J\cup J',J}(\phi) \cup \kappa^*_{J\cup J',J'}(\phi')) \otimes \bigotimes_{J\setminus J'} \theta_j \otimes \bigotimes_{J'\setminus J} \theta'_j \otimes \bigotimes_{j\in J\cap J'} (\theta_j \cup \theta'_j),$$

where $\kappa_{I',I} \colon F_{\cap I'} \to F_{\cap I}$ is the inclusion map for any subsets $I \subseteq I' \subseteq [m]$.

Finally, since there is a graded ring isomorphism

$$H^*((\mathbb{C}P^\infty)^m) \cong \mathbb{Z}[x_1, \dots, x_m],\ \deg(x_1) = \cdots = \deg(x_m) = 2,$$

it is easy to check that $\bigoplus_{J\subseteq[m]} \big(H^*(F_{\cap J}) \otimes \bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)})\big)$ with the product $\circledast$ is isomorphic to the topological face ring $\mathbb{Z}\langle Q\rangle = \bigoplus_{J\subseteq[m]} H^*(F_{\cap J}) \otimes R^J_{\mathbb{Z}}$, where $\bigotimes_{j\in J} \widetilde{H}^*(\mathbb{C}P^\infty_{(j)})$ corresponds to $R^J_{\mathbb{Z}}$; see (6).

By replacing (D^2, S^1) with (D^1, S^0), $(S^1)^m$ with $(\mathbb{Z}_2)^m$ and $\mathbb{C}P^\infty$ with $\mathbb{R}P^\infty$ in the above argument, and by the fact $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$, $\deg(x) = 1$, we obtain the parallel result for $\mathbb{R}\mathscr{Z}_Q$. □

From the canonical fiber bundle associated to the Borel construction in (51), we have a natural $H^*(BT^m)$–module structure on $H^*_{T^m}(\mathscr{Z}_Q)$. By the identification

$$H^*(BT^m) = \mathbb{Z}[x_1, \dots, x_m],$$

we can write the $H^*(BT^m)$–module structure on $H^*_{T^m}(\mathscr{Z}_Q)$ as, for each $1 \le i \le m$,

$$x_i \cdot (\phi \otimes f(x)) = (1 \otimes x_i) \star (\phi \otimes f(x)) = \kappa^*_{J\cup\{i\},J}(\phi) \otimes x_i f(x) \quad \text{(by (8))}, \tag{52}$$

where $\phi \in H^*(F_{\cap J})$ and $f(x) \in R^J_{\mathbb{Z}}$, $J \subseteq [m]$.

Example 5.2 Let P be a simple polytope with facets $F_1, \dots, F_m$. For a subset $J \subseteq [m]$, $F_{\cap J}$ is either empty or a face of P and hence acyclic. So we can write the topological face ring of P as

$$\mathbb{Z}\langle P\rangle \cong \Bigg(\bigoplus_{\substack{F_{\cap J} \neq \varnothing \\ J\subseteq[m]}} R^J_{\mathbb{Z}}, \star \Bigg),$$

where for any $f(x) \in R^J_{\mathbb{Z}}$ and $f'(x) \in R^{J'}_{\mathbb{Z}}$ with $F_{\cap J} \neq \varnothing$ and $F_{\cap J'} \neq \varnothing$,

$$f(x) \star f'(x) = \begin{cases} f(x) f'(x) & \text{if } F_{\cap (J\cup J')} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

According to the linear basis of the face ring $\mathbb{Z}[P]$ in (50), we can easily check that $\mathbb{Z}\langle P\rangle$ is isomorphic to $\mathbb{Z}[P]$.

Theorem 5.3 *Let Q be a nice manifold with corners with m facets. If a subtorus $H \subseteq T^m = (S^1)^m$ acts freely on $\mathfrak{X}_Q$ through the canonical action, the equivariant cohomology ring with $\mathbb{Z}$–coefficients of the quotient space $\mathfrak{X}_Q/H$ with respect to the induced action of T^m/H is isomorphic to the topological face ring $\mathbb{Z}\langle Q\rangle$ of Q.*

Proof Suppose $T^m/H \cong T^k$. Since H acts freely on $\mathfrak{X}_Q$, the Borel constructions of $\mathfrak{X}_Q/H$ and $\mathfrak{X}_Q$ are homotopy equivalent by

$$ET^m \times_{T^m} \mathfrak{X}_Q \cong EH \times (E(T^m/H) \times_{T^m/H} (\mathfrak{X}_Q/H)) \simeq ET^k \times_{T^k} \mathfrak{X}_Q/H. \tag{53}$$

So the equivariant cohomology ring of $\mathfrak{X}_Q/H$ is isomorphic to the equivariant cohomology ring of $\mathfrak{X}_Q$. Then the theorem follows from Theorem 1.7. □

In Theorem 5.3, the group homomorphism $T^m \to T^m/H \cong T^k$ induces a commutative diagram

$$\begin{tikzcd} ET^m \arrow[r] \arrow[d] & E(T^m/H) \arrow[d] \\ BT^m \arrow[r] & B(T^m/H) \end{tikzcd}$$

which, along with the maps in (53), induces the diagram

$$\begin{tikzcd} ET^m \times_{T^m} \mathfrak{X}_Q \arrow[r, "\simeq"] \arrow[d] & E(T^m/H) \times_{T^m/H} (\mathfrak{X}_Q/H) \arrow[d] \\ BT^m \arrow[r] & B(T^m/H) \end{tikzcd} \tag{54}$$

We can describe the natural $H^*(B(T^m/H))$–module structure of the integral equivariant cohomology ring of $\mathfrak{X}_Q/H$ as follows. The inclusion $H \hookrightarrow T^m$ induces a monomorphism $\varphi_H : \mathbb{Z}^{m-k} \to \mathbb{Z}^m$ whose image is a direct summand in $\mathbb{Z}^m$. This determines an integer $m \times (m-k)$ matrix $S = (s_{ij})$ if we choose a basis for each of $\mathbb{Z}^{m-k}$ and $\mathbb{Z}^m$. Then since the image of φ_H is a direct summand in $\mathbb{Z}^m$, there is an integer $k \times m$ matrix $R = (r_{ij})$ of rank k such that $R \cdot S = 0$ which defines the homomorphism $T^m \to T^m/H$.

If we write $H^*(B(T^m/H)) = H^*(BT^k) = \mathbb{Z}[y_1, \dots, y_k]$, it follows from the diagram (54) that the natural $H^*(B(T^m/H))$–module structure of the integral equivariant cohomology ring of $\mathfrak{X}_Q/H$ is determined by the formula in (52) along with the map $H^*(B(T^m/H)) \to H^*(BT^m)$ given by

$$\mathbb{Z}[y_1, \dots, y_k] \to \mathbb{Z}[x_1, \dots, x_m], \quad y_i \mapsto r_{i1}x_1 + \cdots + r_{im}x_m.$$

The above formula is parallel to the formula given in [9, Theorem 7.37] (where Q is a simple polytope).

Remark 5.4 If a subtorus $H \subseteq T^m$ of dimension $m - \dim(Q)$ acts freely on $\mathfrak{X}_Q$ through the canonical action, the quotient space $\mathfrak{X}_Q/H$ with the induced action of $T^m/H \cong T^{\dim(Q)}$ can be considered as a generalization of *quasitoric manifold* over a simple polytope defined by Davis and Januszkiewicz [17].

The following is an application of Theorem 5.3 to locally standard torus actions on closed manifolds. Recall that an action of T^n on a closed $2n$–manifold M^{2n} is called *locally standard* — see [17, Section 1] — if every point in M^{2n} has a T^n–invariant neighborhood that is weakly equivariantly diffeomorphic an open subset of $\mathbb{C}^n$ invariant under the *standard* T^n*–action*,

$$(g_1,\dots,g_n)\cdot(z_1,\dots,z_n)=(g_1z_1,\dots,g_nz_n),\quad g_i\in S^1, z_i\in\mathbb{C}, 1\le i\le n.$$

Corollary 5.5 *Let M^{2n} be a closed smooth $2n$–manifold with a smooth locally standard T^n–action and the free part of the action is a trivial T^n–bundle. Then the integral equivariant cohomology ring $H^*_{T^n}(M^{2n})$ of M^{2n} is isomorphic to the topological face ring $\mathbb{Z}\langle M^{2n}/T^n\rangle$.*

Proof The orbit space $Q=M^{2n}/T^n$ is a smooth nice manifold with corners since the T^n–action is locally standard and smooth. Then Q is triangulable — by [25] — and hence all our theorems can be applied to Q. In addition, using the characteristic function argument in Davis and Januszkiewicz [17] — see also [27, Section 4.2] or [35] — we can prove that M^{2n} is a free quotient space of $\mathcal{Z}_Q$ by a canonical action of some torus. Then this corollary follows from Theorem 5.3. □

Remark 5.6 The equivariant cohomology ring $H^*_{T^n}(M^{2n})$ in the above corollary was also computed by Ayzenberg, Masuda, Park and Zeng [2, Proposition 5.2] under an extra assumption that all the proper faces of M^{2n}/T^n are acyclic. We leave it as an exercise for the reader to check that the formula for $H^*_{T^n}(M^{2n})$ given in [2] is isomorphic to $\mathbb{Z}\langle M^{2n}/T^n\rangle$.

6 Generalizations

Let Q be a nice manifold with corners with facets $\mathcal{F}(Q)=\{F_1,\dots,F_m\}$. Observe that neither in the construction of $\mathcal{Z}_Q$ nor in the proof of Theorems 1.1 and 1.4 do we really use the connectedness of each facet F_j. So we have the following generalization of $\mathcal{Z}_Q$.

Let $\mathcal{J}=\{J_1,\dots,J_k\}$ be a *partition* of $[m]=\{1,\dots,m\}$, ie the J_i are disjoint subsets of $[m]$ with $J_1\cup\dots\cup J_k=[m]$. So $\partial Q=F_{J_1}\cup\dots\cup F_{J_k}$. Moreover, we require $\mathcal{J}$ to satisfy

(55) for any $1\le i\le k$, if $j,j'\in J_i$, then $F_j\cap F_{j'}=\varnothing$.

From Q and the partition $\mathcal{J}$, we can construct the following manifold.

Let $\{e_1,\dots,e_k\}$ be a unimodular basis of $\mathbb{Z}^k$. Let $\mu\colon\mathcal{F}(Q)\to\mathbb{Z}^k$ be the map which sends all the facets in F_{J_i} to e_i for every $1\le i\le k$. Define

$$\mathcal{Z}_{Q,\mathcal{J}}:=Q\times(S^1)^k/\sim,$$

where $(x,g) \sim (x',g')$ if and only if $x = x'$ and $g^{-1}g' \in \mathbb{T}^{\mu}_x$ where $\mathbb{T}^{\mu}_x$ is the subtorus of $(S^1)^k = \mathbb{R}^k/\mathbb{Z}^k$ determined by the linear subspace of $\mathbb{R}^k$ spanned by the set $\{\mu(F_j) \mid x \in F_j\}$. There is a canonical action of $(S^1)^k$ on $\mathfrak{X}_{Q,\mathcal{J}}$ defined by

$$g' \cdot [(x,g)] = [(x,g'g)], \quad x \in Q, \ g, g' \in (S^1)^k. \tag{56}$$

If $\mathcal{J}_0 = \{\{1\},\dots,\{m\}\}$ is the trivial partition of $[m]$, then $\mathfrak{X}_{Q,\mathcal{J}_0} = \mathfrak{X}_Q$.

Note that here $\{F_{J_i}\}$ play the role of facets $\{F_j\}$ in the definition of $\mathfrak{X}_Q$, but F_{J_i} may not be connected. Using the term defined in Davis [14], the decomposition of ∂Q into $\{F_{J_i}\}$ is called a *panel structure on* Q and each F_{J_i} is called a *panel*.

Remark 6.1 For a general partition $\mathcal{J}$ of $[m]$, it is possible that $F_j \cap F_{j'} \neq \varnothing$ for some $j, j' \in J_i$. Although the definition of $\mathfrak{X}_{Q,\mathcal{J}}$ still makes sense in the general setting, the orbit space of the $(S^1)^k$–action on $\mathfrak{X}_{Q,\mathcal{J}}$ may not be Q (as a manifold with corners). It would be Q with some corners smoothed. But for a general partition of $[m]$, one can always reduce to the case where the condition (55) is satisfied by smoothing the corners of the orbit space.

For any subset $\omega \subseteq [k] = \{1,\dots,k\}$, let

$$F_\omega = \bigcup_{i \in \omega} F_{J_i}, \quad F_\varnothing = \varnothing, \quad F_{\cap\omega} = \bigcap_{i\in\omega} F_{J_i}, \quad F_{\cap\varnothing} = Q.$$

Theorem 6.2 *Let Q be a nice manifold with corners with facets $F_1,\dots,F_m$. For any partition $\mathcal{J} = \{J_1,\dots,J_k\}$ of $[m] = \{1,\dots,m\}$,*

$$\mathbf{\Sigma}(\mathfrak{X}_{Q,\mathcal{J}}) \simeq \bigvee_{\omega\subseteq[k]} \mathbf{\Sigma}^{|\omega|+1}(Q/F_\omega), \quad H^p(\mathfrak{X}_{Q,\mathcal{J}}) \cong \bigoplus_{\omega\subseteq[k]} H^{p-|\omega|}(Q,F_\omega) \quad \textit{for all } p \in \mathbb{Z}.$$

Proof We can generalize the rim-cubicalization of Q in Section 2.1 as follows. For any face f of Q, let

$$I^{\mathcal{J}}_f = \{i \in [k] \mid f \subseteq F_{J_i}\} \subseteq [k].$$

Then define

$$\hat{f}^{\mathcal{J}} = f \times \prod_{i\in I^{\mathcal{J}}_f} [0,1]_{(i)} \times \prod_{i \in [k]\setminus I^{\mathcal{J}}_f} 1_{(i)}, \quad \widehat{Q}^{\mathcal{J}} = \bigcup_{f\in\mathcal{S}_Q} \hat{f}^{\mathcal{J}} \subseteq Q \times [0,1]^k.$$

By the same argument as in the proof of Lemma 2.1, we can show that $\widehat{Q}^{\mathcal{J}}$ with faces $\hat{f}^{\mathcal{J}}$ is homeomorphic to Q as a manifold with corners. The partition $\mathcal{J}$ of the facets of Q naturally induces a partition of the corresponding facets of $\widehat{Q}^{\mathcal{J}}$, also denoted by $\mathcal{J}$. So we have $\mathfrak{X}_{\widehat{Q}^{\mathcal{J}},\mathcal{J}} \cong \mathfrak{X}_{Q,\mathcal{J}}$.

For any face f of Q, let

$$(D^2,S^1)^f_{\mathcal{J}} := f \times \prod_{i\in I^{\mathcal{J}}_f} D^2_{(i)} \times \prod_{i\in[k]\setminus I^{\mathcal{J}}_f} S^1_{(i)}, \quad (D^2,S^1)^Q_{\mathcal{J}} := \bigcup_{f\in\mathcal{S}_Q} (D^2,S^1)^f_{\mathcal{J}} \subseteq Q \times (D^2)^k.$$

There is a canonical $(S^1)^k$–action on $(D^2,S^1)^Q_{\mathcal{J}}$ induced from the canonical $(S^1)^k$–action on $Q\times(D^2)^k$. And parallel to Lemma 2.3, we can prove that there is an equivariant homeomorphism from $(D^2,S^1)^Q_{\mathcal{J}}$ to $\mathcal{Z}_{\hat{Q}^{\mathcal{J}},\mathcal{J}}\cong\mathcal{Z}_{Q,\mathcal{J}}$.

For any subset $L\subseteq I^{\mathcal{J}}_f$, let

$$(D^2,S^1)^{(f,L)}_{\mathcal{J}} := f\times\prod_{i\in I^{\mathcal{J}}_f\backslash L} D^2_{(i)}\times\prod_{i\in[k]\backslash(I^{\mathcal{J}}_f\backslash L)} S^1_{(i)}.$$

We can translate the proof of Theorem 1.1 to obtain the desired stable decomposition of $\mathcal{Z}_{Q,\mathcal{J}}\cong(D^2,S^1)^Q_{\mathcal{J}}$ by the following correspondence of symbols:

$$\begin{aligned} \text{Theorem 1.1} &\longrightarrow \text{Theorem 6.2}\\ J\subseteq[m] &\longrightarrow \omega\subseteq[k]\\ F_J &\longrightarrow F_\omega\\ I_f\subseteq[m] &\longrightarrow I^{\mathcal{J}}_f\subseteq[k]\\ D^2_{(j)},S^1_{(j)},\ j\in[m] &\longrightarrow D^2_{(i)},S^1_{(i)},\ i\in[k]\\ (D^2,S^1)^{(f,L)} &\longrightarrow (D^2,S^1)^{(f,L)}_{\mathcal{J}} \end{aligned}$$

□

Remark 6.3 Theorem 6.2 is an analogue of [36, Theorem 1.3].

To describe the cohomology ring of $\mathcal{Z}_{Q,\mathcal{J}}$, let

$$\mathcal{R}^*_{Q,\mathcal{J}} := \bigoplus_{\omega\subseteq[k]} H^*(Q,F_\omega). \tag{57}$$

There is a graded ring structure $\Cup_{\mathcal{J}}$ on $\mathcal{R}^*_{Q,\mathcal{J}}$ defined as follows:

- If $\omega\cap\omega'\neq\varnothing$, then $H^*(Q,F_\omega)\otimes H^*(Q,F_{\omega'})\xrightarrow{\Cup_{\mathcal{J}}} H^*(Q,F_{\omega\cup\omega'})$ is trivial.
- If $\omega\cap\omega'=\varnothing$, then $H^*(Q,F_\omega)\otimes H^*(Q,F_{\omega'})\xrightarrow{\Cup_{\mathcal{J}}} H^*(Q,F_{\omega\cup\omega'})$ is the relative cup product $\cup$.

To describe the equivariant cohomology ring of $\mathcal{Z}_{Q,\mathcal{J}}$, let

$$\boldsymbol{k}^{\mathcal{J}}\langle Q\rangle := \bigoplus_{\omega\subseteq[k]} H^*(F_{\cap\omega};\boldsymbol{k})\otimes R^{\omega}_{\boldsymbol{k}},$$

where the product on $\boldsymbol{k}^{\mathcal{J}}\langle Q\rangle$ is defined in the same way as $\boldsymbol{k}\langle Q\rangle$ in Definition 1.6.

The following theorem generalizes Theorems 1.4 and 1.7. The proof is omitted since it is completely parallel to the proof of these two theorems.

Theorem 6.4 *Let Q be a nice manifold with corners with m facets $F_1,\dots,F_m$ and let $\mathcal{J}=\{J_1,\dots,J_k\}$ be a partition of $[m]$.*

- *There is a ring isomorphism (up to a sign) from $(\mathcal{R}^*_{Q,\mathcal{J}}, \Cup_{\mathcal{J}})$ to the integral cohomology ring of $\mathcal{Z}_{Q,\mathcal{J}}$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of all the elements in $H^*(Q, F_\omega)$ up by $|\omega|$ for every $\omega \subseteq [k]$.*
- *There is a graded ring isomorphism from the equivariant cohomology ring of $\mathcal{Z}_{Q,\mathcal{J}}$ with integral coefficients to $\mathbb{Z}^{\mathcal{J}}\langle Q\rangle$ by choosing $\deg(x_i) = 2$ for all $1 \le i \le k$.*

By combining the constructions in Theorems 4.4 and 6.2, we have the following definitions which provide the most general setting for our study.

Let $\mathcal{J} = \{J_1, \dots, J_k\}$ be a *partition* of $[m] = \{1, \dots, m\}$ and let

$$(\mathbb{X}, \mathbb{A}) = \{(X_i, A_i, a_i)\}_{i=1}^k,$$

where X_i and A_i are CW–complexes with a basepoint $a_i \in A_i \subseteq X_i$.

For any face f of Q, let

$$(\mathbb{X}, \mathbb{A})^f_{\mathcal{J}} := f \times \prod_{i \in I^{\mathcal{J}}_f} X_i \times \prod_{i \in [k]\backslash I^{\mathcal{J}}_f} A_i, \quad (\mathbb{X}, \mathbb{A})^Q_{\mathcal{J}} := \bigcup_{f \in \mathcal{S}_Q} (\mathbb{X}, \mathbb{A})^f_{\mathcal{J}} \subseteq Q \times \prod_{i \in [k]} X_i.$$

The following theorem generalizes Theorems 4.4 and 4.5.

Theorem 6.5 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Let*

$$(\mathbb{X}, \mathbb{A}) = \{(X_i, A_i, a_i)\}_{i=1}^k,$$

where each X_i is contractible and each A_i is either connected or is a disjoint union of a connected CW–complex with its basepoint. Then for any partition $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[m]$, there is a homotopy equivalence

$$\mathbf{\Sigma}((\mathbb{X}, \mathbb{A})^Q_{\mathcal{J}}) \simeq \bigvee_{\omega \subseteq [k]} \mathbf{\Sigma}\Big(Q/F_\omega \wedge \bigwedge_{i \in \omega} A_i\Big).$$

In addition, there is a ring isomorphism

$$\Big(\bigoplus_{\omega \subseteq [k]} \widetilde{H}^*\Big(Q/F_\omega \wedge \bigwedge_{i \in \omega} A_i\Big), \circledast\Big) \to \widetilde{H}^*((\mathbb{X}, \mathbb{A})^Q_{\mathcal{J}}),$$

where $\circledast$ is defined in the same way as in (42).

In particular, for $(\mathbb{D}, \mathbb{S}) = \{(D^{n_i+1}, S^{n_i}, a_i)\}_{i=1}^k$, we can describe the integral cohomology ring of $(\mathbb{D}, \mathbb{S})^Q_{\mathcal{J}}$ explicitly as follows. Define a graded ring structure $\Cup^{(\mathbb{D},\mathbb{S})}_{\mathcal{J}}$ on $\mathcal{R}^*_{Q,\mathcal{J}}$ according to $(\mathbb{D}, \mathbb{S})$ by:

- If $\omega \cap \omega' = \varnothing$ or $\omega \cap \omega' \neq \varnothing$ but $n_i = 0$ for all $i \in \omega \cap \omega'$, then

$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\Cup^{(\mathbb{D},\mathbb{S})}_{\mathcal{J}}} H^*(Q, F_{\omega \cup \omega'})$$

  is the relative cup product.

- If $\omega \cap \omega' \neq \varnothing$ and there exists $n_i \geq 1$ for some $i \in \omega \cap \omega'$, then
$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\Cup_{\mathcal{J}}^{(\mathbb{D},\mathbb{S})}} H^*(Q, F_{\omega \cup \omega'})$$
is trivial.

Theorem 6.6 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. For any partition $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[m]$ and $(\mathbb{D}, \mathbb{S}) = \{(D^{n_i+1}, S^{n_i}, a_i)\}_{i=1}^k$, there is a homotopy equivalence*
$$\Sigma((\mathbb{D}, \mathbb{S})^Q_{\mathcal{J}}) \simeq \bigvee_{\omega \subseteq [k]} \Sigma^{1+\sum_{i \in \omega} n_i}(Q/F_\omega),$$
and there is a ring isomorphism (up to a sign) from $(\mathcal{R}^*_{Q,\mathcal{J}}, \Cup_{\mathcal{J}}^{(\mathbb{D},\mathbb{S})})$ to the integral cohomology ring of $(\mathbb{D}, \mathbb{S})^Q_{\mathcal{J}}$. Moreover, we can make this ring isomorphism degree-preserving by shifting the degrees of the elements in $H^*(Q, F_\omega)$ for every $\omega \subseteq [k]$.

When $(\mathbb{D}, \mathbb{S}) = \{(D^1, S^0, a_0)\}_{i=1}^k$, we denote $(\mathbb{D}, \mathbb{S})^Q_{\mathcal{J}}$ also by $\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}$, which is the real analogue of $\mathcal{Z}_{Q,\mathcal{J}}$. Then we have the following corollary which generalizes Corollary 4.10 and Theorem 1.7.

Corollary 6.7 *Let Q be a nice manifold with corners with facets $F_1, \dots, F_m$. Then for any partition $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[m]$,*
$$\Sigma(\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}) \simeq \bigvee_{\omega \subseteq [k]} \Sigma(Q/F_\omega), \quad H^p(\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}) \cong \bigoplus_{\omega \subseteq [k]} H^p(Q, F_\omega) \quad \text{for all } p \in \mathbb{Z}.$$
Furthermore, the integral cohomology ring of $\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}$ is isomorphic as a graded ring to the ring $(\mathcal{R}^*_{Q,\mathcal{J}}, \cup)$, where $\cup$ is the relative cup product
$$H^*(Q, F_\omega) \otimes H^*(Q, F_{\omega'}) \xrightarrow{\cup} H^*(Q, F_{\omega \cup \omega'}) \quad \text{for all } \omega, \omega' \subseteq [k],$$
and there is a graded ring isomorphism from the equivariant $\mathbb{Z}_2$–cohomology ring of $\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}$ to $\mathbb{Z}_2^{\mathcal{J}}\langle Q\rangle$ by choosing $\deg(x_i) = 1$ for all $1 \leq i \leq k$.

The proofs of Theorems 6.5 and 6.6 and Corollary 6.7 are almost the same as their counterparts in Sections 4 and 5, hence omitted.

For any partition $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[m]$,

- we can think of $\mathcal{Z}_{Q,\mathcal{J}}$ as the quotient space of $\mathcal{Z}_Q$ by the canonical action of an $(m-k)$–dimensional subtorus $\mathbb{T}^{\mathcal{J}}$ of $(S^1)^m$ determined by (see (1))
$$\{\lambda(F_j) - \lambda(F_{j'}) \mid j, j' \text{ belong to the same } J_i \text{ for some } 1 \leq i \leq k\} \subseteq \mathbb{Z}^m;$$
- similarly, we can think of $\mathbb{R}\mathcal{Z}_{Q,\mathcal{J}}$ as the quotient space of $\mathbb{R}\mathcal{Z}_Q$ by the canonical action of a subgroup of rank $m-k$ in $(\mathbb{Z}_2)^m$.

Note that the canonical action of $\mathbb{T}^{\mathscr{J}}$ on $\mathfrak{X}_Q$ may not be free. But when the action is free, the integral equivariant cohomology ring of $\mathfrak{X}_Q/\mathbb{T}^{\mathscr{J}} = \mathfrak{X}_{Q,\mathscr{J}}$ is isomorphic to $\mathbb{Z}\langle Q\rangle$ by Theorem 5.3. So, by Theorem 6.4, $\mathbb{Z}^{\mathscr{J}}\langle Q\rangle$ is isomorphic as a ring to $\mathbb{Z}\langle Q\rangle$ in this case. But this ring isomorphism is not obvious from the algebraic definition of $\mathbb{Z}^{\mathscr{J}}\langle Q\rangle$ and $\mathbb{Z}\langle Q\rangle$.

Remark 6.8 For any partition $\mathscr{J} = \{J_1, \dots, J_k\}$ of $[m]$ with $k = \dim(Q)$, the $\mathfrak{X}_{Q,\mathscr{J}}$ and $\mathbb{R}\mathfrak{X}_{Q,\mathscr{J}}$ can be considered as a generalization of the *pull-back from the linear model* — see [17, Example 1.15] — in the study of quasitoric manifolds and small covers.

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Equivariant Seiberg–Witten–Floer cohomology

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We develop an equivariant version of Seiberg–Witten–Floer cohomology for finite group actions on rational homology 3–spheres. Our construction is based on an equivariant version of the Seiberg–Witten–Floer stable homotopy type, as constructed by Manolescu. We use these equivariant cohomology groups to define a series of d–invariants $d_{G,c}(Y,\mathfrak{s})$ which are indexed by the group cohomology of G. These invariants satisfy a Frøyshov-type inequality under equivariant cobordisms. Lastly, we consider a variety of applications of these d–invariants: concordance invariants of knots via branched covers, obstructions to extending group actions over bounding 4–manifolds, Nielsen realisation problems for 4–manifolds with boundary and obstructions to equivariant embeddings of 3–manifolds in 4–manifolds.

57K31; 57K10, 57K41

1 Introduction

In this paper we develop an equivariant version of Seiberg–Witten–Floer cohomology for rational homology 3–spheres equipped with the action of a finite group. Our approach is modelled on the construction of a Seiberg–Witten–Floer stable homotopy type due to Manolescu [49], which we now briefly recall. Let Y be a rational homology 3–sphere and $\mathfrak{s}$ a spin^c–structure on Y. Given a metric g on Y, the construction of [49] yields an S^1–equivariant stable homotopy type $SWF(Y,\mathfrak{s},g)$. The Seiberg–Witten–Floer cohomology of $(Y,\mathfrak{s})$ is then given (up to a degree shift) by the S^1–equivariant cohomology of $SWF(Y,\mathfrak{s},g)$:

$$HSW^*(Y,\mathfrak{s}) = \widetilde{H}^{*+2n(Y,\mathfrak{s},g)}_{S^1}(SWF(Y,\mathfrak{s},g)),$$

where $n(Y,\mathfrak{s},g)$ is a rational number given by a certain combination of eta invariants.

The stable homotopy type $SWF(Y,\mathfrak{s},g)$ depends on the choice of metric, but only up to a suspension. Given two metrics, g_0 and g_1, one obtains a canonical homotopy equivalence

$$SWF(Y,\mathfrak{s},g_1) \cong \Sigma^{SF(\{D_s\})\mathbb{C}} SWF(Y,\mathfrak{s},g_0), \tag{1-1}$$

where $SF(\{D_s\})$ denotes the spectral flow for the family of Dirac operators $\{D_s\}$ determined by a path of metrics $\{g_s\}$ from g_0 to g_1. The rational numbers $n(Y,\mathfrak{s},g)$ are defined in such a way that they split the spectral flow in the sense that

$$SF(\{D_s\}) = n(Y,\mathfrak{s},g_1) - n(Y,\mathfrak{s},g_0). \tag{1-2}$$

Hence we obtain a canonical isomorphism

$$\widetilde{H}^{*+2n(Y,\mathfrak{s},g_1)}_{S^1}(SWF(Y,\mathfrak{s},g_1)) \cong \widetilde{H}^{*+2n(Y,\mathfrak{s},g_0)}_{S^1}(SWF(Y,\mathfrak{s},g_0)).$$

This shows that the Seiberg–Witten–Floer cohomology $HSW^*(Y,\mathfrak{s})$ does not depend on the choice of metric g.

By working in an appropriately defined S^1–equivariant Spanier–Whitehead category in which suspension by fractional amounts of $\mathbb{C}$ is allowed, Manolescu defined the Seiberg–Witten–Floer homotopy type of $(Y,\mathfrak{s})$ to be

$$SW(Y,\mathfrak{s}) = \Sigma^{-n(Y,\mathfrak{s},g)\mathbb{C}} SWF(Y,\mathfrak{s},g).$$

This is independent of the choice of g by (1-1) and (1-2).

Now suppose that a finite group G acts on Y by orientation-preserving diffeomorphisms which preserve the isomorphism class of $\mathfrak{s}$. Let g be a G–invariant metric on Y. Lifting the action of G to the associated spinor bundle determines an S^1 extension

$$1 \to S^1 \to G_{\mathfrak{s}} \to G \to 1.$$

Manolescu's construction of the stable homotopy type $SWF(Y,\mathfrak{s},g)$ can be carried out $G_{\mathfrak{s}}$–equivariantly, so that $SWF(Y,\mathfrak{s},g)$ may be promoted to a $G_{\mathfrak{s}}$–equivariant stable homotopy type. This is analogous to the construction in [50] of the Pin(2)–equivariant Seiberg–Witten–Floer stable homotopy type of $(Y,\mathfrak{s})$ where $\mathfrak{s}$ is a spin–structure on Y. The main difference is that in our construction, the additional symmetries that comprise the group $G_{\mathfrak{s}}$ come from symmetries of Y rather than internal symmetries of the Seiberg–Witten equations.

We define the *G–equivariant Seiberg–Witten–Floer cohomology* of $(Y,\mathfrak{s})$ to be

$$HSW^*_G(Y,\mathfrak{s}) = \widetilde{H}^{*+2n(Y,\mathfrak{s},g)}_{G_{\mathfrak{s}}}(SWF(Y,\mathfrak{s},g)).$$

The right-hand side is independent of the choice of metric g by much the same argument as in the S^1–equivariant case.

We make some remarks concerning this construction.

(1) Throughout this paper we have chosen to work with cohomology instead of homology. This is simply a matter of preference and we could just as well work with Seiberg–Witten–Floer homology groups.

(2) Instead of Borel equivariant cohomology, we could take co-Borel cohomology or Tate cohomology, which correspond to the different versions of Heegaard Floer cohomology; see Lidman and Manolescu [45, Corollary 1.2.4].

(3) In a similar fashion we can also define the G–equivariant Seiberg–Witten–Floer K–theory

$$KSW^*_G(Y,\mathfrak{s}) = \widetilde{K}^{*+2n(Y,\mathfrak{s},g)}_{G_{\mathfrak{s}}}(SWF(Y,\mathfrak{s},g)).$$

More generally we could use any generalised equivariant cohomology theory in which the Thom isomorphism holds.

(4) We have not attempted to construct a metric independent $G_{\mathfrak{s}}$–equivariant stable homotopy type. To do this one would need to split the equivariant spectral flow $SF_{G_{\mathfrak{s}}}(\{D_s\})$ in the same way that $n(Y,\mathfrak{s},g)$ splits the nonequivariant spectral flow, as in (1-2).

1.1 Main results

Throughout we work with cohomology with coefficients in a field $\mathbb{F}$. To avoid the necessity of local systems we assume that either $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$, or that the order of G is odd (see Section 3.1). We now outline the main properties of equivariant Seiberg–Witten–Floer cohomology.

Module structure $HSW^*_G(Y,\mathfrak{s})$ is a graded module over $H^*_{G_{\mathfrak{s}}}$ (where for a group K we write H^*_K for $H^*_K(\mathrm{pt})$). In particular if $G_{\mathfrak{s}}$ is the trivial extension then $HSW^*_G(Y,\mathfrak{s})$ is a graded module over $H^*_G[U]$, where $\deg(U)=2$.

Theorem 1.1 (spectral sequence) *There is a spectral sequence $E_r^{p,q}$ abutting to $HSW^*_G(Y,\mathfrak{s})$ whose second page is given by*

$$E_2^{p,q}=H^p(BG;HSW^q(Y,\mathfrak{s})).$$

Theorem 1.2 (localisation) *Suppose the extension $G_{\mathfrak{s}}$ is trivial and choose a trivialisation $G_{\mathfrak{s}}\cong S^1\times G$. Then $H^*_{G_{\mathfrak{s}}}\cong H^*_G[U]$ and the localisation $U^{-1}HSW^*_G(Y,\mathfrak{s})$ of $HSW^*_G(Y,\mathfrak{s})$ with respect to U is a free $H^*_G[U,U^{-1}]$–module of rank* 1.

***L*–spaces** We say that Y is an *L–space* with respect to $\mathfrak{s}$ and $\mathbb{F}$ if $HSW^*(Y,\mathfrak{s})$ is isomorphic to a free $\mathbb{F}[U]$–module of rank 1.

Theorem 1.3 *Suppose that $G_{\mathfrak{s}}$ is a split extension. If Y is an L–space with respect to $\mathfrak{s}$ and $\mathbb{F}$, then the spectral sequence given in Theorem 1.1 degenerates at E_2. Moreover,*

$$HSW^*_G(Y,\mathfrak{s})\cong HSW^*(Y,\mathfrak{s})\otimes_{\mathbb{F}}H^*_G.$$

Correction terms Suppose that $G_{\mathfrak{s}}$ is a split extension. For each nonzero $c\in H^*_G$ we obtain an invariant

$$d_{G,c}(Y,\mathfrak{s})\in\mathbb{Q}$$

which may be thought of as a generalisation to the equivariant setting of the d–invariant $d(Y,\mathfrak{s})$. We also set $d_{G,0}(Y,\mathfrak{s})=-\infty$.

Theorem 1.4 *The equivariant d–invariants satisfy the following properties*:

1. $d_{G,1}(Y,\mathfrak{s})\geq d(Y,\mathfrak{s})$, *where* 1 *is the generator of* $H^0_G(pt)$;
2. $d_{G,c_1+c_2}(Y,\mathfrak{s})\leq\max\{d_{G,c_1}(Y,\mathfrak{s}),d_{G,c_2}(Y,\mathfrak{s})\}$;
3. $d_{G,c_1c_2}(Y,\mathfrak{s})\leq\min\{d_{G,c_1}(Y,\mathfrak{s}),d_{G,c_2}(Y,\mathfrak{s})\}$;
4. $d_{G,c_1}(Y,\mathfrak{s})+d_{G,c_2}(\overline{Y},\mathfrak{s})\geq 0$ *whenever* $c_1c_2\neq 0$;

(5) *if Y is an L–space with respect to $\mathfrak{s}$ and $\mathbb{F}$, then $d_{G,c}(Y,\mathfrak{s}) = d(Y,\mathfrak{s})$ for all $c \neq 0$;*

(6) *$d_{G,c}(Y,\mathfrak{s})$ is invariant under equivariant rational homology cobordism.*

We find it convenient to also define corresponding equivariant δ–invariants by setting

$$\delta_{G,c}(Y,\mathfrak{s}) = \tfrac{1}{2} d_{G,c}(Y,\mathfrak{s}).$$

Our primary motivation for considering the equivariant d–invariants is that they are necessary for the formulation of our equivariant generalisation of Frøyshov's inequality described below.

Cobordism maps Suppose that $(W,\mathfrak{s})$ is a G–equivariant cobordism from $(Y_1,\mathfrak{s}_1)$ to $(Y_2,\mathfrak{s}_2)$ (see Section 4.3 for the precise statement). Then W induces a morphism of graded $H^*_{G_\mathfrak{s}}$–modules

$$SW_G(W,\mathfrak{s})\colon HSW^*_G(Y_2,\mathfrak{s}_2) \to HSW_G^{*+b_+(W)-2\delta(W,\mathfrak{s})}(Y_1,\mathfrak{s}_1)$$

where $\delta(W,\mathfrak{s}) = \frac{1}{8}(c_1(\mathfrak{s})^2 - \sigma(W))$.

Theorem 1.5 (equivariant Frøyshov inequality) *Let W be a smooth, compact, oriented 4–manifold with boundary and with $b_1(W) = 0$. Suppose that G acts smoothly on W preserving the orientation and a spin^c–structure $\mathfrak{s}$. Suppose that the extension $G_\mathfrak{s}$ is trivial. Suppose each component of ∂W is a rational homology 3–sphere and that G sends each component of ∂W to itself. Let $e \in H_G^{b_+(W)}$ be the image in $H^*_G(\mathrm{pt};\mathbb{F})$ of the Euler class of any G–invariant maximal positive definite subspace of $H^2(W;\mathbb{R})$. Let $c \in H^*_G$ and suppose that $ce \neq 0$.*

(1) *If $\partial W = Y$ is connected, then*

$$\delta(W,\mathfrak{s}) \le \delta_{G,c}(Y,\mathfrak{s}|_Y) \quad \text{and} \quad \delta_{G,ce}(\overline{Y},\mathfrak{s}|_Y) \le \delta(\overline{W},\mathfrak{s}).$$

(2) *If $\partial W = \overline{Y}_1 \cup Y_2$ has two connected components, then*

$$\delta_{G,ce}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) \le \delta_{G,c}(Y_2,\mathfrak{s}|_{Y_2}).$$

Knot concordance invariants Let K be a knot in S^3 and let $Y = \Sigma_2(K)$ be the double cover of S^3 branched over K. Then Y has an action of $G = \mathbb{Z}_2$ generated by the covering involution. Further, Y has a spin^c–structure $\mathfrak{t}_0$ uniquely determined by the condition that it arises from a spin–structure. Set $\mathbb{F} = \mathbb{Z}_2$. Then $H^*_G \cong \mathbb{F}[Q]$, where $\deg(Q) = 1$. For each $j \ge 0$, we define an invariant $\delta_j(K) \in \mathbb{Z}$ by setting

$$\delta_j(K) = 4\delta_{\mathbb{Z}_2,Q^j}(\Sigma_2(K),\mathfrak{t}_0).$$

Let $\sigma(K)$ and $g_4(K)$ denote the signature and smooth 4–genus of K.

Theorem 1.6 *The invariants $\delta_j(K)$ have the following properties:*

(1) *$\delta_j(K)$ is a knot concordance invariant;*

(2) *$\delta_0(K) \ge \delta(K)$, where $\delta(K)$ is the Manolescu–Owens invariant [51];*

(3) *$\delta_{j+1}(K) \le \delta_j(K)$ for all $j \ge 0$;*

(4) *$\delta_j(K) \geq -\frac{1}{2}\sigma(K)$ for all $j \geq 0$ and $\delta_j(K) = -\frac{1}{2}\sigma(K)$ for $j \geq g_4(K) - \frac{1}{2}\sigma(K)$;*

(5) *$\delta_j(-K) \geq \frac{1}{2}\sigma(K)$ for all $j \geq 0$ and $\delta_j(-K) = \frac{1}{2}\sigma(K)$ for $j \geq g_4(K) + \frac{1}{2}\sigma(K)$;*

(6) *if $\Sigma_2(K)$ is an L–space, then $\delta_j(K) = \delta(K) = -\frac{1}{2}\sigma(K)$ and $\delta_j(-K) = \delta(-K) = \frac{1}{2}\sigma(K)$ for all $j \geq 0$.*

In particular, if K is quasialternating, then $\Sigma_2(K)$ is an L–space; see Ozsváth and Szabó [57]. So we recover the main result of Lisca and Owens [47] that $\delta(K) = -\frac{1}{2}\sigma(K)$ for quasialternating knots.

The concordance invariants $\delta_j(K)$ can also be used to strengthen the inequality $g_4(K) \geq \frac{1}{2}|\sigma(K)|$; see Murasugi [52].

Theorem 1.7 *For a knot K, let $j_+(K)$ be the smallest positive integer such that $\delta_j(K) = -\frac{1}{2}\sigma(K)$ and $j_-(K)$ the smallest positive integer such that $\delta_j(-K) = \frac{1}{2}\sigma(K)$. Then*

$$g_4(K) \geq \max\bigl\{-\tfrac{1}{2}\sigma(K) + j_-(K), \tfrac{1}{2}\sigma(K) + j_+(K)\bigr\}.$$

Corollary 1.8 *If $\delta(K) > -\frac{1}{2}\sigma(K)$ and $\sigma(K) \geq 0$, then*

$$g_4(K) \geq \tfrac{1}{2}|\sigma(K)| + 1$$

Proof If $\delta(K) > -\frac{1}{2}\sigma(K)$, then $\delta_0(K) \geq \delta(K) > -\frac{1}{2}\sigma(K)$, and thus $j_+(K) \geq 1$. Hence

$$g_4(K) \geq \tfrac{1}{2}\sigma(K) + 1 = \tfrac{1}{2}|\sigma(K)| + 1. \qquad \square$$

One can obtain even more knot concordance invariants by considering higher order cyclic branched covers; see Remark 6.7.

1.2 Applications

We outline here some of the applications of equivariant Seiberg–Witten–Floer cohomology. These are considered in more detail in Section 7.

1.2.1 Nonextendable actions (Section 7.2) Let Y be a rational homology 3–sphere equipped with an orientation-preserving action of G and let W be a smooth 4–manifold which bounds Y. The equivariant d–invariants give obstructions to extending the action of G over W.

Example 1.9 The Brieskorn homology sphere $Y = \Sigma(p,q,r)$ where p, q and r are pairwise coprime is the branched cyclic p–fold cover of the torus knot $T_{q,r}$. Let $\tau\colon Y \to Y$ be a generator of the $\mathbb{Z}_p$–action determined by this covering. For certain values of p, q and r it can be shown that Y bounds a contractible 4–manifold. For example, $\Sigma(2,3,13)$ bounds a contractible 4–manifold; see Akbulut and Kirby [2]. It can be shown that τ is smoothly isotopic to the identity; hence it follows that τ can be extended as a diffeomorphism over any 4–manifold bounded by Y. On the other hand we show in Proposition 7.2

that if p is prime then $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s}) = -\lambda(Y)$ is minus the Casson invariant of Y (where $\mathfrak{s}$ is the unique spin^c–structure on Y), which is nonzero. We further show that the nonvanishing of $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s})$ implies that τ cannot be extended as a smooth $\mathbb{Z}_p$–action to any contractible 4–manifold bounded by Y. This partially recovers the nonextendability results of Anvari and Hambleton [6; 7] for Brieskorn homology 3–spheres bounded by contractible 4–manifolds.

On the other hand, our nonextendability result also holds in situations not covered by Anvari and Hambleton. Suppose now that $Y = \Sigma(p,q,r)$ bounds a *rational* homology 4–ball W. For example, Fintushel and Stern showed that $\Sigma(2,3,7)$ bounds a rational homology 4–ball, although it does not bound an integral homology 4–ball [26]. More examples can be found in Akbulut and Larson [4] and Şavk [21]. We show in Section 7.2 that the nonvanishing of $\delta_{\mathbb{Z}_p,1}(\Sigma(p,q,r))$, where p is prime, implies that the $\mathbb{Z}_p$–action cannot be extended to any rational homology 4–ball W bounded by Y, provided that p does not divide the order of $H^2(W;\mathbb{Z})$.

1.2.2 Realisation problems (Section 7.3) Let W be a smooth 4–manifold with boundary an integral homology sphere Y. Suppose that a finite group G acts on $H^2(W;\mathbb{Z})$ preserving the intersection form. We say that the action of G on $H^2(W;\mathbb{Z})$ can be realised by diffeomorphisms if there is a smooth orientation-preserving action of G on W inducing the given action on $H^2(W;\mathbb{Z})$. The equivariant d–invariants give obstructions to realising such actions by diffeomorphism. This extends the nonrealisation results of the first author [10; 11] for closed 4–manifolds to the case of 4–manifolds with nonempty boundary.

Example 1.10 Suppose that $b_1(W) = 0$ and that $H^2(W;\mathbb{Z})$ has no 2–torsion and even intersection form. Suppose that Y is an L–space. Suppose that an action of $G = \mathbb{Z}_p$ on $H^2(W;\mathbb{Z})$ is given, where p is prime and that the subspace of $H^2(W;\mathbb{R})$ fixed by G is negative definite. If $\frac{1}{8}\sigma(W) < -\delta(Y,\mathfrak{s})$ (where $\mathfrak{s}$ is the unique spin^c–structure on Y) then the action of $\mathbb{Z}_p$ on $H^2(W;\mathbb{Z})$ is not realisable by a smooth $\mathbb{Z}_p$–action on W. Note that we are not making any assumptions about the action of $\mathbb{Z}_p$ on the boundary.

