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We show that the mapping cone of a morphism of differential graded Lie algebras, $\chi : L \to M$, can be canonically endowed with an L_∞-algebra structure which at the same time lifts the Lie algebra structure on L and the usual differential on the mapping cone. Moreover, this structure is unique up to isomorphisms of L_∞-algebras.

Introduction

There are several cases where the tangent and obstruction spaces of a deformation theory are the cohomology groups of the mapping cone of a morphism $\chi : L \to M$ of differential graded Lie algebras. It is therefore natural to ask if there exists a canonical differential graded Lie algebra structure on the complex (C_χ, δ), where

$$C_\chi = \bigoplus C_\chi^i, \qquad C_\chi^i = L^i \bigoplus M^{i-1}, \qquad \delta(l, m) = (dl, \chi(l) - dm),$$

such that the projection $C_\chi \to L$ is a morphism of differential graded Lie algebras.

In general we cannot expect the existence of a Lie structure. In fact the canonical bracket

$$l_1 \otimes l_2 \mapsto [l_1, l_2], \qquad m_1 \otimes l_2 \mapsto \tfrac{1}{2}[m_1, \chi(l_2)],$$
$$l_1 \otimes m_2 \mapsto \tfrac{1}{2}(-1)^{\deg(l_1)}[\chi(l_1), m_2], \qquad m_1 \otimes m_2 \mapsto 0$$

satisfies the Leibniz rule with respect to the differential δ but not the Jacobi identity. However, the Jacobi identity for this bracket holds up to homotopy, and so we can look for the weaker requirement of a canonical L_∞ structure on C_χ.

More precisely, let $\mathbb{K}$ be a fixed characteristic zero base field, denote by **DG** the category of differential graded vector spaces, by **DGLA** the category of differential graded Lie algebras, by $\mathbf{L}_\infty$ the category of L_∞ algebras and by $\mathbf{DGLA}^2$ the

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category of morphisms in **DGLA**. The four functors,

$$\begin{array}{cl} \mathbf{DGLA} \to \mathbf{L}_\infty & \text{by natural inclusion,} \\ \mathbf{L}_\infty \to \mathbf{DG} & \text{by forgetting higher brackets,} \\ \mathbf{DGLA}^2 \to \mathbf{DG} & \text{by } \{L \xrightarrow{\chi} M\} \mapsto C_\chi, \\ \mathbf{DGLA} \to \mathbf{DGLA}^2 & \text{by } L \mapsto \{L \to 0\}, \end{array}$$

give a commutative diagram

$$\begin{array}{ccc} \mathbf{DGLA} & \longrightarrow & \mathbf{L}_\infty \\ \downarrow & & \downarrow \\ \mathbf{DGLA}^2 & \xrightarrow{C} & \mathbf{DG}. \end{array}$$

Theorem 1. *There exists a functor* $\widetilde{C} : \mathbf{DGLA}^2 \to \mathbf{L}_\infty$ *making the diagram*

$$\begin{array}{ccc} \mathbf{DGLA} & \longrightarrow & \mathbf{L}_\infty \\ \downarrow & \overset{\widetilde{C}}{\nearrow} & \downarrow \\ \mathbf{DGLA}^2 & \xrightarrow{C} & \mathbf{DG} \end{array}$$

commutative.

Moreover, the functor $\widetilde{C}$ is essentially unique, that is, if $\mathscr{F} : \mathbf{DGLA}^2 \to \mathbf{L}_\infty$ has the same properties, then for every morphism χ of differential graded Lie algebras, the L_∞-algebra $\mathscr{F}(\chi)$ is (noncanonically) isomorphic to $\widetilde{C}(\chi)$.

The L_∞ structure $\widetilde{C}(\chi)$ on the mapping cone of a DGLA morphism $\chi : L \to M$ is actually a particular case of a more general construction of an L_∞ structure on the total complex of a semicosimplicial DGLA. More precisely, the category $\mathbf{DGLA}^2$ of morphisms of DGLAs can be seen as a full subcategory of the category $\mathbf{DGLA}^{\Delta_{\mathrm{mon}}}$ of semicosimplicial DGLAs via the functor

$$\{L \xrightarrow{\chi} M\} \rightsquigarrow \left\{ \; L \underset{\chi}{\overset{0}{\rightrightarrows}} M \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \; \right\}$$

and we have a commutative diagram

$$\begin{array}{ccc} \mathbf{DGLA}^2 & \xrightarrow{\quad\widetilde{C}\quad} & \mathbf{L}_\infty \\ \downarrow & \overset{\widetilde{\mathrm{Tot}}}{\nearrow} & \downarrow \\ \mathbf{DGLA}^{\Delta_{\mathrm{mon}}} & \xrightarrow{\quad\mathrm{Tot}\quad} & \mathbf{DG}. \end{array}$$

The functor $\widetilde{C}$ can be explicitly described. The linear term of the L_∞-algebra $\widetilde{C}(\chi)$ is by construction the differential δ on C_χ, and the quadratic part which turns out to coincide with the naive bracket described at the beginning of the Introduction. An explicit expression for the higher brackets is given in Theorem 5.2.

The second main result of this paper is to prove that the deformation functor $\mathrm{Def}_{\widetilde{C}(\chi)}$ associated with the L_∞ algebra $\widetilde{C}(\chi)$ is isomorphic to the functor Def_χ defined in [Manetti 2005].

Given $\chi : L \to M$, it defines a functor $\mathrm{Def}_\chi : \mathbf{Art} \to \mathbf{Set}$, with $\mathbf{Art}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$,

$$\mathrm{Def}_\chi(A) = \frac{\{(x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid dx + \frac{1}{2}[x, x] = 0,\ e^a * \chi(x) = 0\}}{\text{gauge equivalence}}$$

where $*$ denotes the gauge action in M, and (l_0, e^{m_0}) is defined to be gauge equivalent to (l_1, e^{m_1}) if there exists $(a, b) \in (L^0 \bigoplus M^{-1}) \otimes \mathfrak{m}_A$ such that

$$l_1 = e^a * l_0, \qquad e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}.$$

Theorem 2. *With the notation above, for every morphism of differential graded Lie algebras, $\chi : L \to M$, we have*

$$\mathrm{Def}_{\widetilde{C}(\chi)} \simeq \mathrm{Def}_\chi\,.$$

The importance of Theorem 2 lies in that it allows one to study the functors Def_χ, which are often naturally identified with geometrically defined functors, using the whole machinery of L_∞-algebras. In particular this gives, under some finiteness assumption, the construction and the homotopy invariance of the Kuranishi map [Fukaya 2003; Goldman and Millson 1990; Kontsevich 2003], as well as the local description of the corresponding extended moduli spaces.

Keywords and general notation. We assume that the reader is familiar with the notion and main properties of differential graded Lie algebras and L_∞-algebras (we refer to [Fukaya 2003; Grassi 1999; Kontsevich 2003; Lada and Markl 1995; Lada and Stasheff 1993; Manetti 2004b] as the introduction of such structures); however the basic definitions are recalled in this paper in order to fix notation and terminology.

For the whole paper, $\mathbb{K}$ is a fixed field of characteristic 0 and $\mathbf{Art}$ is the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. For $A \in \mathbf{Art}$ we denote by $\mathfrak{m}_A$ the maximal ideal of A.

1. Conventions on graded vector spaces

In this paper we will work with $\mathbb{Z}$-graded vector spaces. We write a graded vector space as $V = \bigoplus_{n \in \mathbb{Z}} V^n$, and call V^n the degree n component of V; an element v

of V^n is called a degree n homogeneous element of V. The shift functor is defined as $(V[k])^i := V^{i+k}$. We say that a linear map $\varphi: V \to W$ is a degree k map if it is a morphism $V \to W[k]$, that is, if it is a collection of linear maps $\varphi^n: V^n \to W^{n+k}$. The set of degree k liner maps from V to W will be denoted $\mathrm{Hom}^k(V, W)$.

Graded vector spaces form a symmetric tensor category with

$$(V \otimes W)^k = \bigoplus_{i+j=k} V^i \otimes W^j,$$

and $\sigma_{V,W}: V \otimes W \to W \otimes V$ given by $\sigma(v \otimes w) := (-1)^{\deg(v)\cdot\deg(w)} w \otimes v$ on the homogeneous elements. We adopt the convention according to which degrees are "shifted on the left". By this we mean that we have a natural identification, called the *suspension* isomorphism, $V[1] \simeq \mathbb{K}[1] \otimes V$ where $\mathbb{K}[1]$ denotes the graded vector space consisting of the field $\mathbb{K}$ concentrated in degree -1. With this convention, the canonical isomorphism is

$$V \otimes \mathbb{K}[1] \simeq V[1], \qquad v \otimes 1_{[1]} \mapsto (-1)^{\deg(v)} v_{[1]}.$$

More in general we have the following *decalage* isomorphism

$$V_1[1] \otimes \cdots \otimes V_n[1] \xrightarrow{\sim} (V_1 \otimes \cdots \otimes V_n)[n],$$
$$v_{1[1]} \otimes \cdots \otimes v_{n[1]} \mapsto (-1)^{\sum_{i=1}^{n}(n-i)\cdot\deg v_i}(v_1 \otimes \cdots \otimes v_n)_{[n]}.$$

Since graded vector spaces form a symmetric category, for any graded vector space V and any positive integer n we have a canonical representation of the symmetric group S_n on $\otimes^n V$. The space of coinvariants for this action is called the n-th symmetric power of V and is denoted by $\odot^n V$. Twisting the canonical representation of S_n on $\otimes^n V$ by the alternating character $\sigma \mapsto (-1)^\sigma$ and taking the coinvariants one obtains the n-th antisymmetric (or exterior) power of V, denoted by $\bigwedge^n V$. By the naturality of the decalage isomorphism, we have a canonical isomorphism

$$\bigodot^{n}(V[1]) \xrightarrow{\sim} \left(\bigwedge^{n} V\right)[n].$$

Remark 1.1. Using the natural isomorphisms

$$\mathrm{Hom}^i(V, W[l]) \simeq \mathrm{Hom}^{i+l}(V, W)$$

and the decalage isomorphism, we obtain the natural identifications

$$\mathrm{dec}: \mathrm{Hom}^i\left(\bigwedge^{k} V, W\right) \xrightarrow{\sim} \mathrm{Hom}^{i+k-1}\left(\bigodot^{k}(V[1]), W[1]\right),$$
$$\mathrm{dec}(f)(v_{1[1]} \odot \cdots \odot v_{k[1]}) = (-1)^{ki+\sum_{j=1}^{k}(k-j)\cdot\deg(v_j)} f(v_1 \wedge \cdots \wedge v_k)_{[1]}.$$

