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Swan conductors for p-adic differential modules, I: A local construction

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We define a numerical invariant, the differential Swan conductor, for certain differential modules on a rigid analytic annulus over a p-adic field. This gives a definition of a conductor for p-adic Galois representations with finite local monodromy over an equal characteristic discretely valued field, which agrees with the usual Swan conductor when the residue field is perfect. We also establish analogues of some key properties of the usual Swan conductor, such as integrality (the Hasse–Arf theorem), and the fact that the graded pieces of the associated ramification filtration on Galois groups are abelian and killed by p.

Introduction

In this paper, we define a numerical invariant, which we call the differential Swan conductor, for certain differential modules on a rigid analytic annulus over a complete nonarchimedean field of mixed characteristics. We then use this definition to define a differential Swan conductor for p-adic Galois representations with finite local monodromy over an equal characteristic discretely valued field, whose residue field need not be perfect. The latter will coincide with the usual Swan conductor in the case of a perfect residue field.

The construction of the differential Swan conductor proceeds by measuring the failure of convergence of the Taylor isomorphism, or equivalently, the failure of local horizontal sections for the connection to converge on as large a disc as possible. This phenomenon distinguishes the study of differential equations over p-adic fields from its classical analogue, and the relationship with Swan conductors explains the discrepancy in terms of wild ramification in characteristic p. (The analogy between irregularity of connections and wild ramification has been known

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for a while, but recent developments have pushed it further, e.g., construction of a de Rham analogue of local ϵ-factors [Beilinson et al. 2002].)

In the case of Galois representations over an equal characteristic discretely valued field with perfect residue field, the differential interpretation of the Swan conductor is known from the work of several authors, including André, Christol and Mebkhout, Crew, Matsuda, Tsuzuki, and others; see [Kedlaya 2005a, Section 5] for an overview. The question of extending this interpretation to the case of imperfect residue field was first raised by Matsuda [2004], who proposed giving a differential interpretation of the logarithmic conductor of [Abbes and Saito 2002; Abbes and Saito 2003]. Our point of view is a bit different: we first construct a numerical invariant from differential considerations, and check that it has good properties. These include the Hasse–Arf property, i.e., integrality of conductors (Theorem 2.8.2), and the fact that the associated ramification filtration on Galois groups has graded pieces which are elementary abelian (Theorem 3.5.13). Only then do we pose questions about reconciling the definition with other constructions; we do not answer any of these.

In a subsequent paper, we will apply this construction to overconvergent F-isocrystals on varieties over perfect fields of positive characteristic; in particular, the construction applies to discrete representations of the étale fundamental groups of open varieties. We will pay particular attention to how the differential Swan conductor of a fixed isocrystal changes as we vary the choice of a boundary divisor along which to compute the conductor.

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1. Differential fields

We start with a summary of some relevant facts about differential fields and modules. We defer to [Kedlaya 2006b, Section 3] (and other explicitly cited references) for more details.

1.1. *Differential modules and twisted polynomials.*

Hypothesis 1.1.1. Throughout this subsection, let F be a differential field of order 1 and characteristic zero, i.e., a field of characteristic zero equipped with a derivation ∂.

Definition 1.1.2. Let $F\{T\}$ denote the (noncommutative) *ring of twisted polynomials* over F [Ore 1933]; its elements are finite formal sums $\sum_{i \geq 0} a_i T^i$ with $a_i \in F$, multiplied according to the rule $Ta = aT + \partial(a)$ for $a \in F$.

Remark 1.1.3. The opposite ring of $F\{T\}$ is the ring of twisted polynomials for the differential field given by equipping F with the derivation $-\partial$ instead of ∂.

Definition 1.1.4. A *differential module* over F is a finite dimensional F-vector space V equipped with an action of ∂ (subject to the Leibniz rule); any such module inherits a left action of $F\{T\}$ where T acts via ∂. For V a differential module over F, a *cyclic vector* in V is a vector $\mathbf{v} \in V$ such that $\mathbf{v}, \partial(\mathbf{v}), \dots, \partial^{\dim(V)-1}(\mathbf{v})$ form a basis of V. A cyclic vector defines an isomorphism $V \cong F\{T\}/F\{T\}P$ of differential modules for some twisted polynomial $P \in F\{T\}$, where the ∂-action on $F\{T\}/F\{T\}P$ is left multiplication by T.

Lemma 1.1.5. *Every differential module over F contains a cyclic vector.*

Proof. See, e.g., [Dwork et al. 1994, Theorem III.4.2]. □

Hypothesis 1.1.6. For the remainder of this subsection, assume that the differential field F is equipped with a nonarchimedean norm $|\cdot|$, and let V denote a nonzero differential module over F. Write $v(x) = -\log|x|$ for the valuation corresponding to $|\cdot|$.

Definition 1.1.7. Let $|\partial|_F$ denote the operator norm of ∂ on F. Let $|\partial|_{V,\mathrm{sp}}$ denote the *spectral norm* of ∂ on V, i.e., the limit $\lim_{s\to\infty} |\partial^s|_V^{1/s}$ for any fixed F-compatible norm $|\cdot|_V$ on V. Any two such norms on V are equivalent [Schneider 2002, Proposition 4.13], so the spectral norm does not depend on the choice. More explicitly, if one chooses a basis of V and lets D_s denote the matrix via which ∂^s acts on this basis, then

$$\max\{|\partial|_{F,\mathrm{sp}}, |\partial|_{V,\mathrm{sp}}\} = \max\{|\partial|_{F,\mathrm{sp}}, \limsup_{s\to\infty} |D_s|^{1/s}\}, \qquad (1.1.7.1)$$

where the norm applied to D_s is the supremum over entries [Christol and Dwork 1994, Proposition 1.3].

Definition 1.1.8. For $P(T) = \sum_i a_i T^i \in F\{T\}$ a nonzero twisted polynomial, define the *Newton polygon* of P as the lower convex hull of the set $\{(-i, v(a_i))\} \subset \mathbb{R}^2$. This Newton polygon obeys the usual additivity rules *only* for slopes less than $-\log|\partial|_F$ [Kedlaya 2006b, Lemma 3.1.5 and Corollary 3.1.6; Robba 1980, Section 1].

Proposition 1.1.9 (Christol–Dwork). *Suppose $V \cong F\{T\}/F\{T\}P$ and P has least slope r. Then*

$$\max\{|\partial|_F, |\partial|_{V,\mathrm{sp}}\} = \max\{|\partial|_F, e^{-r}\}.$$

Proof. See [Christol and Dwork 1994, Théorème 1.5] or [Kedlaya 2006b, Proposition 3.3.7]. □

Proposition 1.1.10 (Robba). *Suppose that F is complete for its norm. Then any monic twisted polynomial $P \in F\{T\}$ admits a unique factorization*

$$P = P_+ P_m \cdots P_1$$

such that for some $r_1 < \cdots < r_m < -\log|\partial|_F$, each P_i is monic with all slopes equal to r_i, and P_+ is monic with all slopes at least $-\log|\partial|_F$.

Proof. This follows by repeated application of Hensel's lemma for twisted polynomials [Robba 1980]; see also [Kedlaya 2006b, Corollary 3.2.4]. □

1.2. *Differential fields of higher order.*

Hypothesis 1.2.1. Throughout this subsection, let F denote a *differential field of order n*, i.e., a field F equipped with n commuting derivations $\partial_1, \dots, \partial_n$. Assume also that F has characteristic zero and is complete for a nonarchimedean norm $|\cdot|$ with corresponding valuation v. Let V denote a nonzero *differential module* over F, i.e., a nonzero finite dimensional F-vector space equipped with commuting actions of $\partial_1, \dots, \partial_n$. We apply the results of the previous subsection by singling out one of $\partial_1, \dots, \partial_n$.

Definition 1.2.2. Define the *scale* of V as

$$\max\left\{\max\left\{1, \frac{|\partial_i|_{V,\mathrm{sp}}}{|\partial_i|_{F,\mathrm{sp}}}\right\} : i \in \{1, \dots, n\}\right\};$$

note that this quantity is at least 1 by definition, with equality at least when $V = F$. For $i = 1, \dots, n$, we say ∂_i is *dominant for V* if $\max\{1, |\partial_i|_{V,\mathrm{sp}}/|\partial_i|_{F,\mathrm{sp}}\}$ equals the scale of V.

Definition 1.2.3. Let $V_1, \dots, V_m$ be the Jordan–Hölder factors of V (listed with multiplicity). Define the *scale multiset* of V as the multiset of cardinality $\dim_F V$, consisting of the scale of V_j included with multiplicity $\dim_F V_j$, for $j = 1, \dots, m$. Note that the largest element of the scale multiset equals the scale of V.

Remark 1.2.4. If $n = 1$ and $V \cong F\{T\}/F\{T\}P$ for P a twisted polynomial, then Proposition 1.1.10 implies that the multiplicity of any $r < -\log|\partial|_F$ as a slope of the Newton polygon of P coincides with the multiplicity of $e^{-r}/|\partial|_{F,\mathrm{sp}}$ in the scale multiset of V.

Proposition 1.2.5. *Suppose that $|\partial_i|_F/|\partial_i|_{F,\mathrm{sp}} = s_0$ for $i = 1, \dots, n$. Then there is a unique decomposition*

$$V = V_- \oplus \bigoplus_{s > s_0} V_s$$

of differential modules, such that each Jordan–Hölder factor of V_s has scale s, and each Jordan–Hölder factor of V_- has scale at most s_0.

Proof. This may be deduced from Proposition 1.1.10, as in [Kedlaya 2006b, Proposition 3.4.3]. □

Definition 1.2.6. We refer to the decomposition given in Proposition 1.2.5 as the *scale decomposition* of V.

2. Conductors for ∇-modules

In this section, we construct the differential Swan conductor for certain differential modules over p-adic fields. We will perform all of the calculations under the bifurcated Hypothesis 2.1.3; one of the two options therein allows for nonarchimedean fields which are not discretely valued, but restricts their residue fields, while the other is less restrictive on residue fields, but requires the nonarchimedean norms to be discretely valued.

Notation 2.0.1. For S a set or multiset, write $S^p = \{s^p : s \in S\}$. If A, B are two multisets of the same cardinality d, then write $A \geq B$ to mean that for $i = 1, \dots, d$, the i-th largest element of A is greater than or equal to the i-th largest element of B (counting multiplicity).

2.1. *Setup.*

Definition 2.1.1. Given a field K equipped with a (possibly trivial) nonarchimedean norm, for $\rho_1, \dots, \rho_n \in (0, +\infty)$, the $(\rho_1, \dots, \rho_n)$-*Gauss norm* on $K[u_1, \dots, u_n]$ is the norm $|\cdot|_\rho$ given by

$$\left| \sum_I c_I u_1^{i_1} \cdots u_n^{i_n} \right| = \max_I \{|c_I| \rho_1^{i_1} \cdots \rho_n^{i_n}\};$$

this norm extends uniquely to $K(u_1, \dots, u_n)$.

Definition 2.1.2. For ℓ/k an extension of fields of characteristic $p > 0$, a p-*basis* of ℓ over k is a set $B \subset \ell$ with the property that the products $\prod_{b \in B} b^{e_b}$, where $e_b \in \{0, \dots, p-1\}$ for all $b \in B$ and $e_b = 0$ for all but finitely many b, are all distinct and form a basis for ℓ as a vector space over the compositum $k\ell^p$. By a p-*basis* of ℓ, we mean a p-basis of ℓ over ℓ^p.

Hypothesis 2.1.3. For the rest of this section, assume *one* of the following two sets of hypotheses.

