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On the tangent space of the deformation functor of curves with automorphisms

Aristides Kontogeorgis

We provide a method to compute the dimension of the tangent space to the global infinitesimal deformation functor of a curve together with a subgroup of the group of automorphisms. The computational techniques we developed are applied to several examples including Fermat curves, p-cyclic covers of the affine line and to Lehr–Matignon curves.

The aim of this paper is the study of equivariant equicharacteristic infinitesimal deformations of a curve X of genus g, admitting a group of automorphisms. This paper is the result of my attempt to understand the work of J. Bertin and A. Mézard [2000] and of G. Cornelissen and F. Kato [2003].

Let X be a smooth projective algebraic curve, defined over an algebraically closed field of characteristic $p \geq 0$. The infinitesimal deformations of the curve X, without considering compatibility with the group action, correspond to directions on the vector space $H^1(X, \mathcal{T}_X)$ which constitutes the tangent space to the deformation functor of the curve X [Harris and Morrison 1998]. All elements in $H^1(X, \mathcal{T}_X)$ give rise to unobstructed deformations, since X is one-dimensional and the second cohomology vanishes.

In the study of deformations together with the action of a subgroup of the automorphism group, a new deformation functor can be defined. The tangent space of this functor is given by Grothendieck's [1957] equivariant cohomology group $H^1(X, G, \mathcal{T}_X)$; see [Bertin and Mézard 2000, 3.1]. In this case the wild ramification points contribute to the dimension of the tangent space of the deformation functor and also pose several lifting obstructions, related to the theory of deformations of Galois representations.

Bertin and Ménard [2000], after proving a local-global principle, focused on infinitesimal deformations in the case G is cyclic of order p and considered liftings to characteristic zero, while Cornelissen and Kato [2003] considered the case of deformations of ordinary curves without putting any other condition on the automorphism group. The ramification groups of automorphism groups acting on

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ordinary curves have a special ramification filtration, i.e., the p-part of every ramification group is an elementary abelian group, and this makes the computation possible, since elementary abelian group extensions are given explicitly in terms of Artin–Schreier extensions.

In this paper we consider an arbitrary curve X with automorphism group G. By the theory of Galois groups of local fields, the ramification group at every wild ramified point can break to a sequence of extensions of elementary abelian groups [Serre 1979, IV]. We will use this decomposition together with the spectral sequence of Lyndon–Hochschild–Serre in order to reduce the computation to one involving elementary abelian groups.

We are working over an algebraically closed field of positive characterstic and for the sake of simplicity we assume that $p \geq 5$.

The dimension of the tangent space of the deformation functor depends on the group structure of the extensions that appear in the decomposition series of the ramification groups at wild ramified points. We are able to give lower and upper bounds of the dimension of the tangent space of the deformation functor.

In particular, if the decomposition group G_P at a wild ramified point P is the semidirect product of an elementary abelian group with a cyclic group such that there is only a lower jump at the i-th position in the ramification filtration, then we are able to compute exactly the dimension of the local contribution $H^1(G_P, \mathcal{T}_{\mathbb{O}})$ (Proposition 2.9 and Section 3.1).

We begin our exposition in Section 1 by surveying some of the known deformation theory. Next we proceed to the most difficult task, namely the computation of the tangent space of the local deformation functor, by employing the low terms sequence stemming from the Lyndon–Hochschild–Serre spectral sequence.

The dimension of equivariant deformations that are locally trivial, i.e., the dimension of $H^1(X/G, \pi_*^G(\mathcal{T}_X))$ is computed in Section 3. The computational techniques we developed are applied in the case of Fermat curves that are known to have large automorphism group, in the case of p-covers of $\mathbb{P}^1(k)$ and in the case of Lehr–Matignon curves. Moreover, we are able to recover the results of [Cornelissen and Kato 2003] concerning deformations of ordinary curves. Finally, we try to compare our result with the results of R. Pries [2002; 2004] concerning the computation of unobstructed deformations of wild ramified actions on curves.

1. Some deformation theory

There is nothing original in this section, but for the sake of completeness, we present some of the tools we will need for our study. This part is essentially a review of [Bertin and Mézard 2000; Cornelissen and Kato 2003; Mazur 1997].

Let k be an algebraic closed field of characteristic $p \geq 0$. We consider the category $\mathscr{C}$ of local Artin k-algebras with residue field k.

Let X be a nonsingular projective curve defined over the field k, and let G be a fixed subgroup of the automorphism group of X. We will denote by (X, G) the couple of the curve X together with the group G.

A deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

$$\mathscr{X} \to \mathrm{Spec}(A)$$

parametrized by the base scheme $\mathrm{Spec}(A)$, together with a group homomorphism $G \to \mathrm{Aut}_A(\mathscr{X})$ such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathscr{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

Two deformations $\mathscr{X}_1, \mathscr{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ, making the diagram

$$\begin{array}{ccc} \mathscr{X}_1 & \xrightarrow{\psi} & \mathscr{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\, A & \end{array}$$

commutative. The global deformation functor is defined as

$$D_{\mathrm{gl}} : \mathscr{C} \to \mathrm{Sets},\ \mathrm{A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\}$$

Let D be a functor such that $D(k)$ is a single element. If $k[\epsilon]$ is the ring of dual numbers, then the Zariski tangent space t_D of the functor is defined by $t_D := D(k[\epsilon])$. If the functor D satisfies the "Tangent Space Hypothesis", i.e., when the mapping

$$h : D(k[\epsilon] \times_k k[\epsilon]) \to D(k[\epsilon]) \times D(k[\epsilon])$$

is an isomorphism, then the $D(k[\epsilon])$ admits the structure of a k-vector space [Mazur 1997, p. 272]. The tangent space hypothesis is contained in the hypothesis (H3) of Schlessinger, which holds for all the functors in this paper, since all the functors admit versal deformation rings [Schlessinger 1968; Bertin and Mézard 2000, Section 2].

The tangent space $t_{D_{gl}} := D_{\mathrm{gl}}(k[\epsilon])$ of the global deformation functor is expressed in terms of Grothendieck's equivariant cohomology [1957], which combines the construction of group cohomology and sheaf cohomology.

We recall quickly the definition of equivariant cohomology theory: We consider the covering map $\pi : X \to Y = X/G$. For every sheaf F on X we denote by $\pi_*^G(F)$ the sheaf

$$V \mapsto \Gamma(\pi^{-1}(V), F)^G, \text{ where } V \text{ is an open set of } Y.$$

The category of $(G, \mathcal{O}_X)$-modules is the category of $\mathcal{O}_X$-modules with an additional G-module structure. We can define two left exact functors from the category of $(G, \mathcal{O}_X)$-modules, namely

$$\pi_*^G \text{ and } \Gamma^G(X, \cdot),$$

where $\Gamma^G(X, F) = \Gamma(X, F)^G$. The derived functors $R^q\pi_*^G(X, \cdot)$ of the first functor are sheaves of modules on Y, and the derived functors of the second are groups $H^q(X, G, F) = R^q\Gamma^G(X, F)$.

Theorem 1.1 [Bertin and Mézard 2000]. *Let $\mathcal{T}_X$ be the tangent sheaf on the curve X. The tangent space $t_{D_{\mathrm{gl}}}$ to the global deformation functor, is given in terms of equivariant cohomology as $t_{D_{\mathrm{gl}}} = H^1(X, G, \mathcal{T}_X)$. Moreover the following sequence is exact*:

$$0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(X, G, \mathcal{T}_X) \to H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \to 0. \quad (1)$$

For a local ring $k[[t]]$ we define the local tangent space $\mathcal{T}_{\mathcal{O}}$, as the $k[[t]]$-module of k-derivations. The module $\mathcal{T}_{\mathcal{O}} := k[[t]]\, d/dt$, where $\delta = d/dt$ is the derivation such that $\delta(t) = 1$. If G is a subgroup of $\mathrm{Aut}(k[[t]])$, then G acts on $\mathcal{T}_{\mathcal{O}}$ in terms of the adjoint representation. Moreover by [Cornelissen and Kato 2003] there is a bijection

$$D_\rho(k[\epsilon]) \overset{\cong}{\to} H^1(G, \mathcal{T}_{\mathcal{O}}).$$

In order to describe the tangent space of the local deformation space we will compute first the space of tangential liftings, i.e., the space $H^1(G, \mathcal{T}_{\mathcal{O}})$.

This problem was solved in [Bertin and Mézard 2000] when G is a cyclic group of order p, and in [Cornelissen and Kato 2003] when the original curve is ordinary.

We will apply the classification of groups that can appear as Galois groups of local fields in order to reduce the problem to elementary abelian group case.

1.1. *Splitting the branch locus.* Let P be a wild ramified point on the special fibre X, and let $\sigma \in G_j(P)$ where $G_j(P)$ denotes the j-ramification group at P. Assume that we can deform the special fibre to a deformation $\mathcal{X} \to A$, where A is a complete local discrete valued ring that is a k-algebra. Denote by m_A the maximal ideal of A and assume that $A/m_A = k$. Moreover assume that σ acts fibrewise on $\mathcal{X}$. We will follow [Green and Matignon 1998] in expressing the expansion

$$\sigma(T) - T = f_j(T)u(T),$$

where $f_j(T) = \sum_{\nu=0}^{j} a_i T^i$ ($a_i \in m_A$ for $\nu = 0, \dots, j-1, a_j = 1$) is a distinguished Weierstrass polynomial of degree j [Bourbaki 1989, VII, 8, Proposition 6] and $u(T)$ is a unit of $A[[T]]$. The reduction of the polynomial f_j modulo m_A gives the automorphism σ on $G_j(P)$ but σ when lifted on $\mathscr{X}$ has in general more than one fixed points, since $f_j(T)$ might be a reducible polynomial. If $f_j(T)$, gives rise to only one horizontal branch divisor then we say that the corresponding deformation does not split the branch locus.

Moreover, if we reduce $\mathscr{X} \times_A \operatorname{Spec} A/m_A^2$ we obtain an infinitesimal extension that gives rise to a cohomology class in $H^1(G(P), \mathscr{T}_{\mathbb{O}})$ by [Cornelissen and Kato 2003, Proposition 2.3].

On the other hand cohomology classes in $H^1(X/G, \pi_*^G(\mathscr{T}_X))$ induce trivial deformations on formal neighbourhoods of the branch point P [Bertin and Mézard 2000, 3.3.1] and do not split the branch points. In the special case of ordinary curves, the distinction of deformations that do or do not split the branch points does not occur since the polynomials f_j are of degree 1.

1.2. ***Description of the ramification group.*** The finite groups that appear as Galois groups of a local field $k((t))$, where k is algebraically closed of characteristic p are known [Serre 1979].

Let L/K be a Galois extension of a local field K with Galois group G. We consider the ramification filtration of G,

$$G = G_0 \supset G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supset G_{n+1} = \{1\}. \tag{2}$$

The quotient G_0/G_1 is a cyclic group of order prime to the characteristic, G_1 is p-group and for $i \geq 1$ the quotients G_i/G_{i+1} are elementary abelian p-groups. If a curve is ordinary, we know by [Nakajima 1987] that the ramification filtration is short, i.e., $G_2 = \{1\}$, and this gives that G_1 is an elementary abelian group.

We are interested in the ramification filtrations of the decomposition groups acting on the completed local field at wild ramified points. To study this question, we introduce some notation: Consider the set of jumps of the ramification filtration $1 = t_f < t_{f-1} < \cdots < t_1 = n$, such that

$$G_1 = \ldots = G_{t_f} > G_{t_f+1} = \ldots = G_{t_{f-1}} > G_{t_{f-1}+1} \geq \ldots \geq G_{t_1} = G_n > \{1\}, \tag{3}$$

i.e., $G_{t_i} > G_{t_i+1}$. For this sequence it is known that $t_\mu \equiv t_\nu \bmod p$ for all $\mu, \nu \in \{1, \dots, f\}$; see [Serre 1979, Proposition 10, p. 70].

1.3. ***Lyndon–Hochschild–Serre spectral sequences.*** Hochschild and Serre [1953] considered the following problem: Given the short exact sequence of groups

$$1 \to H \to G \to G/H \to 1, \tag{4}$$

and a G-module A, how are the cohomology groups $H^i(G, A)$, $H^i(H, A)$ and $H^i(G/H, A^H)$ related? They gave an answer to the above problem in terms of a spectral sequence. For small values of i this spectral sequence gives us the low-degree exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^{G/H} \\ \xrightarrow{\text{tg}} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, H), \quad (5)$$

where res, tg, inf denote the restriction, transgression and inflation maps.

Lemma 1.2. *Let H be a normal subgroup of G, and let A be a G-module. The group G/H acts on the cohomology group $H^1(H, A)$ in terms of the conjugation action given explicitly on the level of 1-cocycles as follows: Let $\bar{\sigma} = \sigma H \in G/H$. The cocycle*

$$\begin{aligned} d : H &\to A \\ x &\mapsto d(x) \end{aligned}$$

is sent by the conjugation action to the cocycle

$$\begin{aligned} d^{\bar{\sigma}} : H &\to A \\ x &\mapsto \sigma d(\sigma^{-1} x \sigma), \end{aligned}$$

where $\sigma \in G$ is a representative of $\bar{\sigma}$.

Proof. This explicit description of the conjugation action on the level of cocycles is given in Propostion 2-5-1 (p. 79) of [Weiss 1969]. The action is well defined by Corollary 2-3-2 of the same reference. □

Our strategy is to use Equation (5) in order to reduce the problem of computation of $H^1(G, \mathcal{T}_{\mathcal{O}})$ to an easier computation involving only elementary abelian groups.

Lemma 1.3. *Let A be a k-module, where k is a field of characteristic p. For the cohomology groups we have $H^1(G_0, A) = H^1(G_1, A)^{G_0/G_1}$.*

Proof. Consider the short exact sequence

$$0 \to G_1 \to G_0 \to G_0/G_1 \to 0.$$

Equation (5) implies the sequence

$$0 \to H^1(G_0/G_1, A^{G_1}) \to H^1(G_0, A) \to H^1(G_1, A)^{G_0/G_1} \to H^2(G_0/G_1, A^{G_1}).$$

But the order of G_0/G_1 is not divisible by p, and is an invertible element in the k-module A. Thus the groups $H^1(G_0/G_1, A^{G_1})$ and $H^2(G_0/G_1, A^{G_1})$ vanish and the result follows from [Weibel 1994, Corollary 6.59]. □

Lemma 1.4. *If $G = G_i$, $H = G_{i+1}$ are groups in the ramification filtration of the decomposition group at some wild ramified point, and $i \geq 1$ then the conjugation action of G on H is trivial.*

Proof. Let L/K denote a wild ramified extension of local fields with Galois group G, let $\mathbb{O}_L$ denote the ring of integers of L and let m_L be the maximal ideal of $\mathbb{O}_L$. Moreover we will denote by L^* the group of units of the field L. We can define [Serre 1979, Proposition 7, p. 67; Proposition 9, p. 69] injections

$$\theta_0 : \frac{G_0}{G_1} \to L^* \quad \text{and} \quad \theta_i : \frac{G_i}{G_{i+1}} \to \frac{m_L^i}{m_L^{i+1}},$$

with the property

$$\theta_i(\sigma\tau\sigma^{-1}) = \theta_0(\sigma)^i \theta_i(\tau) \quad \text{for all } \sigma \in G_0 \text{ and } \tau \in G_i/G_{i+1}.$$

If $\sigma \in G_{t_j} \subset G_1$ then $\theta_0(\sigma) = 1$ and since θ_i is an injection, the above equation implies that $\sigma\tau\sigma^{-1} = \tau$. Therefore, the conjugation action of an element τ in G_i/G_{i+1} on G_j is trivial, and the result follows. □

1.4. ***Description of the transgression map.*** In this section we will try to determine the kernel of the transgression map. The definition of the transgression map given in (5) is not suitable for computations. We will give an alternative description, following [Neukirch et al. 2000].

Let A be a k-algebra that is acted on by G so that the G action is compatible with the operations on A. Let $\bar{A}$ be the set Map(G, A) of set-theoretic maps of the finite group G to the G-module A. The set $\bar{A}$ can be seen as a G-module by defining the action $f^g(\tau) = gf(g^{-1}\tau)$ for all $g, \tau \in G$. We observe that $\bar{A}$ is projective. The submodule A can be seen as the subset of constant functions. Notice that the induced action of G on the submodule A seen as the submodule of constant functions of $\bar{A}$ coincides with the initial action of G on A. We consider the short exact sequence of G-modules

$$0 \to A \to \bar{A} \to A_1 \to 0. \tag{6}$$

Let $H \lhd G$. By applying the functor of H-invariants to the short exact sequence (6) we obtain the long exact sequence

$$0 \to A^H \to \bar{A}^H \to A_1^H \xrightarrow{\psi} H^1(H, A) \to H^1(H, \bar{A}) = 0, \tag{7}$$

where the last cohomology group is zero since $\bar{A}$ is projective.

We split this four-term sequence into two short exact sequences

$$\begin{gathered} 0 \to A^H \to \bar{A}^H \to B \to 0, \\ 0 \to B \to A_1^H \xrightarrow{\psi} H^1(H, A) \to 0, \end{gathered} \tag{8}$$

where we have defined $B = \ker \psi$. Now we apply the G/H-invariant functor to these two short exact sequences in order to obtain

$$H^i(G/H, B) = H^{i+1}(G/H, A^H),$$

$$0 \to B^{G/H} \to A_1^G \to H^1(H, A)^{G/H} \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\phi} H^1(G/H, A_1^H) \cdots \quad (9)$$

It can be proved (see [Neukirch et al. 2000, Exercise 3, p. 71]) that the composition

$$H^1(H, A)^{G/H} \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\cong} H^2(G/H, A^H)$$

is the transgression map.

Lemma 1.5. *Assume that G is an abelian group. If the quotient G/H is a cyclic group isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and the group G can be written as a direct sum $G = G/H \times H$ then the transgression map is identically zero.*

Proof. Notice that if $A^H = A$ then this lemma can be proved by the explicit form of the transgression map as a cup product; see [Neukirch et al. 2000, Exercise 2, p. 71; Hochschild and Serre 1953].

The study of the kernel of the transgression is reduced to the study of the kernel of δ in (9). We will prove that the map ϕ in (9) is 1-1, and then the desired result will follow by exactness.

Let σ be a generator of the cyclic group $G/H = \mathbb{Z}/p\mathbb{Z}$. We denote by $N_{G/H}$ the norm map $A \to A$, sending

$$A \ni a \mapsto \sum_{g \in G/H} ga = \sum_{\nu=0}^{p-1} \sigma^\nu a.$$

By $I_{G/H}A$ we denote the submodule $(\sigma - 1)A$ and by ${}_{N_{G/H}}A = \{a \in A : N_{G/H}a = 0\}$. Since G/H is a cyclic group we know that

$$H^1(G/H, B) = \frac{{}_{N_{G/H}}B}{I_{G/H}B} \quad \text{and} \quad H^1(G/H, A_1^H) = \frac{{}_{N_{G/H}}A_1^H}{I_{G/H}A_1^H}; \quad (10)$$

see [Serre 1979, VIII 4] and [Weibel 1994, Theorem 6.2.2]. Observe that the map ϕ defined in (9) can be given in terms of (10) as the map sending

$$b \bmod I_{G/H}B \mapsto b \bmod I_{G/H}A_1^H.$$

The map ϕ is well defined since $I_{G/H}B \subset I_{G/H}A_1^H$. The kernel of ϕ is computed:

$$\ker \phi = \frac{{}_{N_{G/H}}B \cap I_{G/H}A_1^H}{I_{G/H}B}.$$

The short exact sequence in (8) is a short exact sequence of $k[G/H]$-modules. This sequence seen as a short exact sequence of k-vector spaces is split, i.e., there is a section $s : H^1(H, A) \to A_1^H$ so that $\psi \circ s = \mathrm{Id}_{H^1(H,A)}$. This section map is only a k-linear map and not apriori compatible with the G/H-action.

Let us study the map ψ more carefully. An element $x \in A_1^H$ is a class $a \bmod A$ where $a \in \bar{A}$, and since $x \in A_1^H$ we have

$$a^h - a = ha - a = c[h] \in A.$$

It is a standard argument that $c[h]$ is an 1-cocycle $c[h] : H \to A$ and the class of this cocycle is defined to be $\psi(x)$. Since the image of $c[h]$ seen as a cocycle $c[h] : H \to \bar{A}$ is trivial, $c[h]$ is a coboundary i.e. we can select $\bar{a}_c \in \bar{A}$ so that

$$c[h] = \bar{a}_c^h - \bar{a}_c. \tag{11}$$

Obviously $\bar{a}_c \bmod A$ is H-invariant and we define one section as

$$s(c[h]) = \bar{a}_c \bmod A.$$

We have assumed that the group G can be written as $G = H \times G/H$ therefore we can write the functions $\bar{a}_c$ as functions of two arguments

$$\bar{a}_c : H \times G/H \to A$$
$$(h, g) \mapsto \bar{a}_c(h, g)$$

Notice that (11) gives us that the for every $h, h_1 \in H$ the quantity $\bar{a}_c(h_1, g_1)^h - \bar{a}_c(h_1, g_1)$ does not depend on $g_1 \in G/H$. Using, this independence of $\bar{a}_c$ on the second argument we can compute that

$$s(c[h]^\sigma) = s(c[h])^\sigma,$$

i.e., the function s is compatible with the G/H-action. But every element $a \in A_1^H$ can be written as $a = b_a + s(\psi(a))$, where $b_a := a - s\psi(a) \in B$, since $\psi(b_a) = 0$. An arbitrary element in $I_{G/H} A_1^H$ is therefore written as

$$(\sigma - 1)a = (\sigma - 1)b_a + s(\sigma \cdot \psi(a) - \psi(a)). \tag{12}$$

If $(\sigma - 1)a \in_{N_{G/H}} B \cap I_{G/H} A_1^H$ we have, since $\mathrm{Im}(s) \cap B = \{0\}$,

$$s(\sigma \cdot \psi(a) - \psi(a)) = 0 \quad \text{if and only if} \quad (\sigma - 1)a = (\sigma - 1)b_a \in I_{G/H} B.$$

Therefore, ϕ is an injection and the desired result follows. □

1.5. ***The G-module structure of*** $\mathcal{T}_{\mathcal{O}}$**.** Our aim is to compute the first order infinitesimal deformations, i.e., the tangent space $D_\rho(k[\epsilon])$ to the infinitesimal deformation functor D_ρ [Mazur 1997, p. 272]. This space can be identified with $H^1(G, \mathcal{T}_{\mathcal{O}})$. The conjugation action on $\mathcal{T}_{\mathcal{O}}$ is defined as follows:

$$\left(f(t)\frac{d}{dt}\right)^\sigma = f(t)^\sigma \sigma \frac{d}{dt} \sigma^{-1} = f(t)^\sigma \sigma\left(\frac{d\sigma^{-1}(t)}{dt}\right)\frac{d}{dt}, \tag{13}$$

where $d\sigma^{-1}/dt$ denotes the operator sending an element $f(t)$ to $(d/dt)f^{\sigma^{-1}}(t)$, i.e. we first compute the action of σ^{-1} on f and then we take the derivative with respect to t. We will approach the cohomology group $H^1(G, \mathcal{T}_{\mathbb{O}})$ using the filtration sequence given in (2) and the low degree terms of the Lyndon–Hochschild–Serre spectral sequence.

The study of the cohomology group $H^1(G, \mathcal{T}_{\mathbb{O}})$ can be reduced to the study of the cohomology groups $H^1(V, \mathcal{T}_{\mathbb{O}})$, where V is an elementary abelian group. These groups can be written as a sequence of Artin–Schreier extensions that have the advantage of the extension and the corresponding actions having a relatively simple explicit form:

Lemma 1.6. *Let L be a an elementary abelian p-extension of the local field $K := k((x))$, with Galois group $G = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$, such that the maximal ideal of $k[[x]]$ is ramified completely and the ramification filtration has no intermediate jumps i.e. is given by $G = G_0 = \cdots = G_n > \{1\} = G_{n+1}$. Then the extension L is given by $K(y_1, \ldots, y_s)$ where $1/y_i^p - 1/y_i = f_i(x)$, where $f_i \in k((x))$ with a pole at the maximal ideal of order n.*

Proof. The desired result follows by the characterization of abelian p-extensions in terms of Witt vectors [Jacobson 1989, 8.11]. Notice that the exponent of the group G is p and we have to consider the image of $W_1(k((x)) = k((x))$, where $W_\lambda(\cdot)$ denotes the Witt ring of order λ as is defined in [Jacobson 1989, 8.26]. □

Lemma 1.7. *Every $\mathbb{Z}/p\mathbb{Z}$-extension $L = K(y)$ of the local field $K := k((x))$, with Galois group $G = \mathbb{Z}/p\mathbb{Z}$, such that the maximal ideal of $k[[x]]$ is ramified completely, is given in terms of an equation $f(1/y) = 1/x^n$, where $f(z) = z^p - z$ is in $k[z]$. The Galois group of the above extension can be identified with the $\mathbb{F}_p$-vector space V of the roots of the polynomial f, and the correspondence is given by*

$$\sigma_v : y \to \frac{y}{1+vy} \quad \text{for } v \in V. \tag{14}$$

Moreover, we can select a uniformization parameter of the local field L such that the automorphism σ_v acts on t as follows:

$$\sigma_v(t) = \frac{t}{(1+vt^n)^{1/n}}.$$

Finally, the ramification filtration is given by $G = G_0 = \cdots = G_n > \{1\} = G_{n+1}$, and $n \not\equiv 0 \bmod p$.

Proof. By the characterization of abelian extensions in terms of Witt vectors we have $f(1/y) = 1/x^n$, where $f(z) = z^p - z \in k[z]$ (see also [Stichtenoth 1993, A.13]). Moreover the Galois group can be identified with the one dimensional $\mathbb{F}_p$-vector space V of roots of f, sending $\sigma_v : y \to y/(1+vy)$.

The filtration of the ramification group G is given by $G \cong G_0 = G_1 = \cdots G_n$, $G_i = \{1\}$ for $i \geq n+1$ [Stichtenoth 1993, Proposition III.7.10, p. 117]. Computation yields

$$x^n = ((1/y)^p - 1/y)^{-1} = \frac{y^p}{1 - y^{p-1}}, \tag{15}$$

hence $v_L(y) = n$, i.e., $y = \epsilon t^n$, where ϵ is a unit in $\mathcal{O}_L$ and t is the uniformization parameter in $\mathcal{O}_L$. Moreover, the polynomial f can be selected so that p does not divide n; see [Stichtenoth 1993, III. 7.8]. Since k is an algebraically closed field, Hensel's lemma implies that every unit in $\mathcal{O}_L$ is an n-th power, therefore we might select the uniformization parameter t such that $y = t^n$, and the desired result follows by (14). □

Lemma 1.8. *Let $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ be an elementary abelian group with ramification filtration*

$$H = H_0 = \cdots = H_n > H_{n+1} = \{1\} \quad \textit{and} \quad H_\kappa = \{1\} \textit{ for } \kappa \geq n+1.$$

The upper ramification filtration in this case coincides with the lower ramification filtration.

Proof. Let m be a natural number. We define the function $\phi : [0, \infty] \to \mathbb{Q}$ so that for $m \leq u < m+1$,

$$\phi(u) = \frac{1}{|H_0|} \sum_{i=1}^{m} |H_i| + (u - m) \frac{|H_{m+1}|}{|H_0|},$$

and since $H_{n+1} = \{1\}$ we compute

$$\phi(u) = \begin{cases} u & \text{if } m+1 \leq n, \\ n + (u - n - 1)/|H_0| & \text{if } m+1 > n. \end{cases}$$

The inverse function ψ is computed by

$$\psi(u) = \begin{cases} u & \text{if } u \leq n, \\ |H_0|u + (-n|H_0| + n + 1) & \text{if } u > n. \end{cases}$$

Therefore, by the definition of the upper ramification filtration we have $H^i = H_{\psi(i)} = H_i$ for $i \leq n$, while for $u > n$ we compute $\psi(u) = |H_0|u - n|H_0| + n \geq n$, thus $H^u = H_{\psi(u)} = \{1\}$. □

Lemma 1.9. *Let $a \in \mathbb{Q}$. Then for every prime p and every $\ell \in \mathbb{N}$ we have*

$$\left\lfloor \left\lfloor \frac{a}{p^\ell} \right\rfloor / p \right\rfloor = \left\lfloor \frac{a}{p^{\ell+1}} \right\rfloor.$$

Proof. This result follows by expressing a as a Laurent p-adic expansion in p, $a = \sum_{\nu=\lambda}^{-1} a_\nu p^\nu + \sum_{\nu=0}^{\infty} a_\nu p^\nu$ and by noticing that $\lfloor a/p^\ell \rfloor$ is the power series $\sum_{\nu=0}^{\infty} a_{\nu+\ell}\, p^\nu$. □

The arbitrary $\sigma_v \in \mathrm{Gal}(L/K)$ sends $t^n \mapsto t^n/(1+vt^n)$, so by computation

$$\frac{d\sigma_v(t)}{dt} = \frac{1}{(1+vt^n)^{(n+1)/n}}.$$

Lemma 1.10. *We consider an Artin–Schreier extension $L/k((x))$ and we keep the notation from Lemma 1.7. Let $\sigma_v \in \mathrm{Gal}(L/K)$. The corresponding action on the tangent space $\mathcal{T}_{\mathbb{O}}$ is given by*

$$\Big(f(t)\frac{d}{dt}\Big)^{\sigma_v} = f(t)^{\sigma_v}(1+vt^n)^{(n+1)/n}\frac{d}{dt}.$$

Proof. We have

$$\frac{d\sigma_v^{-1}(t)}{dt} = \frac{d\sigma_{-v}(t)}{dt} = \frac{1}{(1-vt^n)^{(n+1)/n}}$$

and by computation

$$\sigma_v\Big(\frac{d\sigma_{-v}(t)}{dt}\Big) = (1+vt^n)^{(n+1)/n}. \qquad \square$$

Letting $\mathbb{O} = \mathbb{O}_L$, we will now compute the space of "local modular forms"

$$\mathcal{T}_{\mathbb{O}}^{G_{t_i}} = \{f(t) \in \mathbb{O} : f(t)^{\sigma_v} = f(t)(1+vt^n)^{-(n+1)/n} \text{ for all } \sigma_v \in G_{t_i}\},$$

for $i \geq 1$. First we do the computation for a cyclic p-group.

Lemma 1.11. *Let $L/k((x))$ be an Artin–Schreier extension with Galois group $H = \mathbb{Z}/p\mathbb{Z}$ and ramification filtration*

$$H_0 = H_1 = \ldots = H_n > \{1\}.$$

Let t be the uniformizer of L and denote by $\mathcal{T}_{\mathbb{O}}$ the set of elements of the form $f(t)\,d/dt$, $f(t) \in k[[t]]$, equipped with the conjugation action defined in (13). *The space $\mathcal{T}_{\mathbb{O}}^G$ is G-equivariantly isomorphic to the $\mathbb{O}_K$-module consisting of elements of the form*

$$f(x)x^{n+1-\lfloor (n+1)/p \rfloor}\frac{d}{dx}, \quad f(x) \in \mathbb{O}_K.$$

Proof. Using the description of the action in Lemma 1.10 we see that $\mathcal{T}_{\mathbb{O}}$ is isomorphic to the space of Laurent polynomials of the form $\{f(t)/t^{n+1} : f(t) \in \mathbb{O}\}$, and the isomorphism is compatible with the G-action. Indeed, we observe first that $t^{n+1}\,d/dt$ is a G-invariant element in $\mathcal{T}_{\mathbb{O}}$. Then, for every $f(t)\,d/dt \in \mathcal{T}_{\mathbb{O}}$, the map sending

$$f(t)\frac{d}{dt} = \frac{f(t)}{t^{n+1}}t^{n+1}\frac{d}{dt} \mapsto \frac{f(t)}{t^{n+1}},$$

is a G-equivariant isomorphism.

We have

$$\{f(t)/t^{n+1}, f(t) \in \mathcal{O}\}^G = \{f(t)/t^{n+1}, f(t) \in \mathcal{O}\} \cap k((x)),$$

so the G-invariant space consists of elements $g(x)$ in K such that g seen as an element in L belongs to $\mathcal{T}_{\mathcal{O}}$, i.e., $v_L(g) \geq -(n+1)$. Consider the set of functions $g(x) \in K$ such that $v_L(g) = pv_K(g) \geq -(n+1)$, i.e., $v_K(g) \geq -(n+1)/p$. Since $v_K(g)$ is an integer the last inequality is equivalent to $v_K(g) \geq -\lfloor (n+1)/p \rfloor$.

Now a simple computation with the defining equation of the Galois extension L/K shows that

$$t^{n+1}\frac{d}{dt} = x^{n+1}\frac{d}{dx},$$

and the desired result follows. □

Similarly one can prove the following more general lemma:

Lemma 1.12. *We are using the notation of Lemma 1.11. Let A be the fractional ideal $k[[t]]t^a\, d/dt$, where a is a fixed integer. The G-module A is G-equivariantly isomorphic to $t^{a-(n+1)}k[[t]]$. Moreover, the space A^G is the space of elements of the form*

$$f(x)x^{n+1-\lfloor (n+1-a)/p \rfloor}\frac{d}{dx}.$$

Next we proceed to the more difficult case of elementary abelian p-groups.

Lemma 1.13. *Let $G = \bigoplus_{i=1}^{s} \mathbb{Z}/p\mathbb{Z}$ be the Galois group of the fully ramified elementary abelian extension $L/k((x))$ and assume that the ramification filtration is of the form*

$$G = G_0 = G_1 = \cdots = G_n > \{1\}.$$

Let t denote the uniformizer of L. Denote by $\mathcal{T}_{\mathcal{O}}$ the set of elements of the form $f(t)\, d/dt$, $f(t) \in k[[t]]$, equipped with the conjugation action defined in (13). *The space $\mathcal{T}_{\mathcal{O}}^G$ is G-equivariantly isomorphic to the $\mathcal{O}_K$-module consisted of elements of the form*

$$f(x)x^{n+1-\lfloor n+1/p^s \rfloor}\frac{d}{dx}, \quad f(x) \in \mathcal{O}_K,$$

where $p^s = |G|$.

Proof. We will break the extension $L/k((x))$ to a sequence of extensions $L = L_0 > L_1 > \dots L_s = k((x))$, such that L_i/L_{i+1} is a cyclic p-extension. Denote by π_i the uniformizer of L_i. According to Lemma 1.8 the ramification extension L_i/L_{i+1} is of conductor n, i.e. the conditions of Lemma 1.11 are satisfied. We will prove the result inductively. For the extension L/L_1 the statement is true by Lemma 1.11. Assume that the lemma is true for L/L_i so a $k[[\pi_i]]$ basis of

$$\mathcal{T}_{\mathcal{O}}^{\bigoplus_{\nu=1}^{i} \mathbb{Z}/p\mathbb{Z}}$$

is given by the element $\pi_i^{n+1-\lfloor n+1/p^i \rfloor} d/d\pi_i$. Then Lemma 1.11 implies that a $k[[\pi_{i+1}]]$ basis for

$$\mathcal{T}_{\mathcal{O}}^{\bigoplus_{\nu=1}^{i+1} \mathbb{Z}/p\mathbb{Z}} = \left(\mathcal{T}_{\mathcal{O}}^{\bigoplus_{\nu=1}^{i} \mathbb{Z}/p\mathbb{Z}}\right)^{\mathbb{Z}/p\mathbb{Z}}$$

is given by the element

$$\pi_{i+1}^{n+1-\left\lfloor \left(n+1-(n+1-\lfloor n+1/p^i \rfloor)\right)/p \right\rfloor} \frac{d}{d\pi_{i+1}}.$$

The desired result follows by Lemma 1.9. □

Similarly one can prove the following more general lemma:

Lemma 1.14. *We are using the notation of Lemma 1.13. Let A be the fractional ideal $k[[t]]t^a\, d/dt$, where a is a fixed integer. The G-module A is G-equivariantly isomorphic to $t^{a-(n+1)}k[[t]]$. The space A^G is the space of elements of the form*

$$f(x)x^{n+1-\lfloor (n+1-a)/p^s \rfloor} \frac{d}{dx}.$$

By induction, this computation can be extended to yield:

Proposition 1.15. *Let $L = k((t))$ be a local field acted on by a Galois p-group G with ramification subgroups*

$$G_1 = \ldots = G_{t_f} > G_{t_f+1} = \ldots = G_{t_{f-1}} > G_{t_{f-1}+1} \geq \ldots \geq G_{t_1} = G_n > G_{t_0} = \{1\}.$$

We consider the tower of local fields

$$L^{G_0} = L^{G_1} \subseteq L^{G_i} \subseteq \ldots L^{\{1\}} = L.$$

Let us denote by π_i a local uniformizer for the field L^{G_i}, i.e. $L^{G_i} = k((\pi_i))$. The extension $L^{G_{t_i+1}}/L^{G_{t_i}}$ is Galois with Galois group the elementary abelian group $H(i) := G_{t_i}/G_{t_i+1}$. The ramification filtration of the group $H(i)$ is given by

$$H(i)_0 = H(i)_1 = \ldots = H(i)_{t_i} > H(i)_{t_i+1} = \{1\}$$

and the conductor of the extension is t_i. Let $\mathcal{O}$ be the ring of integers of L. The invariant space $\mathcal{T}_{\mathcal{O}}^{G_{t_i}}$ is the $\mathcal{O}^{G_{t_i}}$-module generated by

$$\pi_i^{\mu_i} \frac{d}{d\pi_i}, \tag{16}$$

where $\mu_0 = 0$ and $\mu_i = t_i + 1 - \left\lfloor \dfrac{-\mu_{i-1} + t_i + 1}{|G_{t_i}|/|G_{t_{i-1}}|} \right\rfloor$.

Proof. The first statements are clear from elementary Galois theory. What needs a proof is the formula for the dimensions μ_i. For $i = 1$, the group $G_{t_1} = G_n$

is elementary abelian and Lemma 1.13 applies, under the assumption $G_{t_0} = \{1\}$. Hence

$$\mathcal{T}_{\mathbb{O}}^{G_{t_1}} = \pi_1^{\,n+1-\left\lfloor \frac{n+1}{|G_{t_1}|/|G_{t_0}|} \right\rfloor} \frac{d}{d\pi_1}.$$

Assume that the formula is correct for i, so

$$\mathcal{T}_{\mathbb{O}}^{G_{t_i}} = \pi^{\mu_i} \frac{d}{d\pi_i}.$$

Then Lemma 1.14 implies that

$$\mathcal{T}_{\mathbb{O}}^{G_{t_{i+1}}} = \left(\mathcal{T}_{\mathbb{O}}^{G_{t_i}}\right)^{G_{t_{i+1}}/G_{t_i}} = \pi_{i+1}^{\mu_{i+1}} \frac{d}{d\pi_{i+1}},$$

where $\mu_{i+1} = n_{i+1} + 1 - \left\lfloor (n_{i+1} + 1 - \mu_i)/|G_{t_{i+1}}/G_{t_i}| \right\rfloor$ and the inductive proof is complete. □

Let $k((t))/k((x))$ be a cyclic extension of local fields of order p, such that the maximal ideal $xk[[x]]$ is ramified completely. For the ramification groups G_i we have

$$\mathbb{Z}/p\mathbb{Z} = G = G_0 = \cdots = G_n > G_{n+1} = \{1\}.$$

Hence, the different exponent is computed $d = (n + 1)(p - 1)$. Let $E = t^a k[[t]]$ be a fractional ideal of $k((t))$. Let $N(E)$ denote the images of elements of E under the norm map corresponding to the group $\mathbb{Z}/p\mathbb{Z}$. It is known that $N(E) = x^{\lfloor (d+a)/p \rfloor} k[[x]]$, and $E \cap k[[x]] = x^{\lceil a/p \rceil} k[[x]]$. The cohomology of cyclic groups is 2-periodic and by [Bertin and Mézard 2000, Proposition 4.1.1] we have

$$\dim_k H^1(G, E) = \dim_k H^2(G, E) = \frac{E \cap k[[x]]}{N(E)} = \left\lfloor \frac{d+a}{p} \right\rfloor - \left\lceil \frac{a}{p} \right\rceil. \tag{17}$$

Remark 1.16. The proposition just quoted actually contains the following formula instead of (17):

$$\dim_k H^1(G, k[[x]] \frac{d}{dx}) = \left\lfloor \frac{2d}{p} \right\rfloor - \left\lceil \frac{d}{p} \right\rceil.$$

But $k[[x]] \dfrac{d}{dx} \cong x^{-n-1} k[[x]]$, and $d = (n + 1)(p - 1)$; thus

$$\left\lfloor \frac{2d}{p} \right\rfloor - \left\lceil \frac{d}{p} \right\rceil = \left\lfloor \frac{d+(n+1)p-n-1}{p} \right\rfloor - \left\lceil \frac{(n+1)p-n-1}{p} \right\rceil$$
$$= \left\lfloor \frac{d-n-1}{p} \right\rfloor - \left\lceil \frac{-n-1}{p} \right\rceil,$$

and the two formulas coincide.