2. Differential graded Lie algebras and L_∞-algebras

A differential graded Lie algebra (DGLA) is a Lie algebra in the category of graded vector spaces, endowed with a compatible degree 1 differential. Via the decalage isomorphisms one can look at the Lie bracket of a DGLA V as a morphism

$$q_2 \in \operatorname{Hom}^1(V[1] \odot V[1], V[1]), \qquad q_2(v_{[1]} \odot w_{[1]}) = (-1)^{\deg(v)}[v, w]_{[1]}.$$

Similarly, the suspended differential $q_1 = d_{[1]} = \mathrm{id}_{\mathbb{K}[1]} \otimes d$ is a degree 1 morphism

$$q_1 : V[1] \to V[1], \qquad q_1(v_{[1]}) = -(dv)_{[1]}.$$

Up to the canonical bijective linear map $V \to V[1]$, $v \mapsto v_{[1]}$, the suspended differential q_1 and the bilinear operation q_2 are written simply as

$$q_1(v) = -dv, \qquad q_2(v \odot w) = (-1)^{\deg_V(v)}[v, w],$$

that is, "the suspended differential is the opposite differential and q_2 is the twisted Lie bracket".

Define morphisms $q_k \in \operatorname{Hom}^1(\odot^k(V[1]), V[1])$ by setting $q_k \equiv 0$, for $k \geq 3$. The map

$$Q^1 = \sum_{n\geq 1} q_n : \bigoplus_{n\geq 1} \bigodot^n V[1] \to V[1]$$

extends to a coderivation of degree 1

$$Q : \bigoplus_{n\geq 1} \bigodot^n V[1] \to \Bigl(\bigoplus_{n\geq 1} \bigodot^n V[1]\Bigr)$$

on the reduced symmetric coalgebra cogenerated by $V[1]$, by the formula

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k,n-k)} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}, \tag{2-1}$$

where $S(k, n-k)$ is the set of unshuffles and $\varepsilon(\sigma) = \pm 1$ is the *Koszul sign*, determined by the relation in $\bigodot^n V[1]$

$$v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma) v_1 \odot \cdots \odot v_n.$$

The axioms of differential graded Lie algebra are then equivalent to Q being a codifferential, that is, $QQ = 0$. This description of differential graded Lie algebra in terms of the codifferential Q is called the Quillen construction [1969]. By dropping the requirement that $q_k \equiv 0$ for $k \geq 3$ one obtains the notion of L_∞-algebra (or strong homotopy Lie algebra); see for example [Lada and Markl 1995; Lada and Stasheff 1993; Kontsevich 2003]. Namely, an L_∞ structure on a graded vector space V is a sequence of linear maps of degree 1,

$$q_k : \bigodot^k V[1] \to V[1], \qquad k \geq 1,$$

such that the induced coderivation Q on the reduced symmetric coalgebra cogenerated by $V[1]$, given by (2-1) is a codifferential, that is, $QQ = 0$. This condition implies $q_1 q_1 = 0$ and therefore an L_∞-algebra is in particular a differential complex. By the preceding discussion, every DGLA can be naturally seen as an L_∞-algebra; namely, a DGLA is an L_∞-algebra with vanishing higher multiplications q_k, $k \geq 3$.

A morphism f_∞ between two L_∞-algebras

$$(V, q_1, q_2, q_3, \dots) \qquad \text{and} \qquad (W, \hat{q}_1, \hat{q}_2, \hat{q}_3, \dots)$$

is a sequence of linear maps of degree 0

$$f_n : \bigodot^n V[1] \to W[1], \qquad n \geq 1,$$

such that the morphism of coalgebras

$$F : \bigoplus_{n\geq 1} \bigodot^n V[1] \to \bigoplus_{n\geq 1} \bigodot^n W[1]$$

induced by $F^1 = \sum_n f_n : \bigoplus_{n\geq 1} \bigodot^n V[1] \to W[1]$ commutes with the codifferentials induced by the two L_∞ structures on V and W [Fukaya 2003; Kontsevich 2003; Lada and Markl 1995; Lada and Stasheff 1993; Manetti 2004b]. An L_∞-morphism f_∞ is called *linear* (sometimes *strict*) if $f_n = 0$ for every $n \geq 2$. Note that a linear map $f_1 : V[1] \to W[1]$ is a linear L_∞-morphism if and only if

$$\hat{q}_n(f_1(v_1) \odot \cdots \odot f_1(v_n)) = f_1(q_n(v_1 \odot \cdots \odot v_n)), \text{ for all } n \geq 1, \ v_1, \dots, v_n \in V[1].$$

The category of L_∞-algebras will be denoted by $\mathbf{L}_\infty$ in this paper. Morphisms between DGLAs are linear morphisms between the corresponding L_∞-algebras, so the category of differential graded Lie algebras is a (nonfull) subcategory of $\mathbf{L}_\infty$.

If f_∞ is an L_∞ morphism between $(V, q_1, q_2, q_3, \dots)$ and $(W, \hat{q}_1, \hat{q}_2, \hat{q}_3, \dots)$, then its linear part $f_1 : V[1] \to W[1]$ satisfies the equation $f_1 \circ q_1 = \hat{q}_1 \circ f_1$, that is, f_1 is a map of differential complexes $(V[1], q_1) \to (W[1], \hat{q}_1)$. An L_∞-morphism f_∞ is called a quasiisomorphism of L_∞-algebras if its linear part f_1 is a quasiisomorphism of differential complexes.

A major result in the theory of L_∞-algebras is the following *homotopical transfer of structure* theorem, dating back to Kadeishvili's work on the cohomology of A_∞ algebras [Kadeishvili 1982]; see also [Huebschmann and Kadeishvili 1991].

Theorem 2.1. *Let* $(V, q_1, q_2, q_3, \dots)$ *be an* L_∞*-algebra and* (C, δ) *be a differential complex. If there exist two morphisms of differential complexes*

$$\imath : (C[1], \delta_{[1]}) \to (V[1], q_1) \qquad \textit{and} \qquad \pi : (V[1], q_1) \to (C[1], \delta_{[1]})$$

such that the composition $\imath\pi$ *is homotopic to the identity, then there exist an* L_∞*-algebra structure* $(C, \langle\ \rangle_1, \langle\ \rangle_2, \dots)$ *on* C *extending its differential complex structure and an* L_∞*-morphism* $\imath_\infty$ *extending* $\imath$*.*

Explicit formulas for the quasiisomorphism $\imath_\infty$ and the brackets $\langle\ \rangle_n$ have been described by Merkulov [1999]; it has then been remarked by Kontsevich and Soibelman [2000; 2001] (see also [Fukaya 2003; Schuhmacher 2004]) that Merkulov's formulas can be nicely written as the summations over rooted trees. Let $K \in \mathrm{Hom}^{-1}(V[1], V[1])$ be an homotopy between $\imath\pi$ and $\mathrm{Id}_{V[1]}$, that is,

$$q_1 K + K q_1 = \imath\pi - \mathrm{Id}_{V[1]},$$

and denote by $\mathcal{T}_{K,n}$ the groupoid whose objects are directed rooted trees with internal vertices of valence at least two and exactly n tail edges. Trees in $\mathcal{T}_{K,n}$ are decorated as follows: each tail edge of a tree in $\mathcal{T}_{K,n}$ is decorated by the operator $\imath$, each internal edge is decorated by the operator K and also the root edge is decorated by the operator K. Every internal vertex v carries the operation q_r, where r is the number of edges having v as endpoint. Isomorphisms between objects in $\mathcal{T}_{K,n}$ are isomorphisms of the underlying trees. Denote the set of isomorphism classes of objects of $\mathcal{T}_{K,n}$ by the symbol $T_{K,n}$. Similarly, let $\mathcal{T}_{\pi,n}$ be the groupoid whose objects are directed rooted trees with the same decoration as $\mathcal{T}_{K,n}$ except for the root edge, which is decorated by the operator π instead of K. The set of isomorphism classes of objects of $\mathcal{T}_{\pi,n}$ is denoted $T_{\pi,n}$.

Via the usual operadic rules, each decorated tree $\Gamma \in \mathcal{T}_{K,n}$ gives a linear map

$$Z_\Gamma(\imath, \pi, K, q_i) : C[1]^{\odot n} \to V[1].$$

Similarly, each decorated tree in $\mathcal{T}_{\pi,n}$ gives rise to a degree 1 multilinear operator from $C[1]$ to itself.

Having introduced these notations, we can write Kontsevich–Soibelman's formulas as follows.