(a) Let K be a field of characteristic zero, complete for a (not necessarily discrete) nonarchimedean norm $|\cdot|$, with residue field k of characteristic $p > 0$. Equip $K(u_1, \dots, u_n)$ with the $(1, \dots, 1)$-Gauss norm. Let ℓ be a finite separable extension of $k(u_1, \dots, u_n)$, and let L be the unramified extension with residue field ℓ of the completion of $K(u_1, \dots, u_n)$.

(b) Let K be a field of characteristic zero, complete for a nonarchimedean norm $|\cdot|$, with discrete value group and residue field k of characteristic $p > 0$. Let L be an extension of K, complete for an extension of $|\cdot|$ with the same value group, whose residue field ℓ admits a finite p-basis $\overline{B} = \{\overline{u_1}, \dots, \overline{u_n}\}$ over k. For $i = 1, \dots, n$, let u_i be a lift of $\overline{u_i}$ to the valuation ring $\mathfrak{o}_L$ of L.

Definition 2.1.4. Under either option in Hypothesis 2.1.3, the module of continuous differentials $\Omega^1_{L/K}$ is generated by $du_1, \dots, du_n$; let $\partial_1, \dots, \partial_n$ denote the dual basis of derivations (that is, $\partial_i = \frac{\partial}{\partial u_i}$).

Remark 2.1.5. Note that $|\partial_i|_L / |\partial_i|_{L,\mathrm{sp}} = |p|^{-1/(p-1)}$ for $i = 1, \dots, n$, so Proposition 1.2.5 applies.

2.2. *Taylor isomorphisms.* The scale of a differential module over L can be interpreted as a normalized radius of convergence for the Taylor series, as follows.

Convention 2.2.1. Let $\mathbb{N}_0$ denote the monoid of nonnegative integers. For $I \in \mathbb{N}_0^n$ and $*$ any symbol, we will write $*^I$ as shorthand for $*_1^{i_1} \cdots *_n^{i_n}$. We also write $I!$ as shorthand for $i_1! \cdots i_n!$.

Definition 2.2.2. Let V be a differential module over L. Define the *formal Taylor isomorphism* on V to be the map $T : V \mapsto V \otimes_L L[\![x_1, \dots, x_n]\!]$ given by

$$T(\mathbf{v}) = \sum_{I \in \mathbb{N}_0^n} \frac{x^I}{I!} \partial^I(\mathbf{v}).$$

We can then interpret the scale of V as the minimum λ such that T takes values in $V \otimes_L R$, for R the subring of $L[\![x_1, \dots, x_n]\!]$ consisting of series convergent on the open polydisc

$$|x_i| < \lambda^{-1} \qquad (i = 1, \dots, n).$$

In particular, if L' is a complete extension of L, and $x_1, \dots, x_n \in L'$ satisfy $|x_i| < \lambda^{-1}$ for λ the scale of V, we obtain by substitution a *concrete Taylor isomorphism*

$$T(\mathbf{v}; x_1, \dots, x_n) : V \to V \otimes_L L'.$$

Remark 2.2.3. If $x_1, \dots, x_n \in L$ satisfy $|x_i| < 1$, then the concrete Taylor isomorphism $T(\cdot\,; x_1, \dots, x_n)$ is defined on L, and is a K-algebra homomorphism carrying u_i to $u_i + x_i$. If V is a differential module of scale λ, and $|x_i| < \lambda^{-1}$ for $i = 1, \dots, n$, then the concrete Taylor isomorphism $T(\cdot\,; x_1, \dots, x_n)$ on V is semilinear over the concrete Taylor isomorphism on L.

Remark 2.2.4. Note that $|\partial^I / I!|_F \leq 1$ for any $I \in \mathbb{N}_0^n$. Hence if $x_1, \dots, x_n \in L$ satisfy $|x_i| < 1$, then for any $f \in L$,

$$|T(f; x_1, \dots, x_n) - f| \leq \max_i \{|x_i|\} \cdot |f|.$$

In particular, suppose $u'_1, \dots, u'_n \in L$ satisfy $|u'_i| = 1$, and the images of $u'_1, \dots, u'_n$ in ℓ form a p-basis of ℓ over k. Then $T(\cdot; x_1, \dots, x_n)$ can also be interpreted as the concrete Taylor isomorphism defined with respect to the dual basis of $du'_1, \dots, du'_n$ and evaluated at $y_1, \dots, y_n$, for $y_i = T(u'_i; x_1, \dots, x_n) - u'_i$. This implies that the scale of a differential module computed with respect to (the dual basis to) $du_1, \dots, du_n$ is no greater than with respect to $du'_1, \dots, du'_n$; by the same calculation in reverse, it follows that the two scales are equal. (Francesco Baldassarri has suggested a coordinate-free definition of the scale that explains this remark; we will follow up on this suggestion elsewhere.)

2.3. *Frobenius descent.* As discovered originally in [Christol and Dwork 1994], in the situations of Hypothesis 2.1.3, one can overcome the limitation on scales imposed by Proposition 1.1.9 by using descent along the substitution $u_i \mapsto u_i^p$.

Definition 2.3.1. Let V be a differential module over L with scale less than $|p|^{-1/(p-1)}$. If K contains a primitive p-th root of unity ζ, we may define an action of the group $(\mathbb{Z}/p\mathbb{Z})^n$ on V using concrete Taylor isomorphisms:

$$\mathbf{v}^J = T(\mathbf{v}; (\zeta^{j_1} - 1)u_1, \dots, (\zeta^{j_n} - 1)u_n) \qquad (J \in (\mathbb{Z}/p\mathbb{Z})^n).$$

Let V_1 be the fixed space under this group action; in particular, taking $V = L$, we obtain a subfield L_1 of L, which we may view as a differential field of order n for the derivations $\partial_{i,1} = \partial/\partial(u_i^p)$. In general, V_1 may be viewed as a differential module over L_1, the natural map $V_1 \otimes_{L_1} L \to V$ is an isomorphism of L-vector spaces (by Hilbert 90), and the actions of ∂_i and $\partial_{i,1}$ on V are related by the formula

$$\partial_{i,1} = \frac{1}{pu_i^{p-1}} \partial_i. \tag{2.3.1.1}$$

We call V_1 the *Frobenius antecedent* of V. If K does not contain a primitive p-th root of unity, we may still define the Frobenius antecedent using Galois descent.

Proposition 2.3.2. *Let V be a differential module over L with scale $s < |p|^{-1/(p-1)}$ and scale multiset S. Then the scale multiset of the Frobenius antecedent of V is S^p.*

Proof. Since any direct sum decomposition commutes with the formation of the Frobenius antecedent V_1, it suffices to check that the scale of V_1 is s^p. Let $T(\mathbf{v})$ be the formal Taylor isomorphism for V, and let $T'(\mathbf{v})$ be the formal Taylor isomorphism for V_1 but with variables $x'_1, \dots, x'_n$.

By [Kedlaya 2005a, Lemma 5.12], for t, t_1 in any nonarchimedean field,

$$|t - t_1| < \lambda^{-1}|t| \implies |t^p - t_1^p| < \lambda^{-p}|t|^p \qquad (1 < \lambda < |p|^{-1/(p-1)}). \tag{2.3.2.1}$$

(We repeat from [Kedlaya 2006b, Lemma 4.4.2] the description of a misprint in the last line of the statement of [Kedlaya 2005a, Lemma 5.12]: one must read

$r^{1/p}\rho^{1/p}$, $r\rho$ for $r\rho^{1/p}$, $r^p\rho$, respectively.). Hence the convergence of the isomorphism $T'(\mathbf{v}; x'_1, \ldots, x'_n)$ for $|x'_i| < \lambda^{-p}$ implies convergence of $T(\mathbf{v}; x_1, \ldots, x_n)$ for $|x_i| < \lambda^{-1}$, so the scale of V_1 is at least s^p. On the other hand, we can obtain T' by averaging T over the action of $(\mathbb{Z}/p\mathbb{Z})^n$, so the scale of V_1 is at most s^p. (Compare [Kedlaya 2005a, Theorem 6.15].) □

Remark 2.3.3. It should also be possible to prove Proposition 2.3.2 by raising both sides of (2.3.1.1) to a large power and comparing the results, but this would appear to be somewhat messy.

Definition 2.3.4. If V is a differential module over F of scale less than

$$|p|^{-1/(p^{m-1}(p-1))},$$

by Proposition 2.3.2, we can iterate the construction of a Frobenius antecedent m times; we call the result the *m-fold Frobenius antecedent* of V.

Remark 2.3.5. Note that it is also possible to construct antecedents one variable at a time; the point is that since the operators ∂_i, ∂_j commute for $i \neq j$, ∂_i continues to act on the antecedent with respect to ∂_j. This will be used in the proof of Proposition 2.5.4.

2.4. ∇-*Modules.*

Notation 2.4.1. Let Γ^* denote the divisible closure of $|K^*|$. We say a subinterval of $(0, +\infty)$ is *aligned* if each endpoint at which it is closed belongs to Γ^*.

Remark 2.4.2. One can drop the word "aligned", and all references to Γ^*, everywhere hereafter if one works with Berkovich analytic spaces [Berkovich 1990] instead of rigid analytic spaces. We omit further details.

Notation 2.4.3. For I an aligned interval and t a dummy variable, let $A_L(I)$ be the rigid analytic (over L) subspace of the affine t-line over L consisting of points with $|t| \in I$; this space is affinoid if I is closed. (We omit the parentheses if I is described explicitly, e.g., if $I = [\alpha, \beta)$, we write $A_L[\alpha, \beta)$ for $A_L(I)$.) For $\rho \in I$, we write $|\cdot|_\rho$ for the ρ-Gauss norm

$$\left| \sum_{i \in \mathbb{Z}} c_i t^i \right|_\rho = \sup_i \{|c_i| \rho^i\};$$

for $\rho \in \Gamma^*$, we may interpret $|\cdot|_\rho$ as the supremum norm on the affinoid space $A_L[\rho, \rho]$.

Lemma 2.4.4. *Let I be an aligned interval. For $\rho, \sigma \in I$ and $c \in [0, 1]$, put $\tau = \rho^c \sigma^{1-c}$. Then for any $f \in \Gamma(A_L(I), \mathbb{O})$,*

$$|f|_\tau \leq |f|_\rho^c |f|_\sigma^{1-c}.$$

Proof. See [Kedlaya 2007, Lemma 3.1.6], [Amice 1975, Corollaire 4.2.8], or [Robba and Christol 1994, Corollaire 5.4.9]. $\square$

Definition 2.4.5. For I an aligned interval, a ∇-*module* on $A_L(I)$ (relative to K) is a coherent locally free sheaf $\mathscr{E}$ on $A_L(I)$ equipped with an integrable K-linear connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{A_L(I)/K}$. (Here $\Omega^1_{A_L(I)/K}$ denotes the sheaf of continuous differentials; it is freely generated over $\mathbb{O}_{A_L(I)}$ by $du_1, \dots, du_n, dt$.) The connection equips $\mathscr{E}$ with actions of the derivations $\partial_i = \frac{\partial}{\partial u_i}$ for $i = 1, \dots, n$ and $\partial_{n+1} = \frac{\partial}{\partial t}$; integrability of the connection is equivalent to commutativity between these actions.

