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In general, not much is known about the arithmetic of K3 surfaces. Once the geometric Picard number, which is the rank of the Néron–Severi group over an algebraic closure of the base field, is high enough, more structure is known and more can be said. However, until recently not a single explicit K3 surface was known to have geometric Picard number one. We give explicit examples of such surfaces over the rational numbers. This solves an old problem that has been attributed to Mumford. The examples we give also contain infinitely many rational points, thereby answering a question of Swinnerton-Dyer and Poonen.

1. Introduction

K3 surfaces are two-dimensional analogues of elliptic curves in the sense that their canonical sheaf is trivial. However, as opposed to elliptic curves, little is known about the arithmetic of K3 surfaces in general. It is for instance an open question if there exists a K3 surface X over a number field such that the set of rational points on X is neither empty, nor dense (which throughout this paper will always refer to the Zariski topology). We will answer a longstanding question regarding the Picard group of a K3 surface. The Picard group of any variety is the group of line bundles on it, up to isomorphism. For a K3 surface X over a field k this is a finitely generated free abelian group, the rank of which is called the Picard number of X. The Picard number of $\bar{X} = X \times_k \bar{k}$, where $\bar{k}$ denotes an algebraic closure of k, is called the geometric Picard number of X. We will give the first known examples of explicit K3 surfaces shown to have geometric Picard number 1.

Bogomolov and Tschinkel [2000] showed an interesting relation between the geometric Picard number of a K3 surface X over a number field K and the arithmetic of X. They proved that if the geometric Picard number is at least 2, then in most cases the rational points on X are potentially dense, which means that there exists a finite field extension L of K such that the set $X(L)$ of L-rational points is Zariski dense in X. However, it is not yet known whether there exists

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any K3 surface over a number field and with geometric Picard number 1 on which the rational points are potentially dense. Neither do we know if there exists a K3 surface over a number field and with geometric Picard number 1 on which the rational points are *not* potentially dense!

In December 2002, at the AIM workshop on rational and integral points on higher-dimensional varieties in Palo Alto, Swinnerton-Dyer and Poonen asked a related question. They asked whether there exists a K3 surface over a number field and with Picard number 1 that contains infinitely many rational points. In this article we will show that such K3 surfaces do indeed exist. It follows from our main theorem.

Theorem 1.1. *In the moduli space of* K3 *surfaces polarized by a very ample divisor of degree* 4, *the set of points parametrizing surfaces defined over* $\mathbb{Q}$ *with geometric Picard number* 1 *and infinitely many rational points is Zariski dense.*

As important as this result is the strategy of its proof. It contains a new way of finding sharp bounds for the geometric Picard number of a surface. This new method is widely applicable. It is based on the older idea that the Néron–Severi group of a surface X defined over a number field injects into the Néron–Severi group of its reduction $X_{\mathfrak{p}}$ at a prime $\mathfrak{p}$ of good reduction. By the Tate conjecture (proven in many cases for K3 surfaces), the geometric Picard number of a K3 surface in positive characteristic is even, and therefore at least 2. We will lower this upper bound for the geometric Picard number of X by comparing the lattice structure on the geometric Néron–Severi group of the reduction of X at two different primes of good reduction. If these both have rank 2 and their discriminants do not differ by a square factor, then there is no 2-dimensional lattice that injects into both, and we may conclude that the geometric Picard number of X equals 1.

Note that a polarization of a K3 surface is a choice of an ample divisor H. The degree of such a polarization is H^2. A K3 surface polarized by a very ample divisor of degree 4 is a smooth quartic surface in $\mathbb{P}^3$. We will prove the main theorem by exhibiting an explicit family of quartic surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ with geometric Picard number 1 and infinitely many rational points. Proving that these surfaces contain infinitely many rational points is the easy part. It is much harder to prove that the geometric Picard number of these surfaces equals 1. It has been known since Max Noether that a general hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ of degree at least 4 has geometric Picard number 1. A modern proof of this fact is given in [Deligne and Katz 1973, Theorem XIX.1.2]. Despite this fact, it has been an old challenge, attributed to Mumford and disposed of in this article, to find even one explicit quartic surface, defined over a number field, of which the geometric Picard number equals 1. Deligne's result does not actually imply that such surfaces exist, as "general" means "up to a countable union of closed subsets of the moduli space." A priori, this could exclude all surfaces defined over $\overline{\mathbb{Q}}$. Although they do not give explicit surfaces with geometric Picard number 1 over number fields either, Terasoma and Ellenberg have proved that they do exist.

Theorem 1.2 [Terasoma 1985]. *For any positive integers* $(n; a_1, \ldots, a_d)$ *not equal to* $(2; 3)$, $(n; 2)$, *or* $(n; 2, 2)$, *and with n even*, *there is a smooth complete intersection X over $\mathbb{Q}$ of dimension n defined by equations of degrees $a_1, \ldots, a_d$ such that the middle geometric Picard number of X is* 1. □

Theorem 1.3 [Ellenberg 2004]. *For every even integer d there exists a number field K and a polarized K3 surface X/K of degree d*, *with geometric Picard number* 1. □

The proofs of Terasoma and Ellenberg are ineffective in the sense that they do not give explicit examples. In principle it might be possible to extend their methods to test whether a given explicit K3 surface has geometric Picard number 1. In practice however, it is an understatement to say that the amount of work involved is not encouraging. The explicit examples we will give to prove the main theorem also prove the case $(n; a_1, \ldots, a_d) = (2; 4)$ of Theorem 1.2 and the case $d = 4$ of Theorem 1.3.