Corollary 1.17. *Let G be an abelian group that can be written as a direct product $G = H_1 \times H_2$ of groups H_1, H_2, and suppose that $H_2 = \mathbb{Z}/p\mathbb{Z}$. The following sequence is exact*:

$$0 \to H^1(H_2, A^{H_1}) \to H^1(H_1 \times H_2, A) \to H^1(H_1, A)^{H_2} \to 0$$

Proof. The group H_2 is cyclic of order p so the transgression map is identically zero by Lemma 1.5 and the desired result follows. □

Remark 1.18. It seems that the result of J. Bertin and A. Mézard, solves the problem of determining the dimension of the k-vector spaces $H^1(\mathbb{Z}/p\mathbb{Z}, A)$ for fractional ideals of $k[[x]]$. But in what follows we have to compute the G/H-invariants of the above cohomology groups, therefore an explicit description of these groups and of the G/H-action is needed.

2. Computing $H^1(\mathbb{Z}/p\mathbb{Z}, A)$

We will need the following

Lemma 2.1. *Let a be a p-adic integer. The binomial coefficient $\binom{a}{i}$ is defined for a, as usual, by*

$$\binom{a}{i} = \frac{a(a-1)\cdot(a-i+1)}{i!}$$

and it is also a p-adic integer [Gouvêa 1997, Lemma 4.5.11]. Moreover, the binomial series is defined

$$(1+t)^a = \sum_{i=0}^{\infty} \binom{a}{i} t^i. \tag{18}$$

Let i be an integer and let $\sum_{\mu=0}^{\infty} b_\mu p^\mu$ and $\sum_{\mu=0}^{\infty} a_\mu p^\mu$ be the p-adic expansions of i and a respectively. The p-adic integer $\binom{a}{i} \not\equiv 0 \bmod p$ if and only if every coefficient $a_i \geq b_i$.

Proof. The only thing that needs a proof is the criterion of the vanishing of the binomial coefficient mod p. If a is a rational integer, then this is a known theorem due to Gauss [Eisenbud 1995, Proposition 15.21]. When a is a p-adic integer we compare the coefficients mod p of the expression

$$(1+t)^a = (1+t)^{\sum_{\mu=0}^{\infty} a_\mu p^\mu} = \prod_{\mu=1}^{\infty} (1+t^{p^\mu})^{a_\mu}$$

and of the binomial expansion in (18) and the result follows. □

Lemma 2.2 (Nakayama map). *Let $G = \mathbb{Z}/p\mathbb{Z}$ be a cyclic group of order p and let $A = t^a k[[t]]$ be a fractional ideal of $k[[t]]$. Let x be a local uniformizer of the* *field* $k((t))^{\mathbb{Z}/p\mathbb{Z}}$. *Let* $\alpha \in H^2(G, A)$, *and let* $u[\sigma, \tau]$ *be any cocycle representing the class* α. *The map*

$$\begin{aligned} \phi : H^2(G, A) &\to \frac{x^{\lceil a/p \rceil} k[[x]]}{x^{\lfloor ((n+1)(p-1)+a)/p \rfloor} k[[x]]} \\ \alpha &\mapsto \sum_{\rho \in G} u[\rho, \tau], \quad \tau \in G, \end{aligned} \tag{19}$$

is well defined and is an isomorphism.

Proof. Let A be a G-module. Denote by $\hat{H}^0(G, A)$ the zeroth Tate cohomology. We use Remark 4-5-7 and theorem 4-5-10 from [Weiss 1969] to prove that the map $H^2(G, A) \ni \alpha \mapsto \sum_{\rho \in G} u[\rho, \tau] \in \hat{H}^0(G, A)$ is well defined and an isomorphism. This map was introduced by T. Nakayama [1936] and used to give an explicit formula for the reciprocity law isomorphism for local class field theory.

We will express $\hat{H}^0(G, A)$ for $G = \mathbb{Z}/p\mathbb{Z}$ with generator σ and $A = t^a k[[t]]$. We know that

$$\hat{H}^0(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = \frac{\ker(\delta)}{N_{\mathbb{Z}/p\mathbb{Z}}(t^a k[[t]])},$$

where $\delta = \sigma - 1$ and $N_{\mathbb{Z}/p\mathbb{Z}} = \sum_{i=0}^{p-1} \sigma^i$. We compute that

$$\begin{aligned} \ker(\delta) &= t^a k[[t]] \cap k((x)) = x^{\lceil a/p \rceil} k[[x]], \\ N_{\mathbb{Z}/p\mathbb{Z}}(t^a k[[t]]) &= x^{\left\lfloor \frac{a+(n+1)(p-1)}{p} \right\rfloor} k[[x]]. \end{aligned}$$

This completes the proof. □

Let $A = t^a k[[t]]$ be a fractional ideal of $k[[t]]$. We consider the fractional ideal $t^{a+n+1} k[[t]]$, and we form the short exact sequence

$$0 \to t^{a+n+1} k[[t]] \to t^a k[[t]] \to M \to 0, \tag{20}$$

where M is an $(n+1)$-dimensional k-vector space with basis

$$\left\{ \frac{1}{t^{-a}}, \frac{1}{t^{-a-1}}, \dots, \frac{1}{t^{-a-n}} \right\}.$$

Let σ_v be the automorphism $\sigma_v(t) = t/(1 + vt^n)^{1/n}$, where $v \in \mathbb{F}_p$. The action of σ_v on $1/t^\mu$ is given by

$$\sigma_v : \frac{1}{t^\mu} \mapsto \frac{(1 + vt^n)^{\mu/n}}{t^\mu} = \frac{1}{t^\mu} \left(\sum_{\nu=0}^{\infty} \binom{\mu/n}{\nu} v^\nu t^{\nu n} \right). \tag{21}$$

The action of $\mathbb{Z}/p\mathbb{Z}$ on the basis elements of M is given by

$$\sigma_v(1/t^\mu) = \begin{cases} 1/t^\mu & \text{if } -a < \mu, \\ 1/t^{-a} - \dfrac{a}{n} v 1/t^{-a-n} & \text{if } \mu = -a. \end{cases} \tag{22}$$

We consider the long exact sequence obtained by applying the G-invariants functor to (20):

$$0 \to t^{a+n+1}k[[t]]^G \to t^a k[[t]]^G \to M^G \stackrel{\delta_1}{\to} H^1(G, t^{a+n+1}k[[t]])$$
$$\to H^1(G, t^a k[[t]]) \to H^1(G, M) \stackrel{\delta_2}{\to} H^2(G, t^{a+n+1}k[[t]]) \to \cdots \quad (23)$$

Lemma 2.3. *Assume that the group $G = \mathbb{Z}/p\mathbb{Z}$ is generated by σ_v. The map δ_1 in* (23) *is onto.*

Proof. By (22) we have

$$\dim_k M^{\mathbb{Z}/p\mathbb{Z}} = \begin{cases} n+1 & \text{if } p \mid a \\ n & \text{if } p \nmid a. \end{cases}$$

Now, if x is a local uniformizer of the field $k((t))^{\mathbb{Z}/p\mathbb{Z}}$, then

$$\left(\frac{1}{t^{-a-(n+1)}}k[[t]]\right)^{\mathbb{Z}/p\mathbb{Z}} = x^{\lceil (a+(n+1))/p \rceil}k[[x]] = \left(\frac{1}{x}\right)^{\lfloor (-a-(n+1))/p \rfloor}k[[x]],$$

and similarly

$$\left(\frac{1}{t^{-a}}k[[t]]\right)^{\mathbb{Z}/p\mathbb{Z}} = \left(\frac{1}{x}\right)^{\lfloor -a/p \rfloor}k[[x]].$$

The image of δ_1 has dimension $\dim_k M^{\mathbb{Z}/p\mathbb{Z}} - \lfloor -a/p \rfloor + \lfloor -a-(n+1)/p \rfloor$. Moreover for the dimension of $H^1(\mathbb{Z}/p\mathbb{Z}, (1/(t^{-a-(n+1)}))k[[t]])$ we compute

$$h := \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \frac{1}{t^{-a-(n+1)}}k[[t]]) = (n+1) - \lceil -a/p \rceil + \lfloor -a-(n+1)/p \rfloor.$$

We now observe that $\dim_k \mathrm{Im}(\delta_1) = h$ by studying separately the cases $p \mid a$ and $p \nmid a$. This finishes the proof. □

Proposition 2.4. *The cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ is isomorphic to*

$$H^1(\mathbb{Z}/p\mathbb{Z}, M) \cong \begin{cases} \bigoplus_{i=-a-n}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k) & \text{if } p \mid a, \\ \bigoplus_{i=-a-n+1}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k) & \text{if } p \nmid a. \end{cases}$$

Proof. Assume that the arbitrary automorphism $\sigma_v \in G = \mathbb{Z}/p\mathbb{Z}$ is given by $\sigma_v(t) = t/(1+vt^n)^{1/n}$ where $v \in \mathbb{F}_p$. Let us write a cocycle d as

$$d\sigma_v = \sum_{i=-a-n}^{-a} \alpha_i(\sigma_v)\frac{1}{t^i}. \quad (24)$$

By computation,

$$d(\sigma_v)^{\sigma_v^\mu} = \sum_{i=-a-n}^{-a} \alpha_i(\sigma_v)\frac{1}{t^i} + \alpha_{-a}(\sigma_v)\frac{-a}{n}\mu v \frac{1}{t^{-a-n}}.$$

We apply the cocycle condition $d(\sigma_v+\sigma_w)=d(\sigma_w)+d(\sigma_v)^{\sigma_w}$ for $d(\sigma_v)$ given in (24) and we obtain the following conditions on the coefficients $\alpha_i(\sigma_v)$:

$$\alpha_i(\sigma_w+\sigma_v)=\alpha_i(\sigma_w)+\alpha_i(\sigma_v) \quad \text{for } i\neq -a-n,$$
$$\alpha_{-a-n}(\sigma_w+\sigma_v)=\alpha_{-a-n}(\sigma_w)+\alpha_{-a-n}(\sigma_v)+\alpha_{-a}(\sigma_v)\frac{-a}{n}w.$$

The last equation allows us to compute the value of α_{-a-n} on any power σ_v^ν of the generator σ_v of $\mathbb{Z}/p\mathbb{Z}$. Indeed, we have

$$\alpha_{-a-n}(\sigma_v^\nu)=\nu\alpha_{-a-n}(\sigma_v)+(\nu-1)\alpha_{-a}(\sigma_v)\frac{-a}{n}v.$$

This proves that the function α_{-a-n} depends only on the selection of $\alpha_{-a-n}(\sigma_v)\in k$.

We will now compute the coboundaries. Let $b=\sum_{i=-a-n}^{-a} b_i/t^i,\ b_i\in k$ be an element in M. By computation,

$$b^{\sigma_v}-b=b_{-a}\frac{-a}{n}v\frac{1}{t^{-a-n}}.$$

For the computation of the cohomology groups we distinguish two cases:

- If $p\mid a$, the $\mathbb{Z}/p\mathbb{Z}$-action on M is trivial, so

$$H^1(\mathbb{Z}/p\mathbb{Z},M)=\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},M)=\bigoplus_{i=-a-n}^{-a}\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},k).$$

  The dimension of $H^1(\mathbb{Z}/p\mathbb{Z},M)$ in this case is $n+1$.

- If $p\nmid a$, the coboundary kills the contribution of the cocycle on the $1/t^{-a-n}$ basis element and the cohomology group is

$$H^1(\mathbb{Z}/p\mathbb{Z},M)=\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},M)=\bigoplus_{i=-a-n+1}^{-a}\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},k). \qquad \square$$

Lemma 2.5. *Assume that $p\geq 3$. Let $e=1$ if $p\nmid a$ and $e=0$ if $p\mid a$. If $n\geq 2$ then an element*

$$\sum_{i=-a-n+e}^{-a} a_i(\cdot)\frac{1}{t^i}\in H^1(\mathbb{Z}/p\mathbb{Z},M)$$

is in the kernel of δ_2 if and only if $a_i(\cdot)\binom{i/n}{p-1}=0$ for all i. If $n=1$ then an element

$$\sum_{i=-a-n+e}^{-a} a_i(\cdot)\frac{1}{t^i}\in H^1(\mathbb{Z}/p\mathbb{Z},M)$$

is in the kernel of δ_2 *if and only if* $a_i(\cdot)\binom{i/n}{p-1} = 0$ *for all* $-a-n+e \le i \le -a$ *and* $a_i(\cdot)\binom{i/n}{2p-2} = 0$ *for all* $-a-n+e \le i \le -a$ *such that* $2(p-1)n - i < (n+1)p + p\lfloor a/p \rfloor$.

Proof. A derivation $a_i(\sigma_v)(1/t^i)$, $-a-n+e \le i \le -a$ representing a cohomology class in $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ is mapped to

$$\begin{aligned}\delta_2\big(a_i(\cdot)\frac{1}{t^i}\big)[\sigma_w,\sigma_v] &= a_i(\sigma_v)\frac{1}{t^i}^{\sigma_w} - a_i(\sigma_v+\sigma_w)\frac{1}{t^i} + a_i(\sigma_w)\frac{1}{t^i} \\ &= \frac{a_i(\sigma_v)}{t^i}\Big(\sum_{\nu=1}^{\infty}\binom{i/n}{\nu} w^\nu t^{n\nu}\Big). \qquad (25)\end{aligned}$$

We now consider the map ϕ defined in (19) in the proof of Lemma 2.2. The map $\delta_2 : H^1(G, M) \to H^2(G, t^{a+n+1}k[[t]])$ is composed with ϕ and the image of $\phi\circ\delta_2$ in $x^{\lceil (a+n+1)/p \rceil}k[[x]]/x^{\lfloor ((n+1)p+a)/p \rfloor}k[[x]]$ is given by

$$\phi\circ\delta_2\big(a_i(\cdot)\frac{1}{t^i}\big) = \sum_{w\in\mathbb{Z}/p\mathbb{Z}} \frac{a_i(\sigma_v)}{t^i}\Big(\sum_{\nu=1}^{\infty}\binom{i/n}{\nu} w^\nu t^{n\nu}\Big).$$

Now recall that

$$\sum_{w\in\mathbb{Z}/p\mathbb{Z}} w^\nu = \begin{cases} 0 & \text{if } p-1 \nmid \nu, \\ -1 & \text{if } p-1 \mid \nu, \end{cases}$$

and every homomorphism $a_i : (\mathbb{Z}/p\mathbb{Z}, \cdot) \to (k, +)$ is given by $a_i(\sigma_w) = \lambda_i w$, where $\lambda_i \in k$. Therefore,

$$\phi\circ\delta_2\big(a_i(\cdot)\frac{1}{t^i}\big) = \sum_{\nu=1,\, p-1\mid\nu}^{\infty} \binom{i/n}{\nu}(-1)a_i(\sigma_v)t^{n\nu-i}. \qquad (26)$$

Observe that $(p-1) \mid \nu$ is equivalent to $\nu = \mu p - \mu$, and since $\nu \ge 1$, we have $\mu \ge 1$. Thus (26) becomes

$$\begin{aligned}&\sum_{\mu=1}^{\infty}\binom{i/n}{\mu p-\mu}(-1)\lambda_i t^{(\mu p-\mu)n-i} \\ &= \binom{i/n}{p-1}(-1)\lambda_i t^{(p-1)n-i} + \binom{i/n}{2p-2}(-1)\lambda_i t^{(2p-2)n-i} + \text{higher order terms.}\end{aligned}$$

Claim 2.6. If $n \ge 2$ and $p \ge 3$ then for all $a \le -i \le a+n$ and for $\mu \ge 2$

$$\mu(p-1)n - i \ge p\Big\lfloor \frac{(n+1)p+a}{p} \Big\rfloor. \qquad (27)$$

If $n=1$ and $p \geq 3$ then (27) holds for $a \leq -i \leq a+n$ and for $\mu \geq 3$. Moreover

$$(p-1)n-i < p\left\lfloor \frac{(n+1)p+a}{p} \right\rfloor,$$

for $a \leq -i \leq a+n$. Indeed, the inequality

$$\mu \geq \frac{n+1}{n}\frac{p}{p-1} \tag{28}$$

holds for $p \geq 3$, $n \geq 2$ and $\mu \geq 2$ or for $p \geq 3$, $n=1$, $\mu \geq 3$. Therefore, (28) implies that

$$n+1+\left\lfloor \frac{a}{p} \right\rfloor - \frac{a}{p} \leq n+1 \leq \mu \frac{p-1}{p} n$$

and therefore

$$(n+1)p+\left\lfloor \frac{a}{p} \right\rfloor p \leq \mu(p-1)n+a \leq \mu(p-1)n-i$$

and the first assertion is proved. For the second assertion, we compute

$$\frac{a}{p} < 1+\left\lfloor \frac{a}{p} \right\rfloor \quad \Rightarrow \quad \frac{a}{p}+n < n+1+\left\lfloor \frac{a}{p} \right\rfloor$$

and thus

$$(p-1)n-i < a+pn < p(n+1)+p\left\lfloor \frac{a}{p} \right\rfloor.$$

Since for elements $g \in k[[x]] \subset k[[t]]$ we have $pv_x(g) = v_t(g)$ we observe that all elements in $k[[t]]$ that have valuation greater or equal to $(n+1)p+\lfloor a/p \rfloor$ are zero in the lift of the ideal $x^{(n+1+\lfloor a/p \rfloor}k[[x]]$ on $k[[t]]$. Therefore Claim 2.6 gives us that for $p \geq 3, n \geq 2$,

$$\phi \circ \delta_2(a_i(\,\cdot\,)\frac{1}{t^i}) = \binom{i/n}{p-1}(-1)\lambda_i t^{(p-1)n-i}$$

so $\sum_{i=-a-n}^{-a} a_i(\,\cdot\,)(1/t^i)$ is in the kernel of δ_2 if and only if

$$\binom{i/n}{p-1}(-1)\lambda_i = 0 \text{ for all } i.$$

The case $n=1$ follows by a similar argument. □

Proposition 2.7. *The cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ is isomorphic to the k-vector space generated by*

$$\left\{ \frac{1}{t^i}, b \leq i \leq -a, \text{ such that } \binom{i/n}{p-1} = 0 \right\},$$

where $b=-a-n$ if $p \mid a$ and $b=-a-n+1$ if $p \nmid a$.

Proof. If $n \geq 2$ and $p \geq 3$ then the result is immediate by the exact sequence (23), Lemma 2.3 and Lemma 2.5 and by the computation of $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ given in Proposition 2.4.

Assume that $n = 1$, and let $a = a_0 + a_1 p + a_2 p^2 + \cdots$ be the p-adic expansion of a. Then the inequality

$$(n+1)p + p\left\lfloor \frac{a}{p} \right\rfloor \leq 2(p-1)n + a \tag{29}$$

holds if $a_0 \neq 0, 1$. Indeed, in this case we have $2/p \leq a/p - \lfloor a/p \rfloor < 1$ and (29) holds. Therefore, for the case $p \mid a$ and $a = 1 + pb$, $b \in \mathbb{Z}$ we have to check the binomial coefficients $\binom{i/n}{2p-2}$ as well. We will prove that in these cases if $\binom{i/n}{p-1} = 0$ then $\binom{i/n}{2p-2} = 0$ and the proof will be complete.

Assume, first that $p \mid a$ and $n = 1$. Then,$-a-1 \leq i \leq -a$, i.e. $i = -a-1$ or $i = -a$. We compute that $\binom{-a}{p-1} = 0$ since there is no constant term in the p-adic expansion of $-a$. Moreover the p-adic expansion of $2p-2$ is computed $2p-2 = p-2+p$, and since $p \neq 2$ we have $\binom{-a}{2p-2} = 0$ as well. For $i = -a-1$ we have $i = p-1+pb$ for some $b \in \mathbb{Z}$; therefore by comparing the p-adic expansions of $-a-1$,$p-1$ we obtain that $\binom{-a-1}{p-1} \neq 0$, and this value of i does not contribute to the cohomology.

Assume now that $a = 1 + pb$, $b \in \mathbb{Z}$. We have $i = -a$ and $-a = p-1+p(b+1)$. Therefore by comparing the p-adic expansions of $-a$, $p-1$ we obtain that $\binom{-a}{p-1} \neq 0$ and this value of i does not contribute to the cohomology. □

Proposition 2.8. *Let $A = t^a k[[t]]$ be a fractional ideal of the local field $k((t))$. Assume that $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ is an elementary abelian group with ramification filtration $H = H_0 = \cdots = H_n > H_{n+1} = \{1\}$. Let π_i be the local uniformizer of the local field $k((t))^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}$, and $a_i = \lceil a_{i-1}/p \rceil$, $a_1 = a$. The cohomology group $H^1(H, A)$ is generated as a k-vector space by the basis elements*

$$\left\{ \bigoplus_{\lambda=1}^{s} \frac{1}{\pi_i^{i_\lambda}}, \quad \begin{matrix} \lambda = 1, \ldots, s \\ b_i \leq i_\lambda \leq -a_i \end{matrix} \quad \text{such that } \binom{i_\lambda/n}{p-1} = 0, \right\}$$

where $b_i = -a_i - n$ if $p \mid a_i$ and $b_i = -a_i - n + 1$ if $p \nmid a_i$. Moreover, let $H(i) := H/\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}$. The groups $H^1(\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ are trivial $H(i)$-modules with respect to the conjugation action.

Proof. For $A = t^a k[[t]]$, we compute the invariants

$$t^a k[[t]] \cap k((t))^{\mathbb{Z}/p\mathbb{Z}} = x^{\lceil a/p \rceil} k[[x]],$$

where x is a local uniformizer for the ring of integers of $k((t))^{\mathbb{Z}/p\mathbb{Z}}$. The modules $A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}$ can be computed recursively:

$$A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}} = \pi_i^{a_i} k[[\pi_i]],$$

where π_i is a uniformizer for the local field $k((t))^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}$ and $a_i = \lceil a_{i-1}/p \rceil$, $a_1 = a$.

To compute the ramification filtration of quotient groups we have to employ the upper ramification filtration for the ramification group [Serre 1979, IV 3, p. 73-74]. But according to Lemma 1.8 the upper ramification filtration coincides with the lower ramification filtration therefore the ramification filtration for the groups $H(i)$ is $H(i)_0 = \cdots = H(i)_n > \{1\}$. For the group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ Corollary 1.17 implies that

$$H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = H^1\Big(\frac{\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}}{\mathbb{Z}/p\mathbb{Z}}, t^a k[[t]]^{\mathbb{Z}/p\mathbb{Z}}\Big) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])^{H/(\mathbb{Z}/p\mathbb{Z})}.$$

By Lemma 1.8 and by the compatibility [Serre 1979, IV 3, p. 73-74] of the upper ramification filtration with quotients, we obtain that the quotient $H/(\mathbb{Z}/p\mathbb{Z})$ has also conductor n. By Lemma 1.2, Lemma 1.4 and by the explicit description of the group $H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ of Proposition 2.7 and by the fact $H/(\mathbb{Z}/p\mathbb{Z})$ is of conductor n, the action of $H/(\mathbb{Z}/p\mathbb{Z})$ on $H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ is trivial. Thus

$$H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = H^1\Big(\frac{H}{\mathbb{Z}/p\mathbb{Z}}, t^a k[[t]]^{\mathbb{Z}/p\mathbb{Z}}\Big) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]]).$$

Moreover the cohomology group $H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ is generated over k by $\langle 1/\pi_1^i \oplus 1/\pi_2^j \rangle$, where $b_1 \leq i \leq -a$, $b_2 \leq j \leq -\lceil a/p \rceil$ and $\binom{i/n}{p-1} = \binom{j/n}{p-1} = 0$.

The desired result follows by induction. □

Proposition 2.9. *Let $A = t^a k[[t]]$ be a fractional ideal of the local field $k((t))$. Assume that $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ is an elementary abelian group with ramification filtration $H = H_0 = \cdots = H_n > H_{n+1} = \{1\}$. The dimension of $H^1(H, A)$ can be computed as*

$$\dim_k H^1(H, A) = \sum_{i=1}^{s} \left(\left\lfloor \frac{(n+1)(p-1)+a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil \right), \tag{30}$$

where a_i are defined recursively by $a_1 = a$ and $a_i = \lceil a_{i-1}/p \rceil$. In particular, if $A = k[[t]]$, we have

$$\dim_k H^1(H, k[[t]]) = s \left\lfloor \frac{(n+1)(p-1)}{p} \right\rfloor. \tag{31}$$

Proof. By induction on the number of direct summands, Corollary 1.17 and Proposition 2.8 we can prove the formula

$$H^1(H, A) = \bigoplus_{i=1}^{s} H^1(\mathbb{Z}/p\mathbb{Z}, A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}). \tag{32}$$

To compute the dimensions of the direct summands $H^1\big(\mathbb{Z}/p\mathbb{Z}, A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}\big)$, for various i we have to compute the ramification filtration for the groups defined as $H(i) = H/\big(\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}\big)$, since $\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z} = H/H(i)$. But the upper ramification filtration coincides with the lower ramification filtration by Lemma 1.8. Thus, the dimension of $H^1(H, A)$ can be computed as

$$\dim_k H^1(H, A) = \sum_{i=1}^{s} \left\lfloor \frac{(n+1)(p-1)+a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil.$$

In particular if $A = k[[t]]$, then

$$\dim_k H^1(H, k[[t]]) = s\left\lfloor \frac{(n+1)(p-1)}{p} \right\rfloor. \qquad \square$$

Let $\kappa_{t_i} = \dim_k \ker\big(\mathrm{tg} : H^1(G_{t_i+1}, \mathcal{T}_{\mathcal{O}}) \to H^2(G_{t_i}/G_{t_i+1}, \mathcal{T}_{\mathcal{O}}^{G_{t_i+1}})\big)$ be the dimension of the kernel of the transgression map. We have

$$0 \leq \kappa_i \leq \dim_k H^1(G_{t_i+1}, \mathcal{T}_{\mathcal{O}})^{G_{t_i}/G_{t_i+1}} \leq \dim_k H^1(G_{t_i+1}, \mathcal{T}_{\mathcal{O}}). \tag{33}$$

This allows us to compute

Proposition 2.10. *Let G be the Galois group of the extensions of local fields L/K, with ramification filtration G_i and let $(t_\lambda)_{1\leq\lambda\leq f}$ be the jump sequence in* (3)*. For the dimension of $H^1(G_1, \mathcal{T}_{\mathcal{O}})$ we have the bound*

$$\begin{aligned} H^1(G_1/G_{t_{f-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{f-1}}}) &\leq \dim_k H^1(G_1, \mathcal{T}_{\mathcal{O}}) \\ &\leq \sum_{i=1}^{f} \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}})^{G_{t_f}/G_{t_i}} \\ &\leq \sum_{i=1}^{f} \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}), \end{aligned} \tag{34}$$

where $G_{n+1} = \{1\}$. The left bound is best possible in the sense that there are ramification filtrations such that the first inequality becomes an equality.

Proof. Using the low-term sequence in (5) we obtain the following inclusion for $i \geq 1$:

$$\begin{aligned} H^1(G_{t_i}, \mathcal{T}_{\mathcal{O}}) &= H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}) + \ker tg \\ &\subseteq H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}) \oplus H^1(G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}})^{G_{t_i}/G_{t_{i-1}}}. \end{aligned} \tag{35}$$

We start our computation from the end of the ramification groups:

$$H^1(G_{t_2}, \mathcal{T}_{\mathcal{O}}) \subseteq H^1(G_{t_2}/G_{t_1}, \mathcal{T}_{\mathcal{O}}^{G_{t_1}}) \oplus H^1(G_{t_1}, \mathcal{T}_{\mathcal{O}})^{G_{t_2}/G_{t_1}}. \tag{36}$$

Observe here that $\mathcal{T}_{\mathbb{O}}$ is not G_{t_1}-invariant so there in no apriori well defined action of G_{t_2}/G_{t_1} on $\mathcal{T}_{\mathbb{O}}$. But since the group G_{t_1} is of conductor n using the explicit form of $H^1(G_{t_1}, \mathcal{T}_{\mathbb{O}})$ we see that $H^1(G_{t_1}, \mathcal{T}_{\mathbb{O}})$ is a trivial G_1-module. Of course this is also clear from the general properties of the conjugation action [Weiss 1969, Corollary 2-3-2]. We move to the next step:

$$H^1(G_{t_3}, \mathcal{T}_{\mathbb{O}}) \subseteq H^1(G_{t_3}/G_{t_2}, \mathcal{T}_{\mathbb{O}}^{G_{t_2}}) \oplus H^1(G_{t_2}, \mathcal{T}_{\mathbb{O}})^{G_{t_3}/G_{t_2}}. \tag{37}$$

The combination of (36) and (37) gives us

$$H^1(G_{t_3}, \mathcal{T}_{\mathbb{O}}) \subseteq H^1\Big(\frac{G_{t_3}}{G_{t_2}}, \mathcal{T}_{\mathbb{O}}^{G_{t_2}}\Big) \oplus H^1\Big(\frac{G_{t_2}}{G_{t_1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_1}}\Big)^{G_{t_3}/G_{t_2}} \oplus \Big(H^1(G_{t_1}, \mathcal{T}_{\mathbb{O}})^{G_{t_2}/G_{t_1}}\Big)^{G_{t_3}/G_{t_2}}.$$

Using induction based on (35) we obtain

$$H^1(G_1, \mathcal{T}_{\mathbb{O}}) \subseteq \bigoplus_{i=1}^{f} H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}}\Big)^{G_{t_f}/G_{t_i}} \subseteq \bigoplus_{i=1}^{f} H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}}\Big),$$

and the desired result follows. □

Notice that in the above proposition $G_{t_{i-1}}$ appears in the ramification filtration of G_0 thus the corollary to Proposition IV.1.3 in [Serre 1979] implies that the ramification filtration of $G_{t_i}/G_{t_{i-1}}$ is constant. Namely, if $Q = G_{t_i}/G_{t_{i-1}}$ the ramification filtration of Q is given by $Q_0 = Q_1 = \cdots = Q_{t_i} > \{1\}$. Therefore, $\delta_{t_1} = \dim_k H^1(G_n, \mathcal{T}_{\mathbb{O}})$, and $\dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}})$ can be computed explicitly by Proposition 2.9 since $G_n = G_{t_1}$, $G_{t_i}/G_{t_{i-1}}$, are elementary abelian groups. Namely we will prove:

Proposition 2.11. *Let $\log_p(\cdot)$ denote the logarithmic function with base p. Let $s(\lambda) = \log_p |G_{t_\lambda}|/|G_{t_{\lambda-1}}|$ and let μ_i be as in Proposition 1.15. Then*

$$\dim_k H^1\Big(\frac{G_{t_\lambda}}{G_{t_{\lambda-1}}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{\lambda-1}}}\Big) = \sum_{i=1}^{s(\lambda)} \Big(\Big\lfloor \frac{(t_\lambda+1)(p-1)+a_i}{p} \Big\rfloor - \Big\lceil \frac{a_i}{p} \Big\rceil\Big),$$

where $a_1 = -t_\lambda - 1 + \mu_{\lambda-1}$, and $a_i = \lceil a_{i-1}/p \rceil$.

Proof. The module $\mathcal{T}_{\mathbb{O}}^{G_{t_{\lambda-1}}}$ is computed in Proposition 1.15 to be isomorphic to $\pi_{\lambda-1}^{\mu_{\lambda-1}}(d/d\pi_{\lambda-1})$, which in turn is $((G_{t_\lambda}/G_{t_{\lambda-1}}))$-equivariantly isomorphic to the module $\pi_{\lambda-1}^{-t_\lambda-1+\mu_{\lambda-1}} k[[\pi_{\lambda-1}]]$. The result follows using Proposition 2.9. □

Remark 2.12. If $n = 1$ (equivalently, if $G_2 = \{1\}$), the left- and right-hand sides of (34) are equal and the bound becomes the formula in [Cornelissen and Kato 2003].

Proposition 2.13. *We will follow the notation of Proposition 2.11. Suppose that for every* i, $G_{t_i}/G_{t_{i-1}}$ *is a cyclic* p*-group. Then*

$$\begin{aligned}\dim_k H^1(G_1,\mathcal{T}_{\mathcal{O}}) &= \sum_{i=1}^{f} \dim_k H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}},\mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}\Big)^{G_{t_f}/G_{t_i}}\\ &\leq \sum_{i=1}^{f}\left(\Big\lfloor\frac{(t_i+1)(p-1)-t_i-1+\mu_{i-1}}{p}\Big\rfloor - \Big\lceil\frac{-t_i-1+\mu_{i-1}}{p}\Big\rceil\right).\end{aligned}$$

Proof. The kernel of the transgression at each step is by Lemma 1.5 the whole $H^1(G_{t_i}/G_{t_{i-1}},\mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}})^{G_{t_f}/G_{t_i}}$. Therefore the right inner inequality in Equation (34) is achieved. The other inequality is trivial by the computation done in Proposition 2.11 but it is far from being best possible. □

3. Global computations

We consider the Galois cover of curves $\pi : X \to Y = X/G$, and let $b_1,\dots,b_r$ be the ramification points of the cover. We will denote by

$$e_0^{(\mu)} \geq e_1^{(\mu)} \geq e_2^{(\mu)} \geq \cdots \geq e_{n_\mu}^{(\mu)} > 1$$

the orders of the higher ramification groups at the point b_μ. The ramification divisor D of the above cover is a divisor supported at the ramification points $b_1,\dots,b_r$ and is equal to

$$D=\sum_{\mu=1}^{r}\sum_{i=0}^{n_\mu}(e_i^{(\mu)}-1)b_\mu .$$

Let Ω_X^1, Ω_Y^1 be the sheaves of holomorphic differentials at X and Y respectively. We have

$$\Omega_X^1 \cong \mathcal{O}_X(D)\otimes\pi^*(\Omega_Y^1)$$

(see [Hartshorne 1977, IV. 2.3]), and, by taking duals,

$$\mathcal{T}_X \cong \mathcal{O}_X(-D)\otimes\pi^*(\mathcal{T}_Y).$$

Thus $\pi_*(\mathcal{T}_X)\cong\mathcal{T}_Y\otimes\pi_*(\mathcal{O}_X(-D))$ and $\pi_*^G(\mathcal{T}_X)\cong\mathcal{T}_Y\otimes(\mathcal{O}_Y\cap\pi_*(\mathcal{O}_X(-D)))$. We compute (similarly with [Cornelissen and Kato 2003, Proposition 1.6]):

$$\pi_*^G(\mathcal{T}_X)=\mathcal{T}_Y\otimes\mathcal{O}_Y\left(-\sum_{\mu=1}^{r}\left\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\right\rceil b_i\right).$$

Therefore, the global contribution to $H^1(G,\mathcal{T}_X)$ is given by

$$H^1(Y,\pi_*^G(\mathcal{T}_X)) \cong H^1\Big(Y,\mathcal{T}_Y\otimes\mathcal{O}_Y\Big(-\sum_{\mu=1}^{r}\cdot\Big\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Big\rceil b_i\cdot\Big)\Big)$$
$$\cong H^0\Big(Y,\Omega_Y^{\otimes 2}\Big(-\sum_{\mu=1}^{r}\Big\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Big\rceil b_i\Big)\Big)$$

and, by the Riemann–Roch formula,

$$\dim_k H^1(Y,\pi_*^G(\mathcal{T}_X)) = 3g_Y-3+\sum_{\mu=1}^{r}\Bigg\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Bigg\rceil. \tag{38}$$

The local contribution can be bounded by Proposition 2.10 and by combining the local and global contributions, we arrive at the desired bound for the dimension.

3.1. ***Examples.*** Let $V=\mathbb{Z}/p\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/p\mathbb{Z}$ be an elementary abelian group acted on by the group $\mathbb{Z}/n\mathbb{Z}$. Assume that $G:=V\rtimes\mathbb{Z}/n\mathbb{Z}$ acts on the local field $k((t))$ and assume that the ramification filtration is given by $G_0>G_1=\cdots=G_j>G_{j+1}=\{1\}$. Let $H:=\mathbb{Z}/p\mathbb{Z}$ be the first summand of V. Let σ be a generator of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and assume that $\sigma(t)=\zeta t$, where ζ is a primitive n-th root of one. Let $A=t^{a'}k[[t]](d/dx)$ and let $A^H=x^a k[[x]](d/dx)$. The inflation-restriction sequence implies the short exact sequence

$$0\to H^1(V/H, x^a k[[x]]\tfrac{d}{dx})\to H^1(V,A)\to H^1(H,A)\to 0.$$

The group $\mathbb{Z}/n\mathbb{Z}$ acts on $t^{a'}k[[t]]$ but there is no apriori well defined action of $\mathbb{Z}/n\mathbb{Z}$ on $x^a k[[x]]\frac{d}{dx}=(t^{a'}k[[t]](d/dt))^H$, since the group H might not be normal in G. Here by this action of $\mathbb{Z}/n\mathbb{Z}$ we mean the natural module action and not the conjugation action on cocycles defined in Lemma 1.2. An element $d\in H^1(G/H, x^a k[[x]]\,d/dx)$ is sent by the inflation map on the 1-cocycle $\mathrm{inf}(d)$ that is a map

$$\mathrm{inf}(d): G\to t^{a'}g(t)\frac{d}{dt}\in t^{a'}k[[t]]\frac{d}{dt},$$

and the action of σ can be considered on the image of the inflation map, sending $\mathrm{inf}(d)(g)\mapsto\sigma(\mathrm{inf}(d(g))$. We observe that $\sigma(\mathrm{inf}d(g)))$ is zero for any $g\in H$, by the definition of the inflation map, therefore there is an element $a\in t^{a'}k[[t]]\,d/dt$ such that

$$\sigma(\mathrm{inf}\,d(g))+a^g-a\in x^a k[[x]]\frac{d}{dx}.$$

We can consider the element $\sigma(\mathrm{inf}(d))+a^g-a=\mathrm{inf}(d')$. This means that although there is no well defined action of $\mathbb{Z}/n\mathbb{Z}$ on $k[[x]]$ we can define $\sigma(d)=d'$ modulo cocycles. In what follows we will try to compute the element $d'\in H^1(G/H, x^a k[[x]]\,d/dx)$.