Definition 2.4.6. For I an aligned interval and $\rho \in I$, let F_ρ be the completion of $L(t)$ for the ρ-Gauss norm, viewed as a differential field of order $n+1$. For $\mathscr{E}$ a nonzero ∇-module on $A_L(I)$, let J be a closed aligned neighborhood of ρ in I, and put

$$\mathscr{E}_\rho = \Gamma(A_L(J), \mathscr{E}) \otimes_{\Gamma(A_L(J), \mathbb{O})} F_\rho,$$

viewed as a differential module over F_ρ; this construction does not depend on J. Define the *radius multiset* of $\mathscr{E}_\rho$, denoted $S(\mathscr{E}, \rho)$, as the multiset of reciprocals of the scale multiset of $\mathscr{E}_\rho$. Define the *(toric) generic radius of convergence* of $\mathscr{E}_\rho$, denoted $T(\mathscr{E}, \rho)$, as the smallest element of $S(\mathscr{E}, \rho)$, i.e., the reciprocal of the scale of $\mathscr{E}_\rho$.

Remark 2.4.7. As in [Kedlaya 2006b], the toric generic radius of convergence is normalized differently from the generic radius of convergence of [Christol and Dwork 1994], which would be multiplied by an extra factor of ρ. Our chief justification for this normalization is "because it works", in the sense of giving the expected answer for Example 3.5.10. We look forward to ongoing work of Baldassarri (compare Remark 2.2.4) for a more intrinsic justification.

Remark 2.4.8. To our knowledge, the consideration of ∇-modules over a rigid analytic annulus, but taking into account derivations of the base field over a subfield, is novel to this paper. It may prove an interesting exercise to transcribe the arguments of [Kedlaya 2005a], such as local duality, as much as possible to this setting.

2.5. *The highest ramification break.*

Definition 2.5.1. Let $\mathscr{E}$ be a ∇-module on $A_L(\epsilon, 1)$ for some $\epsilon \in (0, 1)$. We say $\mathscr{E}$ is *solvable at* 1 if

$$\lim_{\rho \to 1^-} T(\mathscr{E}, \rho) = 1.$$

Hypothesis 2.5.2. For the rest of this subsection, let $\mathscr{E}$ be a ∇-module on $A_L(\epsilon, 1)$ for some $\epsilon \in (0, 1)$, which is solvable at 1.

Lemma 2.5.3. *For each $i \in \{1, \dots, n+1\}$, for $r \in (0, -\log \epsilon)$, put $\rho = e^{-r}$ and let $f_i(r)$ be the negative logarithm of the scale of ∂_i on $\mathscr{E}_\rho$. Then f_i is a concave function of r; in particular,*

$$\log T(\mathscr{E}, e^{-r}) = \min_i \{f_i(r)\}$$

is a concave function of r. (This does not require solvability at 1.)

Proof. It suffices to check concavity on $-\log(J)$ for J an arbitrary closed aligned subinterval of $(\epsilon, 1)$. Since J is closed aligned, $A_L(J)$ is affinoid; by Kiehl's theorem (see for instance [Fresnel and van der Put 2004, Theorem 4.5.2]), $\Gamma(A_L(J), \mathscr{E})$ is a finitely generated module over the ring $\Gamma(A_L(J), \mathscr{O})$. Since that ring is a principal ideal domain [Lazard 1962, Proposition 4, Corollaire], $\Gamma(A_L(J), \mathscr{E})$ is freely generated by some subset $\mathbf{e}_1, \dots, \mathbf{e}_m$. Let $D_{i,l}$ be the matrix over $\Gamma(A_L(J), \mathscr{O})$ via which ∂_i^l acts on $\mathbf{e}_1, \dots, \mathbf{e}_m$. For $\rho, \sigma \in J$ and $c \in [0, 1]$, put $\tau = \rho^c \sigma^{1-c}$. By Lemma 2.4.4, we have

$$|D_{i,l}|_\tau \leq |D_{i,l}|_\rho^c |D_{i,l}|_\sigma^{1-c};$$

taking l-th roots of both sides and taking limits yields

$$\limsup_{l \to \infty} |D_{i,l}|_\tau^{1/l} \leq \Big(\limsup_{l \to \infty} |D_{i,l}|_\rho^{1/l}\Big)^c \Big(\limsup_{l \to \infty} |D_{i,l}|_\sigma^{1/l}\Big)^{1-c}.$$

By (1.1.7.1), this yields the desired result. (Compare [Kedlaya 2006b, Proposition 4.2.6].) □

Proposition 2.5.4. *The function $f(r) = \log T(\mathscr{E}, e^{-r})$ on $(0, -\log \epsilon)$ is piecewise linear, with slopes in $(1/(\operatorname{rank} \mathscr{E})!)\mathbb{Z}$. Moreover, f is linear in a neighborhood of* 0.

Proof. Since f is concave by Lemma 2.5.3, takes nonpositive values, and tends to 0 as $r \to 0^+$, it is everywhere nonincreasing. Hence for sufficiently large integers h, we can choose $\rho_h \in (\epsilon, 1)$ with $T(\mathscr{E}, \rho_h) = |p|^{1/(p^{h-1}(p-1))}$ and $\rho_h < \rho_{h+1}$. Put $r_h = -\log \rho_h$.

We now check piecewise linearity and the slope restriction on (r_{h+1}, r_h); it suffices to check on $-\log(J)$ for J an arbitrary closed aligned subinterval of (ρ_h, ρ_{h+1}). Assume without loss of generality that K contains a primitive p-th root of unity. Put $L_0 = L$. For $l = 1, \dots, h$, let L_l be the subfield of L_{l-1} fixed under the action of $(\mathbb{Z}/p\mathbb{Z})^n$ given in Definition 2.3.1, but with

$$u_1^{p^{l-1}}, \dots, u_n^{p^{l-1}}$$

playing the roles of $u_1, \dots, u_n$. Since $T(\mathscr{E}, \rho) > |p|^{1/(p^{h-1}(p-1))}$ for $\rho \in J$, using Definition 2.3.1 (in the $u_1, \dots, u_n$-directions) and [Kedlaya 2005a, Theorem 6.15] (in the t-direction), we can construct an h-fold Frobenius antecedent $\mathscr{E}_h$ for $\mathscr{E}$, which is defined on $A_{L_h}(J^{p^h})$.

Apply Lemma 1.1.5 to construct a cyclic vector for $\mathscr{E}_h$ over $\operatorname{Frac}\Gamma(A_{L_h}(J^{p^h}), \mathbb{O})$; by writing down the corresponding twisted polynomial $P(T)$ and applying Proposition 1.1.9, we see that for $\sigma \in J^{p^h}$, $T(\mathscr{E}_h, \sigma)$ is piecewise of the form $|g|_\sigma^{1/j}$ for some $g \in \operatorname{Frac}\Gamma(A_{L_h}(J^{p^h}), \mathbb{O})$ and $j \in \{1, \dots, \operatorname{rank}(\mathscr{E})\}$. In particular, for $\sigma = \rho^{p^h}$, this expression is piecewise of the form $(|a|\rho^{ip^h})^{1/j}$ for some $a \in K^*$, $i \in \mathbb{Z}$, and $j \in \{1, \dots, \operatorname{rank}(\mathscr{E})\}$. This proves that on (r_{h+1}, r_h), f is piecewise linear with slopes in $(1/(\operatorname{rank}\mathscr{E})!)\mathbb{Z}$.

To check piecewise linearity in a neighborhood of r_h, note that as we approach r_h from the right, the successive slopes of f that we encounter are increasing but bounded above, and lie in a discrete subset of $\mathbb{R}$. Hence they stabilize, so f is linear in a one-sided neighborhood of r_h. An analogous argument applies again when approaching r_{h+1} from the left, so f is piecewise linear on $[r_{h+1}, r_h]$; taking the union of these intervals, we deduce that f is piecewise linear on $(0, r_h]$ for some h. An analogous argument applies yet again when approaching 0 from the right, yielding the desired result. (Compare [Christol and Mebkhout 2000, Théorème 4.2-1].) □

Corollary 2.5.5. *There exists $b \in \mathbb{Q}_{\geq 0}$ such that $T(\mathscr{E}, \rho) = \rho^b$ for all $\rho \in (\epsilon, 1)$.*

Definition 2.5.6. We will refer to the number b in Corollary 2.5.5 as the *(differential) highest ramification break* of $\mathscr{E}$, denoted $b(\mathscr{E})$.

2.6. *Invariance.*

Definition 2.6.1. Define the *Robba ring* over L as

$$\mathscr{R}_L = \bigcup_{\epsilon \in (0,1)} \Gamma(A_L(\epsilon, 1), \mathbb{O}).$$

The elements of $\mathscr{R}_L$ can be represented as formal Laurent series $\sum_{i \in \mathbb{Z}} c_i t^i$ with $c_i \in L$; let $\mathscr{R}_L^{\text{int}}$ be the subring of series with $|c_i| \leq 1$ for all $i \in \mathbb{Z}$. The ring $\mathscr{R}_L^{\text{int}}$ is local, with maximal ideal consisting of series with $|c_i| < 1$ for all $i \in \mathbb{Z}$, with residue field $\ell((t))$.

We first examine invariance under certain endomorphisms of L, following Definition 2.2.2.

Definition 2.6.2. Choose $u_1', \dots, u_n', t' \in \mathscr{R}_L^{\text{int}}$ such that under the projection $\mathscr{R}_L^{\text{int}} \to \ell((t))$, $u_1' - u_1, \dots, u_n' - u_n$ map into $t\ell[\![t]\!]$ and $t' - t$ maps into $t^2\ell[\![t]\!]$. Then for some $\epsilon \in (0, 1)$, the Taylor series

$$\sum_{I \in \mathbb{N}_0^{n+1}} \frac{(u_1' - u_1)^{i_1} \cdots (u_n' - u_n)^{i_n} (t' - t)^{i_{n+1}}}{I!} \partial^I(f)$$

converges for $f \in \Gamma(A_L(I), \mathcal{O})$ for any closed aligned subinterval I of $(\epsilon, 1)$, so we can use it to define a map $g : \Gamma(A_L(I), \mathcal{O}) \to \Gamma(A_L(I), \mathcal{O})$ such that $g^*(u_i) = u_i'$, $g^*(t) = t'$.

Proposition 2.6.3. *Let g be a map as in Definition 2.6.2. For any ∇-module $\mathscr{E}$ on $A_L(\epsilon, 1)$ which is solvable at* 1, *we have $T(\mathscr{E}, \rho) = T(g^*\mathscr{E}, \rho)$ for all $\rho \in (\epsilon, 1)$ sufficiently close to* 1. *In particular*, *$g^*\mathscr{E}$ is also solvable at* 1, *and $\mathscr{E}$ and $g^*\mathscr{E}$ have the same highest break.*

Proof. By the choice of $u_1', \dots, u_n', t'$, for $\rho \in (0, 1)$ sufficiently close to 1,

$$|u_i' - u_i|_\rho < 1 \qquad (i = 1, \dots, n), \qquad |t' - t|_\rho < \rho.$$

We will prove the claim for such ρ.

