Dual graded graphs for Kac–Moody algebras

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Motivated by affine Schubert calculus, we construct a family of dual graded graphs (Γ_s, Γ_w) for an arbitrary Kac–Moody algebra $\mathfrak{g}$. The graded graphs have the Weyl group W of $\mathfrak{geh}$ as vertex set and are labeled versions of the strong and weak orders of W respectively. Using a construction of Lusztig for quivers with an admissible automorphism, we define folded insertion for a Kac–Moody algebra and obtain Sagan–Worley shifted insertion from Robinson–Schensted insertion as a special case. Drawing on work of Proctor and Stembridge, we analyze the induced subgraphs of (Γ_s, Γ_w) which are distributive posets.

1. Introduction

The Robinson–Schensted correspondence is perhaps the most important algorithm in algebraic combinatorics. It exhibits a bijection between permutations and pairs of standard Young tableaux of the same shape. Stanley [1988] investigated the class of differential posets (also studied in [Fomin 1986]). Fomin [1994] studied the more general notion of a dual graded graph to formalize local conditions which guarantee the existence of a Robinson–Schensted style algorithm.

In this article, we construct a family of dual graded graphs (Γ_s, Γ_w) associated to each Kac–Moody algebra $\mathfrak{g}$. These graded graphs have as vertex set the Weyl group W of $\mathfrak{g}$. The pair $(\Gamma_s, \Gamma_w) = (\Gamma_s(\Lambda), \Gamma_w(K))$ depends on a pair (Λ, K) where Λ is a dominant integral weight and K is a "positive integral" element of the center $Z(\mathfrak{g})$. In every case Γ_w is obtained by labeling the left weak order of W and Γ_s is obtained by labeling the strong Bruhat order of W.

These labelings are motivated by the Schubert calculus for homogeneous spaces associated to the Kac–Moody group G with Lie algebra $\mathfrak{g}$. For $w \in W$, let $\xi^w \in H^*(G/B)$ denote the cohomology Schubert classes of the flag manifold of G. If $\Lambda = \Lambda_i$ is the i-th fundamental weight, then an edge $v \lessdot w$ in $\Gamma_s(\Lambda)$ is labeled with

the coefficient of ξ^w in the product $\xi^{s_i}\xi^v$, also called a Chevalley coefficient. When $\mathfrak{g}$ is of affine type and $K = K_{\mathrm{can}}$ is the canonical central element, the analogous statement holds (Proposition 2.17) for $\Gamma_w(K)$ with the homology Schubert classes $\xi_w \in H_*(\mathrm{Gr})$ of the affine Grassmannian corresponding to $\mathfrak{g}$ replacing the cohomology classes. Thus the combinatorics of these graphs encode computations in Schubert calculus, and the duality of the graded graphs (Γ_s, Γ_w) is a combinatorial skeleton of the duality between cohomology and homology of homogeneous spaces of G.

In the case of the affine Grassmannian, the dual graded graph structure arises from the pair of dual graded Hopf algebras given by $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$: one may define the down operator by the action of the homology class ξ_{s_0} on the Schubert basis of $H^*(\mathrm{Gr})$ and the up operator by multiplication by ξ^{s_i} for any fixed simple reflection s_i. It is a general phenomenon that pairs of dual graded combinatorial Hopf algebras yield dual graded graphs; we shall pursue this in a separate publication [Lam and Shimozono $\geq$ 2007].

Chains in the graded graphs (Γ_s, Γ_w), which we call strong and weak tableaux, are natural generalizations of standard Young tableaux. To go one speculative step further, we believe that the generating functions of an appropriate semistandard notion of strong and weak tableaux would give polynomials which *represent* certain homology and cohomology Schubert classes, in particular for homogeneous spaces corresponding to maximal parabolics, generalizing Schur functions, Schur Q-functions and the like. While this statement is vague in general, it can be made much more precise when $\mathfrak{g}$ is of affine type, and has already been achieved in one case.

In the case that $\mathfrak{g}$ is of the affine type $A_{n-1}^{(1)}$ our construction recovers the dual graded graphs that were implicitly studied in our joint work with Lapointe and Morse [2006]. The weak and strong tableaux in [Lam et al. 2006] are semistandard generalizations of the corresponding objects here; in the same work, an affine insertion algorithm was explicitly constructed for semistandard weak and strong tableaux, and from [Lam 2006; Lam et al. 2006] we know that the corresponding generating functions do indeed represent Schubert classes of the affine Grassmannian of type A. In the limit $n \to \infty$ of the $A_{n-1}^{(1)}$ case, our construction reproduces Young's lattice, which is the self-dual graded graph that gives rise to the Robinson–Schensted algorithm.

Having constructed the Kac–Moody dual graded graphs we study two further aspects of these graphs in detail.

The first aspect is motivated by the relation between the Robinson–Schensted insertion and Sagan–Worley shifted insertion. Using Lusztig's construction [1993] which associates to each symmetrizable generalized Cartan matrix A, a symmetric generalized Cartan matrix B equipped with an admissible automorphism π, we

show that any dual graded graph of the form (Γ_s^A, Γ_w^A) for $\mathfrak{g}(A)$ can be realized in terms of one of the form (Γ_s^B, Γ_w^B) for $\mathfrak{g}(B)$. In particular, for any affine algebra, any of the dual graded graphs (Γ_s, Γ_w) can be realized using a dual graded graph for a simply-laced affine algebra. In particular we obtain a Schensted bijection for type $C_n^{(1)}$, using the insertion algorithm of [Lam et al. 2006] for type $A_{2n-1}^{(1)}$. As n goes to infinity, the type $C_n^{(1)}$ insertion converges to Sagan–Worley insertion [Sagan 1987; Worley 1984]. As a related result, we define a notion of mixed insertion for dual graded graphs equipped with a pair of automorphisms. This generalizes Haiman's variants of Schensted insertion known as left-right, mixed, and doubly dual insertion [Haiman 1989].

The second aspect we investigate are the induced subgraphs of the pair (Γ_s, Γ_w) which are distributive lattices when considered as posets. These are precisely the conditions under which one may describe our strong and weak tableaux by "filling cells with numbers" as in a usual standard Young tableau. Here we draw on [Proctor 1984; 1999; Stembridge 1996], which classify the parabolic quotients of Weyl groups of simple Lie algebras whose left weak orders (or equivalently Bruhat orders) are distributive lattices. We sharpen these results slightly to show that in these cases, the distributivity is compatible with the edge labels of the graphs (Γ_s, Γ_w); see Section 6B. These distributive parabolic quotients have also appeared recently in the geometric work [Thomas and Yong 2006]. They show that in these cases one may use the *jeu-de-taquin* to calculate Schubert structure constants of the cohomology of (co)minuscule flag varieties. We do not recover this result, but we note that their notion of standard tableau, fits into our framework as strong (or weak) tableaux for the distributive parabolic quotients, with the edge labels forgotten.

2. Dual graded graphs for Kac–Moody algebras

2A. *Dual graded graphs.* We recall Fomin's notion of dual graded graphs [Fomin 1994]. A *graded graph* is a directed graph

$$\Gamma = (V, E, h, m)$$

with vertex set V and set of directed edges $E \subset V^2$, together with a grading function $h : V \to \mathbb{Z}_{\geq 0}$, such that every directed edge $(v, w) \in E$ satisfies $h(w) = h(v) + 1$ and has a multiplicity $m(v, w) \in \mathbb{Z}_{\geq 0}$. Forgetting the edge labels m, Γ may be regarded as the Hasse diagram of a graded poset. We shall interpret $m(v, w)$ as making Γ into a directed multigraph in which there are $m(v, w)$ distinct edges from v to w.

Γ is *locally finite* if, for every $v \in V$, there are finitely many $w \in V$ such that $(v, w) \in E$ and finitely many $u \in V$ such that $(u, v) \in E$; we shall assume this condition without further mention. For a graded graph $\Gamma = (V, E, h, m)$ define the

$\mathbb{Z}$-linear *down* and *up* operators $D, U : \mathbb{Z}V \to \mathbb{Z}V$ on the free abelian group $\mathbb{Z}V$ of formal $\mathbb{Z}$-linear combinations of vertices, by

$$U_\Gamma(v) = \sum_{(v,w)\in E} m(v,w)w \quad \text{and} \quad D_\Gamma(w) = \sum_{(v,w)\in E} m(v,w)v.$$

A pair of graded graphs (Γ, Γ') is *dual* if Γ and Γ' have the same vertex sets and grading function but possibly different edge sets and edge multiplicities, such that

$$D_{\Gamma'}U_\Gamma - U_\Gamma D_{\Gamma'} = r \, \mathrm{Id} \tag{1}$$

as $\mathbb{Z}$-linear operators on $\mathbb{Z}V$, for some fixed $r \in \mathbb{Z}_{>0}$. We call r the *differential coefficient*. When $\Gamma = \Gamma'$ and all the edges have multiplicity one, we obtain the r-differential posets of [Stanley 1988].

Remark 2.1. The duality property implies that V is infinite.

Example 2.2. Let $\Gamma = \mathbb{Y}$ be Young's lattice, with $(\lambda, \mu) \in E$ if the diagram of the partition μ is obtained from that of λ by adding a single cell (in which case we say that the cell is λ-*addable* and μ-*removable*), all edge multiplicities are 1, and $h(\lambda) = |\lambda|$ is the number of cells in the diagram of λ. Then $(\mathbb{Y}, \mathbb{Y})$ is a pair of dual graded graphs with differential coefficient 1.

2B. *The labeled Kac–Moody weak and strong orders.* In this section a new family of dual graded graphs is introduced.

Let I be a set of Dynkin nodes and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix (GCM), that is, one with integer entries which satisfies $a_{ii} = 2$ for all $i \in I$, and for all $i \neq j$, $a_{ij} \leq 0$ and $a_{ij} < 0$ if and only if $a_{ji} < 0$. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the Kac–Moody algebra over $\mathbb{C}$ associated to A [Kac 1990], $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra, and $\mathfrak{h}^*$ the dual. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ be the simple roots, $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ the simple coroots, and $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ the fundamental weights, with $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ where $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ is the natural pairing. We assume that the simple roots are linearly independent and the dimension of $\mathfrak{h}$ is chosen to be minimal. Let W be the Weyl group of $\mathfrak{g}$: it has generators s_i for $i \in I$ and relations $s_i^2 = 1$ for $i \in I$ and $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $i \neq j$, where m_{ij} is 2, 3, 4, 6 or ∞ according as $a_{ij}a_{ji}$ is 0, 1, 2, 3 or > 3. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function on W. Let $\Delta_{\mathrm{re}} = W \cdot \{\alpha_i \mid i \in I\}$ be the set of real roots and $\Delta_{\mathrm{re}}^+ = \Delta_{\mathrm{re}} \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\,\alpha_i$ the positive real roots. The associated coroot $\alpha^\vee$ of $\alpha \in \Delta_{\mathrm{re}}^+$ is defined by $\alpha^\vee = u\alpha_i^\vee$, where $u \in W$ and $i \in I$ are such that $\alpha = u\alpha_i$. For $\alpha \in \Delta_{\mathrm{re}}$ let $s_\alpha = us_i u^{-1}$ denote the reflection associated to α. The *strong order* (or Bruhat order) $\leq$ on W is defined by the cover relations $w \lessdot ws_\alpha$ whenever $\ell(ws_\alpha) = \ell(w) + 1$ for some $\alpha \in \Delta_{\mathrm{re}}^+$ and $w \in W$. The *left weak order* $(W, \preceq)$ is the subposet of $(W, \leq)$ generated by the cover relations $w \prec s_i w$ whenever $\ell(s_i w) = \ell(w) + 1$ for some $i \in I$ and $w \in W$. The *left descent set* of v is defined by $\mathrm{Des}(v) = \{i \in I \mid s_i v \prec v\}$.

Given Λ in the set P^+ of dominant integral weights, let $\Gamma_s(\Lambda)$ be the graded graph with vertex set W and edges $(v, w) \in W^2$ such that $v \lessdot w$, with multiplicity $m_\Lambda(v, w) = \langle \alpha^\vee, \Lambda \rangle$, where $\alpha \in \Delta_{\mathrm{re}}^+$ is such that $w = vs_\alpha$. Let $i \in I$ and $u \in W$ be such that $\alpha = u\alpha_i$. Then

$$m_\Lambda(v, w) = \langle u\alpha_i^\vee, \Lambda \rangle = \langle \alpha_i^\vee, u^{-1}\Lambda \rangle. \tag{2}$$

Let $Z^+ = Z^+(\mathfrak{g}(A)) = Z(\mathfrak{g}(A)) \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$, where $Z(\mathfrak{g}(A))$ is the center of $\mathfrak{g}(A)$. If $K \in Z^+$, writing $K = \sum_{i \in I} k_i \alpha_i^\vee$, the vector $(k_i)_{i \in I}$ defines a linear dependence amongst the rows of A.

Given $K \in Z^+$, let $\Gamma_w(K)$ be the graded graph with vertex set W and edges $(v, w) \in W^2$ such that $v \prec w = s_i v$, with multiplicity

$$n_K(v, w) = k_i = \langle K, \Lambda_i \rangle. \tag{3}$$

Both $\Gamma_s(\Lambda)$ and $\Gamma_w(K)$ are graded by the length function.

Theorem 2.3. *Let $(\Lambda, K) \in P^+ \times Z^+$. Then $(\Gamma_s(\Lambda), \Gamma_w(K))$ is a pair of dual graded graphs with differential coefficient $r = \langle K, \Lambda \rangle$.*

Proof. Let $U = U_{\Gamma_s(\Lambda)}$ and $D = D_{\Gamma_w(K)}$. The coefficient of $u \neq v$ in $(DU - UD)v$ is given by

$$\sum_{\substack{(i,\alpha) \in I \times \Delta_{\mathrm{re}}^+ \\ v \lessdot vs_\alpha \\ u = s_i vs_\alpha \prec vs_\alpha}} k_i \langle \alpha^\vee, \Lambda \rangle - \sum_{\substack{(i,\alpha) \in I \times \Delta_{\mathrm{re}}^+ \\ s_i v \prec v \\ s_i v \lessdot s_i vs_\alpha = u}} k_i \langle \alpha^\vee, \Lambda \rangle.$$

This quantity is zero because the indexing sets of both sums coincide, by two versions of [Humphreys 1990, Lemma 5.11].

For every $i \in I$ and $v \in W$, either $v \prec s_i v$ or $s_i v \prec v$ is a covering relation. It follows that the coefficient of v in $(DU - UD)v$ is

$$\sum_{i \in I \setminus \mathrm{Des}(v)} k_i \langle v^{-1}\alpha_i^\vee, \Lambda \rangle - \sum_{i \in \mathrm{Des}(v)} k_i \langle (s_i v)^{-1}\alpha_i^\vee, \Lambda \rangle$$

$$= \sum_{i \in I} k_i \langle v^{-1}\alpha_i^\vee, \Lambda \rangle = \sum_{i \in I} k_i \langle \alpha_i^\vee, v\Lambda \rangle$$

$$= \langle K, v\Lambda \rangle = \langle v^{-1}K, \Lambda \rangle = \langle K, \Lambda \rangle.$$

We have used the W-invariance of $\langle \cdot, \cdot \rangle$ and K. $\qquad \square$

Remark 2.4. For $i \in I$ and $v \in W$, the multiplicity of the edge (v, vs_α) in $\Gamma_s(\Lambda_i)$, is the Chevalley multiplicity, given by the coefficient of ξ^{vs_α} in the product $\xi^{s_i}\xi^v$, where $\xi^v \in H^*(G/B)$ is the Schubert cohomology class for the flag manifold G/B associated with the Kac–Moody algebra $\mathfrak{g}$ [Kostant and Kumar 1986].

In Proposition 2.17 we will relate the multiplicities of the weak graph $\Gamma_w(K)$ with the homology multiplication of the affine Grassmannian, in the case that $\mathfrak{g}(A)$ is of untwisted affine type.

