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Finite descent obstructions and rational points on curves

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Let k be a number field and X a smooth projective k-variety. In this paper, we study the information obtainable from descent via torsors under finite k-group schemes on the location of the k-rational points on X within the adelic points. Our main result is that if a curve C/k maps nontrivially into an abelian variety A/k such that $A(k)$ is finite and $Ш(k, A)$ has no nontrivial divisible element, then the information coming from finite abelian descent cuts out precisely the rational points of C. We conjecture that this is the case for all curves of genus at least 2. We relate finite descent obstructions to the Brauer–Manin obstruction; in particular, we prove that on curves, the Brauer set equals the set cut out by finite abelian descent. Our conjecture therefore implies that the Brauer–Manin obstruction against rational points is the only one on curves.

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1. Introduction

In this paper we explore what can be deduced about the set of rational points on a curve (or a more general variety) from the knowledge of its finite étale coverings.

Given a smooth projective variety X over a number field k and a finite étale, geometrically Galois covering $\pi : Y \to X$, standard descent theory tells us that there are only finitely many twists $\pi_j : Y_j \to X$ of π such that Y_j has points everywhere locally, and then $X(k) = \coprod_j \pi_j(Y_j(k))$. Since $X(k)$ embeds into the adelic points $X(\mathbb{A}_k)$, we obtain restrictions on where the rational points on X can be located inside $X(\mathbb{A}_k)$, that is, we must have

$$X(k) \subset \bigcup_j \pi_j(D_j(\mathbb{A}_k)) =: X(\mathbb{A}_k)^{\pi} .$$

Putting the information from all such finite étale coverings together, we arrive at

$$X(\mathbb{A}_k)^{\text{f-cov}} = \bigcap_{\pi} X(\mathbb{A}_k)^{\pi} .$$

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Since the information we get cannot tell us more than on which connected component a point lies at the infinite places, we make a slight modification by replacing the v-adic component of $X(\mathbb{A}_k)$ with its set of connected components, for infinite places v. In this way, we obtain $X(\mathbb{A}_k)_\bullet$ and $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$.

We can be more restrictive in the kind of coverings we allow. We denote the set cut out by restrictions coming from finite abelian coverings only by $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and the set cut out by solvable coverings by $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$. Then we have the chain of inclusions

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet\,,$$

where $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$; see Section 5 below.

It turns out that the set cut out by the information coming from finite étale abelian coverings on a curve C coincides with the "Brauer set", which is defined using the Brauer group of C,

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\text{Br}}\,.$$

This follows easily from the descent theory of Colliot-Thélène and Sansuc; see Section 7. It should be noted, however, that this result seems to be new. It says that on curves, all the information coming from torsors under groups of multiplicative type is already obtained from torsors under finite abelian group schemes.

In this way, it becomes possible to study the Brauer–Manin obstruction on curves via finite étale abelian coverings. For example, we provide an alternative proof of the main result in Scharaschkin's thesis [1999] characterizing $C(\mathbb{A}_k)_\bullet^{\text{Br}}$ in terms of the topological closure of the Mordell–Weil group in the adelic points of the Jacobian; see Corollary 7.4.

Let us call X "good" if it satisfies $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and "very good" if it satisfies $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Then another consequence is that the Brauer–Manin obstruction is the only obstruction against rational points on a curve that is very good. More precisely, the Brauer–Manin obstruction is the only one against a weak form of weak approximation, namely weak approximation with information at the infinite primes reduced to connected components.

An abelian variety A/k is very good if and only if the divisible subgroup of $Ш(k, A)$ is trivial. For example, if $A/\mathbb{Q}$ is a modular abelian variety of analytic rank zero, then $A(\mathbb{Q})$ and $Ш(\mathbb{Q}, A)$ are both finite, and A is very good. A principal homogeneous space X for A such that $X(k) = \varnothing$ is very good if and only if it represents a nondivisible element of $Ш(k, A)$. See Corollary 6.2 and the text follows it.

