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We construct a singular homology theory on the category of schemes of finite type over a Dedekind domain and verify several basic properties. For arithmetic schemes we construct a reciprocity isomorphism between the integral singular homology in degree zero and the abelianized modified tame fundamental group.

1. Introduction

The objective of this paper is to construct a reasonable singular homology theory on the category of schemes of finite type over a Dedekind domain. Our main criterion for "reasonable" was to find a theory which satisfies the usual properties of a singular homology theory and which has the additional property that, for schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$, the group h_0 serves as the source of a reciprocity map for tame class field theory. In the case of schemes of finite type over finite fields this role was taken over by Suslin's singular homology; see [Schmidt and Spieß 2000]. In this article we motivate and give the definition of the singular homology groups of schemes of finite type over a Dedekind domain and we verify basic properties, e.g. homotopy invariance. Then we present an application to tame class field theory.

The (integral) singular homology groups $h_*(X)$ of a scheme of finite type over a field k were defined by A. Suslin as the homology of the complex $C_*(X)$ whose n-th term is given by

$$C_n(X) = \text{group of finite correspondences } \Delta_k^n \longrightarrow X,$$

where $\Delta_k^n = \mathrm{Spec}(k[t_0, \dots, t_n]/\sum t_i = 1)$ is the n-dimensional standard simplex over k and a finite correspondence is a finite linear combination $\sum n_i Z_i$ where each Z_i is an integral subscheme of $X \times \Delta_k^n$ such that the projection $Z_i \to \Delta_k^n$ is finite and surjective. The differential $d : C_n(X) \to C_{n-1}(X)$ is defined as the alternating sum of the homomorphisms which are induced by the cycle theoretic intersection with the 1-codimensional faces $X \times \Delta_k^{n-1}$ in $X \times \Delta_k^n$. This definition (see [Suslin and Voevodsky 1996]) was motivated by the theorem of Dold–Thom

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in algebraic topology. If X is an integral scheme of finite type over the field $\mathbb{C}$ of complex numbers, then Suslin and Voevodsky show that there exists a natural isomorphism

$$h_*(X, \mathbb{Z}/n\mathbb{Z}) \cong H_*^{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$$

between the algebraic singular homology of X with finite coefficients and the topological singular homology of the space $X(\mathbb{C})$. If X is proper and of dimension d, singular homology is related to the higher Chow groups of [Bloch 1986] by the formula $h_i(X) = \mathrm{CH}^d(X, i)$ [Voevodsky 2000]. A sheafified version of the preceding definition leads to the "triangulated category of motivic complexes" (ibid.), which, mainly due to the work of Voevodsky, Suslin and Friedlander, has become a powerful categorical framework for motivic (co)homology theories.

If the field k is finite and if X is an open subscheme of a projective smooth variety over k, then we have the following relation to class field theory: there exists a natural reciprocity homomorphism

$$\mathrm{rec} : h_0(X) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

from the 0-th singular homology group to the abelianized tame fundamental group of X. The homomorphism rec is injective and has a uniquely divisible cokernel (see [Schmidt and Spieß 2000] or Theorem 8.7 below for a more precise statement).

This connection to class field theory was the main motivation of the author to study singular homology of schemes of finite type over Dedekind domains. Let $S = \mathrm{Spec}(A)$ be the spectrum of a Dedekind domain and let X be a scheme of finite type over S. The naive definition of singular homology as the homology of the complex whose n-th term is the group of finite correspondences $\Delta_S^n \to X$ is certainly not the correct one. For example, according to this definition, we would have $h_*(U) = 0$ for any open subscheme $U \subsetneq S$. Philosophically, a "standard n-simplex" should have dimension n but Δ_S^n is a scheme of dimension $(n+1)$.

If the Dedekind domain A is finitely generated over a field, then one can define the homology of X as its homology regarded as a scheme over this field.

The striking analogy between number fields and function fields in one variable over finite fields, as it is known from number theory, led to the philosophy that it should be possible to consider *any* Dedekind domain A, i.e. also if it is of mixed characteristic, as a curve over a mysterious "ground field" $\mathbb{F}(A)$. In the case $A = \mathbb{Z}$ this "field" is sometimes called the "field with one element" $\mathbb{F}_1$. A more precise formulation of this idea making the philosophy into real mathematics and, in particular, a reasonable intersection theory on "$\mathrm{Spec}(\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z})$" would be of high arithmetic interest. With respect to singular homology, this philosophy predicts that, for a scheme X of finite type over $\mathrm{Spec}(A)$, the groups $h_*(X)$ should be the homology groups of a complex whose n-th term is given as the group of finite

correspondences $\Delta^n_{\mathbb{F}(A)} \to X$. Unfortunately, we do not have a good definition of the category of schemes over $\mathbb{F}(A)$. To overcome this, let us take a closer look on the situation of schemes of finite type over a field.

Let k be a field, C a smooth proper curve over k and let X be any scheme of finite type over k together with a morphism $p : X \to C$. Consider the complex $C_*(X; C)$ whose n-th term is given as

$$C_n(X; C) = \text{free abelian group over closed integral subschemes } Z \subset X \times \Delta^n_k = X \times_C \Delta^n_C \text{ such that the restriction of the projection } X \times_C \Delta^n_C \to \Delta^n_C \text{ to } Z \text{ induces a finite morphism } Z \to T \subset \Delta^n_C \text{ onto a closed integral subscheme } T \text{ of codimension 1 in } \Delta^n_C \text{ intersecting all faces } \Delta^m_C \subset \Delta^n_C \text{ properly.}$$

Then we have a natural inclusion

$$C_*(X) \hookrightarrow C_*(X; C)$$

and the definition of $C_*(X; C)$ only involves the morphism $p : X \to C$ but not the knowledge of k. Moreover, if X is affine, then both complexes coincide.

So, in the general case, having no theory of schemes over "$\mathbb{F}(A)$" at hand, we use the above complex in order to define singular homology. With the case $S = \mathrm{Spec}(\mathbb{Z})$ as the main application in mind, we define the singular homology of a scheme of finite type over the spectrum S of a Dedekind domain as the homology $h_*(X; S)$ of the complex $C_*(X; S)$ whose n-th term is given by

$$C_n(X; S) = \text{free abelian group over closed integral subschemes } Z \subset X \times_S \Delta^n_S \text{ such that the restriction of the projection } X \times_S \Delta^n_S \to \Delta^n_S \text{ to } Z \text{ induces a finite morphism } Z \to T \subset \Delta^n_S \text{ onto a closed integral subscheme } T \text{ of codimension 1 in } \Delta^n_S \text{ intersecting all faces } \Delta^m_S \subset \Delta^n_S \text{ properly.}$$

In this paper we collect evidence that the so-defined groups $h_*(X; S)$ establish a reasonable homology theory on the category of schemes of finite type over S.

The groups $h_*(X; S)$ are covariantly functorial with respect to scheme morphisms and, on the category of smooth schemes over S, they are functorial with respect to finite correspondences. If the structural morphism $p : X \to S$ factors through a closed point P of S, then our singular homology coincides with Suslin's singular homology of X considered as a scheme over the field $k(P)$.

In Section 3, we calculate the singular homology $h_*(X; S)$ if X is regular and of (absolute) dimension 1. The result is similar to that for smooth curves over fields. Let $\bar{X}$ be a regular compactification of X over S and $Y = \bar{X} - X$. Then

$$h_i(X; S) \cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y}),$$

where $\mathbb{G}_{\bar{X},Y} = \ker(\mathbb{G}_{m,\bar{X}} \to i_*\mathbb{G}_{m,Y})$.

In Section 4, we investigate homotopy invariance. We show that the natural projection $X \times_S \mathbb{A}^1_S \to X$ induces an isomorphism on singular homology. We also show that the bivariant singular homology groups $h_*(X, Y; S)$ (see Section 2 for their definition) are homotopy invariant with respect to the second variable.

In Section 5, we give an alternative characterization of the group h_0, which implies, when X is proper over S, a natural isomorphism $h_0(X; S) \cong \mathrm{CH}_0(X)$, where $\mathrm{CH}_0(X)$ is the group of zero-cycles on X modulo rational equivalence. Furthermore, we can verify the exactness of at least a small part of the expected Mayer–Vietoris sequence associated to a Zariski-open cover of a scheme X.

For a proper, smooth (regular?) scheme X of absolute dimension d over the spectrum S of a Dedekind domain, singular homology should be related to motivic cohomology, defined for example by [Voevodsky 1998], by the formula

$$h_i(X; S) \cong H^{2d-i}_{\mathcal{M}ot}(X, \mathbb{Z}(d)).$$

For schemes over a field k, this formula has been proven by Voevodsky under the assumption that k admits resolution of singularities. In the situation of schemes over the spectrum of a Dedekind domain it is true if X is of dimension 1 (Section 3). For a general X it should follow from the fact that each among the following complex homomorphisms is a quasi-isomorphism. The occurring complexes are in each degree the free group over a certain set of cycles and we only write down this set of cycles and also omit the necessary intersection conditions with faces.

$$C_*(X; S)$$
$$\downarrow (1)$$
$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects finitely onto a codimension 1 subscheme in } \mathbb{A}^d \times \Delta^n)$$
$$\uparrow (2)$$
$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects finitely onto a codimension 1 subscheme } T \subset \mathbb{A}^d \times \Delta^n \text{ such that the projection } T \to \Delta^n \text{ is equidimensional of relative dimension } (d-1))$$
$$\downarrow (3)$$
$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ equidimensional of relative dimension } (d-1) \text{ over } \Delta^n)$$
$$\uparrow (4)$$
$$(Z \subset X \times \mathbb{A}^d \times \Delta^n \text{ projects quasifinite and dominant to } X \times \Delta^n)$$
$$\downarrow (5)$$
$$\mathbf{H}_{\mathcal{M}ot}(X, \mathbb{Z}(d)[2d])$$

It follows from the homotopy invariance of the bivariant singular homology groups in the second variable, proven in Section 4, that (1) is a quasi-isomorphism. The statement that the other occurring homomorphisms are also quasi-isomorphisms is completely hypothetical at the moment. However, it is, at least partly, suggested by the proof of the corresponding formula over fields; see [Voevodsky 2000, Theorem 4.3.7; Friedlander and Voevodsky 2000, Theorems 7.1 and 7.4].

We give the following application of singular homology to higher dimensional class field theory. Let X be a regular connected scheme, flat and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Sending a closed point of x of X to its Frobenius automorphism $\mathrm{Frob}_x \in \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$, we obtain a homomorphism

$$r : Z_0(X) \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$$

from the group $Z_0(X)$ of zero-cycles on X to the abelianized étale fundamental group $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$. The homomorphism r is known to have dense image. Assume for simplicity that the set $X(\mathbb{R})$ of real-valued points of X is empty. If X is proper, then r factors through rational equivalence, defining a reciprocity homomorphism $\mathrm{rec} : \mathrm{CH}_0(X) \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$. The main result of the so-called unramified class field theory for arithmetic schemes of Bloch and Kato–Saito [Kato and Saito 1983; Saito 1985] states that rec is an isomorphism of finite abelian groups.

If X is not proper, r no longer factors through rational equivalence. However, consider the composite map

$$r' : Z_0(X) \xrightarrow{r} \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} \twoheadrightarrow \pi_1^{t}(X)^{\mathrm{ab}},$$

where $\pi_1^{t}(X)^{\mathrm{ab}}$ is the quotient of $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ which classifies finite étale coverings of X with at most tame ramification "along the boundary of a compactification" (see Section 6). We show that r' factors through $h_0(X) = h_0(X; \mathrm{Spec}(\mathbb{Z}))$, defining an isomorphisms

$$\mathrm{rec} : h_0(X) \xrightarrow{\sim} \pi_1^{t}(X)^{\mathrm{ab}}$$

of finite abelian groups. Hence the singular homology group $h_0(X)$ takes over the role of $\mathrm{CH}_0(X)$ if the scheme X is not proper.

This article was motivated by the work of A. Suslin, V. Voevodsky and E.M. Friedlander on algebraic cycle theories for varieties over fields. The principal ideas underlying this paper originate from discussions with Michael Spieß during the preparation of our article [Schmidt and Spieß 2000]. The analogy between number fields and function fields in one variable over finite fields predicted that there should be a connection between the, yet to be defined, singular homology groups of a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and its tame fundamental group, similar to that we had proven for varieties over finite fields. The author wants to thank M. Spieß for fruitful discussions and for his remarks on a preliminary version of this paper.

The bulk of this article was part of the author's Habilitationschrift at Heidelberg University, seven years ago. However, I could not decide on publishing the material before the envisaged application to class field theory was established. This is the case now.

2. Preliminaries

Throughout this article we consider the category $\mathrm{Sch}(S)$ of separated schemes of finite type over a regular connected and Noetherian base scheme S. Quite early, we will restrict to the case that S is the spectrum of a Dedekind domain, which is the main case of our arithmetic application. We write $X \times Y = X \times_S Y$ for the fibre product of schemes $X, Y \in \mathrm{Sch}(S)$. Unless otherwise specified, all schemes will be assumed equidimensional.

Slightly modifying the approach of [Fulton 1998, Section 20.1], we define the (absolute) dimension of an integral scheme $X \in \mathrm{Sch}(S)$ in the following way. Let d be the Krull dimension of S, $K(X)$ the field of functions of X and T the closure of the image of X in S. Then we put

$$\dim X = \mathrm{trdeg}(K(X)|K(T)) - \mathrm{codim}_S(T) + d.$$

Examples 2.1. (1) Let $S = \mathrm{Spec}(\mathbb{Z}_p)$ and consider $X = \mathrm{Spec}(\mathbb{Z}_p[T]/pT - 1) \cong \mathrm{Spec}(\mathbb{Q}_p)$, a divisor on $\mathbb{A}^1_S = \mathrm{Spec}(\mathbb{Z}_p[T])$. Then $\dim X = 1$ in our terminology, while $\dim_{\mathrm{Krull}} X = 0$.