2C. *Tableaux and enumeration.* Let $\Gamma = (V, E, h, m)$ be a graded graph and $v, w \in V$. A Γ-*tableau* T of shape v/w is a directed path

$$T = (w = v_0 \xrightarrow{m_1} v_1 \xrightarrow{m_2} \cdots \xrightarrow{m_k} v_k = v)$$

from w to v in Γ, where m_i is an element taken from a set of $m(v_{i-1}, v_i)$ possible markings of the edge (v_{i-1}, v_i). In the multigraph interpretation, m_i indicates which of the $m(v_{i-1}, v_i)$ edges going from v_{i-1} to v_i, is traversed by the path.

Let $v = \text{out}(T)$ and $w = \text{in}(T)$ the outer and inner shapes of T respectively, and denote by $\mathcal{T}(\Gamma)$ the set of Γ-tableaux, and $\mathcal{T}(\Gamma, v/w)$ the subset of those of shape v/w.

Example 2.5. For $\mathbb{Y}$ as in Example 2.2, $\mathbb{Y}$-tableaux are standard Young tableaux.

If Γ has a unique minimum element $\hat{0}$, we say T has shape v if $\text{in}(T) = \hat{0}$.

Theorem 2.6 [Fomin 1994]. *Let (Γ, Γ') be a pair of dual graded graphs with differential coefficient r. Then*

$$r^n n! = \sum_{\substack{v \in V \\ h(v) = n}} f_\Gamma^v f_{\Gamma'}^v \tag{4}$$

where $f_\Gamma^v = |\mathcal{T}(\Gamma, v)|$ and $f_{\Gamma'}^v = |\mathcal{T}(\Gamma', v)|$.

Example 2.7. Let $\Gamma = \Gamma' = \mathbb{Y}$. Then by Examples 2.2 and 2.5, Equation (4) is the well known identity $n! = \sum_\lambda f_\lambda^2$, where λ ranges over the partitions of n and f_λ is the number of standard Young tableaux of shape λ.

The graphs $\Gamma_w(K)$ and $\Gamma_s(\Lambda)$ both have minimum element $\text{id} \in W$. We call $\Gamma_w(K)$-tableaux *(standard) K-weak tableaux* and $\Gamma_s(\Lambda)$-tableaux *(standard) Λ-strong tableaux*.

Corollary 2.8. *Let $\mathfrak{g}$ be a Kac–Moody algebra and $(\Lambda, K) \in P^+ \times Z^+$. Then for each $n \in \mathbb{Z}_{\geq 0}$ we have*

$$r^n n! = \sum_{\substack{w \in W \\ \ell(w) = n}} f_{\text{weak}}^w f_{\text{strong}}^w \tag{5}$$

where $r = \langle K, \Lambda \rangle$, f_{weak}^w is the number of K-weak tableaux of shape w and f_{strong}^w is the number of Λ-strong tableaux of shape w.

In Section 3 standard Young tableaux are realized as special cases of both K-weak and Λ-strong tableaux using affine algebras of type $A^{(1)}$.

2D. *From dual graded graphs to Schensted bijections.* A *differential bijection* for the pair of dual graded graphs (Γ, Γ') is one that exhibits the equality (1); we give a precise definition below. By [Fomin 1995], a differential bijection induces a *Schensted bijection* (see (8)) which describes the enumerative identity (4).

We recall Fomin's theory in more detail. Given $r \in \mathbb{Z}_{\geq 0}$, let $S(r)$ be a set of cardinality r. Sometimes we may write $S(r)$ for a particular set of cardinality r. Let (Γ, Γ') be a pair of dual graded graphs with $\Gamma = (V, E, h, m)$ and $\Gamma' = (V, E', h, m')$ and differential coefficient r. Given $x, y \in V$, define $\mathcal{U}\mathcal{D}_{xy} = \{(z, m, m') \in V \times \mathbb{Z}_{>0}^2 \mid (z, y) \in E, (z, x) \in E', m \leq m(z, y), m' \leq m'(z, x)\}$ and let $\mathcal{D}\mathcal{U}_{xy} = \{(w, M, M') \in V \times \mathbb{Z}_{>0}^2 \mid (x, w) \in E, (y, w) \in E', M \leq m(x, w), M' \leq m'(y, w)\}$. $\mathcal{U}\mathcal{D}_{xy}$ represents the set of marked paths going down one step in Γ' from x to some $z \in V$ and then up one step in Γ from z to y. Similarly, $\mathcal{D}\mathcal{U}_{xy}$ represents the set of marked paths going up one step in Γ from x to some $w \in V$, and then down one step in Γ' from w to y. To cancel the off-diagonal terms in (1), for every $(x, y) \in V^2$ with $x \neq y$, there must be a bijection

$$\Phi_{xy} : \mathcal{U}\mathcal{D}_{xy} \rightarrow \mathcal{D}\mathcal{U}_{xy}, \tag{6}$$

and to obtain agreement of diagonal terms in (1), for each $x \in V$ there must be a bijection

$$\Phi_x : S(r) \sqcup \mathcal{U}\mathcal{D}_x \rightarrow \mathcal{D}\mathcal{U}_x \tag{7}$$

where $\mathcal{D}\mathcal{U}_x = \mathcal{D}\mathcal{U}_{xx}$ and $\mathcal{U}\mathcal{D}_x = \mathcal{U}\mathcal{D}_{xx}$. By definition, a *differential bijection* for (Γ, Γ'), is a collection $\Phi = (\Phi_{xy}; \Phi_x)$ of such bijections Φ_{xy} and Φ_x.

Example 2.9. Let $\Gamma = \mathbb{Y}$ with dual graded graph structure on $(\mathbb{Y}, \mathbb{Y})$ as in Example 2.2. For $\lambda \in \mathbb{Y}$, since all edges have multiplicity 1, $\mathcal{D}\mathcal{U}_\lambda$ is in bijection with λ-addable corner cells and $\mathcal{U}\mathcal{D}_\lambda$ is in bijection with λ-removable corner cells. The λ-addable and λ-removable corner cells of λ are interleaved. Let Φ_λ send the unique element of the set $S(1)$, to the λ-addable corner cell in the first row of λ, and send a λ-removable corner to the nearest λ-addable corner with higher row index. For $\lambda \neq \mu$ the sets $\mathcal{D}\mathcal{U}_{\lambda\mu}$ and $\mathcal{U}\mathcal{D}_{\lambda\mu}$ have the same cardinality, which is either 0 or 1, so there is no choice for the definition of $\Phi_{\lambda\mu}$. This defines a differential bijection Φ for $(\mathbb{Y}, \mathbb{Y})$.

Let $P_n(r)$ be the set of r-colored permutations of n elements. We realize $\sigma \in P_n(r)$ as an $n \times n$ monomial matrix (one with exactly one nonzero element in each row and in each column, whose nonzero entries must be taken from a set $S(r)$ of cardinality r such that $0 \notin S(r)$).

We assume that Γ and Γ' have a common minimum element $\hat{0}$ such that $h(\hat{0}) = 0$.

A *Schensted bijection* I for (Γ, Γ') is a family of bijections (for all $n \in \mathbb{Z}_{\geq 0}$)

$$P_n(r) \to \bigsqcup_{\substack{v \in V \\ h(v)=n}} \mathcal{T}(\Gamma, v) \times \mathcal{T}(\Gamma', v) \tag{8}$$

$$\sigma \mapsto (P, Q).$$

We fix a differential bijection Φ for (Γ, Γ') and define the induced Schensted bijection I_Φ.

Given $\sigma \in P_n(r)$, we shall define a directed graph G with vertices $G_{ij} \in V$ for $0 \leq i, j \leq n$, which is depicted matrix-style. It shall have the property that (1) adjacent vertices $G_{i,j-1}, G_{ij}$ in a row, are either equal, or they form an edge $(G_{i,j-1}, G_{ij}) \in E$ with marking $m \in S(m(G_{i,j-1}, G_{ij}))$, and (2) adjacent vertices $G_{i-1,j}, G_{ij}$ in a column, are either equal, or they form an edge $(G_{i-1,j}, G_{ij}) \in E'$ with marking $m' \in S(m'(G_{i-1,j}, G_{ij}))$. Moreover, (3) $G_{i-1,j} \neq G_{ij}$ (resp. $G_{i,j-1} \neq G_{ij}$) if and only if the unique p such that $\sigma_{pj} \neq 0$ (resp. q such that $\sigma_{iq} \neq 0$) satisfies $p \leq i$ (resp. $q \leq j$). In particular (ignoring the equalities), for each i, the i-th row $G_{i\bullet}$ is a Γ-tableau and for each j, the j-th column $G_{\bullet j}$ is a Γ'-tableau. For the sake of uniform language we shall always imagine that there is a marked edge $G_{i,j-1} \to G_{ij}$ and $G_{i-1,j} \to G_{ij}$, but when the vertices coincide the marked edge degenerates.

G is defined inductively as follows. The north and west edges $G_{0\bullet}$ and $G_{\bullet 0}$ of G are initialized to the empty tableau: $G_{i0} = G_{0j} = \hat{0}$ for all $0 \leq i, j \leq n$. To define the rest of G, it suffices to give a *local rule*, which, given the marked edges

$$G_{i-1,j-1} \overset{m}{\to} G_{i-1,j} \quad \text{and} \quad G_{i-1,j-1} \overset{m'}{\to} G_{i,j-1},$$

and the value σ_{ij}, determines $G_{ij} \in V$ with markings $M \in S(m(G_{i,j-1}, G_{ij}))$ and $M' \in S(m'(G_{i-1,j}, G_{ij}))$.

This is depicted below. Use z, y, x, w to denote $G_{i-1,j-1}, G_{i-1,j}, G_{i,j-1}, G_{ij}$ for convenience and write $c = \sigma_{ij}$. In later examples we shall indicate $\sigma_{ij} = 1$ by the symbol $\otimes$ and $\sigma_{ij} = 0$ by a blank.

$$
\begin{array}{ccc}
G_{i-1,j-1} \xrightarrow{\quad m \quad} G_{i-1,j} & & z \xrightarrow{\quad m \quad} y \\
\Big\downarrow{}^{m'} \quad \sigma_{ij} \quad \Big\downarrow{}^{M'} & & \Big\downarrow{}^{m'} \quad c \quad \Big\downarrow{}^{M'} \\
G_{i,j-1} \dashrightarrow[M]{} G_{ij} & & x \dashrightarrow[M]{} w
\end{array}
\tag{9}
$$

The local rule is defined using Φ.

(1) If $z = x = y$:
    (a) If $c = 0$, set $w = z$.
    (b) If $c \neq 0$, let $\Phi_x(c) = (w, M, M')$.

(2) If $z \neq x = y$ then let $\Phi_x(z, m, m') = (w, M, M')$.

(3) If $z = y$ and $z \neq x$ then let $w = x$ and $M' = m'$.

(4) If $z = x$ and $z \neq y$ then let $w = y$ and $M = m$.

(5) If z, x, y are all distinct, then let $(w, M, M') = \Phi_{xy}(z, m, m')$.

This uniquely determines G [Fomin 1995]. Its south edge $G_{n\bullet}$ is a Γ-tableau P and its east edge $G_{\bullet n}$ is a Γ'-tableau Q, both of a common shape $v = G_{nn} \in V$ with $h(v) = n$. This well-defines a map I_Φ as in (8).

For the inverse of I_Φ, let $v \in V$ be such that $h(v) = n$, and let $(P, Q) \in \mathcal{T}(\Gamma, v) \times \mathcal{T}(\Gamma', v)$. To recover $\sigma \in P_n(r)$, we initialize the south and east edges of G to P and Q respectively. Then for each i, j and two by two subgraph as above, we apply the inverse of the above local rule. Given labeled edges

$$ x \xrightarrow{M} w \quad \text{and} \quad y \xrightarrow{M'} w, $$

it determines $z \in V$ and marked edges

$$ z \xrightarrow{m} y \quad \text{and} \quad z \xrightarrow{m'} x $$

and a value $c \in \{0\} \sqcup S(r)$, such that $c \neq 0$ if and only if $z = x = y \neq w$ and $\Phi_z^{-1}(w, M, M') = c$. The inverse local rule is defined as follows.

(1) If $x = y$:
   (a) If $w = x$, let $z = x$.
   (b) If $w \neq x$:
      (i) If $c := \Phi_x^{-1}(w, M, M') \in S(r)$: let $z = x$.
      (ii) Otherwise $\Phi_x^{-1}(w, M, M') = (z, m, m')$.

(2) If $w = x \neq y$, let $z = y$ and $m' = M'$.

(3) If $w = y \neq x$, let $z = x$ and $m = M$.

(4) If x, y, w are all distinct, let $(z, m, m') = \Phi_{xy}^{-1}(w, M, M')$.

In all cases but (1)(b)(i) let $c = 0$. Using the inverse local rule the rest of G is defined [Fomin 1995] and one obtains a well-defined element $\sigma \in P_n(r)$.

Theorem 2.10 [Fomin 1995]. *Let (Γ, Γ') be a dual graded graph with differential coefficient r. Then for any differential bijection Φ for (Γ, Γ'), the above construction defines a Schensted bijection I_Φ of the form* (8).

We call I_Φ is the *Schensted bijection* induced by the differential bijection Φ.

Example 2.11. The differential bijection Φ of Example 2.9 induces Schensted's row insertion bijection [1961].

Remark 2.12. For the Kac–Moody dual graded graphs $(\Gamma_w(K), \Gamma_s(\Lambda))$ there is a natural choice for the off-diagonal part of the differential bijection. If $v \neq w$ with $v, w \in W$ then Φ_{vw} is essentially obtained from [Humphreys 1990, Lemma 5.11], just as in the proof of Theorem 2.3. For $i \in \mathrm{Des}(v)$, the marked down-then-up path

$$v \xrightarrow{\;m'\;} s_i v \xrightarrow{\;m\;} s_i v s_\alpha = w$$

maps to the marked up-then-down path

$$v \xrightarrow{\;m\;} v s_\alpha \xrightarrow{\;m'\;} s_i v s_\alpha = w.$$

Here m and m' denote edge markings, which in either case are selected from sets of size $m_\Lambda(s_i v, s_i v s_\alpha) = \langle \alpha^\vee, \Lambda \rangle = m_\Lambda(v, v s_\alpha)$ and $n_K(\Lambda_i)$ respectively.

Currently we are not aware of a general rule for Φ_v which exhibits the coefficient of v in $(DU - UD)v$ as $\langle K, \Lambda \rangle$. In Section 3 we shall give a special case where the bijection Φ_v has been constructed explicitly.

2E. *Automorphisms and mixed insertion.* This section is a natural synthesis of the ideas of [Fomin 1994] and [Haiman 1989] which does not seem to have been written down before. We believe this construction is particularly interesting for Kac–Moody dual graded graphs (see also Section 4).

Let (Γ, Γ') be a pair of dual graded graphs with $\Gamma = (V, E, m, h)$ and $\Gamma = (V, E', m', h)$. Say that a permutation $\tau : V \to V$ is an automorphism of (Γ, Γ') if (1) $h \circ \tau = h$, (2) $(x, y) \in E$ if and only if $(\tau(x), \tau(y)) \in E$, and in this case, $m(x, y) = m(\tau(x), \tau(y))$, and (3) $(x, y) \in E'$ if and only if $(\tau(x), \tau(y)) \in E'$, and in this case, $m'(x, y) = m'(\tau(x), \tau(y))$.