The main result of this paper says that if C/k is a curve that has a nonconstant morphism $C \to X$, where X is (very) good and $X(k)$ is finite, then C is (very) good (and $C(k)$ is finite); see Proposition 8.5. This implies that every curve $C/\mathbb{Q}$ whose Jacobian has a nontrivial factor A, namely a modular abelian variety of analytic rank zero, is very good; see Theorem 8.6. As an application, we prove that all modular curves $X_0(N)$, $X_1(N)$ and $X(N)$ (over $\mathbb{Q}$) are very good; see Corollary 8.8. For curves without rational points, we have the following corollary.

Corollary. *If $C/\mathbb{Q}$ has a nonconstant morphism into a modular abelian variety of analytic rank zero, and if $C(\mathbb{Q}) = \varnothing$, then the absence of rational points is explained by the Brauer–Manin obstruction.*

This generalizes a result due to Siksek [2004] by removing all assumptions related to the Galois action on the fibers of the morphism over rational points.

The paper is organized as follows. After a preliminary section (Section 2) setting up the notation, we prove in Section 3 some results on abelian varieties, which will be needed later on, but are also interesting in themselves. Then, in Section 4, we review torsors and twists and set up some categories of torsors for later use. Section 5 introduces the sets cut out by finite descent information, as sketched above, and Section 6 relates this to rational points. Next we study the relationship between our sets $X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}}$ and the Brauer set $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ together with its variants. This is done in Section 7. We then discuss certain inheritance properties of the notion of being "excellent" (which is stronger than "good") in Section 8. This is then the basis for the conjecture formulated and discussed in Section 9.

2. Preliminaries

In the following, k will always denote a number field.

Let X be a smooth projective variety over k. We modify the definition of the set of adelic points of X in the following way.[1]

$$X(\mathbb{A}_k)_\bullet = \prod_{v\nmid\infty} X(k_v) \times \prod_{v\mid\infty} \pi_0(X(k_v)) \,.$$

In other words, the factors at infinite places v are reduced to the set of connected components of $X(k_v)$. We then have a canonical surjection $X(\mathbb{A}_k) \longrightarrow\!\!\!\!\!\rightarrow X(\mathbb{A}_k)_\bullet$. Note that for a zero-dimensional variety (or reduced finite scheme) Z, we have $Z(\mathbb{A}_k) = Z(\mathbb{A}_k)_\bullet$. We will occasionally be a bit sloppy in our notation, pretending that canonical maps like $X(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_K)_\bullet$ (for a finite extension $K \supset k$) or $Y(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_k)_\bullet$ (for a subvariety $Y \subset X$) are inclusions, even though they in general are not at the infinite places. For example, the intersection $X(K) \cap X(\mathbb{A}_k)_\bullet$ means the intersection of the images of both sets in $X(\mathbb{A}_K)_\bullet$.

[1]This notation was introduced by Bjorn Poonen.

If $X = A$ is an abelian variety over k, then

$$\prod_{v\nmid\infty}\{0\} \times \prod_{v\mid\infty} A(k_v)^0 = A(\mathbb{A}_k)_{\mathrm{div}}$$

is exactly the divisible subgroup of $A(\mathbb{A}_k)$. This implies that

$$A(\mathbb{A}_k)_\bullet / nA(\mathbb{A}_k)_\bullet = A(\mathbb{A}_k)/nA(\mathbb{A}_k)$$

and then that

$$A(\mathbb{A}_k)_\bullet = \varprojlim A(\mathbb{A}_k)_\bullet / nA(\mathbb{A}_k)_\bullet = \varprojlim A(\mathbb{A}_k)/nA(\mathbb{A}_k) = \widehat{A(\mathbb{A}_k)}$$

is (isomorphic to) its own componentwise profinite completion and also the componentwise profinite completion of the usual group of adelic points.