By continuity of $T(\mathscr{E}, \rho)$ (implied by Lemma 2.5.3), it suffices to check for $\rho \in \Gamma^*$. There is no loss of generality in enlarging K, so we may in fact assume that there exists $\lambda \in K$ with $|\lambda| = \rho$. In this case, we may put ourselves in the situation of Remark 2.2.4 by considering $\mathscr{E}_\rho$ to be a differential module over the completion of $L(t/\rho)$ for the 1-Gauss norm, comparing the p-bases $u_1, \dots, u_n, t/\lambda$ and $u_1', \dots, u_n', t'/\lambda$. This yields the claim. □

Proposition 2.6.4. *Let $g : A_L(I) \to A_L(I)$ be the map fixing L and pulling t back to t^{p^N} for some positive integer N. Then for any ∇-module $\mathscr{E}$ on $A_L(\epsilon, 1)$, we have $S(\mathscr{E}, \rho) \leq S(g^*\mathscr{E}, \rho^{1/p^N})$ for all $\rho \in (\epsilon, 1)$; moreover*, *if $n = 0$, then $S(\mathscr{E}, \rho) \leq S(g^*\mathscr{E}, \rho^{1/p^N})^{p^N}$.*

Proof. If we compare the scale multisets of ∂_i on $\mathscr{E}_\rho$ and on $(g^*\mathscr{E})_{\rho^{1/p^N}}$, then we get identical results for $i = 1, \dots, n$. For $i = n + 1$, the scale multiset on $\mathscr{E}_\rho$ is at least the p^N-th power of the scale multiset on $(g^*\mathscr{E})_{\rho^{1/p^N}}$, as in the proof of Proposition 2.3.2. This yields the claim. □

Proposition 2.6.5. *Let $g : A_L(I) \to A_L(I)$ be the map fixing L and pulling t back to t^N for some positive integer N coprime to p. Then for any ∇-module $\mathscr{E}$ on $A_L(\epsilon, 1)$, we have $S(\mathscr{E}, \rho) = S(g^*\mathscr{E}, \rho^{1/N})$ for all $\rho \in (\epsilon, 1)$.*

Proof. If we compare the scale multisets of ∂_i on $\mathscr{E}_\rho$ and on $(g^*\mathscr{E})_{\rho^{1/N}}$, then we get identical results for $i = 1, \dots, n$. For $i = n + 1$, we again get identical results by virtue of [Kedlaya 2005a, Lemma 5.11]. □

We next examine what happens when we change the p-basis.

Proposition 2.6.6. *Choose $u_1', \dots, u_n' \in \mathcal{R}_L^{\mathrm{int}}$ such that under the projection $\mathcal{R}_L^{\mathrm{int}} \to \ell((t))$, $u_1', \dots, u_n'$ map to elements of $\ell[\![t]\!]$ lifting a p-basis of ℓ over k. Let $\partial_1', \dots, \partial_n'$ be the derivations dual to the basis $du_1', \dots, du_n'$ of $\Omega^1_{L/K}$. Let $\mathscr{E}$ be a ∇-module on $A_L(\epsilon, 1)$ for some $\epsilon \in (0, 1)$, which is solvable at* 1. *Then for $\rho \in (0, 1)$ sufficiently close to* 1, *the scale of $\mathscr{E}_\rho$ for $\partial_1, \dots, \partial_n, \partial_{n+1}$ is the same as for $\partial_1', \dots, \partial_n', \partial_{n+1}$; in particular*, *the highest break is the same in both cases.*

Proof. If $u'_1, \dots, u'_n \in L$, then we can invoke Remark 2.2.4 to obtain the claim. In general, we may first make a transformation as in the previous sentence, to match up the reductions modulo $t\ell[\![t]\!]$, then invoke Proposition 2.6.3. $\square$

2.7. *The break decomposition.* Retain Hypothesis 2.5.2 throughout this subsection.

Definition 2.7.1. We say that $\mathscr{E}$ has a *uniform break* if for all $\rho \in (0, 1)$ sufficiently close to 1, $S(\mathscr{E}, \rho)$ consists of a single element with multiplicity $\operatorname{rank}(\mathscr{E})$. We write "$\mathscr{E}$ has uniform break b" as shorthand for "$\mathscr{E}$ has a uniform break and its highest ramification break is b".

Theorem 2.7.2. *For some $\eta \in (0, 1)$, there exists a decomposition of ∇-modules (necessarily unique) $\mathscr{E} = \oplus_{b \in \mathbb{Q}_{\geq 0}} \mathscr{E}_b$ over $A_L(\eta, 1)$ such that each $\mathscr{E}_b$ has uniform break b.*

We will prove Theorem 2.7.2 later in this subsection. To begin with, we recall that the case $L = K$ is essentially a theorem of Christol–Mebkhout [2001, Corollaire 2.4-1], from which we will bootstrap to the general case.

Lemma 2.7.3. *Theorem 2.7.2 holds in case $L = K$.*

Proof. This is the conclusion of [Christol and Mebkhout 2001, Corollaire 2.4-1], at least in case K is spherically complete. However, it extends to the general case as follows.

By a straightforward application of Zorn's lemma, we may embed K into a spherically complete field K'. Apply [Christol and Mebkhout 2001, Corollaire 2.4-1] to obtain a break decomposition over $A_{K'}(\eta, 1)$ for some $\eta \in (0, 1)$; let $\mathbf{v} \in \Gamma(A_{K'}(\eta, 1), \mathscr{E}^\vee \otimes \mathscr{E})$ be the projector onto the highest break component.

Now set notation as in the proof of Proposition 2.5.4. The set of $\rho \in (\rho_h, \rho_{h+1})$ for which at least one coefficient $P(T)$ fails to be a unit in $A_{L_h}[\rho^{p^h}, \rho^{p^h}]$ is discrete, so we may choose $\rho \in (\rho_h, \rho_{h+1})$ not of that form. Then Proposition 1.1.10 gives a factorization of $P(T)$ over $A_{L_h}[\rho^{p^h}, \rho^{p^h}]$ (and likewise in the opposite ring); we thus obtain an element $\mathbf{v}'$ of $\Gamma(A_K[\rho, \rho], \mathscr{E}^\vee \otimes \mathscr{E})$ which agrees with $\mathbf{v}$ over $A_{K'}[\rho, \rho]$.

For any closed aligned subinterval J of $(\eta, 1)$ containing ρ, we have

$$\Gamma(A_K[\rho, \rho], \mathbb{O}) \cap \Gamma(A_{K'}(J), \mathbb{O}) = \Gamma(A_K(J), \mathbb{O})$$

inside $\Gamma(A_{K'}[\rho, \rho], \mathbb{O})$. Since $\mathscr{E}^\vee \otimes \mathscr{E}$ is free over $A_K(J)$ (as in the proof of Lemma 2.5.3), this implies that

$$\Gamma(A_K[\rho, \rho], \mathscr{E}^\vee \otimes \mathscr{E}) \cap \Gamma(A_{K'}(J), \mathscr{E}^\vee \otimes \mathscr{E}) = \Gamma(A_K(J), \mathscr{E}^\vee \otimes \mathscr{E}),$$

and so $\mathbf{v} \in \Gamma(A_K(J), \mathcal{E}^\vee \otimes \mathcal{E})$. Running this argument over all possible J, we obtain $\mathbf{v} \in \Gamma(A_K(\eta, 1), \mathcal{E}^\vee \otimes \mathcal{E})$, so $\mathcal{E}$ admits a break decomposition over $A_K(\eta, 1)$ as desired. □

We exploit Lemma 2.7.3 via the following construction.

Definition 2.7.4. Define the *relativization* $\mathcal{F}$ of $\mathcal{E}$ as the ∇-module $\mathcal{E}$ itself, but viewed relative to L instead of K. That is, retain only the action of ∂_{n+1}. (The term "generic fibre" was used in an earlier version of this paper, but we decided to reserve that name for a different concept to appear in a subsequent paper.)

However, we are forced to make a crucial distinction.

Lemma 2.7.5. *For $i \in \{1, \dots, n+1\}$, there exists $\eta \in (0, 1)$ such that one of the following two statements is true.*

- *For all $\rho \in (\eta, 1)$, ∂_i is dominant for $\mathcal{E}_\rho$.*
- *For all $\rho \in (\eta, 1)$, ∂_i is not dominant for $\mathcal{E}_\rho$.*

Proof. Let b denote the highest break of $\mathcal{E}$. Choose $\eta \in (0, 1)$ such that $T(\mathcal{E}, \rho) = \rho^b$ for all $\rho \in (\eta, 1)$. Put

$$f_i(\rho) = \frac{|\partial_i|_{F_\rho,\mathrm{sp}}}{|\partial_i|_{\mathcal{E}_\rho,\mathrm{sp}}};$$

then Lemma 2.5.3 shows that f_i is log-concave. Consequently, if $f_i(\rho) = T(\mathcal{E}, \rho)$ for two different values of ρ, then the same is true for all intermediate values. This proves the claim: if the second statement does not hold, then there exist $\rho \in (0, 1)$ arbitrarily close to 1 such that $f_i(\rho) = T(\mathcal{E}, \rho)$, in which case the first statement holds with η equal to any such ρ. □

Definition 2.7.6. For $i \in \{1, \dots, n+1\}$, we say that ∂_i is *eventually dominant* for $\mathcal{E}$ if the first alternative in Lemma 2.7.5 holds, i.e., if there exists $\eta \in (0, 1)$ such that for all $\rho \in (\eta, 1)$, ∂_i is dominant for $\mathcal{E}_\rho$.

Remark 2.7.7. Note that if ∂_{n+1} is eventually dominant for $\mathcal{E}$, then the highest break term in the decomposition of $\mathcal{F}$ (which is respected by $\partial_1, \dots, \partial_n$ because it is unique) already has a uniform break. Our strategy in case ∂_{n+1} is not eventually dominant for $\mathcal{E}$ is to perform an operation which one might call *rotation* to recover that more favorable situation: namely, we use a concrete Taylor isomorphism to change the embedding of K into L.

In order to perform the rotation suggested in Remark 2.7.7, we need two particular instances of Definition 2.6.2.

Lemma 2.7.8. *For N a nonnegative integer, let $f_N : A_L(0, 1) \to A_L(0, 1)$ be the map fixing L and pulling back t to t^{p^N}. Then for $\rho \in (\epsilon, 1)$, we have the inequality $S(f_N^*\mathcal{E}, \rho^{1/p^N}) \geq S(\mathcal{E}, \rho)$. Moreover, if ∂_i is dominant for $\mathcal{E}_\rho$ for some $i \neq n+1$, then $T(f_N^*\mathcal{E}, \rho^{1/p^N}) = T(\mathcal{E}, \rho)$.*

Proof. The first assertion follows from Proposition 2.6.4. The second follows because if ∂_i is dominant for $\mathcal{E}_\rho$ and $i \neq n+1$, then $T(f_N^*\mathcal{E}, \rho^{1/p^N})$ and $T(\mathcal{E}, \rho)$ can be computed using the same formula. □

Lemma 2.7.9. *Suppose $i \in \{1, \dots, n\}$ is such that ∂_i is eventually dominant for $\mathcal{E}$. Let g_i be the map given by Definition 2.6.2 with*

$$u'_i = u_i + t, \qquad u'_j = u_j \quad (j \neq i), \qquad t' = t.$$

Put $\mathcal{E}' = g_i^*\mathcal{E}$, and let $\mathcal{F}'$ be the relativization of $\mathcal{E}'$. Let b, b_{rel} be the highest breaks of $\mathcal{E}$, $\mathcal{F}$. If $b > b_{\mathrm{rel}} + 1$, then:

- *the highest break of $\mathcal{F}'$ is $b - 1$;*
- *for $\rho \in (0, 1)$ sufficiently close to 1, the multiplicity of ρ^{b-1} in $S(\mathcal{F}', \rho)$ is the same as that of ρ^b in $S(\mathcal{E}, \rho)$.*

Proof. The action of ∂_{n+1} on $g_i^*\mathcal{E}$ is the pullback of the action of $\partial_{n+1} + \partial_i$ on $\mathcal{E}$, so the highest break of $\mathcal{F}'$ is the value of b' satisfying

$$|\partial_{n+1} + \partial_i|_{\mathcal{E}_\rho, \mathrm{sp}} = \rho^{-b'-1}$$

for $\rho \in (0, 1)$ sufficiently close to 1. For such ρ, the spectral norms of ∂_i, ∂_{n+1} on $\mathcal{E}_\rho$ are ρ^{-b}, $\rho^{-b_{\mathrm{rel}}-1}$, respectively. From this the claims are evident. □

Lemma 2.7.10. *Pick $i \in \{1, \dots, n+1\}$ such that ∂_i is eventually dominant for $\mathcal{E}$. Then at least one of the following statements is true.*

- *For $\rho \in (0, 1)$ sufficiently close to 1, the scale multiset of ∂_i on $\mathcal{E}_\rho$ consists of a single element.*
- *There exists $\eta \in (0, 1)$ such that $\mathcal{E}$ is decomposable on $A_L(\eta, 1)$.*

Proof. If $i = n+1$, then the claim follows by Remark 2.7.7, so we assume $i \leq n$. Let b and b_{rel} be the highest breaks of $\mathcal{E}$ and $\mathcal{F}$, respectively. Assume that the first alternative does not hold; this forces $b > 0$.