1.2.3 Equivariant embeddings of 3–manifolds in 4–manifolds (Section 7.4) Let Y be a rational homology 3–sphere equipped with an orientation-preserving action of G. By an equivariant embedding of Y into a 4–manifold X, we mean an embedding $Y \to X$ such that the action of G on Y extends over X.

Example 1.11 Let $Y = \Sigma(2, 2s-1, 2s+1)$ where s is odd, equipped with the involution τ obtained from viewing Y as the branched double cover $\Sigma_2(T_{2s-1,2s+1})$. Then Y embeds in S^4; see Budney and Burton [13, Theorem 2.13]. On the other hand, $\delta_j(Y,\mathfrak{s}) \neq 0$ for some j. We will show that the nonvanishing of this invariant implies that Y cannot be equivariantly embedded in S^4.

It is known that every 3–manifold Y embeds in the connected sum $\#^n(S^2 \times S^2)$ of n copies of $S^2 \times S^2$ for some sufficiently large n [1, Theorem 2.1]. Aceto, Golla and Larson define the *embedding number* $\varepsilon(Y)$ of Y to be the smallest n for which Y embeds in $\#^n(S^2 \times S^2)$. Here we consider an equivariant version of the embedding number. To obtain interesting results we need to make an assumption on the kinds of group actions allowed.

Definition 1.12 Let $G = \mathbb{Z}_p = \langle \tau \rangle$ where p is a prime number. We say that a smooth, orientation-preserving action of G on $X = \#^n(S^2 \times S^2)$ is *admissible* if $H^2(X;\mathbb{Z})^\tau = 0$, where

$$H^2(X;\mathbb{Z})^\tau = \{x \in H^2(X;\mathbb{Z}) \mid \tau(x) = x\}.$$

We define the *equivariant embedding number* $\varepsilon(Y,\tau)$ of (Y,τ) to be the smallest n for which Y embeds equivariantly in $\#^n(S^2 \times S^2)$ for some admissible $\mathbb{Z}_p$–action on $\#^n(S^2 \times S^2)$, if such an embedding exists. We set $\varepsilon(Y,\tau) = \infty$ if there is no such embedding.

Example 1.13 Let $Y = \Sigma(2,3,6n+1) = \Sigma_2(T_{3,6n+1})$ and equip Y with the covering involution τ. We show that

$$2n \le \varepsilon(\Sigma(2,3,6n+1),\tau) \le 12n.$$

Suppose that n is odd. Then from [1, Proposition 3.5], the (nonequivariant) embedding number of $\Sigma(2,3,6n+1)$ is 10. In particular, we see that $\varepsilon(\Sigma(2,3,6n+1),\tau) > \varepsilon(\Sigma(2,3,6n+1))$ for all odd $n > 5$. We also show that

$$\varepsilon(\Sigma(2,3,7),\tau) = 12,$$

whereas $\varepsilon(\Sigma(2,3,7)) = 10$.

1.3 Comparison with other works

In [30], Hendricks, Lipshitz and Sarkar introduce equivariant versions of several types of Floer homology, mostly focusing on the case that the group is $\mathbb{Z}_2$. In particular they define a $\mathbb{Z}_2$–equivariant version of HF^-, which is a module over $H^*_{S^1\times\mathbb{Z}_2}(\mathrm{pt};\mathbb{Z}_2) = \mathbb{Z}_2[U,Q]$. This construction shares many similarities with the equivariant Seiberg–Witten–Floer cohomology constructed in this paper, such as a localisation isomorphism and a spectral sequence relating the equivariant and ordinary Floer homologies. In fact, it seems reasonable to conjecture that our constructions are isomorphic.

In [5], Alfieri, Kang and Stipsicz consider a $\mathbb{Z}_2$–equivariant Heegaard Floer homology $HFB^-(K)$ for a branched double cover $Y = \Sigma_2(K)$ of a knot K, constructed in a manner similar to involutive Heegaard Floer homology — see Hendricks and Manolescu [31] — except that the involution arises from the covering involution on Y. These groups are modules over the ring $\mathbb{Z}_2[U,Q]/(Q^2)$. From this group they obtain knot concordance invariants $\bar{\delta}(K), \underline{\delta}(K)$. A similar approach was taken by Dai, Hedden and Mallick [22] to obtain ι–complexes — see Hendricks, Manolescu and Zemke [32, Definition 8.1] — associated to involutions on Y. Since $\mathbb{Z}_2[U,Q]/(Q^2) = H^*_{S^1\times\mathbb{Z}}(pt;\mathbb{Z}_2)$, we suspect that the group $HFB^-(K)$ may be isomorphic to the $\mathbb{Z}$–equivariant Seiberg–Witten–Floer homology of $\Sigma_2(K)$.

In [46, Remark 3.1], Lidman and Manolescu define equivariant Seiberg–Witten–Floer homology in the special case that G acts freely on Y. Their construction coincides with ours in such cases.

1.4 Structure of the paper

In Section 2 we recall the construction of Seiberg–Witten–Floer spectra using finite-dimensional approximation and the Conley index. In Section 3 we extend this construction to the G–equivariant setting, arriving at the construction of the G–equivariant Seiberg–Witten–Floer cohomology in Section 3.4. In the remainder of Section 3 we introduce the equivariant d–invariants and establish their basic properties. Section 4 is concerned with the behaviour of equivariant Seiberg–Witten–Floer cohomology and the d–invariants under equivariant cobordism. In Section 5 we specialise to the case that G is a cyclic group of prime order. In Section 6 we consider the case of branched double covers of knots with their natural involution to obtain knot concordance invariants. Finally in Section 7 we carry out some explicit computations of d–invariants and consider various applications.

2 Seiberg–Witten–Floer spectra

2.1 Seiberg–Witten trajectories

Throughout we let Y be a rational homology 3–sphere, ie Y is a compact, oriented, smooth 3–manifold with $b_1(Y)=0$. References for the material in this section are [45; 49].

Let g be a Riemannian metric on Y and let $\mathfrak{s}$ be a spin^c–structure with associated spinor bundle S. Let $\rho\colon TY \to \operatorname{End}(S)$ denote Clifford multiplication, satisfying $\rho(v)\rho(w)+\rho(w)\rho(v)=-2g(v,w)$. The spinor bundle S is equipped with a Hermitian metric $\langle\cdot,\cdot\rangle$ which we take to be antilinear in the first variable. Let $\mathfrak{su}(S)$ be the Lie algebra bundle of trace-free skew-adjoint endomorphisms of S and $\mathfrak{sl}(S)$ the Lie algebra bundle of trace-free endomorphisms of S. Then ρ induces an isomorphism $\rho\colon TY \to \mathfrak{su}(S)$ which extends by complexification to an isomorphism $\rho\colon TY_{\mathbb{C}} \to \mathfrak{sl}(S)$ satisfying $\rho(\bar{v})=-\rho(v)^*$. Using the metric g to identify TY and T^*Y we will also view ρ as a map $\rho\colon T^*Y \to \mathfrak{su}(S)$. We extend ρ to 2–forms by the rule $\rho(v\wedge w)=\frac{1}{2}[\rho(v),\rho(w)]$. It follows that $\rho(\lambda)=-\rho(*\lambda)$ for any 2–form λ. Define a Hermitian inner product on $\mathfrak{su}(S)$ by $\langle a,b\rangle=\frac{1}{2}\operatorname{tr}(a^*b)$. Then for any tangent vectors u and v, we have $\langle\rho(u),\rho(v)\rangle=g(\bar{u},v)$. Define a map

$$\tau\colon S\times S \to T^*Y_{\mathbb{C}}$$

by setting

$$\tau(\phi,\psi)=\rho^{-1}(\phi\otimes\psi^*)_0,$$

where $(\phi\otimes\psi^*)_0$ is the trace-free part of $\phi\otimes\psi^*$. That is, if ξ is any spinor, then

$$(\phi\otimes\psi^*)(\xi)=\phi\langle\psi,\xi\rangle-\tfrac{1}{2}\langle\psi,\phi\rangle\xi.$$

Then it follows that

$$\tau(\psi,\phi)=-\overline{\tau(\phi,\psi)},\qquad \tau(a\psi,b\phi)=a\bar{b}\tau(\psi,\phi).$$

In particular, $\tau(\phi,\phi)$ is imaginary and $\tau(c\phi,c\phi)=|c|^2\tau(\phi,\phi)$. We also have the identity

$$\langle\tau(\phi,\phi),v\rangle=\tfrac{1}{2}\langle\phi,\rho(v)\phi\rangle$$

for all spinors ϕ and vectors v.

Let $L = \det(S)$ be the determinant line bundle of S. Let $\Gamma(L)$ denote the space of $U(1)$–connections on L, which is an affine space over $i\Omega^1(Y)$. We will write such a connection as $2A$. Then $2A$ determines a spin$^c(3)$–connection on S whose $\mathfrak{u}(1)$ part is A and whose $\mathfrak{spin}(3)$ part is the Levi-Civita connection. Abusing terminology, we will refer to A as a spin^c–connection.

Given a spin^c–connection A, we let D_A denote the associated Dirac operator on S. Fix a reference spin^c–connection A_0. Then we may write $A = A_0 + a$ for some $a \in i\Omega^1(Y)$. It follows that

$$D_A(\psi) = D_{A_0+a}(\psi) = D_{A_0}(\psi) + \rho(a)\psi.$$

Since $b_1(Y) = 0$, it follows that L admits a flat connection. We will assume that A_0 defines a flat connection on L.

We define the configuration space of Y to be

$$C(Y) = \Gamma(L) \times \Gamma(S).$$

$C(Y)$ depends on g and $\mathfrak{s}$ but we omit this from the notation. $C(Y)$ is an affine space modelled on $i\Omega^1(Y) \oplus \Gamma(S)$. In particular, the tangent space $T_{(A,\phi)}C(Y)$ to any point $(A,\phi) \in C(Y)$ can naturally be identified with $i\Omega^1(Y) \oplus \Gamma(S)$. There is a natural metric on $i\Omega^1(Y) \oplus \Gamma(S)$, the L^2–metric

$$\langle (a_1,\phi_1),(a_2,\phi_2)\rangle_{L^2} = -\int_Y a_1 \wedge *a_2 + \int_Y \operatorname{Re}\langle \phi_1,\phi_2\rangle \, d\mathrm{vol}_Y.$$

This defines a (constant) Riemannian metric on $C(Y)$. We will need to work with Sobolev completions. Given a flat reference spin^c–connection A_0, Sobolev norms are defined using A_0 and g. Fix an integer $k \geq 4$. Later we will work with the L^2_{k+1}–completion of $C(Y)$ and L^2_{k+2}–gauge transformations.

Having fixed a reference connection A_0, we identify $C(Y)$ with $i\Omega^1(Y) \oplus \Gamma(S)$. Thus an element $(A,\phi) \in C(Y)$ will be identified with $(a,\phi) \in i\Omega^1(Y) \oplus \Gamma(S)$, where $A = A_0 + a$. To simplify notation, we will write D_a in place of D_{A_0+a}.

The *Chern–Simons–Dirac functional* $\mathscr{L}\colon C(Y) \to \mathbb{R}$ (with respect to A_0) is defined as

$$\mathscr{L}(a,\phi) = \frac{1}{2}\left(\int_Y \langle \phi, D_a\phi\rangle \, d\mathrm{vol}_Y - \int_Y a \wedge da\right).$$

The gauge group $\mathscr{G} = \mathscr{C}^\infty(Y,S^1)$ acts on $C(Y)$ by

$$u \cdot (a,\phi) = (a - u^{-1}\,du, u\cdot\phi).$$

Observe that $D_{a-u^{-1}du}(u\phi) = uD_a\phi$, so

$$\langle u\phi, D_{a-u^{-1}du}(u\phi)\rangle = \langle u\phi, uD_a\phi\rangle = \langle \phi, D_a\phi\rangle.$$

It follows that $\mathscr{L}$ is gauge invariant and we can regard $\mathscr{L}$ as a function on the quotient space $C(Y)/\mathscr{G}$. The goal of Seiberg–Witten–Floer theory is to construct some sensible notion of Morse homology of $\mathscr{L}$ on

$C(Y)/\mathcal{G}$. Consider the formal L^2–gradient of $\mathcal{L}$; that is, the function $\operatorname{grad}(\mathcal{L})\colon C(Y) \to i\Omega^1(Y) \oplus \Gamma(S)$ such that

$$\langle \operatorname{grad}(\mathcal{L})(a,\phi),(a',\phi')\rangle_{L^2} = \frac{d}{dt}\Big|_{t=0}\big(\mathcal{L}(a,\phi)+t(a',\phi')\big)$$

for all $(a,\phi) \in C(Y)$ and all $(a',\phi') \in i\Omega^1(Y) \oplus \Gamma(S)$. A short calculation gives

$$\operatorname{grad}(\mathcal{L}) = ({*da} + \tau(\phi,\phi), D_a\phi).$$

A critical point of $\mathcal{L}$ is a point where $\operatorname{grad}(\mathcal{L})$ vanishes. So (a,ϕ) is a critical point if and only if it satisfies

$$*da + \tau(\phi,\phi) = 0, \quad D_a\phi = 0.$$

These are the 3–dimensional Seiberg–Witten equations.

A trajectory for the downwards gradient flow is a differentiable map $x\colon \mathbb{R} \to L^2_{k+1}(C(Y))$ such that

$$\frac{d}{dt}x(t) = -\operatorname{grad}(\mathcal{L})(x(t)).$$

If $x(t) = (a(t),\phi(t)) \in L^2_{k+1}(Y, iT^*Y \oplus S)$, then

$$\frac{d}{dt}a(t) = -{*da(t)} - \tau(\phi(t),\phi(t)), \quad \frac{d}{dt}\phi(t) = -D_{a(t)}\phi(t).$$

A key observation is that such trajectories can be reinterpreted as solutions of the 4–dimensional Seiberg–Witten equations on the cylinder $X = \mathbb{R} \times Y$.

Definition 2.1 A Seiberg–Witten trajectory $x(t) = (a(t),\phi(t))$ is said to be of *finite type* if both $\mathcal{L}(x(t))$ and $\|\phi(t)\|_{\mathcal{C}^0}$ are bounded functions of t.

2.2 Restriction to the global Coulomb slice

Define the *global Coulomb slice* (with respect to A_0) to be the subspace

$$V = \operatorname{Ker}(d^*) \oplus \Gamma(S) \subset C(Y).$$

Given $(a,\phi) \in C(Y)$, there exists an element of V which is gauge equivalent to (a,ϕ), namely

$$(a - df, e^f\phi)$$

where $d^*(a - df) = 0$, so $\Delta f = d^*(a)$. If we impose the condition $\int_Y f \, d\mathrm{vol}_Y = 0$, then there is a unique solution to these equations given by $f = Gd^*a$, where G is the Green's operator for the Laplacian $\Delta = dd^*$ on functions.

We have a globally defined map $\Pi\colon C(Y) \to V$, called the *global Coulomb projection*,

$$\Pi(a,\phi) = (a - df, e^f\phi)$$

where $\Delta f = d^*(a)$ and $\int_Y f \, d\mathrm{vol} = 0$.

Restricting to the global Coulomb slice V uses up all of the gauge symmetry except for the S^1 subgroup of constant gauge transformations. Instead of working on $C(Y)$ with full gauge symmetry, we work on V with S^1 symmetry.

As $b_1(Y)=0$, every map $u\colon Y\to S^1$ can be written as $u=e^f$ for some $f\colon Y\to i\mathbb{R}$. Moreover, f is unique up to addition of an integer multiple of $2\pi i$. We define $\mathcal{G}_0$ to be the subgroup of gauge transformations of the form $u=e^f$ for some $f\colon Y\to i\mathbb{R}$ with $\int_Y f\,d\mathrm{vol}=0$. It is easy to see that $\mathcal{G}=\mathcal{G}_0\times S^1$.

We have that $\mathcal{G}_0$ acts freely on $C(Y)$ and the quotient space can be identified with V. This determines a metric $\tilde{g}$ on V as follows. Take the restriction of the L^2–metric on $C(Y)$ to the subbundle of the tangent bundle orthogonal to the gauge orbits. This construction is $\mathcal{G}_0$–invariant and descends to a metric $\tilde{g}$ on V.

The Chern–Simons–Dirac functional $\mathcal{L}$ is gauge invariant; hence the gradient $\mathrm{grad}(\mathcal{L})$ is orthogonal to the gauge orbits. It follows that the projection of $\mathrm{grad}(\mathcal{L})$ to V coincides with taking the gradient of $\mathcal{L}|_V$ with respect to $\tilde{g}$. So the trajectories of $\mathrm{grad}(\mathcal{L})$ on $C(Y)$ project to the trajectories of $\mathcal{L}|_V$, where the gradient of $\mathcal{L}|_V$ is taken using the metric $\tilde{g}$. Thus the trajectories on V have the form

$$\frac{d}{dt}(a(t),\phi(t))=(-*da-\tau(\phi,\phi),-D_a\phi)-(-df,f\phi)$$

for a function $f\colon Y\to i\mathbb{R}$. The function f is uniquely determined by the conditions that $\int_Y f\,d\mathrm{vol}_Y=0$ and that $*da+\tau(\phi,\phi)-df$ is in the kernel of d^*. Hence $df=(1-\pi)\tau(\phi,\phi)$, where π denotes the L^2 orthogonal projection to $\mathrm{Ker}(d^*)$. We have that

$$\frac{d}{dt}(a(t),\phi(t))=(-*da-\pi\tau(\phi,\phi),-D_a\phi-f\phi)=-(l+c)(a,\phi),$$

where

$$l(a,\phi)=(*da,D\phi)$$

is the linear part and

$$c(a,\phi)=(\pi\tau(\phi,\phi),\rho(a)\phi+f\phi)$$

is given by the nonlinear terms.

Let χ denote the gradient of $\mathcal{L}|_V$ with respect to $\tilde{g}$. Then $\chi=l+c$ extends to a map

$$\chi=l+c\colon V_{k+1}\to V_k,$$

where V_k denotes the L^2_k–Sobolev completion of V. The map l is a linear Fredholm operator. Using Sobolev multiplication and an estimate on the unique solution to $df=(1-\pi)\tau(\phi,\phi)$, $\int_Y f\,d\mathrm{vol}_Y=0$, it follows that c viewed as a map $V_{k+1}\to V_{k+1}$ is continuous. Hence $c\colon V_{k+1}\to V_k$ is compact. The flow lines of χ on V will be called Seiberg–Witten trajectories in the Coulomb gauge. We say that such a trajectory $x(t)=(a(t),\phi(t))$ is of finite type if $\mathcal{L}(x(t))$ and $\|\phi(t)\|_{\mathcal{C}^0}$ are bounded independent of t. Clearly the finite type Seiberg–Witten trajectories in the Coulomb gauge are precisely the projection to V of the Seiberg–Witten trajectories in $C(Y)$ of finite type.

2.3 Finite-dimensional approximation

Let V_λ^μ denote the direct sum of all eigenspaces of l in the range $(\lambda, \mu]$ and let $\tilde{p}_\lambda^\mu$ be the L^2–orthogonal projection from V to V_λ^μ. Note that V_λ^μ is a finite-dimensional subspace of V. For technical reasons we replace the projections $\tilde{p}_\lambda^\mu$ with smoothed out versions

$$p_\lambda^\mu = \int_0^1 \rho(\theta)\tilde{p}_{\lambda+\theta}^{\mu-\theta} d\theta$$

where $\rho\colon \mathbb{R} \to \mathbb{R}$ is smooth, nonnegative, nonzero precisely on $(0,1)$ and $\int_{\mathbb{R}} \rho(\theta)d\theta = 1$. This is to make p_λ^μ vary continuously with μ and λ. The reason for doing this is to show that the Conley index is independent of the choices of μ and λ, up to a suspension. This is achieved by continuously increasing or decreasing μ and λ to get a continuous family of flows and using homotopy invariance of the Conley index under continuous deformation of the flow.

Consider the gradient flow equation

$$\frac{d}{dt}x(t) = -(l + p_\lambda^\mu c)x(t),$$

where $x\colon \mathbb{R} \to V_\lambda^\mu$. We call this an *approximate Seiberg–Witten trajectory*.

Let $B(R)$ denote the open ball of radius R in $L_{k+1}^2(V)$. Using the a priori estimates for the Seiberg–Witten equations, it can be shown that there exists an $R > 0$ such that all the finite type trajectories of $l + c$ are in $B(R)$ [49, Proposition 1]. This boundedness property does not necessarily hold for approximate trajectories, since the crucial estimates that hold for the Seiberg–Witten equations do not apply to the approximate trajectories. However, we have the following result which acts as a kind of substitute:

Proposition 2.2 [49, Proposition 3] *For any $-\lambda$ and μ sufficiently large, if an approximate trajectory $x\colon \mathbb{R} \to L_{k+1}^2(V_\lambda^\mu)$ satisfies $x(t) \in \overline{B(2R)}$ for all t, then in fact $x(t) \in B(R)$ for all t.*

This result will allow us to construct the Seiberg–Witten–Floer homotopy type of $(Y, \mathfrak{s})$ using Conley indices.

2.4 The Conley index

Suppose we have a 1–parameter group $\{\varphi_t\}$ of diffeomorphisms of an n–dimensional manifold M (not necessarily compact). The example to keep in mind is the gradient flow of a Morse function. Given a compact subset $N \subseteq M$, the invariant set of N is

$$\mathrm{Inv}(N, \varphi) = \{x \in N \mid \varphi_t(x) \in N \text{ for all } t \in \mathbb{R}\}.$$

A compact subset $N \subseteq M$ is called an *isolating neighbourhood* if $\mathrm{Inv}(N, \varphi) \subseteq \mathrm{int}\, N$. An *isolated invariant set* is a subset $S \subseteq M$ such that $S = \mathrm{Inv}(N, \varphi)$ for some isolating neighbourhood. Note that S must be compact since it is a closed subset of N and N is required to be compact.

Definition 2.3 Let S be an isolated invariant set. An *index pair* (N, L) for S is a pair of compact sets $L \subseteq N \subseteq M$ such that

- $\mathrm{Inv}(N - L, \varphi) = S \subseteq \mathrm{int}(N - L)$;
- L is an exit set for N, that is, for all $x \in N$, if there exists $t > 0$ such that $\varphi_t(x)$ is not in N, then there exists τ with $0 \leq \tau < t$ with $\varphi_\tau(x) \in L$;
- L is positively invariant in N, that is, if $x \in L$ and $t > 0$ and such that $\varphi_s(x) \in N$ for all $0 \leq s \leq t$, then $\varphi_s(x) \in L$ for all $0 \leq s \leq t$.

Any isolated invariant set S admits an index pair (N, L). The *Conley index* of S is the based homotopy type

$$I(S) = (N/L, [L]).$$

The Conley index is independent of the choice of index pair (N, L) in a strong way. Namely for any two pairs (N_1, L_1) and (N_2, L_2), there is a canonical homotopy equivalence $N_1/L_1 \cong N_2/L_2$. The composition of two such canonical homotopy equivalences $N_1/L_1 \cong N_2/L_2$ and $N_2/L_2 \cong N_3/L_3$ coincides up to homotopy with the canonical homotopy equivalence $N_1/L_1 \cong N_3/L_3$ (one says that the collection of Conley indices N/L forms a *connected simple system*). By abuse of terminology, if (N, L) is an index pair for S we say that $I = N/L$ is "the" Conley index of S.

Example 2.4 Consider a Morse function with critical point of index p, say

$$f(x_1, \dots, x_n) = \tfrac{1}{2}(-x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2).$$

The negative gradient of f using the Euclidean metric is

$$-\mathrm{grad}(f)(x) = (x_1, \dots, x_p, -x_{p+1}, \dots, -x_n).$$

It follows that the downwards gradient flow is given by

$$\varphi_t(x) = (e^t x_1, \dots, e^t x_p, e^{-t} x_{p+1}, \dots, e^{-t} x_n).$$

Let $S = \{0\}$ be the critical point. This is an isolated invariant set. In fact, the only invariant point of φ is the origin, so we could take $N = D^p \times D^{n-p}$ as an isolating neighbourhood (where D^j is the closed j–dimensional unit disc). Then $L = S^{p-1} \times D^{n-p}$ is an exit set for N. It is easy to see that (N, L) satisfies the condition for an index pair for S. The Conley index is $I(S) = D^p \times D^{n-p}/(S^{p-1} \times D^{n-p})$, which is homotopy equivalent to S^p, a p–dimensional sphere.

Example 2.5 If M is a compact manifold and φ is a Morse–Smale gradient flow on M, then the set S of all critical points and all flow lines between them is an isolated invariant set. The reduced homology of $I(S)$ is known to be isomorphic to the homology of M.

On the other hand, if M is noncompact, then we cannot take S to be all critical points of M and all flow lines starting or terminating at a critical point, because there could be flow lines going off to $-\infty$ or coming in from $+\infty$ and then S would not be compact.

We also need the equivariant Conley index. Let G be a compact Lie group acting smoothly on M, preserving a flow φ and an isolated invariant set S. It turns out that one can find a G–invariant index pair (N, L) for S and one can define the G–equivariant Conley index to be the pointed G–equivariant homotopy type

$$I_G(S) = (N/L, [L]).$$

It can be shown that this is well defined, up to G–equivariant homotopy equivalence. Moreover $I_G(S)$ has the based homotopy type of a finite G–CW complex.

Example 2.6 Consider again the example of the Morse function on $\mathbb{R}^n$ given by

$$f(x_1, \dots, x_n) = \tfrac{1}{2}(-x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2).$$

Now suppose that G is a compact Lie group which acts linearly on $\mathbb{R}^n$ preserving f. Note that f defines an $O(n-p, p)$–structure on $\mathbb{R}^n$ and the fact that G preserves f just means that the action of G on $\mathbb{R}^n$ factors through a homomorphism $G \to O(n-p, p)$. As G is compact, we may as well assume (after a linear change of coordinates) that G maps to the maximal compact subgroup $O(n-p) \times O(p)$. So we can decompose $\mathbb{R}^n$ as

$$\mathbb{R}^n = V_+ \oplus V_-,$$

where V_+ and V_- are real orthogonal representations of G of dimensions $n-p$ and p respectively. Once again, take $S = \{0\}$ as our isolated invariant set. As our Conley index, we can take $N = D(V_-) \times D(V_+)$ and $L = S(V_-) \times D(V_+)$; hence

$$I_G(S) = D(V_-) \times D(V_+)/S(V_-) \times D(V_+) \cong D(V_-)/S(V_-) \cong (V_-)^+,$$

where $(V_-)^+$ is the one-point compactification of V_-. We see that the action of G on the Conley index is determined by the representation of G on the subspace of the tangent space at the critical point in the direction of the negative eigenvalues of the Hessian of f.

Let G and H be compact Lie groups and suppose that $G \times H$ acts smoothly on M. Suppose that $\{\varphi_t\}$ is a $G \times H$–invariant flow. Then G acts smoothly on the submanifold M^H and the restriction of $\{\varphi_t\}$ defines a G–invariant flow on M^H. In such a situation we can consider the relation between $G \times H$–equivariant Conley indices for the flow on M and G–equivariant Conley indices for the restriction of the flow to M^H.

Proposition 2.7 *Let $G \times H$ act smoothly on M, preserving a flow $\{\varphi_t\}$, and let (N, L) be a $G \times H$–equivariant index pair for an isolated invariant set $S = \mathrm{Inv}(A)$. Then (N^H, L^H) is a G–equivariant index pair for the isolated invariant set $S^H = \mathrm{Inv}(A^H)$. Moreover, $(N/L)^H \cong N^H/L^H$.*

Proof First note that A^H is compact because A is compact. Moreover,

$$\mathrm{Inv}(A^H) = \{a \in A^H \mid \varphi_t(a) \in A^H \text{ for all } t\} = \mathrm{Inv}(A) \cap A^H = S \cap A^H = S^H.$$

Let M be a topological space and let $P, Q \subseteq M$ be subspaces. Give Q the induced topology. Then

$$\operatorname{int}_M(P) \cap Q = \Bigg(\bigcup_{U \subseteq P \text{ open in } M} U\Bigg) \cap Q = \bigcup_{U \subseteq P \text{ open in } M} (U \cap Q) \subseteq \operatorname{int}_Q(P \cap Q).$$

Applying this to $P = A$ and $Q = M^H$, we get

$$\operatorname{int}_M(A) \cap M^H \subseteq \operatorname{int}_{M^H}(A^H).$$

Then since $S \subseteq \operatorname{int}_M(A)$ by the assumption that A is an isolating neighbourhood, it follows that

$$S^H \subseteq \operatorname{int}_M(A) \cap M^H \subseteq \operatorname{int}_{M^H}(A^H).$$

So A^H is an isolating neighbourhood in M^H for S^H.

Now let (N, L) be an index pair for S. So N and L are compact and $L \subseteq N$. This implies that N^H and L^H are compact and $L^H \subseteq N^H$. Next, since

$$\operatorname{Inv}(N - L) = S \subseteq \operatorname{int}_M(N - L),$$

it follows that

$$\begin{aligned}\operatorname{Inv}(N^H - L^H) = S^H = S \cap M^H &\subseteq \operatorname{int}_M(N - L) \cap M^H \\ &\subseteq \operatorname{int}_{M^H}((N - L) \cap M^H) = \operatorname{int}_{M^H}(N^H - L^H).\end{aligned}$$

We verify that L^H is an exit set for N^H. Let $x \in N^H$ and suppose $\varphi_t(x) \notin N^H$ for some $t > 0$. Then it follows that $\varphi_t(x) \notin N$, for if $\varphi_t(x) \in N$, then it would imply that $\varphi_t(x) \in N \cap M^H = N^H$, since φ_t preserves M^H. But L is an exit set for N, so there exists $\tau \in [0, t)$ with $\varphi_\tau(x) \in L$. It follows that $\varphi_\tau(x) \in L \cap M^H = L^H$. Hence L^H is an exit set for N^H.

We check that L^H is positively invariant in N^H. Suppose $x \in L^H$ and there exists a $t > 0$ for which $\varphi_s(x) \in N^H$ for all $s \in [0, t]$. Then since L is positively invariant in N, it follows that $\varphi_s(x) \in L$ for all $s \in [0, t]$. Hence $\varphi_s(x) \in L \cap M^H = L^H$ for all $s \in [0, t]$.

We have verified that (N^H, L^H) is an index pair for S^H. Moreover it is straightforward to check that $(N/L)^H = N^H/L^H$. □

2.5 Equivariant Spanier–Whitehead category

In this section we recall the construction of the category $\mathfrak{C}$ from [49], which is an S^1–equivariant version of the Spanier–Whitehead category. In Section 3.3 we will modify this construction to accommodate a finite group action on Y.

We work with pointed topological spaces with a basepoint-preserving action of S^1. The objects of $\mathfrak{C}$ are triples (X, m, n), where X is a pointed topological space with S^1–action, and $m, n \in \mathbb{Z}$.[1] We further

[1] In [49] n is allowed to take on rational values. This is needed to construct a Seiberg–Witten–Floer spectrum which does not depend on the choice of metric. For our purposes it suffices to consider only integral values of n.

require that X has the S^1–homotopy type of an S^1–CW complex, which holds for Conley indices on manifolds. The set of morphisms between two objects (X,m,n) to (X',m',n') will be denoted by $\{(X,m,n),(X',m',n')\}_{S^1}$ and is defined to be

$$\{(X,m,n),(X',m',n')\}_{S^1} = \operatorname{colim}[(\mathbb{R}^k \oplus \mathbb{C}^l)^+ \wedge X, (\mathbb{R}^{k+m-m'} \oplus \mathbb{C}^{l+n-n'})^+ \wedge X']_{S^1},$$

where $[\cdot,\cdot]_{S^1}$ denotes the set of S^1–equivariant homotopy classes and the colimit is taken over all k and l such that $k \geq m'-m$ and $l \geq n'-n$. The maps that define the colimit are given by suspensions where we smash on the left and for any topological space Z, we let Z^+ denote the one-point compactification with its obvious basepoint.

Any pointed space X with S^1–action defines an object of $\mathfrak{C}$, namely $(X,0,0)$. We often simply write this as X. For any finite-dimensional representation E of S^1, we let Σ^E denote the reduced suspension operation

$$\Sigma^E X = E^+ \wedge X.$$

This operation extends to $\mathfrak{C}$ by taking $\Sigma^E(X,m,n) = (\Sigma^E X, m, n)$. We are mainly interested in the case that E is a real vector space with trivial S^1–action, or E is a complex vector space with S^1 acting by scalar multiplication. If E is a real vector space with trivial action, then one finds that

$$\Sigma^E(X,m,n) \cong (X, m-\dim_{\mathbb{R}}(E), n).$$

The isomorphism depends on a choice of isomorphism $E \cong \mathbb{R}^{\dim_{\mathbb{R}}(E)}$. Up to homotopy there are two choices since $\mathrm{GL}(E,\mathbb{R})$ has two components. If E is a complex vector space and S^1 acts by scalar multiplication, then

$$\Sigma^E(X,m,n) \cong (X,m,n-\dim_{\mathbb{C}}(E)).$$

The isomorphism is unique up to homotopy as $\mathrm{GL}(E,\mathbb{C})$ is connected. We can define desuspension by a real vector space E with trivial S^1–action as

$$\Sigma^{-E}(X,m,n) = ((E)^+ \wedge X, m+2\dim_{\mathbb{R}}(E), n).$$

Then $\Sigma^{-E}\Sigma^E Z \cong Z$ by an isomorphism which is canonical up to homotopy. We can define desuspension by a complex vector space E with S^1 acting by scalar multiplication by

$$\Sigma^{-E}(X,m,n) = (X,m,n+\dim_{\mathbb{C}}(E)).$$

Then $\Sigma^{-E}\Sigma^E Z \cong Z$ by an isomorphism which is canonical up to homotopy.

For $Z = (X,m,n) \in \mathfrak{C}$, we define the reduced equivariant cohomology of Z to be

$$\widetilde{H}^j_{S^1}(Z) = \widetilde{H}^{j+m+2n}_{S^1}(X).$$

The cohomology is well defined as a consequence of the Thom isomorphism.

2.6 Seiberg–Witten–Floer cohomology

Consider as before a rational homology 3–sphere Y and a spin^c–structure $\mathfrak{s}$. Let $R > 0$, λ and μ be as in Proposition 2.2. We want to take the Conley index of the set of all critical points in $B(R)$ and flow lines between them which lie in $B(R)$ for all time for the approximate Seiberg–Witten flow $l + p_\lambda^\mu c$. The problem is that there could be trajectories that go to infinity in a finite amount of time. Hence we do not have a flow $\{\varphi_t\}$ in the sense of a 1–parameter group of diffeomorphisms. To get around this issue, let u_λ^μ be a compactly supported smooth cutoff function which is identically 1 on $B(3R)$. For consistency purposes we assume that $u_\lambda^\mu = u_{\lambda'}^{\mu'}|_{V_\lambda^\mu}$ for $\lambda' \le \lambda$ and $\mu' \ge \mu$. One way of doing this is to take $u_\lambda^\mu(v) = \rho(\|v\|)$, where ρ is smooth, compactly supported and $\rho(t) = 1$ for $t < 3$.

For each μ and λ, the vector field $u_\lambda^\mu(l + p_\lambda^\mu c)$ is compactly supported, so it generates a well-defined flow $\varphi_{\lambda,t}^\mu$ on V_λ^μ. Since $u_\lambda^\mu = 1$ on $\overline{B(2R)}$, Proposition 2.2 still applies to the trajectories of $u_\lambda^\mu(l + p_\lambda^\mu c)$. It follows that

$$\operatorname{Inv}(V_\lambda^\mu \cap \overline{B(2R)}) = S_\lambda^\mu,$$

where S_λ^μ is the set of critical points and flow lines between critical points for the approximate Seiberg–Witten flow $l + p_\lambda^\mu c$ which lie in $B(R)$. Therefore S_λ^μ is an isolated invariant set. Moreover, S^1 preserves the approximate flow; hence we may take the S^1–equivariant Conley index

$$I_\lambda^\mu = I_{S^1}(S_\lambda^\mu).$$

This is an S^1–equivariant homotopy type. However it is not quite an invariant of $(Y, \mathfrak{s})$ because it depends on the choice of metric g as well as the values of λ, μ and R. Note that it is independent of the choice of u_λ^μ because of the assumption that $u_\lambda^\mu = 1$ on $B(3R)$. To get a genuine invariant we must understand how I_λ^μ changes as we vary these parameters.

Let λ^* and μ^* satisfy Proposition 2.2. Suppose that $\lambda' \le \lambda \le \lambda^*$ and $\mu' \ge \mu \ge \mu^*$. We wish to compare the Conley indices I_λ^μ, $I_\lambda^{\mu'}$ and $I_{\lambda'}^\mu$. In other words, what happens if we increase either μ or $-\lambda$, staying in the range where μ and $-\lambda$ are sufficiently large.

We use the following invariance property of the Conley index: Suppose we have a family $\{\varphi_t(s)\}$ of flows depending continuously on $s \in [0,1]$. Suppose that a fixed compact set A is an isolating neighbourhood for all $s \in [0,1]$ and let $S(s) = \operatorname{Inv}(A, \varphi_t(s))$. Then $I(S_0, \varphi_t(0)) \cong I(S_1, \varphi_t(1))$ by a canonical homotopy equivalence.

Consider increasing μ to μ'. The finite energy trajectories of $l + p_\lambda^\mu c p_\lambda^\mu$ in $V_\lambda^{\mu'}$ must actually lie in V_λ^μ. Therefore $A = \overline{B(2R)} \cap V_\lambda^{\mu'}$ is an isolating neighbourhood for S_λ^μ in $V_\lambda^{\mu'}$. Let $\mu(s) = (1-s)\mu + s\mu'$ for $s \in [0,1]$ and let $\tilde{\varphi}_\lambda^{\mu(s)}$ denote the flow of $u_\lambda^{\mu(s)}(l + p_\lambda^{\mu(s)} c p_\lambda^{\mu(s)})$ on $V_\lambda^{\mu'}$. Then for each $s \in [0,1]$, A is an isolating neighbourhood for $S_\lambda^{\mu(s)}$ in $V_\lambda^{\mu'}$ with respect to the flow $\tilde{\varphi}_\lambda^{\mu(s)}$. Hence

$$\operatorname{Inv}(\tilde{\varphi}_\lambda^\mu, A) \cong \operatorname{Inv}(\tilde{\varphi}_\lambda^{\mu'}, A) = I_\lambda^{\mu'}.$$

But $\widetilde{\varphi}^{\mu}$ is easily seen to be homotopic to the product of the flow of $u_{\lambda}^{\mu}(l + p_{\lambda}^{\mu} c)$ on V_{λ}^{μ} with a linear flow on W generated by $l|_W$, where W is the orthogonal complement of V_{λ}^{μ} in $V_{\lambda}^{\mu'}$. The Conley index of a product of flows is just the smash product of Conley indices. Combined with Example 2.6, we see that

$$\mathrm{Inv}(\widetilde{\varphi}_{\lambda}^{\mu}, A) \cong I_{\lambda}^{\mu} \wedge W_{-}^{+},$$

where W_{-} is the part of W spanned by negative eigenvalues of l. But W is contained in the positive eigenvalues of l, so $W_{-} = 0$ and hence

$$I_{\lambda}^{\mu'} \cong I_{\lambda}^{\mu}.$$

Now consider decreasing λ to λ'. An identical argument to the one above gives

$$I_{\lambda'}^{\mu} \cong I_{\lambda}^{\mu} \wedge W_{-}^{+},$$

where W is the orthogonal complement of V_{λ}^{μ} in $V_{\lambda'}^{\mu}$. In this case W is spanned by negative eigenspaces of l, so $W_{-} = W = V_{\lambda'}^{\lambda}$ and

$$I_{\lambda'}^{\mu} \cong I_{\lambda}^{\mu} \wedge (V_{\lambda'}^{\lambda})^{+}.$$

This implies that

$$\Sigma^{-V_{\lambda}^{0}} I_{\lambda}^{\mu}$$

does not depend on the values of μ and λ (provided μ and $-\lambda$ are sufficiently large).

Definition 2.8 Given $(Y, \mathfrak{s})$ and a metric g, we set

$$SWF(Y, \mathfrak{s}, g) = \Sigma^{-V_{\lambda}^{0}(g)} I_{\lambda}^{\mu}(g)$$

for suitably chosen μ, λ and R.

We have established that the homotopy type of $SWF(Y, \mathfrak{s}, g)$ does not depend on the choices of μ and λ, or more precisely, any two choices of μ and λ are related by a canonical homotopy equivalence. One also checks that it does not depend on the choice of R. So up to homotopy, $SWF(Y, \mathfrak{s}, g)$ depends only on Y, $\mathfrak{s}$ and g.

Next we consider varying the metric g. Consider a smooth homotopy g_s for $s \in [0, 1]$ joining two metrics g_0 and g_1, which is constant near $s = 0$. Assuming that the g_s are all sufficiently close to each other in a suitable topology, we can arrange that there exists R, μ^* and λ^* such that Proposition 2.2 is true for all $s \in [0, 1]$ and all μ and λ with $\mu \geq \mu^*$ and $\lambda \leq \lambda^*$. This suffices, as compactness of $[0, 1]$ implies that any smooth path g_s can be broken up into finitely many subpaths over which this assumption holds.

We assume that there exists some $\lambda < \lambda^*$ and $\mu > \mu^*$ such that λ and μ are not eigenvalues of l_s for any $s \in [0, 1]$. This property will hold for all sufficiently small paths. The spaces $(V_{\lambda}^{\mu})_s$ then form a smooth vector bundle over $[0, 1]$. We can trivialise this vector bundle and identify all these spaces with a single V_{λ}^{μ}. Further, we assume that $B(R)_{s_1} \subset B(2R)_{s_2}$ for each $s_1, s_2 \in [0, 1]$. Here we think of the balls as subsets of the same space V_{λ}^{μ}. Once again, this property will hold for all small enough paths. Then

$$\bigcap_{s \in [0,1]} \overline{B(2R)_s}$$

is a compact isolating neighbourhood for S_λ^μ in any metric g_s with the flow $(\varphi_\lambda^\mu)_s$. The Conley index will be independent of s and hence

$$(I_\lambda^\mu)_0 \cong (I_\lambda^\mu)_1.$$

However, we do not have that $\Sigma^{-(V_\lambda^0)_0}(I_\lambda^\mu)_0$ equals $\Sigma^{-(V_\lambda^0)_1}(I_\lambda^\mu)_1$. The reason is that some eigenvalues in (λ,μ) may change sign. On the other hand, any eigenvalue greater than μ or less than λ cannot change sign, by our assumption that μ and λ are not eigenvalues of l_s for any $s\in[0,1]$. Hence the difference between $(V_\lambda^0)_0$ and $(V_\lambda^0)_1$ is given in terms of the spectral flow of the family of operators $\{l_s\}$ for $s\in[0,1]$.

The operator l can be split into real and complex components. The real part has no spectral flow, so we only need to consider the complex part, which is the Dirac operator D_s. The spectral flow can be expressed using the Atiyah–Patodi–Singer (APS) index theorem on the cylinder $X=[0,1]\times Y$; see [8; 9]. Let $\hat{g}$ be the metric on X given by g_s in the vertical direction and $(ds)^2$ in the horizontal direction. Let S_s denote the spinor bundle associated to $(\mathfrak{s}, g_s)$. The bundles S_s can all be identified with $S=S_0$, but with varying Clifford multiplication. The spin^c–structure $\mathfrak{s}$ lifts to a spin^c–structure on X. Let $S^\pm$ denote the spinor bundles of this spin^c–structure. Then $S^\pm$ can be identified with the pullback of S to X. Suppose for each s we have chosen a flat reference connection A_s. Since we have identified S_s with S for all s, we get an induced identification of $L_s=\det(S_s)$ with $L=L_0$. Then $A_s=A_0+i\alpha_s$ for some closed real 1–form α_s. The path of spin^c–connections $\{A_s\}$ fit together to form a spin^c–connection $\widehat{A}$ on the determinant line L pulled back to X. Let $\widehat{D}$ be the Dirac operator determined by $\hat{g}$ and $\widehat{A}$. Then $\widehat{D}(\psi)=\partial_s\psi+D_s\psi$. After a possible reparametrisation we can assume that (g_s,A_s) is constant near the boundary. Applying the APS index theorem to the Dirac operator $\widehat{D}$ on the cylinder $[0,1]\times Y$, one can write the spectral flow $SF(\{D_s\})$ as

$$SF(\{D_s\})=\tfrac{1}{2}(\eta(D_1)-k(D_1))-\tfrac{1}{2}(\eta(D_0)-k(D_0))+\int_{[0,1]\times Y}\big(-\tfrac{1}{24}p_1(\hat{g})+\tfrac{1}{8}c_1(\widehat{A})^2\big),$$

where $\eta(D)$ is the eta invariant of D, $k(D)=\dim_{\mathbb{C}}(\mathrm{Ker}(D))$, $p_1(\hat{g})$ is the first Pontryagin form of $\hat{g}$, and $c_1(\widehat{A})$ is the Chern form $(i/2\pi)F_{2\widehat{A}}$, where $F_{2\widehat{A}}$ is the curvature of the induced connection $2\widehat{A}$ on L. Now since α_s is closed for each s, we get $F_{2\widehat{A}}=ds\wedge 2i\,\partial_s\alpha_s$ and hence $c_1(\widehat{A})^2=0$. So

$$SF(\{D_s\})=\tfrac{1}{2}(\eta(D_1)-k(D_1))-\tfrac{1}{2}(\eta(D_0)-k(D_0))-\frac{1}{24}\int_{[0,1]\times Y}p_1(\hat{g}). \tag{2-1}$$

Let $\eta_{\mathrm{sign}}(g_s)$ denote the eta invariant of the signature operator on Y defined by g_s. Then from the APS index theorem for the signature operator together with the fact that the signature operator has no spectral flow, we find

$$\eta_{\mathrm{sign}}(g_1)-\eta_{\mathrm{sign}}(g_0)=\frac{1}{3}\int_{[0,1]\times Y}p_1(\hat{g}). \tag{2-2}$$

Combining (2-1) and (2-2), we see that

$$SF(\{D_s\})=\tfrac{1}{2}(\eta(D_1)-k(D_1))-\tfrac{1}{2}(\eta(D_0)-k(D_0))-\tfrac{1}{8}(\eta_{\mathrm{sign}}(g_1)-\eta_{\mathrm{sign}}(g_0)),$$

and hence

$$SF(\{D_s\}) = n(Y,\mathfrak{s},g_0) - n(Y,\mathfrak{s},g_1),$$

where we have defined

$$n(Y,\mathfrak{s},g) = \tfrac{1}{2}\eta(D) - \tfrac{1}{2}k(D) - \tfrac{1}{8}\eta_{\mathrm{sign}}(g).$$

We will show that $n(Y,\mathfrak{s},g)$ is a rational number. Let $(W,\mathfrak{s}_W)$ be a spin^c 4–manifold bounding $(Y,\mathfrak{s})$. This always exists because $\Omega_3^{\mathrm{spin}^c} = 0$. Extend the Dirac operator D on Y to a Dirac operator $\widehat{D}$ in the same way as we did for the cylinder $[0,1]\times Y$. The APS index theorem for the Dirac operator and signature operator on W combined give

$$\mathrm{ind}_{\mathrm{APS}}(\widehat{D}) = \tfrac{1}{8}(c_1(\mathfrak{s}_W)^2 - \sigma(W)) + \tfrac{1}{2}(\eta_{\mathrm{dir}} - k) - \tfrac{1}{8}\eta_{\mathrm{sign}}$$

and thus

$$n(Y,\mathfrak{s},g) = \mathrm{ind}_{\mathrm{APS}}(\widehat{D}) - \delta(W,\mathfrak{s}), \tag{2-3}$$

where we set

$$\delta(W,\mathfrak{s}) = \tfrac{1}{8}(c_1(\mathfrak{s}_W)^2 - \sigma(W)).$$

This shows that $n(Y,\mathfrak{s},g)$ is a rational number since $\mathrm{ind}_{\mathrm{APS}}(\widehat{D})$ is an integer and $\delta(W,\mathfrak{s})$ is a rational number.