Proposition 2.2. *In the above set-up the brackets* $\langle\ \rangle_n$*, and the* L_∞ *morphism* $\imath_\infty$ *can be expressed as sums over decorated rooted trees via the formulas*

$$\imath_n = \sum_{\Gamma\in T_{K,n}} \frac{Z_\Gamma(\imath, \pi, K, q_i)}{|\,\mathrm{Aut}\,\Gamma|}, \qquad \langle\ \rangle_n = \sum_{\Gamma\in T_{\pi,n}} \frac{Z_\Gamma(\imath, \pi, K, q_i)}{|\,\mathrm{Aut}\,\Gamma|}, \qquad n \geq 2.$$

3. The suspended mapping cone of $\chi : L \to M$.

The suspended mapping cone of the DGLA morphism $\chi : L \to M$ is the graded vector space

$$C_\chi = \mathrm{Cone}(\chi)[-1],$$

where $\mathrm{Cone}(\chi) = L[1] \bigoplus M$ is the mapping cone of χ. More explicitly,

$$C_\chi = \bigoplus_i C_\chi^i, \qquad C_\chi^i = L^i \bigoplus M^{i-1}.$$

The suspended mapping cone has a natural differential $\delta \in \mathrm{Hom}^1(C_\chi, C_\chi)$ given by

$$\delta(l, m) = (dl, \chi(l) - dm), \qquad l \in L, m \in M.$$

Denote $M[t, dt] = M \otimes \mathbb{K}[t, dt]$ and define, for every $a \in \mathbb{K}$, the evaluation morphism

$$e_a : M[t, dt] \to M, \qquad e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.$$

It is easy to prove that every morphism e_a is a surjective quasiisomorphism of DGLA. The integral operator $\int_a^b : \mathbb{K}[t, dt] \to \mathbb{K}$ extends to a linear map of degree -1

$$\int_a^b : M[t, dt] \to M, \qquad \int_a^b \left(\sum_i t^i m_i + t^i dt \cdot n_i \right) = \sum_i \left(\int_a^b t^i dt \right) n_i.$$

Consider the DGLA

$$H_\chi = \{(l, m) \in L \times M[t, dt] : e_0(m) = 0,\ e_1(m) = \chi(l)\}.$$

The morphism

$$\iota : C_\chi \to H_\chi, \qquad \iota(l, m) = (l, t\chi(l) + dt \cdot m)$$

is an injective quasiisomorphism of complexes. If we denote by

$$\langle\ \rangle_1 \in \mathrm{Hom}^1(C_\chi[1], C_\chi[1]), \quad \text{and} \quad q_1 \in \mathrm{Hom}^1(H_\chi[1], H_\chi[1])$$

the suspended differentials, namely

$$\langle (l, m) \rangle_1 = (-dl, -\chi(l) + dm), \qquad l \in L, m \in M,$$
$$q_1(l, m) = (-dl, -dm),$$

then ι induces naturally an injective quasiisomorphism

$$\iota : C_\chi[1] \to H_\chi[1], \qquad \iota(l, m) = (l, t\chi(l) + dt \cdot m).$$

Consider now the linear maps

$$\pi \in \mathrm{Hom}^0(H_\chi[1], C_\chi[1]), \qquad K \in \mathrm{Hom}^{-1}(H_\chi[1], H_\chi[1])$$

defined as

$$\pi(l, m(t,dt)) = \left(l, \int_0^1 m(t,dt)\right), \qquad K(l,m) = \left(0, \int_0^t m - t\int_0^1 m\right).$$

It is easy to check that π is a morphism of complexes and

$$\pi\imath = \mathrm{Id}_{C_\chi[1]}, \qquad \imath\pi = \mathrm{Id}_{H_\chi[1]} + Kq_1 + q_1K.$$

We are therefore in the hypotheses of Theorem 2.1 and we can transfer the DGLA structure on H_χ to an L_∞ structure on C_χ. We denote by $\widetilde{C}(\chi)$ the induced L_∞ structure on C_χ. The universal formulas for the homotopy transfer described in Proposition 2.2 imply that the above construction is functorial. Namely, for every commutative diagram

$$\begin{CD} L_1 @>{f_L}>> L_2 \\ @V{\chi_1}VV @VV{\chi_2}V \\ M_1 @>{f_M}>> M_2 \end{CD}$$

of morphisms of differential graded Lie algebras, the natural map

$$(f_L, f_M)\colon\ \widetilde{C}(\chi_1) \to \widetilde{C}(\chi_2)$$

is a linear L_∞-morphism. Summing up:

Theorem 3.1. *For any morphism $\chi : L \to M$ of differential graded Lie algebras, let $\widetilde{C}(\chi) = (C_\chi, \hat{Q})$ be the L_∞-algebra structure defined on C_χ by the above construction. Then*

$$\widetilde{C} : \mathbf{DGLA^2} \to \mathbf{L_\infty}$$

is a functor making the diagram

$$\begin{array}{ccc} \mathbf{DGLA} & \longrightarrow & \mathbf{L_\infty} \\ \big\downarrow & \overset{\widetilde{C}}{\nearrow} & \big\downarrow \\ \mathbf{DGLA^2} & \longrightarrow & \mathbf{DG} \end{array}$$

commutative.

Remark 3.2. As an instance of the functoriality, note that the projection on the first factor $p_1 : \widetilde{C}(\chi) \to L$ is a linear morphism of L_∞-algebras. To see this, consider the morphism in $\mathbf{DGLA^2}$

$$\begin{CD} L @>{\mathrm{Id}_L}>> L \\ @V{\chi}VV @VVV \\ M @>>> 0 \end{CD}$$

Remark 3.3. The above construction of the L_∞ structure on C_χ commutes with the tensor products of differential graded commutative algebras. This means that if R is a DGCA, then the L_∞-algebra structure on the suspended mapping cone of $\chi \otimes \mathrm{id}_R : L \otimes R \to M \otimes R$ is naturally isomorphic to the L_∞-algebra $C_\chi \otimes R$.

Remark 3.4. The functorial properties of $\widetilde{C}$ determine the L_∞ structure $\widetilde{C}(\chi)$ up to (noncanonical) isomorphism. Namley, if $\mathscr{F} : \mathbf{DGLA^2} \to \mathbf{L}_\infty$ is a functor such that the diagram

$$\begin{array}{ccc} \mathbf{DGLA} & \longrightarrow & \mathbf{L}_\infty \\ \downarrow & \overset{\mathscr{F}}{\nearrow} & \downarrow \\ \mathbf{DGLA^2} & \longrightarrow & \mathbf{DG} \end{array}$$

commutes, then for every morphism χ of differential graded Lie algebras, the L_∞-algebra $\mathscr{F}(\chi)$ is isomorphic to $\widetilde{C}(\chi)$. To see this, let

$$P = \left\{(l,m) \in L \times M[t,dt] : e_1(m) = \chi(l)\right\}.$$

We have a commutative diagram of morphisms of differential graded Lie algebras

$$\begin{array}{ccccc} L & \xrightarrow{f} & P & \longleftarrow & H_\chi \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \eta} & & \downarrow \\ M & \xrightarrow{\mathrm{Id}_M} & M & \longleftarrow & 0 \end{array}$$

and then two L_∞-morphisms $\mathscr{F}(\chi) \to \mathscr{F}(\eta) \xleftarrow{h_\infty} H_\chi$ whose linear parts are the two injective quasiisomorphisms

$$C_\chi \to C_\eta \xleftarrow{h} H_\chi, \qquad h(l,m) = ((l,m),0).$$

A morphism of complexes $p : C_\eta \to H_\chi$ such that $ph = \mathrm{Id}_{H_\chi}$ can be defined as

$$p((l,m),n) = (l, m + (t-1)e_0(m) + dt \cdot n).$$

The composition of p with the injective quasiisomorphism $C_\chi \to C_\eta$ gives the map $\imath$. By general theory of L_∞ algebras, there exists a (noncanonical) left inverse of h_∞ with linear term equal to p. We therefore get an injective L_∞-quasiisomorphism

$$\hat{\imath}_\infty : \mathscr{F}(\chi) \to H_\chi$$

with linear term $\imath$. The composition of

$$\imath_\infty : \widetilde{C}(\chi) \to H_\chi$$

with a left inverse of $\hat{\imath}_\infty$ is an isomorphism of L_∞-algebras between $\widetilde{C}(\chi)$ and $\mathscr{F}(\chi)$.

4. The case of semicosimplicial DGLAs

The L_∞ structure on the mapping cone of a DGLA morphism described in Section 3 is actually a particular case of a more general construction of an L_∞ structure on the total complex of a semicosimplicial DGLA; see also [Cheng and Getzler 2006], where this construction is described for cosimplicial commutative algebras.

Let Δ_{mon} be the category of finite ordinal sets, with order-preserving injective maps between them. A semicosimplicial differential graded Lie algebra is a covariant functor $\Delta_{\mathrm{mon}} \to \mathbf{DGLA}$. Equivalently, a semicosimplicial DGLA $\mathfrak{g}^\Delta$ is a diagram

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \mathfrak{g}_2 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \cdots$$

where each $\mathfrak{g}_i$ is a DGLA, and for each $i > 0$ there are n morphisms of DGLAs

$$\partial_{k,i} : \mathfrak{g}_{i-1} \to \mathfrak{g}_i, \qquad k = 0, \dots, i,$$

such that $\partial_{k+1,i+i}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \le l$. Therefore, the maps

$$\partial_i = \partial_{i,i} - \partial_{i-1,i} + \cdots + (-1)^i \partial_{0,i}$$

endow the vector space $\bigoplus_i \mathfrak{g}_i$ with the structure of a differential complex. Moreover, being a DGLA, each $\mathfrak{g}_i$ is in particular a differential complex

$$\mathfrak{g}_i = \bigoplus_j \mathfrak{g}_i^j, \qquad d_i : \mathfrak{g}_i^j \to \mathfrak{g}_i^{j+1},$$

and since the maps $\partial_{k,i}$ are morphisms of DGLAs, the space

$$\mathfrak{g}_\bullet^\bullet = \bigoplus_{i,j} \mathfrak{g}_i^j$$

has a natural bicomplex structure. The associated total complex is denoted by $(\mathrm{Tot}(\mathfrak{g}^\Delta), \delta)$, which has no natural DGLA structure. Yet, it can be endowed with a canonical L_∞-algebra structure by homotopy transfer from the homotopy equivalent Thom–Whitney DGLA $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$.

