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We consider mod p modular Galois representations which are unramified at p such that the Frobenius element at p acts through a scalar matrix. The principal result states that the multiplicity of any such representation is bigger than 1.

1. Introduction

A continuous odd irreducible Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is said to be of weight one if it is unramified at p. According to Serre's conjecture (with the minimal weight as defined in [Edixhoven 1992]), all such representations should arise from Katz modular forms of weight 1 over $\overline{\mathbb{F}}_p$ for the group $\Gamma_1(N)$ with N the (prime to p) conductor of ρ. Assuming the modularity of ρ, this is known if $p > 2$ or if $p = 2$ and the restriction of ρ to a decomposition group at 2 is not an extension of twice the same character. A weight 1 Katz modular form over $\overline{\mathbb{F}}_p$ can be embedded into weight p and the same level in two different ways: by multiplication by the Hasse invariant of weight $p-1$ and by applying the Frobenius (see [Edixhoven 2006, Section 4]). Hence, the corresponding eigenform(s) in weight p should be considered as old forms; they lie in the ordinary part.

A modular Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ of conductor N can be realised with a certain multiplicity (see Proposition 4.1) on the p-torsion of $J_1(Np)$ or $J_1(N)$. In this article we prove that this multiplicity is bigger than 1 if ρ is of weight one and Frob_p acts through a scalar matrix. If $p = 2$, we also assume that the corresponding weight 1 form exists. Together with [Buzzard 2001, Theorem 6.1], this completely settles the question of multiplicity one for modular Galois representations. Its study had been started by Mazur and continued among others by Ribet, Gross, Edixhoven and Buzzard. The first example of a modular Galois representation not satisfying multiplicity one was found in [Kilford 2002]. See [Kilford and Wiese 2006] for a more detailed exposition.

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A systematic computational study of the multiplicity of Galois representations of weight one has been carried out in [Kilford and Wiese 2006]. The data gathered suggest that the multiplicity always seems to be 2 if it is not 1. Moreover, the local factors of the Hecke algebras are becoming astonishingly large.

Overview. We give a short overview over the article with an outline of the proof. In Section 2 an isomorphism between a certain part of the p-torsion of a Jacobian of a modular curve with a local factor of a mod p Hecke algebra is established (Proposition 2.2). As an application one obtains a mod p version of the Eichler–Shimura isomorphism (Corollary 2.3). Together with a variant of a well-known theorem by Boston, Lenstra and Ribet (Proposition 4.1) one also gets an isomorphism between a certain kernel in the local mod p Hecke algebra and a part of the corresponding Galois representation. This gives for instance a precise link between multiplicities and properties of the Hecke algebra (Corollary 4.2). In Section 3 it is proved (Theorem 3.1) that under the identification of Section 2, the geometric Frobenius at p on the part of the Galois representation corresponds to the Hecke operator T_p in the Hecke algebra. This relation is exploited in Section 4 to obtain the principal result (Corollary 4.5), a reformulation and a possible application to weight lowering.

Notation. For integers $N \geq 1$ and $k \geq 1$, we let $S_k(\Gamma_1(N))$ be the $\mathbb{C}$-vector space of holomorphic cusp forms and $S_k(\Gamma_1(N), \mathbb{F}_p)$ the $\mathbb{F}_p$-vector space of Katz cusp forms on $\Gamma_1(N)$ of weight k. Whenever $S \subseteq R$ are rings, m is an integer and M is an R-module on which the Hecke and diamond operators act, we let $\mathbb{T}_S^{(m)}(M)$ be the S-subalgebra inside the R-endomorphism ring of M generated by the Hecke operators T_n with $(n, m) = 1$ and the diamond operators. If $\phi : S \to S'$ is a ring homomorphism, we let $\mathbb{T}_\phi^{(m)}(M) := \mathbb{T}_S^{(m)}(M) \otimes_S S'$ or with ϕ understood $\mathbb{T}_{S \to S'}^{(m)}(M)$. If $m = 1$, we drop the superscript.

Every maximal ideal $\overline{\mathfrak{m}} \subseteq \mathbb{T}_{\mathbb{Z} \to \mathbb{F}_p}(S_k(\Gamma_1(N)))$ corresponds to a Galois conjugacy class of cusp forms over $\overline{\mathbb{F}}_p$ of weight k on $\Gamma_1(N)$. One can attach to $\overline{\mathfrak{m}}$ by work of Shimura and Deligne a continuous odd semisimple Galois representation $\rho_{\overline{\mathfrak{m}}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ which is unramified outside Np and satisfies $\mathrm{Tr}(\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_l)) \equiv T_l \mod \overline{\mathfrak{m}}$ and $\mathrm{Det}(\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_l)) \equiv \langle l \rangle l^{k-1} \mod \overline{\mathfrak{m}}$ for all primes $l \nmid Np$ via an embedding $\mathbb{T}_{\mathbb{Z} \to \mathbb{F}_p}(S_k(\Gamma_1(N)))/\overline{\mathfrak{m}} \hookrightarrow \overline{\mathbb{F}}_p$. All Frobenius elements Frob_l are arithmetic ones.

For all the article we fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and a ring surjection $\overline{\mathbb{Z}}_p \to \overline{\mathbb{F}}_p$. If K is a field, we denote by $K(\epsilon) = K[\epsilon]/(\epsilon^2)$ the dual numbers. For a finite flat group scheme G, the Cartier dual is denoted by ${}^tG$. The maximal unramified extension of $\mathbb{Q}_p$ (inside $\overline{\mathbb{Q}}_p$) is denoted by $\mathbb{Q}_p^{\mathrm{nr}}$ and its integer ring by $\mathbb{Z}_p^{\mathrm{nr}}$.

For the conventions on modular curves we follow [Gross 1990]; in particular, we work with μ_N-level structures.

Situations. We shall often assume one of the following two situations. In the applications, the second part will be taken for $p = 2$.

Situation I. Let p be an odd prime and N a positive integer not divisible by p. Define the Hecke algebras

$$\mathbb{T}_{\mathbb{Z}_p} := \mathbb{T}^{(1)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(Np))), \qquad \mathbb{T}'_{\mathbb{Z}_p} := \mathbb{T}^{(p)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(Np))).$$

Let $\mathfrak{m}$ be an ordinary (i.e. $T_p \notin \mathfrak{m}$) maximal ideal of $\mathbb{T}_{\mathbb{Z}_p}$ with residue field $\mathbb{F} = \mathbb{T}_{\mathbb{Z}_p}/\mathfrak{m}$ such that the p-diamond operators give a nontrivial character

$$(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{F}^\times, \quad a \mapsto \langle a \rangle_p.$$

Let $\mathfrak{m}' = \mathfrak{m} \cap \mathbb{T}'_{\mathbb{Z}_p}$ and, more generally, $\mathfrak{m}^{(m)} = \mathfrak{m} \cap \mathbb{T}^{(m)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(Np)))$ for $m \in \mathbb{N}$. Let $\mathbb{T}_{\mathbb{F}_p} := \mathbb{T}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $\mathbb{T}'_{\mathbb{F}_p} := \mathbb{T}'_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Denote the image of $\mathfrak{m}$ in $\mathbb{T}_{\mathbb{F}_p}$ by $\overline{\mathfrak{m}}$ and similarly for $\overline{\mathfrak{m}}'$. Assume that $\rho_{\overline{\mathfrak{m}}}$ is irreducible.

Let furthermore $K = \mathbb{Q}_p(\zeta_p)$ and $\mathcal{O} = \mathbb{Z}_p[\zeta_p]$ with a primitive p-th root of unity ζ_p. Also let $J := J_1(Np)_\mathbb{Q}$ be the Jacobian of $X_1(Np)$ over $\mathbb{Q}$.