Shioda did find explicit examples of surfaces with geometric Picard number 1. In fact, he has shown [1981] that for every prime $m \geq 5$ the surface in $\mathbb{P}^3$ given by

$$w^m + xy^{m-1} + yz^{m-1} + zx^{m-1} = 0$$

has geometric Picard number 1. However, for $m = 4$ this equation determines a K3 surface with maximal geometric Picard number 20, i.e., a singular K3 surface.

Before we prove the main theorem in Section 3, we will recall some definitions and results in Section 2.

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2. Prerequisites

A *lattice* is a free $\mathbb{Z}$-module L of finite rank, endowed with a symmetric, bilinear, nondegenerate map $\langle \_ , \_ \rangle : L \times L \to \mathbb{Q}$, called the *pairing* of the lattice. An *integral lattice* is a lattice with a $\mathbb{Z}$-valued pairing. A lattice L is called *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for every $x \in L$. A *sublattice* of L is a submodule L' of L, such that the induced bilinear pairing on L' is nondegenerate. The *Gram matrix* of a lattice L with respect to a given basis $x = (x_1, \ldots, x_n)$ is $I_x = (\langle x_i, x_j \rangle)_{i,j}$. The *discriminant* of L is defined by $\operatorname{disc} L = \det I_x$ for any basis x of L. For any sublattice L' of finite index in L we have $\operatorname{disc} L' = [L : L']^2 \operatorname{disc} L$. The image of $\operatorname{disc} L$ and $\operatorname{disc} L'$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ is the discriminant of the inner product space $L_{\mathbb{Q}}$, where the inner product is induced by the pairing of L.

Let X be a smooth, projective, geometrically integral surface over a field k and set $\bar{X} = X \times_k \bar{k}$, where $\bar{k}$ denotes an algebraic closure of k. As mentioned in the introduction, the *Picard group* $\operatorname{Pic} X$ of X is the group of line bundles on X up to isomorphism, or equivalently, the group of divisor classes modulo linear equivalence. The divisor classes that become algebraically equivalent to 0 over $\bar{k}$ (see [Hartshorne 1977, exercise V.1.7]) form a subgroup $\operatorname{Pic}^0 X$ of $\operatorname{Pic} X$. The quotient is the *Néron–Severi group* $\operatorname{NS}(X) = \operatorname{Pic} X / \operatorname{Pic}^0 X$, which is a finitely generated abelian group, see [Hartshorne 1977, exercise V.1.7–8], or [Milne 1980, Theorem V.3.25], for surfaces or [Grothendieck et al. 1971, exposé XIII, théorème 5.1] in general. The intersection pairing endows the group $\operatorname{NS}(X)/\operatorname{NS}(X)_{\text{tors}}$ with the structure of a lattice. Its rank is called the *Picard number* of X. The Picard number of $\bar{X}$ is called the *geometric Picard number* of X.

By definition a smooth, projective, geometrically integral surface X is a *K3 surface* if the canonical sheaf ω_X on X is trivial and $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$. Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$. The Betti numbers of a K3 surface are $b_0 = 1$, $b_1 = 0$, $b_2 = 22$, $b_3 = 0$, and $b_4 = 1$.

Lemma 2.1. *If X is a K3 surface, then $\operatorname{Pic}^0 X$ is trivial, the Néron–Severi group $\operatorname{NS}(X) \cong \operatorname{Pic} X$ is torsion free, and the intersection pairing on $\operatorname{NS}(X)$ is even.*

Proof. See [Barth et al. 1984, p. 21 and Proposition VIII.3.2] for characteristic 0 and [Bombieri and Mumford 1977, Theorem 5] for positive characteristic. □

For any scheme Z over $\mathbb{F}_q$, any prime $l \nmid q$, and any integer m, we will use the étale cohomological groups $H^i_{\text{ét}}(Z, \mathbb{Q}_l)$ and their Tate twists $H^i_{\text{ét}}(Z, \mathbb{Q}_l)(m)$ as defined in for instance [Tate 1965, p. 94], Proposition 2.2 describes the behavior of the Néron–Severi group under good reduction. Its corollary will be used to show that the geometric Picard number of a certain surface is equal to 1.

Proposition 2.2 [van Luijk 2007, Proposition 6.2]. *Let A be a discrete valuation ring of a number field L with residue field $k \cong \mathbb{F}_q$. Let S be an integral scheme with a morphism $S \to \operatorname{Spec} A$ that is projective and smooth of relative dimension 2. Assume that the surfaces $\bar{S} = S_{\bar{L}}$ and $\tilde{S} = S_{\bar{k}}$ are integral. Let $l \nmid q$ be a prime number. Then there are natural injective homomorphisms*

$$\operatorname{NS}(\bar{S}) \otimes \mathbb{Q}_l \hookrightarrow \operatorname{NS}(\tilde{S}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)(1) \tag{1}$$

of finite dimensional inner product spaces over $\mathbb{Q}_l$. The first injection is induced by a natural injection $\operatorname{NS}(\bar{S}) \otimes \mathbb{Q} \hookrightarrow \operatorname{NS}(\tilde{S}) \otimes \mathbb{Q}$. The second injection respects the Galois action of $G(\bar{k}/k)$.

