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In 1956, Brauer showed that there is a partitioning of the p-regular conjugacy classes of a group according to the p-blocks of its irreducible characters with close connections to the block theoretical invariants. But an explicit block splitting of regular classes has not been given so far for any family of finite groups. Here, this is now done for the 2-regular classes of the symmetric groups. To prove the result, a detour along the double covers of the symmetric groups is taken, and results on their 2-blocks and the 2-powers in the spin character values are exploited. Surprisingly, it also turns out that for the symmetric groups the 2-block splitting is unique.

1. Introduction

A half-century ago, Richard Brauer [1956] introduced the idea of not only distributing characters into p-blocks but also to associate p-regular conjugacy classes to p-blocks. He showed that it is possible to distribute the p-regular classes in such a way into blocks that it fits with the blocks of irreducible Brauer characters (and suitable subsets of ordinary irreducible characters in the blocks); this is to say that the determinant of the corresponding block part of the Brauer character table (or a suitable part of the ordinary character table) is not congruent to 0 modulo $\mathfrak{p}$ (a prime ideal over p). Given such a splitting of p-regular classes into blocks, Brauer showed that the elementary divisors of the Cartan matrix of a block are then exactly the p-parts in the orders of the centralizers of elements in the classes corresponding to the block. But while it is known how to determine the p-blocks of irreducible characters, for the p-regular classes only the existence of such a block splitting is known by Brauer's work — concrete examples for providing such a distribution for families of groups have not been known so far. Brauer also observed that in general there may be several such block splittings, and there did not seem to be any natural choice for a given finite group.

In the present paper, such an explicit block splitting in the sense of Brauer is exhibited for the conjugacy classes of odd order elements and the 2-blocks of the

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symmetric groups; in fact, it turns out that for these groups this is the unique 2-block splitting of the 2-regular classes. Surprisingly, the strategy employed here takes a detour along the double covers of the symmetric groups and exploits results on the 2-powers in the spin character values of these groups. Also our knowledge on the 2-block distribution of the spin characters is an important ingredient.

Here is a brief outline of the sections. In Section 2, some notation and results for the symmetric groups and its representation theory are collected, and we recall Brauer's results on block splittings for arbitrary finite groups. As already mentioned above, we will not only work in the context of characters of the symmetric group S_n, but we want to use results on the spin characters of the double cover groups $\tilde{S}_n$. For this, we have to introduce further combinatorial notions in Section 3, and in particular we recall the Glaisher bijection between partitions into odd parts and partitions into distinct parts which plays a crucial rôle here; we also review a number of results on spin characters, mostly of the last decade, which will be used in the proof of our main result. In preparation for the application in Section 4, also a new result on spin character values is proved in this section (Theorem 3.4). In the final section, the class labels for the 2-block splitting of S_n are defined; for a 2-block B of S_n we take the 2-regular classes labelled by partitions into odd parts whose Glaisher image has a $\bar{4}$-core corresponding to the 2-block $\tilde{B}$ of $\tilde{S}_n$ containing B (see Definition 4.1). In the main Theorem 4.2 properties of the determinants of the corresponding block character tables are proved which imply that the construction gives indeed a block splitting of the classes; in fact, the proof allows to refine the result on the determinants further to a result on the Smith normal forms given in Theorem 4.3. An analysis of the proof of the main Theorem shows that the information from Section 3 on spin character values exploited there may also be applied to prove uniqueness of our splitting system.

2. Preliminaries

We have to introduce some notation. For the symmetric groups S_n, the corresponding combinatorial notions and their representation theory, we will follow mostly the usual notation in [James and Kerber 1981]; for the double cover groups $\tilde{S}_n$ and the corresponding background we refer the reader to [Hoffman and Humphreys 1992] and [Morris 1962].

Let $n \in \mathbb{N}$. For a partition λ of n, the number of its (nonzero) parts is called its *length* and is denoted by $l(\lambda)$. The complex irreducible character of S_n corresponding to λ is denoted by $[\lambda]$. Given a second partition μ of n,

$$[\lambda](\sigma_\mu)$$

is then the character value on an element σ_μ in S_n of cycle type μ.

Let $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \dots)$ be a partition, written in exponential notation; then we set

$$a_\mu = \prod_{i\geq 1} i^{m_i(\mu)},\ b_\mu = \prod_{i\geq 1} m_i(\mu)!$$

We let z_μ denote the order of the centralizer of an element of cycle type μ in S_n; then $z_\mu = a_\mu b_\mu$.

Let p be a prime. Then a partition is called p-regular if no part is repeated p or more times, and a partition is called p-class regular if no part is divisible by p.

We let $\mathcal{D}(n)$ denote the set of partitions of n into distinct parts; these partitions are thus the 2-regular partitions of n and they are also called bar partitions in connection with the theory of the double cover groups. We let $\mathcal{O}(n)$ denote the set of partitions of n into odd parts; these are thus the 2-class regular partitions of n.

We then define the 2-regular character table of the symmetric group S_n to be

$$X_2 = ([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}(n)\\ \alpha\in\mathcal{O}(n)}}$$

where the partitions are ordered in a suitable way. More generally, the p-regular character table is defined with λ running through the p-regular partitions and α running through the p-class regular partitions. Its determinant has been studied by Olsson, who showed in [2003, Theorem 2] that its absolute value equals the product of all parts of all p-class regular partitions. Hence, $|\det(X_2)| = \prod_{\mu\in\mathcal{O}(n)} a_\mu$, and in particular it is thus known that

$$2 \nmid \det([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}(n)\\ \alpha\in\mathcal{O}(n)}}.$$

Our main result below will provide a block version of this property, by distributing not only the characters but also the 2-regular conjugacy classes into blocks in a suitable way.