Definition 2.9 The *Seiberg–Witten–Floer cohomology* of $(Y,\mathfrak{s},g)$ is defined as

$$HSW^j(Y,\mathfrak{s}) = \widetilde{H}_{S^1}^{j+2n(Y,\mathfrak{s},g)}(SWF(Y,\mathfrak{s},g)),$$

where $j\in\mathbb{Q}$ and as usual the coefficient group $\mathbb{F}$ has been omitted from the notation.

Below we will show that $HSW^*(Y,\mathfrak{s})$ is independent of the choice of metric g (and other auxiliary choices); hence it is a well defined topological invariant of the pair $(Y,\mathfrak{s})$.

Notice that because of the grading shift by $2n(Y,\mathfrak{s},g)$ the cohomology groups $HSW^*(Y,\mathfrak{s})$ are concentrated in rational degrees. It was shown by Lidman and Manolescu [45] that $HSW^*(Y,\mathfrak{s})$ is isomorphic to the Seiberg–Witten monopole Floer cohomology as defined by Kronheimer and Mrowka [39]. Together with the equivalence of monopole Floer homology and Heegaard Floer homology due to the work of Kutluhan, Lee and Taubes [40; 41; 42; 43; 44], Colin, Ghiggini and Honda [16; 17; 15] and Taubes [60; 61; 62; 63; 64], we have isomorphisms

$$HSW^*(Y,\mathfrak{s}) \cong HF^-_{-*}(\overline{Y},\mathfrak{s}) \cong HF^*_+(Y,\mathfrak{s}),$$

where HF^-_* denotes the minus version of Heegaard Floer homology and HF^*_+ denotes the plus version of Heegaard Floer *cohomology* (taken with respect to the same coefficient group $\mathbb{F}$). Here we use a grading convention for HF^- such that $HF^-(S^3)$ starts in degree 0. Through the work of [20; 33; 58], the isomorphism is known to preserve the absolute gradings. Using co-Borel, Tate or nonequivariant cohomologies gives similar isomorphisms to the other versions of Heegaard Floer homology; see [45, Corollary 1.2.4] for the precise statement.

If we have two metrics g_0 and g_1, then the spectral flow of a path joining them satisfies

$$SF(\{D_s\}) = n(Y, \mathfrak{s}, g_1) - n(Y, \mathfrak{s}, g_0).$$

On the other hand, from the definition of spectral flow,

$$SF(\{D_s\}) = \dim(V^0_\lambda(g_0)) - \dim(V^0_\lambda(g_1)).$$

It follows that

$$SWF(Y, \mathfrak{s}, g_1) \cong \Sigma^{SF(\{D_s\})} SWF(Y, \mathfrak{s}, g_0),$$

and hence

$$\widetilde{H}^{j+2SF(\{D_s\})}_{S^1}(SWF(Y, \mathfrak{s}, g_1)) \cong \widetilde{H}^{j}_{S^1}(SWF(Y, \mathfrak{s}, g_0))$$

by the Thom isomorphism. Replacing j by $j + 2n(Y, \mathfrak{s}, g_0)$, we have

$$\widetilde{H}^{j+2n(Y,\mathfrak{s},g_1)}_{S^1}(SWF(Y, \mathfrak{s}, g_1)) \cong \widetilde{H}^{j+2n(Y,\mathfrak{s},g_0)}_{S^1}(SWF(Y, \mathfrak{s}, g_0)).$$

Hence the Seiberg–Witten–Floer cohomology $\widetilde{H}^{j+2n(Y,\mathfrak{s},g)}_{S^1}(SWF(Y, \mathfrak{s}, g))$ is independent of the metric. The above isomorphism is canonical in the sense that it does not depend on the choice of path from g_0 to g_1. This follows from the fact that the space of all metrics on Y is contractible, so any two paths with the same endpoints are homotopic.

2.7 Duality

Definition 2.10 Let V be a finite-dimensional representation of a compact Lie group G. Two pointed, finite G–CW complexes X and X' are equivariantly V*–dual* if there exists a G–map

$$\varepsilon\colon X \wedge X' \to V^+$$

such that for any subgroup $H \subseteq G$, the fixed-point map

$$\varepsilon^H\colon X^H \wedge (X')^H \to (V^H)^+$$

induces a nonequivariant duality between X^H and $(X')^H$, in the sense of nonequivariant Spanier–Whitehead duality.

Consider the Conley index I^μ_λ associated to $(Y, \mathfrak{s}, g)$ for suitably chosen R, μ and λ. One finds that reversing orientation of Y has the effect of reversing the Chern–Simons–Dirac flow. From [19], it follows that $I^\mu_\lambda(Y)$ and $I^\lambda_\mu(\overline{Y})$ are V^μ_λ–dual, so there exists a duality map

$$\varepsilon\colon I^\mu_\lambda(Y) \wedge I^\lambda_\mu(\overline{Y}) \to (V^\mu_\lambda)^+.$$

Notice that $\dim(V_\lambda^0(Y)) + \dim(V_{-\mu}^0(\overline{Y})) - 2k(D) = \dim(V_\lambda^\mu(Y))$, where $k(D)$ is the dimension of the kernel of D. Desuspending, we obtain a duality map

$$\varepsilon\colon SWF(Y, \mathfrak{s}, g) \wedge SWF(\overline{Y}, \mathfrak{s}, g) \to S^{-k(D)\mathbb{C}}.$$

We also have

$$n(Y, \mathfrak{s}, g) + n(\overline{Y}, \mathfrak{s}, g) = -k(D). \tag{2-4}$$

2.8 Fixed points

Definition 2.11 Let $s \geq 0$ be an integer. We say that a finite pointed S^1–CW complex X is of *type SWF at level s* if

- the S^1–fixed-point set X^{S^1} is homotopy equivalent to the sphere $(\mathbb{R}^s)^+$;
- the action of S^1 is free on the complement $X - X^{S^1}$.

Proposition 2.12 *Given $(Y, \mathfrak{s}, g)$, let R, μ and λ be as in Proposition 2.2. Then $\overline{B(2R)} \cap V_\lambda^\mu$ is an isolating neighbourhood for $S_\lambda^\mu = \mathrm{Inv}(\overline{B(2R)} \cap V_\lambda^\mu)$. Let $I_\lambda^\mu = I_{S^1}(S_\lambda^\mu)$ be the Conley index. Then I_λ^μ is of type SWF at level $s = \dim(V_\lambda^0(\mathbb{R}))$, where $V_\lambda^0(\mathbb{R})$ denotes the S^1–invariant part of V_λ^0.*

Proof Let (N, L) be an index pair for S_λ^μ so that $I_\lambda^\mu = N/L$. Then by Proposition 2.7, $(I_\lambda^\mu)^{S^1} = N^{S^1}/L^{S^1}$ is the Conley index of $(S_\lambda^\mu)^{S^1}$. Further, we have that $\overline{B(2R)} \cap V_\lambda^\mu(\mathbb{R})$ is an isolating neighbourhood for $(S_\lambda^\mu)^{S^1}$, where $V_\lambda^\mu(\mathbb{R})$ denotes the S^1–invariant part of V_λ^μ. It is easy to see that $c = 0$ on $V_\lambda^\mu(\mathbb{R})$, where c is the nonlinear part of the Seiberg–Witten flow. Thus the restriction of the approximate Seiberg–Witten flow $u_\lambda^\mu(l + p_\lambda^\mu c)$ to $V_\lambda^\mu(\mathbb{R})$ is the flow $u_\lambda^\mu l$. Restricted to $B(3R)$ this is just the linear flow associated to l. The real part of l has zero kernel, because $b_1(Y) = 0$. It follows that the Conley index of $(S_\lambda^\mu)^{S^1}$ is the Conley index of $\{0\}$ in $V_\lambda^\mu(\mathbb{R})$ with respect to the linear flow of l. This is $(V_\lambda^0(\mathbb{R}))^+$. Thus we have shown that the S^1–fixed-point set of I_λ^μ is homotopy equivalent to $(V_\lambda^0(\mathbb{R}))^+$. Furthermore, S^1 acts freely on $V_\lambda^\mu - V_\lambda^\mu(\mathbb{R})$; hence S^1 acts freely on $N - N^{S^1}$ and therefore also on $(I_\lambda^\mu) - (I_\lambda^\mu)^{S^1}$. □

Using the identities

$$(\mathbb{R}^+ \wedge X)^{S^1} = \mathbb{R}^+ \wedge X^{S^1}, \quad (\mathbb{C}^+ \wedge X)^{S^1} = X^{S^1}, \tag{2-5}$$

we see that

- if X is of type SWF at level s, then $\mathbb{R}^+ \wedge X$ is of type SWF at level $s+1$;
- if X is of type SWF at level s, then $\mathbb{C}^+ \wedge X$ is of type SWF at level s.

Now let $Z = (X, m, n)$ belong to the equivariant Spanier–Whitehead category $\mathfrak{C}$. We say that Z is of type SWF at level s if X is of type SWF of level $s + m$. The above remarks shows that this is a well-defined notion.

We have shown that the Conley index I_λ^μ is of type SWF at level $s = \dim(V_\lambda^0(\mathbb{R}))$, where

$$V_\lambda^0 = V_\lambda^0(\mathbb{R}) \oplus V_\lambda^0(\mathbb{C})$$

denotes the decomposition of V_λ^0 into copies of $\mathbb{R}$ and $\mathbb{C}$. Now we recall that

$$SWF(Y,\mathfrak{s},g) = \Sigma^{-V_\lambda^0} I_\lambda^\mu.$$

It follows that $SWF(Y,\mathfrak{s},g)$ is of type SWF at level 0.

Let X be a space of type SWF at level s. Let $\iota\colon X^{S^1} \to X$ denote the inclusion of the fixed-point set. Using the localisation theorem in equivariant cohomology [23, III (3.8)], it follows that the pullback map $\iota^*\colon \widetilde{H}^*_{S^1}(X) \to \widetilde{H}^*_{S^1}(X^{S^1})$ is not identically zero. Therefore, we may define the d–invariant $d(X)$ of X by

$$d(X) = \min\{ j \mid x \in \mathrm{Im}(\iota^*) \text{ for some } x \in \widetilde{H}^j_{S^1}(X^{S^1}) \text{ with } x \neq 0\}.$$

Note that $d(X)$ could potentially depend on the choice of coefficient group, so we may write the invariant as $d(X;\mathbb{F})$ if we wish to indicate the dependence on $\mathbb{F}$.

We also define the δ–invariant of X by $\delta(X) = \frac{1}{2}d(X)$. Using (2-5) and the Thom isomorphism, one finds

$$d(\mathbb{R}^+ \wedge X) = d(X)+1, \quad d(\mathbb{C}^+ \wedge X) = d(X)+2.$$

Now if $Z = (X,m,n)$ is of type SWF, we define the d–invariant $d(Z)$ of Z to be

$$d(Z) = d(X) - m - 2n \in \mathbb{Z}.$$

From [45, Corollary 1.2.3], it follows that the d–invariant $d(Y,\mathfrak{s})$ as defined by Heegaard Floer homology (with coefficient group $\mathbb{F}$) is given in terms of $SWF(Y,\mathfrak{s},g)$ by

$$d(Y,\mathfrak{s}) = d(SWF(Y,\mathfrak{s},g)) - 2n(Y,\mathfrak{s},g).$$

For notational convenience we also define $\delta(Y,\mathfrak{s}) = \frac{1}{2}d(Y,\mathfrak{s})$.

3 Equivariant Seiberg–Witten–Floer cohomology

3.1 Assumption on G and $\mathbb{F}$

Throughout this paper we will assume that one of the two following conditions hold:

(1) G is an arbitrary finite group and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, or

(2) $\mathbb{F}$ is an arbitrary field and the order of G is odd.

Condition (1) ensures that we do not need to concern ourselves with questions of orientability. Condition (2) ensures that any S^1–central extension $\widetilde{G}$ acts orientation-preservingly on all of its finite-dimensional representations. Hence under either condition, the Thom isomorphism holds without requiring local coefficients,

$$\widetilde{H}^*_{\widetilde{G}}(X) \cong \widetilde{H}^{*+\dim_{\mathbb{R}}(V)}_{\widetilde{G}}(V^+ \wedge X).$$

Here X is any $\widetilde{G}$–space and V is any finite-dimensional representation of $\widetilde{G}$.

3.2 Lifting G–actions

Recall that Y denotes a rational homology 3–sphere. Suppose that a finite group G acts on Y by orientation-preserving diffeomorphisms and suppose G preserves the isomorphism class of a spin^c–structure $\mathfrak{s}$. We will construct a G–equivariant version of the Seiberg–Witten–Floer cohomology of $(Y, \mathfrak{s})$.

Choose a G–invariant metric g on Y and a reference spin^c–connection A_0 such that the connection on the determinant line L is flat. Let $g \in G$ and choose a lift $\hat{g}\colon S \to S$ of g to the spinor bundle S, which is possible since G preserves the isomorphism class of $\mathfrak{s}$. Then $\hat{g}^{-1} A_0 \hat{g} = A_0 + a$ for some $a \in i\Omega^1(Y)$. Since A_0 and $\hat{g}^{-1} A_0 \hat{g}$ are flat, we must have $da = 0$. Moreover, $b_1(Y) = 0$ implies that $a = df$ for some $f\colon Y \to i\mathbb{R}$. Setting $\tilde{g} = e^{-f}\hat{g}$, it follows that $\tilde{g}$ is a lift of g which preserves A_0. Any other lift of g that preserves A_0 is of the form $c\tilde{g}$ with $c \in U(1)$ a constant. Let $G_{\mathfrak{s}}$ denote the set of all possible lifts of elements of G which preserve A_0. Then $G_{\mathfrak{s}}$ is a group and we have a central extension

$$1 \to S^1 \to G_{\mathfrak{s}} \to G \to 1.$$

Now we carry out the construction of the Conley index of a finite-dimensional approximation of the Chern–Simons–Dirac flow $G_{\mathfrak{s}}$–equivariantly, instead of just S^1–equivariantly.

3.3 $G_{\mathfrak{s}}$–equivariant Spanier–Whitehead category

In this section $\widetilde{G}$ denotes any S^1 central extension of G. We will construct a category $\mathfrak{C}(\widetilde{G})$, the $\widetilde{G}$–equivariant version of $\mathfrak{C}$.

Recall from Section 2.5 that the category $\mathfrak{C}$ was constructed so that there exists a desuspension functor Σ^{-V} for any real vector space V with trivial S^1–action or any complex vector space where S^1 acts by scalar multiplication. We now construct a category $\mathfrak{C}(\widetilde{G})$ in which we can desuspend by real representations of $\widetilde{G}$, where S^1 acts trivially, and by complex representations, where S^1 acts by scalar multiplication. We are lead to consider the following two types of finite-dimensional representations of $\widetilde{G}$:

Type (1) V is a real representation of $\widetilde{G}$ and S^1 acts trivially.

Type (2) V is a complex representation $\widetilde{G}$ and S^1 acts on V by scalar multiplication.

Type (1) representations correspond canonically to real representations of G.

Type (2) representations correspond to projective unitary representations of G such that the pullback to G of the central extension $S^1 \to U(n) \to \mathrm{PU}(n)$ gives an extension isomorphic to $\widetilde{G}$. If $\widetilde{G}$ is split, then type (2) representations are in bijection with complex representations of G. However, the bijection depends on a choice of splitting of $\widetilde{G}$.

To define stable homotopy groups, we need to consider suspensions with explicitly chosen representations. In other words, we need to work at the level of representations and not just isomorphism classes. Let $V_1, \dots, V_p$ be a complete set of irreducible representations of type (1), and $W_1, \dots, W_q$ a complete set of irreducible representations of type (2). Any representation of type (1) is isomorphic to a direct sum of copies of $V_1, \dots, V_p$ and likewise any representation of type (2) is a direct sum of copies of $W_1, \dots, W_q$.

If $m = (m_1, \dots, m_p), m' = (m'_1, \dots, m'_p) \in \mathbb{Z}^p$, we say $m \geq m'$ if $m_i \geq m'_i$ for each i. If $m \in \mathbb{Z}^p$ satisfies $m \geq 0$, then we set

$$V(m) = V_1^{\oplus m_1} \oplus \cdots \oplus V_p^{\oplus m_p}.$$

Similarly, if $n = (n_1, \dots, n_q) \in \mathbb{Z}^q$ satisfies $n \geq 0$, then we set

$$W(n) = W_1^{\oplus n_1} \oplus \cdots \oplus W_q^{\oplus n_q}.$$

The category $\mathfrak{C}(\widetilde{G})$ has as objects triples (X, m, n), where

- X is a pointed topological space with a basepoint-preserving $\widetilde{G}$–action and the homotopy type of a $\widetilde{G}$–CW complex;
- $m \in \mathbb{Z}^p$;
- $n \in \mathbb{Z}^q$.

Let (X, m, n) and (X', m', n') be two objects of $\mathfrak{C}(\widetilde{G})$. The set of morphisms from (X, m, n) to (X', m', n'), denoted by $\{(X, m, n), (X', m', n')\}^{\widetilde{G}}$, is defined to be

$$\operatorname*{colim}_{k,l} \big[(V(k))^+ \wedge (W(l))^+ \wedge X, (V(k+m-m'))^+ \wedge (W(l+n-n'))^+ \wedge X'\big]^{\widetilde{G}}.$$

The colimit is taken over all $k \in \mathbb{N}^p$ and $l \in \mathbb{N}^q$ such that $k \geq m' - m$ and $l \geq n' - n$. The maps that define the colimit are given by suspensions where we smash on the left.

Let Y be any pointed $\widetilde{G}$–space. We obtain a functor $Y \wedge \colon \mathfrak{C}(\widetilde{G}) \to \mathfrak{C}(\widetilde{G})$ which is defined on objects by $Y \wedge (X, m, n) = (Y \wedge X, m, n)$ and on morphisms in the evident way. In particular, if V is any finite-dimensional representation of $\widetilde{G}$, we define the reduced suspension

$$\Sigma^V Z = V^+ \wedge Z.$$

We define desuspension by a representation V of type (1) as

$$\Sigma^{-V}(X, m, n) = ((V)^+ \wedge X, m + 2[V], n),$$

where $[V] = (v_1, \dots, v_p)$ and v_i is the multiplicity of V_i in V. Then $\Sigma^{-V}\Sigma^V Z \cong Z$, where the isomorphism is canonical up to homotopy. For any representation W of type (2) we define

$$\Sigma^{-W}(X, m, n) = (X, m, n + [W]),$$

where $[W] = (w_1, \dots, w_q)$ and w_i is the multiplicity of W_i in W. We have that $\Sigma^{-W}\Sigma^W Z \cong Z$ by an isomorphism which is canonical up to homotopy. In fact, such an isomorphism is induced by a choice of isomorphism $W \cong W([W])$. But for any pair of isomorphic *complex* representations, the space of isomorphism is connected (by Schur's lemma it is a torsor for a product of complex general linear groups). Therefore the isomorphism $W \cong W([W])$ is unique up to homotopy.

For $Z = (X, m, n) \in \mathfrak{C}(\widetilde{G})$, we define the reduced equivariant cohomology of Z to be

$$\widetilde{H}^j_{\widetilde{G}}(Z) = \widetilde{H}^{j+|m|+2|n|}_{\widetilde{G}}(X),$$

where $|m|$ and $|n|$ are defined as

$$|m| = \sum_{i=1}^{p} m_i \dim_{\mathbb{R}}(V_i) \quad \text{for } m = (m_1, \dots, m_p), \qquad |n| = \sum_{i=1}^{q} n_i \dim_{\mathbb{C}}(W_i) \quad \text{for } n = (n_1, \dots, n_q).$$

The cohomology is well defined as a consequence of the Thom isomorphism.

3.4 G–equivariant Seiberg–Witten–Floer cohomology

Let Y be a rational homology 3–sphere and G a finite group acting on Y preserving the isomorphism class of a spin^c–structure $\mathfrak{s}$. Let $G_{\mathfrak{s}}$ be the S^1–central extension of G obtained by lifting G to the spinor bundle corresponding to $\mathfrak{s}$. We repeat the construction of the Conley index $I^{\mu}_{\lambda}(g)$ from Section 2.6, except that now we carry out the construction $G_{\mathfrak{s}}$–equivariantly. Restricting to the subgroup $S^1 \subseteq G_{\mathfrak{s}}$, $I^{\mu}_{\lambda}(g)$ agrees with the S^1–equivariant Conley index as previously constructed.

We need to understand how $I^{\mu}_{\lambda}(g)$ depends on μ, λ, the choice of G–invariant metric g, and the constant R. As in the S^1 case, first consider variations of μ and λ. Carrying out a similar argument but $G_{\mathfrak{s}}$–equivariantly, we see that $I^{\mu}_{\lambda}(g)$ simply changes by suspension. Analogous to the nonequivariant case we define

$$SWF(Y, \mathfrak{s}, g) = \Sigma^{-V^0_{\lambda}(g)} I^{\mu}_{\lambda}(g) \in \mathfrak{C}(G_{\mathfrak{s}}),$$

where $V^0_{\lambda}(g)$ is defined as before, but now carries a $G_{\mathfrak{s}}$–action. Note that $V^0_{\lambda}(g)$ is the sum of a representation of type (1) and a representation of type (2), so the desuspension $\Sigma^{-V^0_{\lambda}(g)}$ is defined. Then up to canonical isomorphisms $SWF(Y, \mathfrak{s}, g)$ depends only on the triple $(Y, \mathfrak{s}, g)$.

We consider the dependence of $SWF(Y, \mathfrak{s}, g)$ on the metric g. The argument is much the same as before except done $G_{\mathfrak{s}}$–equivariantly. Let g_0 and g_1 be two G–invariant metrics. The space of such metrics is contractible, so we may choose a path $\{g_s\}$ from g_0 to g_1. Then as in the nonequivariant case, the signature operator has no spectral flow and we have

$$SWF(Y, \mathfrak{s}, g_1) = \Sigma^{SF_{G_{\mathfrak{s}}}(\{D_s\})} SWF(Y, \mathfrak{s}, g_0),$$

where now $SF_{G_{\mathfrak{s}}}(\{D_s\})$ is the equivariant spectral flow of $\{D_s\}$. Thus $SF_{G_{\mathfrak{s}}}(\{D_s\})$ is to be understood as a virtual representation of $G_{\mathfrak{s}}$ [25, Section 2]. Since the S^1 subgroup of $G_{\mathfrak{s}}$ acts by scalar multiplication on spinors, it follows that $SF_{G_{\mathfrak{s}}}(\{D_s\})$ is a type (2) virtual representation. From the Thom isomorphism and the fact that the underlying rank of $SF_{G_{\mathfrak{s}}}(\{D_s\})$ is $SF(\{D_s\}) = n(Y, \mathfrak{s}, g_1) - n(Y, \mathfrak{s}, g_0)$, we obtain a canonical isomorphism

$$\widetilde{H}_{G_{\mathfrak{s}}}^{j+2n(Y,\mathfrak{s},g_1)}(SWF(Y, \mathfrak{s}, g_1)) \cong \widetilde{H}_{G_{\mathfrak{s}}}^{j+2n(Y,\mathfrak{s},g_0)}(SWF(Y, \mathfrak{s}, g_0)).$$

This motivates the following definition:

Definition 3.1 The *G–equivariant Seiberg–Witten–Floer cohomology* of $(Y, \mathfrak{s}, g)$ is defined as

$$HSW_G^j(Y, \mathfrak{s}) = \widetilde{H}_{G_{\mathfrak{s}}}^{j+2n(Y,\mathfrak{s},g)}(SWF(Y, \mathfrak{s}, g)).$$

By the argument above, the $HSW_G^*(Y, \mathfrak{s})$ depends only on $(Y, \mathfrak{s})$ and the G–action.

For a group K we write H_K^* for $H_K^*(\mathrm{pt})$. Since $HSW^*(Y, \mathfrak{s})$ is defined using equivariant cohomology, it is a graded module over the ring $H_{S^1}^* = \mathbb{F}[U]$, where $\deg(U) = 2$. Similarly $HSW_G^*(Y, \mathfrak{s})$ is a graded module over $H_{G_{\mathfrak{s}}}^*$. Restricting from $G_{\mathfrak{s}}$ to S^1, we obtain forgetful maps

$$HSW_G^*(Y, \mathfrak{s}) \to HSW^*(Y, \mathfrak{s}), \quad H_{G_{\mathfrak{s}}}^* \to H_{S^1}^*$$

compatible with the module structures.

Observe that since S^1 is the identity component of $G_{\mathfrak{s}}$, the action of $G_{\mathfrak{s}}$ on $HSW^*(Y, \mathfrak{s})$ descends to an action of G. So we may regard $HSW^*(Y, \mathfrak{s})$ as a G–module.

Theorem 3.2 *There is a spectral sequence $E_r^{p,q}$ abutting to $HSW_G^*(Y, \mathfrak{s})$ whose second page is given by*

$$E_2^{p,q} = H^p(BG; HSW^q(Y, \mathfrak{s})).$$

Proof For a $G_{\mathfrak{s}}$–space M, let $M_{G_{\mathfrak{s}}}$ denote the Borel model for the $G_{\mathfrak{s}}$–action and M_{S^1} the Borel model for the S^1–action obtained by restriction. The composition $M_{G_{\mathfrak{s}}} \to BG_{\mathfrak{s}} \to BG$ is a fibration with fibre M_{S^1}. Applying the Leray–Serre spectral sequence, we get a spectral sequence which abuts to $\widetilde{H}_{G_{\mathfrak{s}}}^*(M)$ and has $E_2^{p,q} = H^p(BG; \widetilde{H}_{S^1}^q(M))$. More generally if M is the formal desuspension of a $G_{\mathfrak{s}}$–space, then via an application of the Thom isomorphism a similar spectral sequence exists. Applying this to $HSW_G^*(Y, \mathfrak{s})$ gives the theorem. □

Definition 3.3 Let Y be a rational homology 3–sphere and $\mathfrak{s}$ a spin^c–structure. We say that Y is an *L–space* (with respect to $\mathfrak{s}$ and $\mathbb{F}$) if the action of U on $HSW^*(Y, \mathfrak{s})$ is injective. Equivalently $HSW^*(Y, \mathfrak{s})$ is a free $\mathbb{F}[U]$–module of rank 1.

Remark 3.4 The usual definition of an L–space is that $HF_{\mathrm{red}}^+(Y, \mathfrak{s}) = 0$ for all spin^c–structures and where the coefficient group is $\mathbb{Z}$. From the universal coefficient theorem it follows that an L–space in this sense is an L–space with respect to any spin^c–structure $\mathfrak{s}$ and any coefficient group $\mathbb{F}$.

Suppose that the extension $G_{\mathfrak{s}}$ is split. A choice of splitting induces an isomorphism $G_{\mathfrak{s}} \cong S^1 \times G$ and an isomorphism $H_{G_{\mathfrak{s}}}^* \cong H_G^*[U]$. We stress that these isomorphisms depend on the choice of splitting.

Theorem 3.5 *Suppose that $G_{\mathfrak{s}}$ is a split extension. If Y is an L–space (with respect to $\mathfrak{s}$ and $\mathbb{F}$), then the spectral sequence given in Theorem 3.2 degenerates at E_2. Moreover,*

$$HSW^*_G(Y,\mathfrak{s}) \cong HSW^*(Y,\mathfrak{s}) \otimes_{\mathbb{F}} H^*_G \cong H^*_G[U]\theta,$$

where θ has degree $d(Y,\mathfrak{s})$.

Proof If Y is an L–space (with respect to $\mathfrak{s}$ and $\mathbb{F}$) then

$$HSW^*(Y,\mathfrak{s}) \cong \mathbb{F}[U]\theta,$$

where θ has degree $d(Y,\mathfrak{s})$. We claim that G acts trivially on $HSW^*(Y,\mathfrak{s})$. This can be seen as follows. First, since $HSW^*(Y,\mathfrak{s})$ is up to a degree shift the S^1–equivariant cohomology of the Conley index $I = I^{\mu}_{\lambda}$, it suffices to prove the result for I. Let $\iota\colon I^{S^1} \to I$ be the inclusion of the S^1 fixed-point set. Since Y is an L–space, U acts injectively on $HSW^*(Y,\mathfrak{s})$. Together with the localisation theorem in equivariant cohomology, this implies that ι^* is injective. Hence it suffices to show that G acts trivially on $\widetilde{H}^*_{S^1}(I^{S^1})$. But I^{S^1} has the homotopy type of a sphere, so if ν is a generator of $\widetilde{H}^*_{S^1}(I^{S^1})$ and $g \in G$, then $g^*(\nu) = \pm\nu$ according to whether or not g acts orientation-preservingly. Our assumptions on G and $\mathbb{F}$ (see Section 3.1) ensures that $g^*(\nu) = \nu$ for all $g \in G$. This proves the claim.

Letting $E_r^{p,q}$ denote the spectral sequence for $HSW^*_G(Y,\mathfrak{s})$, it follows easily that

$$E_2^{p,q} \cong H^*(BG;\mathbb{F}[U]\theta) \cong H^*_G[U]\theta \cong HSW^*(Y,\mathfrak{s}) \otimes_{\mathbb{F}} H^*_G.$$

It remains to show that the differentials $d_2, d_3, \dots$ are all zero. In fact since θ has the lowest q–degree of any term in $E_2^{p,q}$, it follows that $d_j(\theta) = 0$ for all $j \geq 2$. Then since the differentials commute with the $H^*_{G_{\mathfrak{s}}} \cong H^*_G[U]$–module structure, it follows that $d_2, d_3, \dots$ all vanish. □

3.5 Spaces of type G–SWF

We introduce a G–equivariant analogue of spaces of type SWF. We then define a G–equivariant analogue of the d–invariant.

Let $\widetilde{G}$ be an extension of G by S^1. If $\widetilde{G}$ acts on a space X, then we get an induced action of $G = \widetilde{G}/S^1$ on the fixed-point set X^{S^1}. We write $\overline{G} = S^1 \times G$ for the trivial extension of G.

Definition 3.6 Let $s \geq 0$ be an integer. We say that a finite pointed $\widetilde{G}$–CW complex X is of *type G–SWF at level s* if

- the S^1–fixed-point set X^{S^1} is G–homotopy equivalent to a sphere $(V)^+$, where V is a real representation of G of dimension s;
- the action of S^1 is free on the complement $X - X^{S^1}$.

More generally, let V be a finite-dimensional representation which is the direct sum of representations of type (1) and (2). An equivariant spectrum $Z = \Sigma^{-V} X \in \mathfrak{C}(\widetilde{G})$ is said to be of *type G–SWF at level s* if X is G–SWF at level $s + \dim(V^{S^1})$.

Assume that $\widetilde{G}$ is split and choose a splitting $\widetilde{G} \cong \overline{G}$. Let X be a space of type G–SWF at level s. Let $\iota\colon X^{S^1} \to X$ denote the inclusion of the fixed-point set. Recall that $H^*_{S^1} \cong \mathbb{F}[U]$, where $\deg(U) = 2$. Similarly $H^*_{\overline{G}} \cong H^*_G[U]$. The localisation theorem in equivariant cohomology implies that

$$\iota^*\colon U^{-1}\widetilde{H}^*_{\overline{G}}(X) \to U^{-1}\widetilde{H}^*_{\overline{G}}(X^{S^1})$$

is an isomorphism. Note $X^{S^1} \cong (V)^+$, where V is s–dimensional, so

$$\widetilde{H}^*_{\overline{G}}(X^{S^1}) \cong H^*_G[U]\tau,$$

where $\deg(\tau) = s$. Therefore it also follows that

$$U^{-1}\widetilde{H}^*_{\overline{G}}(X^{S^1}) \cong H^*_G[U, U^{-1}]\tau.$$

Then for each $c \in H^*_G$, it follows that there exists an $x \in \widetilde{H}^*_{\overline{G}}(X)$ for which $\iota^*(x) = cU^k\tau$, for some $k \geq 0$. Set $\Lambda_G(X) = \widetilde{H}^*_{\overline{G}}(X^{S^1})$. Then $\Lambda_G(X)$ is a free $H^*_G[U]$–module of rank 1 and $\iota\colon X^{S^1} \to X$ induces a map

$$\iota^*\colon \widetilde{H}^*_{\overline{G}}(X) \to \Lambda_G(X)$$

of $H^*_G[U]$–modules. Introduce a filtration

$$\Lambda_G(X) = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$$

on $\Lambda_G(X)$ by setting

$$F_j = H_G^{*\geq j}\Lambda_G(X),$$

where $H_G^{*\geq j} = \bigoplus_{k\geq j} H^k_G$. This is the filtration induced by the fibration

$$X^{S^1} \times_{\overline{G}} B\overline{G} \to BG.$$

Let τ denote the generator of $\Lambda_G(X)$. Then for $j \geq 0$ we have obvious identifications

$$F_j/F_{j+1} \cong H^j_G[U]\tau.$$

Now let c be a nonzero element in H^*_G of degree $|c| = \deg(c)$. By the discussion above we know that $cU^k\tau$ is in the image of ι^* for some $k \geq 0$. Hence we may define:

Definition 3.7 Let c be a nonzero element in H^*_G of degree $|c| = \deg(c)$. We define $d_{G,c}(X) \in \mathbb{Z}$ by

$$d_{G,c}(X) = \min\{2k+s \mid \iota^*(x) \in F_{|c|} \text{ and } \iota^*(x) = cU^k\tau \bmod F_{|c|+1} \text{ for some } x \in \widetilde{H}^{s+2k+|c|}_{\overline{G}}(X)\}.$$

For convenience we set $d_{G,0}(X) = -\infty$. Then if c is an element of H^*_G, we write $c = c_0 + c_1 + \cdots + c_r$, where $c_i \in H^i_G$ and set

$$d_{G,c}(X) = \max\{d_{G,c_0}(X), \ldots, d_{G,c_r}(X)\}.$$

Note that $d_{G,ac}(X) = d_{G,c}(X)$ for any $a \in \mathbb{F}^*$.

In concrete terms, the condition that $\iota^*(x) \in F_{|c|}$ and $\iota^*(x) = cU^k\tau \mod F_{|c|+1}$ means that $\iota^*(x)$ is of the form

$$\iota^*(x) = cU^k\tau + c_1U^{k-1}\tau + \cdots + c_rU^{k-r}\tau$$

for some $r \geq 0$ and some $c_1, \dots c_r \in H_G^{*\geq(|c|+1)}$.

Remark 3.8 Let X be a space of type G–SWF. The definition of $d_{G,c}(X)$ does not depend on a choice of splitting of $S^1 \to \widetilde{G} \to G$. Indeed, two splittings differ by a homomorphism $\phi\colon G \to S^1$. Let $\alpha = \phi^*(U) \in H_G^2$. The change of splitting acts on $H_G^*[U]$ by sending U to $U + \alpha$. Then since $(U+\alpha)^k = U^k + \cdots$, where $\cdots$ denotes terms involving lower powers of U, it follows that $d_{G,c}(X)$ does not depend on the choice of splitting of $\widetilde{G}$.

Proposition 3.9 *Let X be a space of type G–SWF for the trivial extension. Then for all $c_1, c_2 \in H_G^*$,*

$$d_{G,c_1+c_2}(X) \leq \max\{d_{G,c_1}(X), d_{G,c_2}(X)\}, \quad d_{G,c_1c_2}(X) \leq \min\{d_{G,c_1}(X), d_{G,c_2}(X)\}.$$

Proof Let s be the level of X. First consider the case that c_1 and c_2 are homogeneous, that is, $c_1 \in H_G^{|c_1|}$ and $c_2 \in H_G^{|c_2|}$ for some $|c_1|$ and $|c_2|$. Then by Definition 3.7, there exist $x_1 \in \widetilde{H}_{\widetilde{G}}^{d_{G,c_1}(X)+|c_1|}(X)$ and $x_2 \in \widetilde{H}_{\widetilde{G}}^{d_{G,c_2}(X)+|c_2|}(X)$ such that

$$\iota^*(x_1) = c_1U^{k_1}\tau + \cdots, \quad \iota^*(x_2) = c_2U^{k_2}\tau + \cdots,$$

where $\cdots$ denotes terms that are in the next stage of the filtration and $k_i = \frac{1}{2}(d_{G,c_i}(X) - s)$ for $i = 1, 2$. Note that if c_1 or c_2 are zero then we take x_1 or x_2 to be zero.

If $|c_1| \neq |c_2|$, then by Definition 3.7, we have $d_{G,c_1+c_2}(X) = \max\{d_{G,c_1}(X), d_{G,c_2}(X)\}$. Now suppose that $|c_1| = |c_2|$. Let $k = \max\{k_1, k_2\}$ and set $x = U^{k-k_1}x_1 + U^{k-k_2}x_2 \in \widetilde{H}_{\widetilde{G}}^{2k+s+|c_1|}(X)$. Then

$$\iota^*(x) = (c_1 + c_2)U^k\tau + \cdots$$

and hence, from the definition of $d_{G,c_1+c_2}(X)$,

$$d_{G,c_1+c_2}(X) \leq 2k + s = \max\{2k_1 + s, 2k_2 + s\} = \max\{d_{G,c_1}(X), d_{G,c_2}(X)\}.$$

Next we observe that $c_2x_1 \in \widetilde{H}_{\widetilde{G}}^{d_{G,c_1}(X)+|c_1|+|c_2|}(X)$ and

$$\iota^*(c_2x_1) = (c_1c_2)U^{k_1}\tau,$$

and so it follows that $d_{G,c_1c_2}(X) \leq d_{G,c_1}(X)$. Exchanging the roles of x_1, x_2 and c_1, c_2, we similarly find that $d_{G,c_1c_2}(X) \leq d_{G,c_1}(X)$; hence

$$d_{G,c_1c_2}(X) \leq \min\{d_{G,c_1}(X), d_{G,c_2}(X)\}.$$

Now suppose that c_1 and c_2 are not necessarily homogeneous. We may write $c_1 = a_0 + a_1 + \cdots + a_r$ and $c_2 = b_0 + b_1 + \cdots + b_r$, for some $r \geq 0$, where $a_i, b_i \in H_G^i$. By Definition 3.7,

$$d_{G,c_1}(X) = \max_i\{d_{G,a_i}(X)\}, \quad d_{G,c_2}(X) = \max_i\{d_{G,b_i}(X)\}.$$

Then since $c_1 + c_2 = (a_0 + b_0) + (a_1 + b_1) + \cdots + (a_r + b_r)$,

$$\begin{aligned} d_{G,c_1+c_2}(X) &= \max_i \{d_{G,a_i+b_i}(X)\} \\ &\le \max_i \bigl\{\max\{d_{G,a_i}(X), d_{G,b_i}(X)\}\bigr\} \\ &= \max\bigl\{\max_i\{d_{G,a_i}(X)\}, \max_i\{d_{G,b_i}(X)\}\bigr\} \\ &= \max\{d_{G,c_1}(X), d_{G,c_2}(X)\}. \end{aligned}$$

Next, we have $c_1 c_2 = \sum_{i,j} a_i b_j$ and hence

$$d_{G,c_1c_2}(X) \le \max_{i,j}\{d_{G,a_ib_j}(X)\} \le \max_{i,j}\{d_{G,a_i}(X)\} = \max_i\{d_{G,a_i}(X)\} = d_{G,c_1}(X),$$

where we used $d_{G,a_ib_j}(X) \le d_{G,a_i}(X)$. Similarly we get $d_{G,c_1c_2}(X) \le d_{G,c_2}(X)$, and hence

$$d_{G,c_1c_2}(X) \le \min\{d_{G,c_1}(X), d_{G,c_2}(X)\}.$$

□

Recall that the ordinary (nonequivariant) d–invariant of X, $d(X)$, is defined by

$$d(X) = \min\{j \mid \iota^*(x) \neq 0 \text{ for some } x \in \widetilde{H}^j_{S^1}(X)\}.$$

It is not hard to see that $d(X) = d_{\{e\},1}(X)$, where $\{e\}$ denotes the trivial group and 1 is the generator of $H^0(\mathrm{pt})$.

Proposition 3.10 *Let X be a space of type G–SWF for the trivial extension. Then*

$$d_{G,1}(X) \ge d(X).$$

Proof By the definition of $d_{G,1}(X)$, there exists $x \in \widetilde{H}^{d_{G,1}(X)}_{\overline{G}}(X)$ such that $\iota^*(x) = U^k\tau + \cdots$, where $k = \frac{1}{2}(d_{G,1}(X) - s)$ and s is the level of X. Let $y \in \widetilde{H}^{d_{G,1}(X)}_{S^1}(X)$ be the image of x under the map induced by $S^1 \to \overline{G}$. Then it follows that $\iota^*(y) = U^k\tau \in \widetilde{H}^{d_{G,1}(X)}_{S^1}(X^{S^1})$. In particular, $\iota^*(y) \neq 0$, and hence $d_{G,1}(X) \ge d(X)$ by the definition of $d(X)$. □

Let S^1 act trivially on $\mathbb{R}$ and act by scalar multiplication on $\mathbb{C}$. Let V be a real representation of G. Then $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{R}} V$ and $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ may be regarded as representations of $\overline{G} = S^1 \times G$, where S^1 acts on the first factor and G on the second.

Proposition 3.11 *Let X be a space of type G–SWF for the trivial extension and let V be a finite-dimensional representation of G of type* (1) *or* (2)*, as in Section 3.3. Then for any $c \in H^*_G$,*

$$d_{G,c}(V^+ \wedge X) = d_{G,c}(X) + \dim_{\mathbb{R}}(V).$$

Proof This result follows easily from the Thom isomorphism, together with the fact that in the type (2) case, the $\overline{G}$–equivariant Euler class of V has the form

$$e_{\overline{G}}(V) = U^{\dim(V)} + c_{G,1}(V)U^{\dim(V)-1} + \cdots + c_{G,\dim(V)}(V),$$

where $c_{G,j}(V) \in H^{2j}_G$ denotes the j^{th} G–equivariant Chern class of V. □

If $Z = \Sigma^{-V} X \in \mathfrak{C}(\overline{G})$ is an equivariant spectrum of type G–SWF, we define the $d_{G,c}$–invariant $d_{G,c}(Z)$ of Z to be

$$d_{G,c}(Z) = d_{G,c}(X) - \dim_{\mathbb{R}}(V).$$

This is well defined by Proposition 3.11. We also define a corresponding δ–invariant by setting $\delta_{G,c}(Z) = \frac{1}{2} d_{G,c}(Z)$.

Definition 3.12 Let X and Y be spaces of type G–SWF for the trivial extension of G, where X has level s and Y has level t. Let $f\colon X \to Y$ be an $S^1 \times G$–equivariant map. Consider the restriction

$$f^{S^1}\colon X^{S^1} \to Y^{S^1}$$

of f to the fixed-point set. Note that $\widetilde{H}^*_G(X^{S^1})$ is a free H^*_G–module starting in degree s. Let $\tau_{X^{S^1}}$ denote a generator. Then $\tau_{X^{S^1}}$ is unique up to an element of $\mathbb{F}^*$. Similarly $\widetilde{H}^*_G(Y^{S^1})$ is a free H^*_G–module starting in degree t and we let $\tau_{Y^{S^1}}$ denote a generator. Then there exists a uniquely determined $\mu \in H^{t-s}_G$ such that

$$(f^{S^1})^*(\tau_{Y^{S^1}}) = \mu \tau_{X^{S^1}}.$$

We call $\mu = \deg(f^{S^1})$ the *degree* of f^{S^1}. If we choose different generators for $\widetilde{H}^*_G(X^{S^1})$ or $\widetilde{H}^*_G(Y^{S^1})$, then $\deg(f^{S^1})$ changes by an element of $\mathbb{F}^*$; hence $\deg(f^{S^1})$ is well defined up to multiplication by elements of $\mathbb{F}^*$. If $t < s$, then $\deg(f^{S^1}) = 0$.

Note that suspension does not change the degree of f^{S^1}. Hence we can more generally speak of the degree of f^{S^1} when f is a stable map between spectra of type G–SWF.