Given a differential bijection Φ for (Γ, Γ'), we define its twist Φ^τ by τ as follows. For every $x, y \in V$ there are natural bijections $\tau : \mathcal{UD}_{xy} \to \mathcal{UD}_{\tau(x)\tau(y)}$ given by $(z, m, m') \mapsto (\tau(z), m, m')$ and $\tau : \mathcal{DU}_{xy} \to \mathcal{DU}_{\tau(x)\tau(y)}$ given by $(w, M, M') \mapsto (\tau(w), M, M')$. Let $\tau : S(r) \to S(r)$ be the identity permutation. Then define $\Phi^\tau_{xy} = \tau^{-1} \circ \Phi_{\tau(x)\tau(y)} \circ \tau$. It is easy to verify that Φ^τ is also a differential bijection for (Γ, Γ').

Example 2.13. Let $\Gamma = \Gamma' = \mathbb{Y}$ and $\mathrm{tr} : \mathbb{Y} \to \mathbb{Y}$ the automorphism of (Γ, Γ) that transposes partition diagrams. Let Φ be the differential bijection in Example 2.9. Then $\mathrm{I}_{\Phi^{\mathrm{tr}}}$ is Schensted's column insertion bijection [Schensted 1961].

For the sequel we assume that τ has finite order κ. A Γ-tableau whose edges have an auxiliary marking parameter $p \in S(\kappa) = \{0, 1, \ldots, \kappa - 1\}$ is called a τ-*mixed* Γ-tableau. Let $\mathcal{T}_\tau(\Gamma)$ be the set of τ-mixed Γ-tableaux. Suppose τ' is an automorphism of (Γ, Γ') of order κ'. Let $(\Gamma, \Gamma'; \tau, \tau')$ denote the pair of dual graded graphs given by Γ and Γ' except that Γ-edges (resp. Γ'-edges) are labeled by (m, p) with $p \in S(\kappa)$ (resp. (m', p') with $p' \in S(\kappa')$) and m (resp. m') is a usual

edge label for Γ (resp. Γ'). This multiplies the number of markings for each edge of Γ (resp. Γ') by κ (resp. κ'). The differential coefficient of $(\Gamma, \Gamma'; \tau, \tau')$ is $r\kappa\kappa'$ where r is the differential coefficient of (Γ, Γ').

Let Φ be a differential bijection for (Γ, Γ'). Then there is an obvious differential bijection (also denoted Φ) for $(\Gamma, \Gamma'; \tau, \tau')$, defined by a trivial scaling by κ in Γ and by κ' in Γ'.

We define another bijection

$$P_n(\kappa\kappa'r) \to \bigsqcup_{\substack{x \in V \\ h(x)=n}} \mathcal{T}_\tau(\Gamma, x) \times \mathcal{T}_{\tau'}(\Gamma', x), \tag{10}$$

called (τ, τ')-*mixed insertion* by modifying the process in which we construct the matrix G_{ij} from the colored permutation σ. Instead of using the same differential bijection Φ to compute each G_{ij}, we use twists of Φ by automorphisms that depend on (i, j).

Let $\sigma \in P_n(\kappa\kappa'r)$. Regard σ as a monomial matrix in which each nonzero entry has three labels $(c, p, p') \in S(r) \times S(\kappa) \times S(\kappa')$, where $S(\kappa) = \{0, 1, \ldots, \kappa - 1\}$ and $S(\kappa') = \{0, 1, \ldots, \kappa' - 1\}$. Each horizontal (resp. vertical) edge is marked by a pair (m, p) (resp. (m', p')) where m (resp. m') is the usual marking and $p \in S(\kappa)$ (resp. $p' \in S(\kappa')$). Let $(z, y, x, w) = (G_{i-1,j-1}, G_{i-1,j}, G_{i,j-1}, G_{ij})$ and suppose that

$$z \xrightarrow{(m,p)} y \quad \text{and} \quad z \xrightarrow{(m',p')} x$$

are given, where it is understood that if $z = y$ (resp $z = x$) then (m, p) (resp. (m', p')) need not be specified. Then G_{ij} is determined as before, except that instead of using Φ_{xy} we use the twist

$$\Phi_{xy}^{\tau^k(\tau')^{k'}},$$

where k and k' are as follows. Let (c, p, p') be the nonzero entry of σ in the i-th row, say, σ_{il}. We set $k' = p'$. Separately, let (c, p, p') be the nonzero entry of σ in the j-th column, say, σ_{qj}. We set $k = p$. Note that in the case $q > i$ (resp. $l > j$) the bijection Φ_{xy} is not used in the local rule so the value of k (resp. k') does not affect the algorithm.

In other words, if $\sigma_{ij} = (c, p, p')$ is a nonzero entry, then $(\tau')^{p'}$ acts everywhere to the right in the i-th row and all vertical edges to the right (those of the form $G_{i-1,l} \to G_{il}$ for $l \geq j$) are given the auxiliary marking p', and τ^p acts everywhere below in the j-th column, and all horizontal edges below (those of the form $G_{l,j-1} \to G_{lj}$ for $l \geq i$) are given the auxiliary marking p. The output is the pair $(P, Q) \in \mathcal{T}_\tau(\Gamma, v) \times \mathcal{T}_{\tau'}(\Gamma', v)$ where $v = G_{nn}$ and P and Q are obtained from the south and east edges of G respectively.

Proposition 2.14. (τ, τ')-*mixed insertion gives a well-defined bijection* (10).

Example 2.15. In the context of Example 2.13, (τ, τ')-mixed insertion specializes to the following kinds of insertion algorithms, the first from [Schensted 1961] and the other three from [Haiman 1989].

(1) $(\tau, \tau') = (\mathrm{id}, \mathrm{id})$: Schensted row insertion

(2) $(\tau, \tau') = (\mathrm{id}, \mathrm{tr})$: left-right insertion

(3) $(\tau, \tau') = (\mathrm{tr}, \mathrm{id})$: mixed insertion

(4) $(\tau, \tau') = (\mathrm{tr}, \mathrm{tr})$: doubly mixed insertion

2F. *Restriction to parabolics.* Let $J \subset I$. We say that a weight Λ is supported on J if $\Lambda = \sum_{j \in J} a_j \Lambda_j$. The Kac–Moody dual graded graphs $(\Gamma_s(\Lambda), \Gamma_w(K))$ are compatible with restriction to parabolics. Let $W_J \subset W$ be the parabolic subgroup generated by $\{s_j \mid j \in J\}$ and let W^J be the set of minimal length coset representatives in W/W_J. Note that W^J inherits weak and strong orders from W by restriction.

Proposition 2.16. *Fix $J \subset I$. If Λ is supported on $I \setminus J$ then the restriction of $(\Gamma_s(\Lambda), \Gamma_w(K))$ to W^J is a pair of dual graded graphs with differential coefficient $\langle K, \Lambda \rangle$.*

Proof. Suppose $v \prec w$ is a weak cover. If $w \in W^J$ then $v \in W^J$ since $W^J \subset W$ is a lower order ideal for $\preceq$.

Suppose $w \lessdot v$ and $w \in W^J$ and $v \notin W^J$. Since v has a reduced expression ending in s_j for some $j \in J$ and w is obtained from this reduced expression by omitting a simple generator, we conclude that $v = w s_j$. But Λ is supported on $I \setminus J$, so $\langle \alpha_j^\vee, \Lambda \rangle = 0$.

Combining these two facts we see that the proof of Theorem 2.3 restricts to W^J. □

We shall use the following notation for maximal parabolic subgroups of W. For $i \in I$ we shall write W^i for W^J where $J = I \setminus \{i\}$. We denote by $(\Gamma_s(\Lambda), \Gamma_w(K))^i = (\Gamma_s^i(\Lambda), \Gamma_w^i(K))$, the dual graded graph given by restricting $(\Gamma_s(\Lambda), \Gamma_w(K))$ to W^i.

2G. *The affine case.* If the GCM A is of finite type, then $Z^+ = \{0\}$ and all of the edges of $\Gamma_w(K)$ are labeled 0.

In this section let A be of untwisted affine type. Let $0 \in I$ be the distinguished Kac 0 node and $J = I \setminus \{0\}$. Then W is the affine Weyl group, $W_J = W_{\mathrm{fin}}$ is the finite Weyl group, and we write $W^0 = W^J$. By Proposition 2.16 the restriction of the Kac–Moody dual graded graph to W^0, is a dual graded graph. In this case the weak graph $\Gamma_w(K)$ has an interpretation involving the Schubert calculus of the homology of the affine Grassmannian, and the duality is a combinatorial expression of the pairing between the homology and cohomology of the affine Grassmannian.

For affine algebras, $Z^+ = \mathbb{Z}_{\geq 0} K$ where $K = K_{\text{can}} = \sum_{i \in I} k_i \alpha_i^\vee$ is the canonical central element; the vector $(k_i)_{i \in I}$ is the unique linear dependence of the rows of A given by positive relatively prime integers [Kac 1990]. In this case, since the labels of $\Gamma_w(K)$ are linear in K, without loss of generality we shall only work with K_{can} and define $\Gamma_w := \Gamma_w(K_{\text{can}})$. The edge labels of Γ_w are related to the homology multiplication in affine Grassmannians, as follows.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an untwisted affine algebra. Let $\text{Gr} = \text{Gr}_G$ denote the affine Grassmannian of the simple Lie group G whose Lie algebra $\mathfrak{g}_{\text{fin}}$ is the canonical simple Lie subalgebra of the affine algebra $\mathfrak{g}$. For $w \in W^0$ we let $\xi_w \in H_*(\text{Gr})$ denote the corresponding *homology* Schubert class. Recall the constants $n(w, v)$ from (3).

Proposition 2.17. *Let $\xi_0 = \xi_{s_0}$ be the Schubert class indexed by the unique simple generator $s_0 \notin W_{\text{fin}}$. Then for every $w \in W^0$, we have in $H_*(\text{Gr})$ the identity*

$$\xi_0 \, \xi_w = \sum_v n(w, v) \, \xi_v$$

where $v \in W^0$ runs over the weak covers $w \prec v$ of w.

Proof. We rely on the results of [Lam 2006] which in turn are based on unpublished work of Peterson. Let $S = \text{Sym}((\mathfrak{h}_\mathbb{Z}^*)_{\text{fin}}) = H^{T_{\text{fin}}}(\text{pt})$ denote the symmetric algebra in the weights of the $\mathfrak{g}_{\text{fin}}$ and $\phi_0 : S \to \mathbb{Z}$ denote the evaluation at 0. Let $\mathbb{A}_0$ denote the affine nilCoxeter algebra corresponding to W. As a free $\mathbb{Z}$-module $\mathbb{A}_0$ is spanned by elements $\{A_w \mid w \in W\}$ with multiplication given by

$$A_w \, A_v = \begin{cases} A_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv) \\ 0 & \text{otherwise.} \end{cases}$$

The affine nilHecke algebra $\mathbb{A}$ is the $\mathbb{Z}$-algebra generated by $\mathbb{A}_0$ and S with the additional relation [Lam 2006, Lemma 3.1]

$$A_w \, \lambda = (w \cdot \lambda) A_w + \sum_{w r_\alpha \lessdot w} \langle \lambda, \alpha^\vee \rangle A_{w r_\alpha}, \qquad (11)$$

where α is always taken to be a positive root of W.

Now let

$$\mathbb{B} = \{a \in \mathbb{A}_0 \mid \phi_0(as) = \phi_0(s)a \text{ for any } s \in S\} \subset \mathbb{A}_0$$

denote the affine Fomin–Stanley subalgebra, where $\phi_0 : \mathbb{A} \to \mathbb{A}_0$ is given by $\phi_0(\sum_w a_w A_w) = \sum_w \phi_0(a_w) A_w$. Let $j_0 : H_*(\text{Gr}_G) \to \mathbb{B}$ denote the ring isomorphism [Lam 2006, Theorem 5.5] from the homology of Gr_G to the affine Fomin–Stanley algebra $\mathbb{B}$. We first show that $j_0(\xi_0) = \sum_{i \in I} k_i A_i$, where A_i are the generators of the nilCoxeter algebra and $K_{\text{can}} = \sum_i k_i \alpha_i^\vee$.

By [Lam 2006, Proposition 5.4], the element $j_0(\xi_0)$ is characterized by having unique Grassmannian term A_0, and the property that it lies in $\mathbb{B}$. Since $k_0 = 1$, the unique Grassmannian term property is immediate. Using (11), we calculate that

$$\phi_0(a\alpha_j) = \sum_{i \in I} k_i \langle \alpha_i^\vee, \alpha_j \rangle = \langle K_{\text{can}}, \alpha_j \rangle = 0.$$

Since the α_j span $(\mathfrak{h}_{\mathbb{Z}}^*)_{\text{fin}}$ over $\mathbb{Q}$, we deduce that $\phi_0(as) = \phi_0(s)a$ for any $s \in S$, and thus $j_0(\xi_0) = \sum_{i \in I} k_i A_i$. (This was first pointed out to us by Alex Postnikov.)

By [Lam 2006, Lemma 4.3, Theorem 5.5], we thus have

$$\xi_0 \, \xi_w = j(\xi_0) \cdot \xi_w = \left(\sum_{i \in I} k_i A_i \right) \cdot \xi_w = \sum_{w \prec s_i w} k_i \, \xi_{s_i w}.$$

where we have used the action of $\mathbb{A}_0$ on $H_*(\mathrm{Gr}_G)$ given by

$$A_i \cdot \xi_w = \begin{cases} \xi_{s_i w} & \text{if } s_i w > w, \\ 0 & \text{otherwise;} \end{cases}$$

see [Lam 2006, (3.2)]. Recalling the definition $n(w, s_i w) = k_i$ of the weak graph Γ_w from (3) this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Affine type A and LLMS insertion

For this section let $\mathfrak{g}(A)$ be the affine algebra of type $A_{n-1}^{(1)}$. In this case the combinatorics of the pair of dual graded graphs $(\Gamma_s(\Lambda_i), \Gamma_w)$ was studied extensively in [Lam et al. 2006] and was one of the main motivations of the current work. The affine insertion algorithm of [Lam et al. 2006] (which we shall call LLMS insertion) furnishes an explicit differential bijection for $(\Gamma_s(\Lambda_i), \Gamma_w)$. LLMS insertion involves nontrivial extensions of the notion of tableaux to semistandard weak and strong tableaux, and proves Pieri rules (formulae for certain Schubert structure constants) in the homology $H_*(\mathrm{Gr})$ and cohomology $H^*(\mathrm{Gr})$ of the affine Grassmannian of $\mathrm{SL}(n, \mathbb{C})$ [Lam et al. 2006].

For type $A_{n-1}^{(1)}$ the coefficients of the canonical central element K are all 1. Therefore the weak graph Γ_w has all edge multiplicities equal to 1. Using the rotational symmetry of the Dynkin diagram $A_{n-1}^{(1)}$, we may assume that $\Lambda = \Lambda_0$ and for brevity we write Γ_s for $\Gamma_s(\Lambda_0)$.

Let $I = \{0, 1, \ldots, n-1\}$ and let the Cartan matrix be defined by $a_{i,i+1} = a_{i+1,i} = -1$ for all i, with indices taken modulo n, $a_{ii} = 2$ for all $i \in I$, and $a_{ij} = 0$ otherwise. As in Section 2B the Weyl group is defined by $m_{i,i+1} = 3$ and $m_{ij} = 2$ for $|i - j| \geq 2$.

3A. *Affine permutations.* We use the following explicit realization of the affine symmetric group $W = \widetilde{S}_n$. A bijection $w : \mathbb{Z} \to \mathbb{Z}$ is an affine permutation with period n if $w(i+n) = w(i)$ for each $i \in \mathbb{Z}$ and $\sum_{i=1}^{n} (w(i) - i) = 0$. The set of affine

permutations with period n form a group isomorphic to $\widetilde{S}_n$, with multiplication given by function composition. The reflections t_{ij} in $\widetilde{S}_n$ are indexed by a pair of integers (i, j) satisfying $i < j$ and $i \neq j \mod n$. Suppose $v \lessdot vt_{ij} = w$ is a cover in $\widetilde{S}_n$. Then the edge (v, w) in Γ_s has multiplicity equal to $\#\{k \in \mathbb{Z} \mid v(i) \leq k < v(j) \text{ and } k = 0 \mod n\}$ [Lam et al. 2006].