Assume that the Artin–Schreier extension $k((t))/k((x))$ is given by the equation $1/y^p - 1/y = 1/x^j$. Then, we have computed that if g is a generator if H then

$$g(t) = \frac{t}{(1+t^j)^{1/j}} \quad \text{and} \quad x = \frac{t^p}{(1-t^{j(p-1)})^{1/j}}.$$

The action of σ on x, where x is seen as an element in $k[[t]]$ is given by

$$\sigma(x) = \sigma \frac{t^p}{(1-t^{j(p-1)})^{1/j}} = \zeta^p x \frac{(1-t^{j(p-1)})^{1/j}}{(1-\zeta^{j(p-1)}t^{j(p-1)})^{1/j}} = \zeta^p x u,$$

where

$$u = \frac{(1-t^{j(p-1)})^{1/j}}{(1-\zeta^{j(p-1)}t^{j(p-1)})^{1/j}}$$

is a unit of the form $1+y$, where $y \in t^{2j(p-1)}k[[t]]$. The cohomology group $H^1(V/H, x^a k[[x]])$ is generated by the elements $\{1/x^\mu, b \le \mu \le a, \binom{\mu/n}{p-1} = 0\}$. Each element $1/x^\mu$ is written as $1/x^\mu x^{j+1}\, d/dx$ and it is lifted to

$$1/x^\mu t^{j+1} \frac{d}{dt} \xrightarrow{\sigma} \zeta^{-p\mu+j} 1/x^\mu u^{-\mu} t^{j+1} \frac{d}{dt}.$$

In the above formula we have used the fact that the adjoint action of σ on $t^r\, d/dt$ is given by $\sigma : t^r\, d/dt \to \zeta^{(r-1)} t^r\, d/dt$ [Cornelissen and Kato 2003, 3.7]. Obviously the unit u is not H-invariant but we can add to u a 1-coboundary so that it becomes the H-invariant element $\inf(d')$. This coboundary is of the form $a^g - a$, and obviously $a^g - a$ has to be in $t^{2p-1j}k[[t]]$. This gives us that $(1/x^\mu)' = \zeta^\mu 1/x^\mu + o$, where o is a sum of terms $1/x^\nu$ with $-a < \nu$ and therefore o is cohomologous to zero. Using induction one can prove:

Lemma 3.1. *Let $1/\pi_i^{i_\lambda}$, $\lambda = 1, \ldots, s$, $b_i \le i_\lambda \le -a_i$ so that $\binom{i_\lambda/n}{p-1} = 0$ and $b_i = -a_i - j$ if $p \mid a_i$, $b_i = -a_i - j + 1$ if $p \nmid a_i$ be the basis elements of the cohomology group $H^1(V, \mathcal{T}_\mathcal{O})$. Then the action of the generator $\sigma \in \mathbb{Z}/n\mathbb{Z}$ on $\mathcal{T}_\mathcal{O}$ induces the following action on the basis elements*:

$$\sigma\Big(\frac{1}{\pi_i^\mu}\Big) = \zeta^{-p^i\mu+j} \frac{1}{\pi_i^\mu}.$$

The Fermat curve. The curve

$$F : x_0^n + x_1^n + x_2^n = 0$$

defined over an algebraically closed field k of characteristic p, such that $n-1 = p^a$ is a power of the characteristic is a very special curve. Concerning its automorphism group, the Fermat curve has maximal automorphism group with respect to the genus [Stichtenoth 1973]. Also it leads to Hermitian function fields, that are optimal with respect to the number of $\mathbb{F}_{p^{2a}}$-rational points and Weil's bound.

It is known that the Fermat curve is totally supersingular, i.e., the Jacobian variety $J(F)$ of F has p-rank zero, so this curve cannot be studied by the tools of [Cornelissen and Kato 2003]. The group of automorphism of F was computed in [Leopoldt 1996] to be the projective unitary group $G = PGU(3, q^2)$, where $q = p^a = n - 1$. H. Stichtenoth [Stichtenoth 1973, p. 535] proved that in the extension F/F^G there are two ramified points P, Q and one is wildly ramified and the other is tamely ramified. For the ramification group $G(P)$ of the wild ramified point P, the group $G(P)$ consists of the 3×3 matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \chi & 0 \\ \gamma & -\chi\alpha^q & \chi^{1+q} \end{pmatrix}, \tag{39}$$

where $\chi, \alpha, \gamma \in \mathbb{F}_{q^2}$ and $\gamma + \gamma^q = \chi^{1+q} - 1 - \alpha^{1+q}$. Moreover Leopoldt proves that the order of $G(P)$ is $q^3(q^2-1)$ and the ramification filtration is given by

$$G_0(P) > G_1(P) > G_2(P) = \cdots = G_{1+q}(P) > \{1\},$$

where

$$G_1(P) = \ker(\chi : G_0(P) \to \mathbb{F}^*_{q^2}) \quad \text{and} \quad G_2(P) = \ker(\alpha : G_1(P) \to \mathbb{F}_{q^2}).$$

In this section we will compute the dimension of tangent space of the global deformation functor. Namely, we will prove:

Proposition 3.2. *Let p be a prime number, $p > 3$ let X be the Fermat curve*

$$x_0^{1+p} + x_1^{1+p} + x_2^{1+p} = 0.$$

Then $\dim_k H^1(X, G, \mathcal{T}_X) = 0$.

Proof. By the assumption $q = p$ and by the computations of Leopoldt mentioned above we have $G_2 = \cdots = G_{p+1} = \mathbb{Z}/p\mathbb{Z}$. The different of G_{p+1} is computed $(p+2)(p-1)$. Hence, according to (17),

$$\dim_k H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}}) = \left\lfloor \frac{(p+2)(p-1)-(p+2)}{p} \right\rfloor - \left\lceil \frac{-p-2}{p} \right\rceil = p$$

Proposition 2.7 implies that the set

$$\left\{ \frac{1}{t^i}, 2 \le i \le p+2 \text{ where } \binom{i/(p+1)}{p-1} = 0 \right\}$$

is a k-basis of $H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}})$. Indeed, the group G_{1+p} has conductor $1+p$ and $\mathcal{T}_{\mathcal{O}}$ is G_{1+p}-equivariantly isomorphic to $t^{-p-2}k[[t]]$. Thus following the notation of Proposition 2.7 $-a = p+2$ and $b = 2$. The rational number $(1+p)^{-1}$ has the following p-adic expansion:

$$\frac{1}{1+p} = 1 + (p-1)p + (p-1)p^3 + (p-1)p^5 + \dots$$

and using Lemma 2.1 we obtain that for $2 \le i \le p+2$ the only integer i such that $\binom{i/(p+1)}{p-1} \neq 0$ is $i = p-1$. Thus, the elements

$$\left\{ \frac{1}{t^i} \,\middle|\, 2 \le i \le p+2, i \neq p-1 \right\}$$

form a k-basis of $H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}})$.

Leopoldt in [Leopoldt 1996, 4.1] proves that the $G_0(P)$ acts faithfully on the k-vector space $L((p+1)P)$ that is of dimension 3 with basis functions $1, v, w$ and the representation matrix is given by (39). Moreover, the above functions have t-expansions $v = 1/t^p u$, where u is a unit in $k[[t]]$ and $w = 1/t^{p+1}$, for a suitable choice of the local uniformizer t at the point P. The functions v, w generate the function field corresponding to the Fermat curve and they satisfy the relation $v^n = w^n - (w+1)^n$, therefore one can compute that the unit u can be written as

$$u = 1 + t^{p+1} g, \quad g \in k[[t]].$$

Let σ be an element given by a matrix as in Equation (39). The action of $\sigma \in G_1 = G_1(P)$ on powers of $1/t$ is given by

$$\frac{1}{t^i} = \frac{(1+\gamma t^{p+1} - a^q u t)^{i/(p+1)}}{t^i}, \tag{40}$$

and the action on the basis elements $\{1/t^i,\ 2 \le i \le p+2, i \neq p-1\}$ is given by

$$\frac{1}{t^i} \mapsto \frac{1}{t^i} + \sum_{\nu=1}^{i-2} a^{q\nu} \binom{i/(p+1)}{\nu} \frac{1}{t^{i-\nu}}.$$

The matrix of this action is given by

$$A_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{p+1} & 1 & 0 & 0 & 0 \\ & \frac{3}{p+1} & 1 & 0 & 0 \\ * & * & \ddots & 1 & 0 \\ * & * & * & \frac{p+2}{p+1} & 1 \end{pmatrix}.$$

We observe that $\sigma(1/t^2) = 1/t^2$ and $\sigma(1/t^p) = 1/t^p$, and moreover that all elements below the diagonal of the matrix A_σ are $i/(p+1)$ and are nonzero unless

$i = p$. Therefore the eigenspace of the eigenvalue 1 is 2-dimensional,

$$H^1(G_{1+p}, \mathcal{T}_{\mathcal{O}})^{G_1/G_{1+p}} =_k \left\langle \frac{1}{t^2}, \frac{1}{t^p} \right\rangle$$

is a basis for it. To compute $H^1(G_1(P), \mathcal{T}_{\mathcal{O}})$ we consider the exact sequence

$$1 \to G_2 \to G_1 \xrightarrow{\alpha} G_1/G_2 \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1$$

and the corresponding low-degree-term Lyndon–Hochschild–Serre sequence. The group G_2 is of conductor $p+1$ thus $\mathcal{T}_{\mathcal{O}}^{G_2} = \mathcal{T}_{\mathcal{O}}^{\mathbb{Z}/p\mathbb{Z}}$ is given by Proposition 1.15 ($p > 2$):

$$\mathcal{T}_{\mathcal{O}}^{G_2} = x^{p+2-\lfloor (p+2)/p \rfloor} k[[x]] \frac{d}{dx} = x^{p+1} k[[x]] \frac{d}{dx},$$

where x is a local uniformizer for $\mathcal{O}^{G_2}$. By [Serre 1979, Corollary p. 64] the ramification filtration for G_2/G_1 is

$$G_0/G_2 > G_1/G_2 > \{1\},$$

hence the different for the subgroup $\mathbb{Z}/p\mathbb{Z}$ of G_2/G_1 is $2(p-1)$, and the conductor equals 1. Lemma 1.14 implies $x^{p+1}k[[x]]\, d/dx$ is G_1/G_2-equivariantly isomorphic to $x^{p+1-2}k[[[x]]$. Therefore,

$$H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) = H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]]) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, (x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}})$$

We compute

$$\dim_k H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]]) = \left\lfloor \frac{2(p-1)+p-1}{p} \right\rfloor - \left\lceil \frac{p-1}{p} \right\rceil = 1.$$

At the same time, if π is a local uniformizer for $k((x))^{\mathbb{Z}/p\mathbb{Z}}$ then

$$(x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}} = \pi^{\lceil (p-1)/p \rceil} k[[\pi]] = \pi k[[\pi]]$$

and the dimension of the cohomology group is computed:

$$\begin{aligned} \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, (x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}}) &= \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \pi k[[\pi]]) \\ &= \left\lfloor \frac{2(p-1)+1}{p} \right\rfloor - \left\lceil \frac{1}{p} \right\rceil = 0. \end{aligned}$$

Using the bound for the kernel of the transgression we see that

$$\begin{aligned} 1 = \dim_k H^1\left(\frac{G_1}{G_2}, \mathcal{T}_{\mathcal{O}}^{G_2}\right) &\le \dim_k H^1(G_1, \mathcal{T}_{\mathcal{O}}) \\ &\le \dim_k H^1\left(\frac{G_1}{G_2}, \mathcal{T}_{\mathcal{O}}^{G_2}\right) + \dim_k H^1(G_2, \mathcal{T}_{\mathcal{O}})^{G_1/G_2} = 3. \end{aligned} \tag{41}$$

To compute the action of G_0 on G_1/G_2 we observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ a & \chi & 0 \\ * & -\chi a^p & \chi^{1+p} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ * & -b^p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & \chi^{-1} & 0 \\ * & \chi a^p & \chi^{-1-p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \chi b & 1 & 0 \\ * & -\chi^{p+1} b^p & 1 \end{pmatrix}. \quad (42)$$

If the middle matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ * & -b^p & 1 \end{pmatrix}$$

is an element of $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p \subset \mathbb{F}_{p^2}$ then $b^p = b$. By looking at the computation of (42) we see that the conjugation action of G_0/G_1 to $\mathbb{F}_p$ is given by multiplication $b \mapsto \chi^{1+q} b$. Observe that $(\chi^{1+p})^{p-1} = \chi^{p^2-1} = 1$, thus $\chi^{1+p} \in \mathbb{F}_p$. The action on the cocycles is given by sending the cocycle $d(\tau)$ to $d(\sigma\tau\sigma^{-1})^{\sigma^{-1}}$ therefore the basis cocycle $1/x^{1-p}$ of the one dimensional cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]])\ d$ goes to $\chi^{p(p-1)+1+1+p} d = \chi^{-p^2} d$ under the conjugation action, as one sees by applying Lemma 3.1. Lemma 1.3 implies that

$$H^1(G_1/G_2, \mathcal{T}_{\mathbb{O}}^{G_2})^{G_0/G_1} = 0.$$

Similarly the conjugation action of G_0/G_1 on an element of G_2 can be computed to be multiplication of τ by $\chi^{1+p} \in \mathbb{F}_p$, and the same argument shows that $H^1(G_{1+p}, \mathcal{T}_{\mathbb{O}})^{G_1/G_{1+p}\, G_0/G_1} = 0$.

Finally the global contribution is computed by formula (38)

$$\begin{aligned} \dim_k H^1(F^G, \pi_*(\mathcal{T}_F)) &= -3 + \left\lceil \sum_{i=0}^{p+2} \frac{|G(P)_i| - 1}{|G(P)|} \right\rceil + \left\lceil 1 - \frac{1}{|G(Q)|} \right\rceil \\ &= -3 + 2 + 1 = 0. \end{aligned}$$

The fact that the tangent space of the deformation functor is zero dimensional is compatible with the fact that there is only one isomorphism class of curves C such that $|\mathrm{Aut}(C)| \geq 16 g_C^4$ [Stichtenoth 1973]. □

p–Covers of $\mathbb{P}^1(k)$. We consider curves C_f of the form

$$C_f : w^p - w = f(x),$$

where $f(x)$ is a polynomial of degree m. We will say that such a curve is in reduced form if the polynomial $f(x)$ is of the form

$$f(x) = \sum_{i=1,(i,p)=1}^{m-1} a_i x^i + x^m.$$

Two such curves C_f, C_g in reduced form are isomorphic if and only if $f = g$. The group $G := \mathrm{Gal}\big(C_f/\mathbb{P}^1(k)\big) \cong \mathbb{Z}/p\mathbb{Z}$ acts on C_f. The number of independent monomials $\neq x^m$ in the sums above is given by

$$m - \left\lfloor \frac{m}{p} \right\rfloor - 1, \tag{43}$$

since $\#\{1 \le i \le m,\ p \mid i\} = \lfloor m/p \rfloor$.

We will compute the tangent space of the deformation functor of the curve C_f together with the group C_f. Let P be the point above $\infty \in \mathbb{P}^1(k)$. This is the only point that ramifies in the cover $C_f \to \mathbb{P}^1(k)$, and the group G admits the ramification filtration

$$G_0 = G_1 = G_2 = \cdots = G_m > G_{m+1} = \{1\}.$$

The different is computed $(p-1)(m+1)$ and $\mathcal{T}_{\mathbb{O}} \cong t^{-m-1}k[[t]]$. Thus the space $H^1(G, \mathcal{T}_{\mathbb{O}})$ has dimension

$$d = \left\lfloor \frac{(p-1)(m+1)-(m+1)}{p} \right\rfloor - \left\lceil \frac{-(m+1)}{p} \right\rceil = m+1 - \left\lceil \frac{2m+2}{p} \right\rceil + \left\lfloor \frac{m+1}{p} \right\rfloor.$$

Let $a_0 + a_1 p + a_2 p^2 + \cdots$ be the p-adic expansion of $m+1$. We observe that

$$\left\lceil \frac{2m+2}{p} \right\rceil - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lceil \frac{2a_0}{p} + \sum_{i \ge 1} 2a_i p^{i-1} \right\rceil - \sum_{i \ge 1} a_i p^{i-1},$$

therefore, if $p \nmid m+1$

$$\left\lceil \frac{2m+2}{p} \right\rceil - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lfloor \frac{m+1}{p} \right\rfloor + \delta, \quad \text{where } \delta = \begin{cases} 2 & \text{if } 2a_0 > p, \\ 1 & \text{if } 2a_0 < p. \end{cases}$$

Thus, we have for the dimension

$$d = \begin{cases} m+1 - \lfloor (m+1)/p \rfloor & \text{if } p \mid m+1, \\ m - \lfloor (m+1)/p \rfloor - \delta & \text{otherwise.} \end{cases}$$

Finally, we compute that

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = -3 + \left\lceil \frac{(m+1)(p-1)}{p} \right\rceil = m - 2 - \left\lfloor \frac{m+1}{p} \right\rfloor.$$

Lehr–Matignon curves. Consider the curve

$$C : y^p - y = \sum_{i=0}^{m-1} t_i x^{1+p^i} + x^{1+p^m} \tag{44}$$

defined over an algebraically closed field k of characteristic $p > 2$. Such curves were examined in [van der Geer and van der Vlugt 1992] in connection with coding theory, and their automorphism group was studied in [Lehr and Matignon 2005], (The extreme Fermat curves studied in example 1 can be written in this form after a suitable transformation; see [Stichtenoth 1993, VI.4.3, p. 203].)

Let $n = 1 + p^m$ denote the degree of the right-hand side of (44), and set $H = \mathrm{Gal}(C/\mathbb{P}^1(k))$. The automorphism group G of C can be expressed in the form

$$1 \to H \to G \to V \to 1,$$

where V is the vector space of roots of the additive polynomial

$$\sum_{0 \le i \le m} \left(t_i^{p^{m-i}} Y^{p^{m-i}} + t_i^{p^m} Y^{p^{m+i}} \right) \tag{45}$$

[Lehr and Matignon 2005, Proposition 4.15]. Moreover there is only one point $P \in C$ that ramifies in the cover $C \to C^G$, namely the point above $\infty \in \mathbb{P}^1(k)$.

In order to simplify the calculations we assume that $t_0 = \cdots = t_{m-1} = 0$ so the curve is given by

$$y^p - y = x^{p^m+1}. \tag{46}$$

The polynomial in (45) is given by $Y^{p^{2m}} + Y$ and the vector space V of the roots is $2m$-dimensional. Moreover, according to [Lehr and Matignon 2005] any automorphism σ_v corresponding to $v \in V$ is given by

$$\sigma_v(x) = x + v, \quad \sigma(y) = y + \sum_{\kappa=0}^{m-1} v^{p^{m+k}} x^{p^k}.$$

Observe that w and x have a unique pole of order $p^m + 1$ and p, respectively, at the point above ∞, so we can select the local uniformizer π so that

$$y = \frac{1}{\pi^{p^m+1}}, \quad x = \frac{1}{\pi} u,$$

where u is a unit in $k[[\pi]]$. By replacing x, y in (46) we observe that the unit u is of the form $u = 1 + \pi^{p^m}$.

A simple computation based on the basis $\{1, x, \ldots, x^{p^{m-1}}, y\}$, of the vector space $L\big((1 + p^m)P\big)$ given in [Lehr and Matignon 2005, Proposition 3.3]. shows that the ramification filtration of G is $G = G_0 = G_1 > G_2 = \ldots = G_{p^m+1} > \{1\}$, where $G_2 = H$ and $G_1/G_2 = V$. Using Proposition 2.7 we obtain the basis

$$\left\{ \frac{1}{\pi_1^i} \;\middle|\; 2 \le i \le p^m + 2 \text{ and } \binom{i/(p^m+1)}{p-1} = 0 \right\}$$

for $H^1(G_2, \mathcal{T}_{\mathbb{O}})$. We have to study the action of G_1/G_2 on $H^1(G_2, \mathcal{T}_{\mathbb{O}})$. From the action of σ_v on y we obtain that the action on the basis elements of $H^1(G_{p^m+1}, \mathcal{T}_{\mathbb{O}})$

is given by

$$\sigma_v\Big(\frac{1}{\pi^i}\Big) = \frac{1}{\pi^i} + \frac{\big(\sum_{\kappa=0}^{m-1} u^{p^k} v^{p^{m+\kappa}} \pi^{p^m+1-p^{k+1}}\big)^{i/(p^m+1)}}{\pi^i}$$
$$= \frac{1}{\pi^i} + \frac{i}{p^m+1} v^{p^{2m-1}} \frac{1}{\pi^{i-1}} + \dots .$$

If $p \mid i$, all binomial coefficients $\binom{i/(p^m+1)}{\kappa}$ that contribute a coefficient $1/\pi^\kappa$, $2 \le \kappa \le p^m+2$ are zero. Therefore, the elements $1/\pi^2$, $1/\pi^{\nu p}$ are invariant. Moreover, by writing down the action of σ_v as a matrix we see that there are no other invariant elements, so the dimension is computed ($p > 2$):

$$\dim_k H^1(G_{p^m+1}, \mathcal{T}_{\mathcal{O}})^{G_1/G_{p^m+1}} = 1 + \Big\lfloor \frac{p^m+2}{p} \Big\rfloor = 1 + p^{m-1}.$$

This dimension coincides with the computation done on the Fermat curves $m = 1$.

We proceed by computing $H^1(V, \mathcal{T}_{\mathcal{O}}^H)$. The space $\mathcal{T}_{\mathcal{O}}^H$ is computed by Proposition 2.9

$$x^{p^m+2-\lfloor (p^m+2)/p \rfloor} k[[x]] \frac{d}{dx} = x^{p^m+2-p^{m-1}} k[[x]] \frac{d}{dx}.$$

Thus,

$$\dim_k H^1(V, \mathcal{T}_{\mathcal{O}}^H) = \sum_{\nu=1}^{2m} \left(\Big\lfloor \frac{2(p-1)+a_i}{p} \Big\rfloor - \Big\lceil \frac{a_i}{p} \Big\rceil \right),$$

where $a_1 = p^m - p^{m-1}$, and $a_i = \lceil a_{i-1}/p \rceil$. By computation $a_\nu = p^{m-\nu+1} - p^{m-\nu}$ for $1 \le \nu \le m$, and $a_\nu = 1$ for $\nu > m$. Moreover, an easy computation shows that

$$\Big\lfloor \frac{2(p-1)+a_i}{p} \Big\rfloor - \Big\lceil \frac{a_i}{p} \Big\rceil = \begin{cases} 1 & \text{if } 1 \le \nu < m, \\ 2 & \text{if } \nu = m, \\ 0 & \text{if } m < \nu, \end{cases}$$

thus the dimension of the tangent space is $m + 1$.

We have proved that the dimension of $H^1(G_1, \mathcal{T}_{\mathcal{O}})$ is bounded by

$$m+1 = \dim_k H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) \le H^1(G_1, \mathcal{T}_{\mathcal{O}})$$
$$\le \dim_k H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) + H^1(G_2, \mathcal{T}_{\mathcal{O}})^{G_1/G_2} = 2 + m + p^{m-1}.$$

Unfortunately we cannot be more precise here: an exact computation involves the computation of the kernel of the transgression and such a computation requires new ideas and tools.

To this dimension we must add the contribution of

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + \sum_{\kappa=1}^{r} \left\lceil \sum_{i=0}^{n_\kappa} \frac{(e_i^{(\kappa)} - 1)}{e_0^{(\kappa)}} \right\rceil$$

$$= -3 + \left\lceil 2\frac{p^{m+1}-1}{p^{m+1}} + m\frac{p^m - 1}{p^{m+1}} \right\rceil = -1 + \left\lceil \frac{m}{p} - \frac{2+m}{p^{m+1}} \right\rceil.$$

The latter contribution is strictly positive if $m \gg p$.

Elementary abelian extensions of $\mathbb{P}^1(k)$. Consider the curve C so that $G_0 = (\mathbb{Z}/p\mathbb{Z})^s \rtimes \mathbb{Z}/n\mathbb{Z}$ is the ramification group of wild ramified point, and moreover the ramification filtration is given by

$$G_0 > G_1 = \ldots = G_j > G_{j+1} = \{1\}. \tag{47}$$

An example of such a curve is provided by the curve defined by

$$C : \sum_{i=0}^{s} a_i y^{p^i} = f(x), \tag{48}$$

where f is a polynomial of degree j and all monomial summands $a_k x^k$ of f have exponent congruent to j modulo n. Let V be the $\mathbb{F}_p$-vector space of the roots of the additive polynomial $\sum_{i=0}^{s} a_i y^{p^i}$. Assume that the automorphism group of the curve defined by (48) is $G := V \rtimes \mathbb{Z}/n\mathbb{Z}$. Thus $C \to \mathbb{P}^1(k)$ is Galois cover ramified only above ∞, with ramification group G and ramification filtration is computed to be as in (47).

Let us now return to the general case. Let us denote by V the group $(\mathbb{Z}/p\mathbb{Z})^s$. The group V admits the structure of a $\mathbb{F}_p$ vector space, where $\mathbb{F}_p$ is the finite field with p elements. The conjugation action of $\mathbb{Z}/n\mathbb{Z}$ on V implies a representation

$$\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(V).$$

Since $(n, p) = 1$, Mascke's Theorem gives that V is the direct sum of simple $\mathbb{Z}/n\mathbb{Z}$-modules, i.e., $V = \bigoplus_{i=1}^{r} V_i$. On the other hand, Lemma 1.2 implies that the conjugation action is given by multiplication by ζ^j, where ζ is an appropriate primitive n-th root of one and j is the conductor of the extension. If $\zeta^j \in \mathbb{F}_p$ then all the V_i are one dimensional. In the more general case one has to consider representations

$$\rho_i : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(V_i),$$

where $\dim_{\mathbb{F}_p} V_i = d$. The dimension d is the degree of the extension $\mathbb{F}_q/\mathbb{F}_p$, where $\mathbb{F}_q$ is the smallest field containing ζ^j. Let $e_1^{(i)}, \ldots, e_d^{(i)}$ be an $\mathbb{F}_p$-basis of V_i, and denote by $(a_{\mu\nu}^{(i)})$ the entries of the matrix corresponding to $\rho_i(\sigma)$, where σ is a generator of $\mathbb{Z}/n\mathbb{Z}$. The conjugation action on the arbitrary

$$v = \sum_{i=1}^{r} \sum_{\mu=1}^{d} \lambda_{\mu}^{(i)} e_{\mu}^{(i)} \in V \tag{49}$$

is described by

$$\sigma e_{\mu}^{(i)} \sigma^{-1} = \sum_{\nu=1}^{d} a_{\mu\nu}^{(i)} e_{\nu}^{(i)}. \tag{50}$$

For the computation of $H^1(G, \mathcal{T}_{\mathbb{O}})$, we notice first that the group $H^1(V, \mathcal{T}_{\mathbb{O}})$ can be computed using Proposition 2.8 and the isomorphism $\mathcal{T}_{\mathbb{O}} \cong t^{-j-1}k[[t]]$.

Next we consider the conjugation action of $\mathbb{Z}/n\mathbb{Z}$ on $H^1(V, \mathcal{T}_{\mathbb{O}})$, in order to compute $H^1(G, \mathcal{T}_{\mathbb{O}}) = H^1(V, \mathcal{T}_{\mathbb{O}})^{\mathbb{Z}/n\mathbb{Z}}$. By (32) we have

$$H^1(V, \mathcal{T}_{\mathbb{O}}) = \bigoplus_{\lambda=1}^{s} H^1\big(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_{\mathbb{O}}^{\bigoplus_{\nu=1}^{\lambda-1} \mathbb{Z}/p\mathbb{Z}}\big), \tag{51}$$

i.e., the arbitrary cocycle d representing a cohomology class in $H^1(V, \mathcal{T}_{\mathbb{O}})$ can be written as a sum of cocycles d_ν representing cohomology classes in

$$H^1\big(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_{\mathbb{O}}^{\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}}\big).$$

Let us follow a similar to (49) notation for the decomposition of d, and write $d = \sum_{i=1}^{r} \sum_{\nu=1}^{d} b_{\nu}^{(i)} d_{\nu}^{(i)}$, where $d_{\nu}^{(i)}(e_{\mu}^{(j)}) = 0$ if $i \neq j$ or $\nu \neq \mu$. Therefore,

$$d(\sigma e_{\mu}^{(i)} \sigma^{-1}) = d\left(\sum_{\nu=1}^{d} a_{\mu\nu}^{(i)} e_{\nu}^{(i)}\right) = \sum_{\nu=1}^{d} b_{\nu}^{(i)} a_{\mu\nu}^{(i)} d_{\nu}^{(i)}(e_{\nu}^{(i)}). \tag{52}$$

We have now to compute the $\mathbb{Z}/n\mathbb{Z}$-action on $d_k^{(i)}$. By Lemma 3.1 the element σ acts on the basis elements $1/\pi_i^{\mu}$ of $H^1(V, \mathcal{T}_{\mathbb{O}})$ as follows

$$\sigma\left(\frac{1}{\pi_i^{\mu}}\right) = \zeta^{-p^i \mu + j} \frac{1}{\pi^{\mu}}. \tag{53}$$

By the remarks above we arrive at

$$\sigma(d)(e_{\mu}^{(i)}) := d(\sigma e_{\mu}^{(i)} \sigma^{-1})^{\sigma^{-1}} = \sum_{\nu=1}^{d} b_{\nu}^{(i)} a_{\mu\nu}^{(i)} \zeta^{-c(\nu,i)} d_{\nu}^{(i)}(e_{\nu}^{(i)}), \tag{54}$$

where $c(\nu, i)$ is the appropriate exponent, defined in (53). Let us denote by $A^{(i)}$ the $d \times d$ matrix $(a_{\nu\mu}^{(i)})$. By (54) $\sigma(d)(e_{\mu}^{(i)}) = d(e_{\mu}^{(i)})$ if and only if $b := (b_1^{(i)}, \ldots, b_d^{(i)})$ is a solution of the linear system

$$(A^{(i)} \cdot \mathrm{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \ldots, \zeta^{c(d,i)}) - \mathbb{I}_d)b = 0.$$

This proves that the dimension of the solution space is equal to the dimension of the eigenspace of the eigenvalue 1 of the matrix: $A^{(i)}\operatorname{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \dots, \zeta^{c(d,i)})$.

Moreover using a basis of the form $1, \zeta, \zeta^2, \dots, \zeta^{d-1}$ for the simple space $V(i)$, we obtain that

$$A^{(i)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ 0 & & 1 & 0 & -a_{d-1} \\ 0 & & & 1 & -a_d \end{pmatrix}$$

It can be proved by induction that the characteristic polynomial of $A^{(i)}$ is $x^d + \sum_{\nu=0}^{d-1} a_\nu x^\nu$, and under an appropriate basis change $A^{(i)}$ can be a written in the form $\operatorname{diag}(\zeta^j, \zeta^{2j}, \dots, \zeta^{j(d-1)})$. Moreover, the characteristic polynomial of the matrix $A^{(i)}\operatorname{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \dots, \zeta^{c(d,i)})$ can be computed inductively to be

$$f_i(x) := x^d + \zeta^{c(d,i)} a_{d-1} x^{d-1} + \zeta^{c(d,i)+c(d-1,i)} a_{d-1} + \cdots + \zeta^{\sum_{\nu=2}^{d} c(\nu,i)} x + \zeta^{\sum_{\nu=1}^{d} c(\nu,i)} a_0.$$

If, $f_i(1) \neq 0$, then we set $\delta(i) = 0$. If $f_i(1) = 0$ we set $\delta(i)$ to be the multiplicity of the root 1. The total invariant space has dimension

$$\dim_k H^1(G, \mathcal{T}_{\mathbb{O}}) = \sum_i \delta(i).$$

Comparison with the work of Cornelissen–Kato. We will apply the previous calculation to the case of ordinary curves $j = 1$ and we will obtain the formulas in [Cornelissen and Kato 2003]. We will follow the notation of Proposition 2.8. The number $a_1 = -j - 1 = -2$. Thus, $a_2 = \lceil -2/p \rceil = -\lfloor 2/p \rfloor = 0$ (recall that we have assumed that $p \geq 5$). Furthermore $a_i = 0$ for $i \geq 2$. For the numbers b_i we have $b_1 = -a_1 - j + 1 = 2$, and $b_1 \leq i_1 \leq -a_1$, so there is only one generator, namely $1/\pi_1^2$. Moreover, for $i \geq 2$ we have $b_i = -a_j - j = -1$ and there are two possibilities for $-1 \leq i_\lambda \leq 0 = -a_i$, namely $-1, 0$. But only $\binom{0/n}{p-1} = 0$, and we finally obtain

$$H^1(V, \mathcal{T}_{\mathbb{O}}) \cong \left\langle \frac{1}{\pi_1^2} \right\rangle_k \times \langle 1 \rangle_k \times \cdots \times \langle 1 \rangle_k,$$

a space of dimension $\log_p |V|$.

Let d be the dimension of each simple direct summand of $H^1(V, \mathcal{T}_{\mathbb{O}})$ considered as a $\mathbb{Z}/n\mathbb{Z}$-module. Of course d equals the degree of the extension $\mathbb{F}_p(\zeta)/\mathbb{F}_p$, where ζ is a suitable primitive root of 1. For the matrix $\operatorname{diag}(\zeta^{c(1,i)}, \dots, \zeta^{c(d,i)})$ we have

$$\operatorname{diag}(\zeta^{c(1,i)}, \dots, \zeta^{c(d,i)}) = \begin{cases} \operatorname{diag}(\zeta^2, \zeta, \dots, \zeta) & \text{if } i = 1, \\ \zeta \cdot \mathbb{I}_d & \text{if } i \geq 2. \end{cases}$$

The characteristic polynomial in the first case is computed to be

$$f_1(x) = x^d + \sum_{\nu=1}^{d-1} \zeta^{d-\nu} a_\nu x^\nu + a_0 \zeta^{1+d}.$$

Setting $x = \zeta y$ this becomes

$$\zeta^d \left(y^d + \sum_{\nu=0}^{d-1} a_\nu y^\nu \right) + (\zeta^{d+1} - \zeta^d) a_0.$$

Therefore $y = \zeta$ for $x = 1$, so $f_1(1) = (\zeta^{d+1} - \zeta^d) a_0 \neq 0$, therefore $\delta(1) = 0$.

In the second case, we observe that

$$f_i(x) = x^d + \sum_{\nu=0}^{d-1} \zeta^{d-\nu} a_\nu x^\nu.$$

If we set $x = y/\zeta$, we obtain that 1 is a simple root of f_i, so $\delta(i) = 1$ for $i \geq 2$.

Thus, only the $s/d - 1$ blocks $i \geq 2$ admit invariant elements and

$$\dim_k H^1(V \rtimes \mathbb{Z}/n\mathbb{Z}, \mathcal{T}_{\mathbb{O}}) = s/d - 1.$$

The global contribution can be computed in terms of (38) and gives us that

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + 2r_w + r_t - \sum_{\mu=1}^{r} \left\lceil \frac{1}{n_i} \left(1 + \frac{1}{p^{s_i}} \right) \right\rceil,$$

where n_i is the order of the prime to p part and p^{s_i} is the order of the p-part of the decomposition group at the i-th ramification point of the cover $X \to X/G = Y$. The numbers r_w, r_t are the number of wild,tame ramified points of the above cover, respectively.

Comparison with the work of R. Pries. Consider the curve

$$C : y^p - y = f(x),$$

where $f(x)$ is a polynomial of degree j, $(j, p) = 1$. This, gives rise to a ramified cover of $\mathbb{P}^1(k)$ with ∞ as the unique ramification point. Moreover if all the monomial summands of the polynomial $f(x)$ have exponents congruent to $j \bmod m$, then the curve C admits the group $G := \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$ as a subgroup of the group of automorphisms. R. Pries [2002] constructed a configuration space C of deformations of the above curve and computed the dimension C. More precisely by the term *configuration space* we mean a k-scheme C that represents a contravariant functor F from the category of irreducible k-schemes S to the category of sets so that, there is a morphism $T : \mathrm{Hom}(\cdot, C) \to F(\cdot)$ so that it induces a bijection between the k-points of the configuration space C and $F(\mathrm{Spec}(k))$, and if $\phi_S \in F(S)$ then there is a finite radical morphism $i : S' \to S$ and a unique morphism $f : S' \to C$ such that $T(f) = i^*\phi_S$. Pries considered the functor $F_{G,j}$ from irreducible k-schemes S to sets, defined as follows: $F_{G,j}$ is the set of equivalence classes of G–Galois covers $\phi_S : Y_S \to \underline{\mathrm{Spec}}(\mathbb{O}_S[[u^{-1}]])$ ramified only above the horizontal divisor ∞_S defined by $u^{-1} = 0$ and with constant jump j. Two such covers ϕ_S, ϕ'_S are considered to be equivalent if they are isomorphic after pullback by a finite radical morphism $S' \to S$. We refer the reader to the article of Pries for more information about this configuration space C. Pries [2002] proved that C is of dimension

$$r := \#\{e \in E_0 : \text{ for all } \nu \in \mathbb{N}^+,\, p^\nu e \notin E_0\} \tag{55}$$

where $E_0 := \{e : 1 \le e \le j, e \equiv j \bmod m\}$. Notice that by considering equivalence classes of G-covers we state a local version of the G-deformation problem. Moreover since we assume that the jump remains constant we are considering deformations that do not split the branch locus. It would be interesting to compare the result of Pries to our computation of $H^1(G, \mathcal{T}_\mathbb{O})$ at a wild ramified point.

We calculate $\dim_k(G, \mathcal{T}_\mathbb{O})$ as follows: According to Proposition 2.7 the tangent space of the deformation space is generated as a k-vector space by the elements of the form $1/x^i$ where $b \le i \le j+1$ and

$$b = \begin{cases} 1 & \text{if } p \mid -j-1, \\ 2 & \text{if } p \nmid -j-1. \end{cases}$$

By Lemma 1.4 the $\mathbb{Z}/m\mathbb{Z}$-action on $\mathbb{F}_p$ is given by multiplication by ζ^j where ζ is an appropriate primitive m-th root of unity. This gives us that $\zeta^{jp} = \zeta^j$, i.e. $jp \equiv j \bmod m$. If d_i is the cocycle corresponding to the element $1/x^i$ then

$$d_i(\sigma\tau\sigma^{-1})^{\sigma^{-1}} = \zeta^j d_i(\tau)^{\sigma^{-1}}.$$

But the element $1/x^i$ corresponds to the element $x^{j+1-i}\, d/dx$. The ζ^{-1}-action is given by

$$x^{j+1-i}\frac{d}{dx} \mapsto \zeta^{i-j} x^{j+1-i}\frac{d}{dx}.$$

Therefore, the action of σ on the cocycle corresponding to $1/x^i$ is given by $1/x^i \mapsto \zeta^i(1/x^i)$. Thus, $\dim_k H^1(G, \mathcal{T}_\mathbb{O}) = \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_\mathbb{O})^{\mathbb{Z}/p\mathbb{Z}}$ is equal to

$$\#\Big\{ i : b \le i \le j+1,\ \binom{i/j}{p-1} = 0,\ i \equiv 0 \bmod m. \Big\} \tag{56}$$

By (38) we have

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + \sum_{\kappa=1}^{r} \left\lceil \sum_{i=0}^{n_\kappa} \frac{(e_i^{(\kappa)} - 1)}{e_0^{(\kappa)}} \right\rceil,$$

and by computation we get

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = -3 + \left\lceil 1 + \frac{-1}{mp} + \frac{j(p-1)}{mp} \right\rceil.$$

The dimension formulas of (55) and (56) look quite different, but using Maple[1] we computed the table

p	j	m	r	$\dim_k H^1(G, \mathcal{T}_\mathbb{O})$	$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X))$	$\dim_k D(k[\epsilon])$
13	19	6	3	3	1	4
13	35	6	5	4	9	13
13	51	6	8	8	6	14
13	36	3	12	11	10	21
7	81	3	24	23	22	45
7	90	3	26	26	24	50

We observe that $r+a = \dim_k H^1(G, \mathcal{T}_\mathbb{O})$, where $a = 1, 0$, and also the dimension of $H^1(Y, \pi_*^G(\mathcal{T}_X))$ is near the two values above. By the difference of the formulas and by the fact that all infinitesimal deformations in $H^1(Y, \pi_*^G(\mathcal{T}_X))$ are unobstructed we obtain that the difference in the dimensions r and $\dim_k D(k[\epsilon])$ can be explained either as obstructed deformations or as deformations splitting the branch points; see Section 1.1.