Proposition 3.13 *Let $f\colon X \to Y$ be a $\overline{G}$–equivariant map of spaces of type G–SWF for the trivial extension, where X has level s and Y has level t. Let $\mu = \deg(f^{S^1}) \in H^{t-s}_G$ be the degree of f^{S^1}. Then for any nonzero $c \in H^*_G$,*

$$d_{G,c\mu}(X) - s \le d_{G,c}(Y) - t.$$

Proof We prove the result when $c \in H^{|c|}_G$ is homogeneous. The general case follows easily from this. The inclusion of the fixed-point sets gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X^{S^1} \arrow[u, "\iota"] \arrow[r, "f^{S^1}"] & Y^{S^1} \arrow[u, "\iota"'] \end{tikzcd}$$

Consider the induced commutative diagram in equivariant cohomology

$$\begin{tikzcd} \widetilde{H}^*_{\overline{G}}(Y) \arrow[r, "f^*"] \arrow[d, "\iota^*"] & \widetilde{H}^*_{\overline{G}}(X) \arrow[d, "\iota^*"] \\ \widetilde{H}^*_{\overline{G}}(Y^{S^1}) \arrow[r, "(f^{S^1})^*"] & \widetilde{H}^*_{\overline{G}}(X^{S^1}) \end{tikzcd}$$

From the definition of $d_{G,c}(Y)$, there exists some $x \in \widetilde{H}_{\overline{G}}^{d_{G,c}(Y)+|c|}(Y)$ such that

$$\iota^*(x) = cU^k \tau_{Y^{S^1}} + \cdots ,$$

where $k = \frac{1}{2}(d_{G,c}(Y) - t)$. Then by commutativity of the diagram,

$$\iota^*(f^*(x)) = (f^{S^1})^*(\iota^*(x)) = (f^{S^1})^*(cU^k \tau_{Y^{S^1}} + \cdots) = c\mu U^k \tau_{X^{S^1}} + \cdots .$$

It follows that

$$d_{G,c\mu}(X) \le d_{G,c}(Y) + |c| - |c\mu| = d_{G,c}(Y) - |\mu| = d_{G,c}(Y) - t + s.$$

Hence

$$d_{G,c\mu}(X) - s \le d_{G,c}(Y) - t. \qquad \square$$

3.6 Alternative characterisation of $d_{G,c}$

In this section we will give an alternative characterisation of $d_{G,c}$ which does not directly refer to ι^* and is sometimes more convenient for computations.

Let X be a space of type G–SWF for the trivial extension $\overline{G}$. Set $\Lambda_G^* = \widetilde{H}_{\overline{G}}^*(X^{S^1})$. The inclusion of the fixed points $\iota\colon X^{S^1} \to X$ induces a map $\iota^*\colon \widetilde{H}_{\overline{G}}^*(X) \to \Lambda_G^*$. Recall that Λ_G^* is a free $H_G^*[U]$ module of rank 1. Let τ denote a generator of Λ_G^*, so $\Lambda_G^* \cong H_G^*[U]\tau$. Recall that we have a filtration F_j on Λ_G^* given by $F_j = H_G^{*\ge j}\Lambda_G^*$. Similarly, there is a filtration on $\widetilde{H}_{\overline{G}}^*(X)$ which comes from the spectral sequence for equivariant cohomology. We will denote this filtration by $\mathscr{F}_j$. Then $\iota^*(\mathscr{F}_j) \subseteq F_j$ because the inclusion ι induces a map between spectral sequences.

Let $c \in H_G^*$ be a nonzero element of degree $|c|$. Recall that the invariant $d_{G,c}(X)$ is defined by

$$d_{G,c}(X) = \min\{i \mid \iota^*(x) = cU^k\tau \bmod F_{|c|+1} \text{ for some } x \in \widetilde{H}_{\overline{G}}^i(X) \text{ and } k \ge 0\} - |c|.$$

The localisation theorem in equivariant cohomology implies that upon localising with respect to U, ι^* becomes an isomorphism

$$\iota^*\colon U^{-1}\widetilde{H}_{\overline{G}}^*(X) \to U^{-1}\Lambda_G \cong H_G^*[U, U^{-1}]\tau.$$

In particular, there exists an element $\theta \in \widetilde{H}_{\overline{G}}^{2k+\deg(\tau)}(X)$ such that $\iota^*(\theta) = U^l\tau$ for some $l \ge 0$. Fix a choice of such a θ. The localisation isomorphism implies that $\iota^*(x) = 0$ if and only if $U^k x = 0$ for some $k \ge 0$.

Proposition 3.14 *Let $c \in H_G^*$ be a nonzero element of degree $|c|$. Then*

$$d_{G,c}(X) = \min\{i \mid U^n x = cU^k\theta \bmod \mathscr{F}_{|c|+1} \text{ for some } x \in \widetilde{H}_{\overline{G}}^i(X) \text{ and } n, k \ge 0\} - |c|.$$

Proof Let

$$a_{G,c}(X) = \min\{i \mid U^n x = cU^k\theta \bmod \mathscr{F}_{|c|+1} \text{ for some } x \in \widetilde{H}^i_{\overline{G}}(X) \text{ and } n, k \geq 0\} - |c|.$$

Then we need to show that $d_{G,c}(X) = a_{G,c}(X)$. Suppose $x \in \widetilde{H}^{a_{G,c}(X)+|c|}_{\overline{G}}(X)$ satisfies $U^n x = cU^k\theta \bmod \mathscr{F}_{|c|+1}$ for some $n, k \geq 0$. Then

$$U^n \iota^*(x) = cU^k \iota^*(\theta) = cU^{k+l}\tau \bmod F_{|c|+1}.$$

Since U is injective on Λ_G we must have $k + l \geq n$ and we can cancel U^n from both sides to get

$$\iota^*(x) = cU^{k+l-n}\tau \bmod F_{|c|+1}.$$

Hence $d_{G,c}(X) \leq \deg(x) - |c| = a_{G,c}(X)$. Conversely, let $x \in \widetilde{H}^{d_{G,c}(X)+|c|}_{\overline{G}}(X)$ satisfy $\iota^*(x) = cU^k\tau \bmod F_{|c|+1}$ for some $k \geq 0$. Then

$$\iota^*(x) = cU^k\tau + c_1U^{k-1}\tau + c_2U^{k-2}\tau + \cdots + c_k\tau,$$

where $c_i \in H^{|c|+2i}_G$. Since $\iota^*(\theta) = U^l\tau$, it follows that

$$\iota^*(U^l x) = \iota^*(cU^k\theta + c_1U^{k-1}\theta + \cdots + c_k\theta).$$

Next recall that ι^* is an isomorphism after localising with respect to U. Hence if $\iota^*(y_1) = \iota^*(y_2)$, then $U^n y_1 = U^n y_2$ for some $n \geq 0$ and we have

$$U^{n+l}x = cU^{n+k}\theta + c_1U^{n+k-1}\theta + \cdots + c_kU^n\theta = cU^{n+k}\theta \bmod \mathscr{F}_{|c|+1}.$$

From the definition of $a_{G,c}(X)$, it follows that $a_{G,c}(X) \leq \deg(x) - |c| = d_{G,c}(X)$. We have shown $d_{G,c}(X) \leq a_{G,c}(X)$ and $a_{G,c}(X) \leq d_{G,c}(X)$; hence $d_{G,c}(X) = a_{G,c}(X)$. □

3.7 Equivariant d–invariants for rational homology 3–spheres

We return to the setting that Y is a rational homology 3–sphere, G is a finite group acting on Y preserving the isomorphism class of a spin^c–structure $\mathfrak{s}$. Choose a G–invariant metric g and let $G_\mathfrak{s}$ be the S^1–central extension of G obtained by lifting G to the spinor bundle corresponding to $\mathfrak{s}$. Now suppose that $G_\mathfrak{s}$ is a trivial extension; hence $G_\mathfrak{s} \cong \overline{G}$. From the construction of the Conley index, one finds that $SWF(Y, \mathfrak{s}, g)$ is of type G–SWF at level 0.

Definition 3.15 Let G act on Y and let $\mathfrak{s}$ be a G–invariant spin^c–structure. Suppose that the corresponding S^1–extension $G_\mathfrak{s}$ is trivial and choose an isomorphism of extensions $G_\mathfrak{s} \cong \overline{G}$. For any $c \in H^*_G$ we define the invariant $d_{G,c}(Y, \mathfrak{s})$ by

$$d_{G,c}(Y, \mathfrak{s}) = d_{G,c}(SWF(Y, \mathfrak{s}, g)) - 2n(Y, \mathfrak{s}, g).$$

We also set $\delta_{G,c}(Y, \mathfrak{s}) = \frac{1}{2}d_{G,c}(Y, \mathfrak{s})$.

The definition of $d_{G,c}(Y,\mathfrak{s})$ does not depend on the choice of isomorphism $G_{\mathfrak{s}} \cong \overline{G}$ by Remark 3.8. The definition also does not depend on the choice of metric g as a consequence of the relation

$$SWF(Y,\mathfrak{s},g_1) = \Sigma^{SF_{G_{\mathfrak{s}}}(\{D_s\})} SWF(Y,\mathfrak{s},g_0)$$

and the Thom isomorphism.

We only define the invariants $d_{G,c}(Y,\mathfrak{s})$ in the case that $G_{\mathfrak{s}}$ is a trivial extension. This is because the definition of $d_{G,c}(Y,\mathfrak{s})$ uses localisation by U, but $U \in H^*_{S^1}$ does not necessarily extend to a class in $H^*_{G_{\mathfrak{s}}}$, unless $G_{\mathfrak{s}}$ is a trivial extension.

The inclusion $\iota\colon (V^0_\lambda(\mathbb{R}))^+ \to I^\mu_\lambda$ of the S^1–fixed points of the Conley index desuspends to a map $\iota\colon \Sigma^{-V^0_\lambda(\mathbb{C})} S^0 \to SWF(Y,\mathfrak{s},g)$; hence we get a homomorphism

$$\iota^*\colon HSW^*_G(Y,\mathfrak{s}) \to \Lambda^*_G(Y,\mathfrak{s}),$$

where we have set $\Lambda^*_G(Y,\mathfrak{s}) = \widetilde{H}^{*+2n(Y,\mathfrak{s},g)}_{\overline{G}}(\Sigma^{-V^0_\lambda(\mathbb{C})} S^0)$. This is a free $H^*_G[U]$–module and we let τ denote a generator. As in Section 3.5 we filter $\Lambda^*_G(Y,\mathfrak{s})$ by setting $F_j = H^{*\geq j}_G \Lambda^*_G(Y,\mathfrak{s})$. The construction of ι^* and $\Lambda^*_G(Y,\mathfrak{s})$ depend on the choice of metric g, but the construction for any two metrics are related by a canonical homomorphism. The d–invariants of $(Y,\mathfrak{s})$ are given by

$$d_{G,c}(Y,\mathfrak{s}) = \min\{2k+j \mid \iota^*(x) \in F_{|c|} \text{ and } \iota^*(x) = cU^k\tau \bmod F_{|c|+1} \text{ for some } x \in SWF^{j+2k+|c|}_G(Y,\mathfrak{s})\}.$$

Recall that $d(\overline{Y},\mathfrak{s}) = -d(Y,\mathfrak{s})$. On the other hand, the behaviour of the invariants $d_{G,c}(Y,\mathfrak{s})$ under orientation reversal is not so straightforward. For example, it follows from Proposition 3.10 that

$$-d_{G,1}(\overline{Y},\mathfrak{s}) \leq d(Y,\mathfrak{s}) \leq d_{G,1}(Y,\mathfrak{s}). \tag{3-1}$$

In particular, $d_{G,1}(\overline{Y},\mathfrak{s}) = -d_{G,1}(Y,\mathfrak{s})$ can only occur if $d_{G,1}(Y,\mathfrak{s}) = d(Y,\mathfrak{s})$ and $d_{G,1}(\overline{Y},\mathfrak{s}) = d(\overline{Y},\mathfrak{s})$. From (3-1), we also get that

$$d_{G,1}(Y,\mathfrak{s}) + d_{G,1}(\overline{Y},\mathfrak{s}) \geq 0.$$

We will show in Theorem 4.4 that the invariants $d_{G,c}$ satisfy a stronger positivity condition.

Proposition 3.16 *Let G act on Y and let $\mathfrak{s}$ be a G–invariant* spin^c*–structure. Suppose that the corresponding extension $G_{\mathfrak{s}}$ is trivial. If Y is an L–space (with respect to $\mathfrak{s}$ and $\mathbb{F}$), then for all nonzero $c \in H^*_G$,*

$$d_{G,c}(Y,\mathfrak{s}) = d(Y,\mathfrak{s}).$$

Proof If Y is an L–space (with respect to $\mathfrak{s}$ and $\mathbb{F}$) then

$$HSW^*(Y,\mathfrak{s}) \cong \mathbb{F}[U]\theta,$$

where θ has degree $d(Y,\mathfrak{s})$. From Theorem 3.5, there exists a class $\hat{\theta} \in HSW^*_G(Y,\mathfrak{s})$ which maps to θ under the forgetful map $HSW^*_G(Y,\mathfrak{s}) \to HSW^*(Y,\mathfrak{s})$ and we have that $HSW^*_G(Y,\mathfrak{s})$ is a free $H^*_G[U]$–module generated by $\hat{\theta}$. We must also have that $\iota^*(\hat{\theta}) = U^k\tau \bmod F_1$ for some $k \geq 0$, where τ is a generator of $\Lambda_G(Y,\mathfrak{s})$. This holds because F_1 is the kernel of the forgetful map $\Lambda_G(Y,\mathfrak{s}) \to \Lambda_{\{1\}}(Y,\mathfrak{s})$. So for any nonzero $c \in H^j_G$ we have that $\iota^*(c\hat{\theta}) = c\iota^*(\hat{\theta}) = cU^k\tau \bmod F_{1+|c|}$; hence

$$d_{G,c}(Y,\mathfrak{s}) \leq \deg(c\hat{\theta}) - j = \deg(\hat{\tau}) = d(Y,\mathfrak{s}).$$

That is, $d_{G,c}(Y,\mathfrak{s}) \leq d(Y,\mathfrak{s})$ for all nonzero homogeneous c. On the other hand it is clear that there is no class of lower degree in $HSW^*_G(Y,\mathfrak{s})$ which maps under ι^* to a class of the form $cU^{k'}\tau \bmod F_{1+|c|}$. Hence $d_{G,c}(Y,\mathfrak{s}) = d(Y,\mathfrak{s})$ for all nonzero homogeneous c. Clearly the result extends to all nonzero c. □

4 Behaviour under cobordisms

We show that equivariant cobordisms of rational homology 3–spheres induce maps on equivariant Seiberg–Witten–Floer cohomology. We follow the construction of Manolescu [49], incorporating the corrections due to Khandhawit [37]. Since our construction is a straightforward extension of that of Manolescu and Khandhawit, differing only in the replacement of S^1 by the larger group $G_\mathfrak{s}$, we will be brief.

4.1 Finite-dimensional approximation

Let W be a compact, oriented smooth 4–manifold with boundary $Y = \partial W$ a disjoint union of rational homology spheres $Y = \bigcup_j Y_j$. Assume further that $b_1(W) = 0$ and that W is connected. If $\mathfrak{s}$ is a spin^c–structure on W, then the restriction of $\mathfrak{s}$ to Y determines a spin^c–structure $\mathfrak{s}|_Y$ on Y. Since the boundary of W is a union of rational homology 3–spheres, we have $H^2(W,\partial W;\mathbb{R}) \cong H^2(W;\mathbb{R})$ and by Poincaré–Lefschetz duality we obtain a nondegenerate intersection form on $H^2(W;\mathbb{R})$. Given a metric g on W which is isometric to a product metric in a collar neighbourhood of ∂W, we let $H^+(W)$ denote the space of self-dual L^2–harmonic 2–forms on the cylindrical end manifold $\widehat{W}$ obtained from W by attaching half-infinite cylinders $[0,\infty)\times Y$ to W. It follows from [8, Proposition 4.9] that the natural map $H^+(W) \to H^2(W;\mathbb{R})$ is injective and identifies $H^+(W)$ with a maximal positive definite subspace of $H^2(W;\mathbb{R})$.

Suppose now that G acts smoothly and orientation-preservingly on W and that this action sends each connected component of ∂W to itself. Hence by restriction G acts on each Y_i by orientation-preserving diffeomorphisms. Assume further that G preserves the isomorphism class of a spin^c–structure $\mathfrak{s}$ on W. Set $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$. Then the action of G on Y_i preserves $\mathfrak{s}_i$. Similar to Section 3.2 we obtain an S^1–extension $G_\mathfrak{s}$ of G. Restricting to Y_i, we obtain an isomorphism of extensions $G_\mathfrak{s} \cong G_{\mathfrak{s}_i}$. Hence if $G_\mathfrak{s}$ is split, then it follows that each of the extensions $G_{\mathfrak{s}_i}$ is also split. Moreover a splitting of $G_\mathfrak{s}$ determines corresponding splittings of each $G_{\mathfrak{s}_i}$.

Choose a G–invariant metric g on W which is isometric to a product $(-\epsilon,0]\times Y$ in some equivariant collar neighbourhood of Y (see [34, Theorem 3.5] for existence of equivariant collar neighbourhoods). To see that such a metric exists, first choose a G–invariant metric g_Y on Y. Then choose an arbitrary metric g' on W which equals $(dt)^2 + g_Y$ in some equivariant collar neighbourhood $(-\epsilon, 0] \times Y$. Then let g be obtained from g' by averaging over G. Let $S^{\pm}$ denote the spinor bundles on W corresponding to $\mathfrak{s}$. We note here that under these assumptions G preserves the subspace $H^+(W)$ of $H^2(W;\mathbb{R})$ defined by g. Let $\Omega^1_g(W)$ denote the space of 1–forms on W in double Coulomb gauge with respect to Y [37, Definition 1]. This space is easily seen to be preserved by the action of G on 1–forms. The double Coulomb gauge condition ensures that if $a \in \Omega^1_g(W)$ and $\phi \in \Gamma(S^+)$, then $(a,\phi)|_{Y_j}$ lies in the global Coulomb slice corresponding to Y_j. Let us temporarily assume that $Y = \partial W$ is connected. Let $\widehat{A}$ be a spin^c–connection on W such that in a collar neighbourhood of Y it equals the pullback of A_0. Using the same argument as in Section 3.2, we can assume that $\widehat{A}$ is $G_{\mathfrak{s}}$–invariant. Then using $\widehat{A}$ as a reference connection, we obtain a map which may be thought of as the Seiberg–Witten equations on W together with boundary conditions,

$$SW^{\mu}\colon i\Omega^1_g(W) \oplus \Gamma(S^+) \to i\Omega^2_+(W) \oplus \Gamma(S^-) \oplus V^{\mu}_{-\infty},$$
$$(a,\phi) \mapsto (F^+_{\widehat{A}+a} - \sigma(\phi,\phi), D_{\widehat{A}+a}(\phi), p^{\mu}(a,\phi)|_Y),$$

where p^{μ} is the orthogonal projection from V to $V^{\mu}_{-\infty}$. Taking a finite-dimensional approximation as described in [37; 49], one obtains a map

$$\Psi_{\mu,\lambda,U,U'}\colon (U')^+ \to (U^+) \wedge I^{\mu}_{\lambda},$$

where $U' \subset i\Omega^1_g(W) \oplus \Gamma(S^+)$ and $U \subset i\Omega^2_+(W) \oplus \Gamma(S^-)$ are finite-dimensional G–invariant subspaces which satisfy

$$U \oplus V^0_{\lambda} \oplus \mathrm{Ker}(L^0) \cong U' \oplus \mathrm{Coker}(L^0) \tag{4-1}$$

and L^0 is a Fredholm linear operator defined in [49, Section 9]. Since $SWF(Y,\mathfrak{s},g) = \Sigma^{-V^0_{\lambda}} I^{\mu}_{\lambda}$, we can rewrite the map $\Psi_{\mu,\lambda,U,U'}$ as

$$\Psi_{\mu,\lambda,U,U'}\colon (U')^+ \to (U)^+ \wedge (V^0_{\lambda})^+ \wedge SWF(Y,\mathfrak{s},g).$$

Taking the smash product with $\mathrm{Ker}(L^0)$ and using (4-1), we see that $\Psi_{\mu,\lambda,U,U'}$ is stably equivalent to a map

$$f\colon \mathrm{Ker}(L^0)^+ \to \mathrm{Coker}(L^0)^+ \wedge SWF(Y,\mathfrak{s},g).$$

The real part of L^0 has zero kernel and cokernel isomorphic to $H^+(W)$. The complex part of L^0 can be identified with the Dirac operator $D_{\widehat{A}}$ with Atiyah–Patodi–Singer (APS) boundary conditions. Thus

$$\mathrm{Ker}(L^0) \cong \mathrm{Ker}_{\mathrm{APS}}(D^+_{\widehat{A}}), \quad \mathrm{Coker}(L^0) \cong H^+(W) \oplus \mathrm{Coker}_{\mathrm{APS}}(D^+_{\widehat{A}}),$$

where $\mathrm{Ker}_{\mathrm{APS}}(D^+_{\widehat{A}})$ and $\mathrm{Coker}_{\mathrm{APS}}(D^+_{\widehat{A}})$ denote the kernel and cokernel of $D^+_{\widehat{A}}$ with APS boundary conditions. Hence we obtain a $G_{\mathfrak{s}}$–equivariant map

$$f\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\widehat{A}})^+ \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\widehat{A}})^+ \wedge SWF(Y,\mathfrak{s},g).$$

Note that f is only a map in the stable sense; that is, f is a morphism in the category $\mathfrak{C}(G_{\mathfrak{s}})$.

Recall that the S^1–fixed-point set of I^{μ}_{λ} is $V^0_{\lambda}(\mathbb{R})^+$. The inclusion $(V^0_{\lambda}(\mathbb{R})) \to I^{\mu}_{\lambda}$ of the S^1–fixed points desuspends to a map $\iota\colon S^0 \to SWF(Y,\mathfrak{s},g)$. By restricting to S^1–fixed points we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \arrow[r, "f"] & (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(Y,\mathfrak{s},g) \\
S^0 \arrow[u] \arrow[r, "f^{S^1}"] & (H^+(W))^+ \arrow[u]
\end{tikzcd}$$

Using that the Seiberg–Witten equations reduce to linear equations on the S^1–fixed-point set, one finds that $f^{S^1}\colon S^0 \to (H^+(W))^+$ is the obvious map given by the one-point compactification of the inclusion $\{0\} \to H^+(W)$. Thus according to Definition 3.12, f^{S^1} has degree equal to $e(H^+(W))$, the image of the equivariant Euler class of $H^+(W)$ in $H^*_G(\mathrm{pt};\mathbb{F})$. For instance, if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ then $e(H^+(W))$ is the $b_+(W)^{\text{th}}$ equivariant Stiefel–Whitney class. We will refer to $e(H^+(W))$ as the $\mathbb{F}$*–Euler class of* $H^+(W)$.

So far we have restricted to the case that the boundary ∂W is connected. More generally, if $\partial W = \bigcup_j Y_j$ is a union of rational homology 3–spheres then much the same construction applies. The Conley index I^{μ}_{λ} is now given by the smash product of the Conley indices of each component; hence f is now a map of the form

$$f\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge \bigwedge_j SWF(Y_j,\mathfrak{s}_j,g_j).$$

We still have that the degree of f^{S^1} is $e(H^+(W))$.

4.2 Equivariant Frøyshov inequality

In this section we prove an equivariant generalisation of Frøyshov's inequality [28].

Theorem 4.1 *Let W be a smooth, compact, oriented 4–manifold with boundary and with $b_1(W)=0$. Suppose that G acts smoothly on W preserving the orientation and a spin^c–structure $\mathfrak{s}$. Suppose that the extension $G_{\mathfrak{s}}$ is trivial. Suppose each component of ∂W is a rational homology 3–sphere and that G sends each component of ∂W to itself. Let $e \in H^{b_+(W)}_G$ be the $\mathbb{F}$–Euler class of any G–invariant maximal positive definite subspace of $H^2(W;\mathbb{R})$. Let $c \in H^*_G$ and suppose that $ce \neq 0$.*

(1) *If $\partial W = Y$ is connected, then*

$$\delta(W,\mathfrak{s}) \le \delta_{G,c}(Y,\mathfrak{s}|_Y) \quad \text{and} \quad \delta_{G,ce}(\overline{Y},\mathfrak{s}|_Y) \le \delta(\overline{W},\mathfrak{s}),$$

where we have defined

$$\delta(W,\mathfrak{s}) = \tfrac{1}{8}(c_1(\mathfrak{s})^2 - \sigma(W)).$$

(2) *If $\partial W = \overline{Y}_1 \cup Y_2$ has two connected components, then*

$$\delta_{G,ce}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) \le \delta_{G,c}(Y_2,\mathfrak{s}|_{Y_2}).$$

Proof We will give the proof in the case that W is connected. The general case follows easily from this by applying the theorem to each component of W. To simplify notation we will write $\mathfrak{s}$ instead of $\mathfrak{s}|_Y$ and write g instead of $g|_Y$. In case (1), $\partial W = Y$ is connected.

As in Section 4.1, choosing suitable metrics and reference connections we obtain a stable map

$$f\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(Y,\mathfrak{s},g)$$

such that the degree of f^{S^1} is e. Applying Proposition 3.13 to f, we obtain

$$2\dim_{\mathbb{C}}(\mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})) + b_+(W) \le b_+(W) + 2\dim_{\mathbb{C}}(\mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})) + d_c(Y,\mathfrak{s}) + 2n(Y,\mathfrak{s},g)$$

which simplifies to

$$\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}}) \le \tfrac{1}{2} d_c(Y,\mathfrak{s}) + n(Y,\mathfrak{s},g) = \delta_c(Y,\mathfrak{s}) + n(Y,\mathfrak{s},g).$$

Combined with (2-3) we get $\delta(W,\mathfrak{s}) \le \delta_c(Y,\mathfrak{s})$.

Next recall from Section 2.7 the duality map

$$\varepsilon\colon SWF(Y,\mathfrak{s},g) \wedge SWF(\overline{Y},\mathfrak{s},g) \to S^{-k(D)\mathbb{C}}$$

where $k(D) = \dim_{\mathbb{C}}(\mathrm{Ker}(D))$. By the definition of equivariant duality,

$$\varepsilon^{S^1}\colon SWF(Y,\mathfrak{s},g)^{S^1} \wedge SWF(\overline{Y},\mathfrak{s},g)^{S^1} \to S^0$$

is a nonequivariant duality. It follows that ε^{S^1} has degree 1. Taking the map f, suspending by $SWF(\overline{Y},\mathfrak{s},g)$ and composing with ε, we obtain a stable map

$$h\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(\overline{Y},\mathfrak{s},g) \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge S^{-k(D)\mathbb{C}}$$

such that the degree of h^{S^1} is e. Applying Proposition 3.13 to h, we obtain

$$\begin{aligned} &2\dim_{\mathbb{C}}(\mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}}) + d_{G,ce}(\overline{Y},\mathfrak{s},g) + 2n(\overline{Y},\mathfrak{s},g) + b_+(W) \\ &\qquad\qquad \le b_+(W) + 2\dim_{\mathbb{C}}(\mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}}) - 2k(D) \end{aligned}$$

which simplifies to

$$\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}}) + \delta_{G,ce}(\overline{Y},\mathfrak{s},g) + n(\overline{Y},\mathfrak{s},g) \le -k(D).$$

Using (2-3) and (2-4), we obtain $\delta(W,\mathfrak{s}) \le -\delta_{G,ce}(\overline{Y},\mathfrak{s})$, or equivalently $\delta_{G,ce}(\overline{Y},\mathfrak{s}|_Y) \le \delta(\overline{W},\mathfrak{s})$.

The proof of case (2) is similar. We start with the map

$$f\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(\overline{Y}_1,\mathfrak{s},g) \wedge SWF(Y_2,\mathfrak{s},g).$$

Suspending by $SWF(Y_1,\mathfrak{s},g)$ and applying the duality map corresponding to Y_1 we obtain a map

$$h\colon \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(Y_1,\mathfrak{s},g) \to (H^+(W))^+ \wedge \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(Y_2,\mathfrak{s},g) \wedge S^{-k(D_1)\mathbb{C}},$$

where $k(D_1)$ is the dimension of the kernel of the Dirac operator on Y_1. Applying Proposition 3.13 to this map and simplifying, we obtain the inequality $\delta_{G,ce}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) \le \delta_{G,c}(Y_2,\mathfrak{s}|_{Y_2})$. □

Definition 4.2 Let $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$ be rational homology 3–spheres equipped with spin^c–structures. Suppose that G acts orientation-preservingly on Y_1 and Y_2 and preserves the spin^c–structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$. A *G–equivariant rational homology cobordism from* $(Y_1, \mathfrak{s}_1)$ *to* $(Y_2, \mathfrak{s}_2)$ is a rational homology cobordism W from Y_1 to Y_2 such that the G–action and spin^c–structure $\mathfrak{s}_1 \cup \mathfrak{s}_2$ on ∂W extend over W. We say that $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$ are *G–equivariantly rational homology cobordant* if there exists a G–equivariant rational homology cobordism from $(Y_1, \mathfrak{s}_1)$ to $(Y_2, \mathfrak{s}_2)$.

Similarly we define the notion of a *G–equivariant integral homology cobordism* and say that two integral homology 3–spheres Y_1 and Y_2 on which G acts are *G–equivariantly integral homology cobordant* if there is a G–equivariant integral homology cobordism from Y_1 to Y_2. Note that since Y_1 and Y_2 are integral homology 3–spheres, they have unique spin^c–structures which are automatically G–invariant and any G–equivariant integral homology cobordism from Y_1 to Y_2 has a unique spin^c–structure which restricts on the boundary to the unique spin^c–structures on Y_1 and Y_2.

Corollary 4.3 *The G–equivariant δ–invariants $\delta_{G,c}(Y, \mathfrak{s})$ are invariant under G–equivariant rational homology cobordism; that is, if $(W, \mathfrak{s})$ is a G–equivariant rational homology cobordism from $(Y_1, \mathfrak{s}_1)$ to $(Y_2, \mathfrak{s}_2)$ and if the extensions $G_{\mathfrak{s}_1}$ and $G_{\mathfrak{s}_2}$ are trivial, then $\delta_{G,c}(Y_1, \mathfrak{s}_1) = \delta_{G,c}(Y_2, \mathfrak{s}_2)$ for all $c \in H^*_G$.*

Proof Since W is a rational homology cobordism, we have $H^2(W; \mathbb{R}) = 0$. So $\delta(W, \mathfrak{s}) = 0$ and $e = e(H^+(W)) = 1$. Therefore Theorem 4.1 gives

$$\delta_{G,c}(Y_1, \mathfrak{s}_1) \le \delta_{G,c}(Y_2, \mathfrak{s}_2).$$

Similarly, viewing $\overline{W}$ as a G–equivariant rational homology cobordism from Y_2 to Y_1, we get

$$\delta_{G,c}(Y_2, \mathfrak{s}_2) \le \delta_{G,c}(Y_1, \mathfrak{s}_1).$$

Hence $\delta_{G,c}(Y_1, \mathfrak{s}_1) = \delta_{G,c}(Y_2, \mathfrak{s}_2)$. □

Theorem 4.4 *Let Y be a rational homology 3–sphere, G a finite group acting on Y preserving orientation and the isomorphism class of a spin^c–structure $\mathfrak{s}$ and suppose that $G_{\mathfrak{s}}$ is a trivial extension. Then for any $c_1, c_2 \in H^*_G$ with $c_1 c_2 \neq 0$,*

$$\delta_{c_1}(Y) + \delta_{c_2}(\overline{Y}) \ge 0.$$

Proof The proof is similar to that of Theorem 4.1. Let $W = [0,1] \times Y$ be the trivial cobordism from Y to itself. Choosing suitable metrics and reference connections we obtain a stable map

$$f : \mathrm{Ker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \to \mathrm{Coker}_{\mathrm{APS}}(D^+_{\hat{A}})^+ \wedge SWF(\overline{Y}, \mathfrak{s}, g) \wedge SWF(Y, \mathfrak{s}, g).$$

Note that $H^+(W) = \{0\}$ and hence $e(H^+(W)) = 1$. Applying Proposition 3.13 to this map we see that for any $c_1, c_2 \in H^*_G$ with $c_1 c_2 \neq 0$,

$$\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}}) \le \delta_{G, c_1 c_2}(SWF(\overline{Y}, \mathfrak{s}, g) \wedge SWF(Y, \mathfrak{s}, g)).$$

From the definition of the δ–invariant it is clear that

$$\delta_{G,c_1c_2}(SWF(\overline{Y},\mathfrak{s},g)\wedge SWF(Y,\mathfrak{s},g)) \le \delta_{G,c_1}(SWF(\overline{Y},\mathfrak{s},g)) + \delta_{G,c_2}(SWF(Y,\mathfrak{s},g)),$$

and hence

$$\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}}) \le \delta_{G,c_1}(\overline{Y},\mathfrak{s}) + \delta_{G,c_2}(Y,\mathfrak{s}) + n(\overline{Y},\mathfrak{s},g) + n(Y,\mathfrak{s},g).$$

On the other hand, for $W=[0,1]\times Y$, equation (2-3) reduces to

$$\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}}) = n(\overline{Y},\mathfrak{s},g) + n(Y,\mathfrak{s},g).$$

Hence we obtain $0 \le \delta_{G,c_1}(\overline{Y},\mathfrak{s}) + \delta_{G,c_2}(Y,\mathfrak{s})$. □

4.3 Induced cobordism maps

In this section we show that equivariant cobordisms induce maps on equivariant Seiberg–Witten–Floer cohomology.

Theorem 4.5 *Let W be a smooth, compact, oriented 4–manifold with boundary and with $b_1(W)=0$. Suppose that G acts smoothly on W preserving the orientation and a spin^c–structure $\mathfrak{s}$. Suppose that $\partial W = \overline{Y}_1 \cup Y_2$ where Y_1 and Y_2 are rational homology 3–spheres and set $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$. Suppose G sends Y_i to itself. Then there is a morphism of graded $H^*_{G_\mathfrak{s}}$–modules*

$$SW_G(W,\mathfrak{s})\colon HSW^*_G(Y_2,\mathfrak{s}_2) \to HSW_G^{*+b_+(W)-2\delta(W,\mathfrak{s})}(Y_1,\mathfrak{s}_1)$$

such that the diagram

$$\begin{tikzcd} HSW^*_G(Y_2,\mathfrak{s}_2) \arrow[r, "SW_G(W{,}\mathfrak{s})"] \arrow[d] & HSW_G^{*+b_+(W)-2\delta(W,\mathfrak{s})}(Y_1,\mathfrak{s}_1) \arrow[d] \\ HSW^*(Y_2,\mathfrak{s}_2) \arrow[r, "SW(W{,}\mathfrak{s})"] & HSW^{*+b_+(W)-2\delta(W,\mathfrak{s})}(Y_1,\mathfrak{s}_1) \end{tikzcd}$$

commutes, where the vertical arrows are the forgetful maps to nonequivariant Seiberg–Witten–Floer cohomology and $SW(W,\mathfrak{s})$ is the morphism of Seiberg–Witten–Floer cohomology groups induced by $(W,\mathfrak{s})$.

Proof We give the proof in the case W is connected. The general case follows by a similar argument. As in the proof of Theorem 4.1, choosing suitable metrics and reference connections, we obtain a stable map

$$h\colon S^{\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}})} \wedge SWF(Y_1,\mathfrak{s}_1,g_1) \to (H^+(W))^+ \wedge SWF(Y_2,\mathfrak{s}_2,g_2) \wedge S^{-k(D_1)\mathbb{C}},$$

where $k(D_1)$ is the dimension of the kernel of the Dirac operator on Y_1. The induced map in equivariant cohomology takes the form

$$h^*\colon \widetilde{H}^j_{G_\mathfrak{s}}((H^+(W))^+ \wedge SWF(Y_2,\mathfrak{s}_2,g_2)\wedge S^{-k(D_1)\mathbb{C}}) \to \widetilde{H}^j_{G_\mathfrak{s}}(S^{\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}})} \wedge SWF(Y_1,\mathfrak{s}_1,g_1)).$$

Using the Thom isomorphism, this is equivalent to

$$h^*\colon \widetilde{H}^{j-b_+(W)+2k(D_1)}_{G_{\mathfrak{s}}}(SWF(Y_2,\mathfrak{s}_2,g_2)) \to \widetilde{H}^{j-2\mathrm{ind}_{\mathrm{APS}}(D^+_{\hat{A}})}_{G_{\mathfrak{s}}}(SWF(Y_1,\mathfrak{s}_1,g_1)).$$

By (2-3), $\mathrm{ind}_{\mathrm{APS}} = \delta(W,\mathfrak{s}) + n(Y_2,\mathfrak{s}_2,g_2) + n(\overline{Y}_1,\mathfrak{s}_1,g_1)$ and by (2-4), $n(Y_1,\mathfrak{s}_1,g_1)+n(\overline{Y}_1,\mathfrak{s}_1,g_1) = -k(D_1)$. Replacing j by $j+b_+(W)-2k(D_1)+2n(Y_2,\mathfrak{s}_2,g_2)$, we see that h^* takes the form

$$h^*\colon \widetilde{H}^{j+2n(Y_2,\mathfrak{s}_2,g_2)}_{G_{\mathfrak{s}}}(SWF(Y_2,\mathfrak{s}_2,g_2)) \to \widetilde{H}^{j+b_+(W)-2\delta(W,\mathfrak{s})+2n(Y_1,\mathfrak{s}_1,g_1)}_{G_{\mathfrak{s}}}(SWF(Y_1,\mathfrak{s}_1,g_1)).$$

Then since $HSW^*_G(Y_i,\mathfrak{s}_i) = \widetilde{H}^{*+2n(Y_i,\mathfrak{s}_i,g_i)}_{G_{\mathfrak{s}}}(SWF(Y_i,\mathfrak{s}_i,g_i))$, we see that h^* is equivalent to a map

$$SW_G(W,\mathfrak{s})\colon HSW^*_G(Y_2,\mathfrak{s}_2) \to HSW^{*+b_+(W)-2\delta(W,\mathfrak{s})}_G(Y_1,\mathfrak{s}_1).$$

Since this is a map of equivariant cohomologies induced by an equivariant map of spaces, it follows that $SW_G(W,\mathfrak{s})$ is a morphism of graded $H^*_{G_{\mathfrak{s}}}$–modules. Restricting to the subgroup $S^1 \to G_{\mathfrak{s}}$, we obtain the commutative diagram in the statement of the theorem. □

5 The case $G = \mathbb{Z}_p$

In this section we specialise to the case $G = \mathbb{Z}_p$ and $\mathbb{F} = \mathbb{Z}_p$, where p is a prime number. Then for $p = 2$ we have $H^*_G \cong \mathbb{F}[Q]$, where $\deg(Q) = 1$, and if p is odd we have $H^*_G \cong \mathbb{F}[R,S]/(R^2)$, where $\deg(R) = 1$ and $\deg(S) = 2$. Suppose $G = \langle\tau\rangle$ acts smoothly and orientation-preservingly on a rational homology 3–sphere Y, preserving a spin^c–structure $\mathfrak{s}$. The action of G is equivalent to giving an orientation-preserving diffeomorphism $\tau\colon Y \to Y$ such that $\tau^p = \mathrm{id}$ and $\tau^*(\mathfrak{s}) = \mathfrak{s}$. Choose a lift $\tau' \in G_{\mathfrak{s}}$ of τ. Then $(\tau')^p = \zeta$ for some $\zeta \in S^1$. Replacing τ' by $\tilde{\tau} = \zeta^{-1/p}\tau'$, where $\zeta^{1/p}$ is a p^{th} root of ζ, we see that $\tilde{\tau}^p = \mathrm{id}$. Hence $G_{\mathfrak{s}}$ is a trivial extension.

5.1 δ–invariants

Definition 5.1 If $p = 2$, then for any integer $j \geq 0$, we define $d_j(Y,\mathfrak{s},\tau,2) = d_{\mathbb{Z}_2,Q^j}(Y,\mathfrak{s})$. If p is odd, then for any integer $j \geq 0$, we define $d_j(Y,\mathfrak{s},\tau,p) = d_{\mathbb{Z}_p,S^j}(Y,\mathfrak{s})$. We also set $\delta_j(Y,\mathfrak{s},\tau,p) = \frac{1}{2}d_j(Y,\mathfrak{s},\tau,p)$. When p and τ are understood we will omit them from the notation and simply write $d_j(Y,\mathfrak{s})$ and $\delta_j(Y,\mathfrak{s})$.

In the case p is odd, one may also consider the invariants $d_{\mathbb{Z}_p,RS^j}(Y,\mathfrak{s})$. For simplicity we will not consider these invariants.

Theorem 5.2 *We have the following properties*:

(1) $\delta_0(Y,\mathfrak{s}) \geq \delta(Y,\mathfrak{s})$;

(2) $\delta_{j+1}(Y,\mathfrak{s}) \leq \delta_j(Y,\mathfrak{s})$ *for all* $j \geq 0$;

(3) *the sequence* $\{\delta_j(Y,\mathfrak{s})\}_{j\geq 0}$ *is eventually constant*;

(4) $\delta_j(Y,\mathfrak{s}) + \delta_j(\overline{Y},\mathfrak{s}) \geq 0$ *for all* $j \geq 0$;

(5) *if* Y *is an* L*–space, then* $\delta_j(Y,\mathfrak{s}) = \delta(Y,\mathfrak{s})$ *for all* $j \geq 0$.

Proof Item (1) is a restatement of Proposition 3.10. Item (2) follows from Proposition 3.9, taking $c_1 = Q^j$ and $c_2 = Q$ in the case $p = 2$, and $c_1 = S^j$ and $c_2 = S$ in the case p is odd. Item (4) is a special case of Theorem 4.4. For (3), first note that the difference $\delta_j(Y,\mathfrak{s}) - \delta_{j+1}(Y,\mathfrak{s})$ is always an integer because $\delta_{G,c}(Y,\mathfrak{s}) + n(Y,\mathfrak{s},g) \in \mathbb{Z}$ for any metric g. From (2) and (4) and the fact that $n(Y,\mathfrak{s},g) + n(\overline{Y},\mathfrak{s},g) \in \mathbb{Z}$, it follows that $\delta_j(Y,\mathfrak{s}) + \delta_j(\overline{Y},\mathfrak{s})$ is a nonnegative, decreasing, integer-valued function. Hence the value of $\delta_j(Y,\mathfrak{s}) + \delta_j(\overline{Y},\mathfrak{s})$ must eventually be constant. Using (2) again, it follows that $\delta_j(Y,\mathfrak{s})$ and $\delta_j(\overline{Y},\mathfrak{s})$ are eventually constant. Item (5) is a restatement of Proposition 3.16. □

Next, we specialise Theorem 4.1 to the case $G = \mathbb{Z}_p$.

Theorem 5.3 *Let* W *be a smooth, compact, oriented* 4*–manifold with boundary and with* $b_1(W) = 0$. *Suppose that* $\tau\colon W \to W$ *is an orientation-preserving diffeomorphism of order* p *and* $\mathfrak{s}$ *is a* spin^c*–structure preserved by* τ. *Suppose each component of* ∂W *is a rational homology* 3*–sphere and that* τ *sends each component of* ∂W *to itself. Suppose that the subspace of* $H^2(W;\mathbb{R})$ *fixed by* τ *is negative definite. Then for all* $j \geq 0$:

(1) *If* $\partial W = Y$ *is connected, then*

$$\delta(W,\mathfrak{s}) \leq \delta_j(Y,\mathfrak{s}|_Y) \qquad \text{and} \qquad \delta(\overline{W},\mathfrak{s}) \geq \begin{cases} \delta_{j+b_+(W)}(\overline{Y},\mathfrak{s}|_Y) & \text{if } p = 2, \\ \delta_{j+b_+(W)/2}(\overline{Y},\mathfrak{s}|_Y) & \text{if } p \text{ is odd.} \end{cases}$$

(2) *If* $\partial W = \overline{Y}_1 \cup Y_2$ *has two connected components, then*

$$\delta_j(Y_2,\mathfrak{s}|_{Y_2}) \geq \begin{cases} \delta_{j+b_+(W)}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) & \text{if } p = 2, \\ \delta_{j+b_+(W)/2}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) & \text{if } p \text{ is odd.} \end{cases}$$

Proof Let $H^+(W)$ denote a τ–invariant maximal positive definite subspace of $H^2(W;\mathbb{R})$ (which always exists because $G = \langle\tau\rangle$ is finite) and let e denote the image of the Euler class of $H^+(W)$ in $H^*_{\mathbb{Z}_p}$. To deduce the result from Theorem 4.1, we just need to check that $eQ^j \neq 0$ for all $j \geq 0$ if $p = 2$, and $eS^j \neq 0$ for all $j \geq 0$ if p is odd.

In the case $p = 2$, e is the top Stiefel–Whitney class of $H^+(W)$, which is easily seen to be $Q^{b_+(W)}$ because our assumption that the subspace of $H^2(W;\mathbb{R})$ fixed by τ is negative definite implies that τ acts as -1 on $H^+(W)$. Then clearly $eQ^j \neq 0$ for all $j \geq 0$.

Now suppose p is odd. Let L_i be the complex 1–dimensional representation on which τ acts as multiplication by ζ^i, $\zeta = e^{2\pi i/p}$. Any finite-dimensional real representation of G is the direct sum of a trivial representation and copies of the underlying real representations of the L_i for $1 \leq i \leq p-1$. The hypothesis that the subspace of $H^2(W;\mathbb{R})$ fixed by τ is negative definite means that as a representation of G, $H^+(W)$ contains no trivial summand. Hence $H^+(W)$ admits a complex structure such that

$H^+(W) \cong \bigoplus_{i=1}^{p-1} L_i^{m_i}$ for some integers $m_i \geq 0$. The Euler class of $H^+(W)$ is equal to its top Chern class. Under the map $H^2(\mathbb{Z}_p;\mathbb{Z}) \to H^2(\mathbb{Z}_p;\mathbb{Z}_p)$ one finds that $c_1(L_i)$ gets sent to iS. Hence

$$e = \prod_{i=1}^{p-1} (iS)^{m_i},$$

from which it is clear that $eS^j \neq 0$ for all $j \geq 0$. □

Remark 5.4 Suppose that p is odd. Then as in the proof of Theorem 5.3, $H^+(V)$ admits a complex structure. So if p is odd and the assumptions of Theorem 5.3 hold, then $b_+(W)$ must be even.