This block distribution of conjugacy classes gives a block splitting in the sense of Brauer; we first introduce the general context.

Let G be a finite group, p a prime. Let $\ell(G)$ be the cardinality of the set $\mathrm{Cl}_{p'}(G)$ of p-regular conjugacy classes in G. For each $K \in \mathrm{Cl}_{p'}(G)$ we let x_K denote an element in K. A defect group of K is a Sylow p-subgroup of $C_G(x)$ for some $x \in K$; if this has order p^d, then d is called the p-defect of K. We let $\mathrm{IBr}(G)$ denote the set of modular irreducible characters of G; then

$$\Phi_G = (\varphi(x_K))_{\substack{\varphi\in \mathrm{IBr}(G)\\ K\in\mathrm{Cl}_{p'}(G)}}$$

is the Brauer character table of G. It is well known that the Brauer character table is nonsingular modulo p; that is,

$$\det \Phi_G \not\equiv 0 \pmod{\mathfrak{p}}.$$

Further, we let

$$D = (d_{\chi\,\varphi})_{\substack{\chi \in \operatorname{Irr}(G)\\ \varphi \in \operatorname{IBr}(G)}}$$

denote the p-decomposition matrix for G, and we let $C = D^t D$ denote its Cartan matrix. Let $\operatorname{Bl}_p(G)$ be the set of p-blocks of G. For $B \in \operatorname{Bl}(G)$, $\operatorname{Irr}(B)$ is the set of ordinary irreducible characters in B, $\operatorname{IBr}(B)$ is the set of modular irreducible characters in B, $\ell(B) = |\operatorname{IBr}(B)|$,

$$D(B) = (d_{\chi,\varphi})_{\substack{\chi \in \operatorname{Irr}(B)\\ \varphi \in \operatorname{IBr}(B)}}$$

denotes the p-decomposition matrix for B, and $C(B)$ is the Cartan matrix for B.

Then C and D are the block direct sums of the matrices $C(B)$ and $D(B)$, for $B \in \operatorname{Bl}_p(G)$.

Theorem 2.1 [Brauer 1956, §5]. *There exists a disjoint decomposition of* $\operatorname{Cl}_{p'}(G)$ *into blocks of p-regular conjugacy classes*

$$\operatorname{Cl}_{p'}(G) = \bigcup_{B \in \operatorname{Bl}_p(G)} \operatorname{Cl}_{p'}(B)$$

and a selection of characters $\operatorname{Irr}'(B) \subseteq \operatorname{Irr}(B)$ *for each p-block B of G such that the following conditions are fulfilled*:

(1) $|\operatorname{Cl}_{p'}(B)| = |\operatorname{Irr}'(B)| = \ell(B)$ *for all* $B \in \operatorname{Bl}_p(G)$.

(2) *For* $X_B = (\chi(x_K))_{\substack{\chi \in \operatorname{Irr}'(B)\\ K \in \operatorname{Cl}_{p'}(B)}}$, *we have* $\det X_B \not\equiv 0 \pmod{\mathfrak{p}}$.

(3) *For* $\Phi_B = (\varphi(x_K))_{\substack{\varphi \in \operatorname{IBr}(B)\\ K \in \operatorname{Cl}_{p'}(B)}}$, *we have* $\det \Phi_B \not\equiv 0 \pmod{\mathfrak{p}}$.

(4) *For* $D_B = (d_{\chi\varphi})_{\substack{\chi \in \operatorname{Irr}'(B)\\ \varphi \in \operatorname{IBr}(B)}}$, *we have* $\det D_B \not\equiv 0 \pmod{\mathfrak{p}}$.

Furthermore, the elementary divisors of the Cartan matrix $C(B)$ are then exactly the orders of the p-defect groups of the conjugacy classes in $\operatorname{Cl}_{p'}(B)$, *for all B in* $\operatorname{Bl}_p(G)$.

Note that the properties in (2), (3) and (4) are not independent of each other, as $X_B = D_B \Phi_B$. In particular, if we have a suitable choice $\operatorname{Irr}'(B)$ of characters that satisfies (4), and a suitable choice of classes that satisfies (3), then these together are a suitable choice for (2). If we have a basic set of irreducible characters, i.e., a subset $\operatorname{Irr}'(G) \subseteq \operatorname{Irr}(G)$ giving a $\mathbb{Z}$-basis for the character restrictions to the p-regular classes, then the p-block decomposition of this set will give a suitable choice of sets $\operatorname{Irr}'(B)$ satisfying (4).

We now turn to the symmetric groups. In this case, the so-called Nakayama Conjecture (proved by Brauer and Robinson) gives a combinatorial description for the block distribution of characters. If λ is a partition of n and p a prime, we remove rim hooks of length p from the Young diagram of λ as often as possible; this results in a unique partition $\lambda_{(p)}$ which has no rim hook of length p, called the p-core of λ . The number of rim hooks removed from λ on the way to $\lambda_{(p)}$ is called the p-weight of λ. We refer the reader to [James and Kerber 1981] for more details on this and the following.

"Nakayama Conjecture". Two irreducible characters $[\lambda]$, $[\mu]$ of S_n belong to the same p-block if and only if $\lambda_{(p)} = \mu_{(p)}$.

Hence each p-block B has a well-defined p-weight $w(B)$ and p-core $\kappa(B)$, namely the common p-weight and p-core of all the labels of the irreducible characters in B. Note that then $|\lambda| = pw(B) + |\kappa(B)|$, for all $[\lambda] \in \mathrm{Irr}(B)$.