Situation II. Let p be any prime and N a positive integer not divisible by p. Define the Hecke algebras

$$\mathbb{T}_{\mathbb{Z}_p} := \mathbb{T}^{(1)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(N))), \qquad \mathbb{T}'_{\mathbb{Z}_p} := \mathbb{T}^{(p)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(N))).$$

Let $\mathfrak{m}$ be an ordinary maximal ideal of $\mathbb{T}_{\mathbb{Z}_p}$ with residue field $\mathbb{F} = \mathbb{T}_{\mathbb{Z}_p}/\mathfrak{m}$. Let $\mathfrak{m}' = \mathfrak{m} \cap \mathbb{T}'_{\mathbb{Z}_p}$ and, more generally for $m \in \mathbb{N}$, let

$$\mathfrak{m}^{(m)} = \mathfrak{m} \cap \mathbb{T}^{(m)}_{\mathbb{Z}\to\mathbb{Z}_p}(S_2(\Gamma_1(N))).$$

Let $\mathbb{T}_{\mathbb{F}_p} := \mathbb{T}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $\mathbb{T}'_{\mathbb{F}_p} := \mathbb{T}'_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Denote the image of $\mathfrak{m}$ in $\mathbb{T}_{\mathbb{F}_p}$ by $\overline{\mathfrak{m}}$ and similarly for $\overline{\mathfrak{m}}'$. Assume that $\rho_{\overline{\mathfrak{m}}}$ is irreducible.

Let furthermore $K = \mathbb{Q}_p$ and $\mathcal{O} = \mathbb{Z}_p$. Also let $J := J_1(N)_\mathbb{Q}$ be the Jacobian of $X_1(N)$ over $\mathbb{Q}$.

2. Hecke algebras, Jacobians and p-divisible groups

Assume we are in Situation I or II. The maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\mathbb{Z}_p}$ corresponds to an idempotent $e_\mathfrak{m} \in \mathbb{T}_{\mathbb{Z}_p}$, in the sense that applying $e_\mathfrak{m}$ to any $\mathbb{T}_{\mathbb{Z}_p}$-module is the same as localising the module at $\mathfrak{m}$. Let $\mathcal{G}$ be the p-divisible group $J[p^\infty]_\mathbb{Q}$ over $\mathbb{Q}$. Consider the Tate module $T_p J = T_p \mathcal{G} = \varprojlim J[p^n](\overline{\mathbb{Q}})$. It is a $\mathbb{T}_{\mathbb{Z}_p}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module. The Hecke algebra $\mathbb{T}_{\mathbb{Z}_p}$ acts on $T_p J$ and on $\mathcal{G}$, hence so does the idempotent $e_\mathfrak{m}$. We put $G = e_\mathfrak{m}\mathcal{G}$ and say that this is the *p-divisible group over $\mathbb{Q}$ attached to* $\mathfrak{m}$. We shall mainly be interested in the p-torsion of G. However, making the detour via p-divisible groups allows us to quote the following theorem by Gross.

Theorem 2.1 (Gross). *Assume we are in Situation I or II. Let G be the p-divisible group over $\mathbb{Q}$ attached to $\mathfrak{m}$, as explained above. Let $h = \mathrm{rk}_{\mathbb{Z}_p} \mathbb{T}_{\mathbb{Z}_p,\mathfrak{m}}$, where $\mathbb{T}_{\mathbb{Z}_p,\mathfrak{m}}$ denotes the localisation of $\mathbb{T}_{\mathbb{Z}_p}$ at $\mathfrak{m}$.*

(a) *The p-divisible group G acquires good reduction over $\mathbb{O}$. We write $G_{\mathbb{O}}$ for the corresponding p-divisible group over $\mathbb{O}$. It sits in the exact sequence*

$$0 \to G^0_{\mathbb{O}} \to G_{\mathbb{O}} \to G^e_{\mathbb{O}} \to 0,$$

where $G^e_{\mathbb{O}}$ is étale and $G^0_{\mathbb{O}}$ is of multiplicative type, i.e. its Cartier dual is étale. The exact sequence is preserved by the action of the Hecke correspondences.

(b) *Over $\mathbb{O}[\zeta_N]$ the p-divisible group $G_{\mathbb{O}[\zeta_N]}$ is isomorphic to its Cartier dual ${}^tG_{\mathbb{O}[\zeta_N]}$. This gives isomorphisms of p-divisible groups over $\mathbb{O}[\zeta_N]$*

$$G^e_{\mathbb{O}[\zeta_N]} \cong {}^tG^0_{\mathbb{O}[\zeta_N]} \quad \textit{and} \quad G^0_{\mathbb{O}[\zeta_N]} \cong {}^tG^e_{\mathbb{O}[\zeta_N]}.$$

(c) *We have $G^e_{\overline{\mathbb{F}}_p}[p] \cong (\mathbb{Z}/p\mathbb{Z})^h_{\overline{\mathbb{F}}_p}$ and $G^0_{\overline{\mathbb{F}}_p}[p] \cong \mu^h_{p,\overline{\mathbb{F}}_p}$.*

Proof. The references in this proof are to [Gross 1990].

(a) The statement on the good reduction is Propositions 12.8 (1) and 12.9 (1). The exact sequence is proved in Propositions 12.8 (4) and 12.9 (3). That it is preserved by the Hecke correspondences is a consequence of the fact that there are no nontrivial morphisms from a connected group scheme to an étale one, whence any Hecke correspondence on G can be restricted to G^0.

(b) The Cartier self-duality of G over $K(\zeta_N)$ is also proved in Propositions 12.8 (1) and 12.9 (1). It extends to a self-duality over $\mathbb{O}[\zeta_N]$. The second statement follows as in (a) from the nonexistence of nontrivial morphisms from G^0 to G^e over $\mathbb{O}[\zeta_N]$; this argument gives $G^0 \cong {}^tG^e$. Applying Cartier duality to this, we also get $G^e \cong {}^tG^0$.

(c) By part (b), G^e and G^0 have equal height. That height is equal to h by Propositions 12.8 (1) and 12.9 (1). The statement is now due to the fact that up to isomorphism the given group schemes are the only ones of rank p^h which are killed by p and which are étale or of multiplicative type, respectively. □

The last point makes the ordinarity of $\overline{\mathfrak{m}}$ look like the ordinarity of an abelian variety.

Proposition 2.2. *Assume we are in Situation I or II and let G be the p-divisible group attached to $\mathfrak{m}$. Then we have an isomorphism $G^0[p](\overline{\mathbb{Q}}_p) \cong \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$ of $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules.*

Proof. Taking the p-torsion of the p-divisible groups in Theorem 2.1 (a), one obtains the exact sequence

$$0 \to G^0_{\mathbb{O}}[p](\overline{\mathbb{Q}}_p) \to G_{\mathbb{O}}[p](\overline{\mathbb{Q}}_p) \to G^e_{\mathbb{O}}[p](\overline{\mathbb{Q}}_p) \to 0 \tag{1}$$

of $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules with Galois action. We also spell out the dualities in part (b) of Theorem 2.1, restricted to the p-torsion on $\overline{\mathbb{Q}}_p$-points:

$$\begin{aligned} G^0_{\mathbb{O}[\zeta_N]}[p](\overline{\mathbb{Q}}_p) &\cong \mathrm{Hom}_{\mathrm{gr.sch.}/\overline{\mathbb{Q}}_p}(G^e[p]_{\overline{\mathbb{Q}}_p}, \mu_{p,\overline{\mathbb{Q}}_p}) \\ G^e_{\mathbb{O}[\zeta_N]}[p](\overline{\mathbb{Q}}_p) &\cong \mathrm{Hom}_{\mathrm{gr.sch.}/\overline{\mathbb{Q}}_p}(G^0[p]_{\overline{\mathbb{Q}}_p}, \mu_{p,\overline{\mathbb{Q}}_p}). \end{aligned} \tag{2}$$

These are isomorphisms of $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules, i.e. in particular of $\mathbb{F}_p$-vector spaces. We will from now on identify $\mu_{p,\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p)$ with $\mathbb{F}_p$ and the group homomorphisms on $\overline{\mathbb{Q}}_p$-points above with $\mathbb{F}_p$-linear ones.