Recall that for any scheme Z over $\mathbb{F}_q$ with $q = p^r$ and p prime, the *absolute Frobenius* $F_Z\colon Z \to Z$ of Z acts as the identity on points, and by $f \mapsto f^p$ on the structure sheaf. Set $\Phi_Z = F_Z^r$ and $\overline{Z} = Z \times \overline{\mathbb{F}}_q$. Let Φ_Z^* denote the automorphism on $H^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_l)$ induced by $\Phi_Z \times 1$ acting on $Z \times \overline{\mathbb{F}}_q = \overline{Z}$.

Corollary 2.3. *With the notation as in Proposition 2.2, the ranks of* $\mathrm{NS}(\tilde{S})$ *and* $\mathrm{NS}(\bar{S})$ *are bounded from above by the number of eigenvalues* λ *of* $\Phi^*_{S_k}$ *for which* λ/q *is a root of unity, counted with multiplicity.*

Proof. By Proposition 2.2 any upper bound for the rank of $\mathrm{NS}(\tilde{S})$ is an upper bound for the rank of $\mathrm{NS}(\bar{S})$. Let σ denote the q-th power Frobenius map, i.e., the canonical topological generator of $G(\bar{k}/k)$. For any positive integer m, let σ^* and $\sigma^*(m)$ denote the automorphisms induced on $\mathrm{NS}(\tilde{S}) \otimes \mathbb{Q}_l$ and $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)(m)$ respectively. As all divisor classes are defined over some finite extension of k, some power of Frobenius acts as the identity on $\mathrm{NS}(\tilde{S})$, so all eigenvalues of σ^* acting on $\mathrm{NS}(\tilde{S})$ are roots of unity. It follows from Proposition 2.2 that the rank of $\mathrm{NS}(\tilde{S})$ is bounded from above by the number of roots of $\sigma^*(1)$ that are a root of unity. As the eigenvalues of $\sigma^*(0)$ differ from those of $\sigma^*(1)$ by a factor of q, this equals the number of roots λ of $\sigma^*(0)$ for which λq is a root of unity. The Corollary follows from the fact that $\Phi^*_{S_k}$ acts on $H^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_l)$ as the inverse of $\sigma^*(0)$. See also [van Luijk 2007, Corollary 6.3]. □

Remark 2.4. Tate's conjecture [1965] states that the upper bound mentioned is actually equal to the rank of $\mathrm{NS}(\tilde{S})$. Tate's conjecture has been proven for ordinary K3 surfaces over fields of characteristic $p \geq 5$; see [Nygaard and Ogus 1985, Theorem 0.2].

To find the characteristic polynomial of Frobenius as in Corollary 2.3, we will compute the traces of powers of Frobenius and use Newton's identities, which for convenience we state here (see [Borwein and Erdélyi 1995, p. 5]):

Lemma 2.5 (Newton's identities). *Let* V *be a vector space of dimension* n *and* T *a linear operator on* V. *Let* t_i *denote the trace of* T^i. *Then the characteristic polynomial of* T *is equal to*

$$f_T(x) = \det(x \cdot \mathrm{Id} - T) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n,$$

with the c_i *given recursively by*

$$c_1 = -t_1 \qquad \textit{and} \qquad -kc_k = t_k + \sum_{i=1}^{k-1} c_i t_{k-i}.$$

3. Proof of the main theorem

First we will give a family of smooth quartic surfaces in $\mathbb{P}^3$ with Picard number 1. Let $R = \mathbb{Z}[x, y, z, w]$ be the homogeneous coordinate ring of $\mathbb{P}^3_{\mathbb{Z}}$. Throughout the rest of this article, for any homogeneous polynomial $h \in R$ of degree 4, let $\mathfrak{X}_h$ denote the scheme in $\mathbb{P}^3_{\mathbb{Z}}$ given by

$$wf_1 + 2zf_2 = 3g_1g_2 + 6h, \tag{2}$$

with $f_1, f_2, g_1, g_2 \in R$ equal to

$$\begin{aligned}
f_1 &= x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 \\
&\qquad + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\
f_2 &= xy^2 + xyz - xz^2 - yz^2 + z^3, \\
g_1 &= z^2 + xy + yz, \\
g_2 &= z^2 + xy.
\end{aligned}$$

Its base extensions to $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are denoted X_h and $\overline{X}_h$ respectively.