The situation at $p = 2$ is particularly nice, as we may then easily describe all the 2-core partitions: these are exactly the staircase partitions $\rho_k = (k, k-1, \dots, 2, 1)$, $k \in \mathbb{N}_0$. The removal of a rim hook of length 2 from a partition is just the removal of a "domino piece" from the rim of its Young diagram.

The irreducible characters labelled by the p-regular partitions form a basic set [James and Kerber 1981; Külshammer et al. 2003]; thus with respect to a suitable ordering, the determinant of the corresponding part of the decomposition matrix is 1. We take the corresponding choice $\mathrm{Irr}'(B) \subseteq \mathrm{Irr}(B)$ of characters for Brauer's Theorem in our situation at $p = 2$. This means the following. Let $\mathrm{Bl}_2(n)$ be the set of 2-blocks of S_n. For a given 2-block B we set

$$\mathcal{D}(B) := \{\lambda \in \mathcal{D}(n) \mid [\lambda] \in \mathrm{Irr}(B)\} = \{\lambda \in \mathcal{D}(n) \mid \lambda_{(2)} = \kappa(B)\}.$$

This gives a set partition according to the 2-blocks:

$$\mathcal{D}(n) = \bigcup_{B \in \mathrm{Bl}_2(n)} \mathcal{D}(B).$$

Then $|\mathcal{D}(B)|$ equals $p(w(B))$, the number of partitions of $w(B)$; see [James and Kerber 1981] or [Olsson 1993]. In the notation of Theorem 2.1 we then take $\mathrm{Irr}'(B) = \{[\lambda] \mid \lambda \in \mathcal{D}(B)\}$.

By Brauer's Theorem there must exist a suitable block splitting of the 2-regular conjugacy classes; i.e., there must be a set partition

$$\mathcal{O}(n) = \bigcup_{B \in \mathrm{Bl}_2(n)} \mathcal{O}(B)$$

such that for all $B \in \mathrm{Bl}_2(n)$ we have

$$2 \nmid \det(\varphi^{\lambda}(\sigma_{\mu}))_{\substack{\lambda \in \mathcal{D}(B) \\ \mu \in \mathcal{O}(B)}}, \tag{1}$$

where for $\mu \in \mathcal{D}(n)$ we denote by φ^{μ} the corresponding Brauer character of S_n; note that φ^{μ} belongs to the 2-block B exactly if $\mu_{(2)} = \kappa(B)$. By the remarks above this condition is equivalent to having

$$2 \nmid \det([\lambda](\sigma_{\mu}))_{\substack{\lambda \in \mathcal{D}(B) \\ \mu \in \mathcal{O}(B)}}. \tag{2}$$

As noted above, for any such block splitting, the elementary divisors of the Cartan matrix of B are then the defect group orders of the conjugacy classes labelled by $\mathcal{O}(B)$.

The aim of this article is to define explicit subsets $\mathcal{O}(B)$ of $\mathcal{O}(n)$ satisfying the equivalent conditions (1) and (2), thus giving a 2-block splitting of conjugacy classes for the symmetric groups.

3. Spin characters

We collect here a number of results on spin characters that will be needed in the sequel; the reader is referred to [Hoffman and Humphreys 1992] and [Olsson 1993] for more background on the double cover groups $\tilde{S}_n$ and their representation theory.

The sets $\mathcal{D}^+(n)$ and $\mathcal{D}^-(n)$ are the subsets of partitions $\lambda \in \mathcal{D}(n)$ with $n - l(\lambda)$ even or odd, respectively. For $\mu \in \mathcal{D}^+(n)$, we denote by $\langle\mu\rangle$ the corresponding complex irreducible spin character of $\tilde{S}_n$, for $\mu \in \mathcal{D}^-(n)$, we let $\langle\mu\rangle$ and $\langle\mu\rangle' = \mathrm{sgn}\cdot\langle\mu\rangle$ be the corresponding pair of associate complex irreducible spin characters of $\tilde{S}_n$. We recall that the only conjugacy classes of S_n that split over the double cover groups are those of type $\mathcal{O}$ and of type $\mathcal{D}^-$; the irreducible spin characters vanish on all other conjugacy classes. More precisely, for any such partition α one of the two corresponding conjugacy classes in $\tilde{S}_n$ is chosen in accordance with [Schur 1911], and we denote a corresponding representative by $\tilde{\sigma}_{\alpha}$. While the spin character values on the $\mathcal{D}^-$ classes are known explicitly (but they are in general not integers, and mostly not even real), for the values on the $\mathcal{O}$-classes we have a recursion formula due to A. Morris which is analogous to the Murnaghan–Nakayama formula (and which shows in particular, that these are integers).

In contrast to odd characteristic, the 2-blocks of $\tilde{S}_n$ are mixed, i.e., they contain ordinary as well as spin characters. The simple $\tilde{S}_n$-modules in characteristic 2 may be identified with the simple S_n-modules D^{λ} which are labelled by partitions $\lambda \in \mathcal{D}(n)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{D}(n)$ we set

$$\mathrm{dbl}(\lambda) = \left(\left[\frac{\lambda_1+1}{2}\right], \left[\frac{\lambda_1}{2}\right], \left[\frac{\lambda_2+1}{2}\right], \left[\frac{\lambda_2}{2}\right], \ldots, \left[\frac{\lambda_m+1}{2}\right], \left[\frac{\lambda_m}{2}\right]\right),$$

the *doubling* of λ. For example, the staircase partition $\rho_k = (k, k-1, \dots, 2, 1)$ is the doubling of the partition $\tau_k = (2k-1, 2k-5, \dots)$.