To keep notation simple, we will henceforth set $b'_\pm(W) = b_\pm(W)$ if $p = 2$ and $b'_\pm(W) = \frac{1}{2}b_\pm(W)$ if p is odd. Then (1) and (2) of Theorem 5.3 can be written more uniformly as

$$\delta(W,\mathfrak{s}) \leq \delta_j(Y,\mathfrak{s}|_Y) \quad \text{and} \quad \delta_{j+b'_+(W)}(\overline{Y},\mathfrak{s}|_Y) \leq \delta(\overline{W},\mathfrak{s}),$$

and

$$\delta_{j+b'_+(W)}(Y_1,\mathfrak{s}|_{Y_1}) + \delta(W,\mathfrak{s}) \leq \delta_j(Y_2,\mathfrak{s}|_{Y_2}).$$

Corollary 5.5 *Let W be a smooth, compact, oriented 4–manifold with boundary and with $b_1(W) = 0$. Suppose that $\tau\colon W \to W$ is an orientation-preserving diffeomorphism of order p and $\mathfrak{s}$ is a spin^c–structure preserved by τ. Suppose that $Y = \partial W$ is a rational homology 3–sphere. Suppose that the subspace of $H^2(W;\mathbb{R})$ fixed by τ is zero. Then*

(1) $\delta_j(Y,\mathfrak{s}|_Y) \geq -\frac{1}{8}\sigma(W)$ *for all* $j \geq 0$ *and* $\delta_j(Y,\mathfrak{s}|_Y) = -\frac{1}{8}\sigma(W)$ *for* $j \geq b'_-(W)$;

(2) $\delta_j(\overline{Y},\mathfrak{s}|_Y) \geq \frac{1}{8}\sigma(W)$ *for all* $j \geq 0$ *and* $\delta_j(\overline{Y},\mathfrak{s}|_Y) = \frac{1}{8}\sigma(W)$ *for* $j \geq b'_+(W)$.

Proof It suffices to prove (1) since (2) follows by reversing orientation on W and Y. Since $\mathbb{Z}_p$ preserves $\mathfrak{s}$, it follows that the image of $c_1(\mathfrak{s})$ in real cohomology lies in the subspace of $H^2(W;\mathbb{R})$ fixed by $\mathbb{Z}_p$. By assumption this space is zero; hence $c_1(\mathfrak{s}) = 0$ in real cohomology and hence $c_1(\mathfrak{s})^2 = 0$. So $\delta(W,\mathfrak{s}) = -\frac{1}{8}\sigma(W)$. Then from Theorem 5.3(1), we get $\delta_j(Y,\mathfrak{s}) \geq -\frac{1}{8}\sigma(W)$ for all $j \geq 0$. Reversing orientation on W and Y an applying Theorem 5.3(1), we also get that $\delta_{j+b'_-(W)}(Y,\mathfrak{s}) \leq -\frac{1}{8}\sigma(W)$ for all $j \geq 0$, or equivalently, $\delta_j(Y,\mathfrak{s}) \leq -\frac{1}{8}\sigma(W)$ for all $j \geq b'_-(W)$. Combining inequalities, we see that $\delta_j(Y,\mathfrak{s}) = -\frac{1}{8}\sigma(W)$ for $j \geq b'_-(W)$. □

5.2 Some algebraic results

In this section we collect some algebraic results which will be useful for computing δ invariants.

Let Y be a rational homology 3–sphere, $\tau\colon Y \to Y$ an orientation-preserving diffeomorphism of prime order p and $\mathfrak{s}$ a spin^c–structure preserved by τ. Take $G = \mathbb{Z}_p = \langle\tau\rangle$ and $\mathbb{F} = \mathbb{Z}_p$. Let $\{E_r^{p,q}, d_r\}_{r\geq 2}$ denote the spectral sequence relating equivariant and nonequivariant Seiberg–Witten–Floer cohomologies. Then

$$E_2^{p,q} = H^p(\mathbb{Z}_p, HSW^q(Y,\mathfrak{s})),$$

where $\mathbb{Z}_p$ acts on $HSW^q(Y,\mathfrak{s})$ via the action induced by τ. To simplify notation we will write H^q for $HSW^q(Y,\mathfrak{s})$ and d for $d(Y,\mathfrak{s})$. So $E_2^{p,q} = H^p(\mathbb{Z}_p, H^q)$. For fixed q, H^q is a finite-dimensional representation of $\mathbb{Z}_p$ over $\mathbb{F}$. Moreover, for all sufficiently large k,

$$H^{d+2k} = \mathbb{F}, \quad H^{d+2k+1} = 0. \tag{5-1}$$

Recall that H_G^* is isomorphic to $\mathbb{F}[Q]$ for $p=2$ and to $\mathbb{F}[R,S]/(R^2)$ for odd p. In the case $p=2$ we will set $S=Q^2$, so in all cases $S \in H_G^2$.

Lemma 5.6 *If V is a finite-dimensional representation of $\mathbb{Z}_p$ over $\mathbb{F}=\mathbb{Z}_p$, then*

$$S\colon H^i(\mathbb{Z}_p;V) \to H^{i+1}(\mathbb{Z}_p;V)$$

is surjective for all $i \geq 0$ and an isomorphism for all $i \geq 1$. Furthermore, $\dim_{\mathbb{F}}(H^i(\mathbb{Z}_p;V)) \leq \dim_{\mathbb{F}}(V)$.

Proof Since $\mathbb{Z}_p$ acts freely on S^1, it follows from [12, page 114] that there is an element $\nu \in H^2(\mathbb{Z}_p;\mathbb{Z})$ (independent of V) such that the cup product $\nu\colon H^i(\mathbb{Z}_p;V) \to H^{i+2}(\mathbb{Z}_p;V)$ is an isomorphism for $i>0$ and surjective for $i=0$. Since V is a representation of $\mathbb{Z}_p$ over $\mathbb{F}$, the same statement holds if we replace ν by its image in $H^2(\mathbb{Z}_p;\mathbb{F})$, which must have the form aS for some $a \in \mathbb{F}$. Moreover, $a \neq 0$ follows by considering the case that $V=\mathbb{Z}_p$ is the trivial representation. Hence the cup product $S\colon H^i(\mathbb{Z}_p;V) \to H^{i+2}(\mathbb{Z}_p;V)$ is an isomorphism for $i>0$ and surjective for $i=0$. We have by induction that $\dim_{\mathbb{F}}(H^i(\mathbb{Z}_p;V)) \leq \dim_{\mathbb{F}}(H^0(\mathbb{Z}_p;V))$ if i is even and $\dim_{\mathbb{F}}(H^i(\mathbb{Z}_p;V)) \leq \dim_{\mathbb{F}}(H^1(\mathbb{Z}_p;V))$ if i is odd. Then since $H^0(\mathbb{Z}_p;V)$ and $H^1(\mathbb{Z}_p;V)$ can both be expressed as certain subquotients of V, it follows that $\dim_{\mathbb{F}}(H^i(\mathbb{Z}_p;V)) \leq \dim_{\mathbb{F}}(V)$ for all i. □

Lemma 5.7 *For each $r \geq 2$, the map $S\colon E_r^{p,q} \to E_r^{p+2,q}$ is surjective for all $p \geq 0$ and an isomorphism for all $p \geq r-1$.*

Proof Recall that $E_2^{p,q} = H^p(\mathbb{Z}_p, H^q)$. Hence $S\colon E_2^{p,q} \to E_2^{p+2,q}$ is surjective for all p and an isomorphism for all $p \geq 1$, by Lemma 5.6. This proves the case $r=2$. Now we proceed by induction. Let $r>2$ and suppose that $S\colon E_{r-1}^{p,q} \to E_{r-1}^{p+2,q}$ is surjective for all $p \geq 0$ and an isomorphism for all $p \geq r-1$. Let $x \in E_r^{p+2,q}$. Then $x=[y]$ for some $y \in E_{r-1}^{p+2,q}$ with $d_{r-1}(y)=0$. By the inductive hypothesis $y=Sz$ for some $z \in E_{r-1}^{p,q}$. Then $Sd_{r-1}(z) = d_{r-1}(Sz) = d_{r-1}(y) = 0$. That is, $Sd_{r-1}(z)=0$. However, $d_{r-1}(z) \in E_{r-1}^{p+r-1,q+2-r}$ and $p+r-1 \geq r-2$, so $S\colon E_{r-1}^{p+r-1,q+2-r} \to E_{r-1}^{p+r+1,q+2-r}$ is an isomorphism by the inductive hypothesis. Hence $Sd_{r-1}(z)=0$ implies that $d_{r-1}(z)=0$. So z defines a class $w=[z] \in E_r^{p,q}$. Then $Sw=[Sz]=[y]=x$. Hence $S\colon E_r^{p,q} \to E_r^{p+2,q}$ is surjective for all $p \geq 0$.

Now suppose that $p \geq r-1$ and consider $x \in E_r^{p,q}$ satisfying $Sx=0$. Write $x=[y]$ for some $y \in E_{r-1}^{p,q}$ satisfying $d_{r-1}(y)=0$. Then $0=Sx=S[y]=[Sy]$. Hence $Sy=d_{r-1}(z)$ for some $z \in E_{r-1}^{p-r+3,q+r-2}$. By the inductive hypothesis and since $p-r+3 \geq (r-1)-r+3 = 2$, we have that $z=Sw$ for some $w \in E_{r-1}^{p-r+1,q+r-2}$. Hence $Sy = d_{r-1}(z) = d_{r-1}(Sw) = Sd_{r-1}(w)$. By the inductive hypothesis,

$S: E_{r-1}^{p,q} \to E_{r-1}^{p+2,q}$ is injective; hence $y = d_{r-1}(w)$ and $x = [y] = [d_{r-1}(w)] = 0$. So $S: E_r^{p,q} \to E_r^{p+2,q}$ is injective for $p \geq r-1$. □

From the above lemma, we see that $E_r^{p,q}$ does not depend on p, provided $p \geq r-1$. Let M_r^q denote $E_r^{p,q}$ for $p \geq r-1$. Moreover, since the differentials $\{d_r\}$ for the spectral sequence $E_r^{p,q}$ commute with S, they induce differentials $d_r: M_r^q \to M_r^{q+1-r}$ for which M_{r+1} is the cohomology of $d_r: M_r \to M_r$. Thus M_{r+1} is a subquotient of M_r.

For any module V over $\mathbb{F}[U]$, we define

$$V_{\mathrm{red}} = \{x \in V \mid U^k x = 0 \text{ for some } k \geq 0\} \quad \text{and} \quad V^\infty = V/V_{\mathrm{red}}.$$

Lemma 5.8 *For each $r \geq 2$, the image of the differential d_r is contained in $(E_r^{*,*})_{\mathrm{red}}$.*

Proof By (5-1) there exists a k_0 such that $H^{d+2k} = \mathbb{F}$ and $H^{d+2k+1} = 0$ for all $k \geq k_0$. Hence the action of τ is trivial in these degrees and we have

$$E_2^{p,d+2k} = \mathbb{F}, \quad E_2^{p,d+2k+1} = 0$$

for all $k \geq k_0$. Since $SWF(Y, \mathfrak{s}, g)$ is a space of type $\mathbb{Z}_p$–SWF, the localisation theorem in equivariant cohomology implies that there exists a $k_1 \geq k_0$ such that the generator $x \in E_2^{0,d+2k_1} = \mathbb{F}$ satisfies $d_r(x) = 0$ for all $r \geq 2$. Then if $y \in E_2^{p,q}$ with $q \geq d + 2k_1$, it follows that y is of the form $y = cU^a x$ for some $a \geq 0$, where $c \in H_G^p$. Hence $d_r(y) = 0$ for all $r \geq 2$. Now let $y \in E_r^{p,q}$ where p and q are arbitrary. Then there exists some $a \geq 0$ such that $q + 2a \geq d + 2k_1$; hence $U^a d_r(y) = d_r(U^a y) = 0$. Therefore $d_r(y) \in (E_r^{*,*})_{\mathrm{red}}$. □

Recall that H^∞ is a free $\mathbb{F}[U]$–module of rank 1 with generator in degree d. Hence we may write $H^\infty = \mathbb{F}[U]\theta$ where $\deg(\theta) = d$. Next, observe that $E_2^{0,*}$ is the τ–invariant part of H^*, hence may be regarded as an $\mathbb{F}[U]$–submodule of H^*. Similarly, since $E_{r+1}^{0,*}$ is the kernel of d_r restricted to $E_r^{0,*}$, it follows that $E_{r+1}^{0,*}$ can be identified with an $\mathbb{F}[U]$–submodule of $E_r^{0,*}$. Hence $\{E_r^{0,*}\}$ may be regarded as a decreasing sequence of $\mathbb{F}[U]$–submodules of H^*. Let S_r denote the image of $E_r^{0,*}$ under the quotient map $H^* \to H^\infty = H^*/H_{\mathrm{red}}$. The localisation theorem in equivariant cohomology implies that S_r is nonzero and that the sequence S_r eventually stabilises. Then since S_r is a nonzero graded submodule of $H^\infty = \mathbb{F}[U]\theta$, it follows that $S_r = \mathbb{F}[U]U^{m_r}\theta$ for some nonnegative integer m_r. Note also that the sequence $\{m_r\}$ is increasing and is eventually constant.

Lemma 5.9 *For each $r \geq 2$,*

$$m_{r+1} - m_r \leq \dim_{\mathbb{F}}((M_r)_{\mathrm{red}}) - \dim_{\mathbb{F}}((M_{r+1})_{\mathrm{red}}).$$

Proof The classes $U^{j+m_r}\theta$ with $0 \leq j < m_{r+1} - m_r$ form a basis for S_r/S_{r+1}. Choose a lift $x_r \in E_r^{0,d+2m_r}$ of $U^{m_r}\theta \in S_r$. Then $d_r(U^j x_r) \neq 0$ for $0 \leq j < m_{r+1} - m_r$, for if $d_r(U^j x_r) = 0$ for some $0 \leq j < m_{r+1} - m_r$, then we would have $U^{j+m_r}\theta \in S_{r+1}$. Observe that $d_r(U^j x_r) \in E_r^{r,*}$. By Lemma 5.7 and the definition of M_r, we see that $d_r(U^j x_r)$ can be identified with a nonzero element of M_r. Moreover, $d_r(U^j x_r) \in (M_r)_{\mathrm{red}}$, by Lemma 5.8. Now since the $d_r(U^j x_r)$ are nonzero and have distinct degrees, they span a subspace of $(M_r)_{\mathrm{red}}$ of dimension $m_{r+1} - m_r$. Furthermore, this subspace lies in the image of d_r; hence $m_{r+1} - m_r \leq \dim_{\mathbb{F}}((M_r)_{\mathrm{red}}) - \dim_{\mathbb{F}}((M_{r+1})_{\mathrm{red}})$. □

Proposition 5.10 *Suppose that τ acts trivially on $HSW^*(Y, \mathfrak{s})$. Then*

$$\delta_1(Y, \mathfrak{s}) - \delta(Y, \mathfrak{s}) \leq \dim_{\mathbb{F}}(HSW_{\mathrm{red}}(Y, \mathfrak{s})).$$

Proof Recall that $d = d(Y, \mathfrak{s})$. Hence $\delta(Y, \mathfrak{s}) = \frac{1}{2}d$. From the definition of the invariant $\delta_1(Y, \mathfrak{s})$, it follows that for all sufficiently large r,

$$\delta_1(Y, \mathfrak{s}) = m_r + \delta(Y, \mathfrak{s}).$$

By Lemma 5.9, for each $r \geq 2$,

$$m_{r+1} - m_r \leq \dim_{\mathbb{F}}((M_r)_{\mathrm{red}}) - \dim_{\mathbb{F}}((M_{r+1})_{\mathrm{red}}),$$

and summing from 2 to $r-1$, we get

$$m_r - m_2 \leq \dim_{\mathbb{F}}((M_2)_{\mathrm{red}}).$$

However since τ acts trivially on $HSW^*(Y, \mathfrak{s})$, we have that $E_2^{p,*} = HSW^*(Y, \mathfrak{s})$ for all $p \geq 0$. Hence $m_2 = 0$, $M_2 = HSW^*(Y, \mathfrak{s})$ and $(M_2)_{\mathrm{red}} = HSW_{\mathrm{red}}(Y, \mathfrak{s})$. Taking r sufficiently large,

$$\delta_1(Y, \mathfrak{s}) - \delta(Y, \mathfrak{s}) = m_r = m_r - m_2 \leq \dim_{\mathbb{F}}(HSW_{\mathrm{red}}). \qquad \square$$

6 Branched double covers of knots

6.1 Concordance invariants

Let $K \subset S^3$ be a knot in S^3. Let $Y = \Sigma_2(K)$ be the branched double cover of S^3, branched over K. Let $\pi\colon Y \to S^3$ denote the covering map. One finds that $b_1(Y) = 0$. Manolescu and Owens [51] define a knot invariant

$$\delta(K) = 2d(\Sigma_2(K), \mathfrak{t}_0) = 4\delta(\Sigma_2(K), \mathfrak{t}_0),$$

where $\mathfrak{t}_0$ is the spin^c–structure induced from the unique spin–structure on $\Sigma_2(K)$ (see [51, Section 2] for an explanation of this). It is shown in [51] that $\delta(K)$ is always integer-valued and descends to a surjective group homomorphism $\delta\colon \mathscr{C} \to \mathbb{Z}$, where $\mathscr{C}$ is the smooth concordance group of knots in S^3.

The covering involution on Y determines an action of $G = \mathbb{Z}_2$ on Y preserving $\mathfrak{t}_0$ (by uniqueness of the underlying spin–structure). Hence, for each $j \geq 0$, we may define the knot invariant

$$\delta_j(K) = 2d_j(\Sigma_2(K), \mathfrak{t}_0) = 4\delta_j(\Sigma_2(K), \mathfrak{t}_0).$$

Since $d_j(\Sigma_2(K), \mathfrak{t}_0) - d(\Sigma_2(K), \mathfrak{t}_0) \in 2\mathbb{Z}$, it follows that $\delta_j(K) - \delta(K) \in 4\mathbb{Z}$. Then, since $\delta(K)$ is integer-valued, it follows that the $\delta_j(K)$ are also integer-valued and moreover $\delta_j(K) = \delta(K) \bmod 4$.

Proposition 6.1 *For each $j \geq 0$, $\delta_j(K)$ depends only on the concordance class of K; hence δ_j descends to a concordance invariant $\delta_j : \mathscr{C} \to \mathbb{Z}$.*

Proof For an oriented knot K, recall that $-K$ denotes the knot obtained by reversing orientation on S^3 and K. It follows that $\Sigma_2(-K) = \overline{\Sigma_2(K)}$. A concordance of oriented knots K_1 and K_2 is a smooth embedding of $\Sigma = [0,1] \times S^1$ in $[0,1] \times S^3$ having boundary $-K_1 \cup K_2$. Taking the double cover of $[0,1] \times S^3$ branched along Σ gives a $\mathbb{Z}_2$–equivariant cobordism W from $\Sigma_2(K_1)$ to $\Sigma(K_2)$. From the calculations in [36, Section 3], one sees that W is a rational homology cobordism. We claim that W is spin. To see this, choose a smoothly embedded surface Σ in D^4 whose boundary is K_1. Let W' be the double cover of $D^4 \cup [0,1] \times S^3 \cong D^4$ branched over $\Sigma \cup [0,1] \times S^1$. From [35] we see that W' is spin. Since W is embedded in W', it follows that W' is spin as well. Any spin–structure $\mathfrak{t}$ on W will restrict on each component of the boundary to the unique spin–structure on the branched double cover $\Sigma_2(K_i)$. The result now follows by applying Corollary 4.3 to $(W, \mathfrak{t})$. □

We note that the δ_j are *not* group homomorphisms.

Let $\sigma(K)$ denote the signature of K and $g_4(K)$ the smooth 4–genus. Set $\sigma'(K) = -\frac{1}{2}\sigma(K)$. We also define $b_+(K) = g_4(K) - \sigma'(K)$ and $b_-(K) = g_4(K) + \sigma'(K)$.

Proposition 6.2 *The knot concordance invariants δ_j have the properties*

1. $\delta_0(K) \geq \delta(K)$;
2. $\delta_{j+1}(K) \leq \delta_j(K)$ *for all* $j \geq 0$;
3. $\delta_j(K) \geq \sigma'(K)$ *for all* $j \geq 0$ *and* $\delta_j(K) = \sigma'(K)$ *for* $j \geq b_-(K)$;
4. $\delta_j(-K) \geq -\sigma'(K)$ *for all* $j \geq 0$ *and* $\delta_j(-K) = -\sigma'(K)$ *for* $j \geq b_+(K)$;
5. *if* $\Sigma_2(K)$ *is an L–space, then* $\delta_j(K) = \delta(K)$ *and* $\delta_j(-K) = \delta(-K)$ *for all* $j \geq 0$.

Proof Items (1), (2) and (5) follow from (1), (2) and (5) of Theorem 5.2. For (3) and (4), choose a smooth embedded surface $\Sigma \subset D^4$ in the 4–ball of genus $g_4(K)$ which bounds K. Let W be the double cover of D^4 branched along Σ. From [35] it follows that W is spin. Let $\mathfrak{t}$ be any spin–structure on W. Then $\mathfrak{t}|_{\Sigma_2(K)} = \mathfrak{t}_0$ by uniqueness of $\mathfrak{t}_0$. Next, observe that $H^2(W;\mathbb{R})^{\mathbb{Z}_2} = H^2(D^4;\mathbb{R}) = 0$. Then (3) and (4) follow by applying Corollary 5.5 to $(W, \mathfrak{t})$. □

Corollary 6.3 *If K is a knot such that $\Sigma_2(K)$ is an L–space, then $\delta(K) = \sigma'(K)$.*

Proof This follows by (3) and (5) of Proposition 6.2 □

Remark 6.4 In particular, if K is quasialternating, then $\Sigma_2(K)$ is an L–space [57]. This recovers the main result of [47] that $\delta(K) = \sigma'(K)$ for quasialternating knots.

Theorem 6.5 *For a knot K, let $j_+(K)$ be the smallest positive integer such that $\delta_j(K) = \sigma'(K)$ and $j_-(K)$ the smallest positive integer such that $\delta_j(-K) = -\sigma'(K)$. Then*

$$g_4(K) \geq \max\{\sigma'(K) + j_-(K), -\sigma'(K) + j_+(K)\}.$$

Remark 6.6 Observe that the right-hand side of this inequality is at least $\frac{1}{2}|\sigma(K)|$. Hence we have obtained a strengthening of the well-known inequality $g_4(K) \geq \frac{1}{2}|\sigma(K)|$ [52].

Proof From Proposition 6.2 we have that $\delta_j(K) = \sigma'(K)$ for $j \geq g_4(K)+\sigma'(K)$ and $\delta_j(-K) = -\sigma'(K)$ for $j \geq g_4(K)-\sigma'(K)$. Hence $j_+(K) \leq g_4(K)+\sigma'(K)$ and $j_-(K) \leq g_4(K)-\sigma'(K)$. □

Remark 6.7 In this section we have used branched double covers $\Sigma_2(K)$ of knots equipped with their natural $\mathbb{Z}_2$–action to obtain a sequence of concordance invariants. Similarly, for any odd prime p we may consider the cyclic branched cover $\Sigma_p(K)$ with its natural $\mathbb{Z}_p$–action. Once again there is a canonical spin^c–structure $\mathfrak{t}_0$ [29] and so we may define a sequence of invariants

$$\delta_{(p),j}(K) = 2d_{\mathbb{Z}_p,S^j}(\Sigma_p(K), \mathfrak{t}_0)$$

depending on a prime p and an integer $j \geq 0$. By similar arguments to the $p = 2$ case one finds that these are integer-valued knot concordance invariants of K.

7 Computations and applications

7.1 Brieskorn homology spheres

Let p, q and r be pairwise coprime positive integers and let $Y = \Sigma(p,q,r)$ be the corresponding Brieskorn integral homology 3–sphere. Then Y has a unique spin^c–structure and so when speaking of the Floer homology of Y we omit the mention of the spin^c–structure.

Recall that $\Sigma(p,q,r)$ can be realised as the p–fold cyclic cover of S^3 branched along the torus knot $T_{q,r}$. In particular, this construction defines an action of $\mathbb{Z}_p$ on Y. Let $\tau\colon Y \to Y$ denote the generator of this action. Recall that $\Sigma(p,q,r)$ is obtained by taking the link of the singularity

$$\{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\}.$$

Then τ is given by $(x,y,z) \mapsto (e^{2\pi i/p}x, y, z)$. This map is isotopic to the identity through the homotopy $(x,y,z) \mapsto (e^{2\pi iqrt}x, e^{2\pi iprt}y, e^{2\pi ipqt}z)$ for $t \in [0,(qr)^*]$, where $0 < (qr)^* < p$ denotes the multiplicative inverse of qr mod p. It follows that τ acts trivially on $HF^+(Y)$.

Henceforth we will assume that p is a prime number. Set $\mathbb{F} = \mathbb{Z}_p$ and recall that $H^*_{\mathbb{Z}_p} \cong \mathbb{F}[Q]$ where $\deg(Q) = 1$ if $p = 2$, and $H^*_{\mathbb{Z}_p} \cong \mathbb{F}[R,S]/(R^2)$ where $\deg(R) = 1$ and $\deg(S) = 2$ if p is odd. Let $\mathfrak{s}$ denote the unique spin^c–structure on Y. As in Section 5, we let $\delta_j(Y,\mathfrak{s},\tau,p)$ denote

$\delta_{\mathbb{Z}_p,Q^p}(Y,\mathfrak{s})$ for $p=2$ or $\delta_{\mathbb{Z}_p,S^p}(Y,\mathfrak{s})$ for odd p. We will further abbreviate this to $\delta_j(Y,\tau)$. When $p=2$, $\delta_j(Y,\tau)=\frac{1}{4}\delta_j(T_{q,r})$, where $\delta_j(K)$ denotes the knot concordance invariant introduced in Section 6.1. More generally, $\delta_j(Y,\tau)=\frac{1}{4}\delta_{(p),j}(T_{q,r})$, where $\delta_{(p),j}(K)$ is the knot concordance invariant defined in Remark 6.7.

Example 7.1 Let $(p,q,r)=(2,3,5)$. Then $Y=\Sigma(2,3,5)$ is the Poincaré homology 3–sphere. Since $\Sigma(2,3,5)$ has spherical geometry, it is an L–space [56, Proposition 2.3]. Therefore

$$\delta_j(T_{3,5})=\delta(T_{3,5})=\sigma'(T_{3,5})=4 \quad \text{for all } j\geq 0.$$

The property of being an L–space does not depend on the choice of orientation, so we also have

$$\delta_j(-T_{3,5})=\delta(-T_{3,5})=-4 \quad \text{for all } j\geq 0.$$

The same argument applied to $p=3$ or 5 gives

$$\delta_{(3),j}(T_{2,5})=\delta_{(5),j}(T_{2,3})=4 \quad \text{for all } j\geq 0$$

and

$$\delta_{(3),j}(-T_{2,5})=\delta_{(5),j}(-T_{2,3})=-4 \quad \text{for all } j\geq 0.$$

Proposition 7.2 *Let p, q and r be positive, pairwise coprime integers and assume that p is prime. Then $\delta_j(\Sigma(p,q,r),\tau)=-\lambda(\Sigma(p,q,r))$ for all $j\geq 0$, where $\lambda(\Sigma(p,q,r))$ is the Casson invariant of $\Sigma(p,q,r)$. Furthermore,*

$$\delta_{(p),j}(T_{q,r})=-\frac{1}{2}\sum_{j=1}^{p-1}\sigma_{j/p}(T_{q,r}) \quad \textit{for all } j\geq 0,$$

where $\sigma_\alpha(K)$ is the Tristram–Levine signature of K.

Proof Recall that $Y=\Sigma(p,q,r)$ is the boundary of a negative definite plumbing [53] whose plumbing graph has only one bad vertex in the terminology of [55]. Then it follows from [55, Corollary 1.4] that $HF^+(\overline{Y})$ is concentrated in even degrees. Consequently, $HF^+_{\mathrm{red}}(Y)$ is concentrated in odd degrees. (Note that [55] uses $\mathbb{Z}$ coefficients, but it is shown there that $HF^+_{\mathrm{red}}(Y;\mathbb{Z})$ has no torsion and hence by the universal coefficient theorem, [55, Corollary 1.4] also holds for $\mathbb{Z}_p$ coefficients.) Therefore,

$$\chi(HF^+_{\mathrm{red}}(Y))=\dim_{\mathbb{F}}(HF^+_{\mathrm{red,even}}(Y))-\dim_{\mathbb{F}}(HF^+_{\mathrm{red,odd}}(Y))=-\dim_{\mathbb{F}}(HF^+_{\mathrm{red}}(Y)). \tag{7-1}$$

By [54, Theorem 1.3], $\chi(HF^+_{\mathrm{red}}(Y))$ is related to the Casson invariant $\lambda(Y)$ via the formula

$$\chi(HF^+_{\mathrm{red}}(Y))=\lambda(Y)+\delta(Y). \tag{7-2}$$

Hence

$$\dim_{\mathbb{F}}(HF^+_{\mathrm{red}}(Y))=-\lambda(Y)-\delta(Y). \tag{7-3}$$

Moreover, from [18; 27],

$$\lambda(\Sigma(p,q,r)) = \frac{1}{8}\sum_{j=1}^{p-1} \sigma_{j/p}(T_{q,r}) = \tfrac{1}{8}\sigma(M(p,q,r)),$$

where $M(p,q,r)$ is the Milnor fibre

$$M(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = \delta\} \cap D^6$$

(where δ is a sufficiently small nonzero complex number). Recall that $M(p,q,r)$ is a compact smooth 4–manifold with boundary diffeomorphic to $\Sigma(p,q,r)$. Moreover, $M(p,q,r)$ has the homotopy type of a wedge of 2–spheres, so $b_1(M(p,q,r)) = 0$. Further, $M(p,q,r)$ is a p–fold cyclic cover of D^4 branched along a surface bounding $T_{q,r}$. Hence the action of $\mathbb{Z}_p = \langle \tau \rangle$ on Y extends to $M(p,q,r)$. From [29, Lemma 2.1] it follows that there is a $\mathbb{Z}_p$ invariant spin–structure $\mathfrak{t}_0$ on $M(p,q,r)$. Since $M(p,q,r)$ is a cyclic p–fold cover of D^4, it follows that the subspace of $H^2(M(p,q,r);\mathbb{R})$ fixed by τ is zero. Hence Corollary 5.5 may be applied, giving

$$\delta_j(Y,\tau) \geq -\tfrac{1}{8}\sigma(M(p,q,r)) = -\lambda(Y) \quad \text{for all } j \geq 0.$$

Since τ acts trivially on $HF^+(Y)$, Proposition 5.10 implies that

$$\delta_0(Y,\tau) - \delta(Y) \leq \dim_{\mathbb{F}}(HF^+_{\mathrm{red}}(Y)) = -\lambda(Y) - \delta(Y).$$

Hence $\delta_0(Y,\tau) \leq -\lambda(Y)$. On the other hand, $\delta_0(Y,\tau) \geq \delta_j(Y,\tau) \geq -\lambda(Y)$ for any $j \geq 0$. Hence $\delta_j(Y,\tau) = -\lambda(Y)$ for all $j \geq 0$. Therefore we also have

$$\delta_{(p),j}(T_{q,r}) = 4\delta_j(Y,\tau) = -4\lambda(Y) = -\frac{1}{2}\sum_{j=1}^{p-1} \sigma_{j/p}(T_{q,r})$$

for all $j \geq 0$. □

The above result shows that the values of $\delta_{(p),j}(T_{q,r})$ do not depend on j. In contrast, the values of $\delta_{(p),j}(-T_{q,r})$ usually do depend on j, as the following propositions illustrate.

Proposition 7.3 *Let $(a,b) = (3, 6n-1)$ for $n \geq 1$. Then*

$$\delta(-T_{3,6n-1}) = -4, \quad \sigma'(-T_{3,6n-1}) = -4n$$

and

$$\delta_j(-T_{3,6n-1}) = \begin{cases} -4\left(\lfloor \tfrac{1}{2} j \rfloor + 1\right) & \text{if } 0 \leq j \leq 2n-3, \\ -4n & \text{if } j \geq 2n-2. \end{cases}$$

Proof The case $n = 1$ is already covered in Example 7.1, so we assume $n \geq 2$. Set $Y_{a,b} = \Sigma_2(T_{a,b}) = \Sigma(2,a,b)$ and let τ be the covering involution. Then $\delta_j(-T_{3,6n-1}) = 4\delta_j(\overline{Y}_{3,6n-1})$. From the computations in [54, Section 8] we find that $d(\overline{Y}_{3,6n-1}) = -2$, $SWF^*_{\mathrm{red}}(\overline{Y}_{3,6n-1}) = (\mathbb{F}_{-2})^{n-1}$, where the subscript indicates degree. To simplify notation we let $V = SWF^*_{\mathrm{red}}(\overline{Y}_{3,6n-1}) = (\mathbb{F}_{-2})^{n-1}$. Then

$$E_2^{*,*} \cong \mathbb{F}[U,Q]\theta \oplus V[Q],$$

where the bidegree is given as follows: θ and all elements of V have bidegree $(0,-2)$, U has bidegree $(0,2)$ and Q has bidegree $(1,0)$. Then $E_2^{p,q}=0$ for $q<-2$. It follows that all the differentials in the spectral sequence are zero on θ and on V, since d_r sends $E_r^{p,-2}$ to $E_r^{p+r,-1-r}$ and $-1-r<-2$ for $r\geq 2$. Hence d_r is zero on all of E_r and $E_\infty^{*,*}\cong E_2^{*,*}$. Let $\mathscr{F}_j$ denote the filtration on $HSW^*_{\mathbb{Z}_2}(\overline{Y}_{3,6n-1})$ corresponding to the spectral sequence, so $\mathscr{F}_j/\mathscr{F}_{j+1}\cong E_\infty^{j,*}$. In particular, $\mathscr{F}_1/\mathscr{F}_2\cong\mathbb{F}[U]\theta\oplus V$. Choose lifts of θ and V to $\mathscr{F}_1$. We lift $U^j\theta$ by taking the lift of θ and applying U^j. Hence we obtain a short exact sequence of $\mathbb{F}[U,Q]$–modules

$$0\to\mathbb{F}[U]\oplus V\to HSW^*_{\mathbb{Z}_2}(\overline{Y}_{3,6n-1})\to\mathscr{F}_2\to 0.$$

Next, for each $j\geq 0$, Q induces an isomorphism $Q\colon\mathscr{F}_j/\mathscr{F}_{j+1}\to\mathscr{F}_{j+1}/\mathscr{F}_{j+2}$; hence by applying Q repeatedly to $\mathbb{F}[U]\theta\oplus V$, we obtain a splitting of the filtration $\{\mathscr{F}_j\}$ as $\mathbb{F}[Q]$–modules. The splittings give an isomorphism of $\mathbb{F}[Q]$–modules

$$HSW^*_{\mathbb{Z}_2}(\overline{Y}_{3,6n-1})\cong\mathbb{F}[U,Q]\theta\oplus V[Q].$$

However, this is not necessarily an isomorphism of $\mathbb{F}[U,Q]$–modules. Under this isomorphism, U corresponds to an endomorphism of the form

$$\hat{U}=U_2+QU_1+Q^2U_0+Q^3U_{-1}+\cdots,$$

where $U_j\colon HSW^*(\overline{Y}_{3,6n-1})\to HSW^{*+j}(\overline{Y}_{3,6n-1})$ and $U_2=U$. Since $HSW^*(\overline{Y}_{3,6n-1})$ is concentrated in even degrees, $U_j=0$ for odd j. Moreover, our construction is such that $U_j\theta=0$ for $j\neq 2$. It follows that $U_j=0$ for $j<0$, as V is concentrated in a single degree. So we get

$$\hat{U}=U+Q^2U_0$$

for some $U_0\colon V\to HSW^0(\overline{Y}_{3,6n-1})$.

To simplify notation set $d_j=d_j(\overline{Y}_{3,6n-1})$. Using Proposition 3.14 we obtain the following characterisation of d_j:

$$d_j=\min\bigl\{i\mid\hat{U}^rx=U^mQ^j\theta\ \mathrm{mod}\ Q^{j+1}\text{ for some }x\in HSW^i_{\mathbb{Z}_2}(\overline{Y}_{3,6n-1})\text{ and }r,m\geq 0\bigr\}-j.$$

Recall that $\delta_j(-T_{3,6n-1})=\sigma'(-T_{3,6n-1})=-4n$ for sufficiently large j. Hence $d_j=-2n$ for sufficiently large j. Choose such a j. From the above characterisation of j there exists $x\in HSW^{j-2n}_{\mathbb{Z}_2}(\overline{Y}_{3,6n-1})$ such that $\hat{U}^rx=U^mQ^j\theta+\cdots$ where $\cdots$ denotes terms of higher order in Q. We have that $x=Q^ay$ for some $a\leq j$. Then $\hat{U}^rQ^ay=U^mQ^j\theta+\cdots$. Since Q is injective we may cancel, giving $\hat{U}^ry=U^mQ^{j-a}\theta+\cdots$. If $a=j$, then $\hat{U}^ry=U^m\theta+\cdots$. But $\hat{U}=U+Q^2U_0$, so $\hat{U}^ry=U^ry+\cdots$; hence $U^ry=U^m\theta+\cdots$. From the definition of the usual d–invariant we must have $\deg(y)\geq d(\overline{Y}_{3,6n-1})=-2$. Hence $j-2n=\deg(x)=a+\deg(y)=j-2$, which is a contradiction since we have assumed that $n>1$. It follows that $a<j$. We must have $y\in V$ for if $y=U^b\theta\ \mathrm{mod}\,V$, then we would have $\hat{U}^rx=U^{r+b}Q^a\theta+\cdots$, which contradicts $\hat{U}^rx=U^mQ^j\theta+\cdots$ as $a<j$. Therefore $y\in V$. In particular $\deg(y)=-2$ and $j-2n=\deg(x)=a-2$.

Let b be the smallest positive integer such that $U_0^b y \notin V$. Such a b exists since

$$\hat{U}^r y = (U + Q^2 U_0)^r y = U^m Q^{j-a} \theta + \cdots$$

and U is zero on V. Then it follows that $r \geq b$ and

$$\hat{U}^r y = (U + Q^2 U_0)^r y = U^{r-b} Q^{2b} (U_0^b y) + \cdots = U^m Q^{j-a} \theta + \cdots .$$

Hence $2b = j - a$. So we have shown that $j = a + 2b$ and $j - 2n = a - 2$. Hence $b = n - 1$. But since $\dim_{\mathbb{F}}(V) = n - 1 = b$, it follows that there exists a $v \in V$ such that $v, U_0 v, U_0^2 v, \ldots, U_0^{n-2} v$ is a basis for V and $U_0^{n-1} v = \theta \mod V$. Now it is straightforward to see that the sequence $\{d_j\}$ must have the form $-2, -2, -4, -4, -6, -6, \ldots,$ for $j \leq 2n - 3$ and $d_j = -2n$ for $j \geq 2n - 2$. □

Proposition 7.4 *Let $(a, b) = (3, 6n + 1)$ for $n \geq 1$. Then*

$$\delta(-T_{3,6n+1}) = 0, \quad \sigma'(-T_{3,6n+1}) = -4n$$

and

$$\delta_j(-T_{3,6n+1}) = \begin{cases} -4\lfloor \frac{1}{2} j \rfloor & \text{if } 0 \leq j \leq 2n - 1, \\ -4n & \text{if } j \geq 2n. \end{cases}$$

Proof By [54, Section 8], $d(\overline{Y}_{3,6n+1}) = 0$, $SWF^*_{\mathrm{red}}(\overline{Y}_{3,6n+1}) = (\mathbb{F}_0)^n$ and $\sigma'(-T_{3,6n+1}) = 4n$. From here essentially the same argument as in Proposition 7.3 gives the result. □

Remark 7.5 We can use Theorem 6.5 and the computations in Propositions 7.3 and 7.4 to obtain a lower bound for the 4–genus. From Proposition 7.3, we see that $\sigma'(T_{3,6n-1}) = 4n$ and $j_-(T_{3,6n-1}) = 2n - 2$; hence $g_4(T_{3,6n-1}) \geq 2n - 2 + 4n = 6n - 2$. On the other hand, from the positive solution to the Milnor conjecture [38], we know that $g_4(T_{a,b}) = \frac{1}{2}(a-1)(b-1)$. In particular, $g_4(T_{3,6n-1}) = 6n - 2$. Hence the above estimate for $g_4(T_{3,6n-1})$ is actually sharp.

Similarly, by Proposition 7.4, $\sigma'(T_{3,6n+1}) = 4n$ and $j_-(T_{3,6n+1}) = 2n$. So we obtain an estimate $g_4(T_{3,6n+1}) \geq 6n$. Once again, this estimate is sharp since $g_4(T_{3,6n+1}) = \frac{1}{2}(3-1)(6n+1-1) = 6n$.

7.2 Nonextendable actions

Let Y be a rational homology 3–sphere equipped with an orientation-preserving action of G. Let W be a smooth 4–manifold with boundary Y. In this section we are concerned with the question of whether the G–action can be extended to W. In particular we are interested in finding obstructions to such an extension.

Proposition 7.6 *Let Y be an integral homology 3–sphere and $\mathfrak{s}$ the unique spin^c–structure on Y. Let G act orientation-preservingly on Y and suppose the extension $G_{\mathfrak{s}}$ is trivial. Suppose that Y is the boundary of a contractible 4–manifold W. If the action of G extends over W then $\delta_{G,c}(Y, \mathfrak{s}) = \delta_{G,c}(\overline{Y}, \mathfrak{s}) = 0$ for every nonzero $c \in H^*_G$.*

Proof Suppose that the G–action extends to W. Since W is contractible, there is a unique spin^c–structure $\mathfrak{t}$ on W. By uniqueness it is G–invariant and $\mathfrak{t}|_Y = \mathfrak{s}$. Theorem 4.1 gives $\delta_{G,c}(Y,\mathfrak{s}) \le 0$ and $\delta_{G,c}(Y,\mathfrak{s}) \ge 0$; hence $\delta_{G,c}(Y,\mathfrak{s}) = 0$. Reversing orientations, we also find that $\delta_{G,c}(\overline{Y}\mathfrak{s}) = 0$. □

Example 7.7 Akbulut–Kirby constructed examples of contractible 4–manifolds bounding integral homology spheres, in particular $\Sigma(2,5,7)$, $\Sigma(3,4,5)$ and $\Sigma(2,3,13)$ bound contractible 4–manifolds [2, Theorem 2]. Further examples were given by Casson and Harer, in particular $\Sigma(2,2s-1,2s+1)$ for odd s bounds a contractible 4–manifold [14].

Now let $Y = \Sigma(2,3,13)$ and let τ be the involution obtained by viewing Y as the branched double cover $\Sigma_2(T_{3,13})$. Then $\delta_2(\overline{Y}) = -1$ by Proposition 7.4. Then it follows from Proposition 7.6 that τ does not extend to an involution on any contractible 4–manifold W bounded by Y. On the other hand, since τ is isotopic to the identity, τ does extend to a diffeomorphism on W.

Similarly if we let $Y = \Sigma(2,2s-1,2s+1) = \Sigma_2(T_{2s-1,2s+1})$, where s is odd, and let τ be the covering involution, then Y bounds a contractible 4–manifold W but τ does not extend to an involution on W because $\delta_j(Y) = -\frac{1}{8}\sigma(T_{2s-1,2s+1}) = \frac{1}{2}(s^2-1) \neq 0$ for all $j \ge 0$.

More generally, let $Y = \Sigma(p,q,r)$ where p, q and r are pairwise coprime positive integers. Assume that p is prime and let $\mathbb{Z}_p = \langle \tau \rangle$ act on Y by realising Y as the p–fold cyclic branched cover $\Sigma_p(T_{q,r})$. Then $\delta_0(Y,\tau) = -\lambda(\Sigma(p,q,r))$. From [59, Chapter 19], it can be seen that $\lambda(\Sigma(p,q,r)) < 0$ and hence the $\mathbb{Z}_p$–action on $Y = \Sigma(p,q,r)$ is nonextendable over contractible 4–manifolds bounded by Y. We have thus recovered a special case of the nonextendability results of Anvari and Hambleton [6; 7].

If we relax the condition that W is contractible to being a rational homology 4–ball, then we get a similar result, except that we have to make an assumption on the order of $H^2(W;\mathbb{Z})$.

Proposition 7.8 *Let Y be an integral homology 3–sphere and $\mathfrak{s}$ the unique spin^c–structure on Y. Let $G = \mathbb{Z}_p$ for a prime p act orientation-preservingly on Y, and suppose that the extension $G_{\mathfrak{s}}$ is trivial. Suppose that Y is the boundary of a compact, oriented, smooth rational homology 4–ball W and assume that p does not divide the order of $H^2(W;\mathbb{Z})$. If the action of G extends over W then $\delta_{G,c}(Y,\mathfrak{s}) = \delta_{G,c}(\overline{Y},\mathfrak{s}) = 0$ for every nonzero $c \in H^*_G$.*

Proof The set of spin^c–structures on W has cardinality $|H^2(W;\mathbb{Z})|$ and $G = \mathbb{Z}_p$ acts on this set. By assumption, p does not divide this number and hence there must exist a spin^c–structure $\mathfrak{t}$ whose stabiliser group is not trivial. Since p is prime, this means $\mathfrak{t}$ is fixed by all of G. From here, the rest of the proof is the same as for Proposition 7.6. □

Example 7.9 Let $Y = \Sigma(p,q,r)$ where p, q and r are relatively prime and assume that p is prime. Let $\mathbb{Z}_p$ act on Y as described in Section 7.1. Recall from Proposition 7.2 that $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s}) = -\lambda(\Sigma(p,q,r))$. As in Example 7.7, $\lambda(\Sigma(p,q,r)) < 0$ and hence $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s}) > 0$.

Therefore, by Proposition 7.8, if W is a compact, oriented, smooth rational homology 4–ball bounded by Y and if p does not divide the order of $H^2(W;\mathbb{Z})$, then the action of G does not extend over W. Thus we have obtained a partial extension of the results of Anvari–Hambleton to the case of rational homology 4–balls.