Suppose to begin with that $b > b_{\mathrm{rel}} + 1$. Put $\mathcal{E}' = g_i^*\mathcal{E}$ as in Lemma 2.7.9, and let $\mathcal{F}'$ be the relativization of $\mathcal{E}'$. Then $\mathcal{F}'$ does not have a uniform break, so by Lemma 2.7.3, it splits off a component of uniform break $b - 1$. We conclude that $\mathcal{E}'$ is decomposable on some $A_L(\eta, 1)$, as then is $\mathcal{E}$, as desired.

In the general case, we can always pick N such that $bp^N > b_{\mathrm{rel}} + 1$. By Lemma 2.7.8, $f_N^*\mathcal{E}$ has highest break bp^N, and the first alternative of this lemma also does not hold for $f_N^*\mathcal{E}$. Moreover, by Proposition 2.3.2, the relativization of $f_N^*\mathcal{E}$ has highest break b_{rel}. We may thus apply the previous paragraph to split off a component of $f_N^*\mathcal{E}$ of highest break; since the splitting is unique, it descends down the Galois group of the cover f_N (after adjoining p^N-th roots of unity), so $\mathcal{E}$ is itself decomposable on some $A_L(\eta, 1)$, as desired. □

Proof of Theorem 2.7.2. It suffices to show that if $\mathscr{E}$ is indecomposable over $A_L(\eta, 1)$ for any $\eta \in (0, 1)$ sufficiently close to 1, then $\mathscr{E}$ has a uniform break. This follows from Remark 2.7.7 if ∂_{n+1} is eventually dominant for $\mathscr{E}$, and from Lemma 2.7.10 otherwise. $\square$

It will be useful later to have a more uniform version of the rotation construction used in Section 2.7, which comes at the expense of enlarging the field L. (This generic rotation is inspired by the operation of *generic residual perfection* in [Borger 2004].) The resulting construction will be used to study the graded pieces of the ramification filtration.

Proposition 2.7.11. *Let b be the highest break of $\mathscr{E}$, and suppose $b > 1$. Let L' be the completion of $L(v_1, \dots, v_n)$ for the $(1, \dots, 1)$-Gauss norm, viewed as a differential field of order $2n$ over K. Let $\mathscr{E}'$ be the pullback of $\mathscr{E}$ along the map $f : A_{L'}[0, 1) \to A_L[0, 1)$ given by*

$$f^*(u_i) = u_i^p + v_i t^{p-1} \quad (i = 1, \dots, n), \qquad f^*(t) = \frac{t^p}{1 - t^{p-1}}.$$

Then $\mathscr{E}'$ has highest break $pb - p + 1$. In addition, among the differentials

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}, \frac{\partial}{\partial t},$$

$\frac{\partial}{\partial t}$ (at least) is eventually dominant for $\mathscr{E}'$.

Proof. We first treat the case $n = 0$. In this case, $g^*(t^{-1}) = t^{-p} - t^{-1}$, so this is an instance of [Kedlaya 2005a, Lemma 5.13].

In the general case, writing $\partial_1', \dots, \partial_{n+1}'$ for the actions of $\partial_1, \dots, \partial_{n+1}$ before the pullback, we have

$$\begin{aligned}
\frac{\partial}{\partial u_i} &= p u_i^{p-1} \partial_i', \\
\frac{\partial}{\partial v_i} &= t^{p-1} \partial_i', \\
\frac{\partial}{\partial t} &= \frac{d}{dt}\left(\frac{t^p}{1 - t^{p-1}}\right) \partial_{n+1}' + \sum_{i=1}^n (p-1) v_i t^{p-2} \partial_i'.
\end{aligned}$$

We compute the scale of $\partial/\partial t$ by inspecting each term separately: the contribution from ∂_{n+1}' can be treated as above, and the contribution from ∂_i' can be treated directly after invoking Proposition 2.6.5. This implies that the highest break of $\mathscr{E}'$ is at least $pb - p + 1$, with equality if and only if $\partial/\partial t$ is eventually dominant.

We compute the scale of $\partial/\partial u_i$ as if u_i had pulled back to u_i^p and t to t^p (i.e., as for a Frobenius antecedent). In particular, if $\partial/\partial u_i$ were eventually dominant for

$\mathscr{E}'$, then the highest break of $\mathscr{E}'$ would be at most $b < pb - p + 1$, contradiction. Hence $\partial/\partial u_i$ is not eventually dominant.

We read off the scale of $\partial/\partial v_i$ directly: it is eventually dominant if and only if ∂_i' is, and in any case it cannot mask $\partial/\partial t$. This proves the desired results. □

Remark 2.7.12. The calculations in this subsection may become more transparent when checked against the examples produced by Artin–Schreier covers in positive characteristic, as in Example 3.5.10. Indeed, many of these calculations were conceived with those examples firmly in mind.

2.8. *The differential Swan conductor.* Throughout this subsection, retain Hypothesis 2.5.2.

Definition 2.8.1. By Theorem 2.7.2, there exists a multiset $\{b_1, \dots, b_d\}$ such that for all $\rho \in (0, 1)$ sufficiently close to 1, $S(\mathscr{E}, \rho) = \{\rho^{b_1}, \dots, \rho^{b_d}\}$. We call this multiset the *break multiset* of $\mathscr{E}$, denoted $b(\mathscr{E})$. Define the *(differential) Swan conductor* of $\mathscr{E}$, denoted $\mathrm{Swan}(\mathscr{E})$, as $b_1 + \cdots + b_d$.

Theorem 2.8.2. *The differential Swan conductor of $\mathscr{E}$ is a nonnegative integer.*

Proof. It suffices to check this in case $\mathscr{E}$ is indecomposable over $A_L(\eta, 1)$ for any $\eta \in (0, 1)$ sufficiently close to 1. Choose $i \in \{1, \dots, n+1\}$ such that ∂_i is eventually dominant for $\mathscr{E}$. By Lemma 2.7.10, for $\rho \in (\epsilon, 1)$ sufficiently close to 1, the scale multiset of $\mathscr{E}_\rho$ with respect to ∂_i consists of a single element. That means in the calculation of the Newton polygon in Proposition 2.5.4, the Newton polygon must have only one slope, and so the integer j can be taken to be $\mathrm{rank}(\mathscr{E})$. Consequently, the slopes of the function $f(r) = \log T(\mathscr{E}, e^{-r})$ are always multiples of $1/\mathrm{rank}(\mathscr{E})$, as then is the highest break of $\mathscr{E}$. This proves the desired result. □

Remark 2.8.3. Proposition 2.6.5 implies that pulling $\mathscr{E}$ along the map $t \mapsto t^N$, for N a positive integer coprime to p, has the effect of multiplying $\mathrm{Swan}(\mathscr{E})$ by N. For Galois representations, this will imply that the Swan conductor commutes appropriately with tamely ramified base changes (Theorem 3.5.9).

Remark 2.8.4. In case $L = K$, one can interpret the integrality of $\mathrm{Swan}(\mathscr{E})$ by equating it to a certain local index [Christol and Mebkhout 2001, Théorème 2.3-1]. It would be interesting to give a cohomological interpretation of our more general construction, perhaps by relating it to an appropriate Euler characteristic.

Remark 2.8.5. Liang Xiao points out that one can also prove Theorem 2.8.2 by reduction to the case of perfect residue field, for which one may invoke Remark 2.8.4. The argument is as follows. By Lemma 2.7.10, we may assume that $\mathscr{E}$ and its relativization have respective uniform breaks b, b_{rel}. The perfect residue field case implies that $b_{\mathrm{rel}}\,\mathrm{rank}(\mathscr{E})$ is an integer. If $b \neq b_{\mathrm{rel}}$, we can choose positive integers m_1, m_2 coprime to each other and to p such that $m_i(b - b_{\mathrm{rel}}) > 1$ for $i = 1, 2$. If we pull back along $t \mapsto t^{m_i}$ and then apply the rotation in Lemma 2.7.9, the highest break of the relativization becomes $m_i b - 1$, so $(m_i b - 1)\operatorname{rank}(\mathscr{E})$ is an integer for $i = 1, 2$. This implies that $b \operatorname{rank}(\mathscr{E}) \in \mathbb{Z}$.

3. Differential conductors for Galois representations

In this section, we explain how to define differential Swan conductors for certain p-adic Galois representations of complete discretely valued fields of equal characteristic $p > 0$ (including the discrete representations). This uses a setup for turning representations into differential modules due to [Tsuzuki 1998]. For comments on the mixed characteristic case, see Section 3.7.

3.1. *Preliminaries: Cohen rings.*

Definition 3.1.1. Let k be a field of characteristic $p > 0$. A *Cohen ring* for k is a complete discrete valuation ring C_k with maximal ideal generated by p, equipped with an isomorphism of its residue field with k.

It can be shown that Cohen rings exist and are unique up to noncanonical isomorphism; see [Bourbaki 1983]. One can do better by carrying some extra data.

Definition 3.1.2. Define a *based field* of characteristic $p > 0$ to be a field k equipped with a distinguished p-basis B_k. We view based fields as forming a category whose morphisms from (k, B_k) to (k', B'_k) are morphisms $k \to k'$ of fields carrying B_k into B'_k.

Definition 3.1.3. For (k, B_k) a based field, a *based Cohen ring* for (k, B_k) is a pair (C, B), where C is a Cohen ring for k and B is a subset of C which lifts B_k.

Proposition 3.1.4. *There is a functor from based fields to based Cohen rings which is a quasi-inverse of the residue field functor. In particular, any map between based fields lifts uniquely to given based Cohen rings.*

Proof. This is implicit in Cohen's original paper [Cohen 1946]; an explicit proof is given in [Whitney 2002, Theorem 2.1]. Here is a sketch of another proof. Let W_n be the ring of p-typical Witt vectors of length n over k, let W be the inverse limit of the W_n, let F be the Frobenius on W, and let $[\cdot]$ denote the Teichmüller map. Put $B = \{[\bar{b}] : \bar{b} \in B_k\}$. Let C_n be the image of $F^n(W)[B]$ in W_n. Then the projection $W_{n+1} \to W_n$ induces a surjection $C_{n+1} \to C_n$. Let C be the inverse limit of the C_n; one then verifies that (C, B) is a based Cohen ring for (k, B_k), and functoriality of the construction follows from functoriality of the Witt ring. $\square$

Remark 3.1.5. In fact, [Whitney 2002, Theorem 2.1] asserts something slightly stronger: if (C, B) is a based Cohen ring of (k, B_k), R is any complete local ring with residue field k, and B_R is a lift of B_k to R, then there is a unique ring homomorphism $C \to R$ inducing the identity on k and carrying B to B_R.