The 2-block distribution of the spin characters is described by the following result (which confirmed a conjecture by Knörr and Olsson):

Theorem 3.1 [Bessenrodt and Olsson 1997]. *Let $\lambda \in \mathcal{D}(n)$. Then $\langle\lambda\rangle$ and $[\mathrm{dbl}(\lambda)]$ belong to the same 2-block of $\tilde{S}_n$.*

Thus, the 2-block of $\langle\lambda\rangle$ is determined by the 2-core of $\mathrm{dbl}(\lambda)$. But in fact, the spin combinatorics in this case may also be viewed as a $\bar{4}$-combinatorics (see [Bessenrodt and Olsson 1997] for more details). Indeed, we have a $\bar{4}$-abacus for the bar partitions with one runner for all even parts (the 0-th runner), on which we can slide by steps of 2, and two conjugate runners for the residues 1 and 3 modulo 4. A bar partition is then a $\bar{4}$-core exactly if the 0-th runner is empty (i.e., there are no even parts), at most one of the two conjugate runners is nonempty, and a nonempty runner has only beads at the top; thus the $\bar{4}$-cores are the partitions τ_k defined above. We will denote the $\bar{4}$-core of a bar partition λ by $\lambda_{(\bar{4})}$.

It is well known that $|\mathcal{D}(n)| = |\mathcal{O}(n)|$. In fact, J. W. L. Glaisher [1883] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. In this case, Glaisher's map G is defined as follows. Suppose that $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2, i.e., in its 2-adic decomposition: $m_i = \sum_j 2^{a_{ij}}$. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, sorted in descending order to give a partition. Surprisingly, this map has turned up naturally in connection with spin characters of the symmetric groups (see below).

For any integer $m \geq 0$, let $s(m)$ be the number of summands in the 2-adic decomposition of m. Then for $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ the length of $G(\alpha)$ is $l(G(\alpha)) = \sum_{i \text{ odd}} s(m_i)$. We define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i))$$

and set $\sigma(\alpha) = (-1)^{k_\alpha}$; note that we thus have

$$k_\alpha = l(\alpha) - l(G(\alpha)).$$

We denote by $\mathcal{O}^\varepsilon(n)$ the set of partitions α in $\mathcal{O}(n)$ with the sign of $\sigma(\alpha)$ being $\varepsilon \in \{\pm\}$. With this definition of signs, it is easy to see that the Glaisher map G induces bijections $\mathcal{O}^\epsilon(n) \to \mathcal{D}^\epsilon(n)$; see [Bessenrodt and Olsson 2000].

The integer k_α also comes up naturally in the group-theoretic context. For any nonzero integer m, we denote by $\nu(m)$ the exponent to which 2 divides m; $2^{\nu(m)}$ is the exact 2-power dividing m. Let $\alpha = (1^{m_1(\alpha)}3^{m_3(\alpha)}, \dots) \in \mathcal{O}(n)$, σ_α an element of cycle type α in S_n. Then $\nu(|C_{S_n}(\sigma_\alpha)|) = \prod_{i \text{ odd}} \nu(m_i(\alpha)!) = k_\alpha$. Hence k_α is the 2-defect of K_α, the conjugacy class of S_n labelled by $\alpha \in \mathcal{O}(n)$.

In joint work with J. Olsson, we have previously investigated the 2-powers appearing in the spin character values on a given 2-regular conjugacy class:

Theorem 3.2. *Let* $\alpha \in \mathcal{O}(n)$.

(i) [Bessenrodt and Olsson 2000] *For all* $\lambda \in \mathcal{D}(n)$ *we have*

$$\nu(\langle\lambda\rangle(\tilde{\sigma}_\alpha)) \geq \lfloor k_\alpha/2 \rfloor .$$

(ii) [Bessenrodt and Olsson 2005] *Let* $G(\alpha) \in \mathcal{D}(n)$ *be the Glaisher image of* α. *Then*

$$\nu(\langle G(\alpha)\rangle(\tilde{\sigma}_\alpha)) = \lfloor k_\alpha/2 \rfloor ,$$

and if $\alpha \in \mathcal{O}^-(n)$, *then* $\langle G(\alpha)\rangle$ *and* $\langle G(\alpha)\rangle'$ *are the only spin characters where this equality holds.*

In the case of partitions $\alpha \in \mathcal{O}^+(n)$, we may have spin characters different from $\langle G(\alpha)\rangle$ such that the minimal 2-power is attained on $\tilde{\sigma}_\alpha$. At least we can have nonselfassociate spin characters with this property, but it is not yet clear whether there are also self-associate spin characters satisfying this; see [Bessenrodt and Olsson 2005]. For our later purposes the weaker statement in Theorem 3.4 below suffices. For proving this result, we first have to recall some results due to Stembridge.

Stembridge [1989] has investigated a projective analogue of the outer tensor product, called the reduced Clifford product, and has proved a shifted analogue of the Littlewood–Richardson rule which we will need in the sequel. To state this, we first have to define some further combinatorial notions.