The final ingredient in the proof is that $G^e(\overline{\mathbb{Q}}_p)[\mathfrak{m}] = G^e[p](\overline{\mathbb{Q}}_p)[\overline{\mathfrak{m}}]$ is a one-dimensional $L := \mathbb{T}_{\mathbb{F}_p}/\overline{\mathfrak{m}}$-vector space; see [Gross 1990, Propositions 12.8 (5) and 12.9 (4)]. We quotient the first isomorphism of Equation (2) by $\overline{\mathfrak{m}}$ and obtain

$$G^0[p](\overline{\mathbb{Q}}_p)/\overline{\mathfrak{m}} \cong \mathrm{Hom}_{\mathbb{F}_p}(G^e[p](\overline{\mathbb{Q}}_p)[\overline{\mathfrak{m}}], \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(L, \mathbb{F}_p),$$

which is a 1-dimensional L-vector space. Consequently, Nakayama's Lemma applied to the finitely generated $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-module $G^0[p](\overline{\mathbb{Q}}_p)$ yields a surjection $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}} \twoheadrightarrow G^0[p](\overline{\mathbb{Q}}_p)$. Next we invoke a result from Section 3 of [Kilford and Wiese 2006]. We point out explicitly that all of that section is independent of Section 2 of the same paper, in which Corollary 2.3 is used. From Proposition 3.7 of that paper, it follows that

$$2 \dim_{\mathbb{F}_p} \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}} = \dim_{\mathbb{F}_p} H^1_{\mathrm{par}}(\Gamma, \mathbb{F}_p)_{\overline{\mathfrak{m}}}$$

with $\Gamma = \Gamma_1(Np)$ in Situation I and $\Gamma = \Gamma_1(N)$ in Situation II. At the same time,

$$H^1_{\mathrm{par}}(\Gamma, \mathbb{F}_p)_{\overline{\mathfrak{m}}} \cong J(\mathbb{C})[p]_{\overline{\mathfrak{m}}} \cong G[p](\overline{\mathbb{Q}}_p)$$

(see [Wiese 2007, Proposition 5.3], for example), so we obtain $\dim_{\mathbb{F}_p} \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}} = \dim_{\mathbb{F}_p} G^0[p](\overline{\mathbb{Q}}_p)$ and, thus, $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}} \cong G^0[p](\overline{\mathbb{Q}}_p)$. □

The following result, together with very helpful hints on its proof (amounting to the preceding proposition), was suggested by Kevin Buzzard. See also the discussion before [Emerton 2002, Proposition 6.3] and the letter by Mazur reproduced in the Appendix to [Tilouine 1997].

Corollary 2.3. *Assume we are in Situation I or II and let G be the p-divisible group attached to $\mathfrak{m}$. Then there is an exact sequence*

$$0 \to \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}} \to G[p](\overline{\mathbb{Q}}) \to \mathbb{T}^\vee_{\mathbb{F}_p,\overline{\mathfrak{m}}} \to 0$$

of $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules, where the dual is the $\mathbb{F}_p$-linear dual.

Proof. Substituting the isomorphism of Proposition 2.2 into the second isomorphism of Equation (2) gives

$$G^e[p](\overline{\mathbb{Q}}_p) \cong \mathrm{Hom}(\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}, \mathbb{F}_p)$$

as $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules, whence the corollary follows from Equation (1). □

The following proposition is similar in spirit to Proposition 2.2. It will not be needed in the sequel.

Proposition 2.4. *Assume we are in Situation I or II and let $G = G_{\mathbb{O}}$ be the p-divisible group over $\mathbb{O}$ attached to $\mathfrak{m}$. Then $G^0[p](\mathbb{F}_p(\epsilon))$ and $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$ are isomorphic as $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$-modules.*

Proof. We only give a sketch. Since $G^0[p](\mathbb{F}_p)$ consists of the origin as unique point, $G^0[p](\mathbb{F}_p(\epsilon))$ coincides with the tangent space at 0 of $G^0_{\mathbb{F}_p}[p]$. The latter, however, is equal to the tangent space at 0 of $G_{\mathbb{F}_p}[p]$. On the other hand, its dual, the cotangent space at 0 of $G_{\mathbb{F}_p}[p]$, is isomorphic to $S_k(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$ for some $k \in \{2, \ldots, p+1\}$. In Situation II, $k = 2$ and this result is well-known. In Situation I, we quote [Edixhoven 1992, Equations 6.7.1 and 6.7.2], as well as [Gross 1990, Proposition 8.13] (note that the ordinarity assumption kills the second summand in that proposition). Consequently, $G^0[p](\mathbb{F}_p(\epsilon))$ is isomorphic to the Hecke algebra on $S_k(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$ as a Hecke module. In [Kilford and Wiese 2006, Proposition 2.3], it is shown that this algebra is $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$. □

From Proposition 2.2 and part of the direct proof of Theorem 3.1 we can also derive an isomorphism between $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$ and the image of the reduction map (4).

3. Comparing Frobenius and the Hecke operator T_p

The aim of this section is to discuss and prove the following theorem, which turns out to be an important key to the principal result of this article.

Theorem 3.1. *Assume we are in Situation I or II and let $G^0 = G^0_{\mathbb{O}}$ be the p-divisible group of Theorem 2.1. The action of the geometric Frobenius on the points*

$$G^0_{\mathbb{O}}[p](\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p))$$

is the same as the action of the Hecke operator T_p.

This result is in fact contained in [Gross 1990]. Apart from giving the appropriate citations, we include two more proofs, in the hope that the chosen approaches may find applications in other contexts, too. Due to the Eichler–Shimura congruence relation in Situation II and the reduction of a well-known semistable model of the modular curve in Situation I, for both of these proofs it suffices to compare the geometric Frobenius and Verschiebung on the special fibre of $G^0[p]$. For the first alternative proof, such a comparison is carried out elementarily — roughly speaking — by working with the tangent space at 0 over $\overline{\mathbb{F}}_p$, in order to have an injective reduction map from characteristic zero to the finite field. On the special fibre elementary computations then suffice. For the second alternative proof, a comparison between geometric Frobenius and Verschiebung has been worked out conceptually by Niko Naumann in the Appendix in the context of Fontaine's theory of Honda systems.

Proof by citation. In Situation I, we cite [Gross 1990, Proposition 12.9 (3)], which says that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on the Tate module of G^0 through the p-adic cyclotomic character times $\lambda(T_p^{-1})$, where $\lambda(T_p^{-1})$ is the character sending Frob_p to T_p^{-1}. As we are restricting to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_p))$ and to $G^0[p]$, the cyclotomic character is trivial and the Galois action on

$$G^0[p](\overline{\mathbb{Q}}_p) = \mathrm{Hom}({}^tG^0[p] \times \overline{\mathbb{Q}}_p, \mu_{p,\overline{\mathbb{Q}}_p})$$

is unramified since ${}^tG^0[p]$ is étale.