We will denote by $\widehat{A(k)} = A(k) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ the profinite completion $\varprojlim A(k)/nA(k)$ of the Mordell–Weil group $A(k)$. By a result of Serre [1971, Thm. 3], the natural map $\widehat{A(k)} \to \widehat{A(\mathbb{A}_k)} = A(\mathbb{A}_k)_\bullet$ is an injection and therefore induces an isomorphism with the topological closure $\overline{A(k)}$ of $A(k)$ in $A(\mathbb{A}_k)_\bullet$. We will reprove this in Proposition 3.7 below, and even show something stronger than that; see Theorem 3.10. (Our proof is based on a later result of Serre.) Note that we have an exact sequence

$$0 \longrightarrow A(k)_{\mathrm{tors}} \longrightarrow \widehat{A(k)} \longrightarrow \hat{\mathbb{Z}}^r \longrightarrow 0\,,$$

where r is the Mordell–Weil rank of $A(k)$; in particular,

$$\widehat{A(k)}_{\mathrm{tors}} = A(k)_{\mathrm{tors}}\,.$$

Let $\mathrm{Sel}^{(n)}(k, A)$ denote the n-Selmer group of A over k, as usual sitting in an exact sequence

$$0 \longrightarrow A(k)/nA(k) \longrightarrow \mathrm{Sel}^{(n)}(k, A) \longrightarrow Ш(k, A)[n] \longrightarrow 0\,.$$

If $n \mid N$, we have a canonical map of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(k)/NA(k) & \longrightarrow & \mathrm{Sel}^{(N)}(k, A) & \longrightarrow & Ш(k, A)[N] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cdot N/n} & & \\
0 & \longrightarrow & A(k)/nA(k) & \longrightarrow & \mathrm{Sel}^{(n)}(k, A) & \longrightarrow & Ш(k, A)[n] & \longrightarrow & 0
\end{array}$$

and we can form the projective limit

$$\widehat{\mathrm{Sel}}(k, A) = \varprojlim \mathrm{Sel}^{(n)}(k, A)\,,$$

which sits again in an exact sequence

$$0 \longrightarrow \widehat{A(k)} \longrightarrow \widehat{\mathrm{Sel}}(k, A) \longrightarrow T\,Ш(k, A) \longrightarrow 0\,,$$

where $T\text{Ш}(k, A)$ is the Tate module of $\text{Ш}(k, A)$ (and the exactness on the right follows from the fact that the maps $A(k)/NA(k) \to A(k)/nA(k)$ are surjective). If $\text{Ш}(k, A)$ is finite, or more generally, if the divisible subgroup $\text{Ш}(k, A)_{\text{div}}$ is trivial, then the Tate module vanishes, and $\widehat{\text{Sel}}(k, A) = \widehat{A(k)}$. Note also that since $T\,\text{Ш}(k, A)$ is torsion-free, we have

$$\widehat{\text{Sel}}(k, A)_{\text{tors}} = \widehat{A(k)}_{\text{tors}} = A(k)_{\text{tors}}\,.$$

By the definition of the Selmer group, we get maps

$$\text{Sel}^{(n)}(k, A) \longrightarrow A(\mathbb{A}_k)/nA(\mathbb{A}_k) = A(\mathbb{A}_k)_\bullet/nA(\mathbb{A}_k)_\bullet$$

that are compatible with the projective limit, so we obtain a canonical map

$$\widehat{\text{Sel}}(k, A) \longrightarrow A(\mathbb{A}_k)_\bullet$$

through which the map $\widehat{A(k)} \to A(\mathbb{A}_k)_\bullet$ factors. We will denote the elements of $\widehat{\text{Sel}}(k, A)$ by $\hat{P}$, $\hat{Q}$ and the like, and we will write P_v, Q_v and the like for their images in $A(k_v)$ or $\pi_0(A(k_v))$, so the map $\widehat{\text{Sel}}(k, A) \to A(\mathbb{A}_k)_\bullet$ is specified as $\hat{P} \longmapsto (P_v)_v$. (It will turn out that this map is injective; see Proposition 3.7.)

If X is a k-variety, then we use notation like Pic_X, NS_X, and so forth, to denote the Picard group, Néron–Severi group, and so forth, of X over $\bar{k}$, as a k-Galois module.