Let A' be the ordered alphabet $\{1' < 1 < 2' < 2 < ...\}$. The letters $1', 2', \dots$ are said to be *marked*, the others are *unmarked*. The notation $|a|$ refers to the unmarked version of a letter a in A'. To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq l(\lambda), i \leq j \leq \lambda_i + i - 1\}$$

A shifted tableau T of shape λ is a map $T : Y'(\lambda) \to A'$ such that $T(i, j) \leq T(i+1, j)$, $T(i, j) \leq T(i, j+1)$ for all i, j, and every $k \in \{1, 2, \dots\}$ appears at most once in each column of T, and every $k' \in \{1', 2', \dots\}$ appears at most once in each row of T. For $k \in \{1, 2, \dots\}$, let c_k be the number of boxes (i, j) in $Y'(\lambda)$ such that $|T(i, j)| = k$. Then we say that the tableau T has content $(c_1, c_2, \dots)$. Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape $\lambda \setminus \mu$ if μ is a partition with $Y'(\mu) \subseteq Y'(\lambda)$. For a (possibly skew) shifted tableau S we define its associated word $w(S) = w_1 w_2 \cdots$ by reading the rows of

S from left to right and from bottom to top. By erasing the marks of w, we obtain the word $|w|$.

Given a word $w = w_1 w_2 \ldots$, we define

$$m_i(j) = \text{multiplicity of } i \text{ among } w_{n-j+1}, \ldots, w_n \qquad \text{for } 0 \le j \le n,$$
$$m_i(n+j) = m_i(n) + \text{ multiplicity of } i' \text{ among } w_1, \ldots, w_j \qquad \text{for } 0 < j \le n.$$

This function m_i corresponds to reading the rows of the tableau first from right to left and from top to bottom, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.

The word w satisfies the lattice property if, whenever $m_i(j) = m_{i-1}(j)$, we have

$$w_{n-j} \ne i, i' \qquad \text{if } 0 \le j < n,$$
$$w_{j-n+1} \ne i-1, i' \qquad \text{if } n \le j < 2n.$$

For two partitions μ and ν we denote by $\mu \cup \nu$ the partition which has as its parts all the parts of μ and ν together. Also define

$$\varepsilon_\lambda = \begin{cases} 1 & \text{if } \lambda \in \mathcal{D}^+(n) \\ \sqrt{2} & \text{if } \lambda \in \mathcal{D}^-(n) \end{cases}.$$

We can then state the spin version of the Littlewood–Richardson rule:

Theorem 3.3 [Stembridge 1989, 8.1 and 8.3]. *Let $\mu \in \mathcal{D}(k)$, $\nu \in \mathcal{D}(n-k)$, $\lambda \in \mathcal{D}(n)$, and form the reduced Clifford product $\langle \mu \rangle \times_c \langle \nu \rangle$. Then we have*

$$((\langle \mu \rangle \times_c \langle \nu \rangle) \uparrow^{\tilde{S}_n}, \langle \lambda \rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu \cup \nu}} 2^{(l(\mu)+l(\nu)-l(\lambda))/2} f^\lambda_{\mu\nu},$$

unless λ is odd and $\lambda = \mu \cup \nu$. In that latter case, the multiplicity of $\langle \lambda \rangle$ is 0 or 1, according to the choice of associates.

The coefficient $f^\lambda_{\mu\nu}$ is the number of shifted tableaux S of shape $\lambda \setminus \mu$ and content ν such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \le i \le l(\nu)$.

For further properties of the reduced Clifford product, see [Humphreys 1986; Michler and Olsson 1990; Schur 1911; Stembridge 1989].

Theorem 3.4. *Let $\alpha \in \mathcal{O}^+(n)$, $\lambda \in \mathcal{D}^+(n)$. If $\nu(\langle \lambda \rangle(\tilde{\sigma}_\alpha)) = \lfloor k_\alpha/2 \rfloor$, then $\lambda \trianglerighteq G(\alpha)$. In particular, $G(\alpha)$ is the minimal $\mathcal{D}^+$-partition in lexicographic order where this equality is attained.*

Proof. We recall parts of the proof of [Bessenrodt and Olsson 2005, Theorem 1.2]. Let $\alpha = (i^{m_i})_{i=1,3,\ldots}$, set $\alpha^i = (i^{m_i})$, $a_i = i m_i$, and let $\tilde{S}_a$ be the preimage of the Young subgroup $S_{a_1} \times S_{a_3} \times \ldots$ in $\tilde{S}_n$. Restricting $\langle \lambda \rangle$ to $\tilde{S}_a$ gives

$$\langle \lambda \rangle_{\tilde{S}_a} = \sum_{\mu = (\mu_1, \mu_3, \ldots)} g^\lambda_\mu (\times_c \langle \mu_i \rangle) + \sum_{\mu = (\mu_1, \mu_3, \ldots) \text{n.s.a.}} \tilde{g}^\lambda_\mu (\times_c \langle \mu_i \rangle)',$$

where the g_μ^λ are spin Littlewood–Richardson coefficients, and $\mu = (\mu_1, \mu_3, \dots)$ runs over all sequences with μ_i a partition of a_i. Moreover, μ is nonselfassociate (n.s.a.) if the corresponding reduced Clifford product is nonselfassociate; this is the case if and only if $t_\mu = |\{i \mid \mu_i \in \mathcal{D}^-\}|$ is odd. As we assume that $\lambda \in \mathcal{D}^+$, by [Stembridge 1989] we have $g_\mu^\lambda = \tilde{g}_\mu^\lambda$ for any n.s.a. μ. Thus

$$\langle\lambda\rangle(\tilde{\sigma}_\alpha) = \sum_{\mu=(\mu_1,\mu_3,\dots)\text{ s.a.}} g_\mu^\lambda(\times_c\langle\mu_i\rangle)(\tilde{\sigma}_\alpha) \;+ \sum_{\mu=(\mu_1,\mu_3,\dots)\text{ n.s.a.}} 2g_\mu^\lambda(\times_c\langle\mu_i\rangle)(\tilde{\sigma}_\alpha).$$