Theorem 3.1. *Let $h \in R$ be a homogeneous polynomial of degree 4. Then the quartic surface X_h is smooth over $\mathbb{Q}$ and has geometric Picard number 1. The Picard group* $\operatorname{Pic} \overline{X}_h$ *is generated by a hyperplane section.*

Proof. For $p = 2, 3$, let $X_p/\mathbb{F}_p$ denote the fiber of $\mathfrak{X}_h \to \operatorname{Spec} \mathbb{Z}$ over p. As they are independent of h, one easily checks that X_p is smooth over $\mathbb{F}_p$ for $p = 2, 3$. As the morphism $\mathfrak{X}_h \to \operatorname{Spec} \mathbb{Z}$ is flat and projective, it follows that the generic fiber X_h of $\mathfrak{X}_h \to \operatorname{Spec} \mathbb{Z}$ is smooth over $\mathbb{Q}$ as well; compare [Hartshorne 1977, exercise III.10.2].

We will first show that X_2 and X_3 have geometric Picard number 2. For $p = 2, 3$, let Φ_p denote the absolute Frobenius of X_p. Set $\overline{X}_p = X_p \times \overline{\mathbb{F}}_p$ and let $\Phi_p^*(i)$ denote the automorphism on $H^i_{\text{ét}}(\overline{X}_p, \mathbb{Q}_l)$ induced by $\Phi_p \times 1$ acting on $\overline{X}_p = X_p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Then by Corollary 2.3 the geometric Picard number of X_p is bounded from above by the number of eigenvalues λ of $\Phi_p^*(2)$ for which λ/p is a root of unity. We will find the characteristic polynomial of $\Phi_p^*(2)$ from the traces of its powers. These traces we will compute with the Lefschetz formula

$$\#X_p(\mathbb{F}_{p^n}) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}(\Phi_p^*(i)^n), \tag{3}$$

for which see [Milne 1980, Theorem VI.12.3]. Since X_p is a smooth hypersurface in $\mathbb{P}^3$ of degree 4, it is a K3 surface and its Betti numbers are $b_0 = 1$, $b_1 = 0$, $b_2 = 22$, $b_3 = 0$, and $b_4 = 1$. It follows that $\operatorname{Tr}(\Phi_p^*(i)^n) = 0$ for $i = 1, 3$, and for $i = 0$ and $i = 4$ the automorphism $\Phi_p^*(i)^n$ has only one eigenvalue, which by the

Weil conjectures equals 1 and p^{2n} respectively (see [Deligne 1974, théorème 1.6]). From the Lefschetz formula (3) we conclude $\mathrm{Tr}(\Phi_p^*(2)^n) = \#X_p(\mathbb{F}_{p^n}) - p^{2n} - 1$. After counting points on X_p over $\mathbb{F}_{p^n}$ for $n = 1, \ldots, 11$, this allows us to compute the traces of the first 11 powers of $\Phi_p^*(2)$. With Lemma 2.5 we can then compute the first coefficients of the characteristic polynomial f_p of $\Phi_p^*(2)$, which has degree $b_2 = 22$. We write $f_p = x^{22} + c_1 x^{21} + \cdots + c_{22}$, which by construction is independent of the choice of h, and find this table:

p	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
2	-3	-2	12	0	-32	64	-128	128	256	0	-2048
3	-5	-6	72	27	-891	0	9477	-4374	-78732	19683	708588

The Weil conjectures give a functional equation $p^{22} f_p(x) = \pm x^{22} f_p(p^2/x)$. As in our case (both for $p = 2$ and $p = 3$) the middle coefficient c_{11} of f_p is nonzero, the sign of the functional equation is positive. This functional equation allows us to compute the remaining coefficients of f_p.

If λ is a root of f_p then λ/p is a root of $\tilde{f}_p(x) = p^{-22} f_p(px)$. Hence, the number of roots of $\tilde{f}_p(x)$ that are also a root of unity gives an upper bound for the geometric Picard number of X_p. After full factorization, we find

$$\tilde{f}_2 = \tfrac{1}{2}(x-1)^2\bigl(2x^{20}+x^{19}-x^{18}+x^{16}+x^{14}+x^{11}+2x^{10}+x^9+x^6+x^4-x^2+x+2\bigr),$$
$$\begin{aligned}\tilde{f}_3 = \tfrac{1}{3}(x-1)^2\bigl(3x^{20} + x^{19} - 3x^{18} + x^{17} + 6x^{16} - 6x^{14} + x^{13} + 6x^{12} - x^{11}\\ - 7x^{10} - x^9 + 6x^8 + x^7 - 6x^6 + 6x^4 + x^3 - 3x^2 + x + 3\bigr).\end{aligned}$$

When $p = 2, 3$, the roots of the irreducible factor of $\tilde{f}_p$ of degree 20 are not integral. Therefore these roots are not roots of unity and we conclude that $\tilde{f}_p$ has only two roots that are roots of unity, counted with multiplicities. By Corollary 2.3 this implies that the geometric Picard number of X_p is at most 2.