3B. *Action of $\widetilde{S}_n$ on partitions.*

Given a partition λ, one may associate a bi-infinite binary word $p(\lambda) = p = \cdots p_{-1} p_0 p_1 \cdots$ called its *edge sequence*. The edge sequence $p(\lambda)$ traces the border of the (French) diagram of λ, going from northwest to southeast, such that every letter 0 (resp. 1) represents a south (resp. east) step, and some cell in the i-th diagonal is touched by the steps p_{i-1} and p_i. Here the cell (i, j) lies in row i (where row indices increase from south to north), column j (where column indices increase from west to east), and diagonal $j - i$.

The affine symmetric group $\widetilde{S}_n$ acts on partitions, since elements of $\widetilde{S}_n$ are certain permutations $\mathbb{Z} \to \mathbb{Z}$ and partitions can be identified with their edge sequences, which are certain functions $\mathbb{Z} \to \{0, 1\}$. Then for $i \in \mathbb{Z}/n\mathbb{Z}$, $s_i \lambda$ is obtained by removing from λ every λ-removable cell of residue i, and adding to λ every λ-addable cell of residue i. Here the residue of a box (i, j) is the diagonal index $j - i$ taken modulo n.

3C. *Cores and affine Grassmannian permutations.*

Using the language of cores, we shall describe the combinatorics of the dual graded graph $(\Gamma_s, \Gamma_w)^0$ afforded by Proposition 2.16.

An *n-ribbon* is a skew partition diagram λ/μ (the difference of the diagrams of the partitions λ and μ) consisting of n rookwise connected cells, all with distinct residues. We say that this ribbon is *λ-removable* and *μ-addable*. An *n-core* is a partition that admits no removable n-ribbon. Since the removal of an n-ribbon is the same thing as exchanging bits $p_i = 0$ and $p_{i+n} = 1$ in the edge sequence for some i, it follows that λ is a core if and only if for every i, the sequence $p^{(i)}(\lambda) := \cdots p_{i-2n} p_{i-n} p_i p_{i+n} p_{i+2n} \cdots$ consisting of the subsequence of bits indexed by i mod n, has the form $\cdots 1111100000 \cdots$. We denote the set of n-cores by $\mathscr{C}_n$.

Proposition 3.1 [Lam et al. 2006; Misra and Miwa 1990]. *The map $w \mapsto w \cdot \varnothing$ is a bijection $c : \widetilde{S}_n^0 \to \mathscr{C}_n$. Moreover, for $v, w \in \widetilde{S}_n^0$, we have $v \leq w$ if and only if $c(v) \subseteq c(w)$, and if $v \lessdot w$ then $c(w)/c(v)$ is a disjoint union of translates of some ribbon R, and the number of components of $c(w)/c(v)$ is equal to the multiplicity $m(v, w)$ in Γ_s.*

We say that $\mu \in \mathscr{C}_n$ covers $\lambda \in \mathscr{C}_n$ if $c^{-1}(\mu) \gtrdot c^{-1}(\lambda)$. Thus a standard strong tableau in Γ_s is a sequence $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \mu$ such that λ^i covers λ^{i-1} and one of the components of λ^i/λ^{i-1} has been marked.

It is easy to show that a core cannot have both an addable and a removable cell of the same residue. Thus, in the special case that $v \lessdot s_i v = w$ for some $i \in I$, $c(w)$ is obtained from $c(v)$ by adding all $c(v)$-addable cells of residue i, and the ribbon R of Proposition 3.1 must be a single box. In this case we say that $c(v) \subset c(w)$ is a weak cover.

3D. *LLMS insertion.* In [Lam et al. 2006], for the affine symmetric group $\widetilde{S}_n$, semistandard analogues of weak and strong tableaux were defined (for all of $\widetilde{S}_n$, not just $\widetilde{S}_n^0$), and an RSK correspondence was given between a certain set of bi-words or matrix words, and pairs of tableaux, one semistandard weak and the other semistandard strong. Let us consider the following restriction of this bijection. We first restrict to "standard" tableau pairs, that is, the case in which the tableaux are weak and strong tableaux as defined in Section 2C. Next we take the parabolic restriction from $\widetilde{S}_n$ to $\widetilde{S}_n^0$. Let us denote the restricted bijection by I_{LLMS}.

Let Φ be the differential bijection for $(\Gamma_s, \Gamma_w)^0$ such that $I_\Phi = I_{\mathrm{LLMS}}$. We describe it explicitly.

For $u, v \in \widetilde{S}_n^0$ with $u \neq v$, the off-diagonal part Φ_{uv} of Φ coincides with the natural definition given in Remark 2.12. The diagonal part Φ_v for $v \in \widetilde{S}_n^0$ is specified as follows. Let $\lambda = c(v)$ be the n-core corresponding to v. If $\lambda \subset \mu$ is a weak cover then μ / λ consists of all the λ-addable cells of λ which have a fixed residue. As μ varies over all the weak covers of λ we obtain all the λ-addable cells in this way. Thus there is a natural identification of the set $\mathscr{D}\mathscr{U}_v$ with the set of λ-addable cells. Similarly $\mathscr{U}\mathscr{D}_v$ may be identified with the set of λ-removable cells. This given, we may use the differential bijection denoted Φ_λ in Example 2.9 for Young's lattice. This defines a differential bijection Φ for $(\Gamma_s, \Gamma_w)^0$.

Example 3.2. Figure 1 shows the calculation of I_{LLMS} of the permutation $\sigma = 412635$ (written here in one-line notation; it corresponds to the permutation matrix with ones located positions $(i, \sigma(i))$ for $1 \leq i \leq 6$) for $\widetilde{S}_3$. The symbols $\otimes$ encode σ as described above Equation (9). Each arrow indicates a marked strong cover; the subscript 2 indicates that the marked component is the second from the southeast, and no subscript means the marked component is the southeastmost. Stars in the P tableau indicate the marked components.

Remark 3.3. As n goes to infinity, $(\Gamma_s, \Gamma_w)^0$ converges to the dual graded graph $(\mathbb{Y}, \mathbb{Y})$ of Example 2.2 and LLMS insertion converges to Schensted row insertion [Lam et al. 2006] because the respective differential bijections coincide in the limit.

4. Folding

An automorphism of the GCM $B = (b_{ij} \mid i, j \in J)$ is a permutation π of J such that $b_{\pi(i)\pi(j)} = b_{ij}$ for all $i, j \in J$. The automorphism π is *admissible* if $b_{ij} = 0$ for all i and j in the same π-orbit.
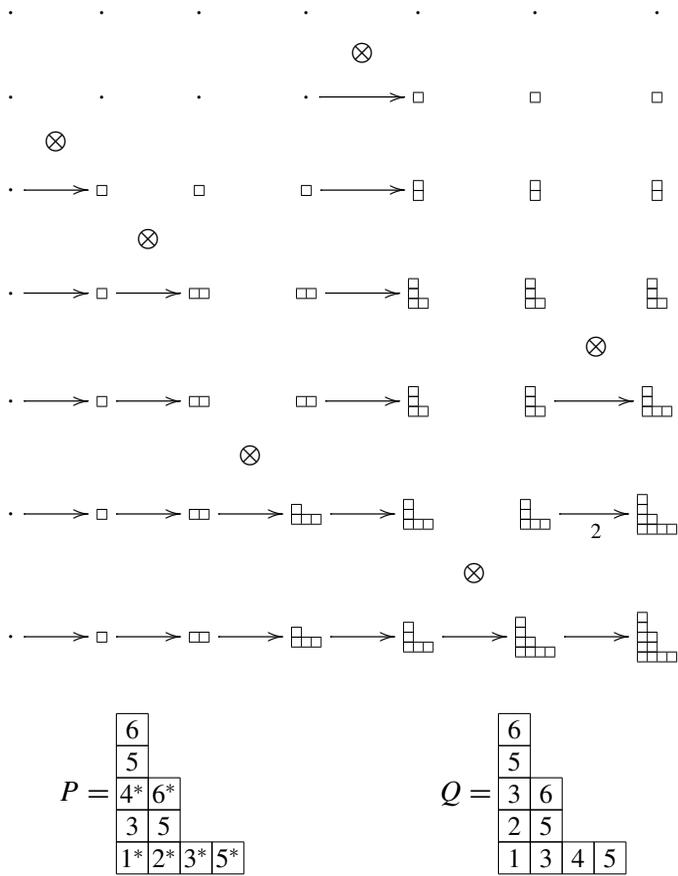
Figure 1. Growth diagram for LLMS insertion of 412635 for $\widetilde{S}_3$.

A GCM $A = (a_{ij} \mid i, j \in I)$ is *symmetric* if it is a symmetric matrix. It is *symmetrizable* if there are positive integers o_i for $i \in I$, such that DA is symmetric, where D is the diagonal matrix with diagonal entries o_i.

Lusztig [1993] showed that every symmetrizable GCM A can be constructed from a symmetric GCM B that is equipped with an admissible automorphism π. We call this construction *folding*.

For A and B related in this manner, we show that the structure of every dual graded graph of the form $(\Gamma_s^A(\Lambda_{i'}), \Gamma_s^A(K))$ for $\mathfrak{g}(A)$, is encoded by some dual graded graph for $\mathfrak{g}(B)$. Thus the combinatorics for $\mathfrak{g}(A)$ is reduced to that of $\mathfrak{g}(B)$. In particular, for any affine algebra $\mathfrak{g}(A)$ there is a simply-laced affine algebra $\mathfrak{g}(B)$ related by folding, so that all affine Schensted bijections can be realized using only the simply-laced affine algebras.

4A. *Folding data.* Let $B = (b_{ij} \mid i, j \in J)$ be a GCM and π an admissible automorphism of B. Let I be a set which indexes the π-orbits of J; we write $O_i \subset J$ for the π-orbit indexed by $i \in I$. Let $o_i = |O_i|$. It is easy to show that the matrix $A = (a_{i'i} \mid i', i \in I)$ defined by

$$a_{i'i} = \frac{o_{i'}}{o_i} \sum_{j \in O_i} b_{j'j} \qquad \text{for any } j' \in O_{i'}, \tag{12}$$

is a well-defined GCM. We say that A is obtained from (B, π) by folding.

Proposition 4.1 [Lusztig 1993, Proposition 14.1.2]. *Given any symmetrizable GCM A, there is a symmetric GCM B with admissible automorphism π, such that A is obtained from (B, π) by folding. In particular, if A is of affine type then B can be taken to be of simply-laced affine type.*

For the Kac–Moody algebras $\mathfrak{g}(A)$ and $\mathfrak{g}(B)$, we denote their weight lattices by P_A and P_B, their coweight lattices by $P_A^\vee$ and $P_B^\vee$ and their coroot lattices by $Q_A^\vee$ and $Q_B^\vee$. For simplicity, we let Λ_i, α_i, $\alpha_i^\vee$ be the fundamental weights, simple roots, and simple coroots for $\mathfrak{g}(A)$ and write ω_i, β_i, $\beta_i^\vee$ for the corresponding data for $\mathfrak{g}(B)$.

Let $P_A' = P_A / (\bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee)^0$ be the weight lattice of A modulo the annihilator of the coroots $\{\alpha_i^\vee\}$. Similarly define P_B'. Note that

$$\alpha_i = \sum_{i' \in I} a_{i'i} \Lambda_{i'} \quad \text{and} \quad \beta_j = \sum_{j' \in J} b_{j'j} \omega_{j'}, \tag{13}$$

where α_i, Λ_i and β_j, ω_j also denote their respective images inside P_A' and P_B'. Set $\kappa = \mathrm{lcm}_{i \in I}(o_i)$, and define $\psi : P_A' \to P_B'$ by

$$\psi(\Lambda_i) = \frac{\kappa}{o_i} \widetilde{\omega}_i, \tag{14}$$

where $\widetilde{\omega}_i = \sum_{j \in O_i} \omega_j$ and $\widetilde{\beta}_i = \sum_{j \in O_i} \beta_j$ for $i \in I$. We have

$$\psi(\alpha_i) = \sum_{i'} a_{i'i} \psi(\Lambda_{i'}) = \sum_{i'} a_{i'i} \frac{\kappa}{o_{i'}} \sum_{j' \in O_{i'}} \omega_{j'}$$

$$= \frac{\kappa}{o_i} \sum_{i'} \sum_{j \in O_i} \sum_{j' \in O_{i'}} b_{j'j} \omega_{j'} = \frac{\kappa}{o_i} \widetilde{\beta}_i, \tag{15}$$

by (13) and (12). Define $\varphi : Q_A^\vee \to Q_B^\vee$ by

$$\varphi(\alpha_i^\vee) = \widetilde{\beta}_i^\vee = \sum_{j \in O_i} \beta_j. \tag{16}$$

4B. *Weyl groups.* For $i \in I$ define

$$f(s_i^A) = \prod_{j \in O_i} s_j^B \in W_B. \tag{17}$$

Since π is admissible the reflections $\{s_j^B \mid j \in O_i\}$ commute with each other so that the product in (17) is independent of the order of its factors.

Steinberg [1968] showed that there is an embedding of Weyl groups $W_A \to W_B$. By [Nanba 2005] it respects the Bruhat order.

Theorem 4.2.

(1) [Steinberg 1968] *There is an injective group homomorphism $f : W_A \to W_B$ defined by* (17), *whose image is the subgroup W_B^{π} of π-fixed elements in W_B, where π acts on W_B by $\pi(s_i) = s_{\pi(i)}$.*

(2) [Nanba 2005, Proposition 3.3 and Theorem 1.2] *$v \le w$ in W_A if and only if $f(v) \le f(w)$ in W_B. Moreover, if $w = s_{i_1} \cdots s_{i_N}$ is a reduced decomposition in W_A then $f(w) = f(s_{i_1})f(s_{i_2}) \cdots f(s_{i_N})$ is a length-additive factorization in W_B.*

Corollary 4.3.

(1) *π acts on $\Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$.*

(2) *Suppose $v \lessdot w = vs_\alpha$ in W_A for $\alpha \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(A))$. Then there is a unique π-orbit $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ such that the reflections $\{s_\gamma \mid \gamma \in O\}$ commute, $f(s_\alpha) = \prod_{\gamma \in O} s_\gamma$, and there is an isomorphism of the boolean lattice of subsets O' of O with the interval $[f(v), f(w)]$ in $(W_B, \le)$ given by $O' \mapsto f(v) \prod_{\gamma \in O'} s_\gamma$. We call O the orbit associated with the cover $v \lessdot w$.*

(3) *Let $w \in W_A$ and $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be such that $f(w)s_\gamma \lessdot f(w)$ in W_B, and let $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be the π-orbit of γ. Then there is a covering relation $v \lessdot w$ of w in W_A of which O is the associated orbit.*

(4) *Let $v \in W_A$, $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be such that $f(v) \lessdot f(v)s_\gamma$, and $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ the π-orbit of γ. Then there is a covering relation $v \lessdot w$ in W_A of which O is the associated cover.*

Proof. For (1), let $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$, with $\gamma = \tilde{u}\beta_j$ for some $\tilde{u} \in W_B$ and $j \in J$. Then $\pi(\gamma) = \pi(\tilde{u})\beta_{\pi(j)} \in \Delta_{\mathrm{re}}(\mathfrak{g}(B))$ and π clearly preserves the set of positive roots, so that π acts on $\Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$.