For every $n \ge 0$, denote by Ω_n the differential graded commutative algebra of polynomial differential forms on the standard n-simplex Δ^n:

$$\Omega_n = \frac{\mathbb{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]}{\left(\sum t_i - 1, \sum dt_i\right)}.$$

Denote by $\delta^{k,n} : \Omega_n \to \Omega_{n-1}$, $k = 0, \dots, n$, the face maps then we have natural morphisms of DGLAs

$$\delta^{k,n} : \Omega_n \otimes \mathfrak{g}_n \to \Omega_{n-1} \otimes \mathfrak{g}_n, \qquad \partial_{k,n} : \Omega_{n-1} \otimes \mathfrak{g}_{n-1} \to \Omega_{n-1} \otimes \mathfrak{g}_n$$

for every $0 \leq k \leq n$. The *Thom–Whitney DGLA* is defined as

$$\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta}) = \Big\{ (x_n)_{n \in \mathbb{N}} \in \bigoplus_n \Omega_n \otimes \mathfrak{g}_n \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1}, \quad \text{for all } 0 \leq k \leq n \Big\}.$$

We denote by d_{TW} the differential of the DGLA $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})$. It is a remarkable fact that the integration maps

$$\int_{\Delta^n} \otimes \, \mathrm{Id} : \Omega_n \otimes \mathfrak{g}_n \to \mathbb{K}[n] \otimes \mathfrak{g}_n = \mathfrak{g}_n[n]$$

give a quasiisomorphism of differential complexes

$$I : (\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta}), d_{TW}) \to (\mathrm{Tot}(\mathfrak{g}^{\Delta}), \delta).$$

Moreover, Dupont has described in [Dupont 1976; Dupont 1978] an explicit morphism of differential complexes

$$E : \mathrm{Tot}(\mathfrak{g}^{\Delta}) \to \mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})$$

and an explicit homotopy

$$h : \mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta}) \to \mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})[-1]$$

such that

$$IE = \mathrm{Id}_{\mathrm{Tot}(\mathfrak{g}^{\Delta})}, \qquad EI - \mathrm{Id}_{\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})} = [h, d_{TW}].$$

We also refer to the papers [Cheng and Getzler 2006; Getzler 2004; Navarro Aznar 1987] for the explicit description of E, h and for the proof of the above identities. Here we point out that E and h are defined in terms of integration over standard simplexes and multiplication with canonical differential forms and in particular, the construction of $\mathrm{Tot}_{TW}(\mathfrak{g}^{\Delta})$, $\mathrm{Tot}(\mathfrak{g}^{\Delta})$, I, E and h is functorial in the category $\mathbf{DGLA}^{\Delta_{\mathrm{mon}}}$ of semicosimplicial DGLAs.

Therefore we are in the position to use the homotopy transfer of L_∞ structures in Theorem 2.1 in order to get a commutative diagram of functors,

$$\begin{array}{ccc} \mathbf{DGLA} & \longrightarrow & \mathbf{L}_\infty \\ \downarrow & \nearrow^{\widetilde{\mathrm{Tot}}} & \downarrow \\ \mathbf{DGLA}^{\Delta_{\mathrm{mon}}} & \xrightarrow{\mathrm{Tot}} & \mathbf{DG}. \end{array}$$

The L_∞ structure $\widetilde{C}(\chi)$ on the mapping cone of a DGLA morphism $\chi : L \to M$ is actually a particular case of this more general construction of the L_∞-algebra $\widetilde{\mathrm{Tot}}(\mathfrak{g}^{\Delta})$. More precisely, the category $\mathbf{DGLA}^2$ of morphisms of DGLAs can be seen as a full subcategory of the category of semicosimplicial DGLAs via the functor

$$\{L \xrightarrow{\chi} M\} \rightsquigarrow \left\{ L \overset{0}{\underset{\chi}{\rightrightarrows}} M \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} 0 \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots \right\},$$

and we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{DGLA}^2 & \xrightarrow{\widetilde{C}} & \mathbf{L}_\infty \\
\downarrow & \overset{\widetilde{\mathrm{Tot}}}{\nearrow} & \downarrow \\
\mathbf{DGLA}^{\Delta_{\mathrm{mon}}} & \xrightarrow{\mathrm{Tot}} & \mathbf{DG}.
\end{array}$$

To check the commutativity of this diagram, one only needs to identify the suspended mapping cone C_χ with the total complex $\mathrm{Tot}(\chi^\Delta)$ of the cosimplicial DGLA

$$L \overset{0}{\underset{\chi}{\rightrightarrows}} M \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} 0 \ ,$$

the Thom–Whitney DGLA $\mathrm{Tot}_{TW}(\chi^\Delta)$ with the DGLA we have called H_χ in the main body of the paper, and the Dupont maps I, E, h with the maps we have denoted ι, π, K.

For instance, to see that $\mathrm{Tot}(\chi^\Delta) \simeq C_\chi$ one only needs to notice that the double complex associated to the cosimplicial DGLA χ^Δ is

$$\begin{array}{ccccc}
\uparrow & & \uparrow & & \uparrow \\
L^{i+1} & \xrightarrow{\chi} & M^{i+1} & \longrightarrow & 0 \\
\uparrow{\scriptstyle d_L} & & \uparrow{\scriptstyle d_M} & & \uparrow \\
L^{i} & \xrightarrow{\chi} & M^{i} & \longrightarrow & 0 \\
\uparrow{\scriptstyle d_L} & & \uparrow{\scriptstyle d_M} & & \uparrow \\
L^{i-1} & \xrightarrow{\chi} & M^{i-1} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow
\end{array}$$

an so the total complex $\mathrm{Tot}(\chi^\Delta)$ is the graded vector space

$$\mathrm{Tot}(\chi^\Delta)^i = L^i \bigoplus M^{i-1}$$

endowed with the total differential

$$\delta : \mathrm{Tot}(\chi^\Delta)^i \to \mathrm{Tot}(\chi^\Delta)^{i+1}, \quad (l, m) \mapsto (d_L l, \chi(l) - d_M m).$$

Therefore, the differential complex $(\mathrm{Tot}(\chi^\Delta), \delta)$ is nothing but the suspended mapping cone C_χ endowed with its usual differential.

Setting $t = t_0 = 1 - t_1$ we get an identification $\Omega_1 \simeq \mathbb{K}[t, dt]$ and therefore the Thom–Whitney complex of the semicosimplicial DGLA

$$L \underset{\chi}{\overset{0}{\rightrightarrows}} M \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; 0$$

is isomorphic to the sub-DGLA of $L \bigoplus (\mathbb{K}[t, dt] \otimes M)$ consisting of the differential forms $(l, m(t, dt))$ such that $m(0) = 0$ and $m(1) = \chi(l)$, that is, $\mathrm{Tot}_{TW}(\chi^\Delta) = H_\chi$.

Moreover, the Dupont maps [Dupont 1976; Navarro Aznar 1987]

$$\begin{aligned} E : \mathrm{Tot}(\chi^\Delta) &\to \mathrm{Tot}_{TW}(\chi^\Delta), \\ (l, m) &\mapsto (l, t_0 \partial_{1,1}(l) - t_1 \partial_{0,1}(l) - (t_0 dt_1 - t_1 dt_0)m) \end{aligned}$$

and

$$\begin{aligned} I : \mathrm{Tot}_{TW}(\chi^\Delta) &\to \mathrm{Tot}(\chi^\Delta), \\ (l, m(t_0, t_1, dt_0, dt_1)) &\mapsto \Big(\int_{\Delta^0} l, \int_{\Delta^1} m(t_0, t_1, dt_0, dt_1)\Big) \end{aligned}$$

are identified with the maps

$$\begin{aligned} \iota : C_\chi &\to H_\chi, \\ (l, m) &\mapsto (l, t\chi(l) + dt \cdot m) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi : H_\chi &\to C_\chi, \\ (l, m(t, dt)) &\mapsto \Big(l, \int_0^1 m(t, dt)\Big). \end{aligned}$$

Finally, we identify the Dupont map $h : \mathrm{Tot}_{TW}(\chi^\Delta) \to \mathrm{Tot}_{TW}(\chi^\Delta)[-1]$ with the map $K : H_\chi \to H_\chi[-1]$. By definition,

$$\begin{aligned} h : \mathrm{Tot}_{TW}(\chi^\Delta) &\to \mathrm{Tot}_{TW}(\chi^\Delta)[-1] \\ (l, m(t_0, t_1, dt_0, dt_1)) &\mapsto (0, t_0 \cdot h_0(m) + t_1 \cdot h_1(m)), \end{aligned}$$

where h_0 and h_1 are the Poincaré homotopies corresponding to the linear contractions of the affine hyperplane $\{t_0 + t_1 = 1\} \subseteq \mathbb{A}^2$ on the points $(1, 0)$ and $(0, 1)$ respectively:

$$h_i(m) = \int_{s \in [0,1]} \phi_i^*(m) \quad \text{with} \quad \begin{aligned} &\phi_0(s; t_0, t_1) = ((1-s)t_0 + s, (1-s)t_1), \\ \text{and} \quad &\phi_1(s; t_0, t_1) = ((1-s)t_0, (1-s)t_1 + s). \end{aligned}$$

Under the identification $\Omega_1 \simeq \mathbb{K}[t, dt]$ above, these homotopies read

$$h_0(m(t, dt)) = \int_t^1 m, \qquad h_1(m(t, dt)) = \int_t^0 m,$$

so

$$t_0 h_0(m) + t_1 h_1(m) = t \int_t^1 m + (1-t) \int_t^0 m = t \int_0^1 m - \int_0^t m.$$

5. A closer look at the L_∞ structure on C_χ

We now look for the explicit expressions for the degree 1 linear maps

$$\langle\ \rangle_n : \bigodot^n C_\chi[1] \to C_\chi[1], \qquad n \geq 2,$$

defining the L_∞ structure $\widetilde{C}(\chi)$, using the Kontsevich–Soibelman formulas described in Proposition 2.2.

The L_∞ structure on the differential graded Lie algebra H_χ is given by the brackets

$$q_k : \bigodot^k (H_\chi[1]) \to H_\chi[1],$$

where $q_k = 0$ for every $k \geq 3$,

$$q_1(l, m(t, dt)) = (-dl, -dm(t, dt))$$

and

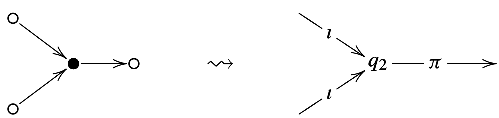
$$q_2\big((l_1, m_1(t, dt)) \odot (l_2, m_2(t, dt))\big) \\ = (-1)^{\deg_{H_\chi}(l_1, m_1(t,dt))}\big([l_1, l_2], [m_1(t, dt), m_2(t, dt)]\big).$$

The properties

$$q_2(\operatorname{Im} K \otimes \operatorname{Im} K) \subseteq \ker \pi \cap \ker K, \qquad q_k = 0 \ \text{ for all } k \geq 3,$$

imply that, fixing the number $n \geq 2$ of tails, there exists at most one isomorphism class of rooted trees giving a nontrivial contribution to $\langle\ \rangle_n$.