Finally, we will denote the absolute Galois group of k by $\mathcal{G}_k$.

3. Some results on abelian varieties

In the following, A is an abelian variety over k of dimension g. For $N \geq 1$, we set $k_N = k(A[N])$ for the N-division field, and $k_\infty = \bigcup_N k_N$ for the division field.

The following lemma, based on a result due to Serre on the image of the Galois group in $\text{Aut}(A_{\text{tors}})$, forms the basis for the results of this section.

Lemma 3.1. *There is some $m \geq 1$ such that m kills all the cohomology groups $H^1(k_N/k, A[N])$.*

Proof. By a result of Serre [2000, p. 60], the image of $\mathcal{G}_k$ in $\text{Aut}(A_{\text{tors}}) = \text{GL}_{2g}(\hat{\mathbb{Z}})$ meets the scalars $\hat{\mathbb{Z}}^\times$ in a subgroup containing $S = (\hat{\mathbb{Z}}^\times)^d$ for some $d \geq 1$. We can assume that d is even.

Now we note that in

$$H^1(k_N/k, A[N]) \overset{\text{inf}}{\hookrightarrow} H^1(k_\infty/k, A[N]) \longrightarrow H^1(k_\infty/k, A_{\text{tors}})\,,$$

the kernel of the second map is killed by $\#A(k)_{\text{tors}}$. Hence it suffices to show that $H^1(k_\infty/k, A_{\text{tors}})$ is killed by some m.

Let $G = \mathrm{Gal}(k_\infty/k) \subset \mathrm{GL}_{2g}(\hat{\mathbb{Z}})$, then $S \subset G$ is a normal subgroup. We have the inflation-restriction sequence

$$H^1(G/S, A_{\mathrm{tors}}^S) \longrightarrow H^1(G, A_{\mathrm{tors}}) \longrightarrow H^1(S, A_{\mathrm{tors}}) .$$

Therefore it suffices to show that there is some integer $D \ge 1$ killing both A_{tors}^S and $H^1(S, A_{\mathrm{tors}}) = H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z})^{2g}$.

For a prime p, we define

$$\nu_p = \min\{v_p(a^d - 1) : a \in \mathbb{Z}_p^\times\} .$$

It is easy to see that when p is odd, we have $\nu_p = 0$ if $p-1$ does not divide d, and $\nu_p = 1 + v_p(d)$ otherwise. Also, $\nu_2 = 1$ if d is odd (which we excluded), and $\nu_2 = 2 + v_2(d)$ otherwise. In particular,

$$D = \prod_p p^{\nu_p}$$

is a well-defined positive integer.

We first show that A_{tors}^S is killed by D. We have

$$A_{\mathrm{tors}}^S = \Bigl(\bigoplus_p (\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d}\Bigr)^{2g} ,$$

and for an individual summand, we see that

$$\begin{aligned}(\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d} &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : (a^d - 1)x = 0 \quad \forall a \in \mathbb{Z}_p^\times\} \\ &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : p^{\nu_p} x = 0\}\end{aligned}$$

is killed by p^{ν_p}, whence the claim.

Now we have to look at $H^1(S, A_{\mathrm{tors}})$. It suffices to consider $H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z})$. We start with

$$H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) = 0 .$$

To see this, note that $(\mathbb{Z}_p^\times)^d$ is procyclic (for odd p, $\mathbb{Z}_p^\times$ is already procyclic; for $p = 2$, $\mathbb{Z}_2^\times$ is $\{\pm 1\}$ times a procyclic group, and the first factor goes away under exponentiation by d, since d was assumed to be even). Let $\alpha \in (\mathbb{Z}_p^\times)^d$ be a topological generator. By evaluating the cocycles at α, we obtain an injection

$$H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow \frac{\mathbb{Q}_p/\mathbb{Z}_p}{(\alpha - 1)(\mathbb{Q}_p/\mathbb{Z}_p)} = \frac{\mathbb{Q}_p/\mathbb{Z}_p}{p^{\nu_p}(\mathbb{Q}_p/\mathbb{Z}_p)} = 0 .$$