In Situation II, we cite [Gross 1990, Proposition 12.8 (4)], and argue as above. Note that by the Eichler–Shimura congruence relation (see the end of the direct proof) the unit u in the citation equals T_p divided by the diamond operator $\langle p\rangle_N$. □

Direct proof. In the proof we prefer to work with the étale Cartier dual of $G^0[p]$ since we find it more convenient for making formulae explicit. So, ${}^tG^0[p] = \mathrm{Spec}(A)$ is a finite étale group scheme over $\mathbb{O}$ such that

$${}^tG^0[p] \times_{\mathbb{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \cong (\mathbb{Z}/p\mathbb{Z})^h_{\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]},$$

i.e. $A \otimes_{\mathbb{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \overset{\alpha}{\cong} \prod \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]$. If $p = 2$, we put $\zeta_2 = -1$. We consider the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p][X]/(X^p-1) \arrow[d] & \mathbb{Z}_p^{\mathrm{nr}}[X,Y]/(X^p-1,Y^p-1) \arrow[l, "\zeta_p \leftarrow\!\shortmid Y"'] \arrow[r, "Y\mapsto 1+\epsilon"] \arrow[d] & \overline{\mathbb{F}}_p(\epsilon)[X]/(X^p-1) \arrow[d] \\
\prod \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] & \prod \mathbb{Z}_p^{\mathrm{nr}}[Y]/(Y^p-1) \arrow[l, "\zeta_p \leftarrow\!\shortmid Y"'] \arrow[r, "Y\mapsto 1+\epsilon"] & \prod \overline{\mathbb{F}}_p(\epsilon).
\end{tikzcd}$$

Any morphism of group schemes ${}^tG^0[p] \times_{\mathbb{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \to \mu_{p,\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]}$ corresponds to a Hopf algebra homomorphism as in the left column. Suppose that it maps X to $(\zeta_p^{i_1}, \dots, \zeta_p^{i_{hp}})$ for $i_j \in \{0, \dots, p-1\}$. It has a unique lifting to a Hopf algebra homomorphism as in the central column if we impose that X maps to $(Y^{i_1}, \dots, Y^{i_{hp}})$. As the referee pointed out, this lift gives the first map in the exact sequence

$$0 \to G^0[p](\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]) \to G^0[p](\mathbb{Z}_p^{\mathrm{nr}}[Y]/(Y^p-1)) \to G^0[p](\mathbb{Z}_p^{\mathrm{nr}}).$$

From the homomorphism in the centre of the diagram we obtain a Hopf algebra homomorphism in the right column, which sends X to $(1+i_1\epsilon, \dots, 1+i_{hp}\epsilon)$. It should be said that the detour via the central column is only necessary for $p=2$, as for $p>2$ one can pass directly from the left hand side column to the right hand side via the map $\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \to \overline{\mathbb{F}}_p(\epsilon)$, sending ζ_p to $1+\epsilon$.

This process gives us an injective reduction map

$$\mathrm{Hom}_{\mathrm{gr.sch./}\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]}({}^tG^0[p] \times_{\mathcal{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p], \mu_{p,\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]}) \\ \to \mathrm{Hom}_{\mathrm{gr.sch./}\overline{\mathbb{F}}_p(\epsilon)}({}^tG^0[p] \times_{\mathcal{O}} \overline{\mathbb{F}}_p(\epsilon), \mu_{p,\overline{\mathbb{F}}_p(\epsilon)}). \quad (3)$$

In terms of points of $G^0[p]$, the reduction map is the composition

$$G^0[p](\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]) \hookrightarrow G^0[p](\mathbb{Z}_p^{\mathrm{nr}}[Y]/(Y^p-1)) \to G^0[p](\overline{\mathbb{F}}_p[\epsilon]). \quad (4)$$

The reduction map is compatible for the action induced by the Hecke correspondences.

Next, we describe the geometric Frobenius on the points $G^0[p](\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p))$ and $G^0[p](\overline{\mathbb{F}}_p(\epsilon))$. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{gr.sch./}\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]}({}^tG^0[p] \times \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p], \mu_{p,\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]}) \arrow[r, "\sim"] \arrow[d, "\sim"] & (A \otimes \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p])^{\mathrm{gl}} \arrow[r, "\sim"] \arrow[dd, hook] & G^0[p](\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]) \arrow[dd, hook] \\
\mathrm{Hom}_{\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]-\mathrm{HA}}(\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p][X]/(X^p-1), A \otimes \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]) \arrow[dr, hook] & & \\
& A \otimes \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \arrow[r, "\sim\ \mathrm{ev}"] & \mathrm{Hom}_{\mathcal{O}}({}^tA, \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]).
\end{tikzcd}$$

It is well-known that a Hopf algebra homomorphism

$$\psi : \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p][X]/(X^p-1) \to A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]$$

is uniquely given by the "group-like element" $\psi(X) = \sum a_i \otimes s_i$, giving the upper left bijection. On the bottom right, we have the evaluation isomorphism

$$A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p] \to \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}), \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p])$$

which is defined by $\mathrm{ev}(a \otimes s)(\varphi) = \varphi(a)s$. We use that as $\mathcal{O}$-modules ${}^tA = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ with $G^0[p] = \mathrm{Spec}({}^tA)$, as well as the freeness of A. It is also well-known that the evaluation map gives rise to the upper right bijection.

Let now ϕ be the geometric Frobenius in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p)/\mathbb{Q}_p(\zeta_p))$. Its action on $\mathrm{Hom}_{\mathcal{O}}({}^tA, \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p])$ is by composition. Via the evaluation map it is clear that ϕ acts on an element $a \otimes s \in A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]$ by sending it to $a \otimes \phi(s)$. Consequently, the morphism ψ^ϕ which is obtained by applying ϕ to ψ is uniquely determined by

$\psi^{\phi}(X) = \sum a_i \otimes \phi(s_i)$. A similar statement holds for the reduction. We note that this implies the compatibility of the reduction map with the ϕ-action.

Next we show that the action of geometric Frobenius on the image of (4) inside the tangent space $G^0[p](\overline{\mathbb{F}}_p(\epsilon))$ coincides with the action induced by Verschiebung on $G^0_{\mathbb{F}_p}[p]$. The étale algebra $A \otimes_{\mathbb{O}} \mathbb{F}_p$ can be written as a product of algebras of the form $\mathbb{F}_p[X]/(f)$ with f an irreducible polynomial. An elementary calculation on the underlying rings gives the commutativity of the diagram

$$\begin{tikzcd} \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) \arrow[r, "F\otimes 1"] \arrow[d, "\alpha"] & \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) \arrow[r, "1\otimes\phi^{-1}"] & \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) \arrow[d, "\alpha"] \\ \prod_{i=1}^{d} \overline{\mathbb{F}}_p(\epsilon) \arrow[rr, "\prod \phi^{-1}"] & & \prod_{i=1}^{d} \overline{\mathbb{F}}_p(\epsilon), \end{tikzcd} \tag{5}$$

where F denotes the absolute Frobenius on ${}^tG^0_{\mathbb{F}_p}[p]$ (defined by $X \mapsto X^p$), which by duality gives the Verschiebung on $G^0_{\mathbb{F}_p}[p]$. We point out that ϕ leaves ϵ invariant.

Let now $\psi : \overline{\mathbb{F}}_p(\epsilon)[X]/(X^p - 1) \to A \otimes_{\mathbb{O}} \overline{\mathbb{F}}_p(\epsilon)$ be an $\overline{\mathbb{F}}_p(\epsilon)$-Hopf algebra homomorphism in the image of (3). It is uniquely given by $\psi(X) = \sum_i a_i \otimes s_i$, and under the identification

$$A \otimes_{\mathbb{O}} \overline{\mathbb{F}}_p(\epsilon) \overset{\alpha}{\cong} \prod_{j=1}^{hp} \overline{\mathbb{F}}_p(\epsilon)$$

we get $\psi(X) = (1 + i_1\epsilon, \ldots, 1 + i_{hp}\epsilon)$ with $i_j \in \mathbb{F}_p$ as we have seen above, which is invariant under the arithmetic Frobenius of the bottom row of (5). Hence, $\phi^{-1}(F(\sum_i a_i \otimes s_i)) = \sum_i a_i \otimes s_i$, so that $F(\sum_i a_i \otimes s_i) = \sum_i a_i \otimes \phi(s_i)$. This proves that the geometric Frobenius and Verschiebung coincide on the image of (4) inside $G^0[p](\overline{\mathbb{F}}_p(\epsilon))$.