Fintushel and Stern [26] showed that $\Sigma(2,3,7)$ bounds a rational homology 4–ball, although it does not bound an integral 4–ball. Akbulut and Larson [4] showed that, for n odd, $\Sigma(2,4n+1,12n+5)$ and $\Sigma(3,3n+1,12n+5)$ bound rational 4–balls but not integral 4–balls. More examples, $\Sigma(2,4n+3,12n+7)$ and $\Sigma(3,3n+2,12n+7)$ for even n, were constructed by Şavk [21]. Taking $p=2$ or 3, the above Brieskorn spheres admit $\mathbb{Z}_p$–actions with nonzero delta invariants, as in Example 7.7. Hence the $\mathbb{Z}_p$–action does not extend to any oriented rational homology 4–ball W with boundary Y, provided the order of $H^2(W;\mathbb{Z})$ is coprime to p. However, it does not seem straightforward to determine whether the above examples are bounded by rational 4–balls satisfying this coprimality condition.

Proposition 7.10 *Let Y be an integral homology 3–sphere and $\mathfrak{s}$ the unique* spin^c*–structure on Y. Let G act orientation-preservingly on Y and suppose that the extension $G_{\mathfrak{s}}$ is trivial. Suppose that Y is the boundary of a smooth, compact, oriented 4–manifold with $b_1(W)=0$ and suppose that $H^2(W;\mathbb{Z})$ has no 2–torsion.*

- *If $H^2(W;\mathbb{R})$ is positive definite and $\delta_{G,1}(Y,\mathfrak{s})>0$, then the G–action on Y cannot be extended to a smooth G–action on W acting trivially on $H^2(W;\mathbb{Z})$.*
- *If $H^2(W;\mathbb{R})$ is negative definite and $\delta_{G,c}(Y,\mathfrak{s})<0$ for some $c\in H_G^*$, then the G–action on Y cannot be extended to a smooth G–action on W acting trivially on $H^2(W;\mathbb{Z})$.*

Proof Suppose the G–action on Y extends to a smooth G–action on W acting trivially on $H^2(W;\mathbb{Z})$. Since $H^2(W;\mathbb{Z})$ has no 2–torsion, a spin^c–structure $\mathfrak{t}$ on W is determined uniquely by $c_1(\mathfrak{t})$. Since G acts trivially on $H^2(W;\mathbb{Z})$, it follows that G preserves every spin^c–structure. Furthermore, $\mathfrak{t}|_Y=\mathfrak{s}$ for any spin^c–structure on W by uniqueness of $\mathfrak{t}$.

If $H^2(W;\mathbb{R})$ is negative definite, then Theorem 4.1 may be applied to any spin^c–structure $\mathfrak{t}$ on W, giving

$$\delta(W,\mathfrak{t})\le\delta_{G,c}(Y,\mathfrak{s})$$

for all $\mathfrak{t}$ and all $c\in H_G^*$. Since Y is an integral homology sphere, the intersection form on the $H^2(W;\mathbb{Z})/\text{torsion}$ is unimodular. By the main theorem of [24], there exists a spin^c–structure $\mathfrak{t}$ such that $\delta(W,\mathfrak{t})\ge 0$. Hence $\delta_{G,c}(Y,\mathfrak{s})\ge 0$. The proof in the case that $H^2(W;\mathbb{R})$ is positive definite is similarly obtained. □

Example 7.11 Consider again $Y=\Sigma(p,q,r)$ with the same $\mathbb{Z}_p$–action. Recall from Proposition 7.2 that $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s})=-\lambda(\Sigma(p,q,r))$. As in Example 7.9, $\delta_{\mathbb{Z}_p,1}(Y,\mathfrak{s})>0$. So by Proposition 7.10, the action of $\mathbb{Z}_p$ on Y cannot be extended to any smooth, compact, oriented 4–manifold W such that $b_1(W)=0$, $H^2(W;\mathbb{Z})$ has no 2–torsion and with $\mathbb{Z}_p$ acting trivially on $H^2(W;\mathbb{Z})$.

7.3 Realisation problems

In this section we are concerned with the following realisation problem. Let W be a smooth 4–manifold with boundary an integral homology sphere Y. Suppose that a finite group G acts on $H^2(W;\mathbb{Z})$ preserving the intersection form. We say that the action of G on $H^2(W;\mathbb{Z})$ can be realised by diffeomorphisms if there is a smooth orientation-preserving action of G on W inducing the given action on $H^2(W;\mathbb{Z})$.

For simplicity we will assume that $G=\mathbb{Z}_p$ for a prime p so that all extensions $G_{\mathfrak{s}}$ are trivial.

Proposition 7.12 *Let W be a smooth, compact, oriented 4–manifold with $b_1(W)=0$ and with boundary $Y=\partial W$ an L–space integral homology sphere. Suppose that an action of $G=\mathbb{Z}_p$ on $H^2(W;\mathbb{Z})$ is given and suppose that $H^2(W;\mathbb{Z})$ has no 2–torsion. Suppose that the subspace of $H^2(W;\mathbb{R})$ fixed by G is negative definite. If the action of G on $H^2(W;\mathbb{Z})$ can be realised by diffeomorphisms, then*

$$\delta(W,\mathfrak{s})\le\delta(Y,\mathfrak{s}|_Y)$$

for every spin^c–structure $\mathfrak{s}$ on W for which $c_1(\mathfrak{s})$ is invariant.

Proof This is essentially a special case of Theorem 5.3. Note that since $H^2(W;\mathbb{Z})$ is assumed to have no 2–torsion, any spin^c–structure $\mathfrak{s}$ for which $c_1(\mathfrak{s})$ is invariant is preserved by G. So if G is realisable by diffeomorphisms, then Theorem 5.3 gives $\delta(W,\mathfrak{s})\le\delta_{G,1}(Y,\mathfrak{s}|_Y)$. But we have assumed that Y is an L–space, so $\delta_{G,1}(Y,\mathfrak{s}|_Y)=\delta(Y,\mathfrak{s}|_Y)$. □

Example 7.13 We consider a specialisation of Proposition 7.12 as follows. Take $G=\mathbb{Z}_p$. Assume Y is an L–space integral homology 3–sphere and let $\mathfrak{s}$ be the unique spin^c–structure. Suppose that W is a smooth, compact, oriented 4–manifold with $b_1(W)=0$ and with boundary Y. Suppose that the intersection form on $H^2(W;\mathbb{Z})$ is even and that $H^2(W;\mathbb{Z})$ has no 2–torsion. Then W is spin and it has a unique spin–structure $\mathfrak{t}$. By uniqueness, the restriction of $\mathfrak{t}$ to the boundary equals $\mathfrak{s}$. Suppose that an action of $G=\mathbb{Z}_p$ on $H^2(W;\mathbb{Z})$ is given and that the subspace of $H^2(W;\mathbb{R})$ fixed by G is negative definite. Then applying Proposition 7.12 to $(W,\mathfrak{t})$, we find that $\delta(W,\mathfrak{t})=-\frac{1}{8}\sigma(W)\le\delta(Y,\mathfrak{s})$. Therefore, if $\frac{1}{8}\sigma(W)<-\delta(Y,\mathfrak{s})$ then the action of $\mathbb{Z}_p$ on $H^2(W;\mathbb{Z})$ is not realisable by a smooth $\mathbb{Z}_p$–action on W.

For example, if $W=K3\,\#\,W_0$ is the connected sum of a $K3$ surface with W_0, the negative definite plumbing of the E_8 graph, then $\partial W=Y=\Sigma(2,3,5)$ is the Poincaré homology 3–sphere which is an L–space. Then W satisfies all the above conditions and $\frac{1}{8}\sigma(W)=-3<\delta(Y,\mathfrak{s})=-1$. Hence for any prime p, any $\mathbb{Z}_p$–action on $H^2(W;\mathbb{Z})$ such that the invariant subspace of $H^2(W;\mathbb{R})$ is negative definite cannot be realised by a smooth $\mathbb{Z}_p$–action on W.

Corollary 7.14 *Let W be a smooth, compact, oriented 4–manifold with $b_1(W)=0$ and with boundary $Y=\partial W$ an L–space integral homology sphere. Suppose that W is spin and that $H^2(W;\mathbb{Z})$ has no 2–torsion. If there is a smooth involution on W which acts as -1 on $H^2(W;\mathbb{R})$, then $\delta(Y,\mathfrak{s})=-\frac{1}{8}\sigma(W)$.*

Proof Since W is spin, there is a spin^c–structure $\mathfrak{s}$ for which $c_1(\mathfrak{s})=0$. Proposition 7.12 then implies that $-\frac{1}{8}\sigma(W)\le\delta(Y,\mathfrak{s})$. The same argument applied to $\overline{W}$ gives $\frac{1}{8}\sigma(W)\le\delta(Y,\mathfrak{s})$. □

7.4 Equivariant embeddings of 3–manifolds in 4–manifolds

Let Y be a rational homology 3–sphere equipped with an orientation-preserving action of G. By an equivariant embedding of Y into a 4–manifold X, we mean an embedding $Y \to X$ such that the action of G on Y extends over X. We consider some existence and nonexistence results for equivariant embeddings.

Proposition 7.15 *Suppose that Y is an integral homology 3–sphere. Let $\mathfrak{s}$ be the unique* spin^c*–structure on Y and assume that $G_{\mathfrak{s}}$ is a trivial extension. If Y can be equivariantly embedded in S^4, then $\delta_{G,c}(Y,\mathfrak{s}) = \delta_{G,c}(\overline{Y},\mathfrak{s}) = 0$ for every nonzero $c \in H^*_G$.*

Proof If Y embeds equivariantly in S^4, then we obtain an equivariant decomposition $S^4 = W_+ \cup_Y W_-$. Mayer–Vietoris and Poincaré–Lefschetz imply that $W_\pm$ are integral homology 4–balls, hence are contractible by Whitehead's theorem. The result now follows from Proposition 7.6. □

Example 7.16 Let $Y = \Sigma(2,3,13)$, equipped with the involution τ obtained from viewing Y as the branched double cover $\Sigma_2(T_{3,13})$. Then Y embeds in S^4 [13, Theorem 2.13]. On the other hand, $\delta_2(\overline{Y},\mathfrak{s}) = -1$ by Proposition 7.4. Hence Y cannot be equivariantly embedded in S^4.

It is known that every 3–manifold Y embeds in the connected sum $\#^n(S^2 \times S^2)$ of n copies of $S^2 \times S^2$ for some sufficiently large n [1, Theorem 2.1]. Aceto, Golla and Larson define the *embedding number* $\varepsilon(Y)$ of Y to be the smallest n for which Y embeds in $\#^n(S^2 \times S^2)$. Here we consider an equivariant version of the embedding number. To obtain interesting results we need to make an assumption on the kinds of group actions allowed.

Definition 7.17 Let $G = \mathbb{Z}_p = \langle \tau \rangle$, where p is a prime number. We say that a smooth, orientation-preserving action of G on $X = \#^n(S^2 \times S^2)$ is *admissible* if $H^2(X;\mathbb{Z})^\tau = 0$, where

$$H^2(X;\mathbb{Z})^\tau = \{x \in H^2(X;\mathbb{Z}) \mid \tau(x) = x\}.$$

One way of constructing admissible actions is as follows. Let X be the p–fold cyclic cover of S^4, branched over an unknotted embedded surface $\Sigma \subset S^4$ of genus g. Then X is diffeomorphic to $\#^{g(p-1)}(S^2 \times S^2)$ [3, Corollary 4.3] and the action of $\mathbb{Z}_p$ on X as a cyclic branched cover is admissible — as can be seen from the proof of Theorem 9.3 in [11].

Let $\mathbb{Z}_p = \langle \tau \rangle$ act on a rational homology 3–sphere Y. We define the *equivariant embedding number* $\varepsilon(Y,\tau)$ of (Y,τ) to be the smallest n for which Y embeds equivariantly in $\#^n(S^2 \times S^2)$ for some admissible $\mathbb{Z}_p$–action on $\#^n(S^2 \times S^2)$, if such an embedding exists. We set $\varepsilon(Y,\tau) = \infty$ if there is no such embedding.

Recall that the *double slice genus* [48, Section 5] $g_{\mathrm{ds}}(K)$ of a knot K in S^3 is defined as the minimal genus of an unknotted compact oriented surface S embedded in S^4 whose intersection with the equator S^3 is K. From the definition, it follows that $2g_4(K) \le g_{\mathrm{ds}}(K) \le 2g_3(K)$, where $g_3(K)$ is the 3–genus of K.

Proposition 7.18 *Let $Y = \Sigma_2(K)$ be the branched double cover of a knot K and let τ be the covering involution on Y. Then $\varepsilon(Y,\tau) \le g_{\mathrm{ds}}(K)$.*

Proof Let S be an unknotted embedded surface in S^4 of genus $g_{\mathrm{ds}}(K)$ intersecting the equator in K. Let W be the double cover of S^4 branched along S. Then by [3, Corollary 4.3], W is diffeomorphic to $\#^{g_{\mathrm{ds}}(K)}(S^2\times S^2)$. The covering involution on W is admissible since W is a branched double cover of S^4. Clearly $Y = \Sigma_2(K) = \partial W$ embeds equivariantly in W and so $\varepsilon(Y,\tau) \le g_{\mathrm{ds}}(K)$. □

Proposition 7.19 *Let $\mathbb{Z}_p = \langle\tau\rangle$ act orientation-preservingly on an integral homology 3–sphere Y. Let $j(Y,\tau)$ be the smallest positive integer such that $\delta_j(Y,\mathfrak{s},\tau,p) + \delta_j(\overline{Y},\mathfrak{s},\tau,p) = 0$, or $j(Y,\tau) = \infty$ if no such j exists. Here $\mathfrak{s}$ is the unique* spin^c*–structure on Y. Then $\varepsilon(Y,\tau) \ge j(Y,\tau)$ if $p = 2$ and $\varepsilon(Y,\tau) \ge 2j(Y,\tau)$ if p is odd.*

Proof To simplify notation we write $\delta_j(Y)$ for $\delta_j(Y,\mathfrak{s},\tau,p)$. Suppose that Y embeds equivariantly in $X = \#^n(S^2\times S^2)$, for an admissible action of τ. Then we obtain an equivariant splitting $X = X_+\cup_Y X_-$. Let $\mathfrak{t}$ be the unique spin–structure on X. By uniqueness, $\mathfrak{t}$ is τ–invariant and $\mathfrak{t}|_Y = \mathfrak{s}$. Corollary 5.5 applied to X_+ gives $\delta_j(Y) = -\frac{1}{8}\sigma(X_+)$ for $j \ge b'_-(X_+)$ and $\delta_j(\overline{Y}) = \frac{1}{8}\sigma(X_+)$ for $j \ge b'_+(X_+)$. Let $n' = n$ if $p = 2$ or $\frac{1}{2}n$ if p is odd. Since $b'_\pm(X_+) \le b'_\pm(X) = n'$, we see that $\delta_j(Y) + \delta_j(\overline{Y}) = 0$ for $j \ge n'$. Hence $j(Y,\tau) \le n'$. Therefore $\varepsilon(Y,\tau) \ge j(Y,\tau)$ if $p = 2$ and $\varepsilon(Y,\tau) \ge 2j(Y,\tau)$ if p is odd. □

Example 7.20 Let $Y = \Sigma(2,3,6n+1) = \Sigma_2(T_{3,6n+1})$ and equip Y with the covering involution τ. Then $g_3(T_{3,6n+1}) = 6n$; hence $\varepsilon(Y,\tau) \le 12n$, by Proposition 7.18. By Proposition 7.4, we see that $j(Y,\tau) = 2n$ and so $\varepsilon(Y,\tau) \ge 2n$. So we have an estimate on the equivariant embedding number of the form

$$2n \le \varepsilon(\Sigma(2,3,6n+1),\tau) \le 12n.$$

Suppose that n is odd. Then by [1, Proposition 3.5], the (nonequivariant) embedding number of $\Sigma(2,3,6n+1)$ is given by $\varepsilon(\Sigma(2,3,6n+1)) = 10$. In particular, $\varepsilon(\Sigma(2,3,6n+1),\tau) > \varepsilon(\Sigma(2,3,6n+1))$ for all odd $n > 5$. Also, since we obviously have $\varepsilon(Y,\tau) \ge \varepsilon(Y)$, we see that

$$10 \le \varepsilon(\Sigma(2,3,7),\tau) \le 12.$$

In fact, we will now prove that $\varepsilon(\Sigma(2,3,7),\tau) = 12$. Suppose that $Y = \Sigma(2,3,7)$ embeds equivariantly in $X = \#^n(S^2\times S^2)$ for some admissible involution, where $n \le 12$. Then we obtain an equivariant splitting $X = X_+\cup_Y X_-$. Since Y is an integral homology sphere, the intersection forms on $X_\pm$ are unimodular. They are also even, since X is spin. Moreover the Rochlin invariant of Y is 1. So the intersection forms of $X_\pm$ must contain at least one E_8 or $-E_8$ summand. Proposition 7.2 implies that $\delta_j(Y) = 1$ for all $j \ge 0$ and Proposition 7.4 implies that $\delta_j(\overline{Y}) = 0$ for $j = 0, 1$ and $\delta_j(\overline{Y}) = -1$ for $j \ge 2$. Since $n \le 12$, Corollary 5.5 applied to $X_\pm$ then implies that the intersection form of X_+ must be of the form $\alpha H \oplus (-E_8)$ for some $\alpha \ge 2$ (where H is the hyperbolic lattice) and similarly the intersection form of X_- must be of the form $\alpha' H \oplus (E_8)$ for some $\alpha' \geq 2$. The intersection form of X is then $(\alpha+\alpha'+8)H$ and so $n = \alpha+\alpha'+8 \geq 2+2+8 = 12$. This proves that

$$\varepsilon(\Sigma(2,3,7),\tau) = 12.$$

Example 7.21 Let $Y = \Sigma(2,3,5) = \Sigma_2(T_{3,5})$ and equip Y with the covering involution τ. Then $g_3(T_{3,5}) = 4$; hence $\varepsilon(Y,\tau) \leq 8$, by Proposition 7.18. On the other hand, $\varepsilon(Y,\tau) \geq \varepsilon(Y)$ and $\varepsilon(Y) = 8$ by [1, Proposition 3.4], so $\varepsilon(\Sigma(2,3,5),\tau) = 8$.

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Constructions stemming from nonseparating planar graphs and their Colin de Verdière invariant

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A planar graph G is said to be nonseparating if there exists an embedding of G in $\mathbb{R}^2$ such that, for any cycle $\mathscr{C} \subset G$, all vertices of $G \setminus \mathscr{C}$ are within the same connected component of $\mathbb{R}^2 \setminus \mathscr{C}$. Dehkordi and Farr classified the nonseparating planar graphs as either outerplanar graphs, subgraphs of wheel graphs, or subgraphs of elongated triangular prisms. We use maximal nonseparating planar graphs to construct examples of maximal linkless graphs and maximal knotless graphs. We show that, for a maximal nonseparating planar graph G with $n \geq 7$ vertices, the complement cG is $(n-7)$–apex. This implies that the Colin de Verdière invariant of the complement cG satisfies $\mu(cG) \leq n-4$. We show this to be an equality. As a consequence, the conjecture of Kotlov, Lovász and Vempala that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$ is true for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating. It also follows that complements of nonseparating planar graphs of order at least nine are intrinsically linked. We prove that the complements of nonseparating planar graphs G of order at least ten are intrinsically knotted.

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1 Introduction

All graphs in this paper are finite and simple. A graph is *intrinsically linked* (IL) if every embedding of it in $\mathbb{R}^3$ (or S^3) contains a nontrivial 2–component link. A graph is *linklessly embeddable* if it is not intrinsically linked (nIL). A graph is *intrinsically knotted* (IK) if every embedding of it in $\mathbb{R}^3$ (or S^3) contains a nontrivial knot. The combined work of Conway and Gordon [1983], Sachs [1984] and Robertson, Seymour and Thomas [Robertson et al. 1993] fully characterize IL graphs: a graph is IL if and only if it contains a graph in the Petersen family as a minor. The Petersen family consists of seven graphs obtained from the complete graph K_6 by ∇Y moves and $Y\nabla$ moves, as described in Figure 1.

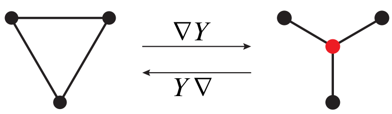


Figure 1: ∇Y and $Y\nabla$ moves.

The ∇Y move and the $Y\nabla$ move preserve the IL property. While K_7 and $K_{3,3,1,1}$ together with many other minor minimal IK graphs have been found [Goldberg et al. 2014; Conway and Gordon 1983; Foisy 2002], a characterization of IK graphs is not fully known. While the ∇Y move preserves the IK property [Motwani et al. 1988], the $Y\nabla$ move doesn't preserve it [Flapan and Naimi 2008]. A graph is said to be k–apex if it can be made planar by removing k vertices. If G and H denote two simple graphs with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively, then the *sum* $G+H$ denotes the simple graph with vertex set $V(G)\sqcup V(H)$ and edge set $E(G)\sqcup E(H)\sqcup L$, where L denotes the set of all edges with one endpoint in $V(G)$ and the other in $V(H)$.

A planar graph G is *nonseparating* if there exists an embedding of G in $\mathbb{R}^2$ such that, for any cycle $\mathscr{C}\subset G$, all vertices of $G\setminus\mathscr{C}$ are within the same connected component of $\mathbb{R}^2\setminus\mathscr{C}$. By [Dehkordi and Farr 2021], a nonseparating planar graph is one of three types:

(1) an outerplanar graph,

(2) a subgraph of a wheel,

(3) a subgraph of an elongated triangular prism.

In Section 2, we consider sums between maximal nonseparating planar graphs and small empty graphs, complete graphs or paths to construct maximal linklessly embeddable graphs and maximal knotlessly embeddable graphs. A simple graph G is called *maximal linklessly embeddable* (maxnIL) if it is not a proper subgraph of a nIL graph of the same order. A simple graph G is called *maximal knotlessly embeddable* (maxnIK) if it is not a proper subgraph of a nIK graph of the same order. Constructions and properties of maxnIL graphs can also be found in [Aires 2021; Naimi et al. 2023], and for maxnIK graphs in [Eakins et al. 2023].

Colin de Verdière [1990] introduced the graph invariant μ, which is based on spectral properties of matrices associated with the graph G. He showed that μ is monotone under taking minors and that planarity is characterized by the inequality $\mu\le 3$. By [Lovász and Schrijver 1998; Robertson et al. 1993], it is known that linkless embeddability is characterized by the inequality $\mu\le 4$. By reformulating the definition of μ in terms of vector labelings, Kotlov, Lovász and Vempala [Kotlov et al. 1997] related the topological properties of a graph to the μ invariant of its complement: for G a simple graph on n vertices,

(1) if G is planar, then $\mu(cG)\ge n-5$;

(2) if G is outerplanar, then $\mu(cG)\ge n-4$;

(3) if G is a disjoint union of paths, then $\mu(cG)\ge n-3$.

For G a graph with n vertices $v_1, v_2, \dots v_n$, cG denotes the *complement of* G in the complete graph K_n. The graph cG has the same set of vertices as G and $E(cG)=\{v_iv_j \mid v_iv_j\notin E(G)\}$.

By [Battle et al. 1962], the complement of a planar graph with nine vertices is not planar. This is also implied by the inequality $\mu(cG)\ge n-5$. Here we show a stronger inequality for maximal nonseparating planar graphs. In Section 3, we prove two theorems.

Theorem 1 *If G is a maximal nonseparating planar graph with $n \geq 7$ vertices, then cG is $(n-7)$–apex.*

Theorem 1 establishes the upper bound $\mu(cG) \leq n-4$ for G a maximal nonseparating planar graph, since $\mu \leq 3$ for planar graphs and adding one vertex increases the value of μ by at most one [van der Holst et al. 1999]. We prove this is an equality.

Theorem 2 *For G a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) = n-4$.*

Kotlov et al. [1997] conjectured that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$. We revisit results about μ to show the conjecture is true for planar graphs and 1–apex graphs. As a consequence of Theorem 2, the conjecture holds for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating. Theorem 2 also implies that, for G a maximal nonseparating planar graph with nine vertices, $\mu(cG) = 5 > 4$, and thus cG is intrinsically linked. While the relationship between the μ invariant and intrinsic linkness is well understood, the same is not true for intrinsic knottedness. The inequality $\mu(cG) \geq n-5$ for planar graphs G implies that complements of planar graphs with ten vertices are intrinsically linked. Theorem 2 establishes that, for G a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, but this does not imply that cG is intrinsically knotted. There are known IK graphs with $\mu = 5$ [Foisy 2003; Mattman et al. 2021], as well as nIK graphs with $\mu = 6$ [Flapan and Naimi 2008]. In Section 4, we do a case-by-case analysis to prove the following theorem:

Theorem 3 *If G is a nonseparating planar graph on ten vertices, then cG is intrinsically knotted.*

Since the complement of a nonseparating planar graph contains the complement of a maximal nonseparating planar graph of the same order as a subgraph, it suffices to prove Theorem 3 for maximal nonseparating planar graphs, namely

(1) maximal outerplanar graphs,
(2) the wheel graph,
(3) elongated triangular prisms.

A similar approach to that presented in Section 4 works to prove that:

(a) If G is a nonseparating planar graph on seven vertices, then cG is not outerplanar.
(b) If G is a nonseparating planar graph on eight vertices, then cG is nonplanar.
(c) If G is a nonseparating planar graph on nine vertices, then cG is intrinsically linked.

For outerplanar graphs G with at most nine vertices, these results can also be obtained using the graph invariant μ, since, for such graphs G, $\mu(cG) \geq n-4$ [Kotlov et al. 1997].

2 MaxnIL and maxnIK graphs

In this section, we use maximal nonseparating planar graphs to build examples of maxnIL and maxnIK graphs. Jørgensen [1989] and Dehkordi and Farr [2021] considered the class of graphs of the type $H + E_2$, where E_2 denotes the graph with two vertices and no edges and H is an elongated prism. Jørgensen proved that these graphs are maximal with no K_6 minors. Dehkordi and Farr proved that these graphs are maxnIL. Here we add to this type of example by taking the sum of maximal nonseparating planar graphs with small empty graphs, complete graphs and paths. Sachs [1984] proved that 1–apex graphs are nIL and 2–apex graphs are nIK. A theorem of Mader [1968] shows that a graph G with n vertices and $4n-9$ edges, with $n \geq 6$, contains a K_6 minor, and a graph G with n vertices and $5n-14$ edges, with $n \geq 7$, contains a K_7 minor. We combine these results into the following useful lemma:

Lemma 4 *A maximal 1–apex graph is maxnIL. A maximal 2–apex graph is maxnIK.*

A vertex of a graph H which is incident to all the other vertices of H is a *cone*. We also say that v cones over the subgraph induced by all the vertices of H minus v. Let W_n denote the wheel graph of order $n \geq 4$. Let P_2 be the graph with vertex set $V(P_2) = \{u, v, w\}$ and edge set $E(P_2) = \{\{u, w\}, \{v, w\}\}$. Let K_3 denote the complete graph on vertices $\{u, v, w\}$. Using Lemma 4, we derive the following result:

Theorem 5 (1) *The graph $G \simeq W_n + E_2$ is maxnIL.*

(2) *If H is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_2$ is a maxnIL graph.*

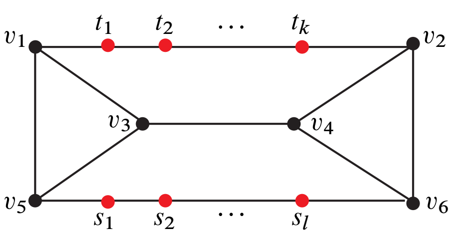
(3) *The graph $G \simeq W_n + P_2$ is maxnIK.*

(4) *If H is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_3$ is a maxnIK graph.*

Proof For the first two cases, the graph G is maximal 1–apex, and thus maxnIL. For the last two cases, the graph G is maximal 2–apex, and thus maxnIK. □

For the elongated prism case, we distinguish two cases, according to the number of nontriangular edges of the triangular prism which are subdivided.

Theorem 6 *Let H denote an elongated prism of order $n \geq 6$ obtained by repeated subdivisions of at most two of three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is a maxnIK graph.*



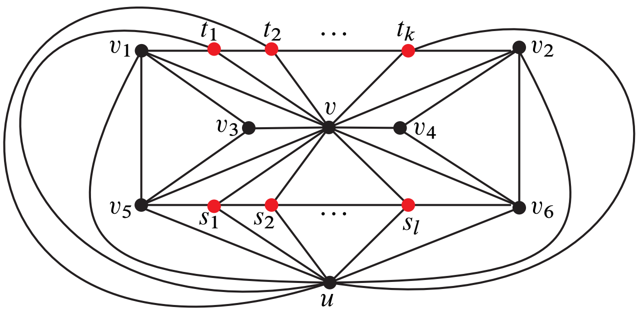


Figure 2: An elongated prism with only two edges subdivided (left) and a planar graph obtained by deleting the vertices t and w of $H + P_2$ (right).

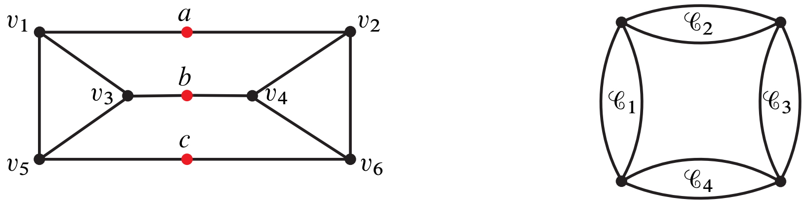


Figure 3: The graph P' obtained by subdividing once each nontriangular edge of the prism graph (left) and the graph D_4 (right).

Proof Assume that H is isomorphic to the graph depicted in Figure 2, left, in which the edge $\{v_3, v_4\}$ is not subdivided. Perform a ∇Y move on the triangle induced by the vertices $\{v_3, v_4, u\}$ by deleting the edges $\{v_3, v_4\}$, $\{v_3, u\}$ and $\{v_4, u\}$ and adding a new vertex t incident to all of $\{v_3, v_4, u\}$ to obtain a new graph G'. This graph is 2–apex, since deleting the vertices t and w gives the planar graph of Figure 2, right. Thus, G' is nIK, and so must be G, as the ∇Y move preserves the IK property [Motwani et al. 1988].

To show that G is maximal nIK, one notices that G is isomorphic to a cone w over $H + E_2$. Since $H + E_2$ is maxnIL by [Dehkordi and Farr 2021], adding any edge to G produces a structure of a cone over an IL graph. This structure will contain a minor isomorphic to a graph in either the K_7 family or the $K_{3,3,1,1}$ family, and will therefore be IK. □

Theorem 7 *Let H denote an elongated prism of order $n \geq 9$ obtained by repeated subdivisions of all three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is an IK graph.*

Proof By repeated edge contractions applied to G, one obtains the minor $S \simeq P' + P_2$, where P' is the graph depicted in Figure 3, left.

Foisy [2002] proved that, if a graph contains a doubly linked D_4 minor in every embedding, the graph must be IK. This result was also proved independently by Taniyama and Yasuhara [2001]. The graph D_4 is depicted in Figure 3, right. An embedding of the graph D_4 is doubly linked if the linking numbers $\mathrm{lk}(C_1, C_3)$ and $\mathrm{lk}(C_2, C_4)$ are both nonzero mod 2. We used a Mathematica program written by Naimi to show that S has a doubly linked D_4 minor in every embedding. □

3 The μ invariant

In this section we determine the value of the μ invariant for complements of maximal nonseparating planar graphs. By [van der Holst et al. 1999], if G is planar with n vertices, then $\mu(cG) \geq n-5$. We first show the inequality $\mu(cG) \leq n-4$ for graphs G which are maximal nonseparating planar. In Theorem 2, we show this is in fact an equality.

Kotlov et al. [1997] conjectured that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$. We review that the conjecture holds for planar graphs and 1–apex graphs. We show that, as a consequence of Theorem 2, the conjecture holds for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating.

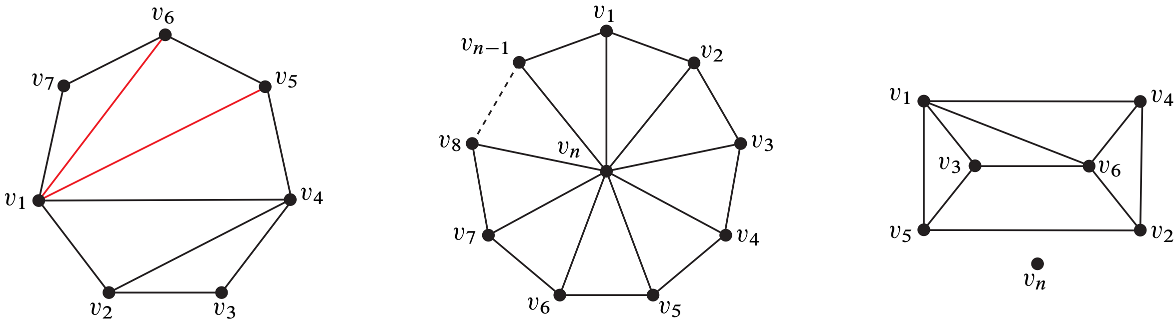


Figure 4: A maximal outerplanar graph with seven vertices (left), the graph G, a wheel with n vertices (center), and $cG \setminus \{v_7, v_8, \dots, v_{n-1}\}$ (right).

Theorem 1 *If G is a maximal nonseparating planar graph with $n \geq 7$ vertices, then cG is $(n-7)$–apex.*

Proof We treat the three types in turn:

Outerplanar case Any maximal outerplanar graph H of order $n \geq 3$ can be represented by a triangulated n–cycle in the plane (with the unbounded face containing all vertices). The n–cycle contains at least one 2–chord, an edge which forms a triangle with two adjacent edges along the cycle. We say that the 2–chord isolates the vertex which is part of the triangle but is not incident to the 2–chord. For example, in Figure 4, left, the 2–chord $v_1 v_6$ isolates the vertex v_7 and the 2–chord $v_1 v_5$ of $H - \{v_7\}$ isolates v_6. The complement of the unique maximal outerplanar graph with five vertices is P_3, a path with three edges, together with an isolated vertex. It follows that the complement of any maximal outerplanar graph with seven vertices is planar, since the deletion of two vertices gives a path with three edges and an isolated vertex. For example, after the deleting the vertices v_7 and v_6, the complement of the graph in Figure 4, left, is the path $v_1 v_3 v_5 v_2$ together with the isolated vertex v_4. Starting with a maximal outerplanar graph with $n \geq 7$ vertices, one can recursively delete $n-7$ isolated vertices and obtain a maximal outerplanar graph of order 7. The same sequence of $n-7$ vertex deletions gives a planar subgraph of cG. Thus, cG is $(n-7)$–apex.

Wheel case Let G be the wheel on n vertices. Then $cG \simeq (K_{n-1} \setminus C_{n-1}) \cup K_1$. Let $\{v_1, v_2, \dots, v_{n-1}\}$ be the vertices of C_{n-1} in consecutive order, as in Figure 4, center. Then $cG \setminus \{v_7, v_8, \dots, v_{n-1}\}$ is a planar graph (the triangular prism added one edge, together with an isolated vertex) and thus cG is $(n-7)$–apex. See Figure 4, right.

Elongated prism case Let G be an elongated prism with $n \geq 7$ vertices. Without loss of generality, let $v_1 v_3 v_5$ be one of two induced triangles of G. Let a, b and c denote their respective neighbors in $V(G) \setminus \{v_1, v_3, v_5\}$, as in Figure 5, left. Deleting all vertices but $\{v_1, v_3, v_5, a, b, c\}$ in cG gives the subgraph of the outerplanar graph with six vertices in Figure 5, right. Deleting any $n-7$ vertices of cG none of which is in the set $\{v_1, v_3, v_5, a, b, c\}$ yields a planar graph, and thus cG is $(n-7)$–apex. □

Corollary 8 *For G a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) \leq n-4$.*

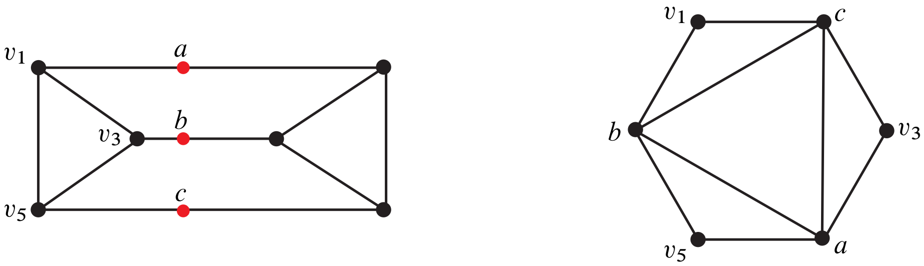


Figure 5: An elongated prism (left) and the subgraph induced by $\{v_1, v_3, v_5, a, b, c\}$ in cG (right).

Proof By Theorem 1, cG is $(n-7)$–apex. Let H be the planar subgraph of cG obtained by deleting $n-7$ vertices. Then $\mu(H) \le 3$ and $\mu(cG) \le 3 + (n-7) = n-4$, since adding one vertex to a graph increases the value of μ by at most one (see [van der Holst et al. 1999, Theorem 2.7]). □

Corollary 8 establishes an upper bound of $n-4$ for the values of μ of complements of maximal nonseparating planar graphs on n vertices. We show that $n-4$ is the actual value of μ. We use [van der Holst et al. 1999, Theorem 5.5], whice says that, for H a graph on n vertices and $\nu(H) := n - \mu(cH) - 1$, the inequality $\nu(H) \le 2$ holds if and only if H does not contain as a subgraph any of the five graphs in Figure 6. We also use that, for a graph G with at least one edge, $\mu(G + K_1) = \mu(G) + 1$ by [van der Holst et al. 1999, Theorem 2.7].

Theorem 2 *For G a maximal nonseparating planar graph with $n \ge 7$ vertices, $\mu(cG) = n-4$.*

Proof Corollary 8 established the inequality $\mu(cG) \le n-4$. Here we show that $\mu(cG) \ge n-4$. If G is outerplanar, then $\mu(cG) \ge n-4$ [Kotlov et al. 1997]. If G is the wheel graph on n vertices, $cG = cC_{n-1} \cup K_1$. By [van der Holst et al. 1999, Theorem 5.5], $\nu(C_{n-1}) \le 2$ and we have

$$\mu(cG) = \mu(cC_{n-1}) = n - 1 - \nu(C_{n-1}) - 1 \ge n-4.$$

For elongated prisms, we distinguish two cases, according to the number of nontriangular edges of the prism which are being subdivided:

Case 1 Consider G the elongated prism in Figure 7, left, with exactly one nontriangular edge of the prism graph subdivided, v_1v_2, If at least two vertices are added along v_1v_2, as in Figure 7, left, consider

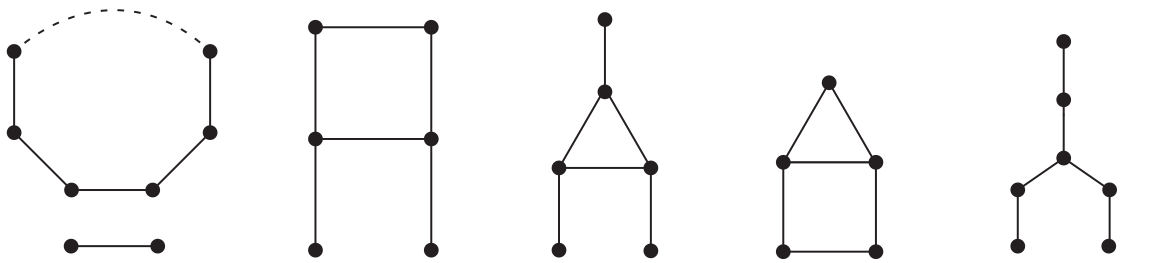


Figure 6: Five graphs.

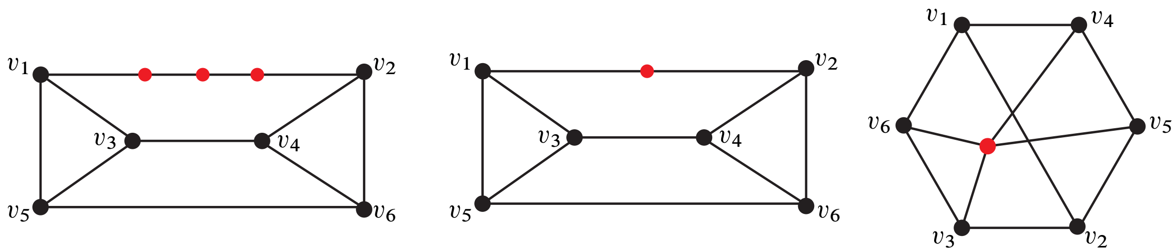


Figure 7: An elongated prism with one nontriangular edge subdivided by more than one vertex (left), an elongated prism with one nontriangular edge subdivided by exactly one vertex (center) and the complement of that graph (right).

the graph $H = G - \{v_1, v_2\}$. Then $\mu(cH) = (n-2) - \nu(H) - 1 \geq n-5$, by [van der Holst et al. 1999, Theorem 5.5]. Since in cG the set of adjacent vertices $\{v_1, v_2\}$ cones over cH, $\mu(cG) \geq n-4$ by [van der Holst et al. 1999, Theorem 2.7]. If only one vertex is added along the one edge, as in Figure 7, center, the set of adjacent vertices $\{v_1, v_2\}$ no longer cones over cH. However, in this case, cG contains a K_4 minor, and thus $\mu(cG) \geq 3$. See Figure 7, right.

Case 2 Assume G is obtained from the triangular prism by subdividing edges v_1v_2 and v_5v_6 along the way, as in Figure 8, left. The graph $H = G - \{v_1, v_6\}$ is a path with $n-2$ vertices, so $\mu(cH) \geq n-5$ [Kotlov et al. 1997]. In cG, the set of adjacent vertices $\{v_1, v_6\}$ cones over cH, yielding $\mu(cG) \geq \mu(cH)+1 \geq n-4$ by [van der Holst et al. 1999, Theorem 2.7]. □

We briefly discuss the state of a conjecture of [Kotlov et al. 1997], that, for a simple graph G on n vertices, $\mu(G) + \mu(cG) \geq n-2$. By [Kotlov et al. 1997; Colin de Verdière 1990; van der Holst et al. 1999], the conjecture holds if either one of G or cG is planar. We note that the conjecture holds if $\mu(G) \geq n-6$ or $\mu(cG) \geq n-6$. Assume $\mu(G) \geq n-6$. If $\mu(cG) \geq 4$, then $\mu(G) + \mu(cG) \geq n-2$; if $\mu(cG) < 4$, $\mu(G)$ is planar, and the conjecture holds.

Proposition 9 *The conjecture holds for* 1*–apex graphs.*

Proof Let G be a 1–apex graphs with n vertices and $H = G - \{v\}$ planar. Then $\mu(cH) \geq (n-1) - 5 = n-6$ [Kotlov et al. 1997]. We have that cH, the complement of H in K_{n-1}, is a subgraph of cG, the complement of G in K_n, since cG may have additional edges incident to v, and so $n-6 \leq \mu(cH) \leq \mu(cG)$. Thus, the conjecture holds for G. □

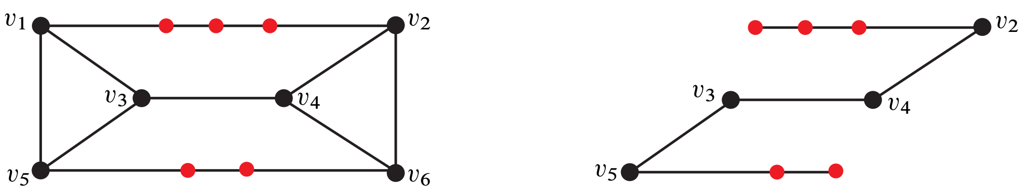


Figure 8: An elongated prism G with two subdivided edges (left) and $H = G - \{v_1, v_6\}$ (right).

Corollary 10 *Let G be a 2–apex graph with n vertices with $H = G - \{u, v\}$ planar nonseparating. Then $\mu(G) + \mu(cG) \geq n - 2$.*

Proof Since H is planar nonseparating, by Theorem 2, $\mu(cH) \geq (n-2) - 4 = n - 6$, with equality if H is maximal. We have that cH, the complement of H in K_{n-2}, is a subgraph of cG, the complement of G in K_n, since cG may have additional edges incident to u and v, and so $\mu(cG) \geq \mu(cH) \geq n - 6$. Thus, the conjecture holds for G. □

4 Graphs of order ten

The relationship between the μ invariant and the property of being intrinsic knotted is not well understood. While Theorem 2 establishes that, for G a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, this information has no bearing on whether cG is intrinsically knotted. Flapan and Naimi [2008] prove that the IK property is not preserved by the $Y\nabla$ move by showing a graph in the K_7 family which is not intrinsically knotted. Since $\mu(K_7) = 6$ and both the ∇Y move and the $Y\nabla$ move preserve μ for $\mu \geq 4$ [van der Holst et al. 1999], this nIK graph has $\mu = 6$. On the other hand, Foisy [2003] and Mattman et al. [2021] provide examples of IK graphs with $\mu = 5$. In this section, we do a case-by-case analysis to prove that, for G a maximal nonseparating planar graph with ten vertices, cG is intrinsically knotted. We recall that the ∇Y move preserves the IK property. In some cases, graphs are shown to be IK because they are obtained through one or more ∇Y moves from IK graphs such as K_7 or $K_{3,3,1,1}$. In other cases, graphs G are shown to be IK because the graphs obtained from G by one or more $Y\nabla$ moves contain K_7 or $K_{3,3,1,1}$ minors.

Lemma 11 *If G is a maximal outerplanar graph with ten vertices, then cG is intrinsically knotted.*

Proof We label the vertices of G by $v_1, v_2, \ldots, v_9, v_{10}$ in clockwise order around the cycle $\mathscr{C}$ bordering the outer face of a planar embedding. See Figure 9. We organize the proof according to the longest chord of $\mathscr{C}$. The length of a chord is defined as the length of the shortest path in $\mathscr{C}$ between the endpoints of the chord. In each case we show the complement cG contains an intrinsically knotted graph as a minor. We remark that, within any triangulation of the disk bounded by $\mathscr{C}$, out of a total of seven chords, at most six have length 2 or 3. Thus there exist chords of length 4 or 5.

Case (a) If the cycle $\mathscr{C}$ has a chord of length 5, we may assume without loss of generality that $v_1v_6 \in E(G)$. Consider the cycles $\mathscr{C}_1 := v_1v_6v_7v_8v_9v_{10}$ and $\mathscr{C}_2 := v_1v_2v_3v_4v_5v_6$. We note that $\mathscr{C}$ necessarily contains a 3–chord or a 4–chord with one endpoint at v_1 or v_6 and the other endpoint among the vertices of $\mathscr{C}_i$ for $i = 1, 2$. We distinguish six cases, according to whether there are any 4–chords at all and whether these chords share one of their ends:

(a1) Assume there exists a 4–chord incident to v_1 or v_6, say $v_1v_5 \in E(G)$.