3.2. *Galois representations and* (ϕ, ∇)*-modules.*

Hypothesis 3.2.1. For the remainder of this section, let R be a complete discrete valuation ring of equal characteristic $p > 0$, with fraction field E and residue field k. Let $k_0 = \bigcap_{n \geq 0} k^{p^n}$ be the maximal perfect subfield of k; note that k_0 embeds canonically into R (whereas if $k \neq k_0$, then k embeds but not canonically).

Convention 3.2.2. Put $G_E = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. Let $\mathbb{O}$ be the integral closure of $\mathbb{Z}_p$ in a finite extension of $\mathbb{Q}_p$, whose residue field $\mathbb{F}_q$ is contained in k. Throughout this section, a "representation" will be a continuous representation $\rho : G_E \to \mathrm{GL}(V)$, where $V = V(\rho)$ is a finite free $\mathbb{O}$-module. (One can also consider representations on finite dimensional $\mathrm{Frac}(\mathbb{O})$-vector spaces, by choosing lattices; for brevity, we stick to statements for integral representations, except for Remark 3.5.11.)

Definition 3.2.3. Fix a based Cohen ring (C_E, B) with residue field E; note that C_E is canonically a $W(\mathbb{F}_q)$-algebra. Put

$$\Gamma = C_E \otimes_{W(\mathbb{F}_q)} \mathbb{O}.$$

Let $\Omega^1_{\Gamma/\mathbb{O}}$ be the completed (for the p-adic topology) direct sum of $\Gamma\, db$ over all $b \in B$, i.e., the inverse limit over n of $\oplus_{b \in B} (\Gamma/p^n\Gamma)\, db$; then there is a canonical derivation $d : \Gamma \to \Omega^1_{\Gamma/\mathbb{O}}$. Note that all of this data stays canonically independent of the choice of B as long as C_E remains fixed.

Definition 3.2.4. A ∇*-module* over Γ is a finite free Γ-module M equipped with an integrable connection $\nabla : M \to M \otimes_\Gamma \Omega^1_{\Gamma/\mathbb{O}}$; integrability means that the composition of ∇ with the map $M \otimes \Omega^1_{\Gamma/\mathbb{O}} \to M \otimes \wedge^2_\Gamma \Omega^1_{\Gamma/\mathbb{O}}$ induced by ∇ is the zero map.

Definition 3.2.5. A *Frobenius lift* on Γ is an endomorphism $\phi : \Gamma \to \Gamma$ fixing $\mathbb{O}$ and lifting the q-power Frobenius map on E. For instance, there is a unique such ϕ carrying b to b^q for each $b \in B$ (induced by the Frobenius action on the construction given in Proposition 3.1.4); we call this ϕ the *standard Frobenius lift with respect to* B. A ϕ*-module* (resp. (ϕ, ∇)*-module*) over Γ is a finite free module (resp. ∇-module) M over Γ equipped with an isomorphism $F : \phi^* M \cong M$ of modules (resp. of ∇-modules); we interpret F as a semilinear action of ϕ on M.

Definition 3.2.6. For any representation ρ, put

$$D(\rho) = (V(\rho) \otimes_{\mathbb{O}} \widehat{\Gamma^{\mathrm{unr}}})^{G_E}.$$

By Hilbert's Theorem 90, the natural map

$$D(\rho) \otimes_\Gamma \widehat{\Gamma^{\mathrm{unr}}} \to V(\rho) \otimes_{\mathbb{O}} \widehat{\Gamma^{\mathrm{unr}}}$$

is a bijection; in particular, $D(\rho)$ is a free Γ-module and $\operatorname{rank}_\Gamma(D(\rho)) = \operatorname{rank}_{\mathbb{O}}(V)$. If we equip Γ^{unr} and its completion with actions of the derivation d and any Frobenius lift ϕ (acting trivially on $V(\rho)$), we obtain by restriction a Frobenius action and connection on $D(\rho)$, turning it into a (ϕ, ∇)-module.

Proposition 3.2.7. *For any Frobenius lift ϕ on Γ, the functor D from representations to ϕ-modules over Γ is an equivalence of categories.*

Proof. Given a ϕ-module M over Γ, put

$$V(M) = (M \otimes_\Gamma \widehat{\Gamma^{\mathrm{unr}}})^{\phi=1}.$$

As in [Fontaine 1990, A1.2.6] or [Tsuzuki 1998, Theorem 4.1.3], one checks that V is a quasi-inverse to D. □

Proposition 3.2.8. *For any Frobenius lift ϕ on Γ, any ϕ-module over Γ admits a unique structure of (ϕ, ∇)-module. Consequently, the functor D from representations to (ϕ, ∇)-modules over Γ is an equivalence of categories.*

Proof. Existence of such a structure follows from Proposition 3.2.7, so we focus on uniqueness. Let M be a (ϕ, ∇)-module over Γ. Let $\partial/\partial b$ be the derivations dual to the db for $b \in B$. Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be a basis of M, and let Φ and N_b be the matrices via which ϕ and $\partial/\partial b$ act on this basis. Then the fact that the ϕ-action on M respects the ∇-module structure implies that

$$N\Phi + \frac{\partial \Phi}{\partial b} = \frac{\partial \phi(b)}{\partial b} \Phi N. \tag{3.2.8.1}$$

Let π be a uniformizer of $\mathbb{O}$; note that $\partial\phi(b)/\phi b \equiv 0 \pmod{\pi}$ because $\phi(b) \equiv b^q \pmod{\pi}$. Consequently, for fixed Φ, if N_b is uniquely determined modulo π^m, then the right side of (3.2.8.1) is determined modulo π^{m+1}, as then is $N_b\Phi$. Since Φ is invertible, N_b is also determined modulo π^{m+1}. By induction, N_b is uniquely determined by Φ for each b, as desired. □

3.3. ***Representations with finite local monodromy.*** We now distinguish the class of representations for which we define differential Swan conductors.

Definition 3.3.1. Let $I_E = \mathrm{Gal}(E^{\mathrm{sep}}/E^{\mathrm{unr}})$ be the inertia subgroup of G_E. We say a representation ρ *has finite local monodromy* if the image of I_E under ρ is finite.

For representations with finite local monodromy, we can refine the construction of the (ϕ, ∇)-module associated to ρ.

Hypothesis 3.3.2. For the remainder of this subsection, assume that k admits a *finite* p-basis. Assume also that the based Cohen ring (C_E, B) has been chosen with $B = B_0 \cup \{t\}$, where t lifts a uniformizer of E, and B_0 lifts elements of R which in turn lift a p-basis of k.

Definition 3.3.3. By the proof of the Cohen structure theorem, or by Remark 3.1.5, there is a unique embedding of k into R whose image contains the image of B_0 under reduction to E. Applying Proposition 3.1.4 to the map $k \to R$, we obtain an embedding of a Cohen ring C_k for k into C_E, the image of which contains B_0. Put

$$\mathbb{O}_k = C_k \otimes_{W(\mathbb{F}_q)} \mathbb{O}.$$

Then each $x \in \Gamma$ can be written formally as a sum $\sum_{i \in \mathbb{Z}} x_i t^i$ with $x_i \in \mathbb{O}_k$, such that for each n, the indices i for which $v_{\mathbb{O}_k}(x_i) \leq n$ are bounded below. For n a nonnegative integer, we define the *partial valuation function* $v_n : \Gamma \to \mathbb{Z} \cup \{\infty\}$ by

$$v_n(x) = \min\{i \in \mathbb{Z} : v_{\mathbb{O}_k}(x_i) \leq n\}.$$

For $r > 0$, put

$$\Gamma^r = \{x \in \Gamma : \lim_{n \to \infty} v_n(x) + rn = \infty\};$$

this is a subring of Γ. Put $\Gamma^\dagger = \bigcup_{r>0} \Gamma^r$; we may speak of ∇-modules over $\Gamma^\dagger$ using the same definition as for Γ, using for the module of differentials

$$\Omega^1_{\Gamma^\dagger/\mathbb{O}_k} = \bigoplus_{b \in B} \Gamma^\dagger \, db.$$

(Here we are using the finiteness of the p-basis to avoid having to worry about a completion.) If ϕ is a Frobenius lift carrying $\Gamma^\dagger$ into itself, we may also define ϕ-modules and (ϕ, ∇)-modules over $\Gamma^\dagger$ as before.

Definition 3.3.4. Since $\mathbb{O}_k \subset \Gamma^\dagger$, we can identify a copy of $\mathbb{O}_k^{\mathrm{unr}}$ inside $(\Gamma^\dagger)^{\mathrm{unr}}$. Using this identification, put

$$\tilde{\Gamma}^\dagger = \widehat{\mathbb{O}_k^{\mathrm{unr}}} \otimes_{\mathbb{O}_k^{\mathrm{unr}}} (\Gamma^\dagger)^{\mathrm{unr}} \subset \widehat{\Gamma^{\mathrm{unr}}}.$$

For ρ a representation, put

$$D^\dagger(\rho) = D(\rho) \cap (V(\rho) \otimes_{\mathbb{O}} \tilde{\Gamma}^\dagger) = (V(\rho) \otimes_{\mathbb{O}} \tilde{\Gamma}^\dagger)^{G_E}.$$

Again, $D^\dagger(\rho)$ inherits a connection, and an action of any Frobenius lift ϕ acting on $\Gamma^\dagger$. Note that the natural map

$$(D^\dagger(\rho) \otimes_{\Gamma^\dagger} \tilde{\Gamma}^\dagger) \to (V(\rho) \otimes_{\mathbb{O}} \tilde{\Gamma}^\dagger)$$

is always injective, and it is surjective if and only if ρ has finite local monodromy.

The following is essentially [Tsuzuki 1998, Theorem 3.1.6].

Proposition 3.3.5. *Let ϕ be a Frobenius lift on Γ acting on $\Gamma^\dagger$. The base change functor from (ϕ, ∇)-modules over $\Gamma^\dagger$ to (ϕ, ∇)-modules over Γ is fully faithful.*

Proof. Using internal Homs, we may rephrase this as follows: if M is a (ϕ, ∇)-module over $\Gamma^\dagger$, then

$$(M \otimes \Gamma)^{\phi=1,\nabla=0} \subset M.$$

In particular, it is sufficient to check this using only the dt component of the connection. In this case, we may replace Γ by the completion of $\Gamma[\phi^{-n}(b) : b \in B_0, n \in \mathbb{Z}_{\geq 0}]$, to get into the case where R has perfect residue field. We may then conclude by applying [Tsuzuki 1996, 4.1.3]. □

The following is essentially [Tsuzuki 1998, Theorem 4.2.6].

Theorem 3.3.6. *Let ϕ be a Frobenius lift on Γ acting on $\Gamma^\dagger$. Then $D^\dagger$ and restriction induce equivalences between the following categories*:

(a) *representations with finite local monodromy*;

(b) (ϕ, ∇)*-modules over* $\Gamma^\dagger$;

(c) ∇*-modules over* $\Gamma^\dagger$ *equipped with* ϕ*-actions over* Γ.

In particular, if a ∇*-module over* $\Gamma^\dagger$ *admits a* ϕ*-action over* Γ, *that action is defined already over* $\Gamma^\dagger$.