(2) The above notion of dimension coincides with the usual Krull dimension if

- S is the spectrum of a field,
- S is the spectrum of a Dedekind domain with infinitely many different prime ideals (e.g. the ring of integers in a number field).

Note that this change in the definition of dimension does not affect the notion of codimension. For a proof of the following lemma we refer to [Fulton 1998, Lemma 20.1].

Lemma 2.2. (i) *Let $U \subset X$ be a nonempty open subscheme. Then*

$$\dim X = \dim U.$$

(ii) *Let Y be a closed integral subscheme of the integral scheme X over S. Then*

$$\dim X = \dim Y + \mathrm{codim}_X(Y).$$

(iii) *If $f : X \to X'$ is a dominant morphism of integral schemes over S, then*

$$\dim X = \dim X' + \mathrm{trdeg}(K(X)|K(X')).$$

In particular, $\dim X' \leq \dim X$ *with equality if and only if $K(X)$ is a finite extension of $K(X')$.*

Recall that a closed immersion $i : Y \longrightarrow X$ is called a *regular imbedding* of codimension d if every point y of Y has an affine neighbourhood U in X such that the ideal in $\mathcal{O}_U$ defining $Y \cap U$ is generated by a regular sequence of length d. We say that two closed subschemes A and B of a scheme X intersect *properly* if

$$\dim W = \dim A + \dim B - \dim X$$

(or, equivalently, $\operatorname{codim}_X W = \operatorname{codim}_X A + \operatorname{codim}_X B$) for every irreducible component W of $A \cap B$. In particular, an empty intersection is proper. Suppose that the immersion $A \to X$ is a regular imbedding. Then an inductive application of Krull's principal ideal theorem shows that every irreducible component of the intersection $A \cap B$ has dimension greater or equal to $\dim A + \dim B - \dim X$. In this case improper intersection means that one of the irreducible components of the intersection has a too large dimension. If B is a cycle of codimension 1, then the intersection is proper if and only if B does not contain an irreducible component of A.

The *group of cycles* $Z^r(X)$ (resp. $Z_r(X)$) of a scheme X is the free abelian group generated by closed integral subschemes of X of codimension r (resp. of dimension r). For a closed immersion $i : Y \to X$, we have obvious maps $i_* : Z_r(Y) \to Z_r(X)$ for all r. If i is a regular imbedding, we have a pullback map

$$i^* : Z^r(X)' \longrightarrow Z^r(Y),$$

where $Z^r(X)' \subset Z^r(X)$ is the subgroup generated by closed integral subschemes of X meeting Y properly. The map i^* is given by

$$i^*(V) = \sum_i n_i W_i,$$

where the W_i are the irreducible components of $i^{-1}(V) = V \cap Y$ and the n_i are the intersection multiplicities. For the definition of these multiplicities we refer to [Fulton 1998, Section 6] (or, alternatively, one can use Serre's Tor-formula [Serre 1965]).

The *standard n-simplex* $\Delta^n = \Delta^n_S$ over S is the closed subscheme in $\mathbb{A}^{n+1}_S$ defined by the equation $t_0 + \cdots + t_n = 1$. We call the sections $v_i : S \to \Delta^n_S$ corresponding to $t_i = 1$ and $t_j = 0$ for $j \neq i$ the *vertices* of Δ^n_S. Each nondecreasing map $\rho : [m] = \{0, 1, \ldots, m\} \longrightarrow [n] = \{0, 1, \ldots, n\}$ induces a scheme morphism

$$\bar{\rho} : \Delta^m \longrightarrow \Delta^n$$

defined by $t_i \mapsto \sum_{\rho(j)=i} t_j$. If ρ is injective, we say that $\rho(\Delta^m_S) \subset \Delta^n_S$ is a *face*. If ρ is surjective, $\bar{\rho}$ is a *degeneracy*. In this way $\Delta^\bullet_S$ becomes a cosimplicial scheme. Further note that all faces are regular imbeddings.

The following definition was motivated in the introduction.

Definition 2.3. For X in $\mathrm{Sch}(S)$ and $n \geq 0$, the group $C_n(X; S)$ is the free abelian group generated by closed integral subschemes Z of $X \times \Delta^n$ such that the restriction of the canonical projection

$$X \times \Delta^n \to \Delta^n$$

to Z induces a finite morphism $p : Z \to T \subset \Delta^n$ onto a closed integral subscheme T of codimension $d = \dim S$ in Δ^n which intersects all faces properly. In particular, such a Z is equidimensional of dimension n.

Remarks 2.4. (1) If the structural morphism $X \to S$ factors through a finite morphism $S' \to S$ with S' regular, then $C_n(X; S) = C_n(X; S')$. In particular, if $S' = \{P\}$ is a closed point of S, i.e. if X is a scheme of finite type over $\mathrm{Spec}(k(P))$, then $C_n(X; S) = C_n(X; k(P))$ is the n-th term of the singular complex of X defined by Suslin.

(2) If S is of dimension 1 (and regular and connected), then a closed integral subscheme T of codimension $d = 1$ in Δ^n_S intersects all faces properly if and only if it does not contain any face. If the image of X in S omits at least one closed point of S, then this condition is automatically satisfied.

Let Z be a closed integral subscheme of $X \times \Delta^n$ which projects finitely and surjectively onto a closed integral subscheme T of codimension d in Δ^n. Assume that T has proper intersection with all faces, i.e. Z defines an element of $C_n(X; S)$. Let $\Delta^m \hookrightarrow \Delta^n$ be a face. Since the projection

$$Z \times_{\Delta^n_X} \Delta^m_X \longrightarrow T \times_{\Delta^n} \Delta^m$$

is finite, each irreducible component of $Z \cap X \times \Delta^m$ has dimension at most m. On the other hand, a face is a regular imbedding and therefore all irreducible components of $Z \cap X \times \Delta^m$ have exact dimension m and project finitely and surjectively onto an irreducible component of $T \cap \Delta^m$. Thus the cycle theoretic inverse image $i^*(Z)$ is well defined and is in $C_m(X; S)$. Furthermore, degeneracy maps are flat, and thus we obtain a simplicial abelian group $C_\bullet(X; S)$. We use the same notation for the associated chain complex which (in the usual way) is constructed as follows.

Consider the 1-codimensional face operators

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n, \quad i = 0, \ldots, n,$$

defined by setting $t_i = 0$, and define the complex (concentrated in positive homological degrees)

$$C_\bullet(X; S), \quad d_n = \sum_{i=0}^{n} (-1)^i (d^i)^* : C_n(X; S) \to C_{n-1}(X; S).$$

Definition 2.5. We call $C_\bullet(X; S)$ the singular complex of X. Its homology groups (or likewise the homotopy groups of $C_\bullet(X; S)$ considered as a simplicial abelian group)

$$h_i(X; S) = H_i(C_\bullet(X; S)) \qquad \big(= \pi_i(C_\bullet(X; S))\big)$$

are called the (integral) singular homology groups of X.

From Remark 2.4(1) above, we obtain:

Lemma 2.6. *Assume that the structural morphism $X \to S$ factors through a finite morphism $S' \to S$ with S' regular. Then for all i,*

$$h_i(X; S) = h_i(X; S').$$

Examples 2.7. (1) If k is a field and $S = \mathrm{Spec}(k)$, then the above definition of $h_i(X)$ coincides with that of the singular homology of X defined by Suslin.

(2) $C_\bullet(X; S)$ is a subcomplex of Bloch's complex $z^r(X, \bullet)$, where $r = \dim X$, and $C_\bullet(S; S)$ coincides with the Bloch complex $z^d(S, \bullet)$. In particular,

$$h_i(S; S) = \mathrm{CH}^d(S, i),$$

where the group on the right is the higher Chow group defined by Bloch. Note that in [Bloch 1986], Bloch defined his higher Chow groups only for equidimensional schemes over a field, but there is no problem with extending his construction at hand.

The push-forward of cycles makes $C_\bullet(X; S)$ and thus also $h_i(X; S)$ covariantly functorial on $\mathrm{Sch}(S)$. Furthermore, it is contravariant under finite flat morphisms. Given a finite flat morphism $f : X' \to X$, we thus have induced maps $f_* : h_\bullet(X'; S) \to h_\bullet(X; S)$ and $f^* : h_\bullet(X; S) \to h_\bullet(X'; S)$, which are connected by the formula

$$f_* \circ f^* = \deg(f) \cdot \mathrm{id}_{h_\bullet(X;S)}.$$

In addition, we introduce bivariant homology groups. Let Y be equidimensional, of finite type and flat over S. If $X \times Y$ is empty, we let $C_\bullet(X, Y; S)$ be the trivial complex. Otherwise, $X \times Y$ it is a scheme of dimension $\dim X + \dim Y - d$ (as before, $d = \dim S$) and we consider the group $C_n(X, Y; S)$ which is the free abelian group generated by closed integral subschemes in $X \times Y \times \Delta^n$ such that the restriction of the canonical projection

$$X \times Y \times \Delta^n \to Y \times \Delta^n$$

to Z induces a finite morphism $p : Z \to T \subset Y \times \Delta^n$ onto a closed integral subscheme T of codimension d in $Y \times \Delta^n$ which intersects all faces $Y \times \Delta^m$ properly. In particular, such a Z is equidimensional of dimension $\dim Y + n - d$. Further, for a closed subscheme $Y' \subset Y$, consider the subgroup $C_n^{Y'}(X, Y; S) \subset C_n(X, Y; S)$,

which is the free abelian group generated by closed integral subschemes of $X \times Y \times \Delta^n$ such that the restriction of the canonical projection

$$X \times Y \times \Delta^n \to Y \times \Delta^n$$

to Z induces a finite morphism $p : Z \to T \subset Y \times \Delta^n$ onto a closed integral subscheme T of codimension d in $Y \times \Delta^n$ which intersects all faces $Y \times \Delta^m$ and all faces $Y' \times \Delta^m$ properly.

In the same way as before, we obtain the complex $C_\bullet(X, Y; S)$, which contains the subcomplex $C_\bullet^{Y'}(X, Y; S)$.

Definition 2.8. We call $C_\bullet(X, Y; S)$ the bivariant singular complex and its homology groups

$$h_i(X, Y; S) = H_i(C_\bullet(X, Y; S))$$

the bivariant singular homology groups.

Note that $C_\bullet(X, S; S) = C_\bullet(X; S)$ and $h_i(X, S; S) = h_i(X; S)$. By pulling back cycles, a flat morphism $Y' \to Y$ induces a homomorphism of complexes

$$C_\bullet(X, Y; S) \longrightarrow C_\bullet(X, Y'; S).$$

If $Y' \hookrightarrow Y$ is a regular imbedding, we get a natural homomorphism

$$C_\bullet^{Y'}(X, Y; S) \longrightarrow C_\bullet(X, Y'; S).$$

Consider the complex of presheaves $\underline{C}_\bullet(X; S)$ which is given on open subschemes $U \subset S$ by

$$U \longmapsto C_\bullet(X, U; S).$$

This is already a complex of Zariski-sheaves on S.

Definition 2.9. By $\underline{h}_i(X; S)$ we denote the cohomology sheaves of the complex $\underline{C}_\bullet(X; S)$. Equivalently, $\underline{h}_i(X; S)$ is the Zariski sheaf on S associated to

$$U \longmapsto h_i(X, U; S).$$

(The sheaves $\underline{h}_i$ play a similar role as Bloch's higher Chow sheaves [Bloch 1986].)

Now assume that X and Y are smooth over S. By $c(X, Y)$ we denote the free abelian group generated by integral closed subschemes $W \subset X \times Y$ which are finite over X and surjective over a connected component of X. An element in $c(X, Y)$ is called a finite correspondence from X to Y. If X_1, X_2, X_3 is a triple of smooth schemes over S, then, by [Voevodsky 2000, Section 2], there exists a natural composition $c(X_1, X_2) \times c(X_2, X_3) \to c(X_1, X_3)$. Therefore one can define a category $\mathrm{SmCor}(S)$ whose objects are smooth schemes of finite type over S and morphisms are finite correspondences. The category $\mathrm{Sm}(S)$ of smooth schemes of

finite type over S admits a natural functor to $\mathrm{SmCor}(S)$ by sending a morphism to its graph.

Let X and Y be smooth over S, let $\phi \in c(X, Y)$ be a finite correspondence and let $\psi \in C_n(X, S)$. Consider the product $X \times Y \times \Delta^n$ and let p_1, p_2, p_3 be the corresponding projections. Then the cycles $(p_1 \times p_3)^*(\psi)$ and $(p_1 \times p_2)^*(\phi)$ are in general position. Let $\psi * \phi$ be their intersection. Since ϕ is finite over X and ψ is finite over Δ^n, we can define the cycle $\phi \circ \psi$ as $(p_2 \times p_3)_*(\phi * \psi)$. The cycle $\phi \circ \psi$ is in $C_n(Y; S)$, and so we obtain a natural pairing $c(X, Y) \times C_\bullet(X; S) \to C_\bullet(Y; S)$. We obtain the following:

Proposition 2.10. *For schemes X, Y that are smooth over S, there exist natural pairings for all i*

$$c(X, Y) \otimes h_i(X; S) \longrightarrow h_i(Y; S)$$

making singular homology into a covariant functor on the category $\mathrm{SmCor}(S)$.

3. Singular homology of curves

We start this section by recalling some notions and lemmas from [Suslin and Voevodsky 1996]. Let X be a scheme and let Y be a closed subscheme of X. Set $U = X - Y$ and denote by $i : Y \longrightarrow X$, $j : U \longrightarrow X$ the corresponding closed and open embeddings.