For (2), there is a unique length-additive factorization $w = u_1 s_i u_2$ in W_A such that $v = u_1 u_2$. We have $\alpha = u\alpha_i$ and $s_\alpha = us_i u^{-1}$ where $u = u_2^{-1}$. Define $O = \{f(u)\beta_j \mid j \in O_i\}$; since $f(u)$ is π-invariant, we have $\pi(f(u)\beta_j) = \pi(f(u))\beta_{\pi(j)} = f(u)\beta_{\pi(j)} \in O(\alpha)$, so that O is a π-orbit. For $j \in O_i$ we have the relation $s_{f(u)\beta_j} = f(u)s_j f(u)^{-1}$, so the reflections $\{s_\gamma \mid \gamma \in O\}$ commute, being conjugate

to commuting reflections $\{s_j \mid j \in O_i\}$. Since f is a homomorphism we have $f(s_\alpha) = f(u)\big(\prod_{j \in O_i} s_j\big) f(u)^{-1} = \prod_{j \in O_i} f(u) s_j f(u)^{-1} = \prod_{\gamma \in O(\alpha)} s_\gamma$.

By Theorem 4.2 $f(w) = f(u_1)\big(\prod_{j \in O_i} s_j\big) f(u_2)$ and $f(v) = f(u_1) f(u_2)$ are length-additive factorizations. It follows that there is an isomorphism of the boolean lattice of subsets $S \subset O_i$ with $[f(v), f(w)]$ where $S \mapsto f(u_1)\big(\prod_{j \in S} s_j\big) f(u_2)$. The desired isomorphism is given by sending $S \rightarrow O'$ where $O' = \{f(u)\beta_j \mid j \in S\}$.

For (3), let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition. Then the image $f(w) = f(s_{i_1}) \cdots f(s_{i_k})$ is length-additive by Theorem 4.2. Therefore the cover $f(w)s_\gamma \lessdot f(w)$ is obtained by removing some unique reflection in $f(s_{i_r})$ for some unique r. Let $u_1 = s_{i_1} \cdots s_{i_{r-1}}$, $i = i_r$, and $u^{-1} = u_2 = s_{i_{r+1}} \cdots s_{i_k}$. Letting $\alpha = u\alpha_i \in \Delta^+_{\mathrm{re}}(\mathfrak{g}(A))$ we find that the π-orbit O of γ is the orbit associated with the cover $v \lessdot w$ where $v = u_1 u_2$.

The proof of (4) is similar. $\qquad\square$

The following result is proved similarly.

Corollary 4.4. *Let $v \lessdot s_i v$ in W_A for some $i \in I$. Then there is a poset isomorphism from the boolean lattice of subsets O' of O_i, to the interval $[f(v), f(s_i v)]$ of $(W_B, \preceq)$ given by $O' \mapsto \big(\prod_{j \in O'} s_j\big) f(v)$.*

4C. *Pairings.* The action of W_B on P_B descends to P'_B, and similarly for W_A.

Theorem 4.5.

(1) *For all $\alpha^\vee \in Q^\vee_A$ and $\lambda \in P'_A$ we have*

$$\langle \varphi(\alpha^\vee), \psi(\lambda) \rangle = \kappa \langle \alpha^\vee, \lambda \rangle. \tag{18}$$

(2) *For all $\Lambda \in P'_A$ and $w \in W_A$ we have*

$$\psi(w\Lambda) = f(w)\psi(\Lambda). \tag{19}$$

(3) *Let $w \in W_A$, $\Lambda \in P'_A$ and $i \in I$. Then*

$$\langle \varphi(\alpha^\vee), f(w)\psi(\Lambda) \rangle = \kappa \langle \alpha^\vee, w\Lambda \rangle \tag{20}$$

and in particular we have

$$\langle \widetilde{\beta}^\vee_i, f(w)\psi(\Lambda) \rangle = \kappa \langle \alpha^\vee_i, w\Lambda \rangle. \tag{21}$$

(4) *The map φ sends $Q^\vee_A \cap Z(\mathfrak{g}(A))$ into $Z(\mathfrak{g}(B))$.*

Proof. By linearity it suffices to check (18) for $\alpha^\vee = \alpha^\vee_i$ and $\lambda = \Lambda_k$ for $i, k \in I$. We have

$$\langle \varphi(\alpha^\vee_i), \psi(\Lambda_k) \rangle = \frac{\kappa}{o_k} \langle \widetilde{\beta}^\vee_i, \widetilde{\omega}_k \rangle = \frac{\kappa}{o_k} o_i \delta_{ik} = \kappa \delta_{ik} = \kappa \langle \alpha^\vee_i, \Lambda_k \rangle.$$

This implies (18).

It suffices to prove (19) for $w = s_{i'}^A$ and $\Lambda = \Lambda_i$. For $i = i'$ we have

$$\psi(s_i \Lambda_i) = \psi(\Lambda_i - \alpha_i) = \frac{\kappa}{o_i}(\widetilde{\omega}_i - \widetilde{\beta}_i)$$

by (14) and (15). Since π is admissible, $\left(\prod_{j \in O_i} s_j^B\right)\widetilde{\omega}_i = \widetilde{\omega}_i - \widetilde{\beta}_i$, giving the required result. The case that $i \neq i'$ is even easier.

Equation (20) follows immediately from (19) and (18) with $\lambda = w\Lambda$. Equation (21) is implied by (20) with $\alpha = \alpha_i$ and (16).

Let $K \in Q_A^\vee$. For any $i \in I$ and $j \in O_i$, by (18), (15), and the π-invariance of $\varphi(K)$, we have

$$\kappa\langle\varphi(K), \beta_j\rangle = \frac{\kappa}{o_i}\langle\varphi(K), \widetilde{\beta}_i\rangle = \kappa\langle K, \alpha_i\rangle. \tag{22}$$

It follows that φ sends $Z(\mathfrak{g}(A))$ into $Z(\mathfrak{g}(B))$. $\qquad\square$

Remark 4.6. Let A be a GCM of affine type, obtained as in [Lusztig 1993] by folding (B, π). Then the canonical central elements and null roots are related by $\varphi(K^A) = K^B$ and $\psi(\delta^A) = r^\vee \delta^B$ where $r^\vee$ is the "twist" of the dual affine root system $X_N^{(r^\vee)}$ to that of A in the nomenclature of [Kac 1990].

4D. *Folding and insertion.* Let $A = (a_{ij} \mid i, j \in I)$ be a GCM with associated Kac–Moody algebra $\mathfrak{g}(A)$, $i' \in I$, and $K \in Z^+(\mathfrak{g}(A))$. Suppose A is obtained by folding the GCM $B = (b_{ij} \mid i, j \in J)$ with admissible automorphism π. Choose $j' \in O_{i'}$. We shall construct the dual graded graph $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$ from the dual graded graph $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K)))$. The construction only requires the subset $W_B^\pi \subset W_B$ of π-invariant vertices, and the edges incident to them, grouped according to their π-orbits.

Remark 4.7. The choice of $j' \in O_{i'}$ is immaterial; if one chooses another element of $O_{i'}$ then the resulting type B structures are transported to each other by a power of the automorphism π.

Proposition 4.8. *Let $v \lessdot w$ in W_A with $\alpha \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(A))$ such that $w = vs_\alpha$ and let $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be the associated orbit of the cover $v \lessdot w$, defined in Corollary 4.3. Then the following sets have the same cardinality.*

(1) *Marked edges $v \xrightarrow{m} w$ in $\Gamma_s^A(\Lambda_{i'})$.*

(2) *The disjoint union over $\gamma \in O$ of the sets $\{f(w)s_\gamma \xrightarrow{M} f(w)\}$ of marked edges going into $f(w)$ in the interval $[f(v), f(w)]$ in $\Gamma_s^B(\omega_{j'})$.*

(3) *The disjoint union over $\gamma \in O$ of the sets $\{f(v) \xrightarrow{M} f(v)s_\gamma\}$ of marked edges coming out of $f(v)$ in the interval $[f(v), f(w)]$ in $\Gamma_s^B(\omega_{j'})$.*

Proof. By Corollary 4.3 the last two sets are in bijection. Let $u \in W_A$ and $i \in I$ be such that $\alpha = u\alpha_i$. Then $O = \{f(u)\beta_j \mid j \in O_i\}$. Using Theorem 4.5 and the π-invariance of $\sum_{\gamma \in O} \gamma^\vee$, we have

$$\sum_{\gamma \in O} m^B_{\omega_{j'}}(f(w)s_\gamma, f(w)) = \sum_{\gamma \in O} \langle \gamma^\vee, \omega_{j'} \rangle = \frac{1}{o_{i'}} \sum_{\gamma \in O} \langle \gamma^\vee, \widetilde{\omega}_{i'} \rangle \qquad (23)$$

$$= \frac{1}{\kappa} \sum_{\gamma \in O} \langle \gamma^\vee, \psi(\Lambda_{i'}) \rangle = \frac{1}{\kappa} \sum_{j \in O_i} \langle f(u)\beta_j^\vee, \psi(\Lambda_{i'}) \rangle$$

$$= \frac{1}{\kappa} \langle \widetilde{\beta}_i^\vee, f(u)^{-1}\psi(\Lambda_{i'}) \rangle$$

$$= \langle \alpha_i^\vee, u^{-1}\Lambda_{i'} \rangle = \langle \alpha^\vee, \Lambda_{i'} \rangle = m^A_{\Lambda_{i'}}(ws_\alpha, w). \qquad \square$$

By definition the graded graph $\Gamma^B_s(\omega_{j'})^\pi$ has vertex set $f(W_A) = W_B^\pi \subset W_B$, grading function $h(f(v)) = \ell(v)$ for $v \in W_A$, and for every cover $v \lessdot w$ in W_A, an edge from $f(v)$ to $f(w)$ whose multiplicity is the common number in Proposition 4.8. It is completely specified by the π-invariant elements of $\Gamma^B_s(\omega_{j'})$ and their incident edges.

Corollary 4.9. *The graded graphs $\Gamma^A_s(\Lambda_{i'})$ and $\Gamma^B_s(\omega_{j'})^\pi$ are isomorphic.*

Proof. By Theorem 4.2 the map $f : W_A \to W_B^\pi$ is a grade- and edge-preserving bijection. The edge multiplicities agree by Proposition 4.8. $\square$

We call $\Gamma^B_s(\omega_{j'})^\tau$ the *folded strong graph* and its tableaux *folded strong tableaux*.

Proposition 4.10. *Let $v \in W_A$ and $i \in I \setminus \mathrm{Des}(v)$ so that $v \prec w = s_i v$. Fix any $j \in O_i$. Then the following sets have the same cardinality.*

(1) *The marked edges $v \xrightarrow{m'} w$ in $\Gamma^A_w(K)$.*

(2) *The marked edges $f(v) \xrightarrow{M'} s_j f(v)$ in $\Gamma^B_w(\varphi(K))$.*

(3) *The marked edges $s_j f(w) \xrightarrow{M'} f(w)$ in $\Gamma^B_w(\varphi(K))$.*

Proof. By a proof similar to the one for (22) and recalling the definition (3) we have

$$n^A_K(v, s_i v) = \langle K, \Lambda_i \rangle = \langle \varphi(K), \omega_j \rangle = n^B_{\varphi(K)}(f(v), s_j f(v)). \qquad (24)$$

This proves the proposition. $\square$

By definition the graded graph $\Gamma^B_w(\varphi(K))^\pi$ has vertex set $f(W_A) = W_B^\pi$, grading function $h(f(w)) = \ell(w)$, and for each $v \in W_A$ and $i \in I \setminus \mathrm{Des}(v)$, an edge from $f(v)$ to $f(s_i v)$ whose multiplicity is the common multiplicity in Proposition 4.10.

Corollary 4.11. *The graded graphs $\Gamma^A_w(K)$ and $\Gamma^B_w(\varphi(K))^\pi$ are isomorphic.*

We call $\Gamma_w^B(\varphi(K))^\pi$ the *folded weak graph*. Its tableaux are called *folded weak tableaux*.

Corollary 4.12. $(\Gamma_s^B(\omega_{j'})^\pi, \Gamma_w^B(\varphi(K))^\pi)$ *and* $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$ *are isomorphic dual graded graphs.*

Proof. This follows immediately from Corollaries 4.9 and 4.11. □

Using a differential bijection Φ^B for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K)))$, we construct a differential bijection for the folded dual graded graph $(\Gamma_s^B(\omega_{j'})^\pi, \Gamma_w^B(\varphi(K))^\pi)$, which, by the identifications given in Propositions 4.8 and 4.10, yields a differential bijection Φ^A for $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.

By Remark 2.12 the off-diagonal part Φ_{vw}^A for $v \neq w$ in W_A, has already been specified.

For the diagonal terms, since $\mathrm{Des}(f(v)) = \bigsqcup_{i \in \mathrm{Des}(v)} O_i$, by (23) in the special case of a cover of the form $s_i v = v s_\alpha \gtrdot v$ with $\alpha = v^{-1} \alpha_i \in \Delta_{\mathrm{re}}^+$ and (24) we have

$$\sum_{i \in I \setminus \mathrm{Des}(v)} m_{\Lambda_{i'}}^A(v, s_i v)\, n_K^A(v, s_i v)$$
$$= \sum_{j \in J \setminus \mathrm{Des}(f(v))} m_{\omega_{j'}}^B(f(v), s_j f(v))\, n_{\varphi(K)}^B(f(v), s_j f(v)). \quad (25)$$

For $i \in \mathrm{Des}(v)$ and $j \in O_i$, we have $v s_\alpha = s_i v \prec v$ where $\alpha = v^{-1} \alpha_i \in -\Delta_{\mathrm{re}}^+$, and an analogous computation yields

$$\sum_{i \in \mathrm{Des}(v)} m_{\Lambda_{i'}}^A(s_i v, v) n_K^A(s_i v, v)$$
$$= \sum_{j \in \mathrm{Des}(f(v))} m_{\omega_{j'}}^B(s_j f(v), f(v))\, n_{\varphi(K)}^B(s_j f(v), f(v)). \quad (26)$$

Using the bijections of Propositions 4.8 and 4.10, we obtain bijections $\mathcal{DU}_v^A \to \mathcal{DU}_{f(v)}^B$ and $\mathcal{UD}_v^A \to \mathcal{UD}_{f(v)}^B$. Under these identifications we obtain a differential bijection Φ^A for $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.

Proposition 4.13. *Let the GCM $A = (a_{ij} \mid i, j \in I)$ be obtained by folding from the GCM $B = (b_{ij} \mid i, j \in J)$ with admissible automorphism π. Then for any $i' \in I$ and $j' \in O_{i'}$, a differential bijection Φ^B for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K))$ restricts to a differential bijection Φ^A for the pair of dual graded graphs $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.*

We shall give an extensive example in Section 5.

Remark 4.14. It is possible to axiomatize conditions for an arbitrary pair of dual graded graphs (Γ, Γ') and an automorphism π of (Γ, Γ') to give rise to a folded insertion in the manner we have described for Kac–Moody graded graphs. The key properties needed are abstract graph-theoretic formulations of Theorem 4.2 and Corollary 4.3. Since we have no interesting examples that do not come from Kac–Moody dual graded graphs, we will not make this precise.

5. Affine type C combinatorics

In this section we consider folded insertion for the affine root system $C_n^{(1)}$. Folding works for the entire Weyl group. However here we shall restrict our discussion to the explicit description of folded insertion for the maximal parabolic quotient of the affine Weyl group given by the dual graded graphs $(\Gamma_s(\Lambda_{i'}), \Gamma_w(K))^{i'}$ using $2n$-cores, where $i' \in I$ is any Dynkin node. In the limit $n \to \infty$ one obtains a new Schensted bijection for every integer i, which for $i = 0$ coincides with "standard" Sagan–Worley insertion into shifted tableaux.

Let $I = \{0, 1, \dots, n\}$ be the Dynkin node set and (a_{ij}) the GCM, with $a_{ii} = 2$ for $i \in I$, $a_{i,i+1} = a_{i+1,i} = -1$ for $1 \le i \le n-2$, $a_{01} = a_{n,n-1} = -1$, $a_{10} = a_{n-1,n} = -2$, and other entries zero. Using the recipe in Section 2B, the Weyl group W_n has generators s_i for $i \in I$ satisfying $s_i^2 = 1$ for $i \in I$ and $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $i \ne j$, where $m_{01} = m_{n-1,n} = 4$, $m_{i,i+1} = 3$ for $1 \le i \le n - 2$, and $m_{ij} = 0$ for $|i - j| \ge 2$.