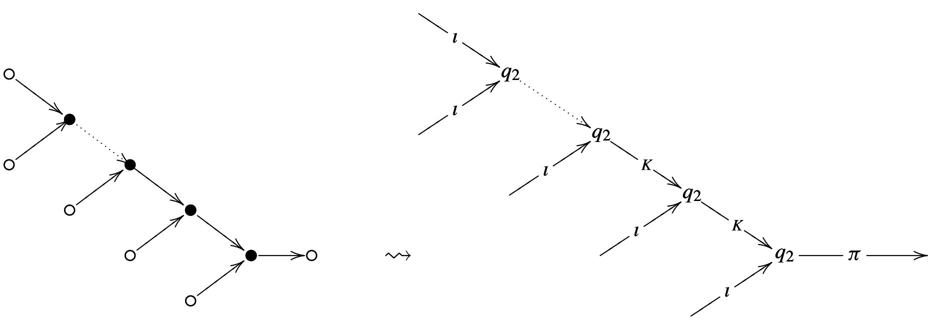
- $n = 2$:



  This graph gives

$$\langle \gamma_1 \odot \gamma_2 \rangle_2 = \pi q_2(\iota(\gamma_1) \odot \iota(\gamma_2)).$$

- $n \geq 3$:



This graph gives, for every $n \geq 3$, the formula

$$
\begin{aligned}
&\langle \gamma_1 \odot \cdots \odot \gamma_n \rangle_n = \\
&= \frac{1}{2} \sum_{\sigma \in S_n} \varepsilon(\sigma) \pi q_2\big(\iota(\gamma_{\sigma(1)}) \odot K q_2\big(\iota(\gamma_{\sigma(2)}) \odot \cdots \odot K q_2(\iota(\gamma_{\sigma(n-1)}) \odot \iota(\gamma_{\sigma(n)})) \cdots\big)\big) \\
&= \sum_{\substack{\sigma \in S_n \\ \sigma(n-1)<\sigma(n)}} \varepsilon(\sigma) \pi q_2\big(\iota(\gamma_{\sigma(1)}) \odot K q_2\big(\iota(\gamma_{\sigma(2)}) \odot \cdots \odot K q_2(\iota(\gamma_{\sigma(n-1)}) \odot \iota(\gamma_{\sigma(n)})) \cdots\big)\big).
\end{aligned}
$$

A more refined description involving the original brackets in the differential graded Lie algebras L and M is obtained by decomposing the symmetric powers of $C_\chi[1]$ into types

$$
\bigodot^n (C_\chi[1]) = \bigodot^n \operatorname{Cone}(\chi) = \bigoplus_{\lambda+\mu=n} \Big(\bigodot^\mu M\Big) \otimes \Big(\bigodot^\lambda L[1]\Big).
$$

The operation $\langle\ \rangle_2$ decomposes into

$$
\begin{aligned}
l_1 \odot l_2 &\mapsto (-1)^{\deg_L(l_1)}[l_1, l_2] \in L, \qquad m_1 \odot m_2 \mapsto 0, \\
m \otimes l &\mapsto \frac{(-1)^{\deg_M(m)+1}}{2}[m, \chi(l)] \in M.
\end{aligned}
$$

For every $n \geq 2$, it is easy to see that $\langle \gamma_1 \odot \cdots \odot \gamma_{n+1} \rangle_{n+1}$ can be nonzero only if the multivector $\gamma_1 \odot \cdots \odot \gamma_{n+1}$ belongs to $\bigodot^n M \otimes L[1]$. For $n \geq 2$, $m_1, \ldots, m_n \in M$, and $l \in L[1]$, the formula for $\langle\ \rangle_{n+1}$ described above becomes

$$
\begin{aligned}
&\langle m_1 \odot \cdots \odot m_n \otimes l \rangle_{n+1} = \\
&\sum_{\sigma \in S_n} \varepsilon(\sigma) \pi q_2\big((dt) m_{\sigma(1)} \odot K q_2\big((dt) m_{\sigma(2)} \odot \cdots \odot K q_2((dt) m_{\sigma(n)} \otimes t\chi(l)) \cdots\big)\big).
\end{aligned}
$$

Define recursively a sequence of polynomials $\phi_i(t) \in \mathbb{Q}[t] \subseteq \mathbb{K}[t]$ and rational numbers I_n by the rule

$$
\phi_1(t) = t, \qquad I_n = \int_0^1 \phi_n(t) dt, \qquad \phi_{n+1}(t) = \int_0^t \phi_n(s) ds - t I_n.
$$

By the definition of the homotopy operator K we have, for every $m \in M$,

$$
K\big((\phi_n(t) dt) m\big) = \phi_{n+1}(t) m.
$$

Therefore, for every $m_1, m_2 \in M$ we have

$$
K q_2\big((dt \cdot m_1) \odot \phi_n(t) m_2\big) = -(-1)^{\deg_M(m_1)} \phi_{n+1}(t)[m_1, m_2].
$$

Therefore, we find

$$\begin{aligned}
&\langle m_1 \odot \cdots \odot m_n \otimes l\ \rangle_{n+1} \\
&= \sum_{\sigma\in S_n} \varepsilon(\sigma)\pi q_2\big(dt\, m_{\sigma(1)} \odot K q_2\big(dt\, m_{\sigma(2)} \odot \cdots \odot K q_2(dt\, m_{\sigma(n)} \otimes t\chi(l)) \cdots\big)\big) \\
&= (-1)^{1+\deg_M(m_{\sigma(n)})} \sum_{\sigma\in S_n} \varepsilon(\sigma)\pi q_2\big(dt\, m_{\sigma(1)} \\
&\qquad\qquad \odot K q_2\big(dt\, m_{\sigma(2)} \odot \cdots \odot \phi_2(t)[m_{\sigma(n)}, \chi(l)] \cdots\big)\big) \\
&= (-1)^{n-1+\sum_{i=2}^n \deg_M(m_{\sigma(i)})} \sum_{\sigma\in S_n} \varepsilon(\sigma)\pi q_2\big(dt\, m_{\sigma(1)} \odot \phi_n(t)[m_{\sigma(2)}, \ldots, \\
&\qquad\qquad [m_{\sigma(n)}, \chi(l)] \cdots]\big) \\
&= (-1)^{n+\sum_{i=1}^n \deg_M(m_i)} I_n \sum_{\sigma\in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots, [m_{\sigma(n)}, \chi(l)] \cdots]],
\end{aligned}$$

which lies in M.

We also have an explicit expression for the coefficients I_n appearing in the formula for $\langle\ \rangle_{n+1}$; in the next lemma we show that they are, up to a sign, the Bernoulli numbers.

Lemma 5.1. *For every $n \geq 1$ we have $I_n = -B_n/n!$, where B_n are the Bernoulli numbers, that is, the rational numbers defined by the series expansion identity*

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x-1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

Proof. Keeping in mind the definition of B_n, we have to prove that

$$1 - \sum_{n=1} I_n x^n = \frac{x}{e^x - 1}.$$

Consider the polynomials $\psi_0(t) = 1$ and $\psi_n(t) = \phi_n(t) - I_n$ for $n \geq 1$. Then, for any $n \geq 1$,

$$\frac{d}{dt}\psi_n(t) = \psi_{n-1}(t), \qquad \int_0^1 \psi_n(t)dt = 0.$$

Setting

$$F(t,x) = \sum_{n=0}^{\infty} \psi_n(t) x^n,$$

we have

$$\frac{d}{dt}F(t,x) = \sum_{n=1}^{\infty} \psi_{n-1}(t)x^n = xF(t,x), \qquad \int_0^1 F(t,x)dt = 1.$$

Therefore, $F(t,x) = F(0,x)e^{tx}$,

$$1 = \int_0^1 F(t,x)dt = F(0,x)\int_0^1 e^{tx}dt = F(0,x)\frac{e^x-1}{x},$$

and then

$$F(0,x) = \frac{x}{e^x-1}.$$

Since $\psi_n(0) = -I_n$ for any $n \geq 1$ we get

$$\frac{x}{e^x-1} = F(0,x) = 1 - \sum_{n=1}^{\infty} I_n x^n.$$

In fact an alternative proof of the equality $I_n = -B_n/n!$ can be done by observing that the polynomials $n!\psi_n(t)$ satisfy the recursive relations of the Bernoulli polynomials; see for example [Remmert 1991]. □

Summing up the results of this section, we have the following explicit description of the L_∞ algebra $\widetilde{C}(\chi)$.

Theorem 5.2. *The L_∞ algebra $\widetilde{C}(\chi)$ is defined by the multilinear maps*

$$\langle\ \rangle_n : \bigodot^n C_\chi[1] \to C_\chi[1]$$

given by

$$\langle (l,m)\rangle_1 = (-dl, -\chi(l) + dm), \qquad \langle l_1 \odot l_2\rangle_2 = (-1)^{\deg_L(l_1)}[l_1, l_2],$$

$$\langle m_1 \odot m_2\rangle_2 = 0, \qquad \langle m \otimes l\rangle_2 = \tfrac{1}{2}(-1)^{\deg_M(m)+1}[m, \chi(l)],$$

$$\langle m_1 \odot \cdots \odot m_n \otimes l_1 \odot \cdots \odot l_k\rangle_{n+k} = 0 \quad \textit{if } n+k \geq 3 \textit{ and } k \neq 1,$$

and

$$\langle m_1 \odot \cdots \odot m_n \otimes l\,\rangle_{n+1} = -(-1)^{\sum_{i=1}^n \deg_M(m_i)} \frac{B_n}{n!} \sum_{\sigma\in S_n} \varepsilon(\sigma)\big[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots, [m_{\sigma(n)}, \chi(l)]\cdots]\big]$$

if $n \geq 2$. Here the B_n are the Bernoulli numbers, that is, the rational numbers defined by the series expansion identity

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x-1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

Remark 5.3. Via the decalage isomorphism $\bigodot^n(C_\chi[1]) \xrightarrow{\sim} (\bigwedge^n C_\chi)[n]$, the linear maps $\langle\ \rangle_n$ defining the L_∞-algebra $\widetilde{C}(\chi)$ correspond to multilinear operations $[\ ]_n$:

$\bigwedge^n C_\chi \to C_\chi[2-n]$ on C_χ. In particular, the linear map $\langle\ \rangle_1$ corresponds to the differential δ on C_χ,

$$\delta : (l, m) \mapsto (dl, \chi(l) - dm),$$

whereas the map $\langle\ \rangle_2$ corresponds to the degree-zero bracket

$$[\ ]_2 : C_\chi \wedge C_\chi \to C_\chi$$

given by

$$[l_1, l_2]_2 = [l_1, l_2], \qquad [m, l]_2 = \tfrac{1}{2}[m, \chi(l)], \qquad [m_1, m_2]_2 = 0.$$

This is precisely the naive bracket described in the Introduction.