Proof. The functor from (a) to (b) is $D^\dagger$, while the functor from (b) to (c) is restriction. The functor from (c) back to (a) will be induced by V; once it is shown to be well-defined, it will be clear that the three functors compose to the identity starting from any point.

To obtain the functor from (c) to (a), we must prove that if M is a ∇-module over $\Gamma^\dagger$ such that $M \otimes \Gamma$ admits a compatible ϕ-action, then the corresponding representation $V(M)$ has finite local monodromy. It suffices to check this after replacing E by a finite extension, which can be chosen to ensure the existence of an isomorphism $(M/2pM) \otimes \Gamma \cong (\Gamma/2p\Gamma)^m$ of ϕ-modules. In this case we claim that $V(M)$ is actually unramified; as in the proof of Proposition 3.3.5, it suffices to check this using only the dt component of ∇, and hence to reduce to the case of R having perfect residue field. This case is treated by the proof of [Tsuzuki 1998, Proposition 5.2.1], but not by its statement (which requires a ϕ-action over $\Gamma^\dagger$); for a literal citation, see [Kedlaya 2006a, Proposition 4.5.1]. □

3.4. (ϕ, ∇)*-Modules over* $\mathcal{R}$. Throughout this subsection, retain Hypothesis 3.3.2, and write L for $\mathrm{Frac}(\mathbb{O}_k)$ and $\mathcal{R}$ for $\mathcal{R}_L$. The choices made so far determine an embedding $\Gamma^\dagger \hookrightarrow \mathcal{R}_L$, and any Frobenius ϕ acting on $\Gamma^\dagger$ extends continuously to $\mathcal{R}_L$ (as in [Kedlaya 2004, Section 2]). We may thus define ϕ-modules, ∇-modules, and (ϕ, ∇)-modules over $\mathcal{R}$ using the same definitions as over Γ.

Remark 3.4.1. From a ∇-module over $\mathcal{R}$, we may construct a ∇-module on $A_L(\epsilon, 1)$ for some $\epsilon \in (0, 1)$. The construction is unique in the following sense: any two such ∇-modules become isomorphic on $A_L(\eta, 1)$ for some $\eta \in (0, 1)$.

Conversely, since any locally free sheaf on $A_L(\eta, 1)$ is freely generated by global sections (because L is spherically complete; see for instance [Kedlaya 2005a, Theorem 3.14]), any ∇-module on $A_L(\eta, 1)$ gives rise to a ∇-module over $\mathcal{R}$.

Remark 3.4.1 is sufficient for the construction of the differential Swan conductor associated to a representation of finite local monodromy. However, for completeness, we record some related facts, including the analogue of the p-adic local monodromy theorem.

Lemma 3.4.2. *Let M be a ϕ-module over $\Gamma^\dagger$ such that $M \otimes \mathcal{R}$ admits the structure of a (ϕ, ∇)-module. Then this structure is induced by a (ϕ, ∇)-module structure on M itself, and so M corresponds to a representation with finite local monodromy.*

Proof. By [Kedlaya 2005b, Proposition 7.1.7], the action of $\partial/\partial t$ on $M \otimes \mathcal{R}$ acts on M itself. Also, for any $b \in B_0$, we may change the p-basis by replacing b by $b+t$, and then the same argument shows that the action of $\frac{\partial}{\partial t} + \frac{\partial}{\partial b}$ on $M \otimes \mathcal{R}$ acts on M itself. (This is another instance of rotation in the sense of Remark 2.7.7.) We conclude that ∇ itself acts on M, so we may apply Theorem 3.3.6 to conclude. $\square$

Definition 3.4.3. A ϕ-module (resp. (ϕ, ∇)-module) M over $\mathcal{R}$ is *unit-root* if it has the form $M_0 \otimes \mathcal{R}$ for some ϕ-module (resp. (ϕ, ∇)-module) M_0 over $\Gamma^\dagger$. By Lemma 3.4.2, a (ϕ, ∇)-module over $\mathcal{R}$ is unit-root if and only if its underlying ϕ-module is unit-root.

Proposition 3.4.4. *The base extension functor from the isogeny category of unit-root ϕ-modules over $\Gamma^\dagger$ (i.e., ϕ-modules over $\Gamma^\dagger[\frac{1}{p}]$ obtained by base extension from $\Gamma^\dagger$) to the category of unit-root ϕ-modules over $\mathcal{R}$ is an equivalence of categories.*

Proof. This is [Kedlaya 2005b, Theorem 6.2.3]. $\square$

Definition 3.4.5. Let $s = c/d$ be a rational number written in lowest terms. A ϕ-module (resp. (ϕ, ∇)-module) M over $\mathcal{R}$ is *pure* (or *isoclinic*) of slope s if there exists a scalar $\lambda \in K^*$ of valuation c such that the ϕ^d-module (resp. (ϕ^d, ∇)-module) obtained from M by twisting the ϕ^d-action by λ^{-1} is unit-root. In particular, by Theorem 3.3.6, the ∇-module structure on M corresponds to a representation with finite local monodromy after replacing $\mathcal{O}$ by a finite extension.

Theorem 3.4.6. *Let M be a ϕ-module (resp. (ϕ, ∇)-module) M over $\mathcal{R}$. Then there exists a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ of M by saturated ϕ-submodules (resp. (ϕ, ∇)-submodules) such that each quotient M_i/M_{i-1} is pure of some slope s_i as a ϕ-module, and $s_1 < \cdots < s_l$.*

Proof. In the ϕ-module case, this is [Kedlaya 2004, Theorem 6.10] or [Kedlaya 2005b, Theorem 6.4.1]. In the (ϕ, ∇)-module case, it suffices to check that the filtration of the underlying ϕ-module is respected by ∇. For this, we proceed as in [Kedlaya 2005b, Theorem 7.1.6]: for each derivation $\partial/\partial b$, we get a morphism of ϕ-modules $M_1 \to (M/M_1) \otimes \mathcal{R}\, db$. The former is pure of slope s_1, whereas the latter admits a slope filtration in which each slope is strictly greater than s_1 (the slope of $\mathcal{R}\, db$ being positive). By [Kedlaya 2005b, Proposition 4.6.4], that morphism is zero, proving that M_1 is respected by each derivation. Hence M_1 is a (ϕ, ∇)-submodule, and repeating the argument on M/M_1 yields the claim. □

Remark 3.4.7. One may apply Theorem 3.3.6 to each individual quotient of the filtration produced by Theorem 3.4.6. (Alternatively, one may project ∇ onto the dt component and directly invoke the p-adic local monodromy theorem; this allows the invocation of [André 2002] or [Mebkhout 2002] in place of [Kedlaya 2004].) It is an interesting question, which we have not considered, whether one can show that the category of (ϕ, ∇)-module M over $\mathcal{R}_L$ is equivalent to a category of representations of G_E times an algebraic group over $\mathrm{Frac}(\mathcal{O})$, as in [Kedlaya 2005a, Theorem 4.45].

3.5. ***Defining the differential Swan conductor.*** In order to use Theorem 3.3.6 to define the differential Swan conductor of a representation $\rho : G_E \to \mathrm{GL}(V)$ with finite local monodromy, we must check that the answer does not depend on the auxiliary choices we made along the way. (Note that the choice of ϕ does not matter: it is only used to define the Frobenius action on $D^\dagger(\rho)$, whereas only the connection is used to compute the conductor.)

Proposition 3.5.1. *Suppose that k admits a finite p-basis. For ρ a representation with finite local monodromy, the isomorphism type of the ∇-module $D^\dagger(\rho)$ does not depend on the choice of the Cohen ring C_E or the lifted p-basis B.*

Proof. By Proposition 3.1.4, the construction of C_E is functorial in pairs $(E, \overline{B})$, where $\overline{B}$ is a p-basis of E. It thus suffices to check that if for $i = 1, 2$, $\overline{B_i}$ is a p-basis of E consisting of a uniformizer $\overline{t_i}$ of R and a lift $\overline{B_{i,0}}$ to R of a p-basis of k over k_0, then the modules $D^\dagger(\rho)$ constructed using lifts of $\overline{B_1}$ and $\overline{B_2}$ are isomorphic, compatibly with some isomorphism of the underlying rings $\Gamma^\dagger$.

Let (C_E, B_1) be a based Cohen ring lifting $(E, \overline{B_1})$; write $C_{k,1}, t_1$ instead of C_k, t. Define $B_{2,0}$ by choosing, for each $\bar{b} \in B_{2,0}$, a lift b of $\bar{b}$ in $C_{k,1}[\![t_1]\!]$. Then choose t_2 to be a lift of $\overline{t_2}$ belonging to $t_1 C_{k,1}[\![t_1]\!]$. We can then view (C_E, B_2) as a based Cohen ring lifting $(E, \overline{B_2})$, containing a Cohen ring $C_{k,2}$ for k.

Since we used the same ring C_E for both lifts, we may identify the two rings Γ. Although $C_{k,1} \neq C_{k,2}$ in general, we did ensure by construction that $C_{k,1}[\![t_1]\!] = C_{k,2}[\![t_2]\!]$. Consequently, the two rings $\Gamma^\dagger$ constructed inside Γ coincide, and we may identify the two copies of $\widetilde{\Gamma}^\dagger$. This gives an identification of the two modules $D^\dagger(\rho)$, as desired. □

Definition 3.5.2. Suppose to start that k is finite over k^p, i.e., any p-basis of k or of E is finite. For $\rho : G_E \to \mathrm{GL}(V)$ a representation with finite local monodromy, with V a finite dimensional $\mathbb{O}$-module, we may now define the *differential highest break*, *differential break multiset*, and *differential Swan conductor* by constructing the (ϕ, ∇)-module $D^\dagger(\rho)$, for some Cohen ring C_E and some lifted p-basis B, and computing the corresponding quantities associated to the underlying ∇-module of $D^\dagger(\rho)$ tensored with the Robba ring $\mathcal{R}_{\mathrm{Frac}(\mathbb{O}_k)}$ (as in Remark 3.4.1). By Proposition 2.6.6 (to change the p-basis of k) and Proposition 3.5.1, this definition depends only on ρ and not on any auxiliary choices. For general k, we may choose a finite subset B_1 of B containing a lift t of a uniformizer of R, project onto the span of the db for $b \in B_1$, and compute a conductor that way; this has the same effect as passing from E to $E_1 = \widehat{E_0}$, where

$$E_0 = E(\bar{b}^{1/p^n} : b \in B \setminus B_1, n \in \mathbb{N}_0).$$

We define the differential Swan conductor of ρ in this case to be the supremum over all choices of B and B_1; it will turn out to be finite (Corollary 3.5.7) and hence integral by Theorem 2.8.2.

Definition 3.5.3. Let E'/E be a finite separable extension, let $\overline{B}$ be a p-basis of E containing a uniformizer $\bar{t}$ of E, and put $\overline{B_0} = \overline{B} \setminus \{\bar{t}\}$. We say a subset $\overline{B_2}$ of $\overline{B_0}$ is a *rectifying set* for E'/E if, putting $E_2 = \widehat{E_0}$ for

$$E_0 = E(\bar{b}^{1/p^n} : \bar{b} \in \overline{B_2}, n \in \mathbb{N}_0),$$

the extension $(E' \otimes_E E_2)/E_2$ has separable residue field extension. Beware that it is not enough for the residue field of E_2 to contain the perfect closure of k in the residue field of E'. For instance, if $p > 2$, $\overline{b_1}, \overline{b_2} \in \overline{B}$, and

$$E' = E[z]/(z^p - z - \overline{b_1}\bar{t}^{-2p} - \overline{b_2}\bar{t}^{-p}),$$

then E' has residue field $k(\overline{b_1}^{1/p})$, but $\overline{B_2} = \{\overline{b_1}\}$ is not a rectifying set because the residue field of $E' \otimes_E E_2$ contains $\overline{b_2}^{1/p}$.