We denote by $\mathrm{Pic}(X, Y)$ (the relative Picard group) the group whose elements are isomorphism classes of pairs of the form (L, ϕ), where L is a line bundle on X and $\phi : L|_Y \cong \mathbb{O}_Y$ is a trivialization of L over Y, and the operation is given by the tensor product. There is an evident exact sequence

$$\Gamma(X, \mathbb{O}_X^\times) \longrightarrow \Gamma(Y, \mathbb{O}_Y^\times) \longrightarrow \mathrm{Pic}(X, Y) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y). \qquad (1)$$

We also use the notation $\mathbb{G}_X$ (or $\mathbb{G}_m$) for the sheaf of invertible functions on X and we write $\mathbb{G}_{X,Y}$ for the sheaf on X which is defined by the exact sequence

$$0 \longrightarrow \mathbb{G}_{X,Y} \longrightarrow \mathbb{G}_X \longrightarrow i_*(\mathbb{G}_Y) \longrightarrow 0.$$

By [Suslin and Voevodsky 1996, Lemma 2.1], there are natural isomorphisms

$$\mathrm{Pic}(X, Y) = H^1_{\mathrm{Zar}}(X, \mathbb{G}_{X,Y}) = H^1_{\mathrm{et}}(X, \mathbb{G}_{X,Y}).$$

Assume that X is integral and denote by K the field of rational functions on X. A relative Cartier divisor on X is a Cartier divisor D such that $\mathrm{supp}(D) \cap Y = \varnothing$. If D is a relative divisor and $Z = \mathrm{supp}(D)$, then $\mathbb{O}_X(D)|_{X-Z} = \mathbb{O}_{X-Z}$. Thus D defines an element in $\mathrm{Pic}(X, Y)$. Denoting the group of relative Cartier divisors by $\mathrm{Div}(X, Y)$, we get a natural homomorphism $\mathrm{Div}(X, Y) \to \mathrm{Pic}(X, Y)$. The image of this homomorphism consists of pairs (L, ϕ) such that ϕ admits an extension to

a trivialization of L over an open neighbourhood of Y. In particular, this map is surjective provided that Y has an affine open neighbourhood. Furthermore, we put

$$\begin{aligned} G &= \{f \in K^\times : f \in \ker(\mathcal{O}_{X,y}^\times \longrightarrow \mathcal{O}_{Y,y}^\times) \text{ for any } y \in Y\} \\ &= \{f \in K^\times : f \text{ is defined and equal to 1 at each point of } Y\}, \end{aligned}$$

The following lemmas are straightforward; see [Suslin and Voevodsky 1996, 2.3, 2.4, 2.5].

Lemma 3.1. *Assume that Y has an affine open neighbourhood in X. Then the following sequence is exact*:

$$0 \longrightarrow \Gamma(X, \mathbb{G}_{X,Y}) \longrightarrow G \longrightarrow \mathrm{Div}(X, Y) \longrightarrow \mathrm{Pic}(X, Y) \longrightarrow 0.$$

Lemma 3.2. *Assume that U is normal and every closed integral subscheme of U of codimension one which is closed in X is a Cartier divisor (this happens for example when U is factorial). Then $\mathrm{Div}(X, Y)$ is the free abelian group generated by closed integral subschemes $T \subset U$ of codimension one which are closed in X.*

Lemma 3.3. *Let X be a scheme. Consider the natural homomorphism*

$$p^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{A}^1_X)$$

which is induced by the projection $p : \mathbb{A}^1_X \to X$. If X is reduced, then p^* is injective. If X is normal, it is an isomorphism.

Proof. Since X is reduced, we have $p_* \mathbb{G}_{\mathbb{A}^1_X} = \mathbb{G}_X$. Therefore the spectral sequence

$$E_2^{ij} = H^i(X, R^j p_* \mathbb{G}_{\mathbb{A}^1_X}) \Longrightarrow H^{i+j}(\mathbb{A}^1_X, \mathbb{G}_{\mathbb{A}^1_X})$$

induces a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{A}^1_X) \to H^0(X, R^1 p_*(\mathbb{G}_{\mathbb{A}^1_X})).$$

This shows the first statement. The stalk of $R^1 p_*(\mathbb{G}_{\mathbb{A}^1_X})$ at a point $x \in X$ is the Picard group of the affine scheme $\mathrm{Spec}(\mathcal{O}_{X,x}[T])$. If X is normal, then this group is trivial by [Bass and Murthy 1967, Proposition 5.5]. This concludes the proof. □

Corollary 3.4. *Assume that X is normal and Y is reduced. Then*

$$\mathrm{Pic}(X, Y) \cong \mathrm{Pic}(\mathbb{A}^1_X, \mathbb{A}^1_Y).$$

Proof. Using the five-lemma, this follows from Lemma 3.3 together with the exact sequence (1). □

In the case that $S = \mathrm{Spec}(k)$ is the spectrum of a field k, our singular homology coincides with that defined by Suslin. For a proof of the next theorem, see [Lichtenbaum 1993].

Theorem 3.5. *Let X be a smooth, geometrically connected curve over k, let $\bar{X}$ be a smooth compactification of X and let $Y = \bar{X} - X$. Then $h_i(X; k) = 0$ for $i \neq 0, 1$ and*

$$\begin{aligned} h_0(X; k) &= \mathrm{Pic}(\bar{X}, Y), \\ h_1(X; k) &= \begin{cases} 0 & \text{if } X \text{ is affine,} \\ k^\times & \text{if } X \text{ is proper.} \end{cases} \end{aligned}$$

Corollary 3.6. *Let X be a smooth curve over a field k, $\bar{X}$ a smooth compactification of X over k and $Y = \bar{X} - X$. Then for all i,*

$$\begin{aligned} h_i(X; k) &\cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y}) \\ &\cong \mathbb{H}^{-i}_{\mathrm{Zar}}\big(\bar{X}, \mathrm{cone}\big(\mathbb{G}_{\bar{X}} \longrightarrow i_{Y*}(\mathbb{G}_Y)\big)\big), \end{aligned}$$

where $\mathbb{H}_{\mathrm{Zar}}$ denotes Zariski hypercohomology.

This corollary is a special case of a general duality theorem proven in [Voevodsky 2000, Theorem 4.3.7] over fields that admit resolution of singularities.

We now consider the case that S is the spectrum of a Dedekind domain, which is the case of main interest for us. The proof of the following theorem is parallel to the proof of Theorem 3.1 of [Suslin and Voevodsky 1996], where the relative singular homology of relative curves was calculated.

Theorem 3.7. *Assume that S is the spectrum of a Dedekind domain and let U be an open subscheme of S. Let $Y \in \mathrm{Sch}(S)$ be regular and flat over S. Setting $Y_U = Y \times U$, suppose that $Y - Y_U$ has an affine open neighbourhood in Y. Then $h_i(U, Y; S) = 0$ for $i \neq 0, 1$ and*

$$\begin{aligned} h_0(U, Y; S) &= \mathrm{Pic}(Y, Y - Y_U), \\ h_1(U, Y; S) &= \Gamma(Y, \mathbb{G}_{Y, Y-Y_U}). \end{aligned}$$

Proof. We may assume that Y is connected. If $Y_U = Y$, then $C_\bullet(U, Y; S)$ coincides with the Bloch complex $z^1(Y, \bullet)$. By [Bloch 1986, Theorem 6.1] (whose proof applies without change to arbitrary regular schemes), we have $h_i(U, Y; S) = 0$ for $i \neq 0, 1$ and

$$\begin{aligned} h_0(U, Y; S) &= \mathrm{Pic}(Y), \\ h_1(U, Y; S) &= \Gamma(Y, \mathbb{G}_Y). \end{aligned}$$

Suppose that $Y_U \subsetneqq Y$. Then an integral subscheme $Z \subset Y_U \times \Delta^n$ is in $C_n(U, Y; S)$ if and only if it is closed and of codimension 1 in $Y \times \Delta^n$. Since Y is regular, such a Z is a Cartier divisor and it automatically has proper intersection with all faces (see Remark 2.4(2)). Thus $C_n(U, Y; S) = \mathrm{Div}(Y, T)$ (see Lemma 3.2). Let $T = Y - Y_U$. If V is an open affine neighbourhood of T in Y, then $V \times \Delta^n$ is an open affine neighbourhood of $T \times \Delta^n$ in $Y \times \Delta^n$. According to Lemma 3.1, we

have an exact sequence of simplicial abelian groups:

$$0 \to A_\bullet \to G_\bullet \to C_\bullet(U, Y; S) \to \mathrm{Pic}(\Delta_Y^\bullet, \Delta_T^\bullet) \to 0, \tag{2}$$

where

$$G_n = \{f \in k(\Delta_Y^n)^\times : f \text{ is defined and equal to 1 at each point of } \Delta_T^n\},$$
$$A_n = \Gamma(\Delta_Y^n, \mathbb{G}_{\Delta_Y^n, \Delta_T^n}).$$

For each n, we have $A_n = A_0 = \Gamma(Y, \mathbb{G}_{Y,T})$ and by Corollary 3.4, we have $\mathrm{Pic}(\Delta_Y^n, \Delta_T^n) = \mathrm{Pic}(Y, T)$. Let us show that the simplicial abelian group $G_\bullet$ is acyclic, i.e. $\pi_*(G_\bullet) = 0$. It suffices to check that for any $f \in G_n$ such that $\delta_i(f) = 1$ for $i = 0, \dots, n$, there exists a $g \in G_{n+1}$ such that $\delta_i(g) = 1$ for $i = 0, \dots, n$ and $\delta_{n+1}(g) = f$. Define functions $g_i \in G_{n+1}$ for $i = 1, \dots, n$ by means of the formula

$$g_i = (t_{i+1} + \cdots + t_{n+1}) + (t_0 + \cdots + t_i) s_i(f).$$

These functions satisfy the following equations:

$$\delta_j(g_i) = \begin{cases} 1 & \text{if } j \neq i, i+1, \\ (t_i + \cdots + t_n) + (t_0 + \cdots + t_{i-1}) f & \text{if } j = i, \\ (t_{i+1} + \cdots + t_n) + (t_0 + \cdots t_i) f & \text{if } j = i+1. \end{cases}$$

In particular, $\delta_0(g_0) = 1$, $\delta_{n+1}(g_n) = f$. Finally, we set

$$g = g_n g_{n-1}^{-1} g_{n-2} \cdots g_0^{(-1)^n}.$$

This function satisfies the conditions we need. Evaluating the 4-term exact sequence (2) above, we obtain the statement of the theorem. □

Corollary 3.8. *Assume that S is the spectrum of a Dedekind domain. Let X be regular and quasifinite over S, $\bar{X}$ a regular compactification of X over S and $Y = \bar{X} - X$. Then for all i,*

$$h_i(X; k) \cong H^{1-i}_{\mathrm{Zar}}(\bar{X}, \mathbb{G}_{\bar{X},Y})$$
$$\cong \mathbb{H}^{-i}_{\mathrm{Zar}}\big(\bar{X}, \mathrm{cone}\big(\mathbb{G}_{\bar{X}} \longrightarrow i_{Y*}(\mathbb{G}_Y)\big)\big),$$

where $\mathbb{H}_{\mathrm{Zar}}$ denotes Zariski hypercohomology.

Proof. We may assume that X is connected. By Zariski's main theorem, X is an open subscheme of the normalization S' of S in the function field of X. As is well known, $S' = \bar{X}$ is again the spectrum of a Dedekind domain and the projection $S' \to S$ is a finite morphism. Therefore the result follows from Lemma 2.6 and from Theorem 3.7 applied to the case $Y = S$. □

Corollary 3.9. *Let S be the spectrum of a Dedekind domain. Assume that X is regular and that the structural morphism $p : X \to S$ is quasifinite. Let $\bar{p} : \bar{X} \to S$ be a regular compactification of X over S and $Y = \bar{X} - X$. Then there is a natural isomorphism*

$$\underline{C}_\bullet(X; S) \cong \bar{p}_* \, \mathbb{G}_{\bar{X},Y}\,[1]$$

in the derived category of complexes of Zariski-sheaves on S.

Proof. We may assume that X is connected and we apply the result of Theorem 3.7 to open subschemes $Y \subset S$. Note that $\bar{X}$ is the normalization of S in the function field of X. The stalk of $\underline{h}_1(X; S)$ at a point $s \in S$ is the relative Picard group of the semilocal scheme $\bar{X} \times_S S_s$ with respect to the finite set of closed points not lying on X. A semilocal Dedekind domain is a principal ideal domain, and the exact sequence (1) from the beginning of this section shows that also the corresponding relative Picard group is trivial. Therefore, the complex of sheaves $\underline{C}_\bullet(X; S)$ has exactly one nontrivial homology sheaf, which is placed in homological degree 1 and is isomorphic to $\bar{p}_* \, \mathbb{G}_{\bar{X},Y}$. □

Let us formulate a few results which easily follow from Theorem 3.7. We hope that these results are (mutatis mutandis) true for regular schemes X of arbitrary dimension. We omit S from the notation, writing $h_*(X)$ for $h_*(X; S)$ and $h_*(X, Y)$ for $h_*(X, Y; S)$

Theorem 3.10. *Let S be the spectrum of a Dedekind domain. Assume that X is regular and quasifinite over S (in particular, $\dim X = 1$). Then the following holds.*

(i) $h_i(X) = \mathbb{H}^{-i}_{\mathrm{Zar}}(S, \underline{C}_\bullet(X; S))$ *for all i.*

(ii) (*Local to global spectral sequence*) *There exists a spectral sequence*

$$E_2^{ij} = H^{-i}_{\mathrm{Zar}}(S, \underline{h}_j(X)) \Rightarrow h_{i+j}(X).$$

(iii) (*Mayer–Vietoris sequence*) *Let $X_1, X_2 \subset X$ be open with $X = X_1 \cup X_2$. Then there is an exact sequence*

$$\begin{aligned} 0 \to h_1(X_1 \cap X_2) \to h_1(X_1) \oplus h_1(X_2) \to h_1(X) \\ \to h_0(X_1 \cap X_2) \to h_0(X_1) \oplus h_0(X_2) \to h_0(X) \to 0. \end{aligned}$$