5A. *Folding for $C_n^{(1)}$*. Let $A = C_n^{(1)}$ and $B = A_{2n-1}^{(1)}$ denote the two GCMs. We use the notation of Section 4.

Let π be the admissible automorphism of B given by $j \mapsto 2n - j$ where indices are taken modulo $2n$. We index the π-orbits by $O_0 = \{0\}$, $O_n = \{n\}$, and $O_i = \{i, 2n - i\}$ for $i \in I \setminus \{0, n\}$. It is easy to check that A is obtained from (B, π) by folding.

Let K be the canonical central element for $C_n^{(1)}$. Let $i' \in I$ and $j' \in O_{i'}$. We define folded insertion for the dual graded graph $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))^{i'}$, realized by LLMS insertion for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K)))$. We call this induced folded insertion the "LLMS insertion for $W_n^{i'}$" (even though it also depends on j').

5B. *$2n$-cores*. As before, fix $i' \in I$ and $j' \in O_{i'}$. The elements of the parabolic quotient $W_n^{i'}$ may be realized by $2n$-cores as follows.

By Proposition 3.1 there is a bijection $c : \widetilde{S}_{2n}^0 \to \mathscr{C}_{2n}$. Using a rotational automorphism of the Dynkin diagram of type $A_{2n-1}^{(1)}$, for any $k \in J$ one may define the k-action of $\widetilde{S}_{2n}$ on $\mathscr{C}_{2n}$, denoted $w \cdot_k \lambda$, which is the same as before except that the diagonal of the cell (i, j) is $j - i + k$. Since the stabilizer of $\varnothing$ under the k-action of $\widetilde{S}_{2n}$ on $\mathscr{C}_{2n}$ is $(\widetilde{S}_{2n})_{J \setminus \{k\}}$, there is a bijection $c_k : \widetilde{S}_{2n}^k \to \widetilde{S}_{2n}/(\widetilde{S}_{2n})_{J \setminus \{k\}} \to \mathscr{C}_{2n}$ defined by $c_k(w) = w \cdot_k \varnothing$.

Define the map $sc_{i'} : W_n \to \mathscr{C}_{2n}$ by $w \mapsto f(w) \cdot_{j'} \varnothing$, where $f : W_n \to \widetilde{S}_{2n}$ is the Weyl group homomorphism of Section 4. Note that $f(W_n^{i'}) \subset \widetilde{S}_{2n}^{J \setminus O_{i'}}$.

Denote by $\mathscr{C}_{2n}^{j'}$ the image of $sc_{i'}$.

The following result is the $C_n^{(1)}$-analogue of (part of) Proposition 3.1.

Proposition 5.1. *The map $sc_{i'}$ restricts to a bijection $W_n^{i'} \to \mathscr{C}_{2n}^{j'}$. For $v, w \in W_n^{i'}$ we have $v \le w$ if and only if $sc_{i'}(v) \subset sc_{i'}(w)$.*

Proof. The stabilizer of $W = W_n$ acting on $\varnothing$ is equal to $W_{I-\{i'\}}$, so the first statement is immediate. Let $v, w \in W_n^{i'}$. The following are equivalent: (1) $v \leq w$; (2) $f(v) \leq f(w)$; (3) $f(v)(\widetilde{S}_{2n})_{J\backslash\{j\}} \leq f(w)(\widetilde{S}_{2n})_{J\backslash\{j\}}$ for all $j \in O_{i'}$; (4) $f(v)(\widetilde{S}_{2n})_{J\backslash\{j'\}} \leq f(w)(\widetilde{S}_{2n})_{J\backslash\{j'\}}$; (5) $sc_{i'}(v) \subset sc_{i'}(w)$. (1) and (2) are equivalent by Theorem 4.2. (2) and (3) are equivalent by Proposition 5.2 below applied to the data $f(v)$, $f(w)$, and $J \setminus O_{i'}$ in $\widetilde{S}_{2n}$. Since $f(v)$ and $f(w)$ are π-invariant, (3) and (4) are equivalent, because $j' \in O_{i'}$ and the condition for j is invariant as j runs over a π-orbit. (4) and (5) are equivalent by Proposition 3.1. $\square$

For a Coxeter group W and a parabolic subgroup W_J, the strong (Bruhat) order denoted $\leq$ on the quotient W/W_J is the partial order naturally induced from the strong order on W^J. The following result is due to [Deodhar 1977].

Proposition 5.2. *Let W be a Coxeter group with simple generators indexed by P and let $Q \subset P$. Suppose $x, y \in W^Q$. Then $x \leq y$ if and only if $xW_{Q'} \leq yW_{Q'}$ for every maximal parabolic subgroup $W_{Q'} \supset W_Q$.*

We describe $\mathscr{C}_{2n}^0$ explicitly in Section 5D, together with an explicit description of LLMS insertion for W_n^0. It would be interesting to obtain an explicit description of $\mathscr{C}_{2n}^{j'}$ for arbitrary j'. The explicit description of the Chevalley coefficients in a manner similar to Proposition 3.1, and of LLMS insertion for $W_n^{i'}$ appears to be rather subtle.

5C. *Large rank limit of folded LLMS insertion.* We now consider the limit of LLMS insertion for $W_n^{i'}$ as n goes to ∞, in such a way that the nodes near 0 in $A_{2n-1}^{(1)}$ are stable; for this purpose we label these nodes $\ldots, -2, -1, 0, 1, 2, \ldots$.

Let $A_{\pm\infty}$ be the Kac–Moody algebra[1] whose Dynkin diagram has vertex set $J_\infty = \mathbb{Z}$, with Cartan matrix (b_{ij}) such that $b_{ii} = 2$ and $b_{i,i+1} = b_{i+1,i} = -1$, and $b_{ij} = 0$ otherwise. Let $S_{\pm\infty}$ be its Weyl group: it has generators s_j for $j \in J_\infty$, with relations $s_j^2 = 1$, $(s_j s_{j+1})^3 = 1$, and $(s_i s_j)^2 = 1$ for $|i - j| \geq 2$. Then $S_{\pm\infty}$ acts on partitions: $s_j \cdot \lambda$ is obtained from λ by adding the unique λ-addable cell in diagonal j if it exists, and removing the unique λ-removable cell in diagonal j if it exists (remembering the shift in diagonal index by j'). Then $S_{\pm\infty}\varnothing = \mathbb{Y}$ is the set of all partitions and there is a bijection $c_{j'} : S_{\pm\infty}^{j'} \cong \mathbb{Y}$.

Let C_∞ be the Kac–Moody algebra with Dynkin node set $I_\infty = \mathbb{Z}_{\geq 0}$ and Cartan matrix a_{ij} with $a_{ii} = 2$ for $i \in I_\infty$, $a_{i,i+1} = a_{i+1,i} = -1$ for $i \in I_\infty \setminus \{0\}$, $a_{01} = -1$ and $a_{10} = -2$. Then its Weyl group W_∞ has generators s_i for $i \in I_\infty$ with relations $s_i^2 = 1$, $(s_0 s_1)^4 = 1$, $(s_i s_{i+1})^3 = 1$ for $i \in I_\infty \setminus \{0\}$, and $(s_i s_j)^2 = 1$ for $|i - j| \geq 2$. As before, there is an injective homomorphism $f : W_\infty \to S_{\pm\infty}$ given by $f(s_0) = s_0$ and $f(s_i) = s_i s_{-i}$ for $i > 0$.

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[1]We allow infinite Dynkin diagrams in a formal manner.

Then W_∞ acts on partitions via f. Define $\mathbb{Y}^{i'} = W_\infty \cdot \varnothing$. The limit of Proposition 5.1 gives a bijection $sc_{i'} : W_\infty^{i'} \cong \mathbb{Y}^{i'}$. The strong and weak orders on $\mathscr{C}_{2n} \cong \widetilde{S}_{2n}^{j'}$ both converge to Young's lattice $\mathbb{Y}$. The weak order on $\mathscr{C}_{2n}^{j'} \cong W_n^{i'}$ converges to the weak order on $\mathbb{Y}^{i'} \cong W_\infty^{i'}$, in which a cover $\lambda \subset s_i \lambda$ adds cells in diagonals i and $-i$ if $i > 0$ or just the cell in diagonal 0 if $i = 0$.

Proposition 5.3. *Suppose $\lambda \subset \mu$ is a strong cover in $\mathbb{Y}^{i'} \cong W_\infty^{i'}$ with $sc_{i'}^{-1}(\lambda) = w \lessdot ws_\alpha = sc_{i'}^{-1}(\mu)$. If s_α is conjugate to s_0 then the Chevalley coefficient $\langle \alpha^\vee, \Lambda_{i'} \rangle$ is equal to 1 and μ/λ has a single connected component which is necessarily a ribbon. Otherwise suppose $f(s_\alpha) = s_\beta s_{\beta'}$. Then the Chevalley coefficient $\langle \alpha^\vee, \Lambda_{i'} \rangle$ is equal 1 or 2 depending on whether one or both of $sc(ws_\beta)$ and $sc(ws_{\beta'})$ strictly contain λ. Furthermore, each strict containment has a single connected component which is a ribbon.*

Proof. That the skew partitions in question contain a single connected component which equals a ribbon follows from the fact that they are obtained by the action of a reflection on a partition, which always changes the shape by a ribbon. This follows from the edge sequence discussion in Section 3.

Suppose s_α is conjugate to s_0 and $f(s_\alpha) = s_\beta$. Every strong cover in $\mathbb{Y}$ has Chevalley coefficient equal to 1 or 0 (since a single box is added). Now write $f(w) = xy$ so that $f(ws_\alpha) = xs_0 y$ in $S_{\pm\infty}$. The Chevalley coefficient $\langle \alpha^\vee, \Lambda_{i'} \rangle$ is equal to the Chevalley coefficient $\langle \beta^\vee, \omega_{j'} \rangle$ which is equal to the Chevalley coefficient of the cover $y \lessdot s_0 y$ in $S_{\pm\infty}$ (with respect to $\omega_{j'}$). Since $\mu \neq \lambda$, we must have $y \cdot \varnothing \subsetneq (s_0 y) \cdot \varnothing$. Thus the required Chevalley coefficient must be nonzero, and hence equal to 1. We have used the calculation $\langle y^{-1} \beta_0^\vee, \omega_{j'} \rangle = \langle \beta_0^\vee, y\omega_{j'} \rangle = \langle \beta_0^\vee, y'\omega_{j'} \rangle$, where $y' = c_{j'}^{-1}(y \cdot \varnothing)$ is the parabolic component" of y.

The proof for $f(s_\alpha) = s_\beta s_{\beta'}$ follows in a similar manner. The only delicate issue is to show that if $\langle \beta^\vee, \omega_{j'} \rangle = 1$ then $\lambda \subset sc(ws_\beta)$ is a strict inclusion. But $\langle \beta^\vee, \omega_{j'} \rangle = 1$ implies that $(s_\beta \cdot \varnothing) \neq \varnothing$ so that $(ws_\beta \cdot \varnothing) \neq (w \cdot \varnothing)$. $\qquad \square$

Figure 3 shows the case of a domino appearing in the strong tableau P, corresponding to the strong cover $s_2 s_0 s_1 \lessdot s_2 s_1 s_0 s_1$ in W_∞^1.

In the case the Chevalley coefficient described in Proposition 5.3 is equal to 2, the difference μ/λ is a union of two ribbons, since s_β and $s_{\beta'}$ commute. We have not shown that these ribbons do not touch, so the difference μ/λ can potentially be written as the union of the two ribbons in two ways, corresponding to the left action of $s_{f(w)\beta} s_{f(w)\beta'}$ and $s_{f(w)\beta'} s_{f(w)\beta}$. To obtain a strong tableau in $\mathbb{Y}^{i'}$, we must mark a ribbon for each strong cover which consists of two ribbons.

For the differential bijection, we note that for $\lambda \in \mathbb{Y}^{i'}$, once again, $\mathscr{U}\mathscr{D}_\lambda$ is in natural bijection with the set of λ-addable corners and $\mathscr{D}\mathscr{U}_\lambda$ is in natural bijection with the set of λ-removable corners; in this context the corners are grouped by diagonals of the form $\pm i$ for various i. Using the differential bijection in Example 2.9,
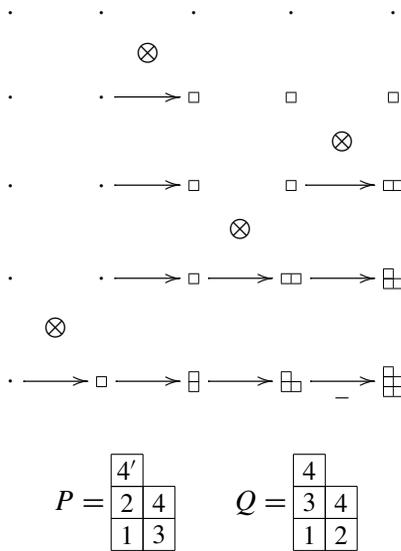
Figure 2. Folded insertion of $\sigma = 2431$ for $i' = j' = 1$.

we obtain a folded insertion for the limit $\mathbb{Y}^{i'}$ of $\mathscr{C}_{2n}^{j'}$. It defines a bijection from permutations $P_n(1)$ to pairs (P, Q) where P and Q are n-step strong and weak tableaux with respect to $\mathbb{Y}^{i'}$.

For example, we let $i' = j' = 1$ and compute the folded insertion of $\sigma = 2431$; see the graph G in Figure 2. For the meaning of $\otimes$ see (9) and Example 3.2. The arrows represent strong covers, and an arrow is labeled with $-$ if the strong cover adds two nonadjacent cells and the marked cell is in a more negative diagonal. The unique arrow labeled with $-$, corresponds to the entry $4'$ in P. Strictly speaking we should mark one ribbon for each number used, but when there is no choice we have omitted the marking.

Again, with $i' = j' = 1$ we compute the folded insertion of 4213; see the graph G in Figure 3. Note that there is a unique strong cover that is not a weak cover, corresponding to the domino in P containing 4s.

5D. Sagan–Worley insertion. We now consider the important case that $i' = 0$. We must have $j' = 0$. The following result is straightforward.

Lemma 5.4. *The set $\mathscr{C}_{2n}^0$ is the subset of $\mathscr{C}_{2n}$ of elements fixed by the transpose* tr.

Using the fact that $f(W_n^0) \subset \widetilde{S}_{2n}^0$, the next result is an easy consequence of Propositions 4.13 and 3.1. A similar statement holds for $i' = n$.

Proposition 5.5. *Suppose $\lambda \subset \mu$ is a strong cover in $\mathscr{C}_{2n}^0 \cong W_n^0$ with $sc_0^{-1}(\mu) = sc_0^{-1}(\lambda)s_\alpha$. Then the Chevalley coefficient $\langle \alpha^\vee, \Lambda_0 \rangle$ is equal to the total number of*
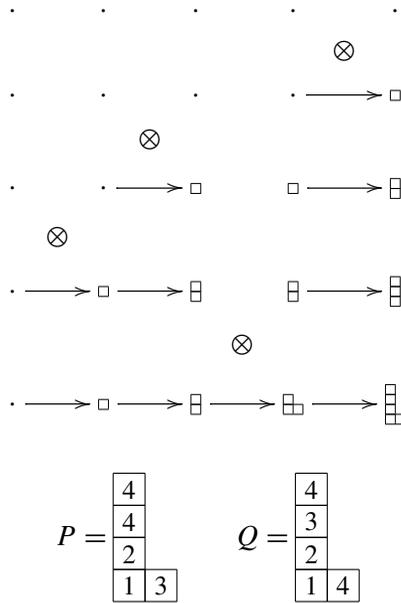
Figure 3. Folded insertion of $\sigma = 4213$ for $i' = j' = 1$.

components of ν/λ for a strong cover $\lambda \subset \nu$ in $\mathscr{C}_{2n} \cong \widetilde{S}_{2n}^0$ satisfying $\lambda \subset \nu \subset \mu$. In particular, if $\lambda \subset \mu$ is also a left weak cover in $\mathscr{C}_{2n}^0 \cong W_n^0$ with $\mu = s_i \lambda$ for $i \in I$, then $\langle \alpha^\vee, \Lambda_0 \rangle$ is equal to the number of λ-addable corner cells of residue i or $-i$ modulo $2n$.