Lemma 3.5.4. *With notation as in Definition 3.5.3, $\overline{B}$ contains a finite rectifying set for E'/E.*

Proof. Use $\overline{B}$ to embed k into E. By induction on the degree of E'/E, we may reduce to the case of an Artin–Schreier extension

$$E' = E[z]/(z^p - z - a_n\bar{t}^{-n} - \cdots - a_1\bar{t}^{-1} - a_0)$$

with $a_i \in k$. In this case, pick any $N \in \mathbb{N}_0$ with $p^N > n$, and write each a_i as a k^{p^N}-linear combination of products of powers of elements of $\overline{B_0}$. Only finitely many elements of $\overline{B_0}$ get used; those form a rectifying set. $\square$

Proposition 3.5.5. *Suppose that there exists a finite separable extension E' of E whose residue field is separable over k, such that ρ is unramified on $G_{E'}$. Then the differential break multiset and Swan conductor of a representation ρ with finite local monodromy can be computed with respect to $\{t\}$, and it agrees with the usual break multiset and Swan conductor.*

Proof. It suffices to consider ρ irreducible and check equality for the highest breaks. Note that the usual highest break is insensitive to further residue field extension, because it can be computed using Herbrand's formalism as in [Serre 1979, Chapter IV]. It thus agrees with the differential highest break computed with respect to $\{t\}$: namely, this claim reduces to the case where k is perfect, for which see [Kedlaya 2005a, Theorem 5.23] and references thereafter.

It remains to show that for any B_1, $\partial/\partial t$ must be eventually dominant. Suppose the contrary, and pick $b \in B_1 \setminus \{t\}$ such that $\partial/\partial b$ is eventually dominant. By a tame base change (invoking Proposition 2.6.5), we can force the gap between the differential highest breaks computed with respect to B_1 and with respect to $\{t\}$ to be greater than 1; then a rotation as in Lemma 2.7.10 sending b to $b+t$ raises the differential highest break computed with respect to $\{t\}$. But that contradicts the previous paragraph: both before and after rotation, the differential highest break computed with respect to $\{t\}$ must coincide with the usual highest break.

We deduce that $\partial/\partial t$ is eventually dominant, proving the claim. □

Corollary 3.5.6. *In the notation of Definition 3.5.2, suppose that there exists a finite separable extension E' of E such that ρ is unramified on $G_{E'}$, and that the image of $B_1 \setminus \{t\}$ in E is a rectifying set for E'/E. Then the differential Swan conductor of ρ computed using $(B \setminus B_1) \cup \{t\}$ is equal to that computed using t.*

Corollary 3.5.7. *In the notation of Definition 3.5.2, suppose that there exists a finite separable extension E' of E such that ρ is unramified on $G_{E'}$, and that the image of $B_1 \setminus \{t\}$ in E is a rectifying set for E'/E. Then the differential Swan conductor of ρ is equal to that computed using B_1.*

For completeness, we record the following observations.

Theorem 3.5.8. *The differential Swan conductor of any representation with finite local monodromy is a nonnegative integer.*

Proof. By Lemma 3.5.4 and Corollary 3.5.7, the conductor can be computed using a finite set B_1; we may thus apply Theorem 2.8.2. □

Theorem 3.5.9. *Let E' be a tamely ramified extension of E of ramification degree m. Let ρ be a representation of G_E with finite local monodromy, and let ρ' be the restriction of ρ to $G_{E'}$. Then* $\mathrm{Swan}(\rho') = m\,\mathrm{Swan}(\rho)$.

Proof. Apply Proposition 2.6.5. □

Example 3.5.10. As an example, consider a nontrivial character of the Artin–Schreier extension $E[z]/(z^p - z - \bar{x})$. The corresponding differential module will be a Dwork isocrystal, i.e., a rank one ∇-module with generator $\mathbf{v}$ such that

$$\nabla(\mathbf{v}) = \pi \mathbf{v} \otimes dx,$$

for x some lift of $\bar{x}$ and π a $(p-1)$-st root of $-p$. One computes that the differential Swan conductor for this character is equal to the least integer $m \geq 0$ such that $v_E(x - y^p + y) \geq -m$ for all $y \in E$. This agrees with the definition given by [Kato 1989] of the Swan conductor of a character; note that the conductor is allowed to be divisible by p if and only if k is imperfect.

Remark 3.5.11. Given a representation $\rho : G_E \to \mathrm{GL}(V)$, where V is a finite dimensional $\mathrm{Frac}(\mathbb{O})$-vector space, we may define a differential Swan conductor for it by picking a ρ-stable $\mathbb{O}$-lattice of V and proceeding as in Definition 3.5.2. Changing the lattice will not change the resulting ∇-module over $\mathcal{R}_{\mathrm{Frac}(\mathbb{O}_k)}$, so we get a well-defined numerical invariant of ρ also.

Defining conductors for Galois representations is tantamount to filtering the Galois group; let us now make this explicit.

Definition 3.5.12. Put $G_E^0 = I_E$. For $r > 0$, let R_r be the set of representations ρ with highest break less than r, and put

$$G_E^r = \bigcap_{\rho \in R_r} (I_E \cap \ker(\rho)).$$

Note that $\rho \in R_r$ if and only if $G_E^r \subseteq I_E \cap \ker(\rho)$; this reduces to the fact that R_r is stable under tensor product and formation of subquotients. We call G_E^r the *differential upper numbering filtration* on G_E. Write G_E^{r+} for the closure of $\bigcup_{s>r} G_E^s$; note that $G_E^r = G_E^{r+}$ for r irrational, because differential highest breaks are always rational numbers.

As in the perfect residue field case, the graded pieces of the upper numbering filtration are particularly simple.

Theorem 3.5.13. *For $r > 0$ rational, G_E^r/G_E^{r+} is abelian and killed by p.*

Proof. Let E' be a finite Galois extension of E with $\mathrm{Gal}(E'/E) = G$; then we obtain an induced filtration on G by taking G^r to be the image of G_E^r under the surjection $G_E \to G$. It suffices to check that G^r/G^{r+} is abelian and killed by p; moreover, we may quotient further to reduce to the case where G^{r+} is the trivial group but G^r is not. Let ρ be the regular representation of G; then ρ has highest break r. Let S be the set of irreducible constituents of ρ of highest break strictly less than r; we are then trying to show that the intersection of $\ker(\psi) \subseteq G$ over all $\psi \in S$ is an elementary abelian p-group.

By Corollary 3.5.7, we may reduce to the case where the lifted p-basis B of Hypothesis 3.3.2 is finite; put $B_0 = \{b_1, \dots, b_n\}$. By making a tame base change, we can force all nonzero ramification breaks to be greater than 1. By another base change (passing from $\operatorname{Frac} C_E$ to the completion of $C_E(v_1, \dots, v_n)$ for the $(1, \dots, 1)$-Gauss norm), we can add extra elements $v_1, \dots, v_n$ to B, then perform the operation described in Proposition 2.7.11. Each nonzero ramification break m before the operation corresponds to the break $pm - p + 1$ afterwards, so the desired result may be checked afterwards. But now $\partial/\partial t$ is dominant on every irreducible component of ρ, so we may reduce to the case of perfect residue field and (by Proposition 3.5.5) the usual upper numbering filtration. In this case, the claim is standard: it follows from the fact that the upper numbering filtration can be constructed by renumbering the lower numbering filtration [Serre 1979, Section IV.3], for which the claim is easy to check [Serre 1979, Section IV.2, Corollary 3 of Proposition 7]. $\square$

Remark 3.5.14. Note that the definition of the differential Swan conductor of a representation is invariant under enlarging $\mathbb{O}$, because the differential Swan conductor of a ∇-module is invariant under enlarging the constant field K.

3.6. ***Reconciliation questions.*** By introducing a numerical invariant of representations and calling it a conductor, one begs various reconciliation questions with other definitions. To begin with, it is known (and was a motivation of our construction) that in the traditional case of a perfect residue field, one computes the right numbers; see Proposition 3.5.5.

In the general case, there is a definition of the "logarithmic conductor" due to [Abbes and Saito 2002; Abbes and Saito 2003]. Following [Matsuda 2004], one is led to ask the following.

Question 3.6.1. For ρ a representation with finite local monodromy, does the differential Swan conductor agree with the Abbes–Saito logarithmic conductor in equal characteristic?

It is easy to check the affirmative answer for Artin–Schreier characters. An affirmative answer in the general case would have the beneficial consequence of verifying the Hasse–Arf theorem for the Abbes–Saito conductor in equal characteristic. Some progress on this question has been made recently by Bruno Chiarellotto and Andrea Pulita, and independently by Liang Xiao.

One might also try to reconcile our definition with conductors for Galois representations over a two-dimensional local field, as in [Zhukov 2000; Zhukov 2003]. In order to formulate a precise question, it may be easiest to pass to the context of considering a representation of the étale fundamental group of a surface and computing its conductor along different boundary divisors. Indeed, this will be the point of view of the sequel to this paper.

There is also a construction of Artin conductors in the imperfect residue field case due to [Borger 2004], by passing from E to a certain extension which is universal for the property of having perfect residue field. Borger's construction does not behave well with respect to tame base extension, but one should get a better invariant by forcing such good behavior (i.e., constructing a logarithmic analogue of Borger's conductor). Indeed, we expect the following.

Conjecture 3.6.2. *For ρ a representation with finite local monodromy, for m a positive integer, let E_m be an extension of E which is tamely ramified of tame degree m. Let $b'(E_m)$ be Borger's Artin conductor of the restriction of ρ to G_{E_m}. Then the limsup of $m^{-1}b'(E_m)$ as $m \to \infty$ equals the differential Swan conductor of ρ.*

Since the Abbes–Saito construction also works in mixed characteristic, one may also be interested in reconciling it there with a differential construction. For more on this possibility, see the next subsection.

3.7. *Comments on mixed characteristic.* It would be interesting to extend the constructions in this paper to the case where R has mixed characteristics. The analogue of the passage from Galois representations to ∇-modules is given by p-adic Hodge theory, specifically via the theory of (ϕ, Γ)-modules over the Robba ring, as in the work of Fontaine, Cherbonnier–Colmez, Berger, et al.

In that context, when R has perfect residue field, Colmez [2003] has given a recipe for reading off the Swan conductor of de Rham representations from the associated (ϕ, Γ)-module . One would like to reformulate this recipe via Berger's construction of the Weil–Deligne representation, which converts the (ϕ, Γ)-module into a (ϕ, ∇)-module over $\mathcal{R}$ [Berger 2002]; however, it is not immediately clear how to do this. The fact that this might even be possible is suggested by work of Marmora [2004], who gives a direct comparison with differential Swan conductors, but only for the Swan conductor of a representation over the maximal p-cyclotomic extension of a given p-adic field.

If one can indeed give a differential definition of the usual Swan conductor of a de Rham representation in the perfect residue field, then it seems likely one can make a differential definition in the imperfect residue field case. Indeed, the construction of (Φ, Γ)-modules has already been generalized to this setting by Morita [2005]. If one can do all this, then one will again encounter the question of reconciliation with the Abbes–Saito constructions; however, it is not clear whether in this case the Hasse–Arf theorem would be any easier on the differential side than on the Abbes–Saito side.

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