(i) If $v_1v_7 \in E(G)$ (see Figure 9, far left), then the complement cG contains as a subgraph the graph obtained through two ∇Y moves from K_7 with vertex set $\{v_2, v_3, v_4, v_8, v_9, v_{10}, v_6\}$: one ∇Y

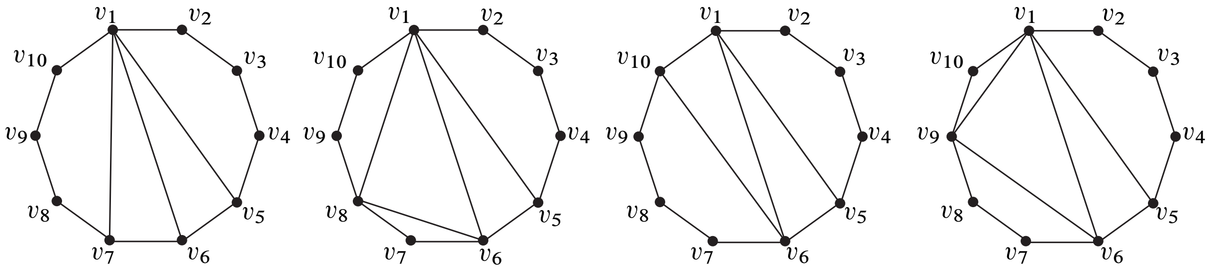


Figure 9: Outerplanar graphs with ten vertices.

move over the triangle $v_2v_3v_4$ with new vertex v_7 and one ∇Y move over the triangle $v_8v_9v_{10}$ with new vertex v_5.

(ii) If $v_1v_7 \notin E(G)$ and $v_1v_8 \in E(G)$ (see Figure 9, center left), then, in cG, delete any edges incident to v_5 except v_5v_8, v_5v_9 and v_5v_{10}, then perform a $Y\nabla$ move at v_5 to create a graph containing the triangle $v_8v_9v_{10}$. This graph contains a $K_{3,3,1,1}$ minor with partition $\{v_2, v_3, v_4\}$, $\{v_6, v_7, v_8\}$, $\{v_9\}$, $\{v_{10}\}$.

(iii) If $v_6v_{10} \in E(G)$ (see Figure 9, center right), then, in cG, delete any edges incident to v_1 except v_1v_7, v_1v_8 and v_1v_9, then perform a $Y\nabla$ move at v_1 to create a graph containing the triangle $v_7v_8v_9$. Further, delete any edges incident to v_6 except v_2v_6, v_3v_6 and v_4v_6, then perform a $Y\nabla$ move at v_6 to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract v_5v_{10} to a new vertex t to obtain a K_7 minor with vertices $\{v_2, v_3, v_4, v_7, v_8, v_9, t\}$.

(iv) If $v_6v_{10} \notin E(G)$ and $v_6v_9 \in E(G)$ (see Figure 9, far right), then, in cG, delete any edges incident to v_6 except v_6v_2, v_6v_3 and v_6v_4, then perform a $Y\nabla$ move at v_6 to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract the edge v_5v_9 to a vertex t, and contract the edge v_1v_7 to a vertex t_7 to obtain a K_7 minor with vertices $\{v_2, v_3, v_4, t_7, v_8, v_{10}, t\}$.

(a2) Assume there is no 4–chord of $\mathscr{C}$ incident to v_1 or v_6. There are two 3–chords of $\mathscr{C}$ incident to v_1 or v_6 and endpoints in each $\mathscr{C}_1$ and $\mathscr{C}_2$. Assume $v_1v_4 \in E(G)$.

(i) If $v_1v_8 \in E(G)$ (see Figure 10, far left), for any choice of edges which triangulate the quadrilaterals $v_1v_2v_3v_4$ and $v_8v_9v_{10}v_1$, the complement cG contains as a subgraph the graph Cousin 12 of

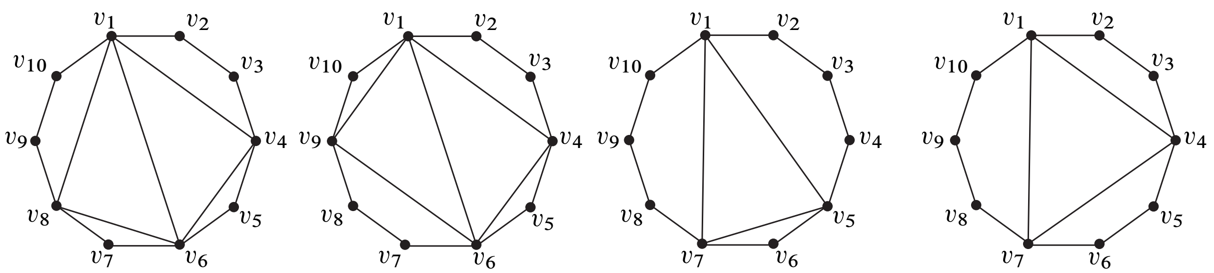


Figure 10: Outerplanar graphs with ten vertices.

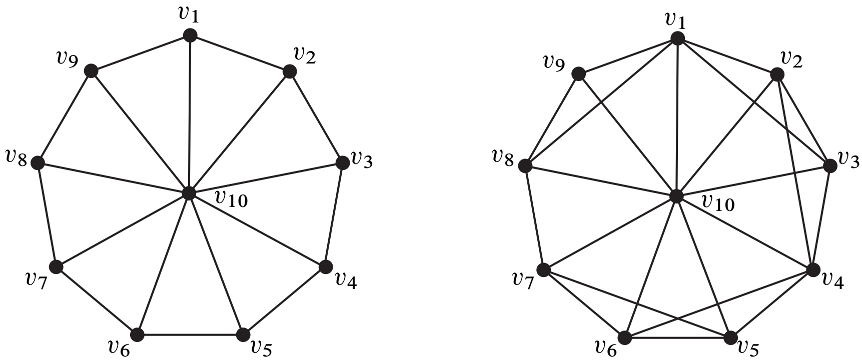


Figure 11: A wheel graph with ten vertices (left) and the complement of $E_9 + e$ in K_{10} (right)

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$K_{3,3,1,1}$ described in [Goldberg et al. 2014]. This is a minor minimal IK graph with nine vertices obtained from $K_{3,3,1,1}$ by two ∇Y moves followed by a $Y\nabla$ move.

(ii) If $v_6v_9 \in E(G)$ (see Figure 10, center left), obtain a K_7 minor of cG by contracting the edges v_1v_8, v_2v_6 and v_4v_9.

Case (b) Assume the cycle $\mathscr{C}$ has no chord of length 5. Then it has at least a chord of length 4. Assume $v_1v_7 \in E(G)$. Up to symmetry, we recognize two cases.

(b1) If $v_1v_5 \in E(G)$ (see Figure 10, center right), then the complement cG contains the graph obtained through two ∇Y moves from K_7 with vertex set $\{v_2, v_3, v_4, v_6, v_8, v_9, v_{10}\}$: one ∇Y move over the triangle $v_2v_3v_4$ with new vertex v_7 and one ∇Y move over the triangle $v_8v_9v_{10}$ with new vertex v_5.

(b2) If $v_1v_4, v_4v_7 \in E(G)$ (see Figure 10, far right), then, in cG, delete any edge incident to v_4 except v_4v_8, v_4v_9 and v_4v_{10}, then perform a $Y\nabla$ move at v_4 to create a graph containing the triangle $v_8v_9v_{10}$. Within this graph, contract the edges v_1v_5 to t_5 and v_2v_7 to t_2 obtain a K_7 with vertex set $\{t_2, v_3, t_5, v_6, v_8, v_9, v_{10}\}$. □

Lemma 12 *If G is a wheel with ten vertices, then cG is intrinsically knotted.*

Proof The graph $E_9 + e$ is a minor minimal intrinsically knotted graph with nine vertices described in [Goldberg et al. 2014]. The complement of $E_9 + e$ in K_{10} contains the 10–wheel as a subgraph. See Figure 11. Thus, the complement cG contains $E_9 + e$ as a subgraph and therefore it is intrinsically knotted. □

Lemma 13 *If G is an elongated triangular prism with ten vertices, then cG is intrinsically knotted.*

Proof An elongated prism with ten vertices is obtained by subdividing the three nontriangular edges of the prism with four vertices. These four vertices can be added in four different ways:

(a) on three different edges,

(b) on two edges with a 2-2 partition,

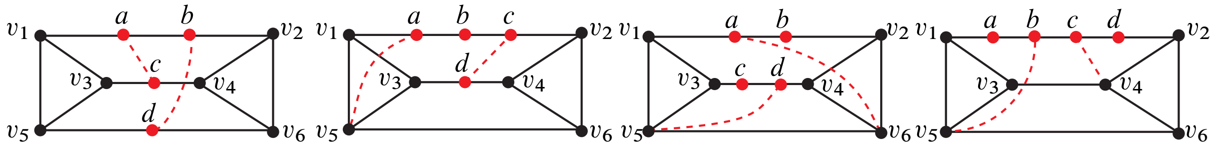

Figure 12: Elongated prisms with ten vertices. Dashed edges are edges of the complement graph.

(c) on two edges with a 3-1 partition,

(d) all on one edge.

See Figure 12. In each case, we show that cG contains a $K_{3,3,1,1}$ minor.

Case (a) The four vertices are added on three different edges of the elongated prism, as in Figure 12, far left. Within cG, contract the edge ac to the vertex t and bd to u to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{t\}$, $\{u\}$.

Case (b) The four vertices are added to two edges of the elongated prism with a 2-2 partition, as in Figure 12, center left. Within cG, contract dv_5 to t_5 and av_6 to t_6 to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, c\}$, $\{v_2, v_4, b\}$, $\{t_5\}$, $\{t_6\}$.

Case (c) The four vertices are added to two edges of the elongated prism with a 3-1 partition, as in Figure 12, center right. Within cG, contract av_5 to t_5 and cd to t to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, t_5\}$, $\{v_2, v_4, v_6\}$, $\{b\}$, $\{t\}$.

Case (d) The four vertices are added all on one edge of the elongated prism, as in Figure 12, far right. Within cG, contract bv_5 to t_5 and cv_4 to t_4 to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, a\}$, $\{v_2, v_6, d\}$, $\{t_4\}$, $\{t_5\}$. □

Since $cG \subseteq cH$ for H a subgraph of G of the same order, Lemmas 11, 12 and 13 give the following theorem:

Theorem 3 *If G is a nonseparating planar graph on ten vertices, then cG is intrinsically knotted.*

Corollary 14 *For $n \geq 10$, the complement of a nonseparating planar graph on n vertices is IK.*

Remark 15 The bound $n \geq 10$ in Corollary 14 is the best possible. If G is the 9–wheel, then $cG \setminus v = K_8 \setminus C_8$. Here v is the isolated point within the complement of the wheel. Since it has 20 edges, $K_8 \setminus C_8$ is 2–apex and it is therefore knotlessly embeddable [Mattman 2011]. As cG is isomorphic to $K_8 \setminus C_8$ with the isolated vertex v added, it is also 2–apex, and thus nIK.

Note that, in the proof of Theorem 3, we've showed that the complements of nonseparating planar graphs of order 10 all have minor minimal intrinsically knotted minors of smaller order. From this it follows that there are no minor minimal intrinsically knotted (MMIK) graphs of order ten or more with nonseparating planar complements. On the other hand, by the combined work of [Blain et al. 2007; Conway and Gordon

1983; Campbell et al. 2008; Foisy 2002; Goldberg et al. 2014; Kohara and Suzuki 1992; Mattman et al. 2017], the eleven MMIK graphs of order at most 9 are known. Considering their complements, there are just four MMIK graphs with nonseparating planar complements.

Corollary 16 *There are exactly four minor minimal intrinsically knotted graphs whose complements are nonseparating planar*: K_7, $K_{3,3,1,1}$, K_7^{∇} (*the graph obtained by performing a single* ∇Y*–move on* K_7) *and* $G_{9,28}$ (*the complement of a* 7*–cycle and an independent edge inside* K_9).

Proof By inspection, the complements of the four graphs are nonseparating. The complements of the remaining seven order 9 graphs are planar, but none of them are nonseparating as:

- They cannot be subgraphs of an elongated prism of order 9 (size 12), since their size (14–15) is too big.
- They all have at least two vertices of degree bigger than 3; thus, they cannot be subgraphs of a wheel graph.
- They all have a K_4 minor; thus, they cannot be outerplanar. □

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Census L–space knots are braid positive, except for one that is not

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Marc Kegel

We exhibit braid positive presentations for all L–space knots in the SnapPy census except one, which is not braid positive. The normalized HOMFLY polynomial of $o9\_30634$, when suitably normalized, is not positive, failing a condition of Ito for braid positive knots.

We generalize this knot to a 1–parameter family of hyperbolic L–space knots that may not be braid positive. Nevertheless, as pointed out by Teragaito, this family yields the first examples of hyperbolic L–space knots whose formal semigroups are actual semigroups, answering a question of Wang. Further, the roots of the Alexander polynomials of these knots are all roots of unity, disproving a conjecture of Li and Ni.

57K10; 57M12, 57R65

1 Introduction

Based on observation, most L–space knots are braid positive. Here *L–space knots* are knots in S^3 with a *positive* Dehn surgery to an L–space (see Ozsváth and Szabó [26]), and a knot that is the closure of a positive braid is *braid positive*. The L–space torus knots are the positive torus knots, and hence they are braid positive. Notably however, the $(2,3)$–cable of the $(2,3)$–torus knot is an L–space knot (see Hedden [16]) that is not braid positive; see eg Dunfield [12, Table 8] and Anderson, Baker, Gao, Kegel, Le, Miller, Onaran, Sangston, Tripp, Wood, and Wright [1, Example 1]. It stands to reason that there probably are other cable L–space knots which are not braid positive. Nevertheless, it was questioned if every hyperbolic L–space knot is braid positive; see eg Hom, Lipschitz, and Ruberman [19, Problem 31(2)].

Dunfield showed that there are exactly 1267 complements of knots in S^3 in the SnapPy census of 1–cusped hyperbolic manifolds that can be triangulated with at most nine ideal tetrahedra [11]. He further determined that (up to mirroring) 635 are not L–space knots, 630 are L–space knots, and left two as undetermined [12]. These last two have been shown to have quasialternating surgeries (see Baker, Kegel, and McCoy [3]) and hence they are L–space knots as well. Thus there are exactly 632 L–space knots in the SnapPy census.

Theorem 1.1 *Every L–space knot in the SnapPy census of up to nine tetrahedra is braid positive except for* $o9\_30634$, *which is not.*

The knot $o9\_30634$ is *nearly braid positive* in the sense that it has a braid presentation that is braid positive except for one strongly quasipositive crossing that jumps over only one strand. We do not know if $o9\_30634$ admits a positive diagram.

Question 1.2 *Is every hyperbolic L–space knot nearly braid positive*?

Proof of Theorem 1.1 In [3] we obtained braid words for every census L–space knot by automating the process from [1]. (An alternative approach is taken by Dunfield, Obeidin, and Rudd [13].) Here, utilizing the braid and simplification methods in SnapPy [10] and Sage [27], we managed to cajole braid positive presentations for all of the knots except for one, $o9\_30634$. The L–space census knots and positive braids with them as closures are detailed in the online supplement and verified in [2].

As one may check, the knot $K = o9\_30634$ is the closure of the 4–braid

$$\beta = [2, 1, 3, 2, 2, 1, 3, 2, 2, 1, 3, 2, -1, 2, 1, 1, 2].$$

Here the list of nonzero integers represents a braid word by letting the integer k stand for the standard generator σ_k or its inverse σ_k^{-1}, depending on whether k is positive or negative.

Ito gives new constraints on a suitably normalized version of the HOMFLY polynomial for positive braids [20]. The Ito-normalized HOMFLY polynomial $\widetilde{P}_K(\alpha, z) = \sum h_{ij} \alpha^i z^{2j}$ of $K = \hat{\beta}$ is represented by the matrix $H = (h_{ij})$ of coefficients

$$H = \begin{pmatrix} 13 & 69 & 133 & 121 & 55 & 12 & 1 \\ 17 & 66 & 83 & 45 & 11 & 1 & 0 \\ 4 & 10 & 6 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the indexing starts at 00, so that $h_{00} = 13$. One may calculate this with Sage (or the knot theory package [21] for Mathematica) from the braid word, using the built-in HOMFLY polynomial and adjusting it to achieve Ito's normalization. The computations can be found at [2].

According to [20, Theorem 2], if a link K is braid positive then the Ito-normalized HOMFLY polynomial should only have nonnegative coefficients. As one observes, the coefficients h_{30} and h_{31} are negative. Hence $o9\_30634$ is not braid positive. □

In Section 2, we generalize the knot $o9\_30634$ to an infinite family of hyperbolic L–space knots that are nearly braid positive but for which Ito's constraints fail to obstruct braid positivity, at least for the examples we managed to calculate. In Section 3, we further extend this family to a doubly infinite family of knots $K_{n,m}$ in hopes of providing more potential examples. While that doesn't quite work out, we highlight several properties of these knots in Proposition 3.1. Notably, we

- show that all but $K_{-1,m}$ and six other exceptional cases of these knots are hyperbolic,
- identify a small Seifert fibered space surgery for each,
- determine that when $n \geq 0$ they are L–space knots if and only if $m \leq 0$,
- compute their Alexander polynomials, and
- examine their structures as positive braids and strongly quasipositive braids.

Lastly, in Section 4 we observe that our infinite family of hyperbolic L–space knots of Section 2 have Alexander polynomials that

- induce formal semigroups that are actually semigroups (which Teragaito pointed out to us), and
- have all their roots on the unit circle, disproving Li and Ni's Conjecture 1.3 in [22].

2 A family of hyperbolic L–space knots that might not be braid positive

Let $\{K_n\}$ be the family of knots that are the closures of the braids

$$\beta_n = [(2,1,3,2)^{2n+1}, -1, 2, 1, 1, 2]$$

and includes our knot $o9\_30634$ as K_1; see Figure 1, bottom right. Observe that β_n gives a strongly quasipositive braid presentation for these knots that is *almost* braid positive — it is braid positive except for one negative crossing.

Proposition 2.1 *For $n \geq 1$, the knots K_n are hyperbolic L–space knots.*

Proof This follows from Lemmas 2.2 and 2.3. □

Lemma 2.2 *For $n \geq 1$, the knots K_n are L–space knots. In particular, the $(8n+6)$–surgery on K_n gives the Seifert fibered L–space $M\left(-1; \frac{1}{2}, (2n+1)/(4n+4), 2/(4n+5)\right)$.*

Proof Figure 2 shows how a strongly invertible surgery description of the knot K_n along with its $(8n+6)$–surgery may be obtained. Figure 3 demonstrates how one may take the quotient and perform rational tangle replacements associated to the surgeries to produce a link whose double branched cover is $(8n+6)$–surgery

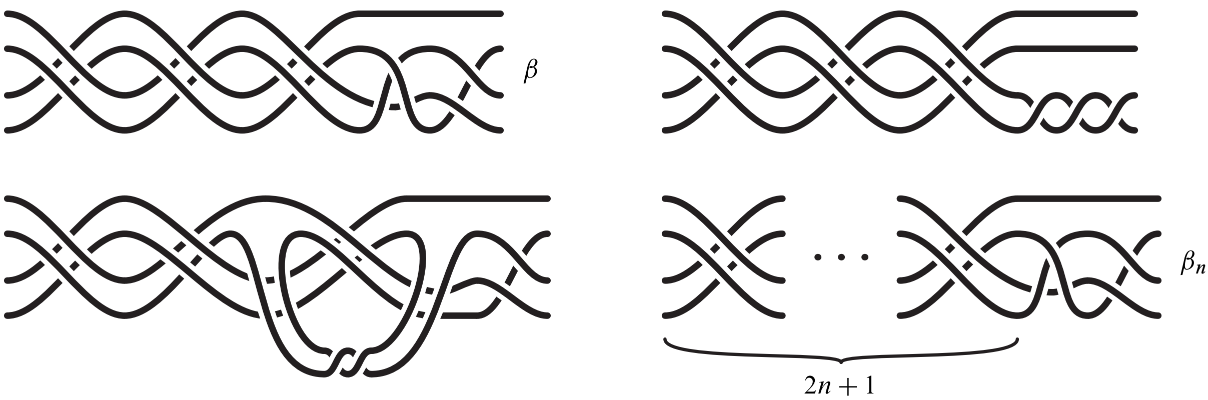

Figure 1: Top left: the braid β is positive except for one strongly quasipositive crossing. Its closure $\hat{\beta}$ is the hyperbolic L–space knot $o9\_30634$, which we show is not braid positive. Bottom left: dragging the base of the strongly quasipositive band of β into the position shown exhibits $\hat{\beta}$ as a positive Hopf basket. Top right: this braid has the $(2,3)$–cable of the $(2,3)$–torus knot as its closure. Bottom right: the closures of the braids β_n are L–space knots that may also fail to be braid positive.

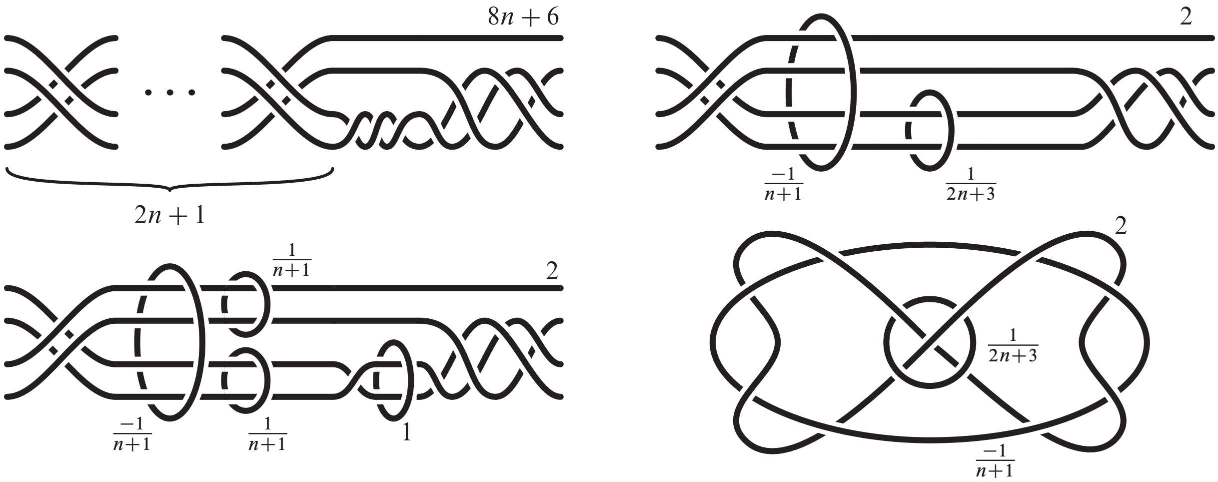

Figure 2: Top left: the braid β_n with a surgery coefficient of $8n+6$ for its closure knot K_n. Bottom left and top right: twists in the braid are expressed and collected into surgeries on unknots. The surgery coefficient on the closure knot is adjusted accordingly. Bottom right: after closure and isotopy, we obtain a surgery description for $(8n+6)$–surgery on K_n.

on K_n. We observe this link to be the Montesinos link $M\big(2/(4n+5), \frac{1}{2}, -(2n+3)/(4n+4)\big)$. Hence its double branched cover is the Seifert fibered space $M_n\big(0; 2/(4n+5), \frac{1}{2}, -(2n+3)/(4n+4)\big)$. Here we use the notation of Lisca and Stipsicz [24] where the Seifert fibered space $M(e_0; r_1, r_2, \dots, r_k)$ is obtained by e_0–surgery on an unknot with k meridians having $(-1/r_i)$–surgery on the i^{th} one.

These Seifert fibered spaces are determined to be L–spaces via [24, Theorem 1]. More specifically, Lisca and Stipsicz [24, Theorem 1] show that the Seifert fibered space $M = M(e_0; r_1, r_2, r_3)$ — with $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$ — is an L–space if and only if either M or $-M$ does not carry a positive transverse contact structure. Then by Lisca and Matić [23], such a Seifert fibered space M carries no positive transverse contact structure if and only if either $e_0 \geq 0$ or $e_0 = -1$ and there exists no coprime integers a and m such that $mr_1 < a < m(1-r_2)$ and $mr_3 < 1$.

Rewriting to apply [24, Theorem 1], we obtain that $M_n = M\big(-1; \frac{1}{2}, (2n+1)/(4n+4), 2/(4n+5)\big)$. Then, since $1-r_2 = (2n+3)/(4n+4)$, we assume for contradiction that there are coprime integers a and m such that $m\frac{1}{2} < a < m(2n+3)/(4n+4)$ and $m2/(4n+5) < 1$. The first gives

$$0 < 2a - m < \frac{m}{2n+2}.$$

The second implies $m < 2n+2+\frac{1}{2}$, so that $m \leq 2n+2$ and

$$\frac{m}{2n+2} \leq 1.$$

Together they yield $0 < 2a-m < 1$. However, since $2a-m$ is an integer, there are no pairs of integers (a, m) that satisfy this equation. This is a contradiction.

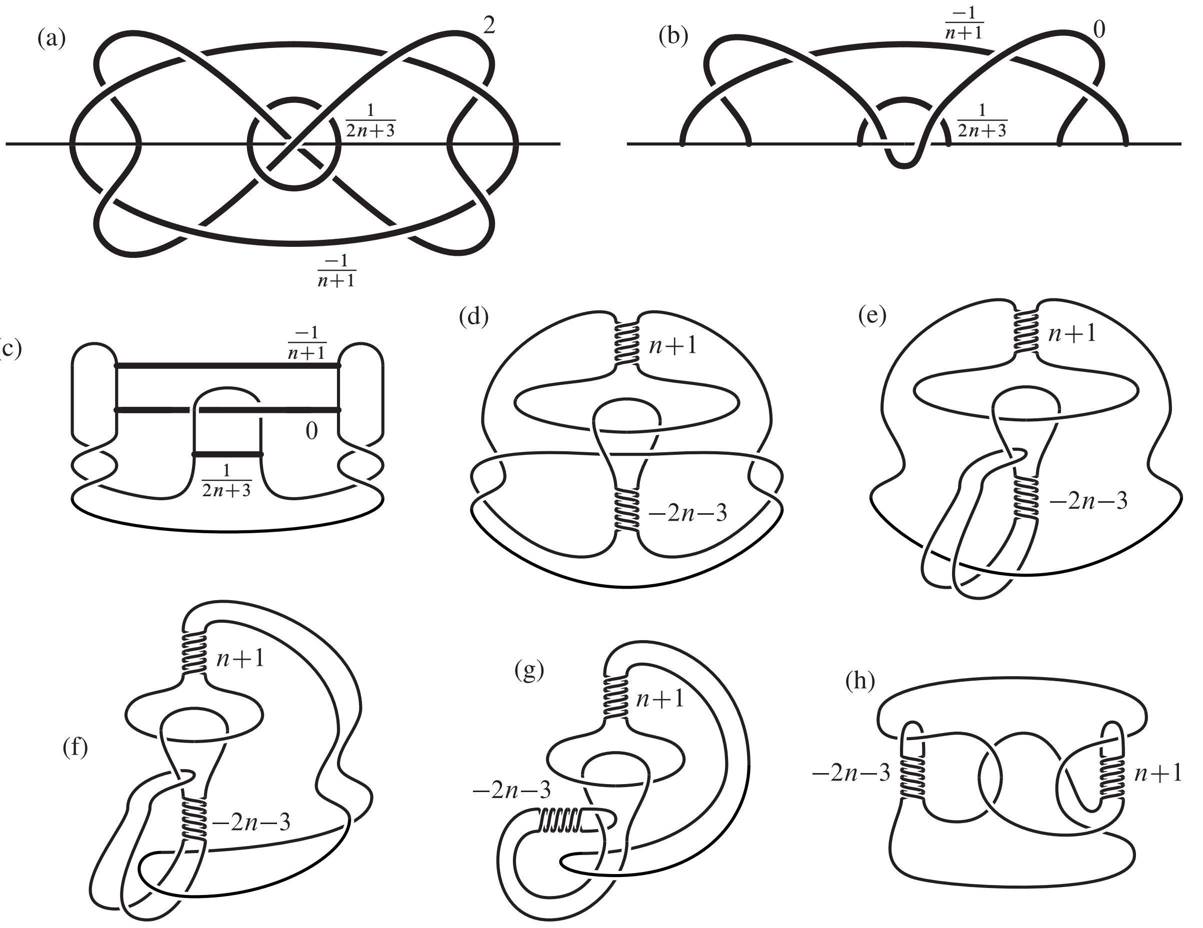

Figure 3: (a) The surgery description from Figure 2, bottom right, is strongly invertible. (b)–(c) The quotient of the surgery description followed by some isotopy to straighten the arcs. (d) Rational tangle replacements along the arcs produce a link whose double branched cover is $(8n+4)$–surgery on K_n. (e)–(h) A sequence of isotopies shows that this link is the Montesinos link $M([0,-2n-3,-2],[0,-2],[0,1,-1,n+1,2]) = M\big(2/(4n+5), \frac{1}{2}, -(2n+3)/(4n+4)\big)$.

Therefore M_n does not carry a positive transverse contact structure, and thus it is an L–space. Hence K_n is an L–space knot for each $n \geq 1$. $\square$

Lemma 2.3 *For $n \geq 1$, the knots K_n are hyperbolic.*

Proof We check that $L12n1739(1,2n+2)(0,0)(-1,n+1)$ has the same exterior as K_n. Via SnapPy we verify that $L12n1739$ is hyperbolic and compute its short slopes of length less than 2π as

$$[(1,0),(-2,1),(-1,1),(0,1),(1,1),(-1,2),(1,2),(-1,3)],$$
$$[(1,0),(-5,1),(-4,1),(-3,1),(-2,1),(-1,1),(0,1),(1,1),(-5,2),(-3,2)],$$
$$[(1,0),(-2,1),(-1,1),(0,1),(1,1),(2,1),(-1,2),(1,2)].$$

Thus for $n > 1$ we fill with slopes longer than 2π and therefore directly get hyperbolic manifolds by Gromov and Thurston's 2π theorem; see for example [7, Theorem 9]. □

Teragaito (personal communication, 2022) suggested an alternative approach to this lemma that does not use SnapPy or any computer calculation. The referee also proposed a similar approach. Since it is more "hands-on", we include a proof along the lines of their suggestions here:

Another proof of Lemma 2.3 As knots in S^3 are either torus knots, satellite knots, or hyperbolic knots by [29], we must show that K_n is neither a torus knot nor a satellite knot.

In the proof of Theorem 4.4 the Alexander polynomial of $K_n = K_{n,0}$ is presented as

$$\Delta_{K_{n,0}} \doteq \frac{(t^{4n+5}+1)(t^{4n+2}+1)}{(t+1)(t^2+1)}.$$

As this is not equivalent to the Alexander polynomial of a torus knot, K_n cannot be a torus knot. (Also, the formal semigroup of K_n has rank 3 as noted in Remark 4.3, whereas the formal semigroup of a torus knot has rank 2.)

So now suppose K_n is a satellite knot. Observe that an unknotting tunnel put at the unique negative crossing for $K_n = \hat{\beta}_n$ in Figure 1, bottom right, shows that K_n has tunnel number 1. Since the bridge index of K_n is at most 4, Morimoto and Sakuma's classification of tunnel number 1 satellite knots [25] tells us that K_n has the 2–bridge torus knot $T(2,q)$ as a companion knot for some odd q and a pattern of wrapping number 2. As K_n is an L–space knot by Lemma 2.2, this pattern must be a braided pattern by [4, Lemma 1.17]. Hence the pattern must be a 2–cable. Thus if K_n is a satellite knot, then it is a 2–cable knot of $T(2,q)$. Indeed, the Alexander polynomial of K_n shown above implies that K_n must be the $(2,4n+5)$–cable of the $T(2,2n+1)$ torus knot. However, the distance of the cabling slope $8n+10$ and the slope $8n+6$ of the Seifert fibered surgery on K_n is $\Delta(8n+10, 8n+6) = 4 > 1$. Thus the cabling torus remains incompressible after surgery; see eg [15, Lemma 7.2]. This contradicts that $(8n+6)$–surgery on K_n produces a small Seifert fibered space. Thus K_n cannot be a satellite knot. □

However, the constraints of Ito on HOMFLY polynomials appear to not obstruct K_n from being braid positive when $n \geq 2$. Using Sage for computations, we see that Ito's constraints on the HOMFLY polynomials of K_n for $n = 2, \dots, 10$ do not obstruct braid positivity for these knots. Furthermore, we have been unsuccessful in finding a braid positive presentation for these knots.

Question 2.4 *Are the knots K_n for $n \geq 2$ braid positive?*

3 A doubly infinite family of knots

From our description of the family of knots K_n in Figure 2, one finds a natural 2–parameter family generalization. While one may initially hope this family yields further examples of hyperbolic L–space knots that fail to be braid positive, we show this is not the case.

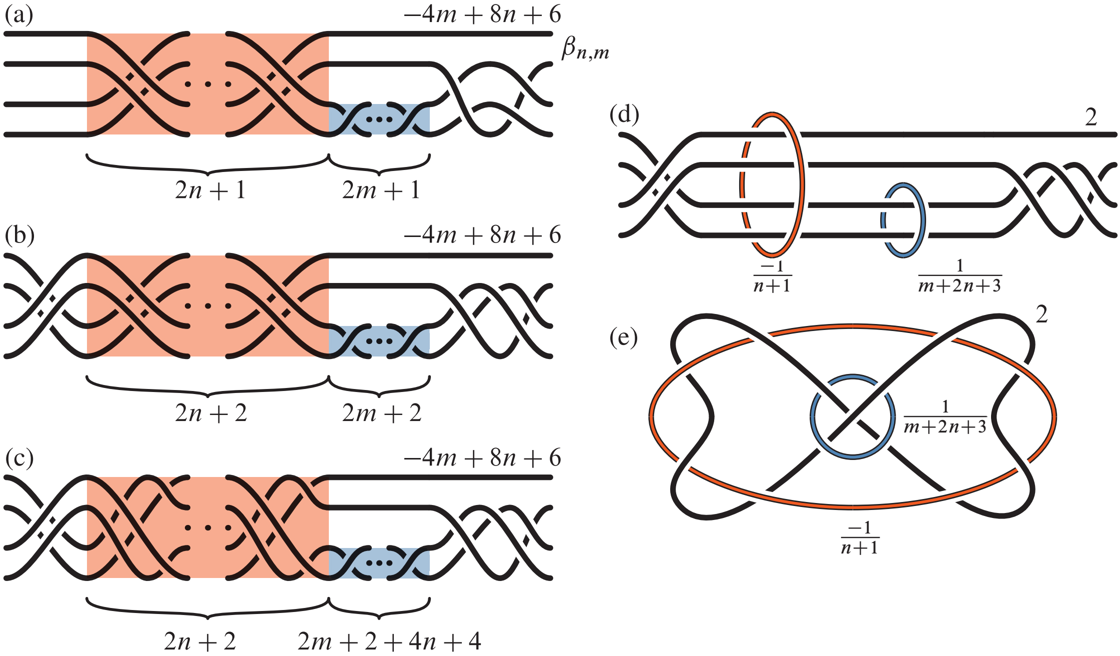

Figure 4: (a) The braid $\beta_{n,m}$ with a surgery coefficient of $-4m+8n+6$ for its closure knot $K_{n,m}$. (b)–(d) Twists in the braid are expressed and collected into surgeries on unknots. The surgery coefficient on the closure knot is adjusted accordingly. (e) After closure and isotopy, we obtain a surgery description for $(-4m+8n+6)$–surgery on $K_{n,m}$.

Proposition 3.1 *Let $\beta_{n,m}$ be the braid indicated in Figure 4(a), and let $K_{n,m} = \hat{\beta}_{n,m}$ be its closure.*

(1) *$K_{n,m}$ is a hyperbolic knot for all $(n, m) \in \mathbb{Z}^2$, except for the pairs*

$$(n, m) \in \{(-1, k) \mid k \in \mathbb{Z}\} \cup \{(0, 0), (0, -1), (0, -2), (-2, 1), (-2, 0), (-2, -1)\}.$$

For each of these pairs, $K_{n,m}$ is a torus knot.

(2) *$(8n+6-4m)$–surgery on $K_{n,m}$ gives the Seifert fibered space*

$$M\left(-1; \frac{1}{2}, \frac{2n+1}{4n+4}, \frac{2}{4n+5+2m}\right).$$

(3) *The Alexander polynomial of $K_{n,m}$ is*

$$\left(t^{m-1}\sum_{i=0}^{n}(t^{-4i-1}-t^{-4i})\right) + \left((-1)^m \sum_{j=-m}^{m}(-t)^j\right) + \left(t^{1-m}\sum_{k=0}^{n}(t^{4k+1}-t^{4k})\right).$$

(4) *Assume $n \geq 0$. Then $K_{n,m}$ is an L–space knot if and only if $m \leq 0$.*

(5) *If $n \geq 0$ and $m < 0$, then $\beta_{n,m}$ is a positive braid and $K_{n,m}$ is a braid positive knot of genus $|m| + 4n + 3$*

(6) *If $2n + 1 \geq m \geq 0$, then $\beta_{n,m}$ is conjugate to a strongly quasipositive braid and $K_{n,m}$ is a strongly quasipositive knot of genus $4n - m + 2$.*

(a) *If* $2n \geq m \geq 0$, *then* $K_{n,m}$ *is a fibered strongly quasipositive knot. Moreover it is a Hopf plumbing basket.*

(b) *If* $2n+1 = m > 0$, *then* $K_{n,m}$ *is a nonfibered strongly quasipositive knot.*

Proof (1) Since the surgery description of $K_{n,m}$ given in Figure 4(e) is on a hyperbolic link, using the 2π theorem a couple of times yields a finite list of pairs (n,m) for which $K_{n,m}$ might not be hyperbolic. A further check in SnapPy confirms that all but five of them are hyperbolic. These remaining five are readily confirmed to be torus knots. The computations are displayed at [2].

(2) Figure 4 shows how to obtain a surgery description on a 3–component link for $(-4m+8n+6)$–surgery on $K_{n,m}$. Figure 5 uses the Montesinos trick to exhibit the result of this surgery description as

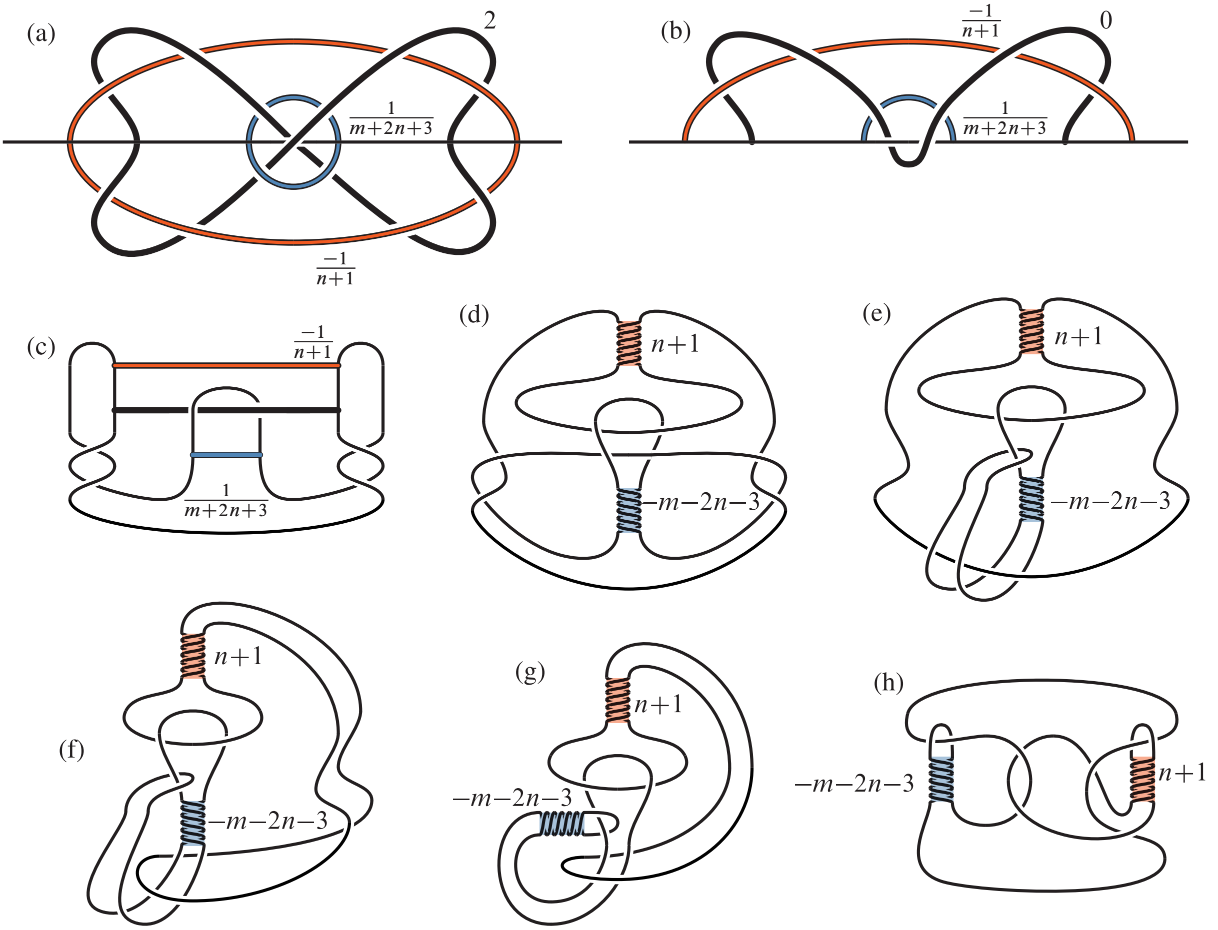

Figure 5: (a) The surgery description from Figure 4(e) is strongly invertible. (b)–(c) The quotient of the surgery description followed by some isotopy to straighten the arcs. (d) Rational tangle replacements along the arcs produce a link whose double branched cover is $(-4m+8n+4)$–surgery on $K_{n,m}$. (e)–(h) A sequence of isotopies shows this link is the Montesinos link $M([0,-m-2n-3,-2],[0,-2],[0,1,-1,n+1,2]) = M\big(2/(2m+4n+5), \frac{1}{2}, -(2n+3)/(4n+4)\big)$.

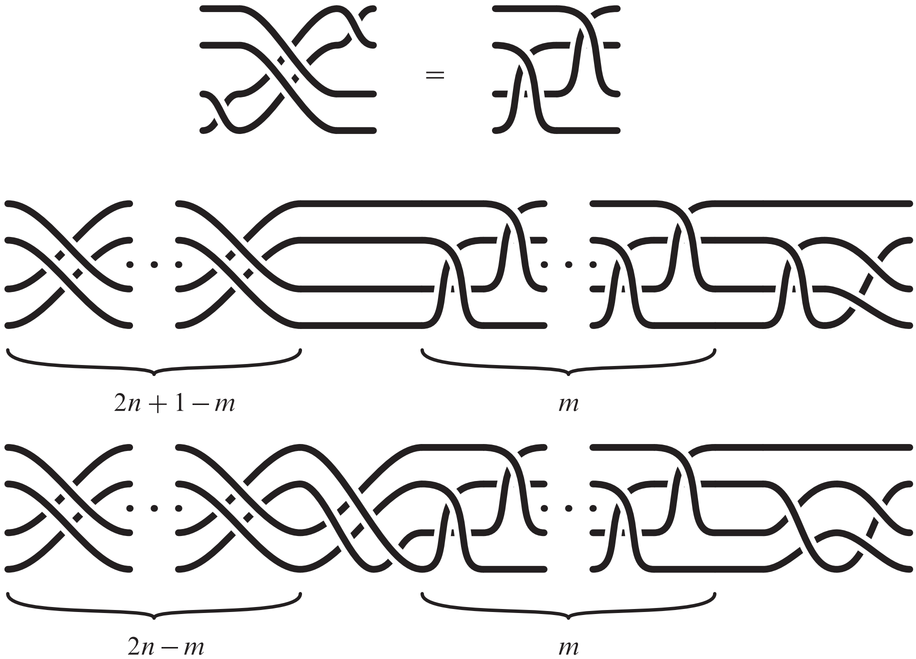

Figure 6: The proof of Proposition 3.1(6).

the double branched cover of the Montesinos link $M([0,-2,-m-2n-3],[0,-2],[0,1,-1,n+1,2])$. This double branched cover is the Seifert fibered space $M\bigl(\frac{1}{2},-(2n+3)/(4n+4),2/(4n+5+2m)\bigr)$, which is equivalent to $M\bigl(-1;\frac{1}{2},(2n+1)/(4n+4),2/(4n+5+2m)\bigr)$.

(5) When $n \geq 0$ and $m < 0$, the braid $\beta_{n,m}$ as described in Figure 4(a) is expressly a positive braid. One counts that it is a braid of index 4 and $4(2n+1)+(1-2m)+4$ crossings. Hence $\chi(K_{n,m}) = -(2|m|+8n+5)$ and $g(K_{n,m}) = |m|+4n+3$.

(6) When $0 \leq m \leq 2n+1$, through braid isotopy and braid conjugacy, we may isotope in pairs $2m$ of the $2m+1$ negative crossings over to m of the $2n+1$ copies of the "2–cabled" positive crossing that appear in $\beta_{n,m}$ so that they appear as in the left-hand side of Figure 6, top. Hence by a further braid isotopy as indicated by Figure 6, each of these $2m$ negative crossings contributes to an SQP band. The final negative crossing also contributes to an SQP band towards the end of the braid, ultimately giving us the strongly quasipositive braid, shown in Figure 6, middle, to which $\beta_{n,m}$ is conjugate. One counts that the braid index is 4 and there are $2m+1$ SQP bands and $4(2n+1-m)+2$ regular crossings. Hence $\chi(K_{n,m}) = -(8n-2m+3)$ and $g(K_{n,m}) = 4n-m+2$.