We now finish the proof. In Situation II, the Eichler–Shimura relation $T_p = \langle p\rangle F + V$ holds on the special fibre of $G[p]$ (see the proof of [Gross 1990, Proposition 12.8 (2)]). Since F is zero on $G^0_{\mathbb{F}_p}[p]$, we get $T_p = V$ on it. We obtain the theorem in this situation since V coincides with ϕ on the image of (4), as we just saw.

In Situation I, we know that $G^0_{\mathbb{F}_p}[p]$ is naturally part of the p-torsion of the Jacobian of the Igusa curve $I_1(N)_{\mathbb{F}_p}$; but on the Igusa curve Verschiebung acts as T_p (see the proof of [Gross 1990, Proposition 12.9 (2)] for both these facts). Hence, we can argue as above and get the theorem also for $p > 2$. □

More conceptual proof. In both situations, Theorem A.1 of Naumann gives an isomorphism between $G^0[p](\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p))$ and the Dieudonné module M attached to the special fibre $G^0_{\mathbb{F}_p}[p]$. Under this isomorphism the geometric Frobenius $\phi \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p)/\mathbb{Q}_p(\zeta_p))$ on $G^0[p](\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p))$ is identified with Verschiebung on the Dieudonné module. The isomorphism is compatible with the Hecke action. Using the same citations as at the end of the direct proof one immediately concludes that the equality $T_p = V$ holds on the Dieudonné module M, finishing the proof. $\square$

Remark 3.2. (a) Conceptually, taking $\mathbb{Z}_p^{\mathrm{nr}}[\zeta_p]$-points is the same as taking $\mathbb{Z}_p^{\mathrm{nr}}$-points of the Weil restriction from $\mathbb{O}$ to $\mathbb{Z}_p$ and similarly for $\mathbb{Q}_p^{\mathrm{nr}}(\zeta_p)$. So, we could have formulated Theorem 3.1 in terms of the Weil restriction.

(b) We point out again that we are using the conventions of [Gross 1990]. Hence, the representation on the Jacobian must be tensored by the corresponding Dirichlet character ϵ (the nebentype) in order to obtain $\rho_{\overline{\mathfrak{m}}}$ (see [Gross 1990, p. 486]).

(c) A theorem by Deligne (see, for instance, [Edixhoven 1992, Theorem 2.5] or [Gross 1990, Proposition 12.1]) describes the restriction of $\rho_{\overline{\mathfrak{m}}}$ to a decomposition group at p in the ordinary case as

$$\begin{pmatrix} \chi_p^{k-1}\lambda(\epsilon(p)/a_p) & * \\ 0 & \lambda(a_p) \end{pmatrix},$$

where χ_p is the mod p cyclotomic character, $\lambda(u)$ is the unramified character sending the arithmetic Frobenius Frob_p to u and $a_p \equiv T_p \mod \overline{\mathfrak{m}}$. When we restrict to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_p))$, the cyclotomic character acts trivially and we see that Theorem 3.1 is in accordance with Deligne's description.

Let f be a Katz eigenform of weight 1 over $\mathbb{F}_p$ with eigenvalue $a_p^{(1)}$ for the weight 1 Hecke operator $T_p^{(1)}$. As explained in [Edixhoven 2006, Section 4], one can embed f into weight p in two different ways. On the span in weight p, the Hecke operator T_p has the eigenvalues a_p and $\epsilon(p)/a_p$ and they satisfy $a_p^{(1)} = a_p + \epsilon(p)/a_p$ (see [Wiese 2007, Proposition 8.4]). The mod p Galois representation attached to f coincides with the one attached to a weight p form. We suppose that this representation is of weight one, which is known for $p > 2$ and for many cases with $p = 2$ and is expected to be true without any exception. Then the characteristic polynomial of Frob_p acting on that representation equals $X^2 - a_p^{(1)}X + \epsilon(p)$ and is thus like any characteristic polynomial of a modular Galois representation at any unramified prime.

4. Application to multiplicities

We first state a slight strengthening of a well-known theorem by Boston, Lenstra and Ribet.

Proposition 4.1 (Boston, Lenstra, Ribet). *Assume we are in Situation I or II. Let m be an integer. Then the $\mathbb{F}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$ is the direct sum of r copies of $\rho_{\overline{\mathfrak{m}}} \otimes \epsilon^{-1}$ for some $r \geq 1$ and Dirichlet character ϵ corresponding to $\overline{\mathfrak{m}}$.*

The integer r is called the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$.

Proof. The same proof as in the original proposition works. More precisely, one considers the two representations $\rho_{\overline{\mathfrak{m}}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ and $\sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}])$. By Chebotarev's density theorem we know that every conjugacy class of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\sigma \otimes \epsilon)$ is hit by a Frobenius element Frob_l for some $l \nmid Npm$.

The Eichler–Shimura congruence relation $T_l = \langle l \rangle F + V$ holds on $J_{\mathbb{F}_l}$ (taking J here over $\mathbb{Z}[\frac{1}{Np}]$) for all primes $l \nmid Npm$. Hence, the minimal polynomial of Frob_l on the Jacobian divides $X^2 - T_l/\langle l \rangle \cdot X + l/\langle l \rangle$. But T_l acts as a_l on $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$ and $X^2 - a_l X + \epsilon(l) l$ (with $T_l \equiv a_l \mod \overline{\mathfrak{m}}$) is the characteristic polynomial of $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_l)$. Consequently, $(\sigma \otimes \epsilon)(g)$ is annihilated by the characteristic polynomial of $\rho_{\overline{\mathfrak{m}}}(g)$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence, Theorem 1 of [Boston et al. 1991] gives the result. □

The notion of multiplicity is sometimes formulated in a way only depending on the representation and not on a particular piece of one Jacobian; see for example [Ribet and Stein 2001, Definition 3.3]. The next corollary says that one can read off multiplicities from properties of Hecke algebras.

Corollary 4.2. *Assume we are in Situation I or II. Let r be the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$. Then*

$$r = \tfrac{1}{2}(\dim_{\mathbb{F}} \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] + 1).$$

Proof. Buzzard [2001] explains the exactness of the sequence

$$0 \to G^0(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \to G(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \to G^e(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \to 0.$$

Via Corollary 2.3 we obtain the exact sequence

$$0 \to \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \to J(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \to \big(\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}\big)^\vee \to 0,$$

from which one reads off the claim by counting dimensions. □

In [Buzzard 2001] Buzzard proved that the multiplicity on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ of $\rho_{\overline{\mathfrak{m}}}$ of weight one is 1 if $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is nonscalar. We include this as a lemma.

Lemma 4.3. *Assume we are in Situation I or II and $\rho_{\overline{\mathfrak{m}}}$ is of weight one.*

If $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is not a scalar matrix, the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is 1.

Proof. We first record that T_p acts as a scalar (in $\mathbb{F}$) on $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$. Suppose that the multiplicity r of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is greater than 1. Then $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] = G^0[p](\overline{\mathbb{Q}})[\overline{\mathfrak{m}}]$ has dimension $2r - 1 > 1$ by (the proof of) Corollary 4.2. Hence, $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ does not act as a scalar on $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$, as it is nonscalar on $J(\overline{\mathbb{Q}})[\mathfrak{m}] \cong \rho_{\overline{\mathfrak{m}}}^r$ by assumption. From Theorem 3.1 we obtain a contradiction, since it implies that T_p does not act as a scalar on $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ either. □

Theorem 4.4. *Assume we are in Situation I or II and $\rho_{\overline{\mathfrak{m}}}$ has weight one and $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is conjugate to $\left(\begin{smallmatrix} a & * \\ 0 & a \end{smallmatrix}\right)$. The following statements are equivalent:*

(a) *The representation $\rho_{\overline{\mathfrak{m}}}$ comes from a Katz cusp form of weight* 1 *on* $\Gamma_1(N)$ *over* $\overline{\mathbb{F}}_p$ *and the multiplicity of* $\rho_{\overline{\mathfrak{m}}}$ *on* $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ *is* 1.