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kontogar@aegean.gr *Department of Mathematics, University of the Aegean, Karlovassi Samos, 83200*
http://eloris.samos.aegean.gr

Surfaces over a p-adic field with infinite torsion in the Chow group of 0-cycles

Masanori Asakura and Shuji Saito

We give an example of a projective smooth surface X over a p-adic field K such that for any prime ℓ different from p, the ℓ-primary torsion subgroup of $\mathrm{CH}_0(X)$, the Chow group of 0-cycles on X, is infinite. A key step in the proof is disproving a variant of the Bloch–Kato conjecture which characterizes the image of an ℓ-adic regulator map from a higher Chow group to a continuous étale cohomology of X by using p-adic Hodge theory. With the aid of the theory of mixed Hodge modules, we reduce the problem to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. Another key ingredient is the injectivity result on the cycle class map for Chow group of 1-cycles on a proper smooth model of X over the ring of integers in K, due to K. Sato and the second author.

1. Introduction

Let X be a smooth projective variety over a base field K and let $\mathrm{CH}^m(X)$ be the Chow group of algebraic cycles of codimension m on X modulo rational equivalence. In case K is a number field, there is a folklore conjecture that $\mathrm{CH}^m(X)$ is finitely generated, which in particular implies that its torsion part $\mathrm{CH}^m(X)_{\mathrm{tor}}$ is finite. The finiteness question has been intensively studied by many authors, particularly for the case $m = 2$ and $m = \dim(X)$; see the nice surveys [Otsubo 2001; Colliot-Thélène 1995].

When K is a p-adic field (namely the completion of a number field at a finite place), Rosenschon and Srinivas [2007] have constructed the first example where $\mathrm{CH}^m(X)_{\mathrm{tor}}$ is infinite. They prove that there exists a smooth projective fourfold X over a p-adic field such that the ℓ-torsion subgroup $\mathrm{CH}_1(X)[\ell]$ (see Notation on p. 166) of $\mathrm{CH}_1(X)$, the Chow group of 1-cycles on X, is infinite for each $\ell \in \{5, 7, 11, 13, 17\}$.

This paper gives an example of a projective smooth surface X over a p-adic field such that for any prime ℓ different from p, the ℓ-primary torsion subgroup

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$\mathrm{CH}_0(X)\{\ell\}$ (see Notation on p. 166) of $\mathrm{CH}_0(X)$, the Chow group of 0-cycles on X, is infinite. Here we note that for X as above, $\mathrm{CH}_0(X)\{\ell\}$ is known to always be of finite cotype over $\mathbb{Z}_\ell$, namely the direct sum of a finite group and a finite number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. This fact follows from Bloch's exact sequence (2-3). Thus our example presents infinite phenomena of different nature from the example in [Rosenschon and Srinivas 2007]. Another noteworthy point is that the phenomena discovered in our example happen rather *generically*.

To make it more precise, we prepare a notion of "generic surfaces" in $\mathbb{P}^3$. Let

$$M \subset \mathbb{P}\big(H^0(\mathbb{P}^3_{\mathbb{Q}}, \mathbb{O}_{\mathbb{P}}(d)\big) \cong \mathbb{P}_{\mathbb{Q}}^{(d+3)(d+2)(d+1)/6-1}$$

be the moduli space over $\mathbb{Q}$ of the nonsingular surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ (the subscript $\mathbb{Q}$ indicates the base field), and let

$$f : \mathscr{X} \longrightarrow M$$

be the universal family over M. For $X \subset \mathbb{P}^3_K$, a nonsingular surface of degree d defined over a field K of characteristic zero, there is a morphism $t : \mathrm{Spec} K \to M$ such that $X \cong \mathscr{X} \times_M \mathrm{Spec} K$. We call X *generic* if t is dominant, that is, t factors through the generic point of M. In other words, X is generic if it is defined by an equation

$$F = \sum_I a_I z^I, \quad (a_I \in K)$$

satisfying the following condition:

(∗) $a_I \neq 0$ for all I and $\{a_I/a_{I_0}\}_{I \neq I_0}$ are algebraically independent over $\mathbb{Q}$ where $I_0 = (1, 0, 0, 0)$.

Here $[z_0 : z_1 : z_2 : z_3]$ is the homogeneous coordinate of $\mathbb{P}^3$, $I = (i_0, \ldots, i_3)$ are multiindices and $z^I = z_0^{i_0} \cdots z_3^{i_3}$.

The main theorem is

Theorem 1.1. *Let K be a finite extension of $\mathbb{Q}_p$ and $X \subset \mathbb{P}^3_K$ a nonsingular surface of degree $d \geq 5$. Suppose that X is generic and has a projective smooth model $X_{\mathbb{O}_K} \subset \mathbb{P}^3_{\mathbb{O}_K}$ over the ring $\mathbb{O}_K$ of integers in K. Let r be the Picard number* (*that is the rank of the Néron–Severi group*) *of the smooth special fiber of $X_{\mathbb{O}_K}$. Then we have*

$$\mathrm{CH}_0(X)\{\ell\} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r-1} \oplus (\textit{finite group})$$

for $\ell \neq p$.

One can construct a surface with infinite torsion in the Chow group of 0-cycles in the following way. Let k be the residue field of K. Let Y be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3_k$ defined by an equation $\sum_I c_I z^I$ $(c_I \in k)$ such that the Picard number $r \geq 2$. There exist such surfaces for each d. (For example if $(p, d) = 1$, one may choose a Fermat type surface defined by $z_0^d - z_1^d + z_2^d - z_3^d$. Then the intersection of Y with the hyperplane $H \subset \mathbb{P}_k^3$ defined by $z_0 - z_1$ is not irreducible, so $r \geq 2$.) Take any lifting $\tilde{c}_I \in O_K$ and choose $a_I \in O_K$ with $\mathrm{ord}(a_I) > 0$ for each index I such that $\{a_I\}_I$ are algebraically independent over $\mathbb{Q}(\tilde{c}_I)$, the subfield of K generated over $\mathbb{Q}$ by $\tilde{c}_I$ for all I. Let $X \subset \mathbb{P}_K^3$ be the surface defined by the equation $\sum_I \tilde{c}_I z^I + \sum_I a_I z^I$. Then it is clear that X is generic and has a smooth projective model over O_K whose the special fiber is Y . Since Y has the Picard number $r \geq 2$, $\mathrm{CH}_0(X)$ has an infinite torsion subgroup by Theorem 1.1. It is proved in [Raskind 1989] that if the special fiber satisfies the Tate conjecture for divisors, the geometric Picard number is congruent to d modulo 2. Thus if d is even, $\mathrm{CH}_0(X)$ has an infinite torsion subgroup after a suitable unramified base change. Theorem 1.1 may be compared with the finiteness results [Colliot-Thélène and Raskind 1991] and [Raskind 1989] on $\mathrm{CH}_0(X)_{\mathrm{tor}}$ for a surface X over a p-adic field under the assumption that $H^2(X, \mathbb{O}_X) = 0$ or, more generally, that the rank of the Néron–Severi group does not change by reduction. For a nonsingular surface $X \subset \mathbb{P}_K^3$ of degree $d \geq 1$, the last condition is satisfied if $d \leq 3$. Hence Theorem 1.1 leaves us an interesting open question whether there is an example of a nonsingular surface $X \subset \mathbb{P}_K^3$ of degree 4 for which $\mathrm{CH}_0(X)\{\ell\}$ is infinite.

A distinguished role is played in the proof of Theorem 1.1 by the ℓ-adic regulator map

$$\rho_X : \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_\ell \longrightarrow H^1_{\mathrm{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) \quad (X_{\overline{K}} = X \times_K \overline{K})$$

from higher Chow group to continuous étale cohomology [Jannsen 1988], where $\overline{K}$ is an algebraic closure of K and ℓ is a prime different from $\mathrm{ch}(K)$. It is known that the image of ρ_X is contained in the subspace

$$H^1_g(\mathrm{Spec}(K), V) \subset H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) \quad \big(V = H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big)$$

introduced by Bloch and Kato [1990]. If $\ell \neq p$ this is obvious since $H^1_g = H^1$ by definition. For $\ell = p$ this is a consequence of a fundamental result in p-adic Hodge theory, which confirms that every representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ arising from the cohomology of a variety over K is a de Rham representation; see the discussion after [Bloch and Kato 1990, (3.7.4)].

When K is a number field or a p-adic field, it is proved in [Saito and Sato 2006a] that in case the image of ρ_X coincides with $H^1_g(\mathrm{Spec}(K), V)$, $\mathrm{CH}^2(X)\{\ell\}$ is finite. Bloch and Kato conjecture that it should be always the case if K is a number field.

The first key step in the proof of Theorem 1.1 is to disprove the variant of the Bloch–Kato conjecture for a generic surface $X \subset \mathbb{P}_K^3$ over a p-adic field K (see Theorem 3.6). In terms of Galois representations of $G_K = \mathrm{Gal}(\overline{K}/K)$, our result implies the existence of a 1-extension of $\mathbb{Q}_\ell$-vector spaces with continuous G_K-

action

$$0 \to H^2\big(X_{\overline{K}}, \mathbb{Q}_\ell(2)\big) \to E \to \mathbb{Q}_l \to 0, \tag{1-1}$$

such that E is a de Rham representation of G_K but that there is no 1-extension of motives over K,

$$0 \to h^2(X)(2) \to M \to h\big(\mathrm{Spec}(K)\big) \to 0,$$

which gives rise to (1-1) under the realization functor. The rough idea of the proof of the first key result is to relate the ℓ-adic regulator map to an analytic regulator map by using the comparison theorem for étale and analytic cohomology and then to show that the analytic regulator map is the zero map. With the aid of the theory of mixed Hodge modules [Saito 1990], this is reduced to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. This is done in Section 3 after in Section 2, we review some basic facts on the cycle class map for higher Chow groups.

Another key ingredient is the injectivity result on the cycle class map for the Chow group of 1-cycles on a proper smooth model of X over the ring O_K of integers in K due to Sato and the second author [Saito and Sato 2006b]. It plays an essential role in deducing the main result, Theorem 1.1 from the first key result, which is done in Section 4.

Finally, in the Appendix, we will apply our method to produce an example of a curve C over a p-adic field such that $SK_1(C)_{\mathrm{tor}}$ is infinite.

Notation. For an abelian group M, we denote by $M[n]$ (respectively M/n) the kernel (respectively cokernel) of multiplication n. For a prime number ℓ we put

$$M\{\ell\} := \bigcup_n M[\ell^n], \qquad M_{\mathrm{tor}} := \bigoplus_\ell M\{\ell\}.$$

For a nonsingular variety X over a field, $\mathrm{CH}^j(X, i)$ denotes Bloch's higher Chow groups. We write $\mathrm{CH}^j(X) := \mathrm{CH}^j(X, 0)$ for the (usual) Chow groups.

2. Review of the cycle class map and ℓ-adic regulator

In this section X denotes a smooth variety over a field K and n denotes a positive integer prime to $\mathrm{ch}(K)$.

By [Geisser and Levine 2001] we have the cycle class map

$$c^{i,j}_{\text{ét}} : \mathrm{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \to H^{2i-j}_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(i)\big),$$

where the right hand side is the étale cohomology of X with coefficients $\mu_n^{\otimes i}$, Tate twist of the sheaf of n-th roots of unity. The left hand side is Bloch's higher Chow group with finite coefficient which fits into the exact sequence

$$0 \to \mathrm{CH}^i(X, j)/n \to \mathrm{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^i(X, j-1)[n] \to 0. \tag{2-1}$$

In this paper we are only concerned with the map

$$c_{\text{ét}} = c_{\text{ét}}^{2,1} : CH^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \to H^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big). \tag{2-2}$$

By [Bloch and Ogus 1974] it is injective and its image is equal to

$$NH^3_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(2)\big) = \mathrm{Ker}\Big(H^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big) \to H^3_{\text{ét}}(\mathrm{Spec}(K(X)\big), \mathbb{Z}/n\mathbb{Z}(2)\big)\Big),$$

where $K(X)$ is the function field of X. In view of (2-1) it implies an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 1)/n \xrightarrow{c_{\text{ét}}} NH^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big) \longrightarrow \mathrm{CH}^2(X)[n] \longrightarrow 0. \tag{2-3}$$

We also need the cycle map to the continuous étale cohomology group

$$c_{\text{cont}} : \mathrm{CH}^2(X, 1) \longrightarrow H^3_{\text{cont}}\big(X, \mathbb{Z}_\ell(2)\big)$$

(see [Jannsen 1988]), where ℓ is a prime different from $\mathrm{ch}(K)$. In case K is a p-adic field, we have

$$H^3_{\text{cont}}\big(X, \mathbb{Z}_\ell(2)\big) = \varprojlim_n H^3_{\text{ét}}\big(X, \mathbb{Z}/\ell^n\mathbb{Z}(2)\big)$$

and c_{cont} is induced by $c_{\text{ét}}$ by passing to the limit. We have the Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i_{\text{cont}}\big(\mathrm{Spec}(K), H^j(X_{\overline{K}}, \mathbb{Z}_\ell(2))\big) \Rightarrow H^{i+j}_{\text{cont}}(X, \mathbb{Z}_\ell(2)). \tag{2-4}$$

If K is finitely generated over the prime subfield and X is proper smooth over K, the Weil conjecture proved by Deligne implies that

$$H^0\big(\mathrm{Spec}(K), H^3(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) = 0.$$

The same conclusion holds if K is a p-adic field and X is proper smooth having good reduction over K. (If $\ell \neq p$ this follows from the proper smooth base change theorem for étale cohomology. If $\ell = p$ one uses comparison theorems between p-adic étale and crystalline cohomology and the Weil conjecture for crystalline cohomology) Thus we get under these assumptions the map

$$\rho_X : \mathrm{CH}^2(X, 1) \longrightarrow H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) \tag{2-5}$$

as the composite of $c_{\text{cont}} \otimes \mathbb{Q}_\ell$ and an edge homomorphism

$$H^3_{\text{cont}}(X, \mathbb{Q}_\ell(2)) \to H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big).$$

For later use, we need an alternative definition of cycle class maps. For an integer $i \geq 1$, we denote by $\mathcal{K}_i$ the sheaf on X_{Zar}, the Zariski site on X, associated to the presheaf $U \mapsto K_i(U)$. By [Landsburg 1991, 2.5], we have canonical isomorphisms

$$\mathrm{CH}^2(X,1) \simeq H^1_{\mathrm{Zar}}(X, \mathcal{K}_2), \quad \mathrm{CH}^2(X,1,\mathbb{Z}/n\mathbb{Z}) \simeq H^1_{\mathrm{Zar}}(X, \mathcal{K}_2/n). \tag{2-6}$$

Let $\epsilon^{\text{ét}} : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the natural map of sites and put

$$\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(r)) = R^i \epsilon^{\text{ét}}_* \mu_n^{\otimes r}.$$

The universal Chern classes in the cohomology groups of the simplicial classifying space for GL_n $(n \geq 1)$ give rise to higher Chern class maps on algebraic K-theory; see [Gillet 1981; Schneider 1988]. It gives rise to a map of sheaves

$$\mathcal{K}_i/n \longrightarrow \mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(i)). \tag{2-7}$$

By [Merkurjev and Suslin 1982] it is an isomorphism for $i = 2$ and induces an isomorphism

$$H^1_{\mathrm{Zar}}(X, \mathcal{K}_2/n) \xrightarrow{\cong} H^1_{\mathrm{Zar}}\big(X, \mathcal{H}^2_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big). \tag{2-8}$$

By the spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Zar}}\big(X, \mathcal{H}^q_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) \Longrightarrow H^{p+q}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(2)),$$

together with the fact

$$H^p_{\mathrm{Zar}}\big(X, \mathcal{H}^q_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) = 0$$

for $p > q$ shown by Bloch and Ogus [1974], we get an injective map

$$H^1_{\mathrm{Zar}}\big(X, \mathcal{H}^2_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) \longrightarrow H^3_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(2)).$$

Again by the Bloch–Ogus theory the image of the above map coincides with the coniveau filtration $NH^3_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(2))$. Combined with (2-6) and (2-8) we thus get the map

$$c_{\text{ét}} : \mathrm{CH}^2(X,1,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^1_{\mathrm{Zar}}(X, \mathcal{K}_2/n) \xrightarrow{\cong} NH^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big) \xrightarrow{\subset} H^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big).$$

This agrees with the map (2-2); see [Colliot-Thélène et al. 1983, Proposition 1].

Now we work over the base field $K = \mathbb{C}$. Let X_{an} be the site on the underlying analytic space $X(\mathbb{C})$ endowed with the ordinary topology. Let $\epsilon^{\mathrm{an}} : X_{\mathrm{an}} \to X_{\mathrm{Zar}}$ be the natural map of sites and put

$$H^i_{\mathrm{an}}(\mathbb{Z}(r)) = R^i \epsilon^{\mathrm{an}}_* \mathbb{Z}(r) \quad \big(\mathbb{Z}(r) = \big(2\pi\sqrt{-1}\big)^r \mathbb{Z}\big).$$

The higher Chern class map then gives a map of sheaves

$$\mathcal{K}_i \longrightarrow \mathcal{H}^i_{\mathrm{an}}(\mathbb{Z}(i)). \tag{2-9}$$

By the same argument as before, it induces a map

$$c_{\mathrm{an}} : \mathrm{CH}^2(X,1) \xrightarrow{\cong} H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) \longrightarrow H^3_{\mathrm{an}}(X(\mathbb{C}),\mathbb{Z}(2)).$$

Lemma 2.1. *The image of* c_{an} *is contained in* $F^2H^3_{\mathrm{an}}(X(\mathbb{C}),\mathbb{C})$, *the Hodge filtration defined in [Deligne 1971]. In particular if* X *is complete, the image is the torsion.*

Proof. Let $\mathcal{H}^r_D(\mathbb{Z}(i))$ be the sheaf on X_{Zar} associated to a presheaf

$$U \mapsto H^r_D(U,\mathbb{Z}(i))$$

where $H^\bullet_D$ denotes Deligne–Beilinson cohomology; see [Esnault and Viehweg 1988, 2.9]. Higher Chern class maps to Deligne–Beilinson cohomology give rise to the map $K_2 \to \mathcal{H}^2_D(\mathbb{Z}(2))$ and c_{an} factors as in the commutative diagram

$$\begin{array}{ccccc} H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) & \longrightarrow & H^1_{\mathrm{Zar}}\big(X,\mathcal{H}^2_D(\mathbb{Z}(2))\big) & \longrightarrow & H^1_{\mathrm{Zar}}\big(X,\mathcal{H}^2_{\mathrm{an}}(\mathbb{Z}(2))\big) \\ & & \downarrow{\scriptstyle a} & & \downarrow \\ & & H^3_D(X,\mathbb{Z}(2)) & \xrightarrow{b} & H^3_{\mathrm{an}}(X(\mathbb{C}),\mathbb{Z}(2)). \end{array}$$

Here the map a is induced from the spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Zar}}\big(X,\mathcal{H}^q_D(\mathbb{Z}(2))\big) \Longrightarrow H^{p+q}_D(X,\mathbb{Z}(2))$$

in view of the fact that $H^p_{\mathrm{Zar}}(X,\mathcal{H}^1_D(\mathbb{Z}(2))) = 0$ for all $p > 0$, since $\mathcal{H}^1_D(\mathbb{Z}(2)) \cong \mathbb{C}/\mathbb{Z}(2)$ (constant sheaf). Since the image of b is contained in $F^2H^3_{\mathrm{an}}(X(\mathbb{C}),\mathbb{C})$ (see [Esnault and Viehweg 1988, 2.10]), so is the image of c_{an}. □

Lemma 2.2. *We have the diagram*

$$\begin{array}{ccc} \mathrm{CH}^2(X,1) & \xrightarrow{c_{\mathrm{an}}} & H^3_{\mathrm{an}}\big(X(\mathbb{C}),\mathbb{Z}(2)\big) \\ \downarrow & & \downarrow \\ \mathrm{CH}^2(X,1,\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{c_{\acute{e}t}} & H^3_{\acute{e}t}\big(X,\mathbb{Z}/n\mathbb{Z}(2)\big). \end{array}$$

Here the right vertical map is the composite

$$H^3_{\mathrm{an}}\big(X(\mathbb{C}),\mathbb{Z}(2)\big) \to H^3_{\mathrm{an}}\big(X(\mathbb{C}),\mathbb{Z}(2)\otimes\mathbb{Z}/n\mathbb{Z}\big) \xrightarrow{\cong} H^3_{\acute{e}t}\big(X,\mathbb{Z}/n\mathbb{Z}(2)\big)$$

and the isomorphism comes from the comparison isomorphism between étale cohomology and ordinary cohomology (SGA 4 1/2 = [Deligne 1977], Arcata, 3.5) *together with the isomorphism*

$$\mathbb{Z}(1) \otimes \mathbb{Z}/n\mathbb{Z} \simeq (\epsilon^{\mathrm{an}})^* \mu_n$$

given by the exponential map.

Proof. This follows from the compatibility of (2-7) and (2-9), namely the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{K}_i & \longrightarrow & \mathcal{H}^i_{\mathrm{an}}\big(\mathbb{Z}(i)\big) \\ \big\downarrow & & \big\downarrow \\ \mathcal{K}_i/n & \longrightarrow & \mathcal{H}^i_{\text{ét}}\big(\mathbb{Z}/n\mathbb{Z}(i)\big), \end{array}$$

and it follows from the compatibility of the universal Chern classes [Gillet 1981; Schneider 1988]. □

3. Counterexample to the Bloch–Kato conjecture over p-adic field

In this section K denotes a p-adic field and let X be a proper smooth surface over K. We fix a prime ℓ (possibly $\ell = p$) and consider the map (2-5)

$$\rho_X : \mathrm{CH}^2(X, 1) \longrightarrow H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) \quad \big(V = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big). \tag{3-1}$$

Define the primitive part $\widetilde{V}$ of V by

$$\widetilde{V} := H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(2))/V_0, \quad V_0 = [H_X] \otimes \mathbb{Q}_\ell(1), \tag{3-2}$$

where $[H_X] \in H^2_{\mathrm{cont}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$ is the cohomology class of a hyperplane section. With the notation

$$\widetilde{V} \simeq \mathrm{Ker}\big(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(2)) \stackrel{\cup[H_X]}{\longrightarrow} H^4_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(3))\big),$$

we get a decomposition as G_K-modules:

$$V = \widetilde{V} \oplus V_0. \tag{3-3}$$

Let $\widetilde{\rho} : \mathrm{CH}^2(X, 1) \longrightarrow H^1_{\mathrm{cont}}(\mathrm{Spec}(K), \widetilde{V})$ be the induced map.

Theorem 3.1. *Let $X \subset \mathbb{P}^3_K$ be a generic smooth surface of degree $d \geq 5$. Then $\widetilde{\rho}$ is the zero map for arbitrary ℓ.*

Remark 3.2. (1) This is an analogue of [Voisin 1995, 1.6], where she worked on Deligne–Beilinson cohomology.

(2) Bloch and Kato [1990] considers regulator maps such as (3-1) for a smooth projective variety over a number field and conjectures that its image coincides with H^1_g. We will see later (see Theorem 3.6) that the variant of the conjecture over a p-adic field is false in general.

(3) The construction of a counterexample mentioned in (2) hinges on the assumption that the surface $X \subset \mathbb{P}^3_K$ is generic. One may still ask whether the image of l-adic regulator map coincides H^1_g for a proper smooth variety X over a p-adic field when X is defined over a number field.

Proof. Let $f : \mathscr{X} \to M$ be as in the introduction and let $t : \mathrm{Spec}(K) \to M$ be a dominant morphism such that $X \simeq \mathscr{X} \times_M \mathrm{Spec}(K)$. For a morphism $S \to M$ of smooth schemes over $\mathbb{Q}$, let $f_S : X_S = \mathscr{X} \times_M S \to S$ be the base change of f. The same construction of (2-5) gives rise to the regulator map

$$\rho_S : \mathrm{CH}^2(X_S, 1) \to H^1_{\mathrm{cont}}(S, V_S),$$

where $V_S = R^2(f_S)_*\mathbb{Q}_l(2)$ is a smooth $\mathbb{Q}_l$-sheaf on S. Define the primitive part of V_S,

$$\widetilde{V}_S = R^2(f_S)_*\mathbb{Q}_l(2)/[H] \otimes \mathbb{Q}_l(1),$$

where $[H] \in H^0(S, R^2(f_S)_*\mathbb{Q}_l(1))$ is the class of a hyperplane section. Let

$$\widetilde{\rho}_S : \mathrm{CH}^2(X_S, 1) \to H^1_{\mathrm{cont}}(S, \widetilde{V}_S)$$

be the induced map. Note that

$$\mathrm{CH}^2(X, 1) = \varinjlim_S \mathrm{CH}^2(X_S, 1),$$

where $S \to M$ ranges over the smooth morphisms which factor $t : \mathrm{Spec}(K) \to M$. We have for such S the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X_S, 1) & \xrightarrow{\widetilde{\rho}_S} & H^1_{\mathrm{cont}}(S, \widetilde{V}_S) \\ \downarrow & & \downarrow \\ \mathrm{CH}^2(X, 1) & \xrightarrow{\widetilde{\rho}} & H^1_{\mathrm{cont}}(\mathrm{Spec}(K), \widetilde{V}). \end{array}$$

Thus it suffices to show

$$H^1_{\mathrm{cont}}(S, \widetilde{V}_S) = 0.$$

Without loss of generality we suppose S is an affine smooth variety over a finite extension L of $\mathbb{Q}$.

Claim 3.3. *Assume $d \geq 4$. The natural map*

$$H^1_{\mathrm{cont}}(S, \widetilde{V}_S) \longrightarrow H^1_{ét}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) \quad (S_{\overline{\mathbb{Q}}} := S \times_L \mathrm{Spec}(\overline{\mathbb{Q}}))$$

is injective.

Indeed, by the Hochschild–Serre spectral sequence, it is enough to see

$$H^0_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) = 0,$$

which follows from [Asakura and Saito 2006b, Theorem 6.1(2)].

By SGA $4\,{}^1\!/_2$, Arcata, Cor. (3.3) and (3.5.1), we have

$$H^1_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) \cong H^1_{\text{ét}}(S_{\mathbb{C}}, \widetilde{V}_S) \simeq H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S) \otimes \mathbb{Q}_l \quad (S_{\mathbb{C}} := S \times_L \operatorname{Spec}(\mathbb{C})),$$

where $\widetilde{V}^{\text{an}}_S$ is the primitive part of $V^{\text{an}}_S = R^2(f^{\text{an}}_S)_*\mathbb{Q}(2)$ with $f^{\text{an}}_S : (X_{S_{\mathbb{C}}})_{\text{an}} \to (S_{\mathbb{C}})_{\text{an}}$, the natural map of sites. By definition $\widetilde{V}^{\text{an}}_S$ is a local system on $S(\mathbb{C})$ whose fiber over $s \in S(\mathbb{C})$ is the primitive part of $H^2_{\text{an}}(X_s(\mathbb{C}), \mathbb{Q}(2))$ for X_s, the fiber of $X_S \to S$ over s. Due to Lemma 2.2, it suffices to show the triviality of the image of the map

$$\widetilde{\rho}^{\text{an}}_S : \mathrm{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S)$$

which is induced from

$$c_{\text{an}} : \mathrm{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H^3_{\text{an}}(X_S(\mathbb{C}), \mathbb{Q}(2))$$

by using the natural map

$$H^3_{\text{an}}(X_S(\mathbb{C}), \mathbb{Q}(2)) \to H^1_{\text{an}}(S(\mathbb{C}), V^{\text{an}}_S)$$

arising from the Leray spectral sequence for $f^{\text{an}}_S : (X_{S_{\mathbb{C}}})_{\text{an}} \to (S_{\mathbb{C}})_{\text{an}}$ and the vanishing $R^3(f^{\text{an}}_S)_*\mathbb{Q}(2) = 0$.

Claim 3.4. *The image of $\widetilde{\rho}^{\text{an}}_S$ is contained in the Hodge filtration*

$$F^2 H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C})$$

defined by the theory of Hodge modules [Saito 1990, §4].

This follows from the functoriality of Hodge filtrations and Lemma 2.1.

It is quite complicated to describe the Hodge filtration on $H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C})$ precisely. However, all that we need is the following property:

Claim 3.5. *For integers m, $p \geq 0$ there is a natural injective map*

$$F^p H^m_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C}) \to H^m_{\text{Zar}}(S_{\mathbb{C}}, G^p DR(\widetilde{V}^{\text{an}}_S))$$

where $G^p DR(\widetilde{V}^{\text{an}}_S)$ is the complex of Zariski sheaves on $S_{\mathbb{C}}$

$$F^p H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \mathcal{O}_{S_{\mathbb{C}}} \to F^{p-1} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \to \cdots \to F^{p-r} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^r_{S_{\mathbb{C}}/\mathbb{C}} \to F^{p-r} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^{r+1}_{S_{\mathbb{C}}/\mathbb{C}} \to \cdots .$$

Here $H^\bullet_{\text{dR}}(X_S/S)$ denotes the de Rham cohomology of X_S/S, and $H^\bullet_{\text{dR}}(X_S/S)_{\text{prim}}$ is its primitive part defined by the same way as before, and the maps are induced from the Gauss–Manin connection thanks to Griffiths transversality.

This follows from [Asakura 2002, Lemma 4.2]. We note that its proof hinges on the theory of mixed Hodge modules. The key points are Deligne's comparison theorem [1970, §6] for algebraic and analytic cohomology of a vector bundle with integrable connection with regular singularities and the degeneration of Hodge spectral sequence for cohomology with coefficients; see [Saito 1990, (4.1.3)].

By the above claims we are reduced to showing the exactness at the middle term of the complex

$$F^2 H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \mathbb{O}_{S_{\mathbb{C}}} \longrightarrow F^1 H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}}. \tag{3-4}$$

This is proved by the infinitesimal method in Hodge theory. We sketch the proof. Let $f : X_S \to S$ be the natural morphism. The assertion follows from the exactness at the middle term of the complex

$$f_*\Omega^2_{X_S/S} \otimes \mathbb{O}_{S_{\mathbb{C}}} \longrightarrow (R^1 f_*\Omega^1_{X_S/S})_{\mathrm{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow R^2 f_*\mathbb{O}_{X_S} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}} \tag{3-5}$$

and the injectivity of the complex

$$f_*\Omega^2_{X_S/S} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow (R^1 f_*\Omega^1_{X_S/S})_{\mathrm{prim}} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}}. \tag{3-6}$$

These complexes are induced by the complex (3-4) by Griffiths transversality. If

$$S = M \subset \mathbb{P}\big(H^0(\mathbb{P}^3_{\mathbb{Q}}, \mathbb{O}_{\mathbb{P}}(d))\big),$$

these assertions are proved as follows. Let $P = \mathbb{C}[z_0, z_1, z_2, z_3]$, and $P^n \subset P$ be the subspace of the homogeneous polynomials of degree n. Take a point $x \in M(\mathbb{C})$ and choose $F \in P^d$ which defines the surface corresponding to x. Let $R = \mathbb{C}[z_0, z_1, z_2, z_3]/(\partial F/\partial z_0, \cdots, \partial F/\partial z_3)$ be the Jacobian ring and $R^n \subset R$ be the image of P^n in R. Then the fibers over x of (3-5) and (3-6) are identified with the Koszul complexes

$$R^{d-4} \longrightarrow R^{2d-4} \otimes (R^d)^* \longrightarrow R^{3d-4} \otimes \overset{2}{\wedge}(R^d)^*, \tag{3-7}$$

$$R^{d-4} \otimes (R^d)^* \longrightarrow R^{2d-4} \otimes \overset{2}{\wedge}(R^d)^* \tag{3-8}$$

where $(R^d)^*$ denotes the dual space of R and the maps are induced from the multiplication $R \otimes R \to R$. Then the Donagi symmetrizer lemma [Green 1994, p. 76] implies that (3-7) is exact at the middle term if $d \geq 5$ and (3-8) is injective if $d \geq 3$, which proves the desired assertion in case $S = M$. The assertion in case S is dominant over M is reduced to the case $S = M$ by an easy argument; see [Asakura and Saito 2006a, §9]. This completes the proof of Theorem 3.1. □

Let $\mathbb{O}_K \subset K$ be the ring of integers and k be the residue field. In order to construct an example where the image of the regulator map

$$\rho_X : \mathrm{CH}^2(X, 1) \xrightarrow{\rho_X} H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) \quad \big(V = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big)$$

is not equal to $H^1_g(\mathrm{Spec}(K), V)$, we now take a proper smooth surface X having good reduction over K so that X has a proper smooth model $X_{\mathbb{O}_K}$ over $\mathrm{Spec}(\mathbb{O}_K)$. We denote the special fiber by Y. By [Langer and Saito 1996, p. 341, diagram below 5.7], there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X,1) & \xrightarrow{\widetilde{\rho}} & H^1_g(\mathrm{Spec}(K),V) \\ \downarrow{\scriptstyle\partial} & & \downarrow \\ \mathrm{CH}^1(Y) & \xrightarrow{\alpha} & H^1_{\mathrm{cont}}(\mathrm{Spec}(K),V)/H^1_f(\mathrm{Spec}(K),V) \end{array} \tag{3-9}$$

where $H^1_f \subset H^1_g \subset H^1_{\mathrm{cont}}$ are the subspaces introduced by Bloch and Kato [1990] and ∂ is a boundary map in localization sequence for higher Chow groups.

Theorem 3.6. *Let $X \subset \mathbb{P}^3_K$ be a generic smooth surface of degree $d \geq 5$. Assume that X has a projective smooth model $X_{\mathbb{O}_K} \subset \mathbb{P}^3_{\mathbb{O}_K}$ over $\mathbb{O}_K$ and let $Y \subset \mathbb{P}^3_k$ be its special fiber.*

(1) *The image of $\partial \otimes \mathbb{Q}$ is contained in the subspace of $\mathrm{CH}^1(Y) \otimes \mathbb{Q}$ generated by the class $[H_Y]$ of a hyperplane section of Y.*

(2) *Let r be the Picard number of Y. Then*

$$\dim_{\mathbb{Q}_\ell}\big(H^1_g(\mathrm{Spec}(K), V)/\mathrm{Image}(\rho_X)\big) \geq r-1.$$

Proof. Recall $V = \widetilde{V} \oplus V_0$, a decomposition as G_K-modules; see (3-3). Let $W \subset CH^2(X, 1)$ be the image of $\mathbb{Z}\cdot[H_X]\otimes K^\times$ under the product map $\mathrm{CH}^1(X)\otimes K^\times \to \mathrm{CH}^2(X, 1)$. Then it is easy to see ρ_X induces an isomorphism

$$W \otimes \mathbb{Q}_\ell \simeq H^1_g(\mathrm{Spec}(K), V_0) = H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V_0)$$

and that $\partial(W) = \mathbb{Z}\cdot[H_Y] \subset \mathrm{CH}^1(Y)$. Hence (1) follows from Theorem 3.1 together with injectivity of α in (3-9), proved by [Langer and Saito 1996, Lemma 5–7].

As for (2) we first note from [Bloch and Kato 1990, 3.9] that

$$\dim_{\mathbb{Q}_\ell}\big(H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V_0)/H^1_f(\mathrm{Spec}(K), V_0)\big) = 1.$$

Moreover the same argument (except using the Tate conjecture) in the last part of [Langer and Saito 1996, §5] shows

$$\dim_{\mathbb{Q}_\ell}(\mathrm{CH}^1(Y) \otimes \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell}(H^1_g(\mathrm{Spec}(K), V)/H^1_f(\mathrm{Spec}(K), V)).$$

Hence (2) follows from (1). □

Remark 3.7. Let the assumption be as in Theorem 3.6. Then

$$\dim_{\mathbb{Q}_\ell}\big(H^1_g(\mathrm{Spec}(K), V)/\mathrm{Image}(\rho_X)\big) \geq \begin{cases} r-1, & \ell \neq p, \\ r-1+(h^{0,2}+h^{1,1}-1)[K:\mathbb{Q}_p], & \ell = p, \end{cases}$$

where $h^{p,q} := \dim_K H^q(X, \Omega^p_{X/K})$ denotes the Hodge number. Moreover the equality holds if and only if the Tate conjecture for divisors on Y holds. This follows from Theorem 3.1 and the computation of $\dim_{\mathbb{Q}_\ell} H^1_g(\mathrm{Spec}(K), V)$ using [Bloch and Kato 1990, 3.8 and 3.8.4]. The details are omitted.

4. Proof of Theorem 1.1

Let K be a p-adic field and $\mathbb{O}_K \subset K$ the ring of integers and k the residue field. Let us consider schemes

$$\begin{array}{ccccc} X & \xrightarrow{j} & X_{\mathbb{O}_K} & \xleftarrow[i]{} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathbb{O}_K) & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where all vertical arrows are projective and smooth of relative dimension 2 and the diagrams are Cartesian. We have a boundary map in localization sequence for higher Chow groups with finite coefficients

$$\partial : \mathrm{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^1(Y)/n.$$

For a prime number ℓ, it induces

$$\partial_\ell : \mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{CH}^1(Y) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $\mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := \varinjlim_n \mathrm{CH}^2(X, 1, \mathbb{Z}/\ell^n\mathbb{Z})$.

Theorem 4.1. *For $\ell \neq p := \mathrm{ch}(k)$, ∂_ℓ is surjective and has finite kernel. Hence we have*

$$\mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r} + (\textit{finite group}),$$

where r is the rank of $\mathrm{CH}^1(Y)$.

Theorem 1.1 is an immediate consequence of Theorem 3.6(1), Theorem 4.1, and the exact sequence (2-1)

$$0 \to \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{CH}^2(X)\{\ell\} \to 0.$$

Proof of Theorem 4.1. Write $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{CH}^2(X, 1, \Lambda) & \xrightarrow{\partial} & \mathrm{CH}^1(Y) \otimes \Lambda & \xrightarrow{i_*} & \mathrm{CH}^2(X_{\mathbb{O}_K}) \otimes \Lambda & \xrightarrow{j^*} & \mathrm{CH}^2(X) \otimes \Lambda \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_3} & & \downarrow{\scriptstyle c_4} \\ H^3_{\text{ét}}(X, \Lambda(2)) & \xrightarrow{\partial_{\text{ét}}} & H^2_{\text{ét}}(Y, \Lambda(1)) & \xrightarrow{i_*} & H^4_{\text{ét}}(X_{\mathbb{O}_K}, \Lambda(2)) & \xrightarrow{j^*_{\text{ét}}} & H^4_{\text{ét}}(X, \Lambda(2)). \end{array}$$

Here the upper exact sequence arises from the localization theory for higher Chow groups with finite coefficient, as in [Levine 2001, Theorem 1.7], and the lower from the localization theory for étale cohomology together with absolute purity [Fujiwara 2002]. The vertical maps are étale cycle class maps. By Equation (2-3), c_1 is injective. Since

$$\mathrm{CH}^1(Y) = H^1(Y, \mathbb{G}_m),$$

c_2 is injective by the Kummer theory. It is shown in [Saito and Sato 2006b] that c_3 is an isomorphism. Hence the diagram reduces the proof of Theorem 4.1 to showing that $\mathrm{Ker}(\partial_{\text{ét}})$ and $\mathrm{Ker}(j^*_{\text{ét}})$ are finite. This is an easy consequence of the proper base change theorem for étale cohomology and the Weil conjecture [Deligne 1980]. For the former we also use an exact sequence

$$H^3_{\text{ét}}(X_{\mathbb{O}_K}, \Lambda(2)) \to H^3_{\text{ét}}(X, \Lambda(2)) \xrightarrow{\partial_{\text{ét}}} H^2_{\text{ét}}(Y, \Lambda(1)). \qquad \square$$

Appendix. SK_1 of curves over p-adic fields

Let C be a proper smooth curve over a field K and consider $\mathrm{CH}^2(C, 1)$. By [Landsburg 1991, 2.5], we have an isomorphism

$$\mathrm{CH}^2(C, 1) \simeq H^1_{\mathrm{Zar}}(C, \mathcal{K}_2) \simeq SK_1(C).$$

By definition

$$SK_1(C) = \mathrm{Coker}\Big(K_2(K(C)) \xrightarrow{\delta} \bigoplus_{x \in C_0} K(x)^\times\Big),$$

where $K(C)$ is the function field of C, C_0 is the set of the closed points of C, and $K(x)$ is the residue field of $x \in C_0$, and δ is given by the tame symbols. The norm maps $K(x)^\times \to K^\times$ for $x \in C_0$ induce

$$N_{C/K} : SK_1(C) \to K^\times.$$

We write $V(C) = \mathrm{Ker}(N_{C/K})$.