Furthermore, when $0 \leq m \leq 2n$ so that $2n-m \geq 0$, we may instead perform braid isotopy and conjugation to arrive at the strongly quasipositive braid shown in Figure 6, bottom. This braid however contains the "dual Garside element" $\delta = \sigma_3\sigma_2\sigma_1$. Hence, as Banfield points out [5], the closure of such an SQP braid is fibered and a Hopf basket.

When $m = 2n+1$, the braid $\beta_{n,2n+1}$ is conjugate to an SQP braid but its closure $K_{n,2n+1}$ might not be fibered. Indeed, we find that the Alexander polynomial of $K_{n,2n+1}$ is not monic, so the closure is not fibered. Explicitly, from our computations of $\Delta_{K_{n,m}}$ for (3) below, we have

$$\begin{aligned}\Delta_{K_{n,2n+1}}(t) &= \frac{t-1}{(t^4-1)(t^2-1)} t(t^2-1)(2-t+t^2+t^{4n+3}-t^{4n+4}+2t^{4n+5})\\ &= t\,\frac{(2t^{4n+6}-2t^2)-(3t^{4n+5}-3t)+(2t^{4n+4}-2)-(t^{4n+3}-t^3)}{t^4-1}\\ &\doteq \frac{(2-3t+2t^2)(t^{4(n+1)}-1)-t^3(t^{4n}-1)}{t^4-1}\\ &= \frac{(2-3t+2t^2)(t^{4(n+1)}-t^{4n}+t^{4n}-1)-t^3(t^{4n}-1)}{t^4-1}\\ &= (2-3t+2t^2)t^{4n}+(2-3t+2t^2-t^3)\frac{t^{4n}-1}{t^4-1},\end{aligned}$$

which has leading coefficient 2.

(3) View the surgery description for $K_{n,m}$ as the link $L = K\cup c\cup c'$ where we do $(-1/(n+1))$–surgery on c and $(1/(m+2n+3))$–surgery on c'. Observe that $c \cup c'$ is the trivial 2–component link, and we may orient the link so that $\operatorname{lk}(K,c) = 4$ and $\operatorname{lk}(K,c') = 2$.

Let E be the exterior of $L = K\cup c\cup c'$. Then $H_1(E) = \langle[\mu_K],[\mu_c],[\mu_{c'}]\rangle \cong \mathbb{Z}^3$ where μ_K, μ_c, and $\mu_{c'}$ are oriented meridians of K, c, and c'. Let λ_K, λ_c, and $\lambda_{c'}$ be their preferred longitudes. Observe that $[\lambda_c] = 4[\mu_K]$ and $[\lambda_{c'}] = 2[\mu_K]$ in $H_1(E)$.

Now consider the family of links $L_{n,m} = K_{n,m}\cup c_n\cup c'_m$ with exterior $E_{n,m}$ obtained from K and the core curves of $(-1/(n+1))$–surgery on c and $(1/(m+2n+3))$–surgery on c'. Thus $E_{n,m} \cong E$ where

$$\mu_{K_{n,m}} = \mu_K, \quad \mu_{c_n} = -\mu_c + (n+1)\lambda_c \quad \text{and} \quad \mu_{c'_m} = \mu_{c'} + (m+2n+3)\lambda_{c'}.$$

Now letting

$$x = [\mu_K], \quad y = [\mu_c], \quad z = [\mu_{c'}], \quad x_{n,m} = [\mu_{K_{n,m}}], \quad y_n = [\mu_{c_n}] \quad \text{and} \quad z_m = [\mu_{c'_m}] \tag{3-1}$$

in the group rings $\mathbb{Z}[H_1(E)]$ and $\mathbb{Z}[H_1(E_{n,m})]$, we have

$$x_{n,m} = x, \quad y_n = y^{-1}x^{4(n+1)} \quad \text{and} \quad z_m = zx^{2(m+2n+3)} \tag{3-2}$$

and hence

$$x = x_{n,m}, \quad y = y_n^{-1}x_{n,m}^{4(n+1)} \quad \text{and} \quad z = z_m x_{n,m}^{-2(m+2n+3)}.$$

Therefore

$$\Delta_{L_{n,m}}(x_{n,m}, y_n, z_m) = \Delta_L(x_{n,m}, y_n^{-1}x_{n,m}^{4(n+1)}, z_m x_{n,m}^{-2(m+2n+3)}). \tag{3-3}$$

Using the Torres formulae [30], one obtains that

$$\Delta_{K_{n,m}}(x_{n,m}) = \frac{x_{n,m}-1}{x_{n,m}^4-1}\Delta_{K_{n,m}\cup c_n}(x_{n,m},1) = \frac{x_{n,m}-1}{(x_{n,m}^4-1)(x_{n,m}^2-1)}\Delta_{K_{n,m}\cup c_n\cup c'_m}(x_{n,m},1,1). \tag{3-4}$$

Hence, using (3-3) and (3-4) where we set $x_{n,m} = t$, $y_n = 1$, and $z_m = 1$, we obtain

$$\Delta_{K_{n,m}}(t) = \frac{t-1}{(t^4-1)(t^2-1)} \Delta_L(t, t^{4(n+1)}, t^{-2(m+2n+3)}).$$

We calculate that

$$\Delta_L(x,y,z) = (x^2-1)(x^3y^2z + x^2y^3z - x^2y^2z + x^2y + xy^2z - xy + x + y).$$

Then

$$\begin{aligned}
\Delta_{K_{n,m}}(t) &= \frac{t-1}{(t^4-1)(t^2-1)} \Delta_L(t, t^{4(n+1)}, t^{-2(m+2n+3)}) \\
&= t^{4n+3-m} \frac{(t-1)(t^{m-4n-2} + t^{-m} - t^{1-m} + t^{2-m} + t^{m+1} - t^{m+2} + t^{m+3} + t^{-m+4n+5})}{(t^4-1)} \\
&\doteq \frac{(t-1)((t^{m-4n-2} - t^{m+2}) + (t^{-m} + t^{2-m} + t^{m+1} + t^{m+3}) + (t^{4n+5-m} - t^{1-m}))}{(t^4-1)} \\
&= \frac{t^{m+2}(t-1)(t^{-4n-4}-1)}{t^4-1} + \frac{(t-1)(t^{-m} + t^{2-m} + t^{m+1} + t^{m+3})}{t^4-1} + \frac{t^{1-m}(t-1)(t^{4n+4}-1)}{t^4-1} \\
&= \left(t^{m-1} \sum_{i=0}^{n} t^{-4i}(t^{-1}-1) \right) + \left(\frac{t^{m+1} - t^m + t^{-m} - t^{-m-1}}{t - t^{-1}} \right) + \left(t^{1-m} \sum_{j=0}^{n} t^{4j}(t-1) \right) \\
&= \left(t^{m-1} \sum_{i=0}^{n} (t^{-4i-1} - t^{-4i}) \right) + \left((-1)^m \sum_{j=-m}^{m} (-t)^j \right) + \left(t^{1-m} \sum_{k=0}^{n} (t^{4k+1} - t^{4k}) \right),
\end{aligned}$$

where the $\doteq$ indicates that we have divided out the unit t^{4n+3-m}.

(4) Using our Alexander polynomial calculations provides obstructions to the knots $K_{n,m}$ for $n > 0$ being L–space knots when $m > 0$. As an example, taking $n > 0$ and $m = 1$ gives

$$\begin{aligned}
\Delta_{K_{n,1}}(t) &= \frac{t-1}{(t^4-1)(t^2-1)} \Delta_L(t, t^{4(n+1)}, t^{-2(2n+4)}) \\
&\doteq \left(\sum_{i=0}^{n} (t^{4i-1} - t^{4i}) \right) + (t^{-1} - 1 + t) + \left(\sum_{k=0}^{n} (t^{4k+1} - t^{4k}) \right).
\end{aligned}$$

One may observe that the constant coefficient is -3. Hence the knots $K_{n,1}$ cannot be L–space knots. Indeed, one may further observe that, when $n > 0$ and $m > 0$, the central terms will have overlap with the end terms to give coefficients ± 2 or ± 3 for terms with degree of small magnitude. Thus none of the knots $K_{n,m}$ with $n > 0$ and $m > 0$ are L–space knots.

In the other direction, where $n > 0$ and $m \leq 0$, we may observe via [23; 24], as in Lemma 2.2, that the Seifert fibered space M resulting from $(8n+6-4m)$–surgery on $K_{n,m}$ is an L–space. For that we need to distinguish several cases. We continue with the notation of Lisca and Stipsicz [24] as in Lemma 2.2.

Since $n > 0$,

$$1 > \tfrac{1}{2} > \frac{2n+1}{4n+4} > 0.$$

So we must reckon with the coefficient

$$\frac{2}{2m+4n+5}=\frac{2}{2(2n+m+1)+3}.$$

If $2n+m+1\geq 1$,

$$1>\tfrac{1}{2}>\frac{2n+1}{4n+4}>\frac{2}{2m+4n+5}>0.$$

If we now assume that there exist coprime integers a and b such that

$$\tfrac{1}{2}b<a<\frac{2n+3}{4n+4}b \quad \text{and} \quad \frac{2}{4n+2m+5}b<1,$$

we conclude from the first inequality that $0<2a-b<b/(2n+2)$ and the second inequality implies that $b\leq 2n+2+m\leq 2n+2$. Putting both together yields the contradiction

$$0<2a-b<\frac{b}{2n+2}\leq 1.$$

Thus M carries no positive transverse contact structure and is therefore an L–space.

If $2n+m+1=0$ we get the Seifert fibered space $M\bigl(-1;\frac{2}{3},\frac{1}{2},(2n+1)/(4n+4)\bigr)$. We assume that there exist coprime integers a and b such that $\frac{2}{3}b<a<\frac{1}{2}b$ and $((2n+1)/(4n+4))b<1$, from which we conclude $4b<6a<3b$ and $b<2+2/(2n+1)\leq 4$, which is a contradiction. Therefore M does not carry a positive transverse contact structure and is thus an L–space.

If $2n+m+1=-1$ we get the Seifert fibered space

$$M\Bigl(-1;\frac{1}{2},\frac{2n+1}{4n+4},2\Bigr)=M\Bigl(1;\frac{1}{2},\frac{2n+1}{4n+4}\Bigr),$$

which is a lens space and hence an L–space.

If $2n+m+1=-2$ we get the Seifert fibered space

$$M\Bigl(-1;\frac{1}{2},\frac{2n+1}{4n+4},-2\Bigr)=M\Bigl(-3;\frac{1}{2},\frac{2n+1}{4n+4}\Bigr),$$

which is a lens space and hence an L–space.

If $2n+m+1\leq -3$ we see that

$$\frac{2}{2m+4n+5}=\frac{2}{2(2n+m+1)+3}\in[-1,0],$$

and thus the correctly normalized Seifert fibered space is

$$M\Bigl(-2;\frac{1}{2},\frac{2n+1}{4n+4},\frac{4n+2m+7}{4n+2m+5}\Bigr),$$

which admits a positive contact structure. Next, we consider

$$-M=M\Bigl(2;-\frac{1}{2},-\frac{2n+1}{4n+4},-\frac{4n+2m+7}{4n+2m+5}\Bigr)=M\Bigl(-1;\frac{1}{2},\frac{2n+3}{4n+4},-\frac{2}{4n+2m+5}\Bigr).$$

If $2n+m+1=-3$, then the correct ordering of the Seifert invariants is $M\bigl(-1;\frac{2}{3},(2n+3)/(4n+4),\frac{1}{2}\bigr)$. We readily see that there exist no coprime integers a and b such that $\frac{2}{3}b<a<((2n+1)/(4n+4))b$ and $\frac{1}{2}b<1$. Thus M carries no positive transverse contact structure and is therefore an L–space. If

$2n+m+1 \le -4$ the Seifert invariants are ordered as $M\big(-1; (2n+3)/(4n+4), \frac{1}{2}, -2/(4n+2m+5)\big)$. We assume that there exist coprime integers a and b such that

$$\frac{2n+3}{4n+4}b < a < \tfrac{1}{2}b \quad \text{and} \quad -\frac{2}{4n+2m+5}b < 1.$$

But putting them together yields the contradiction

$$0 < a - \frac{2n+3}{4n+4}b < -\frac{1}{4n+4}b < 0.$$

Thus M does not admit a positive transverse contact structure and is therefore an L–space. □

Remark 3.2 In the cases of the above proof when $2n+m+1 = -1$ or -2, the knots $K_{n,m}$ have lens space surgeries. These knots can be seen to be Berge knots as follows. With $-m-2n-3 = 1$ or 0, Figure 5(d) can be seen to divide along a horizontal line into two rational tangles. A vertical arc in the middle would be the arc dual to the rational tangle replacement on the 0–framed arc from Figure 5(c). In the double branched cover, this vertical arc will lift to a knot in the lens space with an S^3–surgery. Furthermore, one may observe that this arc lifts to a $(1,1)$–knot in the lens space. Hence the knot $K_{n,m}$ must be a Berge knot [6].

4 Curiosities about the Alexander polynomial of $o9\_30634$ and its generalizations

Like the failure of braid positivity, the hyperbolic L–space knot $o9\_30634$ exhibits two more curious properties that had previously only been observed for L–space knots among iterated cables of torus knots. The first, regarding formal semigroups, Teragaito communicated to us near the completion of the initial preprint. The second, regarding the roots of its Alexander polynomial, came after that. Both actually generalize to the infinite family $\{K_n\}_{n\ge 1}$ as well.

4.1 An infinite family of hyperbolic L–space knots whose formal semigroups are semigroups

Teragaito informed us about the work of Wang [31] on formal semigroups of L–space knots, and that there are only two L–space knots in the SnapPy census whose formal semigroups were actual semigroups. He had also observed that one of these knots appeared to fail to be braid positive. It turns out that this is the knot $o9\_30634$, which we had confirmed to not be braid positive. Upon seeing an early draft of this article, Teragaito further showed that all of our hyperbolic L–space knots K_n have formal semigroups that are semigroups. Below we overview the formal semigroup and then record Teragaito's results in Theorem 4.1.

An algebraic link is defined to be the link of an isolated singularity of a complex curve in $\mathbb{C}^2$. Algebraic knots are known to be iterated cables of torus knots [14] and they are all L–space knots; see [17]. Moreover, one can assign to any algebraic knot K an additive semigroup $S_K < \mathbb{N}_0$ which determines the Heegaard Floer chain complex and is computable from the Alexander polynomial of K; see [8].

In [31] Wang has generalized this definition, but now S_K is not necessarily a semigroup anymore. Let K be an L–space knot with (symmetrized) Alexander polynomial Δ_K. Then the *formal semigroup* $S_K \subset \mathbb{N}_0$ is defined by

$$\frac{t^{g(K)}\Delta_K(t)}{1-t} = \sum_{s\in S_K} t^s,$$

where $g(K)$ denotes the genus of K. (Note that $t^{g(K)}\Delta_K(t)$ is now an ordinary polynomial of degree $2g(K)$.) The set S_K still determines the Heegaard Floer chain complex of K but is not necessarily a semigroup. This is used by Wang to construct an infinite family of L–space knots which are iterated cables of torus knots but not algebraic [31]. On the other hand, it remained open if there exists an L–space knot which is not an iterated cable of torus knots but whose formal semigroup is a semigroup [31, Question 2.8].

Theorem 4.1 (Teragaito, personal communication, 2022) *There exists an infinite family of hyperbolic L–space knots whose formal semigroups are semigroups. More concretely*:

(1) *$o9\_30634$ and $t09847$ are hyperbolic L–space knots whose formal semigroups are semigroups. The formal semigroup of every other L–space knot in the SnapPy census is not a semigroup.*

(2) *The formal semigroups S_{K_n} of the infinite family of hyperbolic L–space knots $\{K_n\}$ from Section 2 are all semigroups.*

Consequently, *the knots $\{K_n\}$ provide an infinite family of knots answering* [31, Question 2.8] *negatively.*

Proof (1) The formal semigroup S_K of an L–space knot is computable from the Alexander polynomial of K; in particular, S_K always contains all natural numbers larger than $g(K)$ and the finitely many other elements of S_K can be read off from the Alexander polynomial. In [2] we present code that computes the formal semigroups of all SnapPy census L–space knots and determines that $o9\_30634$ and $t09847$ are the only ones whose formal semigroups are semigroups.

(2) In Proposition 3.1(3) we have computed the Alexander polynomials of K_n, from which we read off the formal semigroup S_{K_n} to be

$$\{4n+4, 4n+5, 4n+6, 4n+8, 4n+9, 4n+10, 4n+12, 4n+13, 4n+14, \dots, 8n, 8n+1, 8n+2, 8n+4\} \\ \cup \{0, 4, 8, \dots, 4n\} \cup \{4n+2\} \cup \mathbb{N}_{>8n+4},$$

which is a semigroup for any n. □

Remark 4.2 (Teragaito, personal communication, 2022) A braid word of $t09847$ is given by

$$[(2,1,3,2)^3, 1, 2, 1, 1, 2],$$

which is very close to our braid word for $o9\_30634$. One can similarly show that $t09847$ fits into an infinite family of hyperbolic L–space knots with braid words

$$[(2,1,3,2)^{2n+1}, 1, 2, 1, 1, 2]$$

whose formal semigroups are semigroups.

Remark 4.3 The semigroups from Theorem 4.1 and the preceding remark all have rank 3, ie the minimal number of a generating set is 3. On the other hand, Teragaito constructs in [28] an infinite family of hyperbolic L–space knots whose formal semigroups are semigroups of rank 5.

4.2 Two infinite families of hyperbolic L–space knots whose Alexander polynomial roots are all roots of unity

The Alexander polynomial of $o9\_30634 = K_1 = K_{1,0}$ can be written as

$$\Delta_{o9\_30634}(t) \doteq \frac{(t^6+1)(t^9+1)}{(t+1)(t^2+1)}.$$

From this one may observe that all of its roots are roots of unity. Since $o9\_30634$ is a hyperbolic L–space knot, it provides a counterexample to [22, Conjecture 1.3]; see also the discussion surrounding its reference as [18, Conjecture 6.10]. Indeed, we have infinite families of hyperbolic L–space knots that are counterexamples to this conjecture:

Theorem 4.4 *The two infinite families of hyperbolic L–space knots $\{K_n\}_{n\geq 1}$ and $\{K_{n,-1}\}_{n\geq 1}$ consist of knots whose Alexander polynomials have all of their roots on the unit circle.*

Proof Proposition 3.1(1) and (4) show that the knots of $\{K_n\}_{n\geq 1}$ and $\{K_{n,-1}\}_{n\geq 1}$ are hyperbolic L–space knots. Proposition 3.1(3) gives a general formula for $\Delta_{K_{n,m}}(t)$. In the course of that proof, we obtained the first equality below. Dividing out the unit t and rearranging gives the second:

$$\begin{aligned}\Delta_{K_{n,m}}(t) &= t^{4n+3-m}\frac{(t-1)(t^{m-4n-2}+t^{-m}-t^{1-m}+t^{2-m}+t^{m+1}-t^{m+2}+t^{m+3}+t^{-m+4n+5})}{(t^4-1)}\\ &\doteq \frac{(t^{8n+7}+t^{4n+4}-t^{4n+3}+t^{4n+2})t^{-2m}+(t^{4n+3}-t^{4n+4}+t^{4n+5}+1)}{(t+1)(t^2+1)}.\end{aligned}$$

Setting $m=0$ yields

$$\begin{aligned}\Delta_{K_{n,0}}(t) &\doteq \frac{(t^{8n+7}+t^{4n+4}-t^{4n+3}+t^{4n+2})+(t^{4n+3}-t^{4n+4}+t^{4n+5}+1)}{(t+1)(t^2+1)}\\ &= \frac{t^{8n+7}+t^{4n+5}+t^{4n+2}+1}{(t+1)(t^2+1)} = \frac{(t^{4n+5}+1)(t^{4n+2}+1)}{(t+1)(t^2+1)},\end{aligned}$$

while setting $m=-1$ yields

$$\begin{aligned}\Delta_{K_{n,-1}}(t) &\doteq \frac{(t^{8n+9}+t^{4n+6}-t^{4n+5}+t^{4n+4})+(t^{4n+3}-t^{4n+4}+t^{4n+5}+1)}{(t+1)(t^2+1)}\\ &= \frac{t^{8n+9}+t^{4n+6}+t^{4n+3}+1}{(t+1)(t^2+1)} = \frac{(t^{4n+6}+1)(t^{4n+3}+1)}{(t+1)(t^2+1)}.\end{aligned}$$

From these presentations of their Alexander polynomials, one sees that all of their roots are roots of unity. □

Remark 4.5 (1) While we do not yet know if any of the knots in $\{K_n\}_{n\geq 1}$ are braid positive, all of the knots $\{K_{n,-1}\}_{n\geq 1}$ are braid positive by Proposition 3.1(5).

(2) As one may check, the hyperbolic L–space knots $\{K_{n,-2}\}_{n\geq 1}$ have Alexander polynomials with roots that are not roots of unity.

Remark 4.6 In light of Theorem 4.4 and [9, Corollary 1.2], one may hope that at least one of the hyperbolic L–space knots among $\{K_n\}_{n\geq 1}$ and $\{K_{n,-1}\}_{n\geq 1}$ has a double branched cover that is an L–space. This would answer a question of Moore in the negative; see [9, Question 1.3]. However, as one may check, these knots are not *definite*. Indeed, $|\sigma(K_n)| = g(K_n)+2 < 2g(K_n)$ while $|\sigma(K_{n,-1})| = g(K_{n,-1})+3 < 2g(K_{n,-1})$.

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Branched covers and rational homology balls

CHARLES LIVINGSTON

The concordance group of knots in S^3 contains a subgroup isomorphic to $(\mathbb{Z}_2)^\infty$, each element of which is represented by a knot K with the property that, for every $n > 0$, the n–fold cyclic cover of S^3 branched over K bounds a rational homology ball. This implies that the kernel of the canonical homomorphism from the knot concordance group to the infinite direct sum of rational homology cobordism groups (defined via prime-power branched covers) contains an infinitely generated two-torsion subgroup.

57K10, 57M12

1 Introduction

There is a homomorphism

$$\varphi\colon \mathcal{C} \to \prod_{q \in \mathcal{Q}} \Theta^3_{\mathbb{Q}},$$

where $\mathcal{C}$ is the smooth concordance group of knots in S^3, $\mathcal{Q}$ is the set of prime power integers, and $\Theta^3_{\mathbb{Q}}$ is the rational homology cobordism group. For a knot K and $q \in \mathcal{Q}$, the q–component of $\varphi(K)$ is the class of $M_q(K)$, the q–fold cyclic cover of S^3 branched over K.

In [1], Aceto, Meier, A Miller, M Miller, Park, and Stipsicz proved that $\ker\varphi$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^5$. Here we will prove that $\ker\varphi$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^\infty$. Our examples are of the form $K\#{-K^r}$, where $-K$ denotes the concordance inverse of K (the mirror image of K with string orientation reversed), and K^r is formed from K by reversing its string orientation. Such knots are easily seen to be in the kernel of φ; the more difficult work is to find nontrivial examples of order two.

The first known example of a nontrivial element in $\ker\varphi$ was represented by the knot $K_1 = 8_{17}\#{-8^r_{17}}$, which is of order two in $\mathcal{C}$. That K_1 is of order at most two is elementary; that K_1 is nontrivial in $\mathcal{C}$ is one of the main results of Kirk and Livingston in [9], proved using twisted Alexander polynomials.

The results of Kim and Livingston [7] provide an infinitely generated free subgroup of $\ker\varphi$. Conjecturally, $\mathcal{C} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}_2)^\infty$; if true, then $\ker\varphi \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}_2)^\infty$.

1.1 Main result

Figure 1 illustrates a knot P_n in a solid torus, where J_n represents the braid illustrated on the right in the case of $n = 5$; n will always be odd. We let K_n denote the satellite of 8_{17} built from P_n. In standard

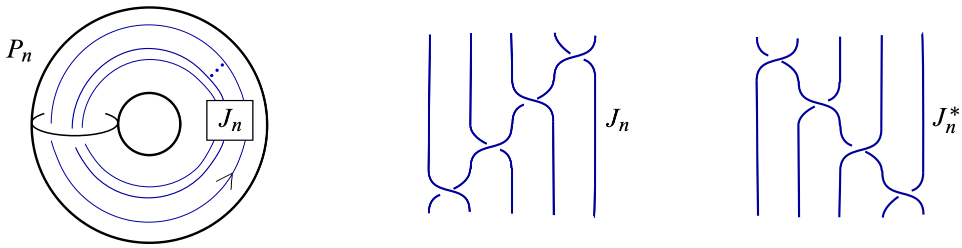

Figure 1: The knot $P_n \subset S^1 \times B^2$, J_n, and J_n^*.

notation, $K_n = P_n(8_{17})$. For future reference, we illustrate the braid J_n^* formed by rotating J_n around the vertical axis.

Theorem 1 *Let $K_n = P_n(8_{17})$. For all odd n, the knot $L_n = K_n \# -K_n^r$ satisfies $2L_n = 0 \in \mathcal{C}$ and $L_n \in \ker\varphi$. There is an infinite set of prime integers $\mathcal{P}$ for which $L_\alpha \neq L_\beta \in \mathcal{C}$ for all $\alpha \neq \beta$ in $\mathcal{P}$. In particular, the set of knots $\{L_n\}_{n\in\mathcal{P}}$ generates a subgroup of $\ker\varphi$ that is isomorphic to $(\mathbb{Z}_2)^\infty$.*

The rest of the paper presents a proof of this theorem. The first two claims are easily dealt with in Sections 2 and 3. The more difficult step of the proof calls on an analysis of twisted Alexander polynomials and their relevance to knot slicing; a review of twisted polynomials is included in Section 4. The proof of Theorem 1 is completed in Section 5, with the exception of a number-theoretic result that is described Appendix A.

Acknowledgements Thanks to Darrell Haile for assisting me in the proof of the number-theoretic result in Appendix A. Allison Miller provided valuable feedback about an early draft of this paper. This work was supported by a grant from the National Science Foundation, NSF-DMS-1505586.

2 Proof that $2L_n = 0 \in \mathcal{C}$

Let $P_n^* \subset S^1 \times B^2$ denote the knot formed using the braid J_n^* in Figure 2. For any knot K, let $P_n^*(K)$ denote the satellite of K built using P_n^*. It should be clear that P_n and P_n^* are orientation-preserving isotopic, and thus for all knots K, $P_n(K) = P_n^*(K)$.

Figure 2 illustrates, for an arbitrary knot K, the connected sum $P_n(K) \# P_n^*(K) = P_n(K) \# P_n(K)$ in the case of $n = 5$. Performing $n-1$ band moves in the evident way yields the $(0,n)$–cable of $K \# K$. Thus, if $K \# K = 0 \in \mathcal{C}$, then the n components of this link can be capped off with parallel copies of the slice disk for $K \# K$, implying that $P_n(K) \# P_n(K) = 0 \in \mathcal{C}$. In particular, $2K_n = 0 \in \mathcal{C}$ and $2K_n^r = 0 \in \mathcal{C}$.

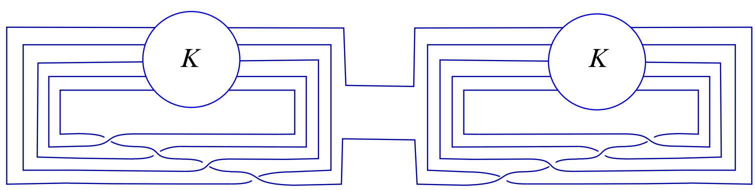

Figure 2: $P_5(K) \# P_5(K)$.

3 Proof that $L_n \in \ker \varphi$

We prove a stronger statement: *for all odd n, and for all positive integers q, $M_q(L_n)$ is a rational homology sphere that represents $0 \in \Theta^3_{\mathbb{Q}}$.*

The q–fold cyclic cover of S^3 branched over $K_n \# -K_n^r$ is the same space as the q–fold cyclic cover of S^3 branched over $K_n \# -K_n$. A slice disk for $K_n \# -K_n$ is built from $(S^3 \times I, K_n \times I)$ by removing a copy of $B^3 \times I$. Taking the q–fold branched cover shows that the q–fold cyclic cover of B^4 branched over that slice disk is diffeomorphic to $M_q(K_n)^* \times I$, where $M_q(K_n)^*$ denotes a punctured copy of $M_q(K_n)$. It remains to show that $M_q(K_n)$ is a rational homology 3–sphere.

A formula of Fox [5] and Goeritz [6] states that the order of the first homology of $M_q(K_n)$ is given by the product of values $\Delta_{K_n}(\omega_q^i)$, where $\Delta_{K_n}(t)$ denotes the Alexander polynomial, ω_q is a primitive q–root of unity, and i runs from 1 to $q-1$.

A result of Seifert [11] shows that $\Delta_{K_n}(t) = \Delta_{8_{17}}(t^n)\Delta_{P_n(U)}$, where U is the unknot. We have that $P_n(U) = U$. The Alexander polynomial for 8_{17} is

$$\Delta_{8_{17}}(t) = 1 - 4t + 8t^2 - 11t^3 + 8t^4 - 4t^5 + t^6.$$

A numeric computation confirms that this polynomial does not have roots on the unit complex circle, and hence $\Delta_{8_{17}}(t^n)$ has no roots on the unit complex circle. From this is follows that $\Delta_{K_n}(\omega_q^i) \neq 0$ for all i; thus the order of the homology of $M_q(K_n)$ is finite.

4 Review of twisted polynomials and 8_{17}

In this section we review twisted Alexander polynomials and their application in [8; 9] showing that $8_{17} \# -8_{17}^r \neq 0 \in \mathcal{C}$.

Let $(X, B) \to (S^3, K)$ be the q–fold cyclic branched cover of a knot K with q a prime power. In particular, X is a rational homology sphere. There is a canonical surjection $\epsilon\colon H_1(X - B) \to \mathbb{Z}$. Suppose that $\rho\colon H_1(X) \to \mathbb{Z}_p$ is a homomorphism for some prime p. Then there is an associated twisted polynomial $\Delta_{K,\epsilon,\rho}(t) \in \mathbb{Q}(\omega_p)[t]$. It is well-defined, up to factors of the form at^k, where $a \neq 0 \in \mathbb{Q}(\omega_p)$. These polynomials are discriminants of Casson–Gordon invariants, first defined in [3].

In the case of $K = 8_{17}$ and $q = 3$, we have $H_1(X) \cong \mathbb{Z}_{13} \oplus \mathbb{Z}_{13}$, and as a $\mathbb{Z}_{13}$–vector space this splits as a direct sum of a 3–eigenspace and a 9–eigenspace under the order three action of the deck transformation. Both eigenspaces are 1–dimensional. We denote this splitting by $E_3 \oplus E_9$. There are corresponding characters ρ_3 and ρ_9 of $H_1(X)$ onto $\mathbb{Z}_{13}$; these are defined as the quotient maps onto $H_1(X)/E_3$ and onto $H_1(X)/E_9$. We let ρ_0 denote the trivial $\mathbb{Z}_{13}$–valued character.

The values of $\Delta_{8_{17},\epsilon,\rho_i}(t)$ are given in [9], duplicated here in Appendix B. For $i = 0$ it is polynomial in $\mathbb{Q}[t]$. For $i = 3$ and $i = 9$ it is in $\mathbb{Q}(\omega_{13})[t]$ and is not in $\mathbb{Q}[t]$. An essential observation is that, for 8_{17}^r, the roles of ρ_3 and ρ_9 are reversed. All three of the polynomials are irreducible in their respective polynomial rings, once any factors of $(1-t)$ and t are removed.

In [9] the proof that $8_{17}\,\#-8^r_{17}$ is not slice comes down to the observation that no product of the form

$$\sigma_\delta(\Delta_{8_{17},\epsilon,\rho_3}(t))\sigma_\gamma(\Delta_{8_{17},\epsilon,\rho_i}(t)) \quad \text{or} \quad \sigma_\delta(\Delta_{8_{17},\epsilon,\rho_9}(t))\sigma_\gamma(\Delta_{8_{17},\epsilon,\rho_j}(t))$$

is of the form $af(t)\overline{f(t^{-1})}(1-t)^j$ for some $f(t)\in\mathbb{Q}(\omega_{13})[t]$. (That is, these products are not *norms* in the polynomial ring $\mathbb{Q}(\omega_{13})[t,t^{-1}]$, modulo powers of $(1-t)$ and t.) Here $i=0$ or $i=9$, and $j=0$ or $j=3$. The number a is in $\mathbb{Q}(\omega)$ and the σ_ν are Galois automorphisms of $\mathbb{Q}(\omega_p)$ (which acts by sending ω_p to ω_p^ν).

Showing that the product of the polynomials does not factor in this way is elementary once it is established that $\Delta_{8_{17},\epsilon,\rho_3}(t)$ and $\Delta_{8_{17},\epsilon,\rho_9}(t)$ are irreducible and not Galois conjugate.

5 Main proof

Using the fact that $-P_n(8_{17})^r = P_n(8_{17})^r$, the knot $L_\alpha\,\#\,L_\beta$ can be expanded as

$$P_\alpha(8_{17})\,\#\,P_\alpha(8_{17})^r\,\#\,P_\beta(8_{17})\,\#\,P_\beta(8_{17})^r.$$

We begin by analyzing the 3–fold cover of S^3 branched over $P_n(8_{17})$, and assume that 3 does not divide n. This cover is $M_3(P_n(8_{17}))$ and we denote the branch set in the cover by $\widetilde{B}$.

There is the obvious separating torus T in $S^3\setminus P_n(8_{17})$. Since 3 does not divide n, T has a connected separating lift $\widetilde{T}\subset M_3(P_n(8_{17}))$. One sees that $\widetilde{T}$ splits $M_3(P_n(8_{17}))$ into two components: X, the 3–fold cyclic cover of $S^3\setminus 8_{17}$, and Y, the 3–fold cyclic branched cover of $S^1\times B^2$, branched over P_n. A simple exercise shows that, since $P_n(U)$ is unknotted, Y is the complement of some knot $\tilde{J}_n\subset S^3$.

A Mayer–Vietoris argument shows that $H_1(M_3(P_n(8_{17})))\cong\mathbb{Z}_{13}\oplus\mathbb{Z}_{13}$ and the two canonical representations ρ_3 and ρ_9 that are defined on X extend trivially on Y, and so to $M_3(P_n(8_{17}))$. We denote these extension by ρ_3' and ρ_9'. Let ϵ' be the canonical surjective homomorphism $\epsilon'\colon H_1(M_3(P_n(8_{17}))\setminus\widetilde{B})\to\mathbb{Z}$. Restricted to X we have $\epsilon'(x)=\epsilon(nx)$, where ϵ was the canonical representation to $\mathbb{Z}$ defined for the cover of $S^3\setminus 8_{17}$.

In [8, Theorem 3.7] there is a discussion of twisted Alexander polynomials of satellite knots in S^3, working in the greater generality of homomorphisms to the unitary group $U(m)$. (A map to $\mathbb{Z}_p$ can be viewed as a representation to $U(1)$.) The proof of that theorem, which relies on the multiplicativity of Reidemeister torsion, applies in the current setting, yielding the following lemma:

Lemma 2 $$\Delta_{P_n(8_{17}),\epsilon',\rho_3'}(t)=\Delta_{8_{17},\epsilon,\rho_3}(t^n)\Delta_{\tilde{J}_n}(t).$$

Similar results hold for the knot $P_n(8_{17})^r$ and for the character ρ_9.

As described in [8; 9], Casson–Gordon theory implies that, if $L_\alpha \# L_\beta$ is slice, then for some 3–eigenvector or for some 9–eigenvector the corresponding twisted Alexander polynomial is a norm; that is, it factors as $at^k f(t)\overline{f(t^{-1})}$, modulo multiples of $(1-t)$. If it is a 3–eigenvector, the relevant polynomial is of the form

$$(1)\quad \Delta(t) = \sigma_a(\Delta_{8_{17},\epsilon,\rho_3}(t^\alpha))^x \sigma_b(\Delta_{8_{17},\epsilon,\rho_9}(t^\alpha))^y \sigma_c(\Delta_{8_{17},\epsilon,\rho_3}(t^\beta))^z \sigma_d(\Delta_{8_{17},\epsilon,\rho_9}(t^\beta))^w (\Delta_{\tilde{J}_\alpha}(t)\Delta_{\tilde{J}_\beta}(t))^2,$$

where one of x, y, z, or w is equal to 1, and each of the others are either 1 or 0.

The four $\mathbb{Q}(\omega_{13})[t]$–polynomials that appear here,

$$\Delta_{8_{17},\epsilon,\rho_3}(t^\alpha),\quad \Delta_{8_{17},\epsilon,\rho_9}(t^\alpha),\quad \Delta_{8_{17},\epsilon,\rho_3}(t^\beta),\quad \text{and}\quad \Delta_{8_{17},\epsilon,\rho_9}(t^\beta),$$

and all their Galois conjugates are easily seen to be distinct for any pair $\alpha \neq \beta$. The following number-theoretic result implies that there is an infinite set of primes $\mathcal{P}$ such that, if $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$, then no product as given in (1) can be a norm in $\mathbb{Q}(\omega_{13})[t]$, proving that the connected sum $L_\alpha \# L_\beta$ is not slice. We will present a proof in Appendix A.

Lemma 3 *Let $f(t) \in \mathbb{Z}(\omega_p)[t]$ be an irreducible monic polynomial. If there exists $\zeta \in \mathbb{C}$ such that $f(\zeta) = 0$ and $\zeta^n \neq 1$ for all $n > 0$, then the set of primes p for which $f(t^p)$ is reducible is finite.*

Proof of Theorem 1 The last factor in (1) involving the $\tilde{J}_n$ is a norm, so it can be ignored in determining if the product is a norm.

A numeric computation shows that the twisted polynomials $\Delta_{8_{17},\epsilon,\rho_i}(t)$ for $i = 3$ and $i = 9$ do not have roots on the unit circle, so Lemma 3 can be applied with $\mathbb{F} = \mathbb{Q}(\omega_{13})$. Let $\mathcal{P}$ be the infinite set of primes with the property that if $p \in \mathcal{P}$, then $\Delta_{8_{17},\epsilon,\rho_3}(t^p)$ and $\Delta_{8_{17},\epsilon,\rho_9}(t^p)$ are irreducible. Consider the case of $x = 1$ in (1). Then, assuming that $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$, the term $\sigma_a(\Delta_{8_{17},\epsilon,\rho_3})(t^\alpha)$ that appears in (1) is relatively prime to the remaining factors, and all the factors are irreducible, modulo powers of t and $1-t$. Hence, the product cannot be of the form $t^k(1-t)^j f(t) f(t^{-1})$ for any $f(t) \in \mathbb{Q}(\omega_{13})[t]$. The cases of y, z, or $w = 1$ are the same. □

Appendix A Factoring $f(t^p)$

In this appendix we prove Lemma 3, stated in somewhat more generality as Lemma 4 below. We first summarize some background material. Further details can be found in any graduate textbook on algebraic number theory.

- $\mathbb{A} \subset \mathbb{C}$ denotes the ring of algebraic integers. This is the ring consisting of all roots of monic polynomials in $\mathbb{Z}[t]$.
- For an extension field $\mathbb{F}/\mathbb{Q}$, the ring of algebraic integers in $\mathbb{F}$ is defined by $\mathcal{O}_\mathbb{F} = \mathbb{F} \cap \mathbb{A}$.
- The property of *transitivity* states that, if $f(t) \in \mathcal{O}_\mathbb{F}[t]$ is monic and $f(\zeta) = 0$, then $\zeta \in \mathbb{A}$.

- $\mathcal{O}_{\mathbb{F}}^{\times}$ is defined to be the set of units in $\mathcal{O}_{\mathbb{F}}$.
- The *norm* of an element $x \in \mathcal{O}_{\mathbb{F}}$ is defined as $N(x) = \prod x_i \in \mathbb{Z}$, where the x_i are the complex Galois conjugates of x. This map satisfies $N(xy) = N(x)N(y)$ for all $x, y \in \mathcal{O}_{\mathbb{F}}$. An element $x \in \mathcal{O}_{\mathbb{F}}$ is in $\mathcal{O}_{\mathbb{F}}^{\times}$ if and only if $N(x) = \pm 1$.
- The *Dirichlet unit theorem* states that, for a finite extension $\mathbb{F}/\mathbb{Q}$, the abelian group $\mathcal{O}_{\mathbb{F}}^{\times}$ is finitely generated and isomorphic to $G \oplus \mathbb{Z}^{r+s-1}$, where G is finite cyclic, r is the number of embeddings of $\mathbb{F}$ in $\mathbb{R}$, and $2s$ is the number of nonreal embeddings of $\mathbb{F}$ in $\mathbb{C}$.

Lemma 4 *Let $\mathbb{F}$ be a finite extension of $\mathbb{Q}$, and let $f(t) \in \mathcal{O}_{\mathbb{F}}[t]$ be an irreducible monic polynomial. If there exists $\zeta \in \mathbb{C}$ such that $f(\zeta) = 0$ and $\zeta^n \neq 1$ for all $n > 0$, then the set of primes p for which $f(t^p)$ is reducible is finite.*

Proof **Step 1** *If $f(\zeta) = 0$, then $\zeta \in \mathcal{O}_{F(\zeta)}$.*

This follows immediately from the assumption that $f(t)$ is monic.

Step 2 *Suppose that $f(t) \in \mathbb{F}[t]$ is irreducible and $f(\zeta) = 0$. If, for some prime p, $f(t^p)$ is reducible over $\mathbb{F}$, then $\zeta = \eta^p$ for some $\eta \in \mathcal{O}_{\mathbb{F}(\zeta)}$.*

Let $\xi \in \mathbb{C}$ satisfy $\xi^p = \zeta$. Since $f(t)$ is irreducible of degree n and $f(t^p)$ is reducible, we have the degrees of extensions satisfying $[\mathbb{F}(\zeta) : \mathbb{F}] = n$ and $[\mathbb{F}(\xi) : \mathbb{F}] < np$. It follows from the multiplicity of degrees of extensions that $[\mathbb{F}(\xi) : \mathbb{F}(\zeta)] < p$.

The polynomial $t^p - \zeta \in \mathbb{F}(\zeta)[t]$ has ξ as a root. For all i, $\omega_p^i \xi$ is also a root, so $t^p - \zeta$ factors completely in $\mathbb{C}[t]$ as

$$t^p - \zeta = (t - \xi)(t - \omega_p \xi) \cdots (t - \omega_p^{p-1} \xi).$$

By the degree calculation just given, $t^p - \zeta$ has an irreducible factor $g(t) \in \mathbb{F}(\zeta)[t]$ of degree $l < p$. We can write $g(t) = \prod (t - \omega_p^i \xi)$, where the product is over some proper subset of $\{0, \dots, p-1\}$. Multiplying this out, one finds that the constant term is of the form $\omega_p^j \xi^l \in \mathbb{F}(\zeta)$ for some j and $l < p$. Since l and p are relatively prime, there are integers u and v such that $ul + vp = 1$. Thus, $(\omega_p^j \xi^l)^u (\xi^p)^v = \omega_p^s \xi$ for some s. In particular, for some s, we have $\omega_p^s \xi \in \mathbb{F}(\zeta)$. We let $\eta = \omega_p^s \xi$ and find that $\eta^p = (\omega_p^s)^p \xi^p = \zeta$. Finally, η satisfies the monic polynomial $f(t^p)$ and thus is in $\mathcal{O}_{\mathbb{F}(\zeta)}$.

Step 3 *The set of primes p such that $\zeta = \eta^p$ for some $\eta \in \mathcal{O}_{\mathbb{F}(\zeta)}$ is finite.*

If $\zeta = \eta^p$, then $N(\zeta) = N(\eta)^p$. If $N(\zeta) \neq \pm 1$, then the set of p for which $N(\zeta) = a^p$ for some integer a is finite.

If $N(\zeta) = \pm 1$, then $\zeta \in \mathcal{O}_{\mathbb{F}(\zeta)}^{\times}$. Hence ζ represents a nontorsion element in a finitely generated abelian group, and thus it has a finite number of roots. □

Comments The argument just given is based on a summary of the proof for the case $\mathbb{F} = \mathbb{Q}$ presented on MathOverflow by Dimitrov [4]. Step 2 is a special case of the *Vahlen–Capelli theorem*, proved in the case of $\mathbb{F} = \mathbb{Q}$ by Vahlen and for fields of characteristic 0 by Capelli [2]. A proof for fields of finite characteristic is given by Rédei [10].

Appendix B Twisted polynomials of 8_{17}

Here are the three needed polynomials. We write ω for ω_{13}.

$$\Delta_{8_{17},\epsilon,\rho_0}(t) = 1-t-34t^2-101t^3-34t^4-t^5+t^6,$$

$$\begin{aligned}
&\Delta_{8_{17},\epsilon,\rho_3}(t)/(1-t)\\
&\quad = 1+t(2\omega+2\omega^2+2\omega^3+4\omega^4+2\omega^5+2\omega^6+\omega^7+\omega^8+2\omega^9+4\omega^{10}+\omega^{11}+4\omega^{12})\\
&\qquad +t^2(-15\omega-10\omega^2-15\omega^3-15\omega^4-10\omega^5-10\omega^6-10\omega^7-10\omega^8-15\omega^9-15\omega^{10}-10\omega^{11}-15\omega^{12})\\
&\qquad +t^3(4\omega+\omega^2+4\omega^3+2\omega^4+\omega^5+\omega^6+2\omega^7+2\omega^8+4\omega^9+2\omega^{10}+2\omega^{11}+2\omega^{12})+t^4,
\end{aligned}$$

$$\begin{aligned}
&\Delta_{8_{17},\epsilon,\rho_9}(t)/(1-t)\\
&\quad = 1+t(6\omega+5\omega^2+6\omega^3+6\omega^4+5\omega^5+5\omega^6+5\omega^7+5\omega^8+6\omega^9+6\omega^{10}+5\omega^{11}+6\omega^{12})\\
&\qquad +t^2(-13\omega-12\omega^2-13\omega^3-13\omega^4-12\omega^5-12\omega^6-12\omega^7-12\omega^8-13\omega^9-13\omega^{10}-12\omega^{11}-13\omega^{12})\\
&\qquad +t^3(6\omega+5\omega^2+6\omega^3+6\omega^4+5\omega^5+5\omega^6+5\omega^7+5\omega^8+6\omega^9+6\omega^{10}+5\omega^{11}+6\omega^{12})+t^4.
\end{aligned}$$

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