Note that besides the hyperplane section H, the surface X_2 also contains the conic C given by $w = g_2 = z^2 + xy = 0$. We have $H^2 = \deg X_2 = 4$ and $H \cdot C = \deg C = 2$. As the genus $g(C)$ of C equals 0 and the canonical divisor K on X_2 is trivial, the adjunction formula $2g(C) - 2 = C \cdot (C + K)$ yields $C^2 = -2$. Thus H and C generate a sublattice of $\mathrm{NS}(\bar{X}_2)$ with Gram matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}.$$

We conclude that the inner product space $\mathrm{NS}(\bar{X}_2)_{\mathbb{Q}}$ has rank 2 and discriminant $-12 \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. Similarly, X_3 contains the line L given by $w = z = 0$, also with genus 0 and thus $L^2 = -2$. The hyperplane section on X_3 and L generate a sublattice of $\mathrm{NS}(\bar{X}_3)$ of rank 2 with Gram matrix

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}.$$

We conclude that the inner product space $\mathrm{NS}(\bar{X}_3)_{\mathbb{Q}}$ also has rank 2, and discriminant $-9 \in \mathbb{Q}^*/\mathbb{Q}^{*2}$.

Let ρ denote the geometric Picard number $\rho = \mathrm{rk}\,\mathrm{NS}(\bar{X}_h)$. It follows from Proposition 2.2 that there is an injection $\mathrm{NS}(\bar{X}_h)_{\mathbb{Q}} \hookrightarrow \mathrm{NS}(\bar{X}_p)_{\mathbb{Q}}$ of inner product spaces for $p = 2, 3$. Hence we get $\rho \leq 2$. If equality held, then both these injections would be isomorphisms and $\mathrm{NS}(\bar{X}_2)_{\mathbb{Q}}$ and $\mathrm{NS}(\bar{X}_3)_{\mathbb{Q}}$ would be isomorphic as inner product spaces. This is not the case because they have different discriminants. We conclude $\rho \leq 1$. As a hyperplane section H on X_h has self intersection $H^2 = 4 \neq 0$, we find $\rho = 1$. Since $\mathrm{NS}(\bar{X}_h)$ is a 1-dimensional even lattice (see Lemma 2.1), the discriminant of $\mathrm{NS}(\bar{X}_h)$ is even. The sublattice of finite index in $\mathrm{NS}(\bar{X}_h)$ generated by H gives

$$4 = \mathrm{disc}\langle H \rangle = [\mathrm{NS}(\bar{X}_h) : \langle H \rangle]^2 \cdot \mathrm{disc}\,\mathrm{NS}(\bar{X}_h).$$

Together with $\mathrm{disc}\,\mathrm{NS}(\bar{X}_h)$ being even this implies $[\mathrm{NS}(\bar{X}_h) : \langle H \rangle] = 1$, so H generates $\mathrm{NS}(\bar{X}_h)$, which is isomorphic to $\mathrm{Pic}\,\bar{X}_h$ by Lemma 2.1. □

Remark 3.2. Corollary 2.3 was pointed out to the author by Jasper Scholten and people have used it before to bound the geometric Picard number of a surface. However, since all nonreal roots of the characteristic polynomial of Frobenius come in conjugate pairs, the upper bound has the same parity as the second Betti number of the surface. For K3 surfaces this means that the upper bound is even, and therefore at least 2. Note that by Tate's conjecture (see Remark 2.4) the actual geometric Picard number of any K3 surface over a field of positive characteristic is at least 2. It is a complete mystery where this second cycle should come from. The strategy of the proof of Theorem 3.1 allows us to sharpen the upper bound in characteristic zero. If the reductions modulo two different primes give the same upper bound r, but the corresponding Néron–Severi groups have discriminants that do not differ by a square factor, then in fact $r - 1$ is an upper bound.

Kloosterman [2005] has used our method to construct an elliptic K3 surface with Mordell–Weil rank 15 over $\bar{\mathbb{Q}}$. In the proof of Theorem 3.1 we were able to compute the discriminant up to squares of the Néron–Severi lattice of $\bar{X}_p$ because we knew a priori a sublattice of finite index. Kloosterman realized that it is not always necessary to know such a sublattice. For an elliptic surface Y over $\bar{\mathbb{F}}_p$, the image in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ of the discriminant of the Néron–Severi lattice can also be deduced from the Artin–Tate conjecture, which has been proved for ordinary K3 surfaces in characteristic $p \geq 5$; see [Nygaard and Ogus 1985, Theorem 0.2] and [Milne 1975, Theorem 6.1]. It allows one to compute the ratio $\operatorname{disc} \mathrm{NS}(Y) \cdot \# \mathrm{Br}(Y)/(\# \mathrm{NS}(Y)_{\mathrm{tors}})^2$ from the characteristic polynomial of Frobenius acting on $H^2_{\text{ét}}(Y, \mathbb{Q}_l)$. For an elliptic surface the Brauer group has square order, so this ratio determines the same element in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ as $\operatorname{disc} \mathrm{NS}(Y)$. Of course this relies on heavy machinery, while our method is essentially elementary.

Remark 3.3. In the proof we counted points over $\mathbb{F}_{p^n}$ for $p = 2, 3$ and $n = 1, \ldots, 11$ in order to find the traces of powers of Frobenius up to the 11-th power. We could have got away with less counting. In both cases $p = 2$ and $p = 3$ we already know a 2-dimensional, Frobenius stable subspace W of $\mathrm{NS}(\bar{X}_p)_{\mathbb{Q}_l} \subset H^2_{\text{ét}}(\bar{X}_p, \mathbb{Q}_l)(1)$, generated by the hyperplane section H and another divisor class. Therefore it suffices to find out the characteristic polynomial of Frobenius acting on the quotient $V = H^2_{\text{ét}}(\bar{X}_p, \mathbb{Q}_l)(1)/W$. This implies it suffices to know the traces of powers of Frobenius acting on V up to the 10-th power.