We then can conclude that $H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p)$ is killed by p^{ν_p}. To see this, write

$$(\hat{\mathbb{Z}}^\times)^d = (\mathbb{Z}_p^\times)^d \times T ,$$

where $T = \prod_{q \neq p} (\mathbb{Z}_q^\times)^d$. Then, by inflation-restriction again, there is an exact sequence

$$0 = H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(T, \mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d},$$

and we have (note that T acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p$)

$$H^1(T, \mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d} = \operatorname{Hom}(T, (\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d}).$$

This group is killed by p^{ν_p}, since $(\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d}$ is. It follows that

$$H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z}) = \bigoplus_p H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p)$$

is killed by $D = \prod_p p^{\nu_p}$.

We therefore find that $H^1(G, A_{\text{tors}})$ is killed by D^2, and that $H^1(k_N/k, A[N])$ is killed by $D^2 \,\#A(k)_{\text{tors}}$, for all N. □

Remark 3.2. A similar statement is proved for elliptic curves in [Viada 2003, Prop. 7].

Lemma 3.3. *For all positive integers N, the map*

$$Sel^{(N)}(k, A) \longrightarrow \operatorname{Sel}^{(N)}(k_N, A)$$

has the kernel killed by m, where m is the number from Lemma 3.1.

Proof. We have the following commutative and exact diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker \arrow[r] \arrow[d] & H^1(k_N/k, A[N]) \arrow[d, "\inf"] \\
0 \arrow[r] & \operatorname{Sel}^{(N)}(k, A) \arrow[r] \arrow[d] & H^1(k, A[N]) \arrow[d, "\operatorname{res}"] \\
0 \arrow[r] & \operatorname{Sel}^{(N)}(k_N, A) \arrow[r] & H^1(k_N, A[N]).
\end{tikzcd}$$

So the kernel in question injects into $H^1(k_N/k, A[N])$, and by Lemma 3.1, this group is killed by m. □

Lemma 3.4. *Let $Q \in \operatorname{Sel}^{(N)}(k, A)$, and let n be the order of mQ, where m is the number from Lemma 3.1. Then the density of places v of k such that v splits completely in k_N/k and such that the image of Q in $A(k_v)/NA(k_v)$ is trivial is at most $1/(n[k_N : k])$.*

Proof. By Lemma 3.3, the kernel of $\mathrm{Sel}^{(N)}(k, A) \to \mathrm{Sel}^{(N)}(k_N, A)$ is killed by m. Hence the order of the image of Q in $\mathrm{Sel}^{(N)}(k_N, A)$ is a multiple of n, the order of mQ. Now consider the following diagram for a place v that splits in k_N and a place w of k_N above it,

$$\begin{tikzcd}
\mathrm{Sel}^{(N)}(k, A) \arrow[r] \arrow[d] & \mathrm{Sel}^{(N)}(k_N, A) \arrow[r, hook] \arrow[d] & H^1(k_N, A[N]) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(G_{k_N}, A[N]) \arrow[d] \\
A(k_v)/NA(k_v) \arrow[r, "\cong"] & A(k_{N,w})/NA(k_{N,w}) \arrow[r, hook] & H^1(k_{N,w}, A[N]) \arrow[r, equal] & \mathrm{Hom}(G_{k_{N,w}}, A[N]) .
\end{tikzcd}$$

Let α be the image of Q in $\mathrm{Hom}(G_{k_N}, A[N])$. Then the image of Q is trivial in $A(k_v)/NA(k_v)$ if and only if α restricts to the zero homomorphism on $G_{k_{N,w}}$. This is equivalent to saying that w splits completely in L/k_N, where L is the fixed field of the kernel of α. Since the order of α is a multiple of n, we have $[L : k_N] \ge n$, and the claim now follows from the Chebotarev Density Theorem. $\square$

Recall the definition of $\widehat{\mathrm{Sel}}(k, A)$ and the natural maps

$$A(k) \hookrightarrow \widehat{A(k)} \hookrightarrow \widehat{\mathrm{Sel}}(k, A) \longrightarrow A(\mathbb{A}_k)_\bullet ,$$

where we denote the rightmost map by

$$\hat{P} \longmapsto (P_v)_v .$$

Also recall that $\widehat{\mathrm{Sel}}(k, A)_{\mathrm{tors}} = A(k)_{\mathrm{tors}}$ under the identification given by the inclusions above.