(b) $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \subsetneq \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$.

(c) T_p *does not act as a scalar on* $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$ (*inside* $J(\overline{\mathbb{Q}})[\mathfrak{m}'] \cong \oplus \rho_{\overline{\mathfrak{m}}}$ *as an* $\mathbb{F}$*-vector space*).

(d) *The multiplicity of* $\rho_{\overline{\mathfrak{m}}}$ *on* $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ *is* 1, *its multiplicity on* $J(\overline{\mathbb{Q}})[\mathfrak{m}']$ *is* 2, *and* $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ *is nonscalar.*

Proof. (a) $\Rightarrow$ (b): By Corollary 4.2, the $\mathbb{F}$-dimension of $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ is 1, hence, so is the dimension of $\mathbb{T}^{\vee}_{\mathbb{F}_p,\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}$. Thus, Nakayama's Lemma yields that $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}$ is Gorenstein, i.e. that it is isomorphic to its dual as a module over itself. By the q-expansion principle, the dual is $S_p(\Gamma_1(N),\mathbb{F}_p)_{\overline{\mathfrak{m}}}$. By [Edixhoven 2006, Propostion 6.2] or [Wiese 2007, Proposition 8.4] the existence of a corresponding weight 1 form is equivalent to $S_p(\Gamma_1(N),\mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$ being 2-dimensional. This establishes (b), since $S_p(\Gamma_1(N),\mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$, which is isomorphic to $(\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}/\overline{\mathfrak{m}})^{\vee}$, is 1-dimensional as an $\mathbb{F}$-vector space.

(b) $\Rightarrow$ (c): This is evident.

(c) $\Rightarrow$ (d): First of all, $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$ is at least 2-dimensional (as an $\mathbb{F}$-vector space). From Theorem 3.1 we know that T_p acts as the inverse of Frob_p on $G^0[p](\overline{\mathbb{Q}})$. We conclude that $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ cannot be scalar. Hence, Lemma 4.3 yields that the multiplicity r of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is equal to 1. If the multiplicity s of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}']$ were bigger than 2, then $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\mathfrak{m}']$ would be at least 4-dimensional by an argument as in the proof of Corollary 4.2. Then it follows that it must contain at least two linearly independent eigenvectors for T_p corresponding to at least two copies of $\rho_{\overline{\mathfrak{m}}}$, contradicting the fact that $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ is 1-dimensional.

(d) $\Rightarrow$ (a): Clearly, $\overline{\mathfrak{m}} \neq \overline{\mathfrak{m}}'$. Hence, $\mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}/\overline{\mathfrak{m}} \neq \mathbb{T}_{\mathbb{F}_p,\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}'$ and, dually,

$$S_p(\Gamma_1(N),\mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \subsetneq S_p(\Gamma_1(N),\mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}'],$$

which implies the existence of a corresponding weight 1 form, again by [Edixhoven 2006, Proposition 6.2] or [Wiese 2007, Proposition 8.4]. □

We now state and prove the principal result of this article.

Corollary 4.5. *Assume we are in Situation I or II and* $\rho_{\overline{\mathfrak{m}}}$ *is of weight one. If* $p=2$, *also assume that a weight* 1 *Katz form of level* N *exists which gives rise to* $\rho_{\overline{\mathfrak{m}}}$.

Then the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ *on* $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ *is* 1 *if and only if* $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ *is nonscalar.*

Proof. By [Edixhoven 1992, Theorem 4.5] together with the remark at the end of the introduction to that article, the existence of the corresponding weight 1 form is also guaranteed for $p > 2$. First suppose that the multiplicity is 1. If $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ has two distinct eigenvalues, then it clearly is nonscalar. If $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is conjugate to $\left(\begin{smallmatrix} a & * \\ 0 & a \end{smallmatrix}\right)$, then Theorem 4.4 shows that $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is nonscalar. On the other hand, if $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is nonscalar, Lemma 4.3 implies that the multiplicity is 1. □

The following corollary gives a different, somewhat cleaner formulation of the results on multiplicities. It suggests that instead of working with the full Hecke algebra, one should restrict to the prime-to-p one.

Corollary 4.6. *Assume we are in Situation I or II. If $p = 2$, also assume that $\rho_{\overline{\mathfrak{m}}}$ is of weight one if and only if there exists a weight 1 Katz form of level N which gives rise to $\rho_{\overline{\mathfrak{m}}}$.*

Then the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}']$ is 1 if and only if $\rho_{\overline{\mathfrak{m}}}$ is ramified at p.

Proof. As in the previous proof, for $p > 2$ by [Edixhoven 1992, Theorem 4.5], together with the remark at the end of the introduction to that article, the existence of a weight 1 form is equivalent to the attached representation being of weight one. If $\rho_{\overline{\mathfrak{m}}}$ is ramified at p, the result follows from Theorem 6.1 of [Buzzard 2001]. For, it gives that $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is isomorphic to precisely one copy of $\rho_{\overline{\mathfrak{m}}}$. Moreover, the localisation at $\overline{\mathfrak{m}}'$ of $\mathbb{T}_{\mathbb{F}_p}$ (as a $\mathbb{T}'_{\mathbb{F}_p}$-module) is equal to $\mathbb{T}'_{\mathbb{F}_p,\overline{\mathfrak{m}}'}$, as otherwise a weight one form would exist e.g. by [Wiese 2007, Proposition 8.1]. Hence, $\overline{\mathfrak{m}} = \overline{\mathfrak{m}}'$ and $J(\overline{\mathbb{Q}})[\mathfrak{m}] = J(\overline{\mathbb{Q}})[\mathfrak{m}']$.

Suppose now that $\rho_{\overline{\mathfrak{m}}}$ is unramified at p. If $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is scalar, it suffices to apply Corollary 4.5. If $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is conjugate to $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ with $b \neq 0$, then the result is obtained from Corollary 4.5 together with the implication $(a) \Rightarrow (d)$ of Theorem 4.4. If, finally, $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ has two distinct eigenvalues, then there are two maximal ideals $\mathfrak{m} = \mathfrak{m}_1$, $\mathfrak{m}_2$ with $\rho_{\overline{\mathfrak{m}}_1} \cong \rho_{\overline{\mathfrak{m}}_2}$, since the operator T_p has two distinct eigenvalues on $S_p(\Gamma_1(N), \mathbb{F}_p)[\overline{\mathfrak{m}}']$ by the formula in [Wiese 2007, Proposition 8.4], namely the same as $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)^{-1}$. Consequently, $J(\overline{\mathbb{Q}})[\mathfrak{m}_1] \oplus J(\overline{\mathbb{Q}})[\mathfrak{m}_2] = J(\overline{\mathbb{Q}})[\mathfrak{m}']$, finishing this proof. □

Corollary 4.7. *Assume we are in Situation I or II and $\rho_{\overline{\mathfrak{m}}}$ is of weight one. Assume also that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}']$ is 2. Then the following statements are equivalent.*

(a) *The multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is 1 and a weight 1 Katz form of level N exists which gives rise to $\rho_{\overline{\mathfrak{m}}}$.*

(b) *$\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ is nonscalar.*

Proof. We have seen the implication (a) $\Rightarrow$ (b) in Corollary 4.5. By Lemma 4.3, we obtain from $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ being nonscalar that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}]$ is 1. From the assumption the inequality $\overline{\mathfrak{m}} \neq \overline{\mathfrak{m}}'$ follows, implying the existence of the weight 1 form as above by [Edixhoven 2006, Proposition 6.2] or [Wiese 2007, Proposition 8.4]. □

If one could prove that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}']$ is always equal to 2 in the unramified situation, Corollary 4.7 would extend weight lowering for $p = 2$ to $\rho_{\overline{\mathfrak{m}}}(\mathrm{Frob}_p)$ being nonscalar.