An extra trick was used for $p = 3$. The family of planes through the line L given by $w = z = 0$ cuts out a fibration of curves of genus 1. We can give all nonsingular fibers the structure of an elliptic curve by quickly looking for a point on it. There are efficient algorithms available in for instance MAGMA to count the number of points on these elliptic curves.

Using these few speed-ups we let a computer run to compute the characteristic polynomial of several random surfaces given by an equation of the form $wf_1 = zf_2$ over $\mathbb{F}_3$ or $wf_1 = g_1g_2$ over $\mathbb{F}_2$, as in (2). If the middle coefficient of the characteristic polynomial was zero, no more effort was spent on trying to find the sign of the functional equation (see proof of Theorem 3.1) and the surface was discarded. After one night two examples over $\mathbb{F}_3$ were found with geometric Picard number 2 and one example over $\mathbb{F}_2$. With the Chinese Remainder Theorem this allows us to construct two families of surfaces with geometric Picard number 1. One of these families consists of the surfaces X_h. A program written in MAGMA that checks the characteristic polynomial of Frobenius on X_2 and X_3 is electronically available from the author upon request.

Remark 3.4. For $p = 2, 3$, let $A_p \subset \mathrm{NS}(\bar{X}_p)$ denote the lattice as described in the proof of Theorem 3.1, i.e., A_2 is generated by a hyperplane section and a conic, and A_3 is generated by a hyperplane section and a line. Then in fact A_p equals $\mathrm{NS}(\bar{X}_p)$ for $p = 2, 3$. Indeed, we have

$$\operatorname{disc} A_p = [\mathrm{NS}(\bar{X}_p) : A_p]^2 \cdot \operatorname{disc} \mathrm{NS}(\bar{X}_p).$$

For $p = 2$ this implies $\operatorname{disc} \mathrm{NS}(\bar{X}_2) = -12$ or $\operatorname{disc} \mathrm{NS}(\bar{X}_2) = -3$. The latter is impossible because modulo 4 the discriminant of an even lattice of rank 2 is congruent to 0 or -1. We conclude $\operatorname{disc}\mathrm{NS}(\bar{X}_2) = -12$, and therefore $[\mathrm{NS}(\bar{X}_2) : A_2] = 1$, so $A_2 = \mathrm{NS}(\bar{X}_2)$.

For $p = 3$ we find $\operatorname{disc}\mathrm{NS}(\bar{X}_3) = -9$ or $\operatorname{disc}\mathrm{NS}(\bar{X}_3) = -1$. Suppose the latter equation held. By the classification of even unimodular lattices we find that $\mathrm{NS}(\bar{X}_3)$ is isomorphic to the lattice with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By a theorem of Van Geemen [2005, 5.4], this is impossible. From this contradiction we conclude $\operatorname{disc}\mathrm{NS}(\bar{X}_3) = -9$ and thus $[\mathrm{NS}(\bar{X}_3) : A_3] = 1$, so $A_3 = \mathrm{NS}(\bar{X}_3)$.

Since there are $\binom{4+3}{3} = 35$ monomials of degree 4 in $\mathbb{Q}[x, y, z, w]$, the quartic surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ are parametrized by the space $\mathbb{P}^{34}_{\mathbb{Q}}$, which we will denote by M. More explicitly, with these 35 monomials as coordinates, the surface given by $F = 0$ with F a homogeneous quartic in $\mathbb{Q}[x, y, z, w]$ corresponds to the point in M whose coordinates are the coefficients in F of the corresponding monomials. Let $M' \cong \mathbb{P}^{27} \subset M$ denote the subvariety of those surfaces X for which the coefficients of the monomials x^4, x^3y, x^3z, y^4, y^3x, y^3z, and x^2z^2 in the defining polynomial of X are all zero. Note that the vanishing of the coefficients of the first six of these monomials is equivalent to the tangency of the plane H_w given by $w = 0$ to the surface X at the points $P = [1:0:0:0]$ and $Q = [0:1:0:0]$. Thus, the vanishing of these coefficients yields a singularity at P and Q in the plane curve $C_X = H_w \cap X$. If the singularity at P in C_X is not worse than a double point, then the vanishing of the coefficient of x^2z^2 is equivalent to the fact that the line given by $y = w = 0$ is one of the limit-tangent lines to C_X at P.