Lemma 3.5. *Let $\hat{Q}_1, \dots, \hat{Q}_s \in \widehat{\mathrm{Sel}}(k, A)$ be elements of infinite order, and let $n \ge 1$. Then there is some N such that the images of $\hat{Q}_1, \dots, \hat{Q}_s$ in $\mathrm{Sel}^{(N)}(k, A)$ all have order at least n.*

Proof. For a fixed $1 \le j \le s$, consider $(n-1)!\hat{Q}_j \neq 0$. There is some N_j such that the image of $(n-1)!\hat{Q}_j$ in $\mathrm{Sel}^{(N_j)}(k, A)$ is nonzero. This implies that the image of $\hat{Q}_j$ has order at least n. Because of the canonical maps $\mathrm{Sel}^{(lN_j)}(k, A) \to \mathrm{Sel}^{(N_j)}(k, A)$, this will also be true for all multiples of N_j. Therefore, any N that is a common multiple of all the N_j will do. $\square$

Proposition 3.6. *Let $Z \subset A$ be a finite subscheme of an abelian variety A over k such that $Z(k) = Z(\bar{k})$. Let $\hat{P} \in \widehat{\mathrm{Sel}}(k, A)$ be such that $P_v \in Z(k_v) = Z(k)$ for a set of places v of k of density* 1. *Then $\hat{P}$ is in the image of $Z(k)$ in $\widehat{\mathrm{Sel}}(k, A)$.*

Proof. In the following, we identify $A(k)$ with its image in $\widehat{\mathrm{Sel}}(k, A)$. We first show that $\hat{P} \in Z(k) + A(k)_{\mathrm{tors}}$. Assume the contrary. Then none of the differences $\hat{P} - Q$ for $Q \in Z(k)$ has finite order. Let $n > \#Z(k)$. Then by Lemma 3.5, we can find a number N such that the image of $m(\hat{P} - Q)$ under $\widehat{\mathrm{Sel}}(k, A) \to \mathrm{Sel}^{(N)}(k, A)$ has order at least n, for all $Q \in Z(k)$.

By Lemma 3.4, the density of places of k such that v splits in k_N and at least one of $\hat{P} - Q$, for $Q \in Z(k)$, maps trivially into $A(k_v)/NA(k_v)$ is at most

$$\frac{\#Z(k)}{n[k_N : k]} < \frac{1}{[k_N : k]}\,.$$

Therefore, there is a set of places v of k of positive density such that v splits completely in k_N/k and such that none of $\hat{P} - Q$ maps trivially into $A(k_v)/NA(k_v)$. This implies $P_v \neq Q$ for all $Q \in Z(k)$, contrary to the assumption on $\hat{P}$ and the fact that $Z(k_v) = Z(k)$.

It therefore follows that $\hat{P} \in Z(k) + A(k)_{\mathrm{tors}} \subset A(k)$. Take a finite place v of k such that $P_v \in Z(k)$ (the set of such places has density 1 by assumption). Then $A(k)$ injects into $A(k_v)$. But the image P_v of $\hat{P}$ under $\widehat{\mathrm{Sel}}(k, A) \to A(k_v)$ is in Z; therefore we must have $\hat{P} \in Z(k)$. □

The following is a simple, but useful consequence.