By [Bessenrodt and Olsson 2005, Proposition 3.3], the 2-value of each Clifford product value is at least $[k_\alpha/2]$; hence we obtain for the n.s.a. μ a contribution of nonminimal 2-value. The same proposition implies that, since $\alpha \in \mathcal{O}^+$, the only Clifford product value which is of 2-value $[k_\alpha/2]$ occurs for the partition sequence $\mu = g(\alpha) = (G(\alpha^1), G(\alpha^3), \dots)$, and thus $g_{g(\alpha)}^\lambda$ has to be odd. In particular, $\langle\lambda\rangle$ is a constituent of $\times_c\langle G(\alpha^i)\rangle\uparrow^{\tilde{S}_n}$. By the spin Littlewood–Richardson rule due to Stembridge, $\langle G(\alpha)\rangle$ is the lowest constituent in this induced character (with respect to dominance, and thus also in lexicographic order). We have already seen before that for this character we have indeed equality on the conjugacy class to α. $\square$

We want to go beyond determinants and study the Smith normal forms of the matrices under consideration. For any integral square matrix X we let $\mathcal{S}(X)$ denote its Smith normal form, i.e., the diagonal matrix with the elementary divisors of X as diagonal elements. The following property of the Smith normal form will be used: If X and Y are square $n \times n$ matrices with relatively prime determinants, then $\mathcal{S}(XY) = \mathcal{S}(X)\,\mathcal{S}(Y)$; see [Newman 1972, Theorem II.15], for instance. For a finite family of numbers c_i, $i \in I$, we mean by $\mathcal{S}(c_i, i \in I)$ the Smith normal form of any diagonal matrix with the given numbers on the diagonal.

We define the reduced spin character table of $\tilde{S}_n$ as the integral square matrix

$$Z_s = (\langle\lambda\rangle(\tilde{\sigma}_\mu))_{\substack{\lambda\in\mathcal{D}(n)\\ \mu\in\mathcal{O}(n)}}.$$

Then we have

Theorem 3.5 [Bessenrodt et al. 2005, Theorem 13]. *The Smith normal form of the reduced spin character table Z_s of $\tilde{S}_n$ is given by*

$$\mathcal{S}(Z_s) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \mathcal{O}(n)) \cdot \mathcal{S}(b_\mu, \mu \in \mathcal{O}(n))_{2'}.$$

In the context of 2-modular representations, we consider the part of the 2-decomposition matrix for $\tilde{S}_n$ corresponding to spin characters. Since the rows corresponding to associate spin characters are equal, this part of the decomposition matrix is determined by the submatrix $D_s = D_s(n)$, where for each $\lambda \in \mathcal{D}(n)$ we keep only one row for each associate class of spin characters. We call D_s the reduced spin 2-decomposition matrix; it is a square matrix of the same size as Z_s.

Theorem 3.6 [Bessenrodt and Olsson 2000]. *Let $\tilde{B} \in \mathrm{Bl}_2(\tilde{S}_n)$, $B \in \mathrm{Bl}_2(S_n)$, $B \subseteq \tilde{B}$. Suppose that $2^{c_1}, 2^{c_2}, \dots, 2^{c_\ell}$ are the elementary divisors of the Cartan matrix $C(B)$. Then the elementary divisors of $D_s(\tilde{B})$ are $2^{[c_1/2]}, 2^{[c_2/2]}, \dots, 2^{[c_\ell/2]}$.*

Now the invariants of the Cartan matrix had been explicitly determined by Olsson (see [Bessenrodt and Olsson 2000] for the correction of the formula misstated in [Olsson 1986]). For $p = 2$, this formula was recast in a nicer combinatorial way by Uno and Yamada; we reformulate this here for our purposes.

Theorem 3.7 [Uno and Yamada 2006]. *Let B be a 2-block of S_n with 2-core $\rho_k = (k, k-1, \dots, 2, 1)$, and let $\tau_k = (2k-1, 2k-5, \dots)$. Then the elementary divisors of the Cartan matrix $C(B)$ are given by*

$$2^{l(\alpha)-l(G(\alpha))},\ \alpha \in \mathcal{O}(n),\ G(\alpha)_{(\bar{4})} = \tau_k.$$

As $k_\alpha = l(\alpha) - l(G(\alpha))$ for any $\alpha \in \mathcal{O}(n)$, we thus conclude

Corollary 3.8. *Let $\tilde{B} \in \mathrm{Bl}_2(\tilde{S}_n)$, $B \in \mathrm{Bl}_2(S_n)$, $B \subseteq \tilde{B}$, τ_k as above. Then*

$$\mathcal{S}(D_s(\tilde{B})) = \mathcal{S}\big(2^{[k_\alpha/2]},\ \alpha \in \mathcal{O}(n),\ G(\alpha)_{(\bar{4})} = \tau_k\big).$$

We observe also that by Brauer's Theorem 2.1, the defect group orders of the classes associated to B in a block splitting thus have to be the numbers 2^{k_α}, $\alpha \in \mathcal{O}(n)$, $\alpha_{(\bar{4})} = \tau_B$. We take this as a hint on how to choose the distribution of the 2-regular conjugacy classes into blocks in the next section.

4. The 2-block splitting for S_n

We fix the following notation.