Proposition 3.5. *There is a nonempty Zariski open subset $U \subset M'$ such that every surface $X \in U$ defined over $\mathbb{Q}$ is smooth and has infinitely many rational points.*

Proof. The singular surfaces in M' form a closed subset of M'. So do the surfaces X for which the intersection $H_w \cap X$ has worse singularities than just two double points at P and Q. Leaving out these closed subsets we obtain an open subset V of M'. Let $X \in V$ be given. The plane quartic curve $C_X = X \cap H_w$ has two double points, so the geometric genus g of the normalization $\tilde{C}_X$ of C_X equals $p_a - 2$, where p_a is the arithmetic genus of C_X; see [Hartshorne 1977, exercise IV.1.8]. As we have $p_a = \frac{1}{2}(4-1)(4-2) = 3$, we get $g = 1$. Now assume X is defined over $\mathbb{Q}$. One of the limit-tangents to C_X at P is given by $w = y = 0$. Its slope, being rational, corresponds to a rational point P' on $\tilde{C}_X$ above P. Fixing this point as the unit element $\mathbb{O} = P'$, the curve $\tilde{C}_X$ obtains the structure of an elliptic curve. Let $D \in \mathrm{Pic}^0(\tilde{C}_X)$ be the pull back under normalization of the divisor $P - Q \in \mathrm{Pic}^0(C_X)$. By the theory of elliptic curves there is a unique point T on $\tilde{C}_X$ such that D is linearly equivalent to $T - \mathbb{O}$; see [Silverman 1986, Proposition

III.3.4]. As D is defined over $\mathbb{Q}$, so is T. By Mazur's theorem (see [Silverman 1986, Theoremm III.7.5] for statement and [Mazur 1977, Theorem 8] for a proof), the point T has finite order if and only if $mT = \mathcal{O}$ for some $m \in \{1, 2, \dots, 10, 12\}$. Note that we have $\operatorname{lcm}(1, 2, \dots, 10, 12) = 2520$. Take for U the complement in V of the closed subset of those X for which we have $2520T = \mathcal{O}$ for the corresponding point T on $\tilde{C}_X$. Then each $X \in U$ contains an elliptic curve with infinitely many rational points. By choosing a Weierstrass equation, one verifies easily that if we take $X = X_h$ with $h = 0$, then the corresponding point T on $\tilde{C}_X$ satisfies $mT \neq \mathcal{O}$ for $m \in \{1, 2, \dots, 10, 12\}$. Therefore, we find $X \in U$, so U is nonempty. □

Remark 3.6. If $\tilde{C}_X$ is the normalization of C_X as in the proof of Proposition 3.5, then generically there is another rational point P'' on $\tilde{C}_X$ above P, besides P'. Generically this point also has infinite order and the Mordell–Weil rank of $\tilde{C}_X$ is at least 2 with independent points P'' and T as in the proof of Proposition 3.5. For $X = X_h$ with $h = 0$ however, the curve $\tilde{C}_X$ is given by

$$3x^2y^2 + xy^2z + 4xyz^2 + 2xz^3 + 5yz^3 + z^4 = 0.$$

As the point $P = [1 : 0 : 0]$ is a cusp, there is only one point above P on $\tilde{C}_X$ here. The conductor of this elliptic curve equals 686004. Both points on $\tilde{C}_X$ above $Q = [0 : 1 : 0]$ are rational and we have an extra rational point $[1 : 1 : -1]$. These generate the full Mordell–Weil group of rank 3.

Remark 3.7. Besides the family X_h (with $h \in U$ as in Proposition 3.5) of surfaces containing an elliptic curve with positive Mordell–Weil rank, we can also find surfaces with infinitely many points on some curve of genus 0. By requiring other coefficients to vanish than is required for M', we can find quartic surfaces Y for which the plane H_w given by $w = 0$ is tangent at $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, and $[0 : 0 : 1 : 0]$. Then the intersection $H_w \cap Y$ has geometric genus 0 and if its normalization has a point defined over $\mathbb{Q}$, then this intersection is birational to $\mathbb{P}^1$. The quartic surface Z given by

$$w(x^3 + y^3 + z^3 + x^2z + xw^2) = 3x^2y^2 - 4x^2yz + x^2z^2 + xy^2z + xyz^2 - y^2z^2 \quad (4)$$

is an example of such a surface. As in the proof of Theorem 3.1, modulo 3 the surface Z contains the line $z = w = 0$. Also, the reduction of Z at $p = 2$ contains a conic again, as the right-hand side of (4) factors over $\mathbb{F}_4$ as $(xy + xz + \zeta yz)(xy + xz + \zeta^2 yz)$, with $\zeta^2 + \zeta + 1 = 0$. An argument very similar to the one in the proof of Theorem 3.1 then shows that Z also has geometric Picard number 1 with the Picard group generated by a hyperplane section. The only difference is that Frobenius does not act trivially on the conic $w = xy + xz + \zeta yz = 0$. The hyperplane section $H_w \cap Z$ is a curve of geometric genus 0, parametrized by

$$[x : y : z : w] = [-(t^2+t-1)(t^2-t-3) : 2(t+2)(t^2+t-1) : 2(t+2)(t^2-t-3) : 0].$$