Remark 5.4. The occurrence of Bernoulli numbers is not surprising. It had already been noticed by K. T. Chen [1957] how Bernoulli numbers are related to the coefficients of the Baker–Campbell–Hausdorff formula.

More recently, the relevance of Bernoulli numbers in deformation theory has been also remarked by Ziv Ran [2004]. In particular, Ran's "JacoBer" complex provides an independent description of the L_∞ structure $\widetilde{C}(\chi)$; see also [Merkulov 2005].

Bernoulli numbers also appear in some expressions of the gauge equivalence in a differential graded Lie algebra [Sullivan 2007; Getzler 2004]. In fact the relation $x = e^a * y$ can be written as

$$x - y = \frac{e^{\mathrm{ad}_a} - 1}{\mathrm{ad}_a}([a, y] - da).$$

Applying to both sides the inverse of the operator $(e^{\mathrm{ad}_a} - 1)/\,\mathrm{ad}_a$ we get

$$da = [a, y] - \sum_{n \geq 0} \frac{B_n}{n!}\,\mathrm{ad}_a^n(x - y).$$

The multilinear brackets $\langle\ \rangle_n$ on $\mathrm{Cone}(\chi) = C_\chi[1]$ can be related to the Koszul (or "higher derived") brackets Φ_n of a differential graded Lie algebra as follows. Let $(M, \partial, [\ ,\ ])$ be a differential graded Lie algebra. The Koszul brackets

$$\Phi_n : \bigodot^n M \to M, \qquad n \geq 1$$

are the degree-1 linear maps defined as $\Phi_1 = 0$ and

$$\Phi_n(m_1 \cdots m_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma)\big[\cdot[[\partial m_{\sigma(1)}, m_{\sigma(2)}], m_{\sigma(3)}], \ldots, m_{\sigma(n)}\big]$$

for $n \geq 2$. Let L be the differential graded Lie subalgebra of M, given by ∂M and let $\chi : L \to M$ be the inclusion. We can identify M with the image of the injective linear map $M \hookrightarrow \mathrm{Cone}(\chi)$ given by $m \mapsto (\partial m, m)$. Then we have $\langle(\partial m, m)\rangle_1 = 0$,

$$\langle(\partial m_1, m_1) \odot (\partial m_2, m_2)\rangle_2 = (\partial \Phi_2(m_1, m_2), \Phi_2(m_1, m_2))$$

and, for $n \geq 2$,

$$\langle(\partial m_1,m_1) \odot \cdots \odot (\partial m_{n+1}, m_{n+1})\rangle_{n+1} = \big(0, B_n(-1)^n(n+1)\Phi_{n+1}(m_1 \odot \cdots \odot m_{n+1})\big).$$

Since the multilinear operations $\langle\ \rangle_n$ define an L_∞-algebra structure on $C_\chi = \mathrm{Cone}(\chi)[-1]$, they satisfy a sequence of quadratic relations. Due to the already mentioned correspondence with the Koszul brackets, these relations are translated into a sequence of differential or quadratic relations between the odd Koszul brackets, defined as $\{m\}_1 = 0$ and

$$\{m_1, \ldots, m_n\}_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma)(-1)^\sigma [\cdot[[\partial m_{\sigma(1)}, m_{\sigma(2)}], m_{\sigma(3)}], \ldots, m_{\sigma(n)}]$$

for $n \geq 2$. For instance, if m_1, m_2, m_3 are homogeneous elements of degree i_1, i_2, i_3 respectively, then

$$\{\{m_1, m_2\}_2, m_3\} + (-1)^{i_1 i_2 + i_1 i_3}\{\{m_2, m_3\}_2, m_1\}_2 + (-1)^{i_2 i_3 + i_1 i_3}\{\{m_3, m_1\}_2, m_2\}_2 = \tfrac{3}{2}\, \partial\{m_1, m_2, m_3\}_3.$$

The occurrence of Bernoulli numbers in the L_∞-type structure defined by the higher Koszul brackets has been recently remarked by K. Bering [2006].

6. The Maurer–Cartan functor

Having introduced an L_∞ structure on C_χ in Section 5, we have a corresponding Maurer–Cartan functor [Fukaya 2003; Kontsevich 2003] $\mathrm{MC}_{C_\chi} : \mathbf{Art} \to \mathbf{Set}$, defined as

$$\mathrm{MC}_{C_\chi}(A) = \Big\{\gamma \in C_\chi[1]^0 \otimes \mathfrak{m}_A : \sum_{n \geq 1} \frac{\langle \gamma^{\odot n}\rangle_n}{n!} = 0\Big\}, \qquad A \in \mathbf{Art}.$$

With $\gamma = (l, m)$, $l \in L^1 \otimes \mathfrak{m}_A$ and $m \in M^0 \otimes \mathfrak{m}_A$, the Maurer–Cartan equation becomes

$$0 = \sum_{n=1}^{\infty} \frac{\langle (l, m)^{\odot n}\rangle_n}{n!}$$

$$= \langle (l,m)\rangle_1 + \tfrac{1}{2}\langle l^{\odot 2}\rangle_2 + \langle m\otimes l\rangle_2 + \tfrac{1}{2}\langle m^{\odot 2}\rangle_2 + \sum_{n\geq 2}\frac{n+1}{(n+1)!}\langle m^{\odot n}\otimes l\rangle_{n+1}$$

$$= -dl - \tfrac{1}{2}[l,l], -\chi(l) + dm - \tfrac{1}{2}[m,\chi(l)] + \sum_{n\geq 2}\frac{1}{n!}\langle m^{\odot n}\otimes l\rangle_{n+1},$$

which lies in $(L^2\oplus M^1)\otimes \mathfrak{m}_A$.

According to Theorem 5.2, since $\deg_M(m) = \deg_{C_{\chi[1]}}(m) = 0$, we have

$$\langle m^{\odot n}\otimes l\rangle_{n+1} = -\frac{B_n}{n!}\sum_{\sigma\in S_n}[m,[m,\ldots,[m,\chi(l)]\cdots]] = -B_n\,\mathrm{ad}_m^n(\chi(l)),$$

where for $a\in M^0\otimes\mathfrak{m}_A$ we denote by $\mathrm{ad}_a : M\otimes\mathfrak{m}_A\to M\otimes\mathfrak{m}_A$ the operator $\mathrm{ad}_a(y) = [a,y]$.

The Maurer–Cartan equation on C_χ is therefore equivalent to

$$\begin{cases} dl + \tfrac{1}{2}[l,l] = 0, \\ \chi(l) - dm + \tfrac{1}{2}[m,\chi(l)] + \displaystyle\sum_{n=2}^{\infty}\frac{B_n}{n!}\,\mathrm{ad}_m^n(\chi(l)) = 0. \end{cases}$$

Since $B_0 = 1$ and $B_1 = -\frac{1}{2}$, we can write the second equation as

$$0 = \chi(l) - dm + \tfrac{1}{2}[m,\chi(l)] + \sum_{n=2}^{\infty}\frac{B_n}{n!}\,\mathrm{ad}_m^n(\chi(l))$$

$$= [m,\chi(l)] - dm + \sum_{n=0}^{\infty}\frac{B_n}{n!}\,\mathrm{ad}_m^n(\chi(l)) = [m,\chi(l)] - dm + \frac{\mathrm{ad}_m}{e^{\mathrm{ad}_m}-1}(\chi(l)).$$

Applying the invertible operator $(e^{\mathrm{ad}_m}-1)/\,\mathrm{ad}_m$ we get

$$0 = \chi(l) + \frac{e^{\mathrm{ad}_m}-1}{\mathrm{ad}_m}([m,\chi(l)] - dm).$$

On the right-hand side of the last formula we recognize the explicit description of the gauge action

$$\exp(M^0\otimes\mathfrak{m}_A)\times M^1\otimes\mathfrak{m}_A \xrightarrow{\ *\ } M^1\otimes\mathfrak{m}_A,$$

$$e^a * y = y + \sum_{n=0}^{+\infty}\frac{\mathrm{ad}_a^n}{(n+1)!}([a,y] - da) = y + \frac{e^{\mathrm{ad}_a}-1}{\mathrm{ad}_a}([a,y]-da).$$

Therefore, the Maurer–Cartan equation for the L_∞-algebra structure on C_χ is equivalent to

$$\begin{cases} dl + \tfrac{1}{2}[l,l] = 0, \\ e^m * \chi(l) = 0. \end{cases}$$

7. Homotopy equivalence and the deformation functor

Recall that the deformation functor associated to an L_∞-algebra $\mathfrak{g}$ is

$$\mathrm{Def}_{\mathfrak{g}} = \mathrm{MC}_{\mathfrak{g}} / \sim,$$

where $\sim$ denotes homotopy equivalence of solutions of the Maurer–Cartan equation: two elements γ_0 and γ_1 of $\mathrm{MC}_{\mathfrak{g}}(A)$ are called homotopy equivalent if there exists an element $\gamma(t, dt) \in \mathrm{MC}_{\mathfrak{g}[t,dt]}(A)$ with $\gamma(0) = \gamma_0$ and $\gamma(1) = \gamma_1$.

Remark 7.1. The homotopy equivalence is an equivalence relation and a proof of this fact can be found in [Manetti 2004b, Ch. 9]. The same conclusion also follows immediately from the more general result [Getzler 2004, Prop. 4.7] that the simplicial set $\{\mathrm{MC}_{\mathfrak{g}\otimes\Omega_n}(A)\}_{n\in\mathbb{N}}$ is a Kan complex, where Ω_n is the DG commutative algebra of polynomial differential forms on the standard n-simplex.