(iv) (*Mayer–Vietoris sequence with respect to the second variable*) *Let $U, V \subset S$ be open. Then there is an exact sequence*

$$\begin{aligned} 0 \to h_1(X, U \cup V) \to h_1(X, U) \oplus h_1(X, V) \to h_1(X, U \cap V) \\ \to h_0(X, U \cup V) \to h_0(X, U) \oplus h_0(X, V) \to h_0(X, U \cap V) \to 0. \end{aligned}$$

Proof. We may assume that X is connected. Let S' be the normalization of S in the function field of X, and we denote by $j_X : X \to S'$ the corresponding open

immersion (compare the proof of Corollary 3.8). Let, for an open subscheme $U \subset S$, U' be its preimage in S'. Then

$$h_i(X, U; S) = h_i(X, U'; S'),$$

and therefore we may assume that $S' = S$ in the proof of (iii) and (iv). Then, by Corollary 3.8, $h_i(X) = H^{1-i}_{\mathrm{Zar}}(S, \mathbb{G}_{S,S-X})$. Assertion (iii) follows by applying the functor $\mathbf{R}\Gamma(S, -)$ to the exact sequence of Zariski sheaves

$$0 \longrightarrow \mathbb{G}_{S,S-X_1\cap X_2} \longrightarrow \mathbb{G}_{S,S-X_1} \oplus \mathbb{G}_{S,S-X_2} \longrightarrow \mathbb{G}_{S,S-X} \longrightarrow 0.$$

For an open subscheme $j_U : U \longrightarrow S$, we denote the sheaf $j_{U,!}j_U^*(\underline{\mathbb{Z}})$ by $\mathbb{Z}_U$. Then, for a sheaf F on S, we have a canonical isomorphism

$$H^i_{\mathrm{Zar}}(U, j_U^* F) \cong \mathrm{Ext}^i_S(\mathbb{Z}_U, F).$$

Applying the functor $\mathbf{R}\mathrm{Hom}_S(-, \mathbb{G}_{S,S-X})$ to the exact sequence of Zariski sheaves

$$0 \longrightarrow \mathbb{Z}_{U\cap V} \longrightarrow \mathbb{Z}_U \oplus \mathbb{Z}_V \longrightarrow \mathbb{Z}_{U\cup V} \longrightarrow 0,$$

Theorem 3.7 implies assertion (iv). From (iv) it follows that the complex $\underline{C}_\bullet(X)$ is pseudo-flasque in the sense of [Brown and Gersten 1973], which shows assertion (i). Finally, (ii) follows from the corresponding hypercohomology spectral sequence converging to $\mathbb{H}^{-i}_{\mathrm{Zar}}(S, \underline{C}_\bullet(X; S))$ and from (i). □

Finally, we deduce an exact Gysin sequence for one-dimensional schemes. In order to formulate it, we need the notion of twists. Let $\mathbb{G}_m$ denote the multiplicative group scheme $\mathbb{A}^1_S - \{0\}$ and let X be any scheme of finite type over S. For $i = 1, \cdots, n$, let $D^i_\bullet(X \times \mathbb{G}_m^{\times(n-1)}; S)$ be the direct summand in $C_\bullet(X \times \mathbb{G}_m^{\times n}; S)$ which is given by the homomorphism

$$\mathbb{G}_m^{\times(n-1)} \longrightarrow \mathbb{G}_m^{\times n}, \quad (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, 1_i, \ldots, x_{n-1})$$

We consider the complex $C_\bullet(X \times \mathbb{G}_m^{\wedge n}; S)$ which is defined as the direct summand of the complex $C_\bullet(X \times \mathbb{G}_m^{\times n}; S)$ complementary to the direct summand $\sum_{i=1}^n D^i_\bullet(X \times \mathbb{G}_m^{\times(n-1)}; S)$; see [Suslin and Voevodsky 2000, Section 3].

Definition 3.11. For $n \geq 0$, we put

$$h_i(X(n); S) = H_{i+n}(C_\bullet(X \times \mathbb{G}_m^{\wedge n}; S)).$$

In particular, we have $h_i(X(0); S) = h_i(X; S)$ for all i and $h_i(X(n); S) = 0$ for $i < -n$. If $X = \{P\}$ is a closed point on S, then (see [Suslin and Voevodsky 2000, Lemma 3.2])

$$h_i(\{P\}(1); S) = \begin{cases} k(P)^\times & \text{for } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The next corollary follows from this and from Theorem 3.7.

Corollary 3.12. *Assume that X is regular and quasifinite over S and that U is an open, dense subscheme in X. Then we have a natural exact sequence*

$$0 \to h_1(U) \to h_1(X) \to h_{-1}((X-U)(1)) \to h_0(U) \to h_0(X) \to 0.$$

4. Homotopy invariance

Throughout this section we fix our base scheme S, which is the spectrum of a Dedekind domain, and we omit it from the notation, writing $h_*(X)$ for $h_*(X; S)$ and $h_*(X, Y)$ for $h_*(X, Y; S)$. Our aim is to prove that the relative singular homology groups $h_*(X, Y)$ are homotopy invariant with respect to both variables.

Theorem 4.1. *Let X and Y be of finite type over S. Then the projection $X \times \mathbb{A}^1 \to X$ induces isomorphisms*

$$h_i(X \times \mathbb{A}^1, Y) \xrightarrow{\sim} h_i(X, Y)$$

for all i.

Let $i_0, i_1 : Y \longrightarrow Y \times \mathbb{A}^1$ be the embeddings defined by the points (i.e. sections over S) 0 and 1 of $\mathbb{A}^1 = \mathbb{A}^1_S$.

Recall that Δ^n has coordinates $(t_0, \ldots, t_n)$ with $\sum t_i = 1$. Vertices are the points (i.e. sections over S) $p_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th place. Consider the linear isomorphisms

$$\theta_i : \Delta^{n+1} \longrightarrow \Delta^n \times \mathbb{A}^1, \quad i = 0, \ldots, n$$

which are defined by taking p_j to $(p_j, 0)$ for $j \le i$ and to $(p_{j-1}, 1)$ if $j > i$. Then consider for each n the formal linear combination

$$T_n = \sum_{i=0}^{n} (-1)^i \theta_i.$$

Let us call a subscheme $F \subset \Delta^n \times \mathbb{A}^1$ a face if it corresponds to a face in Δ^{n+1} under one of the linear isomorphisms θ_i. Using this terminology, T_n defines a homomorphism from a subgroup of $C_n(X, Y \times \mathbb{A}^1)$ to $C_{n+1}(X, Y)$. This subgroup is generated by cycles having good intersection not only with all faces $Y \times \mathbb{A}^1 \times \Delta^m$ but also with all faces of the form $Y \times F$, where F is a face in $\mathbb{A}^1 \times \Delta^n$.

We will deduce Theorem 4.1 from the following proposition.

Proposition 4.2. *The two chain maps*

$$i_{0*}, i_{1*} : C_\bullet(X, Y) \longrightarrow C_\bullet(X \times \mathbb{A}^1, Y)$$

are homotopic. In particular, i_{0*} and i_{1*} induce the same map on homology.

Proof. Let $D \subset \mathbb{A}^1 \times \mathbb{A}^1$ be the diagonal. Consider the map

$$V_n : C_n(X, Y) \longrightarrow C_n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1)$$

which is defined by sending a cycle $Z \subset X \times Y \times \Delta^n$ to the cycle $Z \times D \subset X \times Y \times \Delta^n \times \mathbb{A}^1 \times \mathbb{A}^1$. If Z projects finitely and surjectively onto $T \subset Y \times \Delta^n$, then $Z \times D$ projects finitely and surjectively onto $T \times \mathbb{A}^1 \subset Y \times \Delta^n \times \mathbb{A}^1$. Therefore V_n is well defined. Fortunately, $T \times \mathbb{A}^1$ has proper intersection with all faces $Y \times F$, where F is a face in $\Delta^n \times \mathbb{A}^1$. Therefore the composition

$$T_{n*} \circ V_n : C_n(X, Y) \longrightarrow C_n(X \times \mathbb{A}^1, Y \times A^1) \longrightarrow C_{n+1}(X \times \mathbb{A}^1, Y)$$

is well defined for every n. These maps give the required homotopy. □

Proof of Theorem 4.1. Let $\tau : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the multiplication map. Consider the diagram

$$\begin{array}{ccc} C_\bullet(X \times \mathbb{A}^1, Y) & \xrightarrow{p_*} & C_\bullet(X, Y) \\ \downarrow{\scriptstyle i_{0*}, i_{1*}} & & \downarrow{\scriptstyle i_{0*}, i_{1*}} \\ C_\bullet(X \times \mathbb{A}^1 \times \mathbb{A}^1, Y) & \xrightarrow{\tau_*} & C_\bullet(X \times \mathbb{A}^1, Y). \end{array}$$

We have the following equalities of maps on homology:

$$i_{0*} \circ p_* = \tau_* \circ i_{0*} = \tau_* \circ i_{1*} = \mathrm{id}_{h_\bullet(X,Y)}.$$

Therefore, p_* is injective on homology. But on the other hand, $p \circ i_0 = \mathrm{id}_X$, which shows that p_* is surjective. This concludes the proof. □

Now, exploiting a moving technique of [Bloch 1986], we prove that the bivariant singular homology groups $h_*(X, Y)$ are homotopy invariant with respect to the second variable.

Theorem 4.3. *Assume that S is the spectrum of a Dedekind domain and let X and Y be of finite type over S. Then the projection $p : Y \times \mathbb{A}^1 \to Y$ induces isomorphisms for all i,*

$$h_i(X, Y) \xrightarrow{\sim} h_i(X, Y \times \mathbb{A}^1).$$

A typical intermediate step in proving a theorem like Theorem 4.3 would be to show that the induced chain maps $i_0^*, i_1^* : C_\bullet(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$ are homotopic. However, i_0^*, i_1^* are only defined as homomorphisms on the subcomplex

$$i_0^*, i_1^* : C_\bullet^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y).$$

(The maps $T_n^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_{n+1}(X, Y)$ would define a homotopy $i_0^* \sim i_1^* : C_n(X, Y \times \mathbb{A}^1) \longrightarrow C_n(X, Y)$, if all these maps *would* be defined.)

The proof of Theorem 4.3 will consist of several steps. First, we show that the inclusion

$$C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y\times\mathbb{A}^1)$$

is a quasi-isomorphism. Then we show that the homomorphisms

$$i_0^*, i_1^* : C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$$

induce the same map on homology. Finally, we deduce Theorem 4.3 from these results.

In the proof we will apply a moving technique of [Bloch 1986] which was used there to show the homotopy invariance of the higher Chow groups. As long as we have to deal with cycles of codimension 1, this technique also works in our more general situation (this is the reason for the restriction to the case that S is the spectrum of a Dedekind domain).

We would like to construct a homotopy between the identity of the complex $C_\bullet(X, Y\times\mathbb{A}^1)$ and another map which takes its image in the subcomplex

$$C_\bullet^{Y\times\{0,1\}}(X, Y\times\mathbb{A}^1).$$

What we *can* do is the following:

For a suitable scheme S' over S we construct a homotopy between the pullback map $C_\bullet(X, Y\times\mathbb{A}^1) \longrightarrow C_\bullet(X, Y\times\mathbb{A}^1\times S')$ and another map whose image is contained in the subcomplex

$$C_\bullet^{Y\times\{0,1\}\times S'}(X, Y\times\mathbb{A}^1\times S').$$

(Eventually, we will use $S' = \mathbb{A}^1_S$ but perhaps this would be too many $\mathbb{A}^1$'s in the notation.)

Let (for the moment) $\pi : S' \to S$ be any integral scheme of finite type over S and let t be an element in $\Gamma(S', \mathcal{O}_{S'})$. Consider the action

$$\mathbb{A}^1_{S'} \times_{S'} (Y\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1)_{S'}$$

of the smooth group scheme $\mathbb{A}^1_{S'}$ on $(Y\times\mathbb{A}^1)_{S'}$ given by additive translation

$$a\cdot(y, b)) = (y, a+b)$$

and consider the morphism $\psi : \mathbb{A}^1_{S'} \to \mathbb{A}^1_{S'}$ given by multiplication by t: $a \mapsto ta$. The points 0, 1 of $\mathbb{A}^1_{S'}$ give rise to isomorphisms

$$\psi(0), \psi(1) : (Y\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1)_{S'}$$

($\psi(0)$ is the identity and $\psi(1)$ sends (y, b) to $(y, t+b)$). Furthermore, setting $\phi(y, a, b) = (y, \psi(b)\cdot a, b)$, we obtain an isomorphism

$$\phi : (Y\times\mathbb{A}^1\times\mathbb{A}^1)_{S'} \longrightarrow (Y\times\mathbb{A}^1\times\mathbb{A}^1)_{S'}.$$

We would like to compose the maps

$$C_n(X, Y\times\mathbb{A}^1) \xrightarrow{\pi^*} C_n(X, Y\times\mathbb{A}^1\times S') \xrightarrow{\mathrm{pr}^*} C_n(X, (Y\times\mathbb{A}^1)\times\mathbb{A}^1\times S')$$
$$\xrightarrow{\phi^*} C_n(X, (Y\times\mathbb{A}^1)\times\mathbb{A}^1\times S') \xrightarrow{T_n^*} C_{n+1}(X, Y\times\mathbb{A}^1\times S'),$$

but we are confronted with the problem that the map T_n^* is not defined on the whole group $C_n(X, (Y\times\mathbb{A}^1)\times\mathbb{A}^1\times S')$. The next proposition tells us that the composition is well defined if $S' = \mathbb{A}^1_S = \mathrm{Spec}\, S[t]$.