Let B be a 2-block of S_n, with 2-core $\rho_k = (k, k-1, \dots, 2, 1)$, $k \in \mathbb{N}_0$. Let $\tilde{B}$ be the 2-block of $\tilde{S}_n$ containing B, with corresponding $\bar{4}$-core $\tau_k = (2k-1, 2k-5, \dots)$. As before, we let

$$\mathcal{D}(B) = \{\mu \in \mathcal{D}(n) \mid \mu_{(2)} = \rho_k\}$$

and we set

$$\mathcal{D}(\tilde{B}) = \{\lambda \in \mathcal{D}(n) \mid \lambda_{(\bar{4})} = \tau_k\}.$$

An important point to note here is that these sets of partitions really fit to the 2-block inclusion $B \subseteq \tilde{B}$, as the corresponding characters $[\mu]$, $\mu \in \mathcal{D}(B)$, and $\langle\lambda\rangle$, $\lambda \in \mathcal{D}(\tilde{B})$, belong to the same 2-block $\tilde{B}$ of $\tilde{S}_n$ by Theorem 3.1.

Let $w = w(B)$ be the 2-weight of B. Then

$$|\mathcal{D}(B)| = |\mathcal{D}(\tilde{B})| = p(w);$$

see [Bessenrodt and Olsson 1997] or [Olsson 1993], for example. With this notation, we can now introduce the crucial definition that will provide the 2-block splitting of the 2-regular classes:

Definition 4.1. With $G : \mathcal{O}(n) \to \mathcal{D}(n)$ still denoting the Glaisher map defined in Section 3, we set

$$\mathcal{O}(B) := \{\alpha \in \mathcal{O}(n) \mid G(\alpha)_{(\bar{4})} = \tau_k\} =: \mathcal{O}(\tilde{B}).$$

Thus by definition the Glaisher map restricts to blockwise bijections

$$G : \mathcal{O}(B) \to \mathcal{D}(\tilde{B}).$$

In particular, we thus have $|\mathcal{O}(B)| = \ell(B)$, so the first condition of a block splitting is satisfied for these labelling sets.

We now consider the following parts of the character table and spin character table, respectively, which are all square matrices by the observations made above (note that the spin character table is reduced in the sense that we take only one of a pair of associate spin characters):

$$Z(B) = ([\mu](\sigma_\alpha))_{\substack{\mu\in\mathcal{D}(B)\\ \alpha\in\mathcal{O}(B)}}, \qquad Z_s(\tilde{B}) = (\langle\lambda\rangle(\tilde{\sigma}_\alpha))_{\substack{\lambda\in\mathcal{D}(\tilde{B})\\ \alpha\in\mathcal{O}(B)}}.$$

We also consider the corresponding block part of the Brauer character table:

$$\Phi(B) = (\varphi^\mu(\sigma_\alpha))_{\substack{\mu\in\mathcal{D}(B)\\ \alpha\in\mathcal{O}(B)}}.$$

Finally we define a diagonal matrix associated to B by

$$\Delta(B) = \Delta(2^{[k_\alpha/2]}, \alpha \in \mathcal{O}(B)).$$

After all these preparations, we can now state the following result on the determinants of the matrices defined above, which tells us that the chosen distribution of conjugacy classes given by the sets $\mathcal{O}(B)$ is indeed a 2-block splitting of the 2-regular classes:

Theorem 4.2. *Let $B \subseteq \tilde{B}$ be as above. Then the following holds*:

(i) *The 2-part in the determinant of the block part of the spin character table is given by*

$$\nu(\det Z_s(\tilde{B})) = \sum_{\alpha\in\mathcal{O}(B)} \left[\frac{k_\alpha}{2}\right].$$

(ii) *The odd part of the determinant of the block part of the spin character table satisfies*

$$(\det Z_s(\tilde{B}))_{2'} = \det\Phi(B) = \det Z(B).$$

In particular, the sets $\mathcal{O}(B)$, $B \in \mathrm{Bl}_2(n)$, define a 2-block splitting for S_n.

Proof. (i) By Theorem 3.2 we have

$$\nu(\det Z_s(\tilde{B})) \geq \sum_{\alpha\in\mathcal{O}(B)} \left[\frac{k_\alpha}{2}\right].$$

More precisely, for any bijection $\pi : \mathcal{O}(B) \to \mathcal{D}(\tilde{B})$ we have

$$\nu\Big(\prod_{\alpha\in\mathcal{O}(B)} \langle\pi(\alpha)\rangle(\tilde{\sigma}_\alpha)\Big) \geq \sum_{\alpha\in\mathcal{O}(B)} \left[\frac{k_\alpha}{2}\right].$$

We claim that the Glaisher bijection $G : \mathcal{O}(B) \to \mathcal{D}(\tilde{B})$, $\alpha \mapsto G(\alpha)$ is the *unique* bijection $\mathcal{O}(B) \to \mathcal{D}(\tilde{B})$ such that equality holds. Then the assertion follows by the Leibniz formula for the determinant.

By Theorem 3.2, each such map π has to be the Glaisher map on restriction to $\mathcal{O}^-(B)$, and thus π induces bijections $\mathcal{O}^\varepsilon(B) \to \mathcal{D}^\varepsilon(\tilde{B})$, for both signs $\varepsilon = \pm$. Now we argue by induction on the lexicographic order on $\mathcal{D}^+$. Take $\alpha \in \mathcal{O}^+(B)$ such that $G(\alpha)$ is highest among the partitions in $\mathcal{D}^+(\tilde{B})$. Then by Theorem 3.4, $G(\alpha)$ is the *unique* partition in $\mathcal{D}^+(\tilde{B})$ such that

$$\nu(\langle G(\alpha)\rangle(\tilde{\sigma}_\alpha)) = \left[\frac{k_\alpha}{2}\right]$$

and hence (using again Theorem 3.2) we must have $\pi(\alpha) = G(\alpha)$. Remove α from $\mathcal{O}^+(B)$ and $\pi(\alpha) = G(\alpha)$ from $\mathcal{D}^+(\tilde{B})$ and continue, using Theorem 3.4 in each step. This shows that $\pi = G$, and hence we are done.