We have already described the functor MC_{C_χ} in terms of the Maurer–Cartan equation in L and the gauge action in M. Now we want to prove a similar result for the homotopy equivalence on MC_{C_χ}. We need some preliminary results.

Proposition 7.2. *Let $(L, d, [\ ,\ ])$ be a differential graded Lie algebra such that*

(1) *$L = M \bigoplus C \bigoplus D$ as graded vector spaces,*

(2) *M is a differential graded subalgebra of L, and*

(3) *$d : C \to D[1]$ is an isomorphism of graded vector spaces.*

*Then, for every $A \in$ **Art** there exists a bijection*

$$\alpha : \mathrm{MC}_M(A) \times (C^0 \otimes \mathfrak{m}_A) \xrightarrow{\sim} \mathrm{MC}_L(A), \qquad (x, c) \mapsto e^c * x.$$

Proof. This is essentially proved in [Schlessinger and Stasheff 1979, Section 5] by the induction of the length of A and using the Baker–Campbell–Hausdorff formula. Here we sketch a different proof based on formal theory of deformation functors [Schlessinger 1968; Rim 1972; Fantechi and Manetti 1998; Manetti 1999].

The map α is a natural transformation of homogeneous functors, so it is sufficient to show that α is bijective on tangent spaces and injective on obstruction spaces. Recall that the tangent space of MC_L is $Z^1(L)$, while its obstruction space is $H^2(L)$. The functor $A \mapsto C^0 \otimes \mathfrak{m}_A$ is smooth with tangent space C^0 and therefore tangent and obstruction spaces of the functor

$$A \mapsto \mathrm{MC}_M(A) \times (C^0 \otimes \mathfrak{m}_A)$$

are respectively $Z^1(M) \bigoplus C^0$ and $H^2(M)$. The tangent map is

$$Z^1(M) \bigoplus C^0 \ni (x, c) \mapsto$$
$$e^c * x = x - dc \in Z^1(M) \bigoplus d(C^0) = Z^1(M) \bigoplus D^1 = Z^1(L)$$

and it is an isomorphism. The inclusion $M \hookrightarrow L$ is a quasiisomorphism, therefore the obstruction to lifting x in M is equal to the obstruction to lifting $x = e^0 * x$ in L. We conclude the proof by observing that, according to [Fantechi and Manetti 1998, Prop. 7.5], [Manetti 1999, Lemma 2.20], the obstruction maps of Maurer–Cartan functor are invariant under the gauge action. □

Corollary 7.3. *Let M be a differential graded Lie algebra, $L = M[t, dt]$ and $C \subseteq M[t]$ the subspace consisting of polynomials $g(t)$ with $g(0) = 0$. Then for every $A \in$ **Art** the map $(x, g(t)) \mapsto e^{g(t)} * x$ induces an isomorphism*

$$\mathrm{MC}_M(A) \times (C^0 \otimes \mathfrak{m}_A) \simeq \mathrm{MC}_L(A).$$

Proof. The data M, C and $D = d(C)$ satisfy the condition of Proposition 7.2. □

Corollary 7.4. *Let M be a differential graded Lie algebra. Two elements $x_0, x_1 \in \mathrm{MC}_M(A)$ are gauge equivalent if and only if they are homotopy equivalent.*

Proof. If x_0 and x_1 are gauge equivalent, then there exists $g \in M^0 \otimes \mathfrak{m}_A$ such that $e^g * x_0 = x_1$. Then, by Corollary 7.3. $x(t) = e^{t\,g} * x_0$ is an element of $\mathrm{MC}_{M[t,dt]}(A)$ with $x(0) = x_0$ and $x(1) = x_1$, that is, x_0 and x_1 are homotopy equivalent.

Vice versa, if x_0 and x_1 are homotopy equivalent, there exists

$$x(t) \in \mathrm{MC}_{M[t,dt]}(A)$$

such that $x(0) = x_0$ and $x(1) = x_1$. By Corollary 7.3., there exists $g(t) \in M^0[t] \otimes \mathfrak{m}_A$ with $g(0) = 0$ such that $x(t) = e^{g(t)} * x_0$. Then $x_1 = e^{g(1)} * x_0$, that is, x_0 and x_1 are gauge equivalent. □

Theorem 7.5. *Let $\chi : L \to M$ be a morphism of differential graded Lie algebras and let (l_0, m_0) and (l_1, m_1) be elements of $\mathrm{MC}_{C_\chi}(A)$. Then (l_0, m_0) is homotopically equivalent to (l_1, m_1) if and only if there exists $(a, b) \in C^0_\chi \otimes \mathfrak{m}_A$ such that*

$$l_1 = e^a * l_0, \qquad e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}.$$

Remark 7.6. The condition $e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}$ can be also written as $m_1 \bullet \chi(a) = db \bullet m_0$, where $\bullet$ is the Baker–Campbell–Hausdorff product in the nilpotent Lie algebra $M^0 \otimes \mathfrak{m}_A$.

As a consequence, we get that in this case the homotopy equivalence is induced by a group action, which is false for general L_∞-algebras.

Proof. We shall say that two elements (l_0, m_0), (l_1, m_1) are gauge equivalent if and only if there exists $(a, b) \in C^0_\chi \otimes \mathfrak{m}_A$ such that

$$l_1 = e^a * l_0, \qquad e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}.$$

We first show that homotopy implies gauge. Let (l_0, m_0) and (l_1, m_1) be homotopy equivalent elements of $\mathrm{MC}_{C_\chi}(A)$. Then there exists an element $(\tilde{l}, \tilde{m})$ of

$\mathrm{MC}_{C_\chi[s,ds]}(A)$ with $(\tilde{l}(0),\tilde{m}(0))=(l_0,m_0)$ and $(\tilde{l}(1),\tilde{m}(1))=(l_1,m_1)$. According to Remark 3.3, the Maurer–Cartan equation for $(\tilde{l},\tilde{m})$ is

$$\begin{cases} d\tilde{l}+\frac{1}{2}[\tilde{l},\tilde{l}]=0,\\ e^{\tilde{m}}*\chi(\tilde{l})=0.\end{cases}$$

The first of the two equations above tells us that $\tilde{l}$ is a solution of the Maurer–Cartan equation for $L[s,ds]$. So, by Corollary 7.3, there exists a degree zero element $\lambda(s)$ in $L[s]\otimes\mathfrak{m}_A$ with $\lambda(0)=0$ such that $\tilde{l}=e^{\lambda}*l_0$. Evaluating at $s=1$ we find $l_1=e^{\lambda_1}*l_0$. As a consequence of $\tilde{l}=e^{\lambda}*l_0$, we also have $\chi(\tilde{l})=e^{\chi(\lambda)}*\chi(l_0)$. Set $\tilde{\mu}=\tilde{m}\bullet\chi(\lambda)\bullet(-m_0)$, so that $\tilde{m}=\tilde{\mu}\bullet m_0\bullet(-\chi(\lambda))$ and the second Maurer–Cartan equation is reduced to $e^{\tilde{\mu}}*(e^{m_0}*\chi(l_0))=0$, that is, to $e^{\tilde{\mu}}*0=0$, where we have used the fact that (l_0,m_0) is a solution of the Maurer–Cartan equation in C_χ. This last equation is equivalent to the equation $d\tilde{\mu}=0$ in $C_\chi[s,ds]\otimes\mathfrak{m}_A$. If we write $\tilde{\mu}(s,ds)=\mu^0(s)+ds\,\mu^{-1}(s)$, then the equation $d\tilde{\mu}=0$ becomes

$$\begin{cases} \dot{\mu}^0-d_M\mu^{-1}=0,\\ d_M\mu^0=0,\end{cases}$$

where d_M is the differential in the DGLA M. The solution is, for any fixed μ^{-1},

$$\mu^0(s)=\int_0^s d\sigma\, d_M\mu^{-1}(\sigma)=-d_M\int_0^s d\sigma\,\mu^{-1}(\sigma).$$

Set $\nu=-\int_0^1 ds\,\mu^{-1}(s)$. Then $m_1=\tilde{m}(1)=(d_M\nu)\bullet m_0\bullet(-\chi(\lambda_1))$. In summary, if (l_0,m_0) and (m_1,l_1) are homotopy equivalent, then there exists

$$(d\nu,\lambda_1)\in(dM^{-1}\otimes\mathfrak{m}_A)\times(L^0\otimes\mathfrak{m}_A)$$

such that

$$\begin{cases} l_1=e^{\lambda_1}*l_0,\\ m_1=d\nu\bullet m_0\bullet(-\chi(\lambda_1)),\end{cases}$$

that is, (l_0,m_0) and (m_1,l_1) are gauge equivalent.

We now show that gauge implies homotopy. Assume (l_0,m_0) and (m_1,l_1) are gauge equivalent. Then there exists

$$(d\nu,\lambda_1)\in(dM^{-1}\otimes\mathfrak{m})\times(L^0\otimes\mathfrak{m})$$

such that

$$\begin{cases} l_1=e^{\lambda_1}*l_0,\\ m_1=d\nu\bullet m_0\bullet(-\chi(\lambda_1)).\end{cases}$$

Set $\tilde{l}(s,ds)=e^{s\lambda_1}*l_0$. By Corollary 7.3, $\tilde{l}$ satisfies the equation $d\tilde{l}+\frac{1}{2}[\tilde{l},\tilde{l}]=0$. Set $\tilde{m}=(d(s\nu))\bullet m_0\bullet(-\chi(s\lambda_1))$. Reasoning as above, we find

$$e^{\tilde{m}}*\chi(\tilde{l})=e^{d(s\nu)}*0=0.$$

Therefore, $(\tilde{l}, \tilde{m})$ is a solution of the Maurer–Cartan equation in $C_\chi[s, ds]$. Moreover $\tilde{l}(0) = l_0$, $\tilde{l}(1) = l_1$, $\tilde{m}(0) = m_0$ and $\tilde{m}(1) = dv \bullet m_0 \bullet (-\chi(\lambda_1)) = m_1$, that is, (l_0, m_0) and (m_1, l_1) are homotopy equivalent. □

8. Examples and applications

Let $\chi : L \to M$ be a morphism of differential graded Lie algebras over a field $\mathbb{K}$ of characteristic 0. In the paper [Manetti 2005] one of the authors has introduced, having in mind the example of embedded deformations, the notion of Maurer–Cartan equation and gauge action for the triple (L, M, χ); these notions reduce to the standard Maurer–Cartan equation and gauge action of L when $M = 0$. More precisely, there are two functors of Artin rings $\mathrm{MC}_\chi, \mathrm{Def}_\chi : \mathbf{Art} \to \mathbf{Set}$, defined by

$$MC_\chi(A) = \\ \left\{(x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) : dx + \tfrac{1}{2}[x, x] = 0,\ e^a * \chi(x) = 0\right\},$$

$$\mathrm{Def}_\chi(A) = \frac{\mathrm{MC}_\chi(A)}{\text{gauge equivalence}},$$

where two solutions of the Maurer–Cartan equation are gauge equivalent if they belong to the same orbit of the gauge action

$$(\exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)) \times \mathrm{MC}_\chi(A) \xrightarrow{*} \mathrm{MC}_\chi(A)$$

given by the formula

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}) = (e^l * x, e^{dm \bullet a \bullet (-\chi(l))}).$$

The computations of Sections 6 and 7 show that MC_χ and Def_χ are canonically isomorphic to the functors $\mathrm{MC}_{\widetilde{C}(\chi)}$ and $\mathrm{Def}_{\widetilde{C}(\chi)}$ associated with the L_∞ structure on C_χ.