Proposition 4.4. *Suppose that $S' = \mathbb{A}^1_S = \mathrm{Spec}\, S[t]$. Then the composition*

$$H_n = T_n^*\circ\phi^*\circ\mathrm{pr}^*\circ\pi^* : C_n(X, Y\times\mathbb{A}^1) \longrightarrow C_{n+1}(X, Y\times\mathbb{A}^1\times S')$$

is well defined for every n. The family $\{H_n\}_{n\geq 0}$ defines a homotopy

$$\pi^* = \psi(0)\circ\pi^* \sim \psi(1)\circ\pi^* : C_n(X, Y\times\mathbb{A}^1) \longrightarrow C_n(X, Y\times\mathbb{A}^1\times S').$$

Furthermore, the image of the map $\psi(1)\circ\pi^*$ is contained in the subcomplex

$$C_\bullet^{Y\times\{0,1\}\times S'}(X, Y\times\mathbb{A}^1\times S').$$

Proof. Recall that all groups $C_\bullet$ are relative to the base scheme S which we have omitted from the notation. At the moment, the map H_n is only defined as a map to the group of cycles in $X\times Y\times\mathbb{A}^1\times\Delta^{n+1}\times S'$. If $Z\subset X\times Y\times\mathbb{A}^1\times\Delta^n$ projects finitely and surjectively onto an irreducible subscheme $T\subset Y\times\mathbb{A}^1\times\Delta^n$ of codimension one, then $\phi^*\circ\mathrm{pr}^*\circ\pi^*(Z)$ projects finitely and surjectively onto the irreducible subscheme of codimension one $T' = \phi^*\circ\mathrm{pr}^*\circ\pi^*(T)\subset(Y\times\mathbb{A}^1)\times\Delta^n\times\mathbb{A}^1\times S'$. Therefore, in order to show that $H_n(Z)$ is in $C_{n+1}(X, Y\times\mathbb{A}^1\times S')$, we have to check that $\theta_i^{-1}(T')$ has proper intersection with all faces for $i = 0,\dots,n$. Thus we have to show that T' has proper intersection with all faces $(Y\times\mathbb{A}^1)\times F\times S'$, where F is a face in $\Delta^n\times\mathbb{A}^1$ (as defined above). Since T' has codimension one, this comes down to show that it does not contain any irreducible component of any face (we did not assume Y to be irreducible, but we can silently assume that it is reduced). Consider the projection

$$Y\times\mathbb{A}^1\times\Delta^n\times\mathbb{A}^1\times S' \longrightarrow S'.$$

We can check our condition by considering the fibre over the generic point of S'. More precisely, let k be the function field of S and let $K = k(t)$ be the function field of S'. Let $(Y_1)_k,\dots,(Y_r)_k$ be the irreducible components of Y_k. Then an irreducible subscheme $T'\subset Y\times\mathbb{A}^1\times\Delta^n\times\mathbb{A}^1\times S'$ of codimension one meets all faces $Y\times\mathbb{A}^1\times F\times S'$ (F a face of $\Delta^n\times\mathbb{A}^1$) properly if and only if T_K does not contain $(Y_i\times\mathbb{A}^1)_K\times_K F_K$ for $i = 1,\dots,r$.

Now we arrived exactly at the situation considered in [Bloch 1986, Section 2]. The result follows from [Bloch 1986, Lemma 2.2] by taking $(Y \times \mathbb{A}^1)_k$ for the scheme X of that lemma, taking $\mathbb{A}^1_k$ as the algebraic group G acting on X by additive translation on the second factor and choosing the map $\psi : \mathbb{A}^1_K \to G_K$ of that lemma as the morphism which sends a to ta. The fact that the H_n define the homotopy is a straightforward computation.

It remains to show that the image of the map $\psi(1) \circ \pi^*$ is contained in the subcomplex

$$C_\bullet^{Y \times \{0,1\} \times S'}(X, Y \times \mathbb{A}^1 \times S').$$

But this is a again a condition which says that a subscheme of codimension one does not contain certain subschemes. In the same way as above, this can be verified over the generic fibre of S', and the result follows from the corresponding statement of [Bloch 1986, Lemma 2.2]. □

Corollary 4.5. *The natural inclusion*

$$C_\bullet^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times \mathbb{A}^1)$$

is a quasi-isomorphism.

Proof. Let $S' = \mathbb{A}^1_S$. Then the homomorphism

$$\pi^* : C_\bullet(X, Y \times \mathbb{A}^1)/C_\bullet^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times \mathbb{A}^1 \times S')/C_\bullet^{Y \times \{0,1\} \times S'}(X, Y \times \mathbb{A}^1 \times S')$$

is nullhomotopic (the H_n of Proposition 4.4 give the homotopy). In order to conclude the proof, it suffices to show that the nullhomotopic homomorphism π^* is injective on homology. Suppose that for a cycle z in degree n we have $\pi^*(z) = d_n(w)$. Then we find an $a \in \Gamma(S, \mathcal{O}_S)$ such that the specialization (i.e. $t \mapsto a$) $w(a)$ is well defined. But then $z = d_n(w(a))$. □

Proposition 4.6. *Suppose that* $S' = \mathbb{A}^1_S = \operatorname{Spec} S[t]$. *Then the composition*

$$C_n^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \xrightarrow{\psi(1) \circ \pi^*} C_n^{Y \times \{0,1\} \times S'}(X, Y \times \mathbb{A}^1 \times S') \xrightarrow{T_n^*} C_{n+1}(X, Y \times S')$$

is well defined, giving a homotopy

$$i_0^* \circ \psi(1) \circ \pi^* \sim i_1^* \circ \psi(1) \circ \pi^* : C_\bullet^{Y \times \{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y \times S').$$

Proof. Let again k be the function field of S and let $K = k(t)$ be that of S'. We use the following fact, which is explained in the proof of [Bloch 1986, Corollary 2.6]:

If z_k is a cycle on $Y \times \mathbb{A}^1 \times \Delta^n$ which intersects all faces $(Y \times \mathbb{A}^1 \times \Delta^m)_k$ properly, then $\psi(1) \circ \pi^*(z_k) \subset (Y \times \mathbb{A}^1 \times \Delta^n)_K$ intersects all faces $(Y \times F)_K$ (where F is any face in $\mathbb{A}^1 \times \Delta^n$) properly.

We deduce the statement of Proposition 4.6 from this in the same manner as we deduced Proposition 4.4 from [Bloch 1986, Lemma 2.2]. The fact that the maps $T_n^* \circ \psi(1) \circ \pi^*$ define the homotopy is a straightforward computation. □

Corollary 4.7. *The two maps*

$$i_0^*, i_1^* : C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) \longrightarrow C_\bullet(X, Y)$$

induce the same map on homology.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) & \xrightarrow{\pi^*} & C_\bullet^{Y\times\{0,1\}\times S'}(X, Y \times \mathbb{A}^1 \times S') \\ \downarrow{\scriptstyle i_0^*, i_1^*} & & \downarrow{\scriptstyle i_0^*, i_1^*} \\ C_\bullet(X, Y) & \xrightarrow{\pi^*} & C_\bullet(X, Y \times S'). \end{array}$$

The same specialization argument as in the proof of Corollary 4.5 shows that π^* is injective on homology. Therefore it suffices to show that $i_0^* \circ \pi^* = i_1^* \circ \pi^*$ on homology. By Proposition 4.4, we have a homotopy $\pi^* \sim \psi(1) \circ \pi^*$, and hence it suffices to show that the maps $i_0^* \circ \psi(1) \circ \pi^*$ and $i_1^* \circ \psi(1) \circ \pi^*$ induce the same map on homology. But this follows from Proposition 4.6. □

Now we conclude the proof of Theorem 4.3. First of all, note that

$$p^*(C_\bullet(X, Y)) \subset C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1)$$

and that $i_0^* \circ p^* = \mathrm{id}$, such that p^* is injective on homology. Consider the multiplication map

$$\tau : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1.$$

It is flat and therefore τ^* exists. Consider the diagram

$$\begin{array}{ccc} C_\bullet(X, Y \times \mathbb{A}^1) & \xrightarrow{\tau^*} & C_\bullet(X, Y \times \mathbb{A}^1 \times \mathbb{A}^1) \\ \uparrow{\scriptstyle \text{q.iso.}} & & \uparrow{\scriptstyle \text{q.iso.}} \\ C_\bullet^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1) & \overset{\tau^*}{\cdots\!>} & C_\bullet^{Y\times\mathbb{A}^1\times\{0,1\}}(X, Y \times \mathbb{A}^1 \times \mathbb{A}^1) \\ \downarrow{\scriptstyle i_0^*, i_1^*} & & \downarrow{\scriptstyle i_0^*, i_1^*} \\ C_\bullet(X, Y) & \xrightarrow{p^*} & C_\bullet(X, Y \times \mathbb{A}^1). \end{array}$$

One easily observes that τ^* sends a cycle $z \in C_n^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1)$ to a cycle in $C_n^{Y\times\mathbb{A}^1\times\{0,1\}}(X, Y \times \mathbb{A}^1 \times \mathbb{A}^1)$ and that for such a z the following equalities hold:

$$i_0^* \circ \tau^*(z) = p^* \circ i_0^*(z), \tag{3}$$

$$i_1^* \circ \tau^*(z) = z. \tag{4}$$

By Corollary 4.5, any class in $h_n(X, Y \times \mathbb{A}^1)$ can be represented by an element in $C_n^{Y\times\{0,1\}}(X, Y \times \mathbb{A}^1)$. Therefore (3) shows that in order to prove that p^* is surjective on homology, it suffices to show that $i_0^* \circ \tau^*$ is. But by Corollary 4.7, $i_0^* \circ \tau^*$ induces the same map on homology as $i_1^* \circ \tau^*$, which is the identity, by (4).
□

A naive definition of homotopy between scheme morphisms is the following: Two scheme morphisms $\phi, \psi : X \longrightarrow X'$ are homotopic if there exists a morphism

$$H : X \times \mathbb{A}^1 \longrightarrow X'$$

with $\phi = H \circ i_0$ and $\psi = H \circ i_1$. (This is not an equivalence relation!) The next corollary is an immediate consequence of Proposition 4.2.

Corollary 4.8. *If two morphisms*

$$\phi, \psi : X \longrightarrow X'$$

are homotopic, then they induce the same map on singular homology, i.e. for every scheme Y flat and of finite type over S, the homomorphisms

$$\phi_*, \psi_* : h_i(X, Y) \longrightarrow h_i(X', Y)$$

coincide for all i.

Now we recall the definition of relative singular homology from [Suslin and Voevodsky 1996]. Suppose that Y is an integral scheme and that X is any scheme over Y.

For $n \geq 0$, let $C_n(X/Y)$ be the free abelian group generated by closed integral subschemes of $X \times_Y \Delta_Y^n$ such that the restriction of the canonical projection

$$X \times_Y \Delta_Y^n \longrightarrow \Delta_Y^n$$

to Z induces a finite surjective morphism $p : Z \to \Delta_Y^n$. Let $i : \Delta_Y^m \hookrightarrow \Delta_Y^n$ be a face. Then all irreducible components of $Z \cap X \times_Y \Delta_Y^m$ have the "right" dimension and thus the cycle theoretic inverse image $i^*(Z)$ is well defined and in $C_m(X/Y)$. Furthermore, degeneracy maps are flat, and thus we obtain a simplicial abelian group $C_\bullet(X/Y)$. As above, we use the same notation for the complex of abelian groups obtained by taking the alternating sum of face operators. The groups

$$h_i(X/Y) = H_i(C_\bullet(X/Y))$$

are called the *relative singular homology groups* of X over Y.

We have seen in Section 2 that singular homology is covariantly functorial on the category $\mathrm{SmCor}(X)$ of smooth schemes over S with finite correspondences as morphisms. For $X, Y \in \mathrm{Sm}(S)$ the group of finite correspondences $c(X, Y)$ coincides with $C_0(X \times Y/Y)$ and we call two finite correspondences homotopic

if they have the same image in $h_0(X \times Y/Y)$. The next proposition shows that homotopic finite correspondences induce the same map on singular homology.

Proposition 4.9. *For smooth schemes $X, Y \in \mathrm{Sm}(S)$, the natural pairing*

$$c(X, Y) \otimes h_i(X; S) \to h_i(Y; S)$$

factors through $h_0(X \times Y/Y)$, defining pairings

$$h_0(X \times Y/Y) \otimes h_i(X; S) \longrightarrow h_i(Y; S) \quad \textit{for all } i.$$

Proof. Let $W \subset X \times Y \times \Delta^1 = X \times Y \times \mathbb{A}^1$ define an element in $C_1(X \times Y/Y)$. Let $W^j = i_j^*(W)$, for $j = 0, 1$, so that $d_1(W) = W^0 - W^1 \in C_0(X \times Y/Y)$. Let $\psi \in C_n(X; S)$. We have to show that $(W^0, \psi) = (W^1, \psi)$. Considering W as an element in $C_0(X \times Y \times \mathbb{A}^1/Y \times \mathbb{A}^1)$, the composite (W, ψ) is in $C_n^{\{0,1\}}(Y, \mathbb{A}^1; S)$ and $(W^j, \psi) = i_j^*((W, \psi))$ for $j = 0, 1$. Thus the result follows from Corollary 4.7. □

5. Alternative characterization of h_0

For a noetherian scheme X we have the identification

$$\mathrm{CH}^d(X, 0) = \mathrm{CH}^d(X)$$

between the higher Chow group $\mathrm{CH}^d(X, 0)$ and the group $\mathrm{CH}^d(X)$ of d-codimensional cycles on X modulo rational equivalence (see [Nart 1989, Proposition 3.1]). Fixing the notation and assumptions of the previous sections, we now give an analogous description for the group $h_0(X; S)$.

Let C be an integral scheme over S of absolute dimension 1. Then to every rational function $f \neq 0$ on C, we can attach the zero-cycle $\mathrm{div}(f) \in C_0(C; S)$ (see [Fulton 1998, Chapter I,1.2]). Let $\widetilde{C}$ be the normalization of C in its field of functions. Denoting the normalization morphism by $\phi : \widetilde{C} \to C$, we have $\phi_*(\mathrm{div}(f)) = \mathrm{div}(f)$. If C is regular and connected, then we denote by $P(C)$ the regular compactification of C over S, i.e. the uniquely determined regular and connected scheme of dimension 1 which is proper over S and which contains C as an open subscheme.