The Cremona transformation $[x:y:z:w] \mapsto [yz:xz:xy]$ gives a birational map from this curve to a nonsingular plane curve of degree 2. Coincidentally, it turns out that the curve on Z given by $x = 0$ has a triple point at $[0:0:0:1]$, so it is birational to $\mathbb{P}^1$ as well. It can be parametrized by

$$[x:y:z:w] = [0:1+t^3:t(1+t^3):-t^2].$$

From the local and global Torelli theorem for K3 surfaces [Pyatetskii-Shapiro and Shafarevich 1971] one can find a very precise description of the moduli space of polarized K3 surfaces in general; see [Beauville 1985]. A polarization of a K3 surface Z by a very ample divisor of degree 4 gives an embedding of Z as a smooth quartic surface in $\mathbb{P}^3$ with the very ample divisor corresponding to a hyperplane section. An isomorphism between two smooth quartic surfaces in $\mathbb{P}^3$ that sends one hyperplane section to an other hyperplane section comes from an automorphism of $\mathbb{P}^3$. As any two hyperplane sections are linearly equivalent, we conclude that the moduli space of K3 surfaces polarized by a very ample divisor of degree 4 is isomorphic to the open subset in $M = \mathbb{P}^{34}$ of smooth quartic surfaces modulo the action of PGL(4) by linear transformations of $\mathbb{P}^3$. We are now ready to prove the main theorem of this article.

Proof Theorem 1.1. By the description of the moduli space of K3 surfaces polarized by a very ample divisor of degree 4 given above, it suffices to prove that the set $S \subset M(\mathbb{Q})$ of smooth surfaces with geometric Picard number 1 and infinitely many rational points is Zariski dense in M.

We will first show that $S \cap M'$ is dense in M'. Note that the coefficients of the monomials x^4, x^3y, x^3z, y^4, y^3x, y^3z, and x^2z^2 in $wf_1 + 2zf_2 - 3g_1g_2$ in (2) are zero, so if the coefficients of these monomials in a homogeneous polynomial $h \in R$ of degree 4 are all zero, then X_h is contained in M'. It follows that the set

$$T = M' \cap \{X_h : h \in R, h \text{ homogeneous of degree } 4\}$$

is dense in M'. Let U be as in Proposition 3.5. Then U is a dense open subset of M', so $T \cap U$ is also dense in M'. By Theorem 3.1 and Proposition 3.5 every surface in $T \cap U$ has geometric Picard number 1 and infinitely many rational points. Thus we have an inclusion $T \cap U \subset S \cap M'$, so $S \cap M'$ is dense in M' as well.

Let W denote the vector space of 4×4–matrices over $\mathbb{Q}$ and let T denote the dense open subset of $\mathbb{P}(W)$ corresponding to elements of PGL(4). Let $\varphi\colon T \times M' \to M$ be given by sending (A, X) to $A(X)$. Note that $T(\mathbb{Q}) \times (S \cap M')$ is dense in $T \times M'$ and φ sends $T(\mathbb{Q}) \times (S \cap M')$ to S. Hence, in order to prove that S is dense in M, it suffices to show that φ is dominant, which can be checked after extending to the algebraic closure. A general quartic surface in $\mathbb{P}^3$ has a one-dimensional family of bitangent planes, i.e., planes that are tangent at two different points. This is closely related to the theorem of Bogomolov and Mumford; see the appendix to [Mori and Mukai 1983]. In fact, for a general quartic surface $Y \subset \mathbb{P}^3$, there is such a bitangent plane H, for which the two tangent points are ordinary double points in the intersection $H \cap Y$. Let Y be such a quartic surface and H such a plane, say tangent at P and Q. Then there is a linear transformation that sends H, P, and Q to the plane given by $w = 0$, and the points $[1:0:0:0]$ and $[0:1:0:0]$. Also, one of the limit-tangent lines to the curve $Y \cap H$ at the singular point P can be sent to the line given by $y = w = 0$. This means that there is a linear transformation B that sends Y to an element X in M'. Then $\varphi(B^{-1}, X) = Y$, so φ is indeed dominant. □

Remark 3.8. The explicit polynomials f_1, f_2, g_1, g_2 for X_h in (2) were found by letting a computer pick random polynomials modulo $p = 2$ and $p = 3$ such that the surface X_h with $h = 0$ is contained in M' as in Proposition 3.5. The computer then computed the characteristic polynomial of Frobenius and tested if there were only 2 eigenvalues that were roots of unity, see Remark 3.3.

Remark 3.9. In finding the explicit surfaces X_h not much computing power was needed, as we constructed the surface to have good reduction at small primes p so that counting points over $\mathbb{F}_{p^n}$ was relatively easy. Based on ideas of for instance Alan Lauder, Daqing Wan, Kiran Kedlaya, and Bas Edixhoven, it should be possible to develop more efficient algorithms for finding characteristic polynomials of (K3) surfaces. Together with these algorithms, the method used in the proof of Theorem 3.1 becomes a strong tool in finding Picard numbers of K3 surfaces over number fields.

4. Open problems

We end with the remark that still very little is known about the arithmetic of K3 surfaces, especially those with geometric Picard number 1. We reiterate three questions that remain unsolved.

Question 1. Does there exist a K3 surface over a number field such that the set of rational points is neither empty nor dense?

Question 2. Does there exist a K3 surface over a number field with geometric Picard number 1, such that the set of rational points is potentially dense?

Question 3. Does there exist a K3 surface over a number field with geometric Picard number 1, such that the set of rational points is not potentially dense?

The surfaces exhibited in this paper are candidates to yield affirmative answers to all of these questions, most notably Questions 2 and 3.

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