When K is a p-adic field, it is known by class field theory for curves over a local field [Saito 1985] that $V(C)$ is a direct sum of its maximal divisible subgroup and a finite group. An interesting question is whether the divisible subgroup is uniquely divisible, or equivalently whether $SK_1(C)_{\mathrm{tor}}$ is finite. In case the genus $g(C) = 1$, confirmative results have been obtained in [Sato 1985; Asakura 2006]. The purpose of this section is to show that the method in the previous sections gives rise to an example of a curve C of $g(C) \geq 2$ such that $SK_1(C)_{\mathrm{tor}}$ is infinite.

Let C be as in the beginning of this section and let n be a positive integer prime to $\mathrm{ch}(K)$. We have the cycle class map

$$c_{\text{ét}} : \mathrm{CH}^2(C, 2, \mathbb{Z}/n\mathbb{Z}) \to H^2_{\text{ét}}(C, \mathbb{Z}/n\mathbb{Z}(2)).$$

The main result of [Merkurjev and Suslin 1982] implies that the above map is an isomorphism. In view of the exact sequence (compare (2-1))

$$0 \to \mathrm{CH}^2(C,2)/n \to \mathrm{CH}^2(C,2,\mathbb{Z}/n\mathbb{Z}) \to SK_1(C)[n] \to 0,$$

we get the exact sequence

$$0 \to \mathrm{CH}^2(C,2)/n \to H^2_{\text{ét}}(C,\mathbb{Z}/n\mathbb{Z}(2)) \to SK_1(C)[n] \to 0; \tag{A-2}$$

see [Suslin 1985, 23.4]. We will also use cycle class map to continuous étale cohomology

$$c_{\text{cont}} : \mathrm{CH}^2(C,2)\otimes\mathbb{Q}_\ell \to H^2_{\text{cont}}(C,\mathbb{Q}_\ell(2))$$

where ℓ is any prime number different from $\mathrm{ch}(K)$. When K is a p-adic field, one easily shows

$$H^2_{\text{cont}}(C,\mathbb{Q}_\ell(2)) \simeq H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^1_{\text{ét}}(C_{\overline{K}},\mathbb{Q}_\ell(2))\big) \tag{A-3}$$

by using the Hochschild–Serre spectral sequence (2-4). Hence we get the map

$$\rho_C : \mathrm{CH}^2(C,2)\otimes\mathbb{Q}_\ell \to H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^1_{\text{ét}}(C_{\overline{K}},\mathbb{Q}_\ell(2))\big). \tag{A-4}$$

Note that ρ_C is trivial if C has good reduction and $\ell \neq p$, since the group on the right hand side is trivial. The last fact is a consequence of the proper smooth base change theorem for étale cohomology and the weight argument.

Let M_g be the moduli space of tricanonically embedded projective nonsingular curves of genus $g \geq 2$ over the base field $\mathbb{Q}$ (compare [Deligne and Mumford 1969]), and let $f : \mathscr{C} \to M_g$ be the universal family.

Definition A.2. Let C be a proper smooth curve over a field K of characteristic zero. We say C is generic if there is a dominant morphism $\mathrm{Spec}(K) \to M_g$ such that $C \cong \mathscr{C} \times_{M_g} \mathrm{Spec}(K)$.

Theorem A.3. *Let K be a p-adic field and let C be a generic curve of genus $g \geq 2$ over K. Then ρ_C is the zero map for all ℓ. We have an isomorphism*

$$SK_1(C)_{\text{tor}} \cong H^2_{\acute{e}t}(C,\mathbb{Q}/\mathbb{Z}(2)) \quad \big(:= \varinjlim_n H^2_{\acute{e}t}(C,\mathbb{Z}/n\mathbb{Z}(2))\big).$$

Remark A.4. Theorem A.3 is comparable with the main result of [Green and Griffiths 2002] where they worked on Deligne–Beilinson cohomology.

Proof. The second assertion follows easily from the first in view of Equation (A-2). The first assertion is shown by the same method as the proof of Theorem 3.1, with the following fact from [Green and Griffiths 2002, §3] noted. Let $S \to M_g$ be a dominant smooth morphism, and put $f : C_S := \mathscr{C} \times_{M_g} S \to S$, then the map

$$f_*\Omega^1_{C_S/S} \longrightarrow R^1 f_*\mathscr{O}_{C_S} \otimes \Omega^1_{S/\mathbb{Q}}$$

induced from the Gauss–Manin connection is injective. □

Corollary A.5. *Let C be as in Theorem A.3. Assume the Jacobian variety $J(C)$ has semistable reduction over K. Let $\mathcal{J}$ be the Néron model of J with $\mathcal{J}_s$, its special fiber. Let r be the dimension of the maximal split torus in $\mathcal{J}_s$. For a prime ℓ, we have*

$$SK_1(C)\{\ell\} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{r_\ell} \oplus (\textit{finite group}),$$

where $r_\ell = r$ for $\ell \neq p$ and $r_p = r + 2g[K : \mathbb{Q}_p]$.

For example, $SK_1(C)\{\ell\}$ is infinite for any ℓ if C is a Mumford curve (a proper smooth curve with semistable reduction over K such that the irreducible components are isomorphic to $\mathbb{P}^1_k$ and intersect each other at k-rational points, where k is the residue field of K), which is generic in the sense of Definition A.2.

Corollary A.5 follows from Theorem A.3 and the next result:

Lemma A.6. *Let C be proper smooth curve over a p-adic field K. Assume $J(C)$ has semistable reduction over K and let r_ℓ be as above. Then*

$$\dim_{\mathbb{Q}_\ell} H^2_{\mathrm{cont}}(C, \mathbb{Q}_\ell(2)) = \dim_{\mathbb{Q}_\ell} H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) = r_\ell. \ \big(V = H^1_{\acute{e}t}(C_{\overline{K}}, \mathbb{Q}_\ell(2))\big).$$

Proof. The first equality follows from (A-3). By [Jannsen 1989, p. 354–355, Th. 5 and Cor. 7], we have

$$H^0_{\mathrm{cont}}(\mathrm{Spec}(K), V) = 0, \qquad \dim_{\mathbb{Q}_\ell} H^2_{\mathrm{cont}}(\mathrm{Spec}(K), V) = r.$$

Lemma A.6 now follows from the computation of Euler–Poincaré characteristic given in [Serre 1965, II 5.7]. □

Remark A.7. Using [Bloch and Kato 1990, 3.8.4] and the $\mathrm{Gal}(\overline{K}/K)$-module structure of the Tate module of an abelian variety over K (see [Grothendieck 1972, exposé IX]), one can show that

$$H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) = H^1_g(\mathrm{Spec}(K), V).$$

Hence, if C is a generic curve of genus greater than or equal to 2, then the map ρ_C in Equation (A-4) does not surject onto H^1_g if $r_\ell \geq 1$. This gives another counterexample to a variant of the Bloch–Kato conjecture for p-adic fields.

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asakura@math.kyushu-u.ac.jp *Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan*

sshuji@msb.biglobe.ne.jp *Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan*

ALGEBRA AND NUMBER THEORY 1:2(2007)

Singular homology of arithmetic schemes

Alexander Schmidt

We construct a singular homology theory on the category of schemes of finite type over a Dedekind domain and verify several basic properties. For arithmetic schemes we construct a reciprocity isomorphism between the integral singular homology in degree zero and the abelianized modified tame fundamental group.

1. Introduction

The objective of this paper is to construct a reasonable singular homology theory on the category of schemes of finite type over a Dedekind domain. Our main criterion for "reasonable" was to find a theory which satisfies the usual properties of a singular homology theory and which has the additional property that, for schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$, the group h_0 serves as the source of a reciprocity map for tame class field theory. In the case of schemes of finite type over finite fields this role was taken over by Suslin's singular homology; see [Schmidt and Spieß 2000]. In this article we motivate and give the definition of the singular homology groups of schemes of finite type over a Dedekind domain and we verify basic properties, e.g. homotopy invariance. Then we present an application to tame class field theory.

The (integral) singular homology groups $h_*(X)$ of a scheme of finite type over a field k were defined by A. Suslin as the homology of the complex $C_*(X)$ whose n-th term is given by

$$C_n(X) = \text{group of finite correspondences } \Delta_k^n \longrightarrow X,$$

where $\Delta_k^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/\sum t_i = 1)$ is the n-dimensional standard simplex over k and a finite correspondence is a finite linear combination $\sum n_i Z_i$ where each Z_i is an integral subscheme of $X \times \Delta_k^n$ such that the projection $Z_i \to \Delta_k^n$ is finite and surjective. The differential $d : C_n(X) \to C_{n-1}(X)$ is defined as the alternating sum of the homomorphisms which are induced by the cycle theoretic intersection with the 1-codimensional faces $X \times \Delta_k^{n-1}$ in $X \times \Delta_k^n$. This definition (see [Suslin and Voevodsky 1996]) was motivated by the theorem of Dold–Thom

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in algebraic topology. If X is an integral scheme of finite type over the field $\mathbb{C}$ of complex numbers, then Suslin and Voevodsky show that there exists a natural isomorphism

$$h_*(X, \mathbb{Z}/n\mathbb{Z}) \cong H_*^{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$$

between the algebraic singular homology of X with finite coefficients and the topological singular homology of the space $X(\mathbb{C})$. If X is proper and of dimension d, singular homology is related to the higher Chow groups of [Bloch 1986] by the formula $h_i(X) = \mathrm{CH}^d(X, i)$ [Voevodsky 2000]. A sheafified version of the preceding definition leads to the "triangulated category of motivic complexes" (ibid.), which, mainly due to the work of Voevodsky, Suslin and Friedlander, has become a powerful categorical framework for motivic (co)homology theories.

If the field k is finite and if X is an open subscheme of a projective smooth variety over k, then we have the following relation to class field theory: there exists a natural reciprocity homomorphism

$$\mathrm{rec} : h_0(X) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

from the 0-th singular homology group to the abelianized tame fundamental group of X. The homomorphism rec is injective and has a uniquely divisible cokernel (see [Schmidt and Spieß 2000] or Theorem 8.7 below for a more precise statement).

This connection to class field theory was the main motivation of the author to study singular homology of schemes of finite type over Dedekind domains. Let $S = \mathrm{Spec}(A)$ be the spectrum of a Dedekind domain and let X be a scheme of finite type over S. The naive definition of singular homology as the homology of the complex whose n-th term is the group of finite correspondences $\Delta_S^n \to X$ is certainly not the correct one. For example, according to this definition, we would have $h_*(U) = 0$ for any open subscheme $U \subsetneqq S$. Philosophically, a "standard n-simplex" should have dimension n but Δ_S^n is a scheme of dimension $(n+1)$.

If the Dedekind domain A is finitely generated over a field, then one can define the homology of X as its homology regarded as a scheme over this field.

The striking analogy between number fields and function fields in one variable over finite fields, as it is known from number theory, led to the philosophy that it should be possible to consider *any* Dedekind domain A, i.e. also if it is of mixed characteristic, as a curve over a mysterious "ground field" $\mathbb{F}(A)$. In the case $A = \mathbb{Z}$ this "field" is sometimes called the "field with one element" $\mathbb{F}_1$. A more precise formulation of this idea making the philosophy into real mathematics and, in particular, a reasonable intersection theory on "$\mathrm{Spec}(\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z})$" would be of high arithmetic interest. With respect to singular homology, this philosophy predicts that, for a scheme X of finite type over $\mathrm{Spec}(A)$, the groups $h_*(X)$ should be the homology groups of a complex whose n-th term is given as the group of finite correspondences $\Delta^n_{\mathbb{F}(A)} \to X$. Unfortunately, we do not have a good definition of the category of schemes over $\mathbb{F}(A)$. To overcome this, let us take a closer look on the situation of schemes of finite type over a field.

Let k be a field, C a smooth proper curve over k and let X be any scheme of finite type over k together with a morphism $p : X \to C$. Consider the complex $C_*(X; C)$ whose n-th term is given as

$C_n(X; C) =$ free abelian group over closed integral subschemes $Z \subset X \times \Delta^n_k = X \times_C \Delta^n_C$ such that the restriction of the projection $X \times_C \Delta^n_C \to \Delta^n_C$ to Z induces a finite morphism $Z \to T \subset \Delta^n_C$ onto a closed integral subscheme T of codimension 1 in Δ^n_C intersecting all faces $\Delta^m_C \subset \Delta^n_C$ properly.

Then we have a natural inclusion

$$C_*(X) \hookrightarrow C_*(X; C)$$

and the definition of $C_*(X; C)$ only involves the morphism $p : X \to C$ but not the knowledge of k. Moreover, if X is affine, then both complexes coincide.

So, in the general case, having no theory of schemes over "$\mathbb{F}(A)$" at hand, we use the above complex in order to define singular homology. With the case $S = \mathrm{Spec}(\mathbb{Z})$ as the main application in mind, we define the singular homology of a scheme of finite type over the spectrum S of a Dedekind domain as the homology $h_*(X; S)$ of the complex $C_*(X; S)$ whose n-th term is given by

$C_n(X; S) =$ free abelian group over closed integral subschemes $Z \subset X \times_S \Delta^n_S$ such that the restriction of the projection $X \times_S \Delta^n_S \to \Delta^n_S$ to Z induces a finite morphism $Z \to T \subset \Delta^n_S$ onto a closed integral subscheme T of codimension 1 in Δ^n_S intersecting all faces $\Delta^m_S \subset \Delta^n_S$ properly.

In this paper we collect evidence that the so-defined groups $h_*(X; S)$ establish a reasonable homology theory on the category of schemes of finite type over S.

The groups $h_*(X; S)$ are covariantly functorial with respect to scheme morphisms and, on the category of smooth schemes over S, they are functorial with respect to finite correspondences. If the structural morphism $p : X \to S$ factors through a closed point P of S, then our singular homology coincides with Suslin's singular homology of X considered as a scheme over the field $k(P)$.

In Section 3, we calculate the singular homology $h_*(X; S)$ if X is regular and of (absolute) dimension 1. The result is similar to that for smooth curves over fields. Let $\bar{X}$ be a regular compactification of X over S and $Y = \bar{X} - X$. Then

$$h_i(X; S) \cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y}),$$

where $\mathbb{G}_{\bar{X},Y} = \ker(\mathbb{G}_{m,\bar{X}} \to i_* \mathbb{G}_{m,Y})$.

In Section 4, we investigate homotopy invariance. We show that the natural projection $X \times_S \mathbb{A}^1_S \to X$ induces an isomorphism on singular homology. We also show that the bivariant singular homology groups $h_*(X, Y; S)$ (see Section 2 for their definition) are homotopy invariant with respect to the second variable.

In Section 5, we give an alternative characterization of the group h_0, which implies, when X is proper over S, a natural isomorphism $h_0(X; S) \cong \mathrm{CH}_0(X)$, where $\mathrm{CH}_0(X)$ is the group of zero-cycles on X modulo rational equivalence. Furthermore, we can verify the exactness of at least a small part of the expected Mayer–Vietoris sequence associated to a Zariski-open cover of a scheme X.

For a proper, smooth (regular?) scheme X of absolute dimension d over the spectrum S of a Dedekind domain, singular homology should be related to motivic cohomology, defined for example by [Voevodsky 1998], by the formula

$$h_i(X; S) \cong H^{2d-i}_{\mathcal{M}ot}(X, \mathbb{Z}(d)).$$

For schemes over a field k, this formula has been proven by Voevodsky under the assumption that k admits resolution of singularities. In the situation of schemes over the spectrum of a Dedekind domain it is true if X is of dimension 1 (Section 3). For a general X it should follow from the fact that each among the following complex homomorphisms is a quasi-isomorphism. The occurring complexes are in each degree the free group over a certain set of cycles and we only write down this set of cycles and also omit the necessary intersection conditions with faces.

$$C_*(X; S)$$

$$\Big\downarrow (1)$$

$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects finitely onto a codimension 1 subscheme in } \mathbb{A}^d \times \Delta^n)$$

$$\Big\uparrow (2)$$

$$\begin{gathered}(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects finitely onto a codimension 1 subscheme } T \subset \mathbb{A}^d \times \Delta^n \\ \text{such that the projection } T \to \Delta^n \text{ is equidimensional of relative dimension } (d-1))\end{gathered}$$

$$\Big\downarrow (3)$$

$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ equidimensional of relative dimension } (d-1) \text{ over } \Delta^n)$$

$$\Big\uparrow (4)$$

$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects quasifinite and dominant to } X \times \Delta^n)$$

$$\Big\downarrow (5)$$

$$\mathbf{H}_{\mathcal{M}ot}(X, \mathbb{Z}(d)[2d])$$

It follows from the homotopy invariance of the bivariant singular homology groups in the second variable, proven in Section 4, that (1) is a quasi-isomorphism. The statement that the other occurring homomorphisms are also quasi-isomorphisms is completely hypothetical at the moment. However, it is, at least partly, suggested by the proof of the corresponding formula over fields; see [Voevodsky 2000, Theorem 4.3.7; Friedlander and Voevodsky 2000, Theorems 7.1 and 7.4].

We give the following application of singular homology to higher dimensional class field theory. Let X be a regular connected scheme, flat and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Sending a closed point of x of X to its Frobenius automorphism $\mathrm{Frob}_x \in \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$, we obtain a homomorphism

$$r : Z_0(X) \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$$

from the group $Z_0(X)$ of zero-cycles on X to the abelianized étale fundamental group $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$. The homomorphism r is known to have dense image. Assume for simplicity that the set $X(\mathbb{R})$ of real-valued points of X is empty. If X is proper, then r factors through rational equivalence, defining a reciprocity homomorphism $\mathrm{rec} : \mathrm{CH}_0(X) \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$. The main result of the so-called unramified class field theory for arithmetic schemes of Bloch and Kato–Saito [Kato and Saito 1983; Saito 1985] states that rec is an isomorphism of finite abelian groups.

If X is not proper, r no longer factors through rational equivalence. However, consider the composite map

$$r' : Z_0(X) \xrightarrow{\ r\ } \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} \twoheadrightarrow \pi_1^{t}(X)^{\mathrm{ab}},$$

where $\pi_1^{t}(X)^{\mathrm{ab}}$ is the quotient of $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ which classifies finite étale coverings of X with at most tame ramification "along the boundary of a compactification" (see Section 6). We show that r' factors through $h_0(X) = h_0(X; \mathrm{Spec}(\mathbb{Z}))$, defining an isomorphisms

$$\mathrm{rec} : h_0(X) \xrightarrow{\ \sim\ } \pi_1^{t}(X)^{\mathrm{ab}}$$

of finite abelian groups. Hence the singular homology group $h_0(X)$ takes over the role of $\mathrm{CH}_0(X)$ if the scheme X is not proper.

This article was motivated by the work of A. Suslin, V. Voevodsky and E.M. Friedlander on algebraic cycle theories for varieties over fields. The principal ideas underlying this paper originate from discussions with Michael Spieß during the preparation of our article [Schmidt and Spieß 2000]. The analogy between number fields and function fields in one variable over finite fields predicted that there should be a connection between the, yet to be defined, singular homology groups of a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and its tame fundamental group, similar to that we had proven for varieties over finite fields. The author wants to thank M. Spieß for fruitful discussions and for his remarks on a preliminary version of this paper.

The bulk of this article was part of the author's Habilitationschrift at Heidelberg University, seven years ago. However, I could not decide on publishing the material before the envisaged application to class field theory was established. This is the case now.

2. Preliminaries

Throughout this article we consider the category $\mathrm{Sch}(S)$ of separated schemes of finite type over a regular connected and Noetherian base scheme S. Quite early, we will restrict to the case that S is the spectrum of a Dedekind domain, which is the main case of our arithmetic application. We write $X \times Y = X \times_S Y$ for the fibre product of schemes $X, Y \in \mathrm{Sch}(S)$. Unless otherwise specified, all schemes will be assumed equidimensional.

Slightly modifying the approach of [Fulton 1998, Section 20.1], we define the (absolute) dimension of an integral scheme $X \in \mathrm{Sch}(S)$ in the following way. Let d be the Krull dimension of S, $K(X)$ the field of functions of X and T the closure of the image of X in S. Then we put

$$\dim X = \operatorname{trdeg}(K(X)|K(T)) - \operatorname{codim}_S(T) + d.$$

Examples 2.1. (1) Let $S = \mathrm{Spec}(\mathbb{Z}_p)$ and consider $X = \mathrm{Spec}(\mathbb{Z}_p[T]/pT - 1) \cong \mathrm{Spec}(\mathbb{Q}_p)$, a divisor on $\mathbb{A}^1_S = \mathrm{Spec}(\mathbb{Z}_p[T])$. Then $\dim X = 1$ in our terminology, while $\dim_{\mathrm{Krull}} X = 0$.

(2) The above notion of dimension coincides with the usual Krull dimension if
- S is the spectrum of a field,
- S is the spectrum of a Dedekind domain with infinitely many different prime ideals (e.g. the ring of integers in a number field).

Note that this change in the definition of dimension does not affect the notion of codimension. For a proof of the following lemma we refer to [Fulton 1998, Lemma 20.1].

Lemma 2.2. (i) *Let $U \subset X$ be a nonempty open subscheme. Then*

$$\dim X = \dim U.$$

(ii) *Let Y be a closed integral subscheme of the integral scheme X over S. Then*

$$\dim X = \dim Y + \operatorname{codim}_X(Y).$$

(iii) *If $f : X \to X'$ is a dominant morphism of integral schemes over S, then*

$$\dim X = \dim X' + \operatorname{trdeg}(K(X)|K(X')).$$

In particular, $\dim X' \le \dim X$ *with equality if and only if $K(X)$ is a finite extension of $K(X')$.*

Recall that a closed immersion $i : Y \longrightarrow X$ is called a *regular imbedding* of codimension d if every point y of Y has an affine neighbourhood U in X such that the ideal in $\mathbb{O}_U$ defining $Y \cap U$ is generated by a regular sequence of length d. We say that two closed subschemes A and B of a scheme X intersect *properly* if

$$\dim W = \dim A + \dim B - \dim X$$

(or, equivalently, $\operatorname{codim}_X W = \operatorname{codim}_X A + \operatorname{codim}_X B$) for every irreducible component W of $A \cap B$. In particular, an empty intersection is proper. Suppose that the immersion $A \to X$ is a regular imbedding. Then an inductive application of Krull's principal ideal theorem shows that every irreducible component of the intersection $A \cap B$ has dimension greater or equal to $\dim A + \dim B - \dim X$. In this case improper intersection means that one of the irreducible components of the intersection has a too large dimension. If B is a cycle of codimension 1, then the intersection is proper if and only if B does not contain an irreducible component of A.

The *group of cycles* $Z^r(X)$ (resp. $Z_r(X)$) of a scheme X is the free abelian group generated by closed integral subschemes of X of codimension r (resp. of dimension r). For a closed immersion $i : Y \to X$, we have obvious maps $i_* : Z_r(Y) \to Z_r(X)$ for all r. If i is a regular imbedding, we have a pullback map

$$i^* : Z^r(X)' \longrightarrow Z^r(Y),$$

where $Z^r(X)' \subset Z^r(X)$ is the subgroup generated by closed integral subschemes of X meeting Y properly. The map i^* is given by

$$i^*(V) = \sum_i n_i W_i,$$

where the W_i are the irreducible components of $i^{-1}(V) = V \cap Y$ and the n_i are the intersection multiplicities. For the definition of these multiplicities we refer to [Fulton 1998, Section 6] (or, alternatively, one can use Serre's Tor-formula [Serre 1965]).

The *standard n-simplex* $\Delta^n = \Delta^n_S$ over S is the closed subscheme in $\mathbb{A}^{n+1}_S$ defined by the equation $t_0 + \cdots + t_n = 1$. We call the sections $v_i : S \to \Delta^n_S$ corresponding to $t_i = 1$ and $t_j = 0$ for $j \neq i$ the *vertices* of Δ^n_S. Each nondecreasing map $\rho : [m] = \{0, 1, \ldots, m\} \longrightarrow [n] = \{0, 1, \ldots, n\}$ induces a scheme morphism

$$\bar{\rho} : \Delta^m \longrightarrow \Delta^n$$

defined by $t_i \mapsto \sum_{\rho(j)=i} t_j$. If ρ is injective, we say that $\rho(\Delta^m_S) \subset \Delta^n_S$ is a *face*. If ρ is surjective, $\bar{\rho}$ is a *degeneracy*. In this way $\Delta^\bullet_S$ becomes a cosimplicial scheme. Further note that all faces are regular imbeddings.

The following definition was motivated in the introduction.

Definition 2.3. For X in $\mathrm{Sch}(S)$ and $n \geq 0$, the group $C_n(X;S)$ is the free abelian group generated by closed integral subschemes Z of $X \times \Delta^n$ such that the restriction of the canonical projection

$$X \times \Delta^n \to \Delta^n$$

to Z induces a finite morphism $p: Z \to T \subset \Delta^n$ onto a closed integral subscheme T of codimension $d = \dim S$ in Δ^n which intersects all faces properly. In particular, such a Z is equidimensional of dimension n.

Remarks 2.4. (1) If the structural morphism $X \to S$ factors through a finite morphism $S' \to S$ with S' regular, then $C_n(X;S) = C_n(X;S')$. In particular, if $S' = \{P\}$ is a closed point of S, i.e. if X is a scheme of finite type over $\mathrm{Spec}(k(P))$, then $C_n(X;S) = C_n(X;k(P))$ is the n-th term of the singular complex of X defined by Suslin.

(2) If S is of dimension 1 (and regular and connected), then a closed integral subscheme T of codimension $d = 1$ in Δ^n_S intersects all faces properly if and only if it does not contain any face. If the image of X in S omits at least one closed point of S, then this condition is automatically satisfied.

Let Z be a closed integral subscheme of $X \times \Delta^n$ which projects finitely and surjectively onto a closed integral subscheme T of codimension d in Δ^n. Assume that T has proper intersection with all faces, i.e. Z defines an element of $C_n(X;S)$. Let $\Delta^m \hookrightarrow \Delta^n$ be a face. Since the projection

$$Z \times_{\Delta^n_X} \Delta^m_X \longrightarrow T \times_{\Delta^n} \Delta^m$$

is finite, each irreducible component of $Z \cap X \times \Delta^m$ has dimension at most m. On the other hand, a face is a regular imbedding and therefore all irreducible components of $Z \cap X \times \Delta^m$ have exact dimension m and project finitely and surjectively onto an irreducible component of $T \cap \Delta^m$. Thus the cycle theoretic inverse image $i^*(Z)$ is well defined and is in $C_m(X;S)$. Furthermore, degeneracy maps are flat, and thus we obtain a simplicial abelian group $C_\bullet(X;S)$. We use the same notation for the associated chain complex which (in the usual way) is constructed as follows.

Consider the 1-codimensional face operators

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n, \quad i = 0, \ldots, n,$$

defined by setting $t_i = 0$, and define the complex (concentrated in positive homological degrees)

$$C_\bullet(X;S), \quad d_n = \sum_{i=0}^{n} (-1)^i (d^i)^* : C_n(X;S) \to C_{n-1}(X;S).$$

Definition 2.5. We call $C_\bullet(X; S)$ the singular complex of X. Its homology groups (or likewise the homotopy groups of $C_\bullet(X; S)$ considered as a simplicial abelian group)

$$h_i(X; S) = H_i(C_\bullet(X; S)) \qquad \big(= \pi_i(C_\bullet(X; S))\big)$$

are called the (integral) singular homology groups of X.

From Remark 2.4(1) above, we obtain:

Lemma 2.6. *Assume that the structural morphism $X \to S$ factors through a finite morphism $S' \to S$ with S' regular. Then for all i,*

$$h_i(X; S) = h_i(X; S').$$

Examples 2.7. (1) If k is a field and $S = \mathrm{Spec}(k)$, then the above definition of $h_i(X)$ coincides with that of the singular homology of X defined by Suslin.

(2) $C_\bullet(X; S)$ is a subcomplex of Bloch's complex $z^r(X, \bullet)$, where $r = \dim X$, and $C_\bullet(S; S)$ coincides with the Bloch complex $z^d(S, \bullet)$. In particular,

$$h_i(S; S) = \mathrm{CH}^d(S, i),$$

where the group on the right is the higher Chow group defined by Bloch. Note that in [Bloch 1986], Bloch defined his higher Chow groups only for equidimensional schemes over a field, but there is no problem with extending his construction at hand.

The push-forward of cycles makes $C_\bullet(X; S)$ and thus also $h_i(X; S)$ covariantly functorial on $\mathrm{Sch}(S)$. Furthermore, it is contravariant under finite flat morphisms. Given a finite flat morphism $f : X' \to X$, we thus have induced maps $f_* : h_\bullet(X'; S) \to h_\bullet(X; S)$ and $f^* : h_\bullet(X; S) \to h_\bullet(X'; S)$, which are connected by the formula

$$f_* \circ f^* = \deg(f) \cdot \mathrm{id}_{h_\bullet(X;S)}.$$

In addition, we introduce bivariant homology groups. Let Y be equidimensional, of finite type and flat over S. If $X \times Y$ is empty, we let $C_\bullet(X, Y; S)$ be the trivial complex. Otherwise, $X \times Y$ it is a scheme of dimension $\dim X + \dim Y - d$ (as before, $d = \dim S$) and we consider the group $C_n(X, Y; S)$ which is the free abelian group generated by closed integral subschemes in $X \times Y \times \Delta^n$ such that the restriction of the canonical projection

$$X \times Y \times \Delta^n \to Y \times \Delta^n$$

to Z induces a finite morphism $p : Z \to T \subset Y \times \Delta^n$ onto a closed integral subscheme T of codimension d in $Y \times \Delta^n$ which intersects all faces $Y \times \Delta^m$ properly. In particular, such a Z is equidimensional of dimension $\dim Y + n - d$. Further, for a closed subscheme $Y' \subset Y$, consider the subgroup $C_n^{Y'}(X, Y; S) \subset C_n(X, Y; S)$, which is the free abelian group generated by closed integral subschemes of $X \times Y \times \Delta^n$ such that the restriction of the canonical projection

$$X \times Y \times \Delta^n \to Y \times \Delta^n$$

to Z induces a finite morphism $p : Z \to T \subset Y \times \Delta^n$ onto a closed integral subscheme T of codimension d in $Y \times \Delta^n$ which intersects all faces $Y \times \Delta^m$ and all faces $Y' \times \Delta^m$ properly.

In the same way as before, we obtain the complex $C_\bullet(X, Y; S)$, which contains the subcomplex $C_\bullet^{Y'}(X, Y; S)$.

Definition 2.8. We call $C_\bullet(X, Y; S)$ the bivariant singular complex and its homology groups

$$h_i(X, Y; S) = H_i(C_\bullet(X, Y; S))$$

the bivariant singular homology groups.

Note that $C_\bullet(X, S; S) = C_\bullet(X; S)$ and $h_i(X, S; S) = h_i(X; S)$. By pulling back cycles, a flat morphism $Y' \to Y$ induces a homomorphism of complexes

$$C_\bullet(X, Y; S) \longrightarrow C_\bullet(X, Y'; S).$$

If $Y' \hookrightarrow Y$ is a regular imbedding, we get a natural homomorphism

$$C_\bullet^{Y'}(X, Y; S) \longrightarrow C_\bullet(X, Y'; S).$$

Consider the complex of presheaves $\underline{C}_\bullet(X; S)$ which is given on open subschemes $U \subset S$ by

$$U \longmapsto C_\bullet(X, U; S).$$

This is already a complex of Zariski-sheaves on S.

Definition 2.9. By $\underline{h}_i(X; S)$ we denote the cohomology sheaves of the complex $\underline{C}_\bullet(X; S)$. Equivalently, $\underline{h}_i(X; S)$ is the Zariski sheaf on S associated to

$$U \longmapsto h_i(X, U; S).$$

(The sheaves $\underline{h}_i$ play a similar role as Bloch's higher Chow sheaves [Bloch 1986].)

Now assume that X and Y are smooth over S. By $c(X, Y)$ we denote the free abelian group generated by integral closed subschemes $W \subset X \times Y$ which are finite over X and surjective over a connected component of X. An element in $c(X, Y)$ is called a finite correspondence from X to Y. If X_1, X_2, X_3 is a triple of smooth schemes over S, then, by [Voevodsky 2000, Section 2], there exists a natural composition $c(X_1, X_2) \times c(X_2, X_3) \to c(X_1, X_3)$. Therefore one can define a category SmCor(S) whose objects are smooth schemes of finite type over S and morphisms are finite correspondences. The category Sm(S) of smooth schemes of finite type over S admits a natural functor to $\mathrm{SmCor}(S)$ by sending a morphism to its graph.

Let X and Y be smooth over S, let $\phi \in c(X, Y)$ be a finite correspondence and let $\psi \in C_n(X, S)$. Consider the product $X \times Y \times \Delta^n$ and let p_1, p_2, p_3 be the corresponding projections. Then the cycles $(p_1 \times p_3)^*(\psi)$ and $(p_1 \times p_2)^*(\phi)$ are in general position. Let $\psi * \phi$ be their intersection. Since ϕ is finite over X and ψ is finite over Δ^n, we can define the cycle $\phi \circ \psi$ as $(p_2 \times p_3)_*(\phi * \psi)$. The cycle $\phi \circ \psi$ is in $C_n(Y; S)$, and so we obtain a natural pairing $c(X, Y) \times C_\bullet(X; S) \to C_\bullet(Y; S)$. We obtain the following:

Proposition 2.10. *For schemes X, Y that are smooth over S, there exist natural pairings for all i*

$$c(X, Y) \otimes h_i(X; S) \longrightarrow h_i(Y; S)$$

making singular homology into a covariant functor on the category $\mathrm{SmCor}(S)$.

3. Singular homology of curves

We start this section by recalling some notions and lemmas from [Suslin and Voevodsky 1996]. Let X be a scheme and let Y be a closed subscheme of X. Set $U = X - Y$ and denote by $i : Y \longrightarrow X$, $j : U \longrightarrow X$ the corresponding closed and open embeddings.

We denote by $\mathrm{Pic}(X, Y)$ (the relative Picard group) the group whose elements are isomorphism classes of pairs of the form (L, ϕ), where L is a line bundle on X and $\phi : L|_Y \cong \mathbb{O}_Y$ is a trivialization of L over Y, and the operation is given by the tensor product. There is an evident exact sequence

$$\Gamma(X, \mathbb{O}_X^\times) \longrightarrow \Gamma(Y, \mathbb{O}_Y^\times) \longrightarrow \mathrm{Pic}(X, Y) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y). \qquad (1)$$

We also use the notation $\mathbb{G}_X$ (or $\mathbb{G}_m$) for the sheaf of invertible functions on X and we write $\mathbb{G}_{X,Y}$ for the sheaf on X which is defined by the exact sequence

$$0 \longrightarrow \mathbb{G}_{X,Y} \longrightarrow \mathbb{G}_X \longrightarrow i_*(\mathbb{G}_Y) \longrightarrow 0.$$

By [Suslin and Voevodsky 1996, Lemma 2.1], there are natural isomorphisms

$$\mathrm{Pic}(X, Y) = H^1_{\mathrm{Zar}}(X, \mathbb{G}_{X,Y}) = H^1_{\mathrm{et}}(X, \mathbb{G}_{X,Y}).$$

Assume that X is integral and denote by K the field of rational functions on X. A relative Cartier divisor on X is a Cartier divisor D such that $\mathrm{supp}(D) \cap Y = \varnothing$. If D is a relative divisor and $Z = \mathrm{supp}(D)$, then $\mathbb{O}_X(D)|_{X-Z} = \mathbb{O}_{X-Z}$. Thus D defines an element in $\mathrm{Pic}(X, Y)$. Denoting the group of relative Cartier divisors by $\mathrm{Div}(X, Y)$, we get a natural homomorphism $\mathrm{Div}(X, Y) \to \mathrm{Pic}(X, Y)$. The image of this homomorphism consists of pairs (L, ϕ) such that ϕ admits an extension to a trivialization of L over an open neighbourhood of Y. In particular, this map is surjective provided that Y has an affine open neighbourhood. Furthermore, we put

$$\begin{aligned} G &= \{f \in K^\times : f \in \ker(\mathbb{O}_{X,y}^\times \longrightarrow \mathbb{O}_{Y,y}^\times) \text{ for any } y \in Y\} \\ &= \{f \in K^\times : f \text{ is defined and equal to 1 at each point of } Y\}, \end{aligned}$$

The following lemmas are straightforward; see [Suslin and Voevodsky 1996, 2.3, 2.4, 2.5].

Lemma 3.1. *Assume that Y has an affine open neighbourhood in X. Then the following sequence is exact*:

$$0 \longrightarrow \Gamma(X, \mathbb{G}_{X,Y}) \longrightarrow G \longrightarrow \mathrm{Div}(X, Y) \longrightarrow \mathrm{Pic}(X, Y) \longrightarrow 0.$$

Lemma 3.2. *Assume that U is normal and every closed integral subscheme of U of codimension one which is closed in X is a Cartier divisor* (*this happens for example when U is factorial*). *Then* $\mathrm{Div}(X, Y)$ *is the free abelian group generated by closed integral subschemes $T \subset U$ of codimension one which are closed in X.*

Lemma 3.3. *Let X be a scheme. Consider the natural homomorphism*

$$p^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{A}_X^1)$$

which is induced by the projection $p : \mathbb{A}_X^1 \to X$. If X is reduced, *then p^* is injective. If X is normal*, *it is an isomorphism.*

Proof. Since X is reduced, we have $p_*\mathbb{G}_{\mathbb{A}_X^1} = \mathbb{G}_X$. Therefore the spectral sequence

$$E_2^{ij} = H^i(X, R^j p_*\mathbb{G}_{\mathbb{A}_X^1}) \Longrightarrow H^{i+j}(\mathbb{A}_X^1, \mathbb{G}_{\mathbb{A}_X^1})$$

induces a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{A}_X^1) \to H^0(X, R^1 p_*(\mathbb{G}_{\mathbb{A}_X^1})).$$

This shows the first statement. The stalk of $R^1 p_*(\mathbb{G}_{\mathbb{A}_X^1})$ at a point $x \in X$ is the Picard group of the affine scheme $\mathrm{Spec}(\mathbb{O}_{X,x}[T])$. If X is normal, then this group is trivial by [Bass and Murthy 1967, Proposition 5.5]. This concludes the proof. □

Corollary 3.4. *Assume that X is normal and Y is reduced. Then*

$$\mathrm{Pic}(X, Y) \cong \mathrm{Pic}(\mathbb{A}_X^1, \mathbb{A}_Y^1).$$

Proof. Using the five-lemma, this follows from Lemma 3.3 together with the exact sequence (1). □

In the case that $S = \mathrm{Spec}(k)$ is the spectrum of a field k, our singular homology coincides with that defined by Suslin. For a proof of the next theorem, see [Lichtenbaum 1993].

Theorem 3.5. *Let X be a smooth, geometrically connected curve over k, let $\bar{X}$ be a smooth compactification of X and let $Y = \bar{X} - X$. Then $h_i(X; k) = 0$ for $i \neq 0, 1$ and*

$$h_0(X; k) = \operatorname{Pic}(\bar{X}, Y),$$
$$h_1(X; k) = \begin{cases} 0 & \text{if } X \text{ is affine,} \\ k^\times & \text{if } X \text{ is proper.} \end{cases}$$

Corollary 3.6. *Let X be a smooth curve over a field k, $\bar{X}$ a smooth compactification of X over k and $Y = \bar{X} - X$. Then for all i,*

$$h_i(X; k) \cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y}) \cong \mathbb{H}^{-i}_{\mathrm{Zar}}\big(\bar{X}, \operatorname{cone}\big(\mathbb{G}_{\bar{X}} \longrightarrow i_{Y*}(\mathbb{G}_Y)\big)\big),$$

where $\mathbb{H}_{\mathrm{Zar}}$ denotes Zariski hypercohomology.