With this terminology, for an integral scheme C of absolute dimension 1, elements in the function field $k(C)$ are in 1-1 correspondence to morphisms $P(\widetilde{C}) \to \mathbb{P}^1_S$, which are not $\equiv \infty$.

Theorem 5.1. *The group $h_0(X; S)$ is the quotient of the group of zero-cycles on X modulo the subgroup generated by elements of the form* $\mathrm{div}(f)$, *where*

- *C is a closed integral curve on X,*
- *f is a rational function on C which, considered as a rational function on $P(\widetilde{C})$, is defined and $\equiv 1$ at every point of $P(\widetilde{C}) - \widetilde{C}$.*

Proof. We may suppose that X is reduced. Let $Z \subset X \times \Delta^1$ be an integral curve such that the projection $Z \to \Delta^1$ induces a finite and surjective morphism of Z onto a closed integral subscheme T of codimension 1 in Δ^1. Embed Δ^1 linearly to $\mathbb{P}^1 = \mathbb{P}^1_S$ by sending $(0, 1)$ to $0 = (0 : 1)$ and $(1, 0)$ to $\infty = (1 : 0)$. Since $Z \to \Delta^1$ is finite, the projection $Z \to \mathbb{P}^1$ corresponds to a rational function g on Z which is defined and $\equiv 1$ at every point of $P(\widetilde{Z}) - Z$. Let $\bar{Z}$ be the closure of Z in $X \times \mathbb{P}^1$, and let $\bar{C}$ be the image of $\bar{Z}$ under the (proper) projection $X \times \mathbb{P}^1 \to X$, considered as a reduced (hence integral) subscheme of X.

We have to consider two cases:

(1) If $\bar{C} = P$ is a closed point on X, then $Z = \{P\} \times \Delta^1$ and $d_1(Z) = 0$.

(2) If $\bar{C}$ is an integral curve, then the image C of Z under $X \times \mathbb{P}^1 \to X$ is an open subscheme of $\bar{C}$. Consider the extension of function fields

$$k(Z)|k(C)$$

and let $f \in k(C)$ be the norm of g with respect to this extension. Then f is defined and $\equiv 1$ at every point of $P(\widetilde{C}) - C$ and

$$\operatorname{div}(f) = \delta_0(Z) - \delta_1(Z) = d(Z).$$

If X is of dimension 1, the last equality follows from [Nart 1989, Proposition 1.3]. The general case can be reduced to this by replacing X by $\bar{C}$. Considering f as a rational function on $\bar{C}$, it satisfies the assumption of the theorem.

It remains to show the other direction. Let C and f be as in the theorem. We have to show that $\operatorname{div}(f) \in C_0(X; S)$ is a boundary. To see this, interpret f as a nonconstant morphism $U \to \mathbb{P}^1$ defined on an open subscheme $U \subset C$ and let $\bar{Z}$ be the closure of the graph of this morphism in $X \times \mathbb{P}^1$. The scheme $\bar{Z}$ is integral, of dimension 1 and projects birationally and properly onto C. Consider again the open linear embedding $\Delta^1 \subset \mathbb{P}^1$ which is defined by sending $(0, 1)$ to 0 and $(1, 0)$ to ∞ and let $Z = \bar{Z} \cap X \times \Delta^1$. The properties of f imply that the induced projection $Z \to \Delta^1$ is finite and surjective onto a closed subscheme of codimension 1 in Δ^1, thus defining an element of $C_1(X; S)$. Finally note that $d(Z) = \delta_0(Z) - \delta_1(Z) = \operatorname{div}(f)$. □

This immediately implies:

Corollary 5.2. *If X is proper over S, then*

$$h_0(X; S) = \mathrm{CH}_0(X).$$

Corollary 5.3. *The natural homomorphism $\bigoplus_{i_C} d(C_1(C; S)) \xrightarrow{i_{C*}} d(C_1(X; S))$ is surjective, where $i_C : C \to X$ runs through all S-morphisms from a regular scheme C over S of dimension 1 to X.*

Proof. By Theorem 5.1, $d(C_1(X;S)$ is generated by elements of the form $\mathrm{div}(f)$, where f is a rational function on an integral curve on X satisfying an additional property. The normalization $\widetilde{C}$ of C is a regular scheme of dimension 1 and let $i:\widetilde{C}\to X$ the associated morphism. Considering f as a rational function on $\widetilde{C}$, we have the equality

$$i_*(\mathrm{div}(f))=\mathrm{div}(f).$$

By the additional property of f, the associated line bundle $\mathscr{L}(\mathrm{div}(f))$ over the compactification $P(\widetilde{C})$ together with its canonical trivialization over $P(\widetilde{C})-\widetilde{C}$ defines the trivial element in $\mathrm{Pic}(P(\widetilde{C}),P(\widetilde{C})-\widetilde{C})$. Therefore, the calculation of singular homology of regular schemes of dimension 1 (see Theorems 3.5 and 3.7), shows that $\mathrm{div}(f)$ is in $d(C_1(C;S))$. This finishes the proof. □

Now we can prove the exactness of a part of the Mayer–Vietoris sequence for X of arbitrary dimension.

Proposition 5.4. *Let $S=U\cup V$ be a covering by Zariski-open subschemes U and V. Then the natural sequence*

$$h_0(X;S)\longrightarrow h_0(X;U)\oplus h_0(X;V)\longrightarrow h_0(X;U\cap V)\longrightarrow 0$$

is exact.

Proof. First of all, the homomorphism

$$C_0(X;U)\oplus C_0(X;V)\longrightarrow C_0(X;U\cap V)$$

is surjective, and therefore so is $h_0(X;U)\oplus h_0(X;V)\longrightarrow h_0(X;U\cap V)$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
d(C_1(X;S)) & \longrightarrow & d(C_1(X;U))\oplus d(C_1(X;V)) & \longrightarrow & d(C_1(X;U\cap V)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_0(X;S) & \hookrightarrow & C_0(X;U)\oplus C_0(X;V) & \longrightarrow & C_0(X;U\cap V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
h_0(X;S) & \longrightarrow & h_0(X;U)\oplus h_0(X;V) & \longrightarrow & h_0(X;U\cap V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The middle row and the middle and right columns are exact. Therefore the snake lemma shows that the lower line is exact if and only if the homomorphism

$$d(C_1(X;U)) \oplus d(C_1(X;V)) \longrightarrow d(C_1(X;U\cap V)) \tag{5}$$

is surjective. By Theorem 3.10 (iv), we observe that (5) is surjective if X is regular and of dimension 1. For a general X, put

$$X' = X \times_S (U \cap V).$$

Then the commutative diagram

$$\begin{array}{ccc} C_1(X';U) \oplus C_1(X';V) & \longrightarrow & C_1(X';U\cap V) \\ \downarrow & & \| \\ C_1(X;U) \oplus C_1(X;V) & \longrightarrow & C_1(X;U\cap V) \end{array}$$

shows that, in order to show the surjectivity of (5), we may suppose that $X = X'$. Now the statement follows from Corollary 5.3, using the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i_C} d(C_1(C;U)) \oplus d(C_1(C;V)) & \xrightarrow{i_{C*}} & d(C_1(X;U)) \oplus d(C_1(X;V)) \\ \downarrow\!\!\downarrow & & \downarrow \\ \bigoplus_{i_C} d(C_1(C;U\cap V)) & \xrightarrow{i_{C*}} & d(C_1(X;U\cap V)). \end{array}$$

This concludes the proof. □

A similar argument shows:

Proposition 5.5. *Let $X = X_1 \cup X_2$ be a covering by Zariski open subschemes X_1 and X_2. Then the natural sequence*

$$h_0(X_1 \cap X_2; S) \longrightarrow h_0(X_1; S) \oplus h_0(X_2; S) \longrightarrow h_0(X; S) \longrightarrow 0$$

is exact.

Proof. We omit the base scheme S from our notation. The homomorphism

$$C_0(X_1) \oplus C_0(X_2) \longrightarrow C_0(X)$$

is surjective, and therefore so is

$$h_0(X_1) \oplus h_0(X_2) \longrightarrow h_0(X).$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
d(C_1(X_1\cap X_2)) & \longrightarrow & d(C_1(X_1))\oplus d(C_1(X_2)) & \longrightarrow & d(C_1(X)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_0(X_1\cap X_2) & \hookrightarrow & C_0(X_1)\oplus C_0(X_2) & \longrightarrow & C_0(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
h_0(X_1\cap X_2) & \longrightarrow & h_0(X_1)\oplus h_0(X_2) & \longrightarrow & h_0(X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The middle row and the middle and right columns are exact. Therefore the snake lemma shows that the lower line is exact if and only if the homomorphism

$$d(C_1(X_1))\oplus d(C_1(X_2)) \longrightarrow d(C_1(X)) \tag{6}$$

is surjective. By Theorem 3.10 (iii), we observe that (6) is surjective if X is regular and of dimension 1.

For a morphism $i : C \to X$ we use the notation $C_1 = i^{-1}(X_1)$ and $C_2 = i^{-1}(X_2)$, thus $C = C_1 \cup C_2$ is a Zariski open covering.

Now the required statement for arbitrary X follows from Corollary 5.3, using the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i_C} d(C_1(C_1))\oplus d(C_1(C_2)) & \xrightarrow{i_{C*}} & d(C_1(X_1))\oplus d(C_1(X_2)) \\
\downarrow & & \downarrow \\
\bigoplus_{i_C} d(C_1(C)) & \xrightarrow{i_{C*}} & d(C_1(X)).
\end{array}$$

This concludes the proof. □

We conclude this section with a surjectivity result.

Proposition 5.6. *Let X be regular and let U be a dense open subscheme in X. Then the natural homomorphism*

$$h_0(U; S) \longrightarrow h_0(X; S)$$

is surjective.

Proof. Let P be a 0-dimensional point on X which is not contained in U. We have to show that the image of P in $h_0(X;S)$ is equal to the image of a finite linear combination $\sum n_i P_i$ with $P_i \in U$ for all i. Choose a one-dimensional subscheme C on X such that P is a regular point on C and such that C is not contained in $X-U$. We find such a curve, since X is regular: Indeed, $\mathbb{O}_{X,P}$ is a d-dimensional regular local ring, with $d = \dim X$. Let $\mathfrak{m}$ be the maximal ideal and $\mathfrak{a}$ the ideal defining the closed subset $(X-U) \cap \operatorname{Spec}(\mathbb{O}_{X,P})$. Choose elements $\bar{x}_1, \ldots, \bar{x}_{d-1}$ in $\mathfrak{m}/\mathfrak{m}^2$ which span a $(d-1)$-dimensional subspace which does not contain $\mathfrak{a} + \mathfrak{m}/\mathfrak{m}$. Lifting $\bar{x}_1, \ldots, \bar{x}_{d-1}$ to a regular sequence $x_1, \ldots, x_{d-1} \in \mathbb{O}_{X,P}$, the ideal $(x_1, \ldots, x_{d-1})$ is a prime ideal of height $(d-1)$ which does not contain $\mathfrak{a}$. Finally, extend $\bar{x}_1, \ldots, x_{d-1}$ to an affine open neighbourhood of P in X and choose C as the closure of their zero-locus.

Consider the normalization $\widetilde{C}$ of C and let $P(\widetilde{C})$ be a regular compactification over S. Let $P(\widetilde{C}) - \widetilde{C} = \{P_1, \ldots, P_r\}$ and let $P_{r+1}, \ldots, P_s$ be the finitely many closed points on $\widetilde{C}$ mapping to $C \cap (X-U)$. Let $\widetilde{P}$ be the unique point on $\widetilde{C}$ projecting to $P \in C$. Let $D = \{P_1, \ldots, P_s, \widetilde{P}\}$ and consider the ring $A = \mathbb{O}_{P(\widetilde{C}),D}$, which is a semilocal principal ideal domain. We find an element $f \in A$ which has exact valuation 1 at $\widetilde{P}$ and which is $\equiv 1$ at each P_i, $i = 1, \ldots, n$. Then $(\operatorname{div} f) \subset X$ is of the form $P + \sum Q_i$ with $Q_i \in U$. □

6. Review of tame coverings

The concept of tame ramification stems from number theory: A finite extension of number fields $L|K$ is called tamely ramified at a prime $\mathfrak{P}$ of L if the associated extension of completions $L_{\mathfrak{P}}|K_{\mathfrak{P}}$ is a tamely ramified extension of local fields. The latter means that the ramification index is prime to the characteristic of the residue field. It is a classical result that composites and towers of tamely ramified extensions are again tamely ramified. This concept generalizes to separable extensions of arbitrary discrete valuation fields by requiring that the associated residue field extensions are separable.

Let from now on S be the spectrum of an excellent Dedekind domain and let $X \in \operatorname{Sch}(S)$. Our aim is to say when a finite étale covering $Y \to X$ is tame. Here "tame" means tamely ramified along the boundary of a compactification $\bar{X}$ of X over S. If $\bar{X}$ is regular and $D = \bar{X} - X$ is a normal crossing divisor, then one can use the approach of [Grothendieck 1971; Grothendieck and Murre 1971]:

Definition 6.1 [Grothendieck and Murre 1971, 2.2.2]. A finite étale covering $Y \to X$ is called tame (along D) if the extension of function fields $k(Y)|k(X)$ is tamely ramified at the discrete valuations associated to the irreducible components of D.

Even if one restricts attention to regular schemes, one is confronted with the following problems:

- If X is regular, we do not know whether there exists a regular compactification with an NCD as its boundary,
- The notion of tameness might depend on the choice of the compactification $\bar{X}$ of X.
- Even if the first two questions can be answered in a positive way, there is no obvious functoriality for the tame fundamental group (already for an open immersion).

All these problems are void in the case of a regular curve C, where a canonical compactification $\bar{C}$ exists. Starting from the therefore obvious notion of tame coverings of regular curves, G. Wiesend [2006] proposed the following definition.

Definition 6.2. Let X be a separated integral scheme of finite type over S. A finite étale covering $Y \to X$ is called *tame* if for every integral curve $C \subset X$ with normalization $\widetilde{C} \to C$ the base change

$$Y \times_X \widetilde{C} \longrightarrow \widetilde{C}$$

is a tame covering of the regular curve $\widetilde{C}$.