Thus for $i' = 0$, a *folded strong tableau* of shape $\lambda \in \mathscr{C}_{2n}^0$ is a sequence of strong covers in $\mathscr{C}_{2n}^0 \cong W_n^0$ from $\varnothing$ to λ, such that every cover has a marked component. A *folded weak tableau* of shape $\lambda \in \mathscr{C}_{2n}^0$ is a sequence of weak covers in $\mathscr{C}_{2n}^0 \cong W_n^0$ going from $\varnothing$ to λ; no marking is necessary. With these explicit descriptions we have the following:

Corollary 5.6. *LLMS insertion induces a bijection from the set of permutations $P_n(1)$ to pairs (P, Q) of tableaux of the same shape $\lambda \in \mathscr{C}_{2n}^0$, where $\ell(sc_0^{-1}(\lambda)) = n$, P is a folded strong tableau and Q a folded weak tableau.*

Again we now consider the $n \to \infty$ limit. In this case the limit of $\mathscr{C}_{2n}^0$ is the set $\mathbb{Y}^0$ of partitions fixed under the transpose. The strong and weak orders both converge to the same order on $\mathbb{Y}^0$. Since added cells are in transpose-symmetric positions, when marking a strong cover of the form $\lambda \lessdot s_i \lambda$, one must mark either the added cell in diagonal i or $-i$ if $i > 0$.

Folded weak tableaux Q are in obvious bijection with standard shifted tableaux Q^*, given by taking only the part on one side of the diagonal. A similarly obvious

bijection exists from folded strong tableaux to standard shifted tableaux in which off-diagonal entries may or may not have a mark; we choose the bijection so that a mark in a shifted tableau P^* indicates that the corresponding cell with negative diagonal index is marked in the folded strong tableau P.

Thus we have the correct kinds of tableaux to compare with Sagan–Worley insertion.

Example 5.7. Let $n = 7$ and $\sigma = 2673541$. See Example 3.2 for the way to interpret σ and the symbols $\otimes$. We draw the graph G_{ij} for the folded insertion of σ. We draw arrows to represent strong covers and place an asterisk on an edge if the marked cell is on a negative diagonal. P and Q are the folded strong and weak tableaux respectively. P^* and Q^* are the shifted tableaux corresponding to P and Q.

By reformulating Sagan–Worley insertion using Fomin's setup, we obtain the following theorem. Note the exchange of P^* and Q^*.

Theorem 5.8. *Let σ map to (P, Q) under folded insertion. Then the pair (Q^*, P^*) of shifted tableaux, is the image of σ^{-1} under Sagan–Worley insertion.*

Remark 5.9. In [1989, Proposition 6.2], Haiman relates Sagan–Worley shifted insertion with left-right insertion. It is natural to ask whether one can connect Sections 2E and 4D in a similar manner. Unfortunately, a straightforward generalization of Haiman's result does not appear to be possible. For example when π has order 2, one would need to relate the left-right (or mixed) insertion of a colored permutation on $2r$ letters with folded insertion of a permutation on r letters. Length considerations show that this can be done only if each orbit of π on J has order 2, which nearly never happens in our setup.

6. Distributive parabolic quotients

6A. *Proctor's classification.* Let W be a finite irreducible Weyl group with simple generators $\{s_i \mid i \in I\}$ and set of reflections T. Recall the notations W_J and W^J from before Proposition 2.16. We have

$$W^J = \{w \in W \mid w < ws_i \text{ for any } i \in J\}.$$

Proctor [1984] classified the cases when W^J is a distributive lattice under the weak order. In all such cases, Stembridge [1996] showed that the weak and strong orders agree on W^J and that W_J is a maximal parabolic subgroup of W, that is, $J = I \setminus \{i\}$ for some $i \in I$. We call such $W^J := W^i$ *distributive parabolic quotients.*

Theorem 6.1 [Proctor 1984]. *The distributive parabolic quotients are:*

(1) $W \simeq A_n$; $J = I \setminus \{i\}$ *for any $i \in I$.*
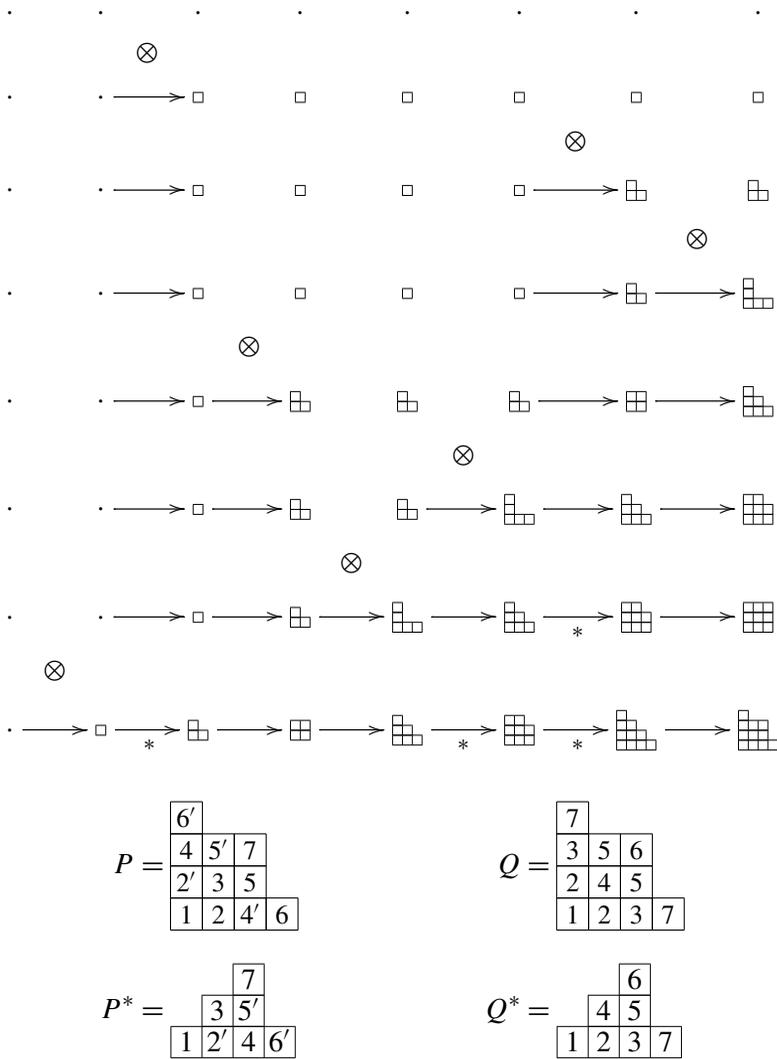
Figure 4. Growth diagram illustrating Theorem 5.8.

(2) $W \simeq B_n$; $W_J \simeq B_{n-1}$ or $W_J \simeq A_{n-1}$.

(3) $W \simeq D_n$; $W_J \simeq D_{n-1}$ or $W_J \simeq A_{n-1}$.

(4) $W \simeq G_2$; $J = I \setminus \{i\}$ for any $i \in I$.

In [Stembridge 1996], it is shown that these cases are also exactly the parabolic quotients W^J of Weyl groups such that every element $w \in W^J$ is fully commutative, that is, every two reduced decompositions of w can be obtained from each other using just the relations of the form $s_i s_j = s_j s_i$ for $i, j \in I$.

6B. *Distributive labeled posets.* We need a slightly more precise form of the results of Proctor and Stembridge. If Q is a finite poset we let $J(Q)$ denote the poset of (lower) order ideals of Q. The poset $J(Q)$ is a distributive lattice and the fundamental theorem of finite distributive posets [Stanley 1999] says that the correspondence $Q \mapsto J(Q)$ is a bijection between finite posets and finite distributive lattices. Suppose P is a finite poset and $\omega : \{x \lessdot y\} \to A$ is a labeling of the edges of the Hasse diagram of P with elements of some set A. We call (P, ω) an edge-labeled poset. We say that (P, ω) is a *distributively labeled lattice* if

(1) $P = J(Q)$ is a distributive lattice; and

(2) there is a vertex (element) labeling $\pi : Q \to A$ such that

$$\omega(I \setminus \{q\} \lessdot I) = \pi(q)$$

for any $I \in J(Q)$ and q maximal in I.

If W is a Weyl group, we may label the edges of the Hasse diagram of the weak order $(W, \prec)$ with simple reflections: the cover $w \prec s_i w$ is labeled with s_i. We denote the resulting edge-labeled poset by W_{weak}. Similarly define W_{strong} to be the strong order where $w \lessdot wt$ is labeled with $t \in T$. These labeled posets restrict to give labeled posets W^i_{weak} and W^i_{strong}. Note that each cover relation in W^i under either order is itself a cover relation in W. Thus W^i_{weak} and W^i_{strong} are induced subgraphs of W_{weak} and W_{strong}.

Theorem 6.2. *Suppose W^i is a distributive parabolic quotient. Then the strong and weak orders on W^i coincide. In particular W^i_{weak} and W^i_{strong} are distributively labeled lattices.*

Stembridge [1996, Theorem 2.2] proved that W^i_{weak} is a distributively labeled lattice. For the sake of completeness we give a self-contained proof of Theorem 6.2.

6C. *Cominuscule parabolic quotients.* Let Φ be an irreducible finite root system and W be its Weyl group. Let $\Phi = \Phi^+ \sqcup \Phi^-$ denote the decomposition of the roots into the disjoint subsets of positive and negative roots. Let $\theta = \sum_{i \in I} a_i \alpha_i$ denote the highest root of Φ. We say that $i \in I$ is *cominuscule* if $a_i = 1$.

It can be checked case-by-case using Theorem 6.1 that the distributive parabolic quotients W^i correspond to cominuscule nodes $i \in I$ except in the cases $W = G_2$, $W^i = B_n/A_{n-1}$ or $W^i = C_n/C_{n-1}$. In the latter two cases, one may use the isomorphic quotients given by their duals C_n/A_{n-1} and B_n/B_{n-1}, which are cominuscule.

For now we suppose that a cominuscule node $i \in I$ has been fixed. If α and β are two roots, we say $\alpha \geq \beta$ if $\alpha - \beta$ is a sum of positive roots. Recall that θ is the unique maximal root under this order. Let $\Phi^{(i)}$ denote the poset of positive roots

which lie above α_i. Clearly $\theta \in \Phi^{(i)}$. The inversion set of $w \in W$ is defined by

$$\text{Inv}(w) = \{\alpha \in \Phi^+ \mid w\alpha < 0\}.$$

Lemma 6.3. *Suppose* $\alpha, \beta \in \Phi^{(i)}$. *Write* $s_\alpha\beta = \beta + k\alpha$ *where* $k = -\langle \alpha^\vee, \beta \rangle$. *Then* $k \in \{0, -1, -2\}$ *and the following facts hold*:

(1) *If* α *and* β *are incomparable then* $s_\alpha\beta = \beta$.

(2) *If* $\alpha > \beta$ *then* $s_\alpha\beta$ *is equal to one of the following*: (i) β; (ii) $-\gamma$ *where* $\gamma \in \Phi^+ \setminus \Phi^{(i)}$; *or* (iii) $-\gamma$ *where* $\gamma > \alpha$.

Proof. To obtain the bounds on k we observe that for all roots $\gamma \in \Phi$, $-\theta \leq \gamma \leq \theta$, so that the coefficient of α_i in γ, lies between the corresponding coefficients in $-\theta$ and θ, which are -1 and 1 by the assumption that i is cominuscule.

Suppose that α and β are incomparable. Then $\beta - \alpha$ is neither positive nor negative and hence not a root. Since the roots in Φ occur in strings, we must have $k = 0$.

If $\alpha > \beta$, the three cases correspond to $k = 0$, $k = -1$, and $k = -2$. $\square$

For our results on distributive parabolic quotients, we require the following result, which is a slight strengthening of [Thomas and Yong 2006, Proposition 2.1, Lemma 2.2]. We include a self-contained proof, part of which is the same as the proof of [Thomas and Yong 2006, Proposition 2.1]. In particular we prove directly that the edge labeled poset W^i_{strong} defined in Section 6B is a distributively labeled lattice.

Proposition 6.4. *The map* $w \longmapsto \text{Inv}(w)$ *defines an isomorphism of posets* $\text{Inv}|_{W^i} : (W^i, \leq) \to J(\Phi^{(i)})$. *Moreover, if* $u \lessdot w$ *for* $u, w \in W^i$, *then writing* $w = us_\alpha$ *for* $\alpha \in \Phi^+$, *we have* $\alpha \in \Phi^{(i)}$ *and* $\text{Inv}(w) = \text{Inv}(u) \sqcup \{\alpha\}$.

Proof. Let $w \in W^i$. First we show that $\text{Inv}(w) \subset \Phi^{(i)}$. Suppose that $\gamma \in \text{Inv}(w) \setminus \Phi^{(i)}$. If $\gamma = \alpha_k$ where $k \neq i$ this means $ws_k < w$ which contradicts the assumption that $w \in W^i$. Otherwise $\gamma = \delta + \rho$ where $\delta, \rho \in \Phi^+ \setminus \Phi^{(i)}$. Since $w\gamma < 0$ we have $w\delta < 0$ or $w\rho < 0$ so the same argument applies. Repeating we obtain a contradiction.

Now we show that $\text{Inv}(w) \in J(\Phi^{(i)})$. Suppose $\alpha \in \text{Inv}(w)$ and $\beta < \alpha$. Then $\gamma = \alpha - \beta \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i in γ is zero. Since $\text{Inv}(w) \subset \Phi^{(i)}$, we have $\gamma \notin \text{Inv}(w)$, that is, $w\alpha - w\beta = w\gamma > 0$. Since $w\alpha < 0$ this shows that $w\beta < 0$ as desired. Thus $\text{Inv}|_{W^i}$ is well-defined.

Next we show that $\text{Inv}|_{W^i}$ sends covers to covers. Let $u \lessdot w$ with $u, w \in W^i$ and $\alpha \in \Phi^+$ such that $w = us_\alpha$. Then $0 > w\alpha = -u\alpha$ so $\alpha \in \text{Inv}(w) \setminus \text{Inv}(u)$. For all $\beta \in \text{Inv}(u)$, since $\text{Inv}(u) \in J(\Phi^{(i)})$, $\alpha > \beta$ or $\alpha \ngtr \beta$. Either way we have $w\beta = us_\alpha\beta < 0$, since by Lemma 6.3, $s_\alpha\beta$ is either equal to β or $-\gamma$ for

$\gamma \in \Phi^+ \setminus \mathrm{Inv}(u)$. That is, $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$. Since $|\mathrm{Inv}(w)| = |\mathrm{Inv}(u)| + 1$ it follows that $\mathrm{Inv}(w) = \mathrm{Inv}(u) \sqcup \{\alpha\}$, so that $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$ is a covering relation in $J(\Phi^{(i)})$.

Next we show that every covering relation in $J(\Phi^{(i)})$ is the image of a covering relation in W^i, and in particular, that $\mathrm{Inv}|_{W^i}$ is onto. An arbitrary covering relation in $J(\Phi^{(i)})$ is given by $S \setminus \{\alpha\} \subset S$ where $S \in J(\Phi^{(i)})$ and α is maximal in S.