This corollary is a special case of a general duality theorem proven in [Voevodsky 2000, Theorem 4.3.7] over fields that admit resolution of singularities.

We now consider the case that S is the spectrum of a Dedekind domain, which is the case of main interest for us. The proof of the following theorem is parallel to the proof of Theorem 3.1 of [Suslin and Voevodsky 1996], where the relative singular homology of relative curves was calculated.

Theorem 3.7. *Assume that S is the spectrum of a Dedekind domain and let U be an open subscheme of S. Let $Y \in \mathrm{Sch}(S)$ be regular and flat over S. Setting $Y_U = Y \times U$, suppose that $Y - Y_U$ has an affine open neighbourhood in Y. Then $h_i(U, Y; S) = 0$ for $i \neq 0, 1$ and*

$$h_0(U, Y; S) = \operatorname{Pic}(Y, Y - Y_U),$$
$$h_1(U, Y; S) = \Gamma(Y, \mathbb{G}_{Y, Y-Y_U}).$$

Proof. We may assume that Y is connected. If $Y_U = Y$, then $C_\bullet(U, Y; S)$ coincides with the Bloch complex $z^1(Y, \bullet)$. By [Bloch 1986, Theorem 6.1] (whose proof applies without change to arbitrary regular schemes), we have $h_i(U, Y; S) = 0$ for $i \neq 0, 1$ and

$$h_0(U, Y; S) = \operatorname{Pic}(Y),$$
$$h_1(U, Y; S) = \Gamma(Y, \mathbb{G}_Y).$$

Suppose that $Y_U \subsetneq Y$. Then an integral subscheme $Z \subset Y_U \times \Delta^n$ is in $C_n(U, Y; S)$ if and only if it is closed and of codimension 1 in $Y \times \Delta^n$. Since Y is regular, such a Z is a Cartier divisor and it automatically has proper intersection with all faces (see Remark 2.4(2)). Thus $C_n(U, Y; S) = \operatorname{Div}(Y, T)$ (see Lemma 3.2). Let $T = Y - Y_U$. If V is an open affine neighbourhood of T in Y, then $V \times \Delta^n$ is an open affine neighbourhood of $T \times \Delta^n$ in $Y \times \Delta^n$. According to Lemma 3.1, we have an exact sequence of simplicial abelian groups:

$$0 \to A_\bullet \to G_\bullet \to C_\bullet(U, Y; S) \to \mathrm{Pic}(\Delta_Y^\bullet, \Delta_T^\bullet) \to 0, \qquad (2)$$

where

$$G_n = \{f \in k(\Delta_Y^n)^\times : f \text{ is defined and equal to 1 at each point of } \Delta_T^n\},$$
$$A_n = \Gamma(\Delta_Y^n, \mathbb{G}_{\Delta_Y^n, \Delta_T^n}).$$

For each n, we have $A_n = A_0 = \Gamma(Y, \mathbb{G}_{Y,T})$ and by Corollary 3.4, we have $\mathrm{Pic}(\Delta_Y^n, \Delta_T^n) = \mathrm{Pic}(Y, T)$. Let us show that the simplicial abelian group $G_\bullet$ is acyclic, i.e. $\pi_*(G_\bullet) = 0$. It suffices to check that for any $f \in G_n$ such that $\delta_i(f) = 1$ for $i = 0, \ldots, n$, there exists a $g \in G_{n+1}$ such that $\delta_i(g) = 1$ for $i = 0, \ldots, n$ and $\delta_{n+1}(g) = f$. Define functions $g_i \in G_{n+1}$ for $i = 1, \ldots, n$ by means of the formula

$$g_i = (t_{i+1} + \cdots + t_{n+1}) + (t_0 + \cdots + t_i)s_i(f).$$

These functions satisfy the following equations:

$$\delta_j(g_i) = \begin{cases} 1 & \text{if } j \neq i, i+1, \\ (t_i + \cdots + t_n) + (t_0 + \cdots + t_{i-1})f & \text{if } j = i, \\ (t_{i+1} + \cdots + t_n) + (t_0 + \cdots t_i)f & \text{if } j = i+1. \end{cases}$$

In particular, $\delta_0(g_0) = 1$, $\delta_{n+1}(g_n) = f$. Finally, we set

$$g = g_n g_{n-1}^{-1} g_{n-2} \cdots g_0^{(-1)^n}.$$

This function satisfies the conditions we need. Evaluating the 4-term exact sequence (2) above, we obtain the statement of the theorem. □

Corollary 3.8. *Assume that S is the spectrum of a Dedekind domain. Let X be regular and quasifinite over S, $\bar{X}$ a regular compactification of X over S and $Y = \bar{X} - X$. Then for all i,*

$$h_i(X; k) \cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y})$$
$$\cong \mathbb{H}^{-i}_{\mathrm{Zar}}\big(\bar{X}, \mathrm{cone}\big(\mathbb{G}_{\bar{X}} \longrightarrow i_{Y*}(\mathbb{G}_Y)\big)\big),$$

where $\mathbb{H}_{\mathrm{Zar}}$ denotes Zariski hypercohomology.

Proof. We may assume that X is connected. By Zariski's main theorem, X is an open subscheme of the normalization S' of S in the function field of X. As is well known, $S' = \bar{X}$ is again the spectrum of a Dedekind domain and the projection $S' \to S$ is a finite morphism. Therefore the result follows from Lemma 2.6 and from Theorem 3.7 applied to the case $Y = S$. □

Corollary 3.9. *Let S be the spectrum of a Dedekind domain. Assume that X is regular and that the structural morphism $p : X \to S$ is quasifinite. Let $\bar{p} : \bar{X} \to S$ be a regular compactification of X over S and $Y = \bar{X} - X$. Then there is a natural isomorphism*

$$\underline{C}_\bullet(X; S) \cong \bar{p}_* \mathbb{G}_{\bar{X},Y}[1]$$

in the derived category of complexes of Zariski-sheaves on S.

Proof. We may assume that X is connected and we apply the result of Theorem 3.7 to open subschemes $Y \subset S$. Note that $\bar{X}$ is the normalization of S in the function field of X. The stalk of $\underline{h}_1(X; S)$ at a point $s \in S$ is the relative Picard group of the semilocal scheme $\bar{X} \times_S S_s$ with respect to the finite set of closed points not lying on X. A semilocal Dedekind domain is a principal ideal domain, and the exact sequence (1) from the beginning of this section shows that also the corresponding relative Picard group is trivial. Therefore, the complex of sheaves $\underline{C}_\bullet(X; S)$ has exactly one nontrivial homology sheaf, which is placed in homological degree 1 and is isomorphic to $\bar{p}_* \mathbb{G}_{\bar{X},Y}$. □

Let us formulate a few results which easily follow from Theorem 3.7. We hope that these results are (mutatis mutandis) true for regular schemes X of arbitrary dimension. We omit S from the notation, writing $h_*(X)$ for $h_*(X; S)$ and $h_*(X, Y)$ for $h_*(X, Y; S)$

Theorem 3.10. *Let S be the spectrum of a Dedekind domain. Assume that X is regular and quasifinite over S (in particular,* $\dim X = 1$*). Then the following holds.*

(i) $h_i(X) = \mathbb{H}^{-i}_{\mathrm{Zar}}(S, \underline{C}_\bullet(X; S))$ *for all i.*

(ii) (*Local to global spectral sequence*) *There exists a spectral sequence*

$$E_2^{ij} = H^{-i}_{\mathrm{Zar}}(S, \underline{h}_j(X)) \Rightarrow h_{i+j}(X).$$

(iii) (*Mayer–Vietoris sequence*) *Let $X_1, X_2 \subset X$ be open with $X = X_1 \cup X_2$. Then there is an exact sequence*

$$0 \to h_1(X_1 \cap X_2) \to h_1(X_1) \oplus h_1(X_2) \to h_1(X) \to h_0(X_1 \cap X_2) \to h_0(X_1) \oplus h_0(X_2) \to h_0(X) \to 0.$$

(iv) (*Mayer–Vietoris sequence with respect to the second variable*) *Let $U, V \subset S$ be open. Then there is an exact sequence*

$$0 \to h_1(X, U \cup V) \to h_1(X, U) \oplus h_1(X, V) \to h_1(X, U \cap V) \to h_0(X, U \cup V) \to h_0(X, U) \oplus h_0(X, V) \to h_0(X, U \cap V) \to 0.$$

Proof. We may assume that X is connected. Let S' be the normalization of S in the function field of X, and we denote by $j_X : X \to S'$ the corresponding open immersion (compare the proof of Corollary 3.8). Let, for an open subscheme $U \subset S$, U' be its preimage in S'. Then

$$h_i(X, U; S) = h_i(X, U'; S'),$$

and therefore we may assume that $S' = S$ in the proof of (iii) and (iv). Then, by Corollary 3.8, $h_i(X) = H^{1-i}_{\mathrm{Zar}}(S, \mathbb{G}_{S,S-X})$. Assertion (iii) follows by applying the functor $\mathbf{R}\Gamma(S, -)$ to the exact sequence of Zariski sheaves

$$0 \longrightarrow \mathbb{G}_{S,S-X_1\cap X_2} \longrightarrow \mathbb{G}_{S,S-X_1} \oplus \mathbb{G}_{S,S-X_2} \longrightarrow \mathbb{G}_{S,S-X} \longrightarrow 0.$$

For an open subscheme $j_U : U \longrightarrow S$, we denote the sheaf $j_{U,!}j_U^*(\mathbb{Z})$ by $\mathbb{Z}_U$. Then, for a sheaf F on S, we have a canonical isomorphism

$$H^i_{\mathrm{Zar}}(U, j_U^* F) \cong \mathrm{Ext}^i_S(\mathbb{Z}_U, F).$$

Applying the functor $\mathbf{R}\mathrm{Hom}_S(-, \mathbb{G}_{S,S-X})$ to the exact sequence of Zariski sheaves

$$0 \longrightarrow \mathbb{Z}_{U\cap V} \longrightarrow \mathbb{Z}_U \oplus \mathbb{Z}_V \longrightarrow \mathbb{Z}_{U\cup V} \longrightarrow 0,$$

Theorem 3.7 implies assertion (iv). From (iv) it follows that the complex $\underline{C}_\bullet(X)$ is pseudo-flasque in the sense of [Brown and Gersten 1973], which shows assertion (i). Finally, (ii) follows from the corresponding hypercohomology spectral sequence converging to $\mathbb{H}^{-i}_{\mathrm{Zar}}(S, \underline{C}_\bullet(X; S))$ and from (i). □

Finally, we deduce an exact Gysin sequence for one-dimensional schemes. In order to formulate it, we need the notion of twists. Let $\mathbb{G}_m$ denote the multiplicative group scheme $\mathbb{A}^1_S - \{0\}$ and let X be any scheme of finite type over S. For $i = 1, \cdots, n$, let $D^i_\bullet(X \times \mathbb{G}_m^{\times(n-1)}; S)$ be the direct summand in $C_\bullet(X \times \mathbb{G}_m^{\times n}; S)$ which is given by the homomorphism

$$\mathbb{G}_m^{\times(n-1)} \longrightarrow \mathbb{G}_m^{\times n}, \quad (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, 1_i, \ldots, x_{n-1})$$

We consider the complex $C_\bullet(X \times \mathbb{G}_m^{\wedge n}; S)$ which is defined as the direct summand of the complex $C_\bullet(X \times \mathbb{G}_m^{\times n}; S)$ complementary to the direct summand $\sum_{i=1}^n D^i_\bullet(X \times \mathbb{G}_m^{\times(n-1)}; S)$; see [Suslin and Voevodsky 2000, Section 3].

Definition 3.11. For $n \geq 0$, we put

$$h_i(X(n); S) = H_{i+n}(C_\bullet(X \times \mathbb{G}_m^{\wedge n}; S)).$$

In particular, we have $h_i(X(0); S) = h_i(X; S)$ for all i and $h_i(X(n); S) = 0$ for $i < -n$. If $X = \{P\}$ is a closed point on S, then (see [Suslin and Voevodsky 2000, Lemma 3.2])

$$h_i(\{P\}(1); S) = \begin{cases} k(P)^\times & \text{for } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The next corollary follows from this and from Theorem 3.7.

Corollary 3.12. *Assume that X is regular and quasifinite over S and that U is an open, dense subscheme in X. Then we have a natural exact sequence*

$$0 \to h_1(U) \to h_1(X) \to h_{-1}((X-U)(1)) \to h_0(U) \to h_0(X) \to 0.$$

4. Homotopy invariance

Throughout this section we fix our base scheme S, which is the spectrum of a Dedekind domain, and we omit it from the notation, writing $h_*(X)$ for $h_*(X; S)$ and $h_*(X, Y)$ for $h_*(X, Y; S)$. Our aim is to prove that the relative singular homology groups $h_*(X, Y)$ are homotopy invariant with respect to both variables.

Theorem 4.1. *Let X and Y be of finite type over S. Then the projection $X \times \mathbb{A}^1 \to X$ induces isomorphisms*

$$h_i(X \times \mathbb{A}^1, Y) \xrightarrow{\sim} h_i(X, Y)$$

for all i.

Let $i_0, i_1 : Y \longrightarrow Y \times \mathbb{A}^1$ be the embeddings defined by the points (i.e. sections over S) 0 and 1 of $\mathbb{A}^1 = \mathbb{A}^1_S$.

Recall that Δ^n has coordinates $(t_0, \dots, t_n)$ with $\sum t_i = 1$. Vertices are the points (i.e. sections over S) $p_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th place. Consider the linear isomorphisms

$$\theta_i : \Delta^{n+1} \longrightarrow \Delta^n \times \mathbb{A}^1, \quad i = 0, \dots, n$$

which are defined by taking p_j to $(p_j, 0)$ for $j \le i$ and to $(p_{j-1}, 1)$ if $j > i$. Then consider for each n the formal linear combination

$$T_n = \sum_{i=0}^{n} (-1)^i \theta_i.$$

Let us call a subscheme $F \subset \Delta^n \times \mathbb{A}^1$ a face if it corresponds to a face in Δ^{n+1} under one of the linear isomorphisms θ_i. Using this terminology, T_n defines a homomorphism from a subgroup of $C_n(X, Y \times \mathbb{A}^1)$ to $C_{n+1}(X, Y)$. This subgroup is generated by cycles having good intersection not only with all faces $Y \times \mathbb{A}^1 \times \Delta^m$ but also with all faces of the form $Y \times F$, where F is a face in $\mathbb{A}^1 \times \Delta^n$.

We will deduce Theorem 4.1 from the following proposition.

Proposition 4.2. *The two chain maps*

$$i_{0*}, i_{1*} : C_\bullet(X, Y) \longrightarrow C_\bullet(X \times \mathbb{A}^1, Y)$$

are homotopic. In particular, i_{0*} and i_{1*} induce the same map on homology.

Proof. Let $D \subset \mathbb{A}^1 \times \mathbb{A}^1$ be the diagonal. Consider the map

$$V_n : C_n(X, Y) \longrightarrow C_n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1)$$

which is defined by sending a cycle $Z \subset X \times Y \times \Delta^n$ to the cycle $Z \times D \subset X \times Y \times \Delta^n \times \mathbb{A}^1 \times \mathbb{A}^1$. If Z projects finitely and surjectively onto $T \subset Y \times \Delta^n$, then $Z \times D$ projects finitely and surjectively onto $T \times \mathbb{A}^1 \subset Y \times \Delta^n \times \mathbb{A}^1$. Therefore V_n is well defined. Fortunately, $T \times \mathbb{A}^1$ has proper intersection with all faces $Y \times F$, where F is a face in $\Delta^n \times \mathbb{A}^1$. Therefore the composition

$$T_{n*} \circ V_n : C_n(X, Y) \longrightarrow C_n(X \times \mathbb{A}^1, Y \times A^1) \longrightarrow C_{n+1}(X \times \mathbb{A}^1, Y)$$

is well defined for every n. These maps give the required homotopy. □

Proof of Theorem 4.1. Let $\tau : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the multiplication map. Consider the diagram

$$\begin{array}{ccc} C_\bullet(X \times \mathbb{A}^1, Y) & \xrightarrow{p_*} & C_\bullet(X, Y) \\ \big\downarrow{\scriptstyle i_{0*}, i_{1*}} & & \big\downarrow{\scriptstyle i_{0*}, i_{1*}} \\ C_\bullet(X \times \mathbb{A}^1 \times \mathbb{A}^1, Y) & \xrightarrow{\tau_*} & C_\bullet(X \times \mathbb{A}^1, Y). \end{array}$$

We have the following equalities of maps on homology:

$$i_{0*} \circ p_* = \tau_* \circ i_{0*} = \tau_* \circ i_{1*} = \mathrm{id}_{h_\bullet(X,Y)}.$$

Therefore, p_* is injective on homology. But on the other hand, $p \circ i_0 = \mathrm{id}_X$, which shows that p_* is surjective. This concludes the proof. □

Now, exploiting a moving technique of [Bloch 1986], we prove that the bivariant singular homology groups $h_*(X, Y)$ are homotopy invariant with respect to the second variable.

Theorem 4.3. *Assume that S is the spectrum of a Dedekind domain and let X and Y be of finite type over S. Then the projection $p : Y \times \mathbb{A}^1 \to Y$ induces isomorphisms for all i,*

$$h_i(X, Y) \xrightarrow{\sim} h_i(X, Y \times \mathbb{A}^1).$$

A typical intermediate step in proving a theorem like Theorem 4.3 would be to show that the induced chain maps $i_0^*, i_1^* : C_\bullet(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$ are homotopic. However, i_0^*, i_1^* are only defined as homomorphisms on the subcomplex

$$i_0^*, i_1^* : C_\bullet^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y).$$

(The maps $T_n^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_{n+1}(X, Y)$ would define a homotopy $i_0^* \sim i_1^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_n(X, Y)$, if all these maps *would* be defined.)

The proof of Theorem 4.3 will consist of several steps. First, we show that the inclusion

$$C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y\times\mathbb{A}^1)$$

is a quasi-isomorphism. Then we show that the homomorphisms

$$i_0^*, i_1^* : C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$$

induce the same map on homology. Finally, we deduce Theorem 4.3 from these results.

In the proof we will apply a moving technique of [Bloch 1986] which was used there to show the homotopy invariance of the higher Chow groups. As long as we have to deal with cycles of codimension 1, this technique also works in our more general situation (this is the reason for the restriction to the case that S is the spectrum of a Dedekind domain).

We would like to construct a homotopy between the identity of the complex $C_\bullet(X, Y\times\mathbb{A}^1)$ and another map which takes its image in the subcomplex

$$C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1).$$

What we *can* do is the following:

For a suitable scheme S' over S we construct a homotopy between the pullback map $C_\bullet(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y\times\mathbb{A}^1\times S')$ and another map whose image is contained in the subcomplex

$$C_\bullet^{Y\times\{0,1\}\times S'}(X, Y\times\mathbb{A}^1\times S').$$

(Eventually, we will use $S' = \mathbb{A}^1_S$ but perhaps this would be too many $\mathbb{A}^1$'s in the notation.)

Let (for the moment) $\pi : S' \to S$ be any integral scheme of finite type over S and let t be an element in $\Gamma(S', \mathbb{O}_{S'})$. Consider the action

$$\mathbb{A}^1_{S'} \times_{S'} (Y\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1)_{S'}$$

of the smooth group scheme $\mathbb{A}^1_{S'}$ on $(Y\times\mathbb{A}^1)_{S'}$ given by additive translation

$$a\cdot(y, b)) = (y, a+b)$$

and consider the morphism $\psi : \mathbb{A}^1_{S'} \to \mathbb{A}^1_{S'}$ given by multiplication by t: $a \mapsto ta$. The points 0, 1 of $\mathbb{A}^1_{S'}$ give rise to isomorphisms

$$\psi(0), \psi(1) : (Y\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1)_{S'}$$

($\psi(0)$ is the identity and $\psi(1)$ sends (y, b) to $(y, t+b)$). Furthermore, setting $\phi(y, a, b) = (y, \psi(b)\cdot a, b)$, we obtain an isomorphism

$$\phi : (Y\times\mathbb{A}^1\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1\times\mathbb{A}^1)_{S'}.$$

We would like to compose the maps

$$C_n(X, Y \times \mathbb{A}^1) \xrightarrow{\pi^*} C_n(X, Y \times \mathbb{A}^1 \times S') \xrightarrow{\mathrm{pr}^*} C_n(X, (Y \times \mathbb{A}^1) \times \mathbb{A}^1 \times S')$$
$$\xrightarrow{\phi^*} C_n(X, (Y \times \mathbb{A}^1) \times \mathbb{A}^1 \times S') \xrightarrow{T_n^*} C_{n+1}(X, Y \times \mathbb{A}^1 \times S'),$$

but we are confronted with the problem that the map T_n^* is not defined on the whole group $C_n(X, (Y \times \mathbb{A}^1) \times \mathbb{A}^1 \times S')$. The next proposition tells us that the composition is well defined if $S' = \mathbb{A}^1_S = \operatorname{Spec} S[t]$.

Proposition 4.4. *Suppose that* $S' = \mathbb{A}^1_S = \operatorname{Spec} S[t]$. *Then the composition*

$$H_n = T_n^* \circ \phi^* \circ \mathrm{pr}^* \circ \pi^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_{n+1}(X, Y \times \mathbb{A}^1 \times S')$$

is well defined for every n. The family $\{H_n\}_{n \geq 0}$ *defines a homotopy*

$$\pi^* = \psi(0) \circ \pi^* \sim \psi(1) \circ \pi^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_n(X, Y \times \mathbb{A}^1 \times S').$$

Furthermore, the image of the map $\psi(1) \circ \pi^*$ *is contained in the subcomplex*

$$C_\bullet^{Y \times \{0,1\} \times S'}(X, Y \times \mathbb{A}^1 \times S').$$

Proof. Recall that all groups $C_\bullet$ are relative to the base scheme S which we have omitted from the notation. At the moment, the map H_n is only defined as a map to the group of cycles in $X \times Y \times \mathbb{A}^1 \times \Delta^{n+1} \times S'$. If $Z \subset X \times Y \times \mathbb{A}^1 \times \Delta^n$ projects finitely and surjectively onto an irreducible subscheme $T \subset Y \times \mathbb{A}^1 \times \Delta^n$ of codimension one, then $\phi^* \circ \mathrm{pr}^* \circ \pi^*(Z)$ projects finitely and surjectively onto the irreducible subscheme of codimension one $T' = \phi^* \circ \mathrm{pr}^* \circ \pi^*(T) \subset (Y \times \mathbb{A}^1) \times \Delta^n \times \mathbb{A}^1 \times S'$. Therefore, in order to show that $H_n(Z)$ is in $C_{n+1}(X, Y \times \mathbb{A}^1 \times S')$, we have to check that $\theta_i^{-1}(T')$ has proper intersection with all faces for $i = 0, \dots, n$. Thus we have to show that T' has proper intersection with all faces $(Y \times \mathbb{A}^1) \times F \times S'$, where F is a face in $\Delta^n \times \mathbb{A}^1$ (as defined above). Since T' has codimension one, this comes down to show that it does not contain any irreducible component of any face (we did not assume Y to be irreducible, but we can silently assume that it is reduced). Consider the projection

$$Y \times \mathbb{A}^1 \times \Delta^n \times \mathbb{A}^1 \times S' \longrightarrow S'.$$

We can check our condition by considering the fibre over the generic point of S'. More precisely, let k be the function field of S and let $K = k(t)$ be the function field of S'. Let $(Y_1)_k, \dots, (Y_r)_k$ be the irreducible components of Y_k. Then an irreducible subscheme $T' \subset Y \times \mathbb{A}^1 \times \Delta^n \times \mathbb{A}^1 \times S'$ of codimension one meets all faces $Y \times \mathbb{A}^1 \times F \times S'$ (F a face of $\Delta^n \times \mathbb{A}^1$) properly if and only if T_K does not contain $(Y_i \times \mathbb{A}^1)_K \times_K F_K$ for $i = 1, \dots, r$.

Now we arrived exactly at the situation considered in [Bloch 1986, Section 2]. The result follows from [Bloch 1986, Lemma 2.2] by taking $(Y \times \mathbb{A}^1)_k$ for the scheme X of that lemma, taking $\mathbb{A}^1_k$ as the algebraic group G acting on X by additive translation on the second factor and choosing the map $\psi : \mathbb{A}^1_K \to G_K$ of that lemma as the morphism which sends a to ta. The fact that the H_n define the homotopy is a straightforward computation.

It remains to show that the image of the map $\psi(1) \circ \pi^*$ is contained in the subcomplex

$$C_\bullet^{Y\times\{0,1\}\times S'}(X, Y \times \mathbb{A}^1 \times S').$$

But this is a again a condition which says that a subscheme of codimension one does not contain certain subschemes. In the same way as above, this can be verified over the generic fibre of S', and the result follows from the corresponding statement of [Bloch 1986, Lemma 2.2]. □

Corollary 4.5. *The natural inclusion*

$$C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times \mathbb{A}^1)$$

is a quasi-isomorphism.

Proof. Let $S' = \mathbb{A}^1_S$. Then the homomorphism

$$\pi^* : C_\bullet(X, Y \times \mathbb{A}^1)/C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times \mathbb{A}^1 \times S')/C_\bullet^{Y\times\{0,1\}\times S'}(X, Y \times \mathbb{A}^1 \times S')$$

is nullhomotopic (the H_n of Proposition 4.4 give the homotopy). In order to conclude the proof, it suffices to show that the nullhomotopic homomorphism π^* is injective on homology. Suppose that for a cycle z in degree n we have $\pi^*(z) = d_n(w)$. Then we find an $a \in \Gamma(S, \mathcal{O}_S)$ such that the specialization (i.e. $t \mapsto a$) $w(a)$ is well defined. But then $z = d_n(w(a))$. □

Proposition 4.6. *Suppose that* $S' = \mathbb{A}^1_S = \operatorname{Spec} S[t]$. *Then the composition*

$$C_n^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) \xrightarrow{\psi(1)\circ\pi^*} C_n^{Y\times\{0,1\}\times S'}(X, Y\times\mathbb{A}^1\times S') \xrightarrow{T_n^*} C_{n+1}(X, Y\times S')$$

is well defined, giving a homotopy

$$i_0^* \circ \psi(1) \circ \pi^* \sim i_1^* \circ \psi(1) \circ \pi^* : C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times S').$$

Proof. Let again k be the function field of S and let $K = k(t)$ be that of S'. We use the following fact, which is explained in the proof of [Bloch 1986, Corollary 2.6]:

If z_k is a cycle on $Y \times \mathbb{A}^1 \times \Delta^n$ which intersects all faces $(Y \times \mathbb{A}^1 \times \Delta^m)_k$ properly, then $\psi(1) \circ \pi^*(z_k) \subset (Y \times \mathbb{A}^1 \times \Delta^n)_K$ intersects all faces $(Y \times F)_K$ (where F is any face in $\mathbb{A}^1 \times \Delta^n$) properly.

We deduce the statement of Proposition 4.6 from this in the same manner as we deduced Proposition 4.4 from [Bloch 1986, Lemma 2.2]. The fact that the maps $T_n^* \circ \psi(1) \circ \pi^*$ define the homotopy is a straightforward computation. □

Corollary 4.7. *The two maps*

$$i_0^*, i_1^* : C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$$

induce the same map on homology.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) & \xrightarrow{\pi^*} & C_\bullet^{Y\times\{0,1\}\times S'}(X, Y\times\mathbb{A}^1\times S') \\ \downarrow{\scriptstyle i_0^*, i_1^*} & & \downarrow{\scriptstyle i_0^*, i_1^*} \\ C_\bullet(X, Y) & \xrightarrow{\pi^*} & C_\bullet(X, Y\times S'). \end{array}$$

The same specialization argument as in the proof of Corollary 4.5 shows that π^* is injective on homology. Therefore it suffices to show that $i_0^* \circ \pi^* = i_1^* \circ \pi^*$ on homology. By Proposition 4.4, we have a homotopy $\pi^* \sim \psi(1) \circ \pi^*$, and hence it suffices to show that the maps $i_0^* \circ \psi(1) \circ \pi^*$ and $i_1^* \circ \psi(1) \circ \pi^*$ induce the same map on homology. But this follows from Proposition 4.6. □

Now we conclude the proof of Theorem 4.3. First of all, note that

$$p^*(C_\bullet(X, Y)) \subset C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1)$$

and that $i_0^* \circ p^* = \mathrm{id}$, such that p^* is injective on homology. Consider the multiplication map

$$\tau : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1.$$

It is flat and therefore τ^* exists. Consider the diagram

$$\begin{array}{ccc} C_\bullet(X, Y\times\mathbb{A}^1) & \xrightarrow{\tau^*} & C_\bullet(X, Y\times\mathbb{A}^1\times\mathbb{A}^1) \\ \uparrow{\scriptstyle \text{q.iso.}} & & \uparrow{\scriptstyle \text{q.iso.}} \\ C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) & \overset{\tau^*}{\dashrightarrow} & C_\bullet^{Y\times\mathbb{A}^1\times\{0,1\}}(X, Y\times\mathbb{A}^1\times\mathbb{A}^1) \\ \downarrow{\scriptstyle i_0^*, i_1^*} & & \downarrow{\scriptstyle i_0^*, i_1^*} \\ C_\bullet(X, Y) & \xrightarrow{p^*} & C_\bullet(X, Y\times\mathbb{A}^1). \end{array}$$

One easily observes that τ^* sends a cycle $z \in C_n^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1)$ to a cycle in $C_n^{Y\times\mathbb{A}^1\times\{0,1\}}(X, Y\times\mathbb{A}^1\times\mathbb{A}^1)$ and that for such a z the following equalities hold:

$$i_0^* \circ \tau^*(z) = p^* \circ i_0^*(z), \tag{3}$$

$$i_1^* \circ \tau^*(z) = z. \tag{4}$$

By Corollary 4.5, any class in $h_n(X, Y \times \mathbb{A}^1)$ can be represented by an element in $C_n^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1)$. Therefore (3) shows that in order to prove that p^* is surjective on homology, it suffices to show that $i_0^* \circ \tau^*$ is. But by Corollary 4.7, $i_0^* \circ \tau^*$ induces the same map on homology as $i_1^* \circ \tau^*$, which is the identity, by (4).
□

A naive definition of homotopy between scheme morphisms is the following: Two scheme morphisms $\phi, \psi : X \longrightarrow X'$ are homotopic if there exists a morphism

$$H : X \times \mathbb{A}^1 \longrightarrow X'$$

with $\phi = H \circ i_0$ and $\psi = H \circ i_1$. (This is not an equivalence relation!) The next corollary is an immediate consequence of Proposition 4.2.

Corollary 4.8. *If two morphisms*

$$\phi, \psi : X \longrightarrow X'$$

are homotopic, then they induce the same map on singular homology, i.e. for every scheme Y flat and of finite type over S, the homomorphisms

$$\phi_*, \psi_* : h_i(X, Y) \longrightarrow h_i(X', Y)$$

coincide for all i.

Now we recall the definition of relative singular homology from [Suslin and Voevodsky 1996]. Suppose that Y is an integral scheme and that X is any scheme over Y.

For $n \geq 0$, let $C_n(X/Y)$ be the free abelian group generated by closed integral subschemes of $X \times_Y \Delta_Y^n$ such that the restriction of the canonical projection

$$X \times_Y \Delta_Y^n \longrightarrow \Delta_Y^n$$

to Z induces a finite surjective morphism $p : Z \to \Delta_Y^n$. Let $i : \Delta_Y^m \hookrightarrow \Delta_Y^n$ be a face. Then all irreducible components of $Z \cap X \times_Y \Delta_Y^m$ have the "right" dimension and thus the cycle theoretic inverse image $i^*(Z)$ is well defined and in $C_m(X/Y)$. Furthermore, degeneracy maps are flat, and thus we obtain a simplicial abelian group $C_\bullet(X/Y)$. As above, we use the same notation for the complex of abelian groups obtained by taking the alternating sum of face operators. The groups

$$h_i(X/Y) = H_i(C_\bullet(X/Y))$$

are called the *relative singular homology groups* of X over Y.

We have seen in Section 2 that singular homology is covariantly functorial on the category $\mathrm{SmCor}(X)$ of smooth schemes over S with finite correspondences as morphisms. For $X, Y \in \mathrm{Sm}(S)$ the group of finite correspondences $c(X, Y)$ coincides with $C_0(X \times Y/Y)$ and we call two finite correspondences homotopic if they have the same image in $h_0(X \times Y/Y)$. The next proposition shows that homotopic finite correspondences induce the same map on singular homology.

Proposition 4.9. *For smooth schemes $X, Y \in \mathrm{Sm}(S)$, the natural pairing*

$$c(X, Y) \otimes h_i(X; S) \to h_i(Y; S)$$

factors through $h_0(X \times Y/Y)$, defining pairings

$$h_0(X \times Y/Y) \otimes h_i(X; S) \longrightarrow h_i(Y; S) \quad \text{for all } i.$$

Proof. Let $W \subset X \times Y \times \Delta^1 = X \times Y \times \mathbb{A}^1$ define an element in $C_1(X \times Y/Y)$. Let $W^j = i_j^*(W)$, for $j = 0, 1$, so that $d_1(W) = W^0 - W^1 \in C_0(X \times Y/Y)$. Let $\psi \in C_n(X; S)$. We have to show that $(W^0, \psi) = (W^1, \psi)$. Considering W as an element in $C_0(X \times Y \times \mathbb{A}^1/Y \times \mathbb{A}^1)$, the composite (W, ψ) is in $C_n^{\{0,1\}}(Y, \mathbb{A}^1; S)$ and $(W^j, \psi) = i_j^*((W, \psi))$ for $j = 0, 1$. Thus the result follows from Corollary 4.7. □

5. Alternative characterization of h_0

For a noetherian scheme X we have the identification

$$\mathrm{CH}^d(X, 0) = \mathrm{CH}^d(X)$$

between the higher Chow group $\mathrm{CH}^d(X, 0)$ and the group $\mathrm{CH}^d(X)$ of d-codimensional cycles on X modulo rational equivalence (see [Nart 1989, Proposition 3.1]). Fixing the notation and assumptions of the previous sections, we now give an analogous description for the group $h_0(X; S)$.

Let C be an integral scheme over S of absolute dimension 1. Then to every rational function $f \neq 0$ on C, we can attach the zero-cycle $\mathrm{div}(f) \in C_0(C; S)$ (see [Fulton 1998, Chapter I,1.2]). Let $\widetilde{C}$ be the normalization of C in its field of functions. Denoting the normalization morphism by $\phi : \widetilde{C} \to C$, we have $\phi_*(\mathrm{div}(f)) = \mathrm{div}(f)$. If C is regular and connected, then we denote by $P(C)$ the regular compactification of C over S, i.e. the uniquely determined regular and connected scheme of dimension 1 which is proper over S and which contains C as an open subscheme.

With this terminology, for an integral scheme C of absolute dimension 1, elements in the function field $k(C)$ are in 1-1 correspondence to morphisms $P(\widetilde{C}) \to \mathbb{P}^1_S$, which are not $\equiv \infty$.

Theorem 5.1. *The group $h_0(X; S)$ is the quotient of the group of zero-cycles on X modulo the subgroup generated by elements of the form* $\mathrm{div}(f)$, *where*

- *C is a closed integral curve on X,*
- *f is a rational function on C which, considered as a rational function on $P(\widetilde{C})$, is defined and $\equiv 1$ at every point of $P(\widetilde{C}) - \widetilde{C}$.*

Proof. We may suppose that X is reduced. Let $Z \subset X \times \Delta^1$ be an integral curve such that the projection $Z \to \Delta^1$ induces a finite and surjective morphism of Z onto a closed integral subscheme T of codimension 1 in Δ^1. Embed Δ^1 linearly to $\mathbb{P}^1 = \mathbb{P}^1_S$ by sending (0, 1) to $0 = (0:1)$ and (1, 0) to $\infty = (1:0)$. Since $Z \to \Delta^1$ is finite, the projection $Z \to \mathbb{P}^1$ corresponds to a rational function g on Z which is defined and $\equiv 1$ at every point of $P(\widetilde{Z}) - Z$. Let $\bar{Z}$ be the closure of Z in $X \times \mathbb{P}^1$, and let $\bar{C}$ be the image of $\bar{Z}$ under the (proper) projection $X \times \mathbb{P}^1 \to X$, considered as a reduced (hence integral) subscheme of X.

We have to consider two cases:

(1) If $\bar{C} = P$ is a closed point on X, then $Z = \{P\} \times \Delta^1$ and $d_1(Z) = 0$.

(2) If $\bar{C}$ is an integral curve, then the image C of Z under $X \times \mathbb{P}^1 \to X$ is an open subscheme of $\bar{C}$. Consider the extension of function fields

$$k(Z)|k(C)$$

and let $f \in k(C)$ be the norm of g with respect to this extension. Then f is defined and $\equiv 1$ at every point of $P(\widetilde{C}) - C$ and

$$\operatorname{div}(f) = \delta_0(Z) - \delta_1(Z) = d(Z).$$

If X is of dimension 1, the last equality follows from [Nart 1989, Proposition 1.3]. The general case can be reduced to this by replacing X by $\bar{C}$. Considering f as a rational function on $\bar{C}$, it satisfies the assumption of the theorem.

It remains to show the other direction. Let C and f be as in the theorem. We have to show that $\operatorname{div}(f) \in C_0(X; S)$ is a boundary. To see this, interpret f as a nonconstant morphism $U \to \mathbb{P}^1$ defined on an open subscheme $U \subset C$ and let $\bar{Z}$ be the closure of the graph of this morphism in $X \times \mathbb{P}^1$. The scheme $\bar{Z}$ is integral, of dimension 1 and projects birationally and properly onto C. Consider again the open linear embedding $\Delta^1 \subset \mathbb{P}^1$ which is defined by sending (0, 1) to 0 and (1, 0) to ∞ and let $Z = \bar{Z} \cap X \times \Delta^1$. The properties of f imply that the induced projection $Z \to \Delta^1$ is finite and surjective onto a closed subscheme of codimension 1 in Δ^1, thus defining an element of $C_1(X; S)$. Finally note that $d(Z) = \delta_0(Z) - \delta_1(Z) = \operatorname{div}(f)$. □

This immediately implies:

Corollary 5.2. *If X is proper over S, then*

$$h_0(X; S) = \mathrm{CH}_0(X).$$

Corollary 5.3. *The natural homomorphism $\bigoplus_{i_C} d(C_1(C; S)) \xrightarrow{i_{C*}} d(C_1(X; S))$ is surjective, where $i_C : C \to X$ runs through all S-morphisms from a regular scheme C over S of dimension 1 to X.*

Proof. By Theorem 5.1, $d(C_1(X; S)$ is generated by elements of the form $\mathrm{div}(f)$, where f is a rational function on an integral curve on X satisfying an additional property. The normalization $\widetilde{C}$ of C is a regular scheme of dimension 1 and let $i : \widetilde{C} \to X$ the associated morphism. Considering f as a rational function on $\widetilde{C}$, we have the equality

$$i_*(\mathrm{div}(f)) = \mathrm{div}(f).$$

By the additional property of f, the associated line bundle $\mathscr{L}(\mathrm{div}(f))$ over the compactification $P(\widetilde{C})$ together with its canonical trivialization over $P(\widetilde{C}) - \widetilde{C}$ defines the trivial element in $\mathrm{Pic}(P(\widetilde{C}), P(\widetilde{C}) - \widetilde{C})$. Therefore, the calculation of singular homology of regular schemes of dimension 1 (see Theorems 3.5 and 3.7), shows that $\mathrm{div}(f)$ is in $d(C_1(C; S))$. This finishes the proof. □

Now we can prove the exactness of a part of the Mayer–Vietoris sequence for X of arbitrary dimension.