This definition has the advantage of making no use of a compactification of X. Furthermore, it is obviously stable under base change. However, it is difficult to decide whether a given étale covering is actually tame. For coverings of normal schemes, several authors [Abbes 2000; Chinburg and Erez 1992; Schmidt 2002] have made suggestions for a definition of tameness which all come down to the following notion, which we want to call numerically tameness here.

Definition 6.3. Let $\bar{X} \in \mathrm{Sch}(S)$ be normal connected and proper, and let $X \subset \bar{X}$ be an open subscheme. Let $Y \to X$ be a finite étale Galois covering and let $\bar{Y}$ be the normalization of $\bar{X}$ in the function field $k(Y)$ of Y. We say that $Y \to X$ is *numerically tame* (along $D = \bar{X} - X$) if the order of the inertia group $T_x(\bar{Y}|\bar{X}) \subset \mathrm{Gal}(\bar{Y}|\bar{X}) = \mathrm{Gal}(Y|X)$ of each closed point $x \in D$ (see [Bourbaki 1964, Chapter 5, Section 2.2] for the definition of inertia groups) is prime to the residue characteristic of x. A finite étale covering $Y \to X$ is called numerically tame if it can be dominated by a numerically tame Galois covering.

Proposition 6.4. *Let $\bar{X} \in \mathrm{Sch}(S)$ be normal connected and proper, and let $X \subset \bar{X}$ be an open subscheme. If the finite étale covering $Y \to X$ is numerically tame (along $\bar{X} - X$), then it is tame.*

Proof. For regular curves the notions of tameness and of numerically tameness obviously coincide. Therefore the statement of the proposition follows from the fact that numerically tame coverings are stable under base change; see [Schmidt 2002]. □

Theorem 6.5 [Wiesend 2006, Theorem 2]. *Assume that $\overline{X}$ is regular and that $D = \overline{X} - X$ is an NCD. Then, for a finite étale covering $Y \to X$, there is equivalence between*:

(i) *$Y \to X$ is tame according to Definition 6.1.*

(ii) *$Y \to X$ is tame (according to Definition 6.2).*

(iii) *$Y \to X$ is numerically tame.*

Remark 6.6. The equivalence of (i) and (iii) had already been shown in [Schmidt 2002].

Theorem 6.7 [Wiesend 2006, Theorem 2]. *Assume that $\overline{X}$ is regular (but make no assumption on $D = \overline{X} - X$). If a numerically tame covering $Y \to X$ can be dominated by a Galois covering with nilpotent Galois group, then it is tame.*

In particular, for nilpotent coverings of a regular scheme X the notion of numerically tameness does not depend on the choice of a regular compactification $\overline{X}$ (if it exists). This had already been shown in [Schmidt 2002]. A counterexample with non-nilpotent Galois group can be found in [Wiesend 2006, Remark 3].

7. Finiteness results for tame fundamental groups

The tame coverings of a connected integral scheme $X \in \mathrm{Sch}(S)$ satisfy the axioms of a Galois category [Wiesend 2006, Proposition 1]. After choosing a geometric point $\bar{x}$ of X we have the fibre functor $(Y \to X) \mapsto \mathrm{Mor}_X(\bar{x}, Y)$ from the category of tame coverings of X to the category of sets, whose automorphisms group is called the *tame fundamental group* $\pi_1^t(X, \bar{x})$. It classifies finite connected tame coverings of X. We have an obvious surjection

$$\pi_1^{\mathrm{et}}(X, \bar{x}) \twoheadrightarrow \pi_1^t(X, \bar{x}),$$

which is an isomorphism if X is proper. Assume that X is normal, connected and let $\overline{X}$ be a normal compactification. Then, replacing tame coverings by numerically tame coverings, we obtain in an analogous way the *numerically tame fundamental group* $\pi_1^{\mathrm{nt}}(\overline{X}, \overline{X} - X, \bar{x})$, which classifies finite connected numerically tame coverings of X (along $\overline{X} - X$). By Proposition 6.4 we have a surjection

$$\varphi : \pi_1^t(X, \bar{x}) \twoheadrightarrow \pi_1^{\mathrm{nt}}(\overline{X}, \overline{X} - X, \bar{x}),$$

which, by Theorem 6.7, induces an isomorphism on the maximal pro-nilpotent factor groups if $\overline{X}$ is regular. If, in addition, $\overline{X} - X$ is a normal crossing divisor then φ is an isomorphism by Theorem 6.5. The fundamental groups of a connected scheme X with respect to different base points are isomorphic, and the isomorphism is canonical up to inner automorphisms. Therefore, when working with the maximal abelian quotient of the étale fundamental group (tame fundamental group,

n.t. fundamental group) of a connected scheme, we are allowed to omit the base point from notation.

Now we specialize to the case $S = \mathrm{Spec}(\mathbb{Z})$, i.e. to arithmetic schemes. In [Schmidt 2002] we proved the finiteness of the abelianized numerically tame fundamental group $\pi_1^{\mathrm{nt}}(\overline{X}, \overline{X} - X)^{\mathrm{ab}}$ of a connected normal scheme, flat and of finite type over $\mathrm{Spec}(\mathbb{Z})$ with respect to a normal compactification $\overline{X}$. The proof given there can be adapted to apply also to the larger group $\pi_1^t(X)^{\mathrm{ab}}$.

Theorem 7.1. *Let X be a connected normal scheme, flat and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Then the abelianized tame fundamental group $\pi_1^t(X)^{\mathrm{ab}}$ is finite.*

For the proof we need the following two lemmas. The first one extends [Schmidt 2002, Corollary 2.6] from numerical tameness to tameness.

Lemma 7.2. *Let $X \in \mathrm{Sch}(S)$ be normal and connected, p a prime number and $Y \to X$ a finite étale Galois covering whose Galois group is a finite p-group. Let $\overline{X}$ be a normal compactification of X and assume there exists a prime divisor D on $\overline{X}$ which is ramified in $k(Y)|k(X)$ and which contains a closed point of residue characteristic p. Then $Y \to X$ is not tame.*

Proof. The statement of the lemma is part of the proof of [Wiesend 2006, Theorem 2]. □

Lemma 7.3. *Let A be a strictly henselian discrete valuation ring with perfect (hence algebraically closed) residue field and with quotient field k. Let $k_\infty|k$ be a $\mathbb{Z}_p$-extension. Let $K|k$ be a regular field extension and let $B \subset K$ be a discrete valuation ring dominating A. Then B is ramified in $Kk_\infty|K$.*

Proof. See [Schmidt 2002, Lemma 3.2]. □

Proof of Theorem 7.1. The proof is a modification of the proof of [Schmidt 2002, Theorem 3.1]. Let $\overline{X}$ be a normal compactification of X over $\mathrm{Spec}(\mathbb{Z})$. Let k be the normalization of $\mathbb{Q}$ in the function field of X and put $S = \mathrm{Spec}(\mathbb{O}_k)$. Then the natural projection $\overline{X} \to \mathrm{Spec}(\mathbb{Z})$ factors through S.

Since X is normal, for any open subscheme V of X the natural homomorphism $\pi_1^{\mathrm{et}}(V) \to \pi_1^{\mathrm{et}}(X)$ is surjective. Therefore also the homomorphism

$$\pi_1^t(V)^{\mathrm{ab}} \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

is surjective and so we may replace X by a suitable open subscheme and assume that X is smooth over S. Let $T \subset S$ be the image of X. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}(X/T) & \longrightarrow & \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} & \longrightarrow & \pi_1^{\mathrm{et}}(T)^{\mathrm{ab}} \\
 & & \downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
0 & \longrightarrow & \mathrm{Ker}^t(X/T) & \longrightarrow & \pi_1^t(X)^{\mathrm{ab}} & \longrightarrow & \pi_1^t(T)^{\mathrm{ab}}
\end{array}$$

where the groups $\mathrm{Ker}(X/T)$ and $\mathrm{Ker}^t(X/T)$ are defined by the exactness of the corresponding rows, and the two right vertical homomorphisms are surjective. By [Katz and Lang 1981, Theorem 1], the group $\mathrm{Ker}(X/T)$ is finite. By classical one-dimensional class field theory, the group $\pi_1^t(T)^{\mathrm{ab}}$ is finite (it is the Galois group of the ray class field of k with modulus $\prod_{\mathfrak{p}\notin T}\mathfrak{p}$). The kernel of $\pi_1^{\mathrm{et}}(T)^{\mathrm{ab}} \to \pi_1^t(T)^{\mathrm{ab}}$ is generated by the ramification groups of the primes of S which are not in T. Denoting the product of the residue characteristics of these primes by N, we see that $\pi_1^{\mathrm{et}}(T)^{\mathrm{ab}}$ is the product of a finite group and a topologically finitely generated pro-N group. Therefore the same is also true for $\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ and for $\pi_1^t(X)^{\mathrm{ab}}$. Hence it suffices to show that the cokernel C of the induced map $\mathrm{Ker}(X/T) \to \mathrm{Ker}^t(X/T)$ is a torsion group.

Let K be the function field of X and let k_1 be the maximal abelian extension of k such that the normalization X_{Kk_1} of X in the composite Kk_1 is ind-tame over X. By [Katz and Lang 1981, Lemma 2, (2)], the normalization of T in k_1 is ind-étale over T. Let $k_2|k$ be the maximal subextension of $k_1|k$ such that the normalization T_{k_2} of T in k_2 is tame over T. Then $G(k_2|k) = \pi_1^t(T)^{\mathrm{ab}}$ and $C \cong G(k_1|k_2)$.

In order to show that C is a torsion group, we therefore have to show that $k_1|k_2$ does not contain a $\mathbb{Z}_p$-extension of k_2 for any prime number p. Since $k_2|k$ is a finite extension and $k_1|k$ is abelian, this is equivalent to the assertion that $k_1|k$ contains no $\mathbb{Z}_p$-extension of k for any prime number p.

Let p be a prime number and suppose that $k_\infty|k$ is a $\mathbb{Z}_p$-extension such that the normalization X_{Kk_∞} is ind-tame over X. A $\mathbb{Z}_p$-extension of a number field is unramified outside p and there exists at least one ramified prime dividing p; see e.g. [Neukirch et al. 2000, (10.3.20)(ii)]. Let k' be the maximal unramified subextension of $k_\infty|k$ and let S' be the normalization of S in k'. Then the base change $\bar{X}' = \bar{X} \times_S S' \to X$ is étale. Hence $\bar{X}'$ is normal and the preimage X' of X is smooth and geometrically connected over k'. So, after replacing k by k', we may suppose that $k_\infty|k$ is totally ramified at a prime $\mathfrak{p}|p$, $\mathfrak{p}\in S-T$. Considering the base change to the strict henselization of S at $\mathfrak{p}$ and applying Lemma 7.3, we see that each vertical divisor of $\bar{X}$ in the fibre over $\mathfrak{p}$ ramifies in Kk_∞. Replacing $\bar{X}$ by its normalization in a suitable finite subextension of Kk_∞, we obtain a contradiction using Lemma 7.2. □

Next we consider the case $S = \mathrm{Spec}(\mathbb{F})$, i.e. varieties over a finite field $\mathbb{F}$. In this case we have the degree map

$$\deg : \pi_1^t(X)^{\mathrm{ab}} \longrightarrow \pi_1^t(S)^{\mathrm{ab}} \cong \mathrm{Gal}(\bar{\mathbb{F}} \mid \mathbb{F}) \cong \widehat{\mathbb{Z}},$$

and we denote the kernel of this degree map by $(\pi_1^t(X)^{\mathrm{ab}})^0$. The image of deg is an open subgroup of $\widehat{\mathbb{Z}}$ and is therefore isomorphic to $\widehat{\mathbb{Z}}$. As $\widehat{\mathbb{Z}}$ is a projective profinite

group, we have a (noncanonical) isomorphism

$$\pi_1^t(X)^{\mathrm{ab}} \cong (\pi_1^t(X)^{\mathrm{ab}})^0 \times \widehat{\mathbb{Z}}.$$

Let p be the characteristic of the finite field $\mathbb{F}$. If X is an open subscheme of a smooth proper variety $\overline{X}$, then we have a decomposition

$$(\pi_1^t(X)^{\mathrm{ab}})^0 \cong (\pi_1^{\mathrm{et}}(X)^{\mathrm{ab}})^0(\text{prime-to-}p\text{-part}) \oplus (\pi_1^{\mathrm{et}}(\overline{X})^{\mathrm{ab}})^0(p\text{-part}),$$

and both summands are known to be finite. The finiteness statement for $(\pi_1^t(X)^{\mathrm{ab}})^0$ can be generalized to normal schemes.

Theorem 7.4. *Let X be a normal connected variety over a finite field. Then the group $(\pi_1^t(X)^{\mathrm{ab}})^0$ is finite.*

Proof (sketch). We may replace X by a suitable open subscheme and therefore assume that there exists a smooth morphism $X \longrightarrow C$ to a smooth projective curve. Then we proceed in an analogous way as in the proof of Theorem 7.1 using the fact that a global field of positive characteristic has exactly one unramified $\widehat{\mathbb{Z}}$-extension, which is obtained by base change from the constant field. □

8. Tame class field theory

In this section we construct a reciprocity homomorphism from the singular homology group $h_0(X)$ to the abelianized tame fundamental group of an arithmetic scheme X. A sketch of the results of this section is contained in [Schmidt 2007].

Let for the whole section $S = \mathrm{Spec}(\mathbb{Z})$ and let $X \in \mathrm{Sch}(\mathbb{Z})$ be connected and regular. If X has $\mathbb{R}$-valued points, we have to modify the tame fundamental group in the following way.