Example 8.1. Let X be a compact complex manifold and let $Z \subset X$ be a smooth subvariety. Denote by Θ_X the holomorphic tangent sheaf of X and by $N_{Z|X}$ the normal sheaf of Z in X.

Consider the short exact sequence of complexes

$$0 \to \ker \pi \xrightarrow{\chi} A_X^{0,*}(\Theta_X) \xrightarrow{\pi} A_Z^{0,*}(N_{Z|X}) \to 0.$$

It is proved in [Manetti 2005] that there exists a natural isomorphism between Def_χ and the functor of embedded deformations of Z in X. Therefore, the L_∞ algebra $\widetilde{C}(\chi)$ governs the embedded deformations in this case.

The DGLA $A^{0,*}_Z(\Theta_Z)$ governs the deformations of Z; the natural transformation

$$\mathrm{Def}_{\widetilde{C}(\chi)} = \mathrm{Def}_\chi \to \mathrm{Def}_{A^{0,*}_Z(\Theta_Z)},$$
$$\{\text{Embedded deformations of } Z\} \to \{\text{Deformations of } Z\},$$

is induced by the morphism in **DGLA²** given by the diagram

$$\begin{array}{ccc} \ker\pi & \longrightarrow & A^{0,*}_Z(\Theta_Z) \\ {\scriptstyle\chi}\downarrow & & \downarrow \\ A^{0,*}_X(\Theta_X) & \longrightarrow & 0. \end{array}$$

The next result was proved in [Manetti 2005] using the theory of extended deformation functors. Here we can prove it in a more standard way.

Theorem 8.2. *Consider a commutative diagram*

$$\begin{array}{ccc} L_1 & \xrightarrow{f_L} & L_2 \\ {\scriptstyle\chi_1}\downarrow & & \downarrow{\scriptstyle\chi_2} \\ M_1 & \xrightarrow{f_M} & M_2 \end{array}$$

of morphisms of differential graded Lie algebras and assume that

$$(f_L, f_M) : C_{\chi_1} \to C_{\chi_2}$$

is a quasiisomorphism of complexes (for example, if both f_L and f_M are quasiisomorphisms). Then the natural transformation $\mathrm{Def}_{\chi_1} \to \mathrm{Def}_{\chi_2}$ *is an isomorphism.*

Proof. The map $(f_L, f_M) : \widetilde{C}(\chi_1) \to \widetilde{C}(\chi_2)$ is a linear quasiisomorphism of L_∞-algebras and then induces an isomorphism of the associated deformation functors [Kontsevich 2003]. □

Example 8.3. It is shown in [Fiorenza and Manetti 2006] how the L_∞ structures $\widetilde{C}(\chi)$ are related to the period maps of a compact Kähler manifold X. Denote by $A_X = F^0 \supseteq F^1 \supseteq \cdots$, the Hodge filtration of differential forms on X, that is, for every $p \geq 0$,

$$F^p = \bigoplus_{i \geq p} \bigoplus_j A^{i,j}_X.$$

For a fixed nonnegative integer p one considers the inclusion of differential graded Lie algebras

$$\left\{ f \in \mathrm{Hom}^*(A_X, A_X) : f(F^p) \subseteq F^p \right\} \xrightarrow{\chi} \mathrm{Hom}^*(A_X, A_X).$$

The contraction of differential forms with vector fields

$$\boldsymbol{i} : A^{0,*}_X(\Theta_X) \to \mathrm{Hom}^*(A_X, A_X)[-1],$$

and the holomorphic Lie derivative

$$\boldsymbol{l} : A_X^{0,*}(\Theta_X) \to \{ f \in \mathrm{Hom}^*(A_X, A_X) : f(F^p) \subseteq F^p \}$$

define a linear map $\mathfrak{p}^p = (\boldsymbol{l}, \boldsymbol{i}) : A_X^{0,*}(\Theta_X) \to C_\chi$, which is actually a linear L_∞-morphism

$$\mathfrak{p}^p : A_X^{0,*}(\Theta_X) \to \widetilde{C}(\chi).$$

The induced morphism of deformation functors

$$\mathcal{P}^p : \mathrm{Def}_X \to \mathrm{Def}_\chi \simeq \mathrm{Grass}_{H^*(F^p), H^*(A_X)}$$

is the infinitesimal p-th period map of the Kähler manifold X. As immediate corollaries of this L_∞-algebra interpretation of period maps, one recovers Griffiths' description of the differential of the period map, namely

$$d\mathcal{P}^p = \boldsymbol{i} : H^1(X, T_X) \to \bigoplus_i \mathrm{Hom}\Big(F^p H^i(X, \mathbb{C}), \frac{H^i(X, \mathbb{C})}{F^p H^i(X, \mathbb{C})} \Big),$$

and a proof of the so-called Kodaira's Principle [Clemens 2005; Manetti 2004a; Ran 1999] that obstructions to deformations of X are contained in the kernel of

$$\boldsymbol{i} : H^2(X, T_X) \to \bigoplus_i \mathrm{Hom}\Big(F^p H^i(X, \mathbb{C}), \frac{H^{i+1}(X, \mathbb{C})}{F^p H^{i+1}(X, \mathbb{C})} \Big),$$

for every $p \geq 0$.

Example 8.4. Let $\pi : A \to B$ be a surjective morphism of associative $\mathbb{K}$-algebras and denote by I its kernel. The algebra B is an A-module via π; this makes B a trivial I-module. Let K be the suspended Hochschild complex

$$K = \mathrm{Hoch}^\bullet(I, B)[-1].$$

The differential d of K is identically zero if and only if $I \cdot I = 0$.

The natural map

$$\alpha : \mathrm{Hoch}^\bullet(A, A) \to K[1] = \mathrm{Hoch}^\bullet(I, B)$$

is a surjective morphism of complexes, and its kernel

$$\ker \alpha = \{ f : f(I^{\otimes}) \subseteq I \}$$

is a Lie subalgebra of $\mathrm{Hoch}^\bullet(A, A)$ endowed with the Hochschild bracket. Denote by $\chi : \ker \alpha \hookrightarrow \mathrm{Hoch}^\bullet(A, A)$ the inclusion. Since χ is injective, the projection on the second factor induces a quasiisomorphism of differential complexes

$$\mathrm{pr}_2 : C_\chi \to \mathrm{Coker}(\chi)[-1] \simeq K,$$

where the isomorphism on the right is induced by the map α. Therefore we have a canonical L_∞ structure (defined up to homotopy) on K. This gives a Lie structure on the cohomology of K, which is not trivial in general. Consider for instance the exact sequence

$$0 \to \mathbb{K}\varepsilon \to \mathbb{K}[\varepsilon]/(\varepsilon^2) \xrightarrow{\pi} \mathbb{K} \to 0$$

and take $f \in K^1 = H^1(K)$ with $f(\varepsilon) = 1$. Choose as a lifting the linear map $g : \mathbb{K}[\varepsilon]/(\varepsilon^2) \to \mathbb{K}[\varepsilon]/(\varepsilon^2)$ such that $g(1) = 0$ and $g(\varepsilon) = 1$. Then

$$dg(\varepsilon \otimes \varepsilon) = 2\varepsilon$$

and so $dg \in \ker\alpha$. Therefore, (dg, g) is a closed element of C^1_χ representing the cohomology class $f \in H^1(K)$ and so

$$[f, f] = \alpha(\mathrm{pr}_2([(dg, g), (dg, g)]_2)) = \alpha([g, dg]).$$

One computes

$$\begin{aligned}
[f, f](\varepsilon \otimes \varepsilon) &= \pi([g, dg](\varepsilon \otimes \varepsilon)) \\
&= \pi\big(g(dg(\varepsilon \otimes \varepsilon)) - dg(g(\varepsilon) \otimes \varepsilon) + dg(\varepsilon \otimes g(\varepsilon))\big) \\
&= \pi(g(2\varepsilon) - dg(1 \otimes \varepsilon) + dg(\varepsilon \otimes 1)) = 2.
\end{aligned}$$

Hence $[f, f] \neq 0$.

On the other hand, if $A = B \bigoplus I$ as an associative $\mathbb{K}$-algebra, then the L_∞ structure on K is trivial. Indeed, as $K[1]$ is considered to be a DGLA with trivial bracket, the obvious map

$$K[1] = \mathrm{Hoch}^\bullet(I, B) \to \mathrm{Hoch}^\bullet(A, A)$$

gives a commutative diagram of morphisms of DGLAs

$$\begin{array}{ccc}
0 & \to & \ker\alpha \\
\big\downarrow & & \big\downarrow \chi \\
K[1] & \to & \mathrm{Hoch}^\bullet(A, A)
\end{array}$$

such that the composition $K \to C_\chi \to K$ is the identity. Therefore the L_∞-algebra structure induced on K is isomorphic to $\widetilde{C}(0 \hookrightarrow K[1])$, which is a trivial L_∞-algebra.

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