Proposition 5.4. *Let $S = U \cup V$ be a covering by Zariski-open subschemes U and V. Then the natural sequence*

$$h_0(X; S) \longrightarrow h_0(X; U) \oplus h_0(X; V) \longrightarrow h_0(X; U \cap V) \longrightarrow 0$$

is exact.

Proof. First of all, the homomorphism

$$C_0(X; U) \oplus C_0(X; V) \longrightarrow C_0(X; U \cap V)$$

is surjective, and therefore so is $h_0(X; U) \oplus h_0(X; V) \longrightarrow h_0(X; U \cap V)$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
d(C_1(X; S)) & \longrightarrow & d(C_1(X; U)) \oplus d(C_1(X; V)) & \longrightarrow & d(C_1(X; U \cap V)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_0(X; S) & \hookrightarrow & C_0(X; U) \oplus C_0(X; V) & \longrightarrow & C_0(X; U \cap V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
h_0(X; S) & \longrightarrow & h_0(X; U) \oplus h_0(X; V) & \longrightarrow & h_0(X; U \cap V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The middle row and the middle and right columns are exact. Therefore the snake lemma shows that the lower line is exact if and only if the homomorphism

$$d(C_1(X;U)) \oplus d(C_1(X;V)) \longrightarrow d(C_1(X;U\cap V)) \tag{5}$$

is surjective. By Theorem 3.10 (iv), we observe that (5) is surjective if X is regular and of dimension 1. For a general X, put

$$X' = X \times_S (U \cap V).$$

Then the commutative diagram

$$\begin{array}{ccc} C_1(X';U)\oplus C_1(X';V) & \longrightarrow & C_1(X';U\cap V) \\ \big\downarrow & & \big\| \\ C_1(X;U)\oplus C_1(X;V) & \longrightarrow & C_1(X;U\cap V) \end{array}$$

shows that, in order to show the surjectivity of (5), we may suppose that $X = X'$. Now the statement follows from Corollary 5.3, using the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i_C} d(C_1(C;U)) \oplus d(C_1(C;V)) & \xrightarrow{i_{C*}} & d(C_1(X;U)) \oplus d(C_1(X;V)) \\ \big\downarrow\!\!\!\!\!\!\downarrow & & \big\downarrow \\ \bigoplus_{i_C} d(C_1(C;U\cap V)) & \xrightarrow{i_{C*}}\!\!\!\!\!\rightarrow & d(C_1(X;U\cap V)). \end{array}$$

This concludes the proof. □

A similar argument shows:

Proposition 5.5. *Let $X = X_1 \cup X_2$ be a covering by Zariski open subschemes X_1 and X_2. Then the natural sequence*

$$h_0(X_1 \cap X_2; S) \longrightarrow h_0(X_1; S) \oplus h_0(X_2; S) \longrightarrow h_0(X; S) \longrightarrow 0$$

is exact.

Proof. We omit the base scheme S from our notation. The homomorphism

$$C_0(X_1) \oplus C_0(X_2) \longrightarrow C_0(X)$$

is surjective, and therefore so is

$$h_0(X_1) \oplus h_0(X_2) \longrightarrow h_0(X).$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
d(C_1(X_1 \cap X_2)) & \longrightarrow & d(C_1(X_1)) \oplus d(C_1(X_2)) & \longrightarrow & d(C_1(X)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_0(X_1 \cap X_2) & \hookrightarrow & C_0(X_1) \oplus C_0(X_2) & \longrightarrow & C_0(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
h_0(X_1 \cap X_2) & \longrightarrow & h_0(X_1) \oplus h_0(X_2) & \longrightarrow & h_0(X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The middle row and the middle and right columns are exact. Therefore the snake lemma shows that the lower line is exact if and only if the homomorphism

$$d(C_1(X_1)) \oplus d(C_1(X_2)) \longrightarrow d(C_1(X)) \tag{6}$$

is surjective. By Theorem 3.10 (iii), we observe that (6) is surjective if X is regular and of dimension 1.

For a morphism $i : C \to X$ we use the notation $C_1 = i^{-1}(X_1)$ and $C_2 = i^{-1}(X_2)$, thus $C = C_1 \cup C_2$ is a Zariski open covering.

Now the required statement for arbitrary X follows from Corollary 5.3, using the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i_C} d(C_1(C_1)) \oplus d(C_1(C_2)) & \xrightarrow{i_{C*}} & d(C_1(X_1)) \oplus d(C_1(X_2)) \\
\downarrow\!\!\downarrow & & \downarrow \\
\bigoplus_{i_C} d(C_1(C)) & \xrightarrow{i_{C*}}\!\!\!\!\rightarrow & d(C_1(X)).
\end{array}$$

This concludes the proof. □

We conclude this section with a surjectivity result.

Proposition 5.6. *Let X be regular and let U be a dense open subscheme in X. Then the natural homomorphism*

$$h_0(U; S) \longrightarrow h_0(X; S)$$

is surjective.

Proof. Let P be a 0-dimensional point on X which is not contained in U. We have to show that the image of P in $h_0(X; S)$ is equal to the image of a finite linear combination $\sum n_i P_i$ with $P_i \in U$ for all i. Choose a one-dimensional subscheme C on X such that P is a regular point on C and such that C is not contained in $X - U$. We find such a curve, since X is regular: Indeed, $\mathcal{O}_{X,P}$ is a d-dimensional regular local ring, with $d = \dim X$. Let $\mathfrak{m}$ be the maximal ideal and $\mathfrak{a}$ the ideal defining the closed subset $(X - U) \cap \operatorname{Spec}(\mathcal{O}_{X,P})$. Choose elements $\bar{x}_1, \dots, \bar{x}_{d-1}$ in $\mathfrak{m}/\mathfrak{m}^2$ which span a $(d-1)$-dimensional subspace which does not contain $\mathfrak{a} + \mathfrak{m}/\mathfrak{m}$. Lifting $\bar{x}_1, \dots, \bar{x}_{d-1}$ to a regular sequence $x_1, \dots, x_{d-1} \in \mathcal{O}_{X,P}$, the ideal $(x_1, \dots, x_{d-1})$ is a prime ideal of height $(d-1)$ which does not contain $\mathfrak{a}$. Finally, extend $\bar{x}_1, \dots, x_{d-1}$ to an affine open neighbourhood of P in X and choose C as the closure of their zero-locus.

Consider the normalization $\widetilde{C}$ of C and let $P(\widetilde{C})$ be a regular compactification over S. Let $P(\widetilde{C}) - \widetilde{C} = \{P_1, \dots, P_r\}$ and let $P_{r+1}, \dots, P_s$ be the finitely many closed points on $\widetilde{C}$ mapping to $C \cap (X - U)$. Let $\widetilde{P}$ be the unique point on $\widetilde{C}$ projecting to $P \in C$. Let $D = \{P_1, \dots, P_s, \widetilde{P}\}$ and consider the ring $A = \mathcal{O}_{P(\widetilde{C}),D}$, which is a semilocal principal ideal domain. We find an element $f \in A$ which has exact valuation 1 at $\widetilde{P}$ and which is $\equiv 1$ at each P_i, $i = 1, \dots, n$. Then $(\operatorname{div} f) \subset X$ is of the form $P + \sum Q_i$ with $Q_i \in U$. □

6. Review of tame coverings

The concept of tame ramification stems from number theory: A finite extension of number fields $L|K$ is called tamely ramified at a prime $\mathfrak{P}$ of L if the associated extension of completions $L_{\mathfrak{P}}|K_{\mathfrak{P}}$ is a tamely ramified extension of local fields. The latter means that the ramification index is prime to the characteristic of the residue field. It is a classical result that composites and towers of tamely ramified extensions are again tamely ramified. This concept generalizes to separable extensions of arbitrary discrete valuation fields by requiring that the associated residue field extensions are separable.

Let from now on S be the spectrum of an excellent Dedekind domain and let $X \in \operatorname{Sch}(S)$. Our aim is to say when a finite étale covering $Y \to X$ is tame. Here "tame" means tamely ramified along the boundary of a compactification $\bar{X}$ of X over S. If $\bar{X}$ is regular and $D = \bar{X} - X$ is a normal crossing divisor, then one can use the approach of [Grothendieck 1971; Grothendieck and Murre 1971]:

Definition 6.1 [Grothendieck and Murre 1971, 2.2.2]. A finite étale covering $Y \to X$ is called tame (along D) if the extension of function fields $k(Y)|k(X)$ is tamely ramified at the discrete valuations associated to the irreducible components of D.

Even if one restricts attention to regular schemes, one is confronted with the following problems:

- If X is regular, we do not know whether there exists a regular compactification with an NCD as its boundary,
- The notion of tameness might depend on the choice of the compactification $\bar{X}$ of X.
- Even if the first two questions can be answered in a positive way, there is no obvious functoriality for the tame fundamental group (already for an open immersion).

All these problems are void in the case of a regular curve C, where a canonical compactification $\bar{C}$ exists. Starting from the therefore obvious notion of tame coverings of regular curves, G. Wiesend [2006] proposed the following definition.

Definition 6.2. Let X be a separated integral scheme of finite type over S. A finite étale covering $Y \to X$ is called *tame* if for every integral curve $C \subset X$ with normalization $\widetilde{C} \to C$ the base change

$$Y \times_X \widetilde{C} \longrightarrow \widetilde{C}$$

is a tame covering of the regular curve $\widetilde{C}$.

This definition has the advantage of making no use of a compactification of X. Furthermore, it is obviously stable under base change. However, it is difficult to decide whether a given étale covering is actually tame. For coverings of normal schemes, several authors [Abbes 2000; Chinburg and Erez 1992; Schmidt 2002] have made suggestions for a definition of tameness which all come down to the following notion, which we want to call numerically tameness here.

Definition 6.3. Let $\bar{X} \in \mathrm{Sch}(S)$ be normal connected and proper, and let $X \subset \bar{X}$ be an open subscheme. Let $Y \to X$ be a finite étale Galois covering and let $\bar{Y}$ be the normalization of $\bar{X}$ in the function field $k(Y)$ of Y. We say that $Y \to X$ is *numerically tame* (along $D = \bar{X} - X$) if the order of the inertia group $T_x(\bar{Y}|\bar{X}) \subset \mathrm{Gal}(\bar{Y}|\bar{X}) = \mathrm{Gal}(Y|X)$ of each closed point $x \in D$ (see [Bourbaki 1964, Chapter 5, Section 2.2] for the definition of inertia groups) is prime to the residue characteristic of x. A finite étale covering $Y \to X$ is called numerically tame if it can be dominated by a numerically tame Galois covering.

Proposition 6.4. *Let $\bar{X} \in \mathrm{Sch}(S)$ be normal connected and proper, and let $X \subset \bar{X}$ be an open subscheme. If the finite étale covering $Y \to X$ is numerically tame (along $\bar{X} - X$), then it is tame.*

Proof. For regular curves the notions of tameness and of numerically tameness obviously coincide. Therefore the statement of the proposition follows from the fact that numerically tame coverings are stable under base change; see [Schmidt 2002]. □

Theorem 6.5 [Wiesend 2006, Theorem 2]. *Assume that $\bar{X}$ is regular and that $D = \bar{X} - X$ is an NCD. Then, for a finite étale covering $Y \to X$, there is equivalence between*:

(i) *$Y \to X$ is tame according to Definition 6.1.*

(ii) *$Y \to X$ is tame* (*according to Definition 6.2*).

(iii) *$Y \to X$ is numerically tame.*

Remark 6.6. The equivalence of (i) and (iii) had already been shown in [Schmidt 2002].

Theorem 6.7 [Wiesend 2006, Theorem 2]. *Assume that $\bar{X}$ is regular* (*but make no assumption on $D = \bar{X} - X$*). *If a numerically tame covering $Y \to X$ can be dominated by a Galois covering with nilpotent Galois group*, *then it is tame.*

In particular, for nilpotent coverings of a regular scheme X the notion of numerically tameness does not depend on the choice of a regular compactification $\bar{X}$ (if it exists). This had already been shown in [Schmidt 2002]. A counterexample with non-nilpotent Galois group can be found in [Wiesend 2006, Remark 3].

7. Finiteness results for tame fundamental groups

The tame coverings of a connected integral scheme $X \in \mathrm{Sch}(S)$ satisfy the axioms of a Galois category [Wiesend 2006, Proposition 1]. After choosing a geometric point $\bar{x}$ of X we have the fibre functor $(Y \to X) \mapsto \mathrm{Mor}_X(\bar{x}, Y)$ from the category of tame coverings of X to the category of sets, whose automorphisms group is called the *tame fundamental group* $\pi_1^t(X, \bar{x})$. It classifies finite connected tame coverings of X. We have an obvious surjection

$$\pi_1^{\mathrm{et}}(X, \bar{x}) \twoheadrightarrow \pi_1^t(X, \bar{x}),$$

which is an isomorphism if X is proper. Assume that X is normal, connected and let $\bar{X}$ be a normal compactification. Then, replacing tame coverings by numerically tame coverings, we obtain in an analogous way the *numerically tame fundamental group* $\pi_1^{\mathrm{nt}}(\bar{X}, \bar{X} - X, \bar{x})$, which classifies finite connected numerically tame coverings of X (along $\bar{X} - X$). By Proposition 6.4 we have a surjection

$$\varphi : \pi_1^t(X, \bar{x}) \twoheadrightarrow \pi_1^{\mathrm{nt}}(\bar{X}, \bar{X} - X, \bar{x}),$$

which, by Theorem 6.7, induces an isomorphism on the maximal pro-nilpotent factor groups if $\bar{X}$ is regular. If, in addition, $\bar{X} - X$ is a normal crossing divisor then φ is an isomorphism by Theorem 6.5. The fundamental groups of a connected scheme X with respect to different base points are isomorphic, and the isomorphism is canonical up to inner automorphisms. Therefore, when working with the maximal abelian quotient of the étale fundamental group (tame fundamental group, n.t. fundamental group) of a connected scheme, we are allowed to omit the base point from notation.

Now we specialize to the case $S = \mathrm{Spec}(\mathbb{Z})$, i.e. to arithmetic schemes. In [Schmidt 2002] we proved the finiteness of the abelianized numerically tame fundamental group $\pi_1^{\mathrm{nt}}(\bar{X}, \bar{X} - X)^{\mathrm{ab}}$ of a connected normal scheme, flat and of finite type over $\mathrm{Spec}(\mathbb{Z})$ with respect to a normal compactification $\bar{X}$. The proof given there can be adapted to apply also to the larger group $\pi_1^t(X)^{\mathrm{ab}}$.

Theorem 7.1. *Let X be a connected normal scheme, flat and of finite type over* $\mathrm{Spec}(\mathbb{Z})$. *Then the abelianized tame fundamental group $\pi_1^t(X)^{\mathrm{ab}}$ is finite.*

For the proof we need the following two lemmas. The first one extends [Schmidt 2002, Corollary 2.6] from numerical tameness to tameness.

Lemma 7.2. *Let $X \in \mathrm{Sch}(S)$ be normal and connected, p a prime number and $Y \to X$ a finite étale Galois covering whose Galois group is a finite p-group. Let $\bar{X}$ be a normal compactification of X and assume there exists a prime divisor D on $\bar{X}$ which is ramified in $k(Y)|k(X)$ and which contains a closed point of residue characteristic p. Then $Y \to X$ is not tame.*

Proof. The statement of the lemma is part of the proof of [Wiesend 2006, Theorem 2]. □

Lemma 7.3. *Let A be a strictly henselian discrete valuation ring with perfect (hence algebraically closed) residue field and with quotient field k. Let $k_\infty|k$ be a $\mathbb{Z}_p$-extension. Let $K|k$ be a regular field extension and let $B \subset K$ be a discrete valuation ring dominating A. Then B is ramified in $Kk_\infty|K$.*

Proof. See [Schmidt 2002, Lemma 3.2]. □

Proof of Theorem 7.1. The proof is a modification of the proof of [Schmidt 2002, Theorem 3.1]. Let $\bar{X}$ be a normal compactification of X over $\mathrm{Spec}(\mathbb{Z})$. Let k be the normalization of $\mathbb{Q}$ in the function field of X and put $S = \mathrm{Spec}(\mathcal{O}_k)$. Then the natural projection $\bar{X} \to \mathrm{Spec}(\mathbb{Z})$ factors through S.

Since X is normal, for any open subscheme V of X the natural homomorphism $\pi_1^{\mathrm{et}}(V) \to \pi_1^{\mathrm{et}}(X)$ is surjective. Therefore also the homomorphism

$$\pi_1^t(V)^{\mathrm{ab}} \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

is surjective and so we may replace X by a suitable open subscheme and assume that X is smooth over S. Let $T \subset S$ be the image of X. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}(X/T) & \longrightarrow & \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} & \longrightarrow & \pi_1^{\mathrm{et}}(T)^{\mathrm{ab}} \\
 & & \downarrow & & \twoheaddownarrow & & \twoheaddownarrow \\
0 & \longrightarrow & \mathrm{Ker}^t(X/T) & \longrightarrow & \pi_1^t(X)^{\mathrm{ab}} & \longrightarrow & \pi_1^t(T)^{\mathrm{ab}}
\end{array}$$

where the groups $\mathrm{Ker}(X/T)$ and $\mathrm{Ker}^t(X/T)$ are defined by the exactness of the corresponding rows, and the two right vertical homomorphisms are surjective. By [Katz and Lang 1981, Theorem 1], the group $\mathrm{Ker}(X/T)$ is finite. By classical one-dimensional class field theory, the group $\pi_1^t(T)^{\mathrm{ab}}$ is finite (it is the Galois group of the ray class field of k with modulus $\prod_{\mathfrak{p}\notin T}\mathfrak{p}$). The kernel of $\pi_1^{\mathrm{et}}(T)^{\mathrm{ab}} \to \pi_1^t(T)^{\mathrm{ab}}$ is generated by the ramification groups of the primes of S which are not in T. Denoting the product of the residue characteristics of these primes by N, we see that $\pi_1^{\mathrm{et}}(T)^{\mathrm{ab}}$ is the product of a finite group and a topologically finitely generated pro-N group. Therefore the same is also true for $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ and for $\pi_1^t(X)^{\mathrm{ab}}$. Hence it suffices to show that the cokernel C of the induced map $\mathrm{Ker}(X/T) \to \mathrm{Ker}^t(X/T)$ is a torsion group.

Let K be the function field of X and let k_1 be the maximal abelian extension of k such that the normalization X_{Kk_1} of X in the composite Kk_1 is ind-tame over X. By [Katz and Lang 1981, Lemma 2, (2)], the normalization of T in k_1 is ind-étale over T. Let $k_2|k$ be the maximal subextension of $k_1|k$ such that the normalization T_{k_2} of T in k_2 is tame over T. Then $G(k_2|k) = \pi_1^t(T)^{\mathrm{ab}}$ and $C \cong G(k_1|k_2)$.

In order to show that C is a torsion group, we therefore have to show that $k_1|k_2$ does not contain a $\mathbb{Z}_p$-extension of k_2 for any prime number p. Since $k_2|k$ is a finite extension and $k_1|k$ is abelian, this is equivalent to the assertion that $k_1|k$ contains no $\mathbb{Z}_p$-extension of k for any prime number p.

Let p be a prime number and suppose that $k_\infty|k$ is a $\mathbb{Z}_p$-extension such that the normalization X_{Kk_∞} is ind-tame over X. A $\mathbb{Z}_p$-extension of a number field is unramified outside p and there exists at least one ramified prime dividing p; see e.g. [Neukirch et al. 2000, (10.3.20)(ii)]. Let k' be the maximal unramified subextension of $k_\infty|k$ and let S' be the normalization of S in k'. Then the base change $\bar{X}' = \bar{X} \times_S S' \to X$ is étale. Hence $\bar{X}'$ is normal and the preimage X' of X is smooth and geometrically connected over k'. So, after replacing k by k', we may suppose that $k_\infty|k$ is totally ramified at a prime $\mathfrak{p}|p$, $\mathfrak{p} \in S - T$. Considering the base change to the strict henselization of S at $\mathfrak{p}$ and applying Lemma 7.3, we see that each vertical divisor of $\bar{X}$ in the fibre over $\mathfrak{p}$ ramifies in Kk_∞. Replacing $\bar{X}$ by its normalization in a suitable finite subextension of Kk_∞, we obtain a contradiction using Lemma 7.2. □

Next we consider the case $S = \mathrm{Spec}(\mathbb{F})$, i.e. varieties over a finite field $\mathbb{F}$. In this case we have the degree map

$$\deg : \pi_1^t(X)^{\mathrm{ab}} \longrightarrow \pi_1^t(S)^{\mathrm{ab}} \cong \mathrm{Gal}(\bar{\mathbb{F}} \mid \mathbb{F}) \cong \widehat{\mathbb{Z}},$$

and we denote the kernel of this degree map by $(\pi_1^t(X)^{\mathrm{ab}})^0$. The image of deg is an open subgroup of $\widehat{\mathbb{Z}}$ and is therefore isomorphic to $\widehat{\mathbb{Z}}$. As $\widehat{\mathbb{Z}}$ is a projective profinite group, we have a (noncanonical) isomorphism

$$\pi_1^t(X)^{\mathrm{ab}} \cong (\pi_1^t(X)^{\mathrm{ab}})^0 \times \widehat{\mathbb{Z}}.$$

Let p be the characteristic of the finite field $\mathbb{F}$. If X is an open subscheme of a smooth proper variety $\bar{X}$, then we have a decomposition

$$(\pi_1^t(X)^{\mathrm{ab}})^0 \cong (\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}})^0(\text{prime-to-}p\text{-part}) \oplus (\pi_1^{\mathrm{et}}(\bar{X})^{\mathrm{ab}})^0(p\text{-part}),$$

and both summands are known to be finite. The finiteness statement for $(\pi_1^t(X)^{\mathrm{ab}})^0$ can be generalized to normal schemes.

Theorem 7.4. *Let X be a normal connected variety over a finite field. Then the group $(\pi_1^t(X)^{\mathrm{ab}})^0$ is finite.*

Proof (sketch). We may replace X by a suitable open subscheme and therefore assume that there exists a smooth morphism $X \longrightarrow C$ to a smooth projective curve. Then we proceed in an analogous way as in the proof of Theorem 7.1 using the fact that a global field of positive characteristic has exactly one unramified $\widehat{\mathbb{Z}}$-extension, which is obtained by base change from the constant field. □

8. Tame class field theory

In this section we construct a reciprocity homomorphism from the singular homology group $h_0(X)$ to the abelianized tame fundamental group of an arithmetic scheme X. A sketch of the results of this section is contained in [Schmidt 2007].

Let for the whole section $S = \mathrm{Spec}(\mathbb{Z})$ and let $X \in \mathrm{Sch}(\mathbb{Z})$ be connected and regular. If X has $\mathbb{R}$-valued points, we have to modify the tame fundamental group in the following way.

We consider the full subcategory of the category of tame coverings of X which consists of that coverings in which every $\mathbb{R}$-valued point of X splits completely. After choosing a geometric point $\bar{x}$ of X we have the fibre functor $(Y \to X) \mapsto \mathrm{Mor}_X(\bar{x}, Y)$, and its automorphism group $\tilde{\pi}_1^t(X, \bar{x})$ is called the *modified tame fundamental group* of X. It classifies connected tame coverings of X in which every $\mathbb{R}$-valued point of X splits completely. We have an obvious surjection

$$\pi_1^t(X, \bar{x}) \twoheadrightarrow \tilde{\pi}_1^t(X, \bar{x})$$

which is an isomorphism if $X(\mathbb{R}) = \varnothing$.

For $x \in X(\mathbb{R})$ let $\sigma_x \in \pi_1^t(X)^{\mathrm{ab}}$ be the image of the complex conjugation $\sigma \in \mathrm{Gal}(\mathbb{C}|\mathbb{R})$ under the natural map $x_* : \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \to \pi_1^t(X)^{\mathrm{ab}}$. By [Saito 1985, Lemma 4.9 (iii)], the map

$$X(\mathbb{R}) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}, \quad x \longmapsto \sigma_x,$$

is locally constant for the norm topology on $X(\mathbb{R})$. Therefore the kernel of the homomorphism

$$\pi_1^t(X)^{\mathrm{ab}} \twoheadrightarrow \tilde{\pi}_1^t(X)^{\mathrm{ab}}$$

is an $\mathbb{F}_2$-vector space of dimension less or equal the number of connected components of $X(\mathbb{R})$.

Let $x \in X$ be a closed point. We have a natural isomorphism

$$\pi_1^{\mathrm{et}}(\{x\}) \cong \mathrm{Gal}(\overline{k(x)}|k(x)) \cong \widehat{\mathbb{Z}},$$

and we denote the image of the (arithmetic) Frobenius automorphism Frob $\in G(\overline{k(x)}|k(x))$ under the natural homomorphism $\pi_1^{\mathrm{et}}(\{x\})^{\mathrm{ab}} \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ by Frob_x.

In the following we omit the base scheme $\mathrm{Spec}(\mathbb{Z})$ from notation, writing $C_\bullet(X)$ for $C_\bullet(X; \mathrm{Spec}(\mathbb{Z}))$ and similar for homology. Recall that $C_0(X) = Z_0(X)$ is the group of zero-cycles on X. Sending x to Frob_x, we obtain a homomorphism

$$r : C_0(X) \longrightarrow \pi_1(X)^{\mathrm{ab}},$$

which is known to have dense image [Lang 1956; Raskind 1995, Lemma 1.7]. Our next goal is to show:

Theorem 8.1. *The composite map*

$$C_0(X) \xrightarrow{r} \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} \longrightarrow \tilde{\pi}_1^t(X)^{\mathrm{ab}}$$

factors through $h_0(X)$, thus defining a reciprocity homomorphism

$$\mathrm{rec} : h_0(X) \longrightarrow \tilde{\pi}_1^t(X)^{\mathrm{ab}},$$

which has a dense image.

In order to prove Theorem 8.1, let us apply Theorem 3.7 to the case of rings of integers of algebraic number fields. Let k be a finite extension of $\mathbb{Q}$ and let Σ be a finite set of nonarchimedean primes of k. Let $\mathcal{O}_{k,\Sigma}$ be the ring of Σ-integers of k and let $E_k^{1,\Sigma}$ be the subgroup of elements in the group of global units E_k which are $\equiv 1$ at every prime $\mathfrak{p} \in \Sigma$. Let r_1 and r_2 be the number of real and complex places of k. If $\mathfrak{m}$ is a product of primes of k, then we denote by $C_{\mathfrak{m}}(k)$ the ray class group of k with modulus $\mathfrak{m}$.

Proposition 8.2. *For $X = \mathrm{Spec}(\mathcal{O}_{k,\Sigma})$, we have $h_i(X) = 0$ for $i \neq 0, 1$,*

(i) $h_0(X) = C_{\mathfrak{m}}(k)$ *with* $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$, *and*

(ii) $h_1(X) = E_k^{1,\Sigma} \cong (\text{finite group}) \oplus \mathbb{Z}^{r_1+r_2-1}$.

In particular, $h_0(X)$ is finite and $h_1(X)$ is finitely generated. If Σ contains at least two primes with different residue characteristics, the finite summand in (ii) *vanishes.*

Proof. The vanishing of $h_i(X)$ for $i \neq 0, 1$ follows from Theorem 3.7. A straightforward computation shows that for $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$

$$C_{\mathfrak{m}}(k) \cong \mathrm{Pic}(\mathrm{Spec}(\mathbb{O}_k), \Sigma),$$

and the finiteness of $C_{\mathfrak{m}}(k)$ is well-known. The group $E_k^{1,\Sigma}$ is of finite index in the full unit group E_k. Therefore the remaining statement in (ii) follows from Dirichlet's unit theorem. Furthermore, a root of unity congruent to 1 modulo two primes of different residue characteristics equals 1. □

By Theorem 3.5, we have an analogous statement for smooth curves over finite fields.

Proposition 8.3. *Let X be a smooth, geometrically connected curve over a finite field $\mathbb{F}$ and let $\bar{X}$ be the uniquely defined smooth compactification of X. Let $\Sigma = \bar{X} - X$ and let k be the function field of X. Then we have $h_i(X) = 0$ for $i \neq 0, 1$,*

(i) *$h_0(X) = C_{\mathfrak{m}}(k)$ with $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$, and*

(ii) $h_1(X) = \begin{cases} 0 & \text{if } \Sigma \neq \varnothing, \\ \mathbb{F}^\times & \text{if } \Sigma = \varnothing. \end{cases}$

In particular, $h_i(X)$ is finite for all i.

Proof of Theorem 8.1. Using Propositions 8.2 and 8.3, classical (one-dimensional) class field theory for global fields shows the statement in the case $\dim X = 1$. In order to show the general statement, it suffices by Corollary 5.3 to show that for any morphism $f : C \to X$ from a regular curve C to X and for any $x \in d(C_1(C))$, we have $r(f_*(x)) = 0$. This follows from the corresponding result in dimension 1 and from the commutative diagram

$$\begin{array}{ccccc} d(C_1(C)) & \longrightarrow & C_0(C) & \xrightarrow{r_C} & \tilde{\pi}_1^t(C)^{\mathrm{ab}} \\ \downarrow & & \downarrow & & \downarrow \\ d(C_1(X)) & \longrightarrow & C_0(X) & \xrightarrow{r_X} & \tilde{\pi}_1^t(X)^{\mathrm{ab}}. \end{array}$$

□

In order to investigate the reciprocity map, we use Wiesend's version of higher dimensional class field theory [Wiesend 2007]. We start with the arithmetic case, i.e. when X is flat over $\mathrm{Spec}(\mathbb{Z})$. In this case $\tilde{\pi}_1^t(X)^{\mathrm{ab}}$ is finite by Theorem 7.1.

Theorem 8.4. *Let X be a regular, connected scheme, flat and of finite type over* $\mathrm{Spec}(\mathbb{Z})$. *Then the reciprocity homomorphism*

$$\mathrm{rec}_X : h_0(X) \longrightarrow \tilde{\pi}_1^t(X)^{\mathrm{ab}}$$

is an isomorphism of finite abelian groups.

Remark 8.5. If X is proper, then $h_0(X) \cong \mathrm{CH}_0(X)$ and $\widetilde{\pi}_1^t(X)^{\mathrm{ab}} \cong \widetilde{\pi}_1^{\mathrm{et}}(X)^{\mathrm{ab}}$, and we recover the unramified class field theory for arithmetic schemes of Bloch and Kato–Saito [Kato and Saito 1983; Saito 1985].

Proof of Theorem 8.4. Recall the definition of Wiesend's idèle group $\mathcal{I}_X$. It is defined by

$$\mathcal{I}_X := Z_0(X) \oplus \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} k(C)_v^\times.$$

Here C runs through all closed integral subschemes of X of dimension 1, C_∞ is the finite set of places (including the archimedean ones if C is horizontal) of the global field $k(C)$ with center outside C and $k(C)_v$ is the completion of $k(C)$ with respect to v. $\mathcal{I}_X$ becomes a topological group by endowing the group $Z_0(X)$ of zero cycles on X with the discrete topology, the groups $k(C)_v^\times$ with their natural locally compact topology and the direct sum with the direct sum topology.[1]

The idèle class group $\mathcal{C}_X$ is defined as the cokernel of the natural map

$$\bigoplus_{C \subset X} k(C)^\times \longrightarrow \mathcal{I}_X.$$

which is given for a fixed $C \subset X$ by the divisor map $k(C)^\times \to Z_0(C) \to Z_0(X)$ and the diagonal map $k(C)^\times \to \bigoplus_{v \in C_\infty} k(C)_v^\times$. $\mathcal{C}_X$ is endowed the quotient topology of $\mathcal{I}_X$.

We consider the quotient $\mathcal{C}_X^t$ of $\mathcal{C}_X$ obtained by cutting out the 1-unit groups at all places outside X. More precisely, let for $v \in C_\infty$, $U^1(k(C)_v)$ be the group of principal units in the local field $k(C)_v$. We make the notational convention $U^1(K) = K^\times$ for the archimedean local fields $K = \mathbb{R}, \mathbb{C}$. Then

$$\mathcal{U}_X^t := \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} U^1(k(C)_v)$$

is an open subgroup of the idèle group $\mathcal{I}_X$ and we put

$$\mathcal{C}_X^t := \mathrm{coker}\Big(\bigoplus_{C \subset X} k(C)^\times \longrightarrow \mathcal{I}_X / \mathcal{U}_X^t\Big).$$

Consider the map

$$R: \ \mathcal{I}_X \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$$

which is given by the map $r : Z_0(X) \to \pi_1(X)^{\mathrm{ab}}$ defined above and the reciprocity maps of local class field theory

$$\rho_v : \ k(C)_v^\times \longrightarrow \pi_1^{\mathrm{et}}(\mathrm{Spec}(k(C)_v))^{\mathrm{ab}}$$

[1] The topology of a finite direct sum is just the product topology, and the topology of an infinite direct sum is the direct limit topology of the finite partial sums.

followed by the natural maps $\pi_1^{\mathrm{et}}(\mathrm{Spec}(k(C)_v))^{\mathrm{ab}} \to \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ for all $C \subset X$, $v \in C_\infty$. By [Wiesend 2007, Theorem 1 (a)], the homomorphism R induces an isomorphism

$$\rho:\ \mathscr{C}_X^t \xrightarrow{\sim} \tilde{\pi}_1^t(X)^{\mathrm{ab}}.$$

Now we consider the obvious map

$$\phi:\ Z_0(X) \longrightarrow \mathscr{C}_X^t.$$

The kernel of ϕ is the subgroup in $Z_0(X)$ generated by elements of the form $\mathrm{div}(f)$ where $C \subset X$ is a closed curve and f is an invertible rational function on C which is in $U^1(k(C)_v)$ for all $v \in C_\infty$. By Theorem 5.1 we obtain $\ker(\phi) = d_1(C_1(X))$. Therefore ϕ induces an injective homomorphism

$$i:\ h_0(X) \hookrightarrow \mathscr{C}_X^t$$

with $\rho \circ i = \mathrm{rec}$. As ρ is injective, rec is injective, and hence an isomorphism. □

Finally, assume that $\bar{X}$ is regular, flat and proper over $\mathrm{Spec}(\mathbb{Z})$, let $D \subset X$ be a divisor and $X = \bar{X} - D$. In [Schmidt 2005] we introduced the relative Chow group of zero cycles $\mathrm{CH}_0(\bar{X}, D)$ and constructed, under a mild technical assumption, a reciprocity isomorphism $\mathrm{rec}' : \mathrm{CH}_0(\bar{X}, D) \xrightarrow{\sim} \tilde{\pi}_1^t(X)^{\mathrm{ab}}$. By [Schmidt 2005, Proposition 2.4], there exists natural projection $\pi : h_0(X) \twoheadrightarrow \mathrm{CH}_0(\bar{X}, D)$ with $\mathrm{rec} = \mathrm{rec}' \circ \pi$. We obtain the

Theorem 8.6. *Let $\bar{X}$ be a regular, connected scheme, flat and proper over* $\mathrm{Spec}(\mathbb{Z})$, *such that its generic fibre $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is projective over $\mathbb{Q}$. Let D be a divisor on $\bar{X}$ whose vertical irreducible components are normal schemes. Put $X = \bar{X} - D$. Then the natural homomorphism*

$$h_0(X) \longrightarrow \mathrm{CH}_0(\bar{X}, D)$$

is an isomorphism of finite abelian groups.

Finally, we deal with the geometric case. The next theorem was proved in 1999 by M. Spieß and the author under the assumption that X has a smooth projective compactification; see [Schmidt and Spieß 2000]. Now we get rid of this assumption.

Theorem 8.7. *Let X be a smooth, connected variety over a finite field $\mathbb{F}$. Then the reciprocity homomorphism*

$$\mathrm{rec}_X : h_0(X) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

is injective. The image of rec_X *consists of all elements whose degree in* $\mathrm{Gal}(\bar{\mathbb{F}}|\mathbb{F})$ *is an integral power of the Frobenius automorphism. In particular, the cokernel*

$\operatorname{coker}(\operatorname{rec}_X) \cong \widehat{\mathbb{Z}}/\mathbb{Z}$ *is uniquely divisible. The induced map on the degree-zero parts* $\operatorname{rec}_X^0 : h_0(X)^0 \xrightarrow{\sim} (\pi_1^t(X)^{\mathrm{ab}})^0$ *is an isomorphism of finite abelian groups.*

Proof. The proof is strictly parallel to the proof of Theorem 8.4, using Theorem 5.1 and the tame version of Wiesend's class field theory for smooth varieties over finite fields [Wiesend 2007, Theorem 1 (b)]. □

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alexander.schmidt@mathematik.uni-regensburg.de
Universität Regensburg, NWF I-Mathematik,
D-93040 Regensburg, Germany
www.mathematik.uni-regensburg.de/Schmidt

The 2-block splitting in symmetric groups

Christine Bessenrodt

In 1956, Brauer showed that there is a partitioning of the p-regular conjugacy classes of a group according to the p-blocks of its irreducible characters with close connections to the block theoretical invariants. But an explicit block splitting of regular classes has not been given so far for any family of finite groups. Here, this is now done for the 2-regular classes of the symmetric groups. To prove the result, a detour along the double covers of the symmetric groups is taken, and results on their 2-blocks and the 2-powers in the spin character values are exploited. Surprisingly, it also turns out that for the symmetric groups the 2-block splitting is unique.

1. Introduction

A half-century ago, Richard Brauer [1956] introduced the idea of not only distributing characters into p-blocks but also to associate p-regular conjugacy classes to p-blocks. He showed that it is possible to distribute the p-regular classes in such a way into blocks that it fits with the blocks of irreducible Brauer characters (and suitable subsets of ordinary irreducible characters in the blocks); this is to say that the determinant of the corresponding block part of the Brauer character table (or a suitable part of the ordinary character table) is not congruent to 0 modulo $\mathfrak{p}$ (a prime ideal over p). Given such a splitting of p-regular classes into blocks, Brauer showed that the elementary divisors of the Cartan matrix of a block are then exactly the p-parts in the orders of the centralizers of elements in the classes corresponding to the block. But while it is known how to determine the p-blocks of irreducible characters, for the p-regular classes only the existence of such a block splitting is known by Brauer's work — concrete examples for providing such a distribution for families of groups have not been known so far. Brauer also observed that in general there may be several such block splittings, and there did not seem to be any natural choice for a given finite group.

In the present paper, such an explicit block splitting in the sense of Brauer is exhibited for the conjugacy classes of odd order elements and the 2-blocks of the

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symmetric groups; in fact, it turns out that for these groups this is the unique 2-block splitting of the 2-regular classes. Surprisingly, the strategy employed here takes a detour along the double covers of the symmetric groups and exploits results on the 2-powers in the spin character values of these groups. Also our knowledge on the 2-block distribution of the spin characters is an important ingredient.

Here is a brief outline of the sections. In Section 2, some notation and results for the symmetric groups and its representation theory are collected, and we recall Brauer's results on block splittings for arbitrary finite groups. As already mentioned above, we will not only work in the context of characters of the symmetric group S_n, but we want to use results on the spin characters of the double cover groups $\tilde{S}_n$. For this, we have to introduce further combinatorial notions in Section 3, and in particular we recall the Glaisher bijection between partitions into odd parts and partitions into distinct parts which plays a crucial rôle here; we also review a number of results on spin characters, mostly of the last decade, which will be used in the proof of our main result. In preparation for the application in Section 4, also a new result on spin character values is proved in this section (Theorem 3.4). In the final section, the class labels for the 2-block splitting of S_n are defined; for a 2-block B of S_n we take the 2-regular classes labelled by partitions into odd parts whose Glaisher image has a $\bar{4}$-core corresponding to the 2-block $\tilde{B}$ of $\tilde{S}_n$ containing B (see Definition 4.1). In the main Theorem 4.2 properties of the determinants of the corresponding block character tables are proved which imply that the construction gives indeed a block splitting of the classes; in fact, the proof allows to refine the result on the determinants further to a result on the Smith normal forms given in Theorem 4.3. An analysis of the proof of the main Theorem shows that the information from Section 3 on spin character values exploited there may also be applied to prove uniqueness of our splitting system.