By induction there is a $u \in W^i$ such that $\mathrm{Inv}(u) = S \setminus \{\alpha\}$. Let $w = us_\alpha$. It suffices to show that

$$\mathrm{Inv}(w) = S \quad \text{and} \quad w \in W^i.$$

The second claim follows from the first since none of the α_k for $k \neq i$ lie in $\mathrm{Inv}(w)$. For the first claim, since $\alpha \in \Phi^{(i)} \setminus \mathrm{Inv}(u)$, we may argue as before to show that $S = \mathrm{Inv}(u) \sqcup \{\alpha\} \subset \mathrm{Inv}(w)$.

For the opposite inclusion, suppose $\beta \in \Phi^+ \setminus S$. We must show that $w\beta > 0$. Write $s_\alpha\beta = \beta + k\alpha$ for $k \in \mathbb{Z}$. If $k = 0$ then we are done as before. If $k > 0$ then $s_\alpha\beta > \alpha$, so that $s_\alpha\beta \in \Phi^+ \setminus S$ since S is an order ideal. But then $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. So we may assume that $k < 0$.

Suppose first that $\beta \in \Phi^{(i)}$. We may assume that α and β are comparable by Lemma 6.3. Since S is an order ideal we have $\beta > \alpha$. If $k = -1$ then $s_\alpha\beta = \beta - \alpha \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i is 1 in both α and β. In particular $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. Otherwise $k = -2$. Then $s_\alpha\beta = \beta - 2\alpha < 0$. We have $0 < \beta - \alpha < \alpha$ and $-s_\alpha\beta = 2\alpha - \beta = \alpha - (\beta - \alpha) < \alpha$. Since S is an order ideal it follows that $-s_\alpha\beta \in \mathrm{Inv}(u)$ and $w\beta = us_\alpha\beta > 0$ as desired.

Otherwise $\beta \in \Phi^+ \setminus \Phi^{(i)}$. Since i is cominuscule we have $k \in \{-1, 0, 1\}$. We assume $k = -1$ as the other cases were already done. Then $s_\alpha\beta = \beta - \alpha < 0$ since its coefficient of α_i is -1. Moreover $\alpha - \beta \in \Phi^{(i)}$. Since $\alpha > \alpha - \beta$ and S is an order ideal, it follows that $\alpha - \beta \in \mathrm{Inv}(u)$. Therefore $w\beta = us_\alpha\beta > 0$ as desired.

We have shown that every cover in $J(\Phi^{(i)})$ is the image under $\mathrm{Inv}|_{W^i}$ of a cover in $(W^i, \leq)$.

The bijectivity of $\mathrm{Inv}|_{W^i}$ follows by induction and the explicit description of the image of a cover under $\mathrm{Inv}|_{W^i}$.   □

Proof of Theorem 6.2. For the case $W = G_2$, both labeled posets W^i_{weak} and W^i_{strong} are chains, so the result follows immediately. Thus we may assume that W^i is a cominuscule parabolic quotient.

For W^i_{strong} the result follows from Proposition 6.4. We label the vertices of $\Phi^{(i)}$ by reflections, defining $\pi : \Phi^{(i)} \to T$ by $\pi(\alpha) = s_\alpha$. Each cover $w \lessdot ws_\alpha$ in W^i_{strong} corresponds to adding $\alpha \in \Phi^{(i)}$ to $\mathrm{Inv}(w)$. Thus the edge label of $w \lessdot ws_\alpha$ agrees with the vertex label $\pi(\alpha) = s_\alpha$.

For the weak order W^i_{weak} let us consider two covers $w \lessdot ws_\alpha = s_\beta w$ and $v \lessdot vs_\alpha = s_{\beta'} v$ which have the same label s_α in W^i_{strong}. We claim that $s_\beta = s_{\beta'} = s_k$ for some $k \in I$. The elements w and v differ by right multiplication by some s_γ's where

$\gamma \in \Phi^{(i)}$ is incomparable with α; this is accomplished by passing between w or v to the element $u \in W^i$ such that $\mathrm{Inv}(u) = \mathrm{Inv}(w) \cap \mathrm{Inv}(v)$. By Lemma 6.3 these s_γ's commute with s_α, and so $w\alpha = v\alpha$. This gives us a map $f : \Phi^{(i)} \to \Phi^+$ defined by $f(\alpha) = \beta = w\alpha$, which does not depend on $w \in W^i$ as long as $w \lessdot ws_\alpha$.

To show that $f(\alpha)$ is simple for each $\alpha \in \Phi^{(i)}$, consider a reduced word $ws_\alpha = s_{k_1} s_{k_2} \cdots s_{k_l}$. We know that $w^{(r)} = s_{k_r} \cdots s_{k_l} \in W^i$ and that $\mathrm{Inv}(w^{(r)})$ differs from $\mathrm{Inv}(w^{(r+1)})$ by some root in $\Phi^{(i)}$ since $w^{(r+1)} \lessdot w^{(r)}$. For some value $r = r^*$, this root is α and by the well-definedness just proved $f(\alpha) = \alpha_{k_{r^*}}$, since $w^{(r^*)} = w^{(r^*+1)} s_\alpha$. This shows that the strong order and weak order on W^i coincide, and that W^i_{weak} is isomorphic to the poset of order ideals of $\Phi^{(i)}$ where $\Phi^{(i)}$ is labeled with $\pi(\alpha) = f(\alpha)$. $\qquad\square$

7. Distributive subgraphs of Kac–Moody graded graphs

In this section we apply Theorem 6.2 to the dual graded graphs constructed in Section 2.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody algebra associated to the generalized Cartan matrix A and let W be its Weyl group. Let $W_{\mathrm{fin}} \subset W$ be a finite parabolic subgroup corresponding to some index set $I' \subset I$. Now suppose that W_{fin} has a distributive parabolic quotient as in Theorem 6.1 corresponding to $J = I' \setminus \{i\} \subset I'$. We let $W^J \subset W_{\mathrm{fin}}$ denote the distributive parabolic quotient (we use W^J instead of W^i in this section since W^J is not a maximal parabolic quotient of W, but of W_{fin}).

Now let $(\Lambda, K) \in P^+ \times Z^+$ and $(\Gamma_s(\Lambda), \Gamma_w(K))$ be the pair of dual graded graphs constructed in Section 2. By restricting to the set of vertices $W^J \subset W_{\mathrm{fin}} \subset W$ we obtain the induced pair of graded graphs $(\Gamma_s(\Lambda), \Gamma_w(K))^J$. These graded graphs are not dual (see Remark 2.1) but they still have rich combinatorics.

The distributive lattice $(W^J, \leq)$ has two edge labelings. Recall that in W^J_{strong}, the edge $v \lessdot w = vs_\alpha$ is labeled either by the reflection s_α, while in the strong Kac–Moody subgraph $\Gamma^J_s(\Lambda)$, the edge $v \lessdot w = vs_\alpha$ is labeled by the integer $\langle \alpha^\vee, \Lambda \rangle$. Similarly the distributive lattice $(W^J, \preceq)$ has two edge labelings; in W^J_{weak}, the edge $v \prec s_j v$ is labeled by the simple reflection s_j, while in the weak Kac–Moody subgraph $\Gamma^J_w(K)$, the edge $v \prec s_j v$ is labeled by the integer $\langle K, \Lambda_j \rangle$. The following result is an immediate consequence of Theorem 6.2.

Theorem 7.1. *The induced graded subgraphs $\Gamma^J_s(\Lambda)$ and $\Gamma^J_w(K)$ are distributively labeled lattices.*

Thus $\Gamma^J_s(\Lambda)$ (resp. $\Gamma^J_w(K)$) can be thought of as the poset of order ideals in some integer labeled poset P^J (resp. Q^J). The Λ-strong and K-weak tableaux can be thought of as linear extensions of P^J and Q^J with additional markings.

In the rest of the paper, we give examples of the posets P^J and Q^J and relate them to classically understood tableaux. In each case we let $\mathfrak{g}$ be of untwisted affine

Root system	Dynkin Diagram
A_n	
$C_n, n \geq 3$	
$D_n, n \geq 4$	
E_6	
E_7	
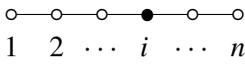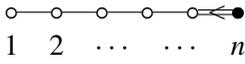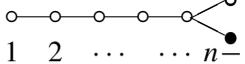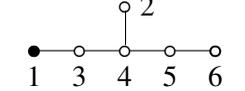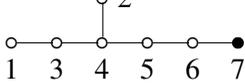
Figure 5. Some cominuscule parabolic quotients.

type, $I_{\mathrm{fin}} = I \setminus \{0\}$ and $J = I \setminus \{i\}$ for a fixed node $i \in I_{\mathrm{fin}}$ to be specified. We use the canonical central element $K_{\mathrm{can}} = \sum_{i \in I} a_i^\vee \alpha_i^\vee$ for K and Λ_i for the dominant weight. In this case P^J and Q^J are both labelings of the poset $\Phi^{(i)} \subset \Phi^+$ for the simple Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ whose Dynkin diagram is the subdiagram of that of $\mathfrak{g}$ given by removing the 0 node. These examples, with the exception of G_2, can be viewed as providing some additional data for the posets $\Phi^{(i)}$, whose unlabeled versions were given explicitly in [Thomas and Yong 2006]. As in that reference, we rotate the labeled Hasse diagrams clockwise by 45 degrees so that the minimal element is in the southwest corner. In the following, we let V_{weak}^J, V_{strong}^J denote the vertex-labeled posets such that $W_{\mathrm{weak}}^J = J(V_{\mathrm{weak}}^J)$ and $W_{\mathrm{strong}}^J = J(V_{\mathrm{strong}}^J)$.

7A. *Type $A_n^{(1)}$*. Let $i \in I_{\mathrm{fin}}$ be arbitrary. The poset $\Phi^{(i)}$ consists of elements $\alpha_{p,q} = \alpha_p + \cdots + \alpha_q$ for $1 \leq p \leq i \leq q \leq n$. The weak labeling of $\Phi^{(i)}$ is given by $\alpha_{p,q} \mapsto s_{p+q-i}$. For example, for $n = 7$ and $i = 3$ and abbreviating $\alpha_{p,q}$ by pq and s_j by j, the labelings of $\Phi^{(i)}$ by positive roots and simple reflections are given by

$$V_{\mathrm{strong}}^J = \begin{array}{|c|c|c|c|c|} \hline 13 & 14 & 15 & 16 & 17 \\ \hline 23 & 24 & 25 & 26 & 27 \\ \hline 33 & 34 & 35 & 36 & 37 \\ \hline \end{array} \qquad V_{\mathrm{weak}}^J = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 & 5 & 6 \\ \hline 3 & 4 & 5 & 6 & 7 \\ \hline \end{array}$$

All labelings in P^J and Q^J are given by the constant 1. The resulting strong and weak tableaux are usual standard tableaux.

7B. *Type $C_n^{(1)}$*. Let $i = n$. Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = 2e_n$ where e_i is the i-th standard basis element of the weight lattice $\mathbb{Z}^n$. Then $\Phi^{(n)}$ consists of

the roots $\alpha_{i,j} = e_i + e_j$ for $1 \leq i \leq j \leq n$. We have $a_i^\vee = 1$ for all i. For $n = 4$ we have

$$
V_{\text{strong}}^J =
\begin{array}{|c|c|c|c|}
\hline
14 & 13 & 12 & 11 \\
\hline
24 & 23 & 22 \\
\cline{1-3}
34 & 33 \\
\cline{1-2}
44 \\
\cline{1-1}
\end{array}
\qquad
V_{\text{weak}}^J =
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
2 & 3 & 4 \\
\cline{1-3}
3 & 4 \\
\cline{1-2}
4 \\
\cline{1-1}
\end{array}
\qquad
P^J =
\begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 1 \\
\hline
2 & 2 & 1 \\
\cline{1-3}
2 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}
\qquad
Q^J =
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
\cline{1-3}
1 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}
$$

The strong tableaux are shifted standard tableaux with two kinds of markings on off-diagonal entries; these are the standard recording tableaux for shifted insertion [Sagan 1987]. The weak tableaux are standard shifted tableaux.

7C. Type $D_n^{(1)}$. Let $i = n$. Letting $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_{n-1} + e_n$, the roots of $\Phi^{(n)}$ are given by $\alpha_{p,q} = e_p + e_q$ for $1 \leq p < q \leq n$. We have $a_j^\vee = 1$ for $j \in \{0, 1, n-1, n\}$ and $a_j^\vee = 2$ otherwise. For $n = 5$ we give the labelings of $\Phi^{(n)}$ below. Note the 1 in the upper left corner of Q^J.

$$
V_{\text{strong}}^J =
\begin{array}{|c|c|c|c|}
\hline
15 & 14 & 13 & 12 \\
\hline
25 & 24 & 23 \\
\cline{1-3}
35 & 34 \\
\cline{1-2}
45 \\
\cline{1-1}
\end{array}
\qquad
V_{\text{weak}}^J =
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
2 & 3 & 5 \\
\cline{1-3}
3 & 4 \\
\cline{1-2}
5 \\
\cline{1-1}
\end{array}
\qquad
P^J =
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
\cline{1-3}
1 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}
\qquad
Q^J =
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 1 \\
\hline
2 & 2 & 1 \\
\cline{1-3}
2 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}
$$

7D. Type E. The computations in this section were made using Stembridge's Coxeter/Weyl package [Stembridge 2004]. In both of the following cases, P^J has all labels 1.

For $E_6^{(1)}$ and $i = 1$ with the Dynkin labeling in Figure 5, we have

$$
V_{\text{weak}}^J =
\begin{array}{ccccc}
& & 1 & 3 & 4 & 5 & 6 \\
& & 3 & 4 & 2 \\
& 2 & 4 & 5 \\
1 & 3 & 4 & 5 & 6
\end{array}
\qquad
Q^J =
\begin{array}{ccccc}
& & 1 & 2 & 3 & 2 & 1 \\
& & 2 & 3 & 2 \\
& 2 & 3 & 2 \\
1 & 2 & 3 & 2 & 1
\end{array}
$$

For $E_7^{(1)}$ and $i = 7$ with the Dynkin labeling in Figure 5, we have

$$
V_{\text{weak}}^J =
\begin{array}{ccccc}
& & & 7 \\
& & & 6 \\
& & & 5 \\
& & 2 & 4 \\
7 & 6 & 5 & 4 & 3 \\
6 & 5 & 4 & 3 & 1 \\
5 & 4 & 2 \\
2 & 4 & 3 \\
7 & 6 & 5 & 4 & 3 & 1
\end{array}
\qquad
Q^J =
\begin{array}{ccccc}
& & & 1 \\
& & & 2 \\
& & & 3 \\
& & 2 & 4 \\
1 & 2 & 3 & 4 & 3 \\
2 & 3 & 4 & 3 & 2 \\
3 & 4 & 2 \\
2 & 4 & 3 \\
1 & 2 & 3 & 4 & 3 & 2
\end{array}
$$

7E. Type $G_2^{(1)}$. This case does *not* correspond to a cominiscule root. Pick $i = 1$ and let α_1, α_2 be the two simple roots, so that the highest root is $3\alpha_1 + 2\alpha_2$. Then

$a_1^\vee = 1$ and $a_2^\vee = 2$. Abbreviating the reflection $s_{p\alpha_1+q\alpha_2}$ by pq, we have:

$$V_{\text{strong}}^J = \boxed{1}\boxed{31}\boxed{21}\boxed{32}\boxed{11} \qquad\qquad P^J = \boxed{1}\boxed{3}\boxed{2}\boxed{3}\boxed{1}$$

$$V_{\text{weak}}^J = \boxed{1}\boxed{2}\boxed{1}\boxed{2}\boxed{1} \qquad\qquad Q^J = \boxed{1}\boxed{2}\boxed{1}\boxed{2}\boxed{1}$$

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