Appendix

by Niko Naumann

Let p be a prime, $A := \mathbb{Z}_p$, $A' := \mathbb{Z}_p[\zeta_p]$, $K := \mathbb{Q}_p$, $K' := \mathbb{Q}_p(\zeta_p)$ and $K' \subseteq \overline{K}$ an algebraic closure. We have the inertia subgroup $I \subseteq G_{K'} := \mathrm{Gal}(\overline{K}/K')$ and for a $G_{K'}$-module V we denote by τ the geometric Frobenius acting on the inertia invariants V^I. If G/A' is a finite flat group-scheme, always assumed to be commutative, we denote by M the Dieudonné module of its special fiber and by $V : M \to M$ the Verschiebung.

Theorem A.1. *Let G/A' be a finite flat group-scheme which is connected with étale Cartier dual and annihilated by multiplication with p. Then $G(\overline{K})^I = G(\overline{K})$ and there is an isomorphism $\phi : G(\overline{K})^I \to M$ of $\mathbb{F}_p$-vector spaces such that $\phi \circ \tau = V \circ \phi$.*

The assumption that $pG = 0$ cannot be dropped:

Proposition A.2. *For every $n \geq 2$ there is a finite flat group-scheme G/A' of order p^n which is connected with an étale dual and such that $G(\overline{K})^I \simeq \mathbb{Z}/p\mathbb{Z}$ with τ acting trivially and $V \neq 1$ on the Dieudonné module of the special fiber of G.*

Proof of Theorem A.1. Denoting by G' the Cartier dual of G/A' we have an isomorphism of $G_{K'}$-modules

$$G(\overline{K}) \simeq \mathrm{Hom}(G'(\overline{K}), \mu_{p^\infty}(\overline{K})) \overset{(pG'=0)}{=} \mathrm{Hom}(G'(\overline{K}), \mu_p(\overline{K})).$$

Since $G'(\overline{K})$ is unramified because G/A' is étale and $\mu_p(\overline{K})$ is unramified because $\zeta_p \in K'$ we see that $G(\overline{K})^I = G(\overline{K})$. Letting p^n denote the order of G we have $\dim_{\mathbb{F}_p} G(\overline{K})^I = \dim_{\mathbb{F}_p} G(\overline{K}) = n = \dim_{\mathbb{F}_p} M$.

In the rest of the proof we use the explicit quasi-inverse to J.-M. Fontaine's functor associating with G a finite Honda system in order to determine the action of τ on $G(\overline{K})^I$ [Fontaine 1977; Conrad 1999].

Let (M, L) be the finite Honda system over A' associated with G/A'. Recall that M is the Dieudonné module of the special fiber of G and $L \subseteq M_{A'}$ is an A'-submodule where $M_{A'}$ is an A'-module functorially associated with M [Fontaine 1977, Chapter IV, Section 2]. We claim that $L = M_{A'}$: Let $\mathrm{m} \subseteq A'$ denote the maximal ideal. Using the notation of [Conrad 1999, Section 2], the defining epimorphism of A'-modules $M_{A'} \to \mathrm{coker}(\mathcal{F}_M)$ factors through an epimorphism $M_{A'}/\mathrm{m}M_{A'} \to \mathrm{coker}(\mathcal{F}_M)$ because $\mathrm{m} \cdot \mathrm{coker}(\mathcal{F}_M) = 0$ [Conrad 1999, Lemma 2.4].

Denoting by l the length of a module we have

$$l_{A'}(\operatorname{coker}(\mathscr{F}_M)) = l_A(\ker F) = l_A(\ker(p : M \to M)) = l_A(M) = n$$

where the first equality follows from [Conrad 1999, 2.4], the second because $\ker F = \ker(p : M \to M)$ since V is bijective, and the third since $pM = 0$. On the other hand, the canonical morphism of A'-modules $\iota_M : M \otimes_A A' \to M_{A'}$ is an isomorphism by [Fontaine 1977, Chapter IV, Proposition 2.5] using again that V is bijective. Thus

$$l_{A'}(M_{A'}/\mathrm{m}M_{A'}) = l_{A'}(M \otimes_A A'/\mathrm{m}) = l_{A'}(M/pM) = l_A(M) = n$$

and $M_{A'}/\mathrm{m}M_{A'} \xrightarrow{\simeq} \operatorname{coker}(\mathscr{F}_M)$. Since $L/\mathrm{m}L \xrightarrow{\simeq} \operatorname{coker}(\mathscr{F}_M)$ holds for every finite Honda system we see that the inclusion $L \subseteq M_{A'}$ induces an isomorphism $L/\mathrm{m}L \xrightarrow{\simeq} M_{A'}/\mathrm{m}M_{A'}$ and Nakayama's lemma implies that $L = M_{A'}$.

Fix $\pi \in \overline{K}$ with $\pi^{p-1} = -p$, then $K' = K(\pi)$: This is obvious for $p = 2$ and for $p \neq 2$ it follows from local class field theory and the norm computation $\mathrm{N}_K^{K'}(\zeta_p - 1) = \mathrm{N}_K^{K(\pi)}(\pi) = p$. Note that $\pi \in A'$ is a local uniformizer. Let K'^{ur} denote the completion of the maximal unramified extension of K' inside $\overline{K}$ and $\mathscr{O} \subseteq K'^{ur}$ its ring of integers.

By [Fontaine 1977, Remarque on p. 218] and the fact that $L = M_{A'}$ we see that reduction induces an isomorphism

$$\begin{aligned} G(\overline{K})^I &= G(K'^{ur}) = G(\mathscr{O}) \\ &\xrightarrow{\simeq} \left\{\phi \in \operatorname{Hom}_{\mathrm{D}_{\mathbb{F}_p}}(M, \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O})) \mid w'^c \circ \phi_{A'} = 0\right\} \end{aligned} \tag{1}$$

where $\mathrm{D}_{\mathbb{F}_p} = \mathbb{F}_p[F, V]$ is the Dieudonné ring, CW denotes Witt covectors [Fontaine 1977, Chapter II, Section 1],

$$w'^c : \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O})_{A'} \to K'^{ur}/\pi^2\mathscr{O}$$

is as in [Fontaine 1977, Chapter IV, Section 3] and $\phi_{A'} : M_{A'} \to \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O})_{A'}$ is induced by ϕ. By construction of w'^c we have for $\phi \in \operatorname{Hom}_{\mathrm{D}_{\mathbb{F}_p}}(M, \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O}))$ a commutative diagram

$$\begin{tikzcd}
M_{A'} \arrow[r, "\phi_{A'}"] & \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O})_{A'} \arrow[r, "w'^c"] & K'^{ur}/\pi^2\mathscr{O} \\
 & \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O}) \otimes_A A' \arrow[u, "\iota_{\mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O})}"] \arrow[ur, "\tilde{w}"] & \mathrm{CW}_{\mathbb{F}_p}(\pi\mathscr{O}/\pi^2\mathscr{O}) \arrow[l] \arrow[u, "w^c"'] \\
M \otimes_A A' \arrow[uu, "\iota_M", "\simeq"'] \arrow[ur, "\phi\otimes 1"] & M \arrow[l] \arrow[ur, "\phi"] &
\end{tikzcd}$$

in which

$$w^c((x_{-n})_{n\geq 0}) = \sum_{n=0}^{\infty} p^{-n} \hat{x}_{-n}^{p^n}$$

with $\hat{x}_{-n} \in \pi\mathbb{O}$ lifting x_{-n}, $\tilde{w} = w^c \otimes 1$ is the A'-linear extension of w^c and $\iota_{\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})}$ is surjective by [Fontaine 1977, Chapter IV, Proposition 2.5] since $\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})$ is V-divisible. It is easy to see that we have

$$w'^c \circ \phi_{A'} = 0 \Leftrightarrow w^c \circ \phi = 0. \tag{2}$$

Combining (2) and (1) we obtain an isomorphism

$$G(\overline{K})^I \stackrel{\simeq}{\to} \{\phi \in \mathrm{Hom}_{\mathrm{D}_{\mathbb{F}_p}}(M, \mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})) \mid w^c \circ \phi = 0\}. \tag{3}$$