2. Preliminaries

We have to introduce some notation. For the symmetric groups S_n, the corresponding combinatorial notions and their representation theory, we will follow mostly the usual notation in [James and Kerber 1981]; for the double cover groups $\tilde{S}_n$ and the corresponding background we refer the reader to [Hoffman and Humphreys 1992] and [Morris 1962].

Let $n \in \mathbb{N}$. For a partition λ of n, the number of its (nonzero) parts is called its *length* and is denoted by $l(\lambda)$. The complex irreducible character of S_n corresponding to λ is denoted by $[\lambda]$. Given a second partition μ of n,

$$[\lambda](\sigma_\mu)$$

is then the character value on an element σ_μ in S_n of cycle type μ.

Let $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \dots)$ be a partition, written in exponential notation; then we set

$$a_\mu = \prod_{i\geq 1} i^{m_i(\mu)},\ b_\mu = \prod_{i\geq 1} m_i(\mu)!$$

We let z_μ denote the order of the centralizer of an element of cycle type μ in S_n; then $z_\mu = a_\mu b_\mu$.

Let p be a prime. Then a partition is called p-regular if no part is repeated p or more times, and a partition is called p-class regular if no part is divisible by p.

We let $\mathcal{D}(n)$ denote the set of partitions of n into distinct parts; these partitions are thus the 2-regular partitions of n and they are also called bar partitions in connection with the theory of the double cover groups. We let $\mathcal{O}(n)$ denote the set of partitions of n into odd parts; these are thus the 2-class regular partitions of n.

We then define the 2-regular character table of the symmetric group S_n to be

$$X_2 = ([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}(n)\\ \alpha\in\mathcal{O}(n)}}$$

where the partitions are ordered in a suitable way. More generally, the p-regular character table is defined with λ running through the p-regular partitions and α running through the p-class regular partitions. Its determinant has been studied by Olsson, who showed in [2003, Theorem 2] that its absolute value equals the product of all parts of all p-class regular partitions. Hence, $|\det(X_2)| = \prod_{\mu\in\mathcal{O}(n)} a_\mu$, and in particular it is thus known that

$$2 \nmid \det([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}(n)\\ \alpha\in\mathcal{O}(n)}}.$$

Our main result below will provide a block version of this property, by distributing not only the characters but also the 2-regular conjugacy classes into blocks in a suitable way.

This block distribution of conjugacy classes gives a block splitting in the sense of Brauer; we first introduce the general context.

Let G be a finite group, p a prime. Let $\ell(G)$ be the cardinality of the set $\mathrm{Cl}_{p'}(G)$ of p-regular conjugacy classes in G. For each $K \in \mathrm{Cl}_{p'}(G)$ we let x_K denote an element in K. A defect group of K is a Sylow p-subgroup of $C_G(x)$ for some $x \in K$; if this has order p^d, then d is called the p-defect of K. We let $\mathrm{IBr}(G)$ denote the set of modular irreducible characters of G; then

$$\Phi_G = (\varphi(x_K))_{\substack{\varphi\in \mathrm{IBr}(G)\\ K\in\mathrm{Cl}_{p'}(G)}}$$

is the Brauer character table of G. It is well known that the Brauer character table is nonsingular modulo p; that is,

$$\det \Phi_G \not\equiv 0 \pmod{\mathfrak{p}}.$$

Further, we let

$$D = (d_{\chi\,\varphi})_{\substack{\chi \in \mathrm{Irr}(G)\\ \varphi \in \mathrm{IBr}(G)}}$$

denote the p-decomposition matrix for G, and we let $C = D^t D$ denote its Cartan matrix. Let $\mathrm{Bl}_p(G)$ be the set of p-blocks of G. For $B \in \mathrm{Bl}(G)$, $\mathrm{Irr}(B)$ is the set of ordinary irreducible characters in B, $\mathrm{IBr}(B)$ is the set of modular irreducible characters in B, $\ell(B) = |\mathrm{IBr}(B)|$,

$$D(B) = (d_{\chi,\varphi})_{\substack{\chi \in \mathrm{Irr}(B)\\ \varphi \in \mathrm{IBr}(B)}}$$

denotes the p-decomposition matrix for B, and $C(B)$ is the Cartan matrix for B.

Then C and D are the block direct sums of the matrices $C(B)$ and $D(B)$, for $B \in \mathrm{Bl}_p(G)$.

Theorem 2.1 [Brauer 1956, §5]. *There exists a disjoint decomposition of* $\mathrm{Cl}_{p'}(G)$ *into blocks of p-regular conjugacy classes*

$$\mathrm{Cl}_{p'}(G) = \bigcup_{B \in \mathrm{Bl}_p(G)} \mathrm{Cl}_{p'}(B)$$

and a selection of characters $\mathrm{Irr}'(B) \subseteq \mathrm{Irr}(B)$ *for each* p*-block* B *of* G *such that the following conditions are fulfilled*:

(1) $|\mathrm{Cl}_{p'}(B)| = |\mathrm{Irr}'(B)| = \ell(B)$ *for all* $B \in \mathrm{Bl}_p(G)$.

(2) *For* $X_B = (\chi(x_K))_{\substack{\chi \in \mathrm{Irr}'(B)\\ K \in \mathrm{Cl}_{p'}(B)}}$, *we have* $\det X_B \not\equiv 0 \pmod{\mathfrak{p}}$.

(3) *For* $\Phi_B = (\varphi(x_K))_{\substack{\varphi \in \mathrm{IBr}(B)\\ K \in \mathrm{Cl}_{p'}(B)}}$, *we have* $\det \Phi_B \not\equiv 0 \pmod{\mathfrak{p}}$.

(4) *For* $D_B = (d_{\chi\varphi})_{\substack{\chi \in \mathrm{Irr}'(B)\\ \varphi \in \mathrm{IBr}(B)}}$, *we have* $\det D_B \not\equiv 0 \pmod{\mathfrak{p}}$.

Furthermore, the elementary divisors of the Cartan matrix $C(B)$ *are then exactly the orders of the* p*-defect groups of the conjugacy classes in* $\mathrm{Cl}_{p'}(B)$, *for all* B *in* $\mathrm{Bl}_p(G)$.

Note that the properties in (2), (3) and (4) are not independent of each other, as $X_B = D_B\,\Phi_B$. In particular, if we have a suitable choice $\mathrm{Irr}'(B)$ of characters that satisfies (4), and a suitable choice of classes that satisfies (3), then these together are a suitable choice for (2). If we have a basic set of irreducible characters, i.e., a subset $\mathrm{Irr}'(G) \subseteq \mathrm{Irr}(G)$ giving a $\mathbb{Z}$-basis for the character restrictions to the p-regular classes, then the p-block decomposition of this set will give a suitable choice of sets $\mathrm{Irr}'(B)$ satisfying (4).

We now turn to the symmetric groups. In this case, the so-called Nakayama Conjecture (proved by Brauer and Robinson) gives a combinatorial description for the block distribution of characters. If λ is a partition of n and p a prime, we remove rim hooks of length p from the Young diagram of λ as often as possible; this results in a unique partition $\lambda_{(p)}$ which has no rim hook of length p, called the p-core of λ . The number of rim hooks removed from λ on the way to $\lambda_{(p)}$ is called the p-weight of λ. We refer the reader to [James and Kerber 1981] for more details on this and the following.

"Nakayama Conjecture". Two irreducible characters $[\lambda]$, $[\mu]$ of S_n belong to the same p-block if and only if $\lambda_{(p)} = \mu_{(p)}$.

Hence each p-block B has a well-defined p-weight $w(B)$ and p-core $\kappa(B)$, namely the common p-weight and p-core of all the labels of the irreducible characters in B. Note that then $|\lambda| = pw(B) + |\kappa(B)|$, for all $[\lambda] \in \mathrm{Irr}(B)$.

The situation at $p = 2$ is particularly nice, as we may then easily describe all the 2-core partitions: these are exactly the staircase partitions $\rho_k = (k, k-1, \dots, 2, 1)$, $k \in \mathbb{N}_0$. The removal of a rim hook of length 2 from a partition is just the removal of a "domino piece" from the rim of its Young diagram.

The irreducible characters labelled by the p-regular partitions form a basic set [James and Kerber 1981; Külshammer et al. 2003]; thus with respect to a suitable ordering, the determinant of the corresponding part of the decomposition matrix is 1. We take the corresponding choice $\mathrm{Irr}'(B) \subseteq \mathrm{Irr}(B)$ of characters for Brauer's Theorem in our situation at $p = 2$. This means the following. Let $\mathrm{Bl}_2(n)$ be the set of 2-blocks of S_n. For a given 2-block B we set

$$\mathcal{D}(B) := \{\lambda \in \mathcal{D}(n) \mid [\lambda] \in \mathrm{Irr}(B)\} = \{\lambda \in \mathcal{D}(n) \mid \lambda_{(2)} = \kappa(B)\}.$$

This gives a set partition according to the 2-blocks:

$$\mathcal{D}(n) = \bigcup_{B \in \mathrm{Bl}_2(n)} \mathcal{D}(B).$$

Then $|\mathcal{D}(B)|$ equals $p(w(B))$, the number of partitions of $w(B)$; see [James and Kerber 1981] or [Olsson 1993]. In the notation of Theorem 2.1 we then take $\mathrm{Irr}'(B) = \{[\lambda] \mid \lambda \in \mathcal{D}(B)\}$.

By Brauer's Theorem there must exist a suitable block splitting of the 2-regular conjugacy classes; i.e., there must be a set partition

$$\mathcal{O}(n) = \bigcup_{B \in \mathrm{Bl}_2(n)} \mathcal{O}(B)$$

such that for all $B \in \mathrm{Bl}_2(n)$ we have

$$2 \nmid \det(\varphi^\lambda(\sigma_\mu))_{\substack{\lambda\in\mathcal{D}(B)\\ \mu\in\mathcal{O}(B)}}, \tag{1}$$

where for $\mu \in \mathcal{D}(n)$ we denote by φ^μ the corresponding Brauer character of S_n; note that φ^μ belongs to the 2-block B exactly if $\mu_{(2)} = \kappa(B)$. By the remarks above this condition is equivalent to having

$$2 \nmid \det([\lambda](\sigma_\mu))_{\substack{\lambda\in\mathcal{D}(B)\\ \mu\in\mathcal{O}(B)}}. \tag{2}$$

As noted above, for any such block splitting, the elementary divisors of the Cartan matrix of B are then the defect group orders of the conjugacy classes labelled by $\mathcal{O}(B)$.

The aim of this article is to define explicit subsets $\mathcal{O}(B)$ of $\mathcal{O}(n)$ satisfying the equivalent conditions (1) and (2), thus giving a 2-block splitting of conjugacy classes for the symmetric groups.

3. Spin characters

We collect here a number of results on spin characters that will be needed in the sequel; the reader is referred to [Hoffman and Humphreys 1992] and [Olsson 1993] for more background on the double cover groups $\tilde{S}_n$ and their representation theory.

The sets $\mathcal{D}^+(n)$ and $\mathcal{D}^-(n)$ are the subsets of partitions $\lambda \in \mathcal{D}(n)$ with $n - l(\lambda)$ even or odd, respectively. For $\mu \in \mathcal{D}^+(n)$, we denote by $\langle\mu\rangle$ the corresponding complex irreducible spin character of $\tilde{S}_n$, for $\mu \in \mathcal{D}^-(n)$, we let $\langle\mu\rangle$ and $\langle\mu\rangle' = \mathrm{sgn}\cdot\langle\mu\rangle$ be the corresponding pair of associate complex irreducible spin characters of $\tilde{S}_n$. We recall that the only conjugacy classes of S_n that split over the double cover groups are those of type $\mathcal{O}$ and of type $\mathcal{D}^-$; the irreducible spin characters vanish on all other conjugacy classes. More precisely, for any such partition α one of the two corresponding conjugacy classes in $\tilde{S}_n$ is chosen in accordance with [Schur 1911], and we denote a corresponding representative by $\tilde{\sigma}_\alpha$. While the spin character values on the $\mathcal{D}^-$ classes are known explicitly (but they are in general not integers, and mostly not even real), for the values on the $\mathcal{O}$-classes we have a recursion formula due to A. Morris which is analogous to the Murnaghan–Nakayama formula (and which shows in particular, that these are integers).

In contrast to odd characteristic, the 2-blocks of $\tilde{S}_n$ are mixed, i.e., they contain ordinary as well as spin characters. The simple $\tilde{S}_n$-modules in characteristic 2 may be identified with the simple S_n-modules D^λ which are labelled by partitions $\lambda \in \mathcal{D}(n)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{D}(n)$ we set

$$\mathrm{dbl}(\lambda) = \left(\left[\frac{\lambda_1+1}{2}\right], \left[\frac{\lambda_1}{2}\right], \left[\frac{\lambda_2+1}{2}\right], \left[\frac{\lambda_2}{2}\right], \ldots, \left[\frac{\lambda_m+1}{2}\right], \left[\frac{\lambda_m}{2}\right]\right),$$

the *doubling* of λ. For example, the staircase partition $\rho_k = (k, k-1, \dots, 2, 1)$ is the doubling of the partition $\tau_k = (2k-1, 2k-5, \dots)$.

The 2-block distribution of the spin characters is described by the following result (which confirmed a conjecture by Knörr and Olsson):

Theorem 3.1 [Bessenrodt and Olsson 1997]. *Let* $\lambda \in \mathcal{D}(n)$. *Then* $\langle\lambda\rangle$ *and* $[\mathrm{dbl}(\lambda)]$ *belong to the same* 2*-block of* $\tilde{S}_n$.

Thus, the 2-block of $\langle\lambda\rangle$ is determined by the 2-core of $\mathrm{dbl}(\lambda)$. But in fact, the spin combinatorics in this case may also be viewed as a $\bar{4}$-combinatorics (see [Bessenrodt and Olsson 1997] for more details). Indeed, we have a $\bar{4}$-abacus for the bar partitions with one runner for all even parts (the 0-th runner), on which we can slide by steps of 2, and two conjugate runners for the residues 1 and 3 modulo 4. A bar partition is then a $\bar{4}$-core exactly if the 0-th runner is empty (i.e., there are no even parts), at most one of the two conjugate runners is nonempty, and a nonempty runner has only beads at the top; thus the $\bar{4}$-cores are the partitions τ_k defined above. We will denote the $\bar{4}$-core of a bar partition λ by $\lambda_{(\bar{4})}$.

It is well known that $|\mathcal{D}(n)| = |\mathcal{O}(n)|$. In fact, J. W. L. Glaisher [1883] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. In this case, Glaisher's map G is defined as follows. Suppose that $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2, i.e., in its 2-adic decomposition: $m_i = \sum_j 2^{a_{ij}}$. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, sorted in descending order to give a partition. Surprisingly, this map has turned up naturally in connection with spin characters of the symmetric groups (see below).

For any integer $m \geq 0$, let $s(m)$ be the number of summands in the 2-adic decomposition of m. Then for $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ the length of $G(\alpha)$ is $l(G(\alpha)) = \sum_{i \text{ odd}} s(m_i)$. We define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i))$$

and set $\sigma(\alpha) = (-1)^{k_\alpha}$; note that we thus have

$$k_\alpha = l(\alpha) - l(G(\alpha)).$$

We denote by $\mathcal{O}^\varepsilon(n)$ the set of partitions α in $\mathcal{O}(n)$ with the sign of $\sigma(\alpha)$ being $\varepsilon \in \{\pm\}$. With this definition of signs, it is easy to see that the Glaisher map G induces bijections $\mathcal{O}^\epsilon(n) \to \mathcal{D}^\epsilon(n)$; see [Bessenrodt and Olsson 2000].

The integer k_α also comes up naturally in the group-theoretic context. For any nonzero integer m, we denote by $\nu(m)$ the exponent to which 2 divides m; $2^{\nu(m)}$ is the exact 2-power dividing m. Let $\alpha = (1^{m_1(\alpha)}3^{m_3(\alpha)}, \dots) \in \mathcal{O}(n)$, σ_α an element of cycle type α in S_n. Then $\nu(|C_{S_n}(\sigma_\alpha)|) = \prod_{i \text{ odd}} \nu(m_i(\alpha)!) = k_\alpha$. Hence k_α is the 2-defect of K_α, the conjugacy class of S_n labelled by $\alpha \in \mathcal{O}(n)$.

In joint work with J. Olsson, we have previously investigated the 2-powers appearing in the spin character values on a given 2-regular conjugacy class:

Theorem 3.2. *Let $\alpha \in \mathcal{O}(n)$.*

(i) [Bessenrodt and Olsson 2000] *For all $\lambda \in \mathcal{D}(n)$ we have*

$$\nu(\langle\lambda\rangle(\tilde{\sigma}_\alpha)) \geq \lfloor k_\alpha/2 \rfloor .$$

(ii) [Bessenrodt and Olsson 2005] *Let $G(\alpha) \in \mathcal{D}(n)$ be the Glaisher image of α. Then*

$$\nu(\langle G(\alpha)\rangle(\tilde{\sigma}_\alpha)) = \lfloor k_\alpha/2 \rfloor ,$$

and if $\alpha \in \mathcal{O}^-(n)$, then $\langle G(\alpha)\rangle$ and $\langle G(\alpha)\rangle'$ are the only spin characters where this equality holds.

In the case of partitions $\alpha \in \mathcal{O}^+(n)$, we may have spin characters different from $\langle G(\alpha)\rangle$ such that the minimal 2-power is attained on $\tilde{\sigma}_\alpha$. At least we can have non-selfassociate spin characters with this property, but it is not yet clear whether there are also self-associate spin characters satisfying this; see [Bessenrodt and Olsson 2005]. For our later purposes the weaker statement in Theorem 3.4 below suffices. For proving this result, we first have to recall some results due to Stembridge.

Stembridge [1989] has investigated a projective analogue of the outer tensor product, called the reduced Clifford product, and has proved a shifted analogue of the Littlewood–Richardson rule which we will need in the sequel. To state this, we first have to define some further combinatorial notions.

Let A' be the ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$. The letters $1', 2', \dots$ are said to be *marked*, the others are *unmarked*. The notation $|a|$ refers to the unmarked version of a letter a in A'. To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq l(\lambda), i \leq j \leq \lambda_i + i - 1\}$$

A shifted tableau T of shape λ is a map $T : Y'(\lambda) \to A'$ such that $T(i, j) \leq T(i+1, j)$, $T(i, j) \leq T(i, j+1)$ for all i, j, and every $k \in \{1, 2, \dots\}$ appears at most once in each column of T, and every $k' \in \{1', 2', \dots\}$ appears at most once in each row of T. For $k \in \{1, 2, \dots\}$, let c_k be the number of boxes (i, j) in $Y'(\lambda)$ such that $|T(i, j)| = k$. Then we say that the tableau T has content $(c_1, c_2, \dots)$. Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape $\lambda \setminus \mu$ if μ is a partition with $Y'(\mu) \subseteq Y'(\lambda)$. For a (possibly skew) shifted tableau S we define its associated word $w(S) = w_1 w_2 \cdots$ by reading the rows of

S from left to right and from bottom to top. By erasing the marks of w, we obtain the word $|w|$.

Given a word $w = w_1 w_2 \ldots$, we define

$$m_i(j) = \text{multiplicity of } i \text{ among } w_{n-j+1}, \ldots, w_n \qquad \text{for } 0 \le j \le n,$$
$$m_i(n+j) = m_i(n) + \text{ multiplicity of } i' \text{ among } w_1, \ldots, w_j \quad \text{for } 0 < j \le n.$$

This function m_i corresponds to reading the rows of the tableau first from right to left and from top to bottom, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.

The word w satisfies the lattice property if, whenever $m_i(j) = m_{i-1}(j)$, we have

$$w_{n-j} \neq i, i' \qquad \text{if } 0 \le j < n,$$
$$w_{j-n+1} \neq i-1, i' \quad \text{if } n \le j < 2n.$$

For two partitions μ and ν we denote by $\mu \cup \nu$ the partition which has as its parts all the parts of μ and ν together. Also define

$$\varepsilon_\lambda = \begin{cases} 1 & \text{if } \lambda \in \mathcal{D}^+(n) \\ \sqrt{2} & \text{if } \lambda \in \mathcal{D}^-(n) \end{cases}.$$

We can then state the spin version of the Littlewood–Richardson rule:

Theorem 3.3 [Stembridge 1989, 8.1 and 8.3]. *Let $\mu \in \mathcal{D}(k)$, $\nu \in \mathcal{D}(n-k)$, $\lambda \in \mathcal{D}(n)$, and form the reduced Clifford product $\langle\mu\rangle \times_c \langle\nu\rangle$. Then we have*

$$((\langle\mu\rangle \times_c \langle\nu\rangle) \uparrow^{\tilde{S}_n}, \langle\lambda\rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu\cup\nu}} 2^{(l(\mu)+l(\nu)-l(\lambda))/2} f^\lambda_{\mu\nu},$$

unless λ is odd and $\lambda = \mu \cup \nu$. In that latter case, the multiplicity of $\langle\lambda\rangle$ is 0 *or* 1, *according to the choice of associates.*

The coefficient $f^\lambda_{\mu\nu}$ is the number of shifted tableaux S of shape $\lambda \backslash \mu$ and content ν such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \le i \le l(\nu)$.

For further properties of the reduced Clifford product, see [Humphreys 1986; Michler and Olsson 1990; Schur 1911; Stembridge 1989].

Theorem 3.4. *Let $\alpha \in \mathcal{O}^+(n)$, $\lambda \in \mathcal{D}^+(n)$. If $\nu(\langle\lambda\rangle(\tilde{\sigma}_\alpha)) = \lfloor k_\alpha/2 \rfloor$, then $\lambda \trianglerighteq G(\alpha)$. In particular, $G(\alpha)$ is the minimal $\mathcal{D}^+$-partition in lexicographic order where this equality is attained.*

Proof. We recall parts of the proof of [Bessenrodt and Olsson 2005, Theorem 1.2]. Let $\alpha = (i^{m_i})_{i=1,3,\ldots}$, set $\alpha^i = (i^{m_i})$, $a_i = im_i$, and let $\tilde{S}_a$ be the preimage of the Young subgroup $S_{a_1} \times S_{a_3} \times \ldots$ in $\tilde{S}_n$. Restricting $\langle\lambda\rangle$ to $\tilde{S}_a$ gives

$$\langle\lambda\rangle_{\tilde{S}_a} = \sum_{\mu=(\mu_1,\mu_3,\ldots)} g^\lambda_\mu(\times_c\langle\mu_i\rangle) + \sum_{\mu=(\mu_1,\mu_3,\ldots)\text{n.s.a.}} \tilde{g}^\lambda_\mu(\times_c\langle\mu_i\rangle)',$$

where the g^λ_μ are spin Littlewood–Richardson coefficients, and $\mu = (\mu_1, \mu_3, \dots)$ runs over all sequences with μ_i a partition of a_i. Moreover, μ is nonselfassociate (n.s.a.) if the corresponding reduced Clifford product is nonselfassociate; this is the case if and only if $t_\mu = |\{i \mid \mu_i \in \mathcal{D}^-\}|$ is odd. As we assume that $\lambda \in \mathcal{D}^+$, by [Stembridge 1989] we have $g^\lambda_\mu = \tilde{g}^\lambda_\mu$ for any n.s.a. μ. Thus

$$\langle\lambda\rangle(\tilde{\sigma}_\alpha) = \sum_{\mu=(\mu_1,\mu_3,\dots)\text{ s.a.}} g^\lambda_\mu(\times_c\langle\mu_i\rangle)(\tilde{\sigma}_\alpha) + \sum_{\mu=(\mu_1,\mu_3,\dots)\text{ n.s.a.}} 2g^\lambda_\mu(\times_c\langle\mu_i\rangle)(\tilde{\sigma}_\alpha).$$

By [Bessenrodt and Olsson 2005, Proposition 3.3], the 2-value of each Clifford product value is at least $[k_\alpha/2]$; hence we obtain for the n.s.a. μ a contribution of nonminimal 2-value. The same proposition implies that, since $\alpha \in \mathcal{O}^+$, the only Clifford product value which is of 2-value $[k_\alpha/2]$ occurs for the partition sequence $\mu = g(\alpha) = (G(\alpha^1), G(\alpha^3), \dots)$, and thus $g^\lambda_{g(\alpha)}$ has to be odd. In particular, $\langle\lambda\rangle$ is a constituent of $\times_c\langle G(\alpha^i)\rangle\uparrow^{\tilde{S}_n}$. By the spin Littlewood–Richardson rule due to Stembridge, $\langle G(\alpha)\rangle$ is the lowest constituent in this induced character (with respect to dominance, and thus also in lexicographic order). We have already seen before that for this character we have indeed equality on the conjugacy class to α. □

We want to go beyond determinants and study the Smith normal forms of the matrices under consideration. For any integral square matrix X we let $\mathcal{S}(X)$ denote its Smith normal form, i.e., the diagonal matrix with the elementary divisors of X as diagonal elements. The following property of the Smith normal form will be used: If X and Y are square $n \times n$ matrices with relatively prime determinants, then $\mathcal{S}(XY) = \mathcal{S}(X)\,\mathcal{S}(Y)$; see [Newman 1972, Theorem II.15], for instance. For a finite family of numbers c_i, $i \in I$, we mean by $\mathcal{S}(c_i, i \in I)$ the Smith normal form of any diagonal matrix with the given numbers on the diagonal.

We define the reduced spin character table of $\tilde{S}_n$ as the integral square matrix

$$Z_s = (\langle\lambda\rangle(\tilde{\sigma}_\mu))_{\substack{\lambda\in\mathcal{D}(n)\\ \mu\in\mathcal{O}(n)}}.$$

Then we have

Theorem 3.5 [Bessenrodt et al. 2005, Theorem 13]. *The Smith normal form of the reduced spin character table Z_s of $\tilde{S}_n$ is given by*

$$\mathcal{S}(Z_s) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \mathcal{O}(n)) \cdot \mathcal{S}(b_\mu, \mu \in \mathcal{O}(n))_{2'}.$$

In the context of 2-modular representations, we consider the part of the 2-decomposition matrix for $\tilde{S}_n$ corresponding to spin characters. Since the rows corresponding to associate spin characters are equal, this part of the decomposition matrix is determined by the submatrix $D_s = D_s(n)$, where for each $\lambda \in \mathcal{D}(n)$ we keep only one row for each associate class of spin characters. We call D_s the reduced spin 2-decomposition matrix; it is a square matrix of the same size as Z_s.

Theorem 3.6 [Bessenrodt and Olsson 2000]. *Let* $\tilde{B} \in \mathrm{Bl}_2(\tilde{S}_n)$, $B \in \mathrm{Bl}_2(S_n)$, $B \subseteq \tilde{B}$. *Suppose that* $2^{c_1}, 2^{c_2}, \ldots, 2^{c_\ell}$ *are the elementary divisors of the Cartan matrix* $C(B)$. *Then the elementary divisors of* $D_s(\tilde{B})$ *are* $2^{[c_1/2]}, 2^{[c_2/2]}, \ldots, 2^{[c_\ell/2]}$.

Now the invariants of the Cartan matrix had been explicitly determined by Olsson (see [Bessenrodt and Olsson 2000] for the correction of the formula misstated in [Olsson 1986]). For $p = 2$, this formula was recast in a nicer combinatorial way by Uno and Yamada; we reformulate this here for our purposes.

Theorem 3.7 [Uno and Yamada 2006]. *Let* B *be a* 2*-block of* S_n *with* 2*-core* $\rho_k = (k, k-1, \ldots, 2, 1)$, *and let* $\tau_k = (2k-1, 2k-5, \ldots)$. *Then the elementary divisors of the Cartan matrix* $C(B)$ *are given by*

$$2^{l(\alpha)-l(G(\alpha))},\ \alpha \in \mathcal{O}(n),\ G(\alpha)_{(\bar{4})} = \tau_k.$$

As $k_\alpha = l(\alpha) - l(G(\alpha))$ for any $\alpha \in \mathcal{O}(n)$, we thus conclude

Corollary 3.8. *Let* $\tilde{B} \in \mathrm{Bl}_2(\tilde{S}_n)$, $B \in \mathrm{Bl}_2(S_n)$, $B \subseteq \tilde{B}$, τ_k *as above. Then*

$$\mathcal{S}(D_s(\tilde{B})) = \mathcal{S}\big(2^{[k_\alpha/2]},\ \alpha \in \mathcal{O}(n),\ G(\alpha)_{(\bar{4})} = \tau_k\big).$$

We observe also that by Brauer's Theorem 2.1, the defect group orders of the classes associated to B in a block splitting thus have to be the numbers 2^{k_α}, $\alpha \in \mathcal{O}(n)$, $\alpha_{(\bar{4})} = \tau_B$. We take this as a hint on how to choose the distribution of the 2-regular conjugacy classes into blocks in the next section.

4. The 2-block splitting for S_n

We fix the following notation.

Let B be a 2-block of S_n, with 2-core $\rho_k = (k, k-1, \ldots, 2, 1)$, $k \in \mathbb{N}_0$. Let $\tilde{B}$ be the 2-block of $\tilde{S}_n$ containing B, with corresponding $\bar{4}$-core $\tau_k = (2k-1, 2k-5, \ldots)$. As before, we let

$$\mathcal{D}(B) = \{\mu \in \mathcal{D}(n) \mid \mu_{(2)} = \rho_k\}$$

and we set

$$\mathcal{D}(\tilde{B}) = \{\lambda \in \mathcal{D}(n) \mid \lambda_{(\bar{4})} = \tau_k\}.$$

An important point to note here is that these sets of partitions really fit to the 2-block inclusion $B \subseteq \tilde{B}$, as the corresponding characters $[\mu]$, $\mu \in \mathcal{D}(B)$, and $\langle\lambda\rangle$, $\lambda \in \mathcal{D}(\tilde{B})$, belong to the same 2-block $\tilde{B}$ of $\tilde{S}_n$ by Theorem 3.1.

Let $w = w(B)$ be the 2-weight of B. Then

$$|\mathcal{D}(B)| = |\mathcal{D}(\tilde{B})| = p(w);$$

see [Bessenrodt and Olsson 1997] or [Olsson 1993], for example. With this notation, we can now introduce the crucial definition that will provide the 2-block splitting of the 2-regular classes:

Definition 4.1. With $G : \mathcal{O}(n) \to \mathcal{D}(n)$ still denoting the Glaisher map defined in Section 3, we set

$$\mathcal{O}(B) := \{\alpha \in \mathcal{O}(n) \mid G(\alpha)_{(\bar{4})} = \tau_k\} =: \mathcal{O}(\tilde{B}).$$

Thus by definition the Glaisher map restricts to blockwise bijections

$$G : \mathcal{O}(B) \to \mathcal{D}(\tilde{B}).$$

In particular, we thus have $|\mathcal{O}(B)| = \ell(B)$, so the first condition of a block splitting is satisfied for these labelling sets.

We now consider the following parts of the character table and spin character table, respectively, which are all square matrices by the observations made above (note that the spin character table is reduced in the sense that we take only one of a pair of associate spin characters):

$$Z(B) = ([\mu](\sigma_\alpha))_{\substack{\mu\in\mathcal{D}(B)\\ \alpha\in\mathcal{O}(B)}}, \qquad Z_s(\tilde{B}) = (\langle\lambda\rangle(\tilde{\sigma}_\alpha))_{\substack{\lambda\in\mathcal{D}(\tilde{B})\\ \alpha\in\mathcal{O}(B)}}.$$

We also consider the corresponding block part of the Brauer character table:

$$\Phi(B) = (\varphi^\mu(\sigma_\alpha))_{\substack{\mu\in\mathcal{D}(B)\\ \alpha\in\mathcal{O}(B)}}.$$

Finally we define a diagonal matrix associated to B by

$$\Delta(B) = \Delta(2^{[k_\alpha/2]}, \alpha \in \mathcal{O}(B)).$$

After all these preparations, we can now state the following result on the determinants of the matrices defined above, which tells us that the chosen distribution of conjugacy classes given by the sets $\mathcal{O}(B)$ is indeed a 2-block splitting of the 2-regular classes:

Theorem 4.2. *Let $B \subseteq \tilde{B}$ be as above. Then the following holds*:

(i) *The 2-part in the determinant of the block part of the spin character table is given by*

$$\nu(\det Z_s(\tilde{B})) = \sum_{\alpha\in\mathcal{O}(B)} \left[\frac{k_\alpha}{2}\right].$$

(ii) *The odd part of the determinant of the block part of the spin character table satisfies*

$$(\det Z_s(\tilde{B}))_{2'} = \det \Phi(B) = \det Z(B).$$

In particular, the sets $\mathcal{O}(B)$, $B \in \mathrm{Bl}_2(n)$, define a 2-block splitting for S_n.

Proof. (i) By Theorem 3.2 we have

$$\nu(\det Z_s(\tilde{B})) \geq \sum_{\alpha\in\mathcal{O}(B)} \left[\frac{k_\alpha}{2}\right].$$

More precisely, for any bijection $\pi : \mathcal{O}(B) \to \mathcal{D}(\tilde{B})$ we have

$$\nu\Bigg(\prod_{\alpha \in \mathcal{O}(B)} \langle \pi(\alpha) \rangle (\tilde{\sigma}_\alpha) \Bigg) \geq \sum_{\alpha \in \mathcal{O}(B)} \left[\frac{k_\alpha}{2} \right].$$

We claim that the Glaisher bijection $G : \mathcal{O}(B) \to \mathcal{D}(\tilde{B})$, $\alpha \mapsto G(\alpha)$ is the *unique* bijection $\mathcal{O}(B) \to \mathcal{D}(\tilde{B})$ such that equality holds. Then the assertion follows by the Leibniz formula for the determinant.

By Theorem 3.2, each such map π has to be the Glaisher map on restriction to $\mathcal{O}^-(B)$, and thus π induces bijections $\mathcal{O}^\varepsilon(B) \to \mathcal{D}^\varepsilon(\tilde{B})$, for both signs $\varepsilon = \pm$. Now we argue by induction on the lexicographic order on $\mathcal{D}^+$. Take $\alpha \in \mathcal{O}^+(B)$ such that $G(\alpha)$ is highest among the partitions in $\mathcal{D}^+(\tilde{B})$. Then by Theorem 3.4, $G(\alpha)$ is the *unique* partition in $\mathcal{D}^+(\tilde{B})$ such that

$$\nu(\langle G(\alpha) \rangle (\tilde{\sigma}_\alpha)) = \left[\frac{k_\alpha}{2} \right]$$

and hence (using again Theorem 3.2) we must have $\pi(\alpha) = G(\alpha)$. Remove α from $\mathcal{O}^+(B)$ and $\pi(\alpha) = G(\alpha)$ from $\mathcal{D}^+(\tilde{B})$ and continue, using Theorem 3.4 in each step. This shows that $\pi = G$, and hence we are done.

(ii) Let

$$D_s(\tilde{B}) = (\tilde{d}_{\lambda\mu})_{\substack{\lambda \in \mathcal{D}(\tilde{B}) \\ \mu \in \mathcal{D}(B)}}$$

be the reduced spin 2-decomposition matrix for the spin characters in $\tilde{B}$ (taking only one of an associate pair). By Theorem 3.1 we have

$$Z_s(\tilde{B}) = D_s(\tilde{B}) \Phi(B).$$

By Corollary 3.8 and part (i) we know that

$$|\det D_s(\tilde{B})| = \prod_{\alpha \in \mathcal{O}(B)} 2^{[k_\alpha/2]} = (\det Z_s(\tilde{B}))_2,$$

and hence the first equality in (ii) follows.
With $D_B = (d_{\lambda\mu})_{\substack{\lambda \in \mathcal{D}(B) \\ \mu \in \mathcal{D}(B)}}$ denoting the upper square part of the 2-decomposition matrix for B (with the usual order where the characters to regular partitions come first) we also have

$$Z(B) = D_B \, \Phi(B).$$

As D_B is well-known to be a lower unitriangular matrix, this immediately implies $\det \Phi(B) = \det Z(B)$. □

We can also deduce further information on the Smith normal forms of the matrices defined above; these may also be considered as block versions of some results in [Bessenrodt et al. 2005].

Theorem 4.3. *Let $B \subseteq \tilde{B}$ be as above.*

(i) *The 2-part of the Smith normal form of $Z_s(\tilde{B})$ is given by*

$$\mathcal{S}(Z_s(\tilde{B}))_2 = \mathcal{S}(\Delta(B)).$$

(ii) *The odd part of the Smith normal form of $Z_s(\tilde{B})$ satisfies*

$$\mathcal{S}(Z_s(\tilde{B}))_{2'} = \mathcal{S}(\Phi(B)) = \mathcal{S}(Z(B)).$$

Proof. We have already seen above that

$$Z_s(\tilde{B}) = D_s(\tilde{B})\,\Phi(B).$$

By Corollary 3.8 and Theorem 4.2 we know that $D_s(\tilde{B})$ and $\Phi(B)$ have coprime determinants, and more precisely, we then obtain

$$\mathcal{S}(Z_s(\tilde{B}))_2 = \mathcal{S}(D_s(\tilde{B})) = \mathcal{S}(\Delta(B)),$$
$$\mathcal{S}(Z_s(\tilde{B}))_{2'} = \mathcal{S}(\Phi(B)).$$

Since $Z(B) = D_B\,\Phi(B)$ and $\det D_B = 1$, this immediately implies $\mathcal{S}(\Phi(B)) = \mathcal{S}(Z(B))$. □

Theorem 4.4. *The block splitting of the 2-regular classes given by the sets $\mathcal{O}(B)$, $B \in \mathrm{Bl}_2(n)$, is the unique block splitting in the sense of Brauer* (i.e., such that Theorem 2.1(3) is satisfied).

Proof. We keep our previous choice of characters $\mathrm{Irr}'(B) \subseteq \mathrm{Irr}(B)$, i.e., we take the ordinary characters labelled by $\mathcal{D}(B)$, and we take the spin characters labelled by $\mathcal{D}(\tilde{B})$. For any choice $\mathcal{O}(B)'$, $B \in \mathrm{Bl}_2(n)$, of blocks of labels of the 2-regular conjugacy classes, we have the analogous equality

$$Z_s(\tilde{B})' = D_s(\tilde{B})\Phi(B)'$$

and hence $\det Z_s(\tilde{B})' = \det D_s(\tilde{B}) \det \Phi(B)'$. Thus the sets $\mathcal{O}(B)'$ correspond to a splitting system, i.e., condition (3) in Brauer's Theorem is satisfied for all B, if and only if

$$(\det Z_s(\tilde{B})')_2 = (\det D_s(\tilde{B}))_2 = \prod_{\alpha \in \mathcal{O}(B)} 2^{[k_\alpha/2]} \quad \text{for all } B \in \mathrm{Bl}_2(n).$$

As in the proof of Theorem 4.2 we again consider the bijections on the block level that give a summand with minimal 2-power in the Leibniz formula for the determinant and use Theorem 3.2; one immediately observes that we must have for any 2-block B of S_n:

$$\mathcal{O}^-(B)' = G^{-1}(\mathcal{D}^-(B)) = \mathcal{O}^-(B),$$

i.e., the $\mathbb{O}^-$-part of the blocks in a block splitting of regular classes is uniquely determined. In the next step, arguing similarly as before with Theorem 3.4 along the lexicographic ordering on the $\mathcal{D}^+$-partitions (and considering all blocks in this argument simultaneously) one also obtains

$$\mathbb{O}^+(B)' = \mathbb{O}^+(B) \quad \text{for all } B \in \mathrm{Bl}_2(n).$$

Thus the block splitting $\mathbb{O}(B)$, $B \in \mathrm{Bl}_2(n)$, constructed above is the unique block splitting of the 2-regular classes of S_n. □

Remark. While there is a nice formula for the determinant of the whole regular character table of S_n (see Section 2), we do not have a formula for the determinant of the block character table. A first guess might be that it is again the product of the parts of the corresponding labelling $\mathbb{O}$-partitions (or related to this), but examples show that this is *not* the case — in fact, primes $> n$ may appear in the determinant.

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bessen@math.uni-hannover.de *Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany*
http://www-ifm.math.uni-hannover.de/~bessen/

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