We consider the full subcategory of the category of tame coverings of X which consists of that coverings in which every $\mathbb{R}$-valued point of X splits completely. After choosing a geometric point $\bar{x}$ of X we have the fibre functor $(Y \to X) \mapsto \mathrm{Mor}_X(\bar{x}, Y)$, and its automorphism group $\widetilde{\pi}_1^t(X, \bar{x})$ is called the *modified tame fundamental group* of X. It classifies connected tame coverings of X in which every $\mathbb{R}$-valued point of X splits completely. We have an obvious surjection

$$\pi_1^t(X, \bar{x}) \twoheadrightarrow \widetilde{\pi}_1^t(X, \bar{x})$$

which is an isomorphism if $X(\mathbb{R}) = \varnothing$.

For $x \in X(\mathbb{R})$ let $\sigma_x \in \pi_1^t(X)^{\mathrm{ab}}$ be the image of the complex conjugation $\sigma \in \mathrm{Gal}(\mathbb{C}|\mathbb{R})$ under the natural map $x_* : \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \to \pi_1^t(X)^{\mathrm{ab}}$. By [Saito 1985, Lemma 4.9 (iii)], the map

$$X(\mathbb{R}) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}, \quad x \longmapsto \sigma_x,$$

is locally constant for the norm topology on $X(\mathbb{R})$. Therefore the kernel of the homomorphism

$$\pi_1^t(X)^{\mathrm{ab}} \twoheadrightarrow \widetilde{\pi}_1^t(X)^{\mathrm{ab}}$$

is an $\mathbb{F}_2$-vector space of dimension less or equal the number of connected components of $X(\mathbb{R})$.

Let $x \in X$ be a closed point. We have a natural isomorphism

$$\pi_1^{\mathrm{et}}(\{x\}) \cong \mathrm{Gal}(\overline{k(x)}|k(x)) \cong \widehat{\mathbb{Z}},$$

and we denote the image of the (arithmetic) Frobenius automorphism $\mathrm{Frob} \in G(\overline{k(x)}|k(x))$ under the natural homomorphism $\pi_1^{\mathrm{et}}(\{x\})^{\mathrm{ab}} \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ by Frob_x.

In the following we omit the base scheme $\mathrm{Spec}(\mathbb{Z})$ from notation, writing $C_\bullet(X)$ for $C_\bullet(X; \mathrm{Spec}(\mathbb{Z}))$ and similar for homology. Recall that $C_0(X) = Z_0(X)$ is the group of zero-cycles on X. Sending x to Frob_x, we obtain a homomorphism

$$r : C_0(X) \longrightarrow \pi_1(X)^{\mathrm{ab}},$$

which is known to have dense image [Lang 1956; Raskind 1995, Lemma 1.7]. Our next goal is to show:

Theorem 8.1. *The composite map*

$$C_0(X) \xrightarrow{r} \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} \longrightarrow \widetilde{\pi}_1^t(X)^{\mathrm{ab}}$$

factors through $h_0(X)$, thus defining a reciprocity homomorphism

$$\mathrm{rec} : h_0(X) \longrightarrow \widetilde{\pi}_1^t(X)^{\mathrm{ab}},$$

which has a dense image.

In order to prove Theorem 8.1, let us apply Theorem 3.7 to the case of rings of integers of algebraic number fields. Let k be a finite extension of $\mathbb{Q}$ and let Σ be a finite set of nonarchimedean primes of k. Let $\mathbb{O}_{k,\Sigma}$ be the ring of Σ-integers of k and let $E_k^{1,\Sigma}$ be the subgroup of elements in the group of global units E_k which are $\equiv 1$ at every prime $\mathfrak{p} \in \Sigma$. Let r_1 and r_2 be the number of real and complex places of k. If $\mathfrak{m}$ is a product of primes of k, then we denote by $C_{\mathfrak{m}}(k)$ the ray class group of k with modulus $\mathfrak{m}$.

Proposition 8.2. *For $X = \mathrm{Spec}(\mathbb{O}_{k,\Sigma})$, we have $h_i(X) = 0$ for $i \neq 0, 1$,*

(i) *$h_0(X) = C_{\mathfrak{m}}(k)$ with $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$, and*

(ii) *$h_1(X) = E_k^{1,\Sigma} \cong$ (finite group) $\oplus\, \mathbb{Z}^{r_1+r_2-1}$.*

In particular, $h_0(X)$ is finite and $h_1(X)$ is finitely generated. If Σ contains at least two primes with different residue characteristics, the finite summand in (ii) *vanishes.*

Proof. The vanishing of $h_i(X)$ for $i \neq 0, 1$ follows from Theorem 3.7. A straightforward computation shows that for $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$

$$C_{\mathfrak{m}}(k) \cong \mathrm{Pic}(\mathrm{Spec}(\mathbb{O}_k), \Sigma),$$

and the finiteness of $C_{\mathfrak{m}}(k)$ is well-known. The group $E_k^{1,\Sigma}$ is of finite index in the full unit group E_k. Therefore the remaining statement in (ii) follows from Dirichlet's unit theorem. Furthermore, a root of unity congruent to 1 modulo two primes of different residue characteristics equals 1. □

By Theorem 3.5, we have an analogous statement for smooth curves over finite fields.

Proposition 8.3. *Let X be a smooth, geometrically connected curve over a finite field $\mathbb{F}$ and let $\bar{X}$ be the uniquely defined smooth compactification of X. Let $\Sigma = \bar{X} - X$ and let k be the function field of X. Then we have $h_i(X) = 0$ for $i \neq 0, 1$,*

(i) $h_0(X) = C_{\mathfrak{m}}(k)$ *with* $\mathfrak{m} = \prod_{\mathfrak{p}\in\Sigma} \mathfrak{p}$, *and*

(ii) $h_1(X) = \begin{cases} 0 & \text{if } \Sigma \neq \varnothing, \\ \mathbb{F}^\times & \text{if } \Sigma = \varnothing. \end{cases}$

In particular, $h_i(X)$ is finite for all i.

Proof of Theorem 8.1. Using Propositions 8.2 and 8.3, classical (one-dimensional) class field theory for global fields shows the statement in the case $\dim X = 1$. In order to show the general statement, it suffices by Corollary 5.3 to show that for any morphism $f : C \to X$ from a regular curve C to X and for any $x \in d(C_1(C))$, we have $r(f_*(x)) = 0$. This follows from the corresponding result in dimension 1 and from the commutative diagram

$$\begin{array}{ccccc} d(C_1(C)) & \longrightarrow & C_0(C) & \xrightarrow{r_C} & \widetilde{\pi}_1^t(C)^{\mathrm{ab}} \\ \downarrow & & \downarrow & & \downarrow \\ d(C_1(X)) & \longrightarrow & C_0(X) & \xrightarrow{r_X} & \widetilde{\pi}_1^t(X)^{\mathrm{ab}}. \end{array}$$

□

In order to investigate the reciprocity map, we use Wiesend's version of higher dimensional class field theory [Wiesend 2007]. We start with the arithmetic case, i.e. when X is flat over $\mathrm{Spec}(\mathbb{Z})$. In this case $\widetilde{\pi}_1^t(X)^{\mathrm{ab}}$ is finite by Theorem 7.1.

Theorem 8.4. *Let X be a regular, connected scheme, flat and of finite type over* $\mathrm{Spec}(\mathbb{Z})$. *Then the reciprocity homomorphism*

$$\mathrm{rec}_X : h_0(X) \longrightarrow \widetilde{\pi}_1^t(X)^{\mathrm{ab}}$$

is an isomorphism of finite abelian groups.

Remark 8.5. If X is proper, then $h_0(X) \cong \mathrm{CH}_0(X)$ and $\widetilde{\pi}_1^t(X)^{\mathrm{ab}} \cong \widetilde{\pi}_1^{\mathrm{et}}(X)^{\mathrm{ab}}$, and we recover the unramified class field theory for arithmetic schemes of Bloch and Kato–Saito [Kato and Saito 1983; Saito 1985].

Proof of Theorem 8.4. Recall the definition of Wiesend's idèle group $\mathcal{I}_X$. It is defined by

$$\mathcal{I}_X := Z_0(X) \oplus \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} k(C)_v^\times.$$

Here C runs through all closed integral subschemes of X of dimension 1, C_∞ is the finite set of places (including the archimedean ones if C is horizontal) of the global field $k(C)$ with center outside C and $k(C)_v$ is the completion of $k(C)$ with respect to v. $\mathcal{I}_X$ becomes a topological group by endowing the group $Z_0(X)$ of zero cycles on X with the discrete topology, the groups $k(C)_v^\times$ with their natural locally compact topology and the direct sum with the direct sum topology.[1]

The idèle class group $\mathcal{C}_X$ is defined as the cokernel of the natural map

$$\bigoplus_{C \subset X} k(C)^\times \longrightarrow \mathcal{I}_X.$$

which is given for a fixed $C \subset X$ by the divisor map $k(C)^\times \to Z_0(C) \to Z_0(X)$ and the diagonal map $k(C)^\times \to \bigoplus_{v \in C_\infty} k(C)_v^\times$. $\mathcal{C}_X$ is endowed the quotient topology of $\mathcal{I}_X$.

We consider the quotient $\mathcal{C}_X^t$ of $\mathcal{C}_X$ obtained by cutting out the 1-unit groups at all places outside X. More precisely, let for $v \in C_\infty$, $U^1(k(C)_v)$ be the group of principal units in the local field $k(C)_v$. We make the notational convention $U^1(K) = K^\times$ for the archimedean local fields $K = \mathbb{R}, \mathbb{C}$. Then

$$\mathcal{U}_X^t := \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} U^1(k(C)_v)$$

is an open subgroup of the idèle group $\mathcal{I}_X$ and we put

$$\mathcal{C}_X^t := \mathrm{coker}\Big(\bigoplus_{C \subset X} k(C)^\times \longrightarrow \mathcal{I}_X/\mathcal{U}_X^t\Big).$$

Consider the map

$$R : \mathcal{I}_X \longrightarrow \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$$

which is given by the map $r : Z_0(X) \to \pi_1(X)^{\mathrm{ab}}$ defined above and the reciprocity maps of local class field theory

$$\rho_v : k(C)_v^\times \longrightarrow \pi_1^{\mathrm{et}}(\mathrm{Spec}(k(C)_v))^{\mathrm{ab}}$$

[1] The topology of a finite direct sum is just the product topology, and the topology of an infinite direct sum is the direct limit topology of the finite partial sums.

followed by the natural maps $\pi_1^{\mathrm{et}}(\mathrm{Spec}(k(C)_v))^{\mathrm{ab}} \to \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}}$ for all $C \subset X$, $v \in C_\infty$. By [Wiesend 2007, Theorem 1 (a)], the homomorphism R induces an isomorphism

$$\rho : \mathscr{C}_X^t \xrightarrow{\sim} \widetilde{\pi}_1^t(X)^{\mathrm{ab}}.$$

Now we consider the obvious map

$$\phi : Z_0(X) \longrightarrow \mathscr{C}_X^t.$$

The kernel of ϕ is the subgroup in $Z_0(X)$ generated by elements of the form $\mathrm{div}(f)$ where $C \subset X$ is a closed curve and f is an invertible rational function on C which is in $U^1(k(C)_v)$ for all $v \in C_\infty$. By Theorem 5.1 we obtain $\ker(\phi) = d_1(C_1(X))$. Therefore ϕ induces an injective homomorphism

$$i : h_0(X) \hookrightarrow \mathscr{C}_X^t$$

with $\rho \circ i = \mathrm{rec}$. As ρ is injective, rec is injective, and hence an isomorphism. □

Finally, assume that $\bar{X}$ is regular, flat and proper over $\mathrm{Spec}(\mathbb{Z})$, let $D \subset X$ be a divisor and $X = \bar{X} - D$. In [Schmidt 2005] we introduced the relative Chow group of zero cycles $\mathrm{CH}_0(\bar{X}, D)$ and constructed, under a mild technical assumption, a reciprocity isomorphism $\mathrm{rec}' : \mathrm{CH}_0(\bar{X}, D) \xrightarrow{\sim} \widetilde{\pi}_1^t(X)^{\mathrm{ab}}$. By [Schmidt 2005, Proposition 2.4], there exists natural projection $\pi : h_0(X) \twoheadrightarrow \mathrm{CH}_0(\bar{X}, D)$ with $\mathrm{rec} = \mathrm{rec}' \circ \pi$. We obtain the

Theorem 8.6. *Let $\bar{X}$ be a regular, connected scheme, flat and proper over* $\mathrm{Spec}(\mathbb{Z})$, *such that its generic fibre $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is projective over $\mathbb{Q}$. Let D be a divisor on $\bar{X}$ whose vertical irreducible components are normal schemes. Put $X = \bar{X} - D$. Then the natural homomorphism*

$$h_0(X) \longrightarrow \mathrm{CH}_0(\bar{X}, D)$$

is an isomorphism of finite abelian groups.

Finally, we deal with the geometric case. The next theorem was proved in 1999 by M. Spieß and the author under the assumption that X has a smooth projective compactification; see [Schmidt and Spieß 2000]. Now we get rid of this assumption.

Theorem 8.7. *Let X be a smooth, connected variety over a finite field $\mathbb{F}$. Then the reciprocity homomorphism*

$$\mathrm{rec}_X : h_0(X) \longrightarrow \pi_1^t(X)^{\mathrm{ab}}$$

is injective. The image of rec_X *consists of all elements whose degree in* $\mathrm{Gal}(\bar{\mathbb{F}}|\mathbb{F})$ *is an integral power of the Frobenius automorphism. In particular, the cokernel*

$\operatorname{coker}(\mathrm{rec}_X) \cong \widehat{\mathbb{Z}}/\mathbb{Z}$ *is uniquely divisible. The induced map on the degree-zero parts* $\mathrm{rec}_X^0 : h_0(X)^0 \xrightarrow{\sim} (\pi_1^t(X)^{\mathrm{ab}})^0$ *is an isomorphism of finite abelian groups.*

Proof. The proof is strictly parallel to the proof of Theorem 8.4, using Theorem 5.1 and the tame version of Wiesend's class field theory for smooth varieties over finite fields [Wiesend 2007, Theorem 1 (b)]. □

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