Now we need to study $\ker(w^c)$. We will use the isomorphism of $\overline{\mathbb{F}}_p$-vector spaces

$$\pi\mathbb{O}/\pi^2\mathbb{O} \stackrel{:\pi}{\to} \mathbb{O}/\pi\mathbb{O} \simeq \overline{\mathbb{F}}_p \tag{4}$$

to describe elements of $\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})$ as covectors $(y_{-n})_{n\geq 0}$ with $y_{-n} \in \overline{\mathbb{F}}_p$. Of course, since (4) is not multiplicative, some care has to be taken with this. We denote by $\sigma : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$, $\sigma(x) = x^p$ the absolute Frobenius and claim that

$$\ker(w^c) = \{(y_{-n})_n \mid y_{-n} \in \overline{\mathbb{F}}_p\,,\ y_{-1} = y_0^{\sigma^{-1}}\}. \tag{5}$$

To see this, let $(x_{-n})_n \in \mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})$ be given, choose $\hat{x}_{-n} \in \pi\mathbb{O}$ lifting x_{-n} and write $\hat{x}_{-n} = \pi\hat{y}_{-n}$ with $\hat{y}_{-n} \in \mathbb{O}$. Then we compute in $K'^{ur}/\pi^2\mathbb{O}$:

$$w^c((x_{-n})) = \sum_{n=0}^{\infty} p^{-n}(\pi\hat{y}_{-n})^{p^n} \stackrel{(\pi^{p-1}=-p)}{=} \sum_{n=0}^{\infty} (-1)^n \pi^{p^n - n(p-1)} \hat{y}_{-n}^{p^n} = \pi(\hat{y}_0 - \hat{y}_{-1}^p),$$

using that $p^n - n(p-1) \geq 2$ for all $n \geq 2$. Now (5) is obvious.

Next, we claim that the subset

$$\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O}) \supseteq \mathcal{M} := \{(y_0^{\sigma^{-n}})_{n\geq 0} \mid y_0 \in \overline{\mathbb{F}}_p\} \tag{6}$$

is a $\mathrm{D}_{\mathbb{F}_p}$-submodule. First note that $F = 0$ on $\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})$ so we will consider it as a $\mathrm{D}_{\mathbb{F}_p}/F = \mathbb{F}_p[V]$-module in the following. Since all products in $\pi\mathbb{O}/\pi^2\mathbb{O}$ are zero we have

$$(x_{-n}) + (y_{-n}) = (x_{-n} + y_{-n})$$

in $\mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})$ and $\mathcal{M}$ is indeed a $\mathbb{F}_p$-submodule, visibly stable under V.

We claim that the inclusion (6) induces an isomorphism

$$\mathrm{Hom}_{\mathbb{F}_p[V]}(M, \mathcal{M}) \stackrel{\simeq}{\to} \{\phi \in \mathrm{Hom}_{\mathrm{D}_{F_p}}(M, \mathrm{CW}_{\mathbb{F}_p}(\pi\mathbb{O}/\pi^2\mathbb{O})) \mid w^c \circ \phi = 0\}. \tag{7}$$

Since $\mathcal{M} \subseteq \ker(w^c)$ by (5) we only need to see that an $\mathbb{F}_p[V]$-linear morphism

$$\phi : M \to \mathrm{CW}_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$$

with $\phi(M) \subseteq \ker(w^c)$ factors through $\mathcal{M}$: For every $m \in M$ and $n \geq 0$ we have, writing $\phi(m) =: (y_{-n})$ with $y_{-n} \in \overline{\mathbb{F}}_p$,

$$0 = w^c(\phi(V^n m)) = w^c(V^n(\phi(m))) = w^c((\dots, y_{-n-1}, y_{-n})),$$

thus $y_{-n-1} = y_{-n}^{\sigma^{-1}}$ by (5) and as this is true for every $n \geq 0$ we get $\phi(m) \in \mathcal{M}$.

To proceed, note that

$$\mathcal{M} \to \overline{\mathbb{F}}_p\,,\ (y_0^{\sigma^{-n}}) \mapsto y_0 \tag{8}$$

is an isomorphism of $\mathbb{F}_p[V]$-modules if one defines $V(\alpha) := \alpha^{\sigma^{-1}}$ for $\alpha \in \overline{\mathbb{F}}_p$. Denoting by $\Phi : G(\overline{K})^I \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}}_p)$ the isomorphism obtained by combining (3), (7) and (8), by construction we have a commutative diagram

$$\begin{tikzcd} G(\overline{K})^I \arrow[r, "\Phi"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}}_p) \arrow[d, "\mathrm{Hom}(V,\overline{\mathbb{F}}_p)"] \\ G(\overline{K})^I \arrow[r, "\Phi"] & \mathrm{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}}_p). \end{tikzcd} \tag{9}$$

Let e_i (resp. ϕ_i) ($1 \leq i \leq n$) be an $\mathbb{F}_p$-basis of M (resp. $\mathrm{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}}_p)$) and define $Ve_i =: \sum_j a_{ij} e_j$, hence $A := (a_{ij}) \in \mathrm{Gl}_n(\mathbb{F}_p)$, $\psi_i := \mathrm{Hom}(V, \overline{\mathbb{F}}_p)(\phi_i) =: \sum_j b_{ij}\phi_j$, hence $B := (b_{ij}) \in \mathrm{Gl}_n(\mathbb{F}_p)$ and $C := (\phi_i(e_j)) \in \mathrm{Gl}_n(\overline{\mathbb{F}}_p)$. By definition, A is a representing matrix of $V : M \to M$ and by (9) B is a representing matrix for τ. So we will be done if we can show that A and B are conjugate over $\mathbb{F}_p$.

From the computation $\psi_i(e_j) = \phi_i(Ve_j) = \sum_k a_{jk}\phi_i(e_k) = \sum_k b_{ik}\phi_k(e_j)$ we obtain ${}^tA = C^{-1}BC$. Now recall that over every field κ two square matrices with coefficients in κ which are conjugate over an algebraic closure of κ are conjugate over κ and, furthermore, that every square matrix with coefficient in κ is conjugate, over κ, to its transpose. Hence A is indeed conjugate to B over $\mathbb{F}_p$. □

Remark A.3. Inspecting the proof we see that for G/A' connected with étale dual (not necessarily annihilated by p) we have a commutative diagram

$$\begin{tikzcd} G(\overline{K})^I \arrow[r, "\Phi", "\simeq"'] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}}_p) \arrow[d, "\mathrm{Hom}(V,\overline{\mathbb{F}}_p)"] \\ G(\overline{K})^I \arrow[r, "\Phi", "\simeq"'] & \mathrm{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}}_p). \end{tikzcd}$$

Proof of Proposition A.2. Define a finite Honda system over A' by

$$M := \mathbb{Z}/p^n\mathbb{Z}, \quad 1 \neq V \in 1 + p(\mathbb{Z}/p^n\mathbb{Z}) \subseteq (\mathbb{Z}/p^n\mathbb{Z})^* = \mathrm{Aut}_{\mathbb{Z}_p}(M),$$
$$F := pV^{-1},$$
$$L := M_{A'}.$$

It is easy to see that this is indeed a finite Honda system. For the corresponding group G/A' we have by Remark A.3

$$G(\overline{K})^I \simeq \mathrm{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p^{V=1} = \mathbb{F}_p$$

with trivial geometric Frobenius. Note that V is the identity on M/FM, but $V \neq 1$. □

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