(ii) Let

$$D_s(\tilde{B}) = (\tilde{d}_{\lambda\mu})_{\substack{\lambda\in\mathcal{D}(\tilde{B})\\ \mu\in\mathcal{D}(B)}}$$

be the reduced spin 2-decomposition matrix for the spin characters in $\tilde{B}$ (taking only one of an associate pair). By Theorem 3.1 we have

$$Z_s(\tilde{B}) = D_s(\tilde{B})\Phi(B).$$

By Corollary 3.8 and part (i) we know that

$$|\det D_s(\tilde{B})| = \prod_{\alpha\in\mathcal{O}(B)} 2^{[k_\alpha/2]} = (\det Z_s(\tilde{B}))_2,$$

and hence the first equality in (ii) follows.
With $D_B = (d_{\lambda\mu})_{\substack{\lambda\in\mathcal{D}(B)\\ \mu\in\mathcal{D}(B)}}$ denoting the upper square part of the 2-decomposition matrix for B (with the usual order where the characters to regular partitions come first) we also have

$$Z(B) = D_B\,\Phi(B).$$

As D_B is well-known to be a lower unitriangular matrix, this immediately implies $\det\Phi(B) = \det Z(B)$. □

We can also deduce further information on the Smith normal forms of the matrices defined above; these may also be considered as block versions of some results in [Bessenrodt et al. 2005].

Theorem 4.3. *Let $B \subseteq \tilde{B}$ be as above.*

(i) *The* 2*-part of the Smith normal form of $Z_s(\tilde{B})$ is given by*

$$\mathcal{S}(Z_s(\tilde{B}))_2 = \mathcal{S}(\Delta(B)).$$

(ii) *The odd part of the Smith normal form of $Z_s(\tilde{B})$ satisfies*

$$\mathcal{S}(Z_s(\tilde{B}))_{2'} = \mathcal{S}(\Phi(B)) = \mathcal{S}(Z(B)).$$

Proof. We have already seen above that

$$Z_s(\tilde{B}) = D_s(\tilde{B})\,\Phi(B).$$

By Corollary 3.8 and Theorem 4.2 we know that $D_s(\tilde{B})$ and $\Phi(B)$ have coprime determinants, and more precisely, we then obtain

$$\mathcal{S}(Z_s(\tilde{B}))_2 = \mathcal{S}(D_s(\tilde{B})) = \mathcal{S}(\Delta(B)),$$
$$\mathcal{S}(Z_s(\tilde{B}))_{2'} = \mathcal{S}(\Phi(B)).$$

Since $Z(B) = D_B\,\Phi(B)$ and $\det D_B = 1$, this immediately implies $\mathcal{S}(\Phi(B)) = \mathcal{S}(Z(B))$. □

Theorem 4.4. *The block splitting of the* 2*-regular classes given by the sets $\mathcal{O}(B)$, $B \in \mathrm{Bl}_2(n)$, is the unique block splitting in the sense of Brauer* (i.e., such that Theorem 2.1(3) is satisfied).

Proof. We keep our previous choice of characters $\mathrm{Irr}'(B) \subseteq \mathrm{Irr}(B)$, i.e., we take the ordinary characters labelled by $\mathcal{D}(B)$, and we take the spin characters labelled by $\mathcal{D}(\tilde{B})$. For any choice $\mathcal{O}(B)'$, $B \in \mathrm{Bl}_2(n)$, of blocks of labels of the 2-regular conjugacy classes, we have the analogous equality

$$Z_s(\tilde{B})' = D_s(\tilde{B})\Phi(B)'$$

and hence $\det Z_s(\tilde{B})' = \det D_s(\tilde{B}) \det \Phi(B)'$. Thus the sets $\mathcal{O}(B)'$ correspond to a splitting system, i.e., condition (3) in Brauer's Theorem is satisfied for all B, if and only if

$$(\det Z_s(\tilde{B})')_2 = (\det D_s(\tilde{B}))_2 = \prod_{\alpha \in \mathcal{O}(B)} 2^{[k_\alpha/2]} \quad \text{for all } B \in \mathrm{Bl}_2(n).$$

As in the proof of Theorem 4.2 we again consider the bijections on the block level that give a summand with minimal 2-power in the Leibniz formula for the determinant and use Theorem 3.2; one immediately observes that we must have for any 2-block B of S_n:

$$\mathcal{O}^-(B)' = G^{-1}(\mathcal{D}^-(B)) = \mathcal{O}^-(B),$$

i.e., the $\mathbb{O}^-$-part of the blocks in a block splitting of regular classes is uniquely determined. In the next step, arguing similarly as before with Theorem 3.4 along the lexicographic ordering on the $\mathcal{D}^+$-partitions (and considering all blocks in this argument simultaneously) one also obtains

$$\mathbb{O}^+(B)' = \mathbb{O}^+(B) \quad \text{for all } B \in \mathrm{Bl}_2(n).$$

Thus the block splitting $\mathbb{O}(B)$, $B \in \mathrm{Bl}_2(n)$, constructed above is the unique block splitting of the 2-regular classes of S_n. □

Remark. While there is a nice formula for the determinant of the whole regular character table of S_n (see Section 2), we do not have a formula for the determinant of the block character table. A first guess might be that it is again the product of the parts of the corresponding labelling $\mathbb{O}$-partitions (or related to this), but examples show that this is *not* the case — in fact, primes $> n$ may appear in the determinant.

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