Proposition 3.7. *If S is a set of places of k of density* 1, *then*

$$\widehat{\mathrm{Sel}}(k, A) \longrightarrow \prod_{v \in S} A(k_v)/A(k_v)^0$$

is injective. (*Note that $A(k_v)^0 = 0$ for v finite.*) *In particular*,

$$\widehat{A(k)} \longrightarrow \prod_{v \in S} A(k_v)/A(k_v)^0$$

is injective, *and the canonical map $\widehat{A(k)} \to A(\mathbb{A}_k)_\bullet$ induces an isomorphism between $\widehat{A(k)}$ and $\overline{A(k)}$, the topological closure of $A(k)$ in $A(\mathbb{A}_k)_\bullet$.*

This is essentially Serre's result in [Serre 1971, Thm. 3].

Proof. Let $\hat{P}$ be in the kernel. Then we can apply Proposition 3.6 with $Z = \{0\}$, and we find that $\hat{P} = 0$.

In the last statement, it is clear that the image of the map is $\overline{A(k)}$, whence the result. □

From now on, we will identify $\widehat{\mathrm{Sel}}(k, A)$ with its image in $A(\mathbb{A}_k)_\bullet$. We then have a chain of inclusions

$$A(k) \subset \overline{A(k)} \subset \widehat{\mathrm{Sel}}(k, A) \subset A(\mathbb{A}_k)_\bullet, \quad \text{and} \quad \widehat{\mathrm{Sel}}(k, A)/\overline{A(k)} \cong T\text{Ш}(k, A)$$

vanishes if and only if the divisible subgroup of Ш(k, A) is trivial.

We can prove a result stronger than the above. For a finite place v of k, we denote by $\mathbb{F}_v$ the residue class field at v. If v is a place of good reduction for A, then it makes sense to speak of $A(\mathbb{F}_v)$, the group of $\mathbb{F}_v$-points of A. There is a canonical map

$$\widehat{\mathrm{Sel}}(k, A) \longrightarrow A(k_v) \longrightarrow A(\mathbb{F}_v)\,.$$

Lemma 3.8. *Let $0 \neq \hat{Q} \in \widehat{\mathrm{Sel}}(k, A)$. Then there is a set of (finite) places v of k (of good reduction for A) of positive density such that the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is nontrivial.*

Proof. First assume that $\hat{Q} \notin A(k)_{\mathrm{tors}}$. Then $m\hat{Q} \neq 0$, so there is some N such that $m\hat{Q}$ has nontrivial image in $\mathrm{Sel}^{(N)}(k, A)$ (where m is, as usual, the number from Lemma 3.1). By Lemma 3.4, we find that there is a set of places v of k of positive density such that $Q_v \notin NA(k_v)$. Excluding the finitely many places dividing $N\infty$ or of bad reduction for A does not change this density. For v in this reduced set, we have $A(k_v)/NA(k_v) \cong A(\mathbb{F}_v)/NA(\mathbb{F}_v)$, and so the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is not in $NA(\mathbb{F}_v)$, let alone zero.

Now consider the case that $\hat{Q} \in A(k)_{\mathrm{tors}} \setminus \{0\}$. We know that for all but finitely many finite places v of good reduction, $A(k)_{\mathrm{tors}}$ injects into $A(\mathbb{F}_v)$, so in this case, the statement is even true for a set of places of density 1. □

Remark 3.9. Note that the corresponding statement for points $Q \in A(k)$ is trivial; indeed, there are only finitely many finite places v of good reduction such that Q maps trivially into $A(\mathbb{F}_v)$. To see this, consider some projective model of A; then Q and 0 are two distinct points in projective space. They will reduce to the same point mod v if and only if v divides certain nonzero numbers (2×2 determinants formed with the coordinates of the two points). The lemma above says that things can not go wrong too badly when we replace $A(k)$ by its completion $\widehat{A(k)}$ or even $\widehat{\mathrm{Sel}}(k, A)$.

Theorem 3.10. *Let S be a set of finite places of k of good reduction for A and of density* 1. *Then the canonical homomorphisms*

$$\widehat{\mathrm{Sel}}(k, A) \longrightarrow \prod_{v \in S} A(\mathbb{F}_v) \quad \textit{and} \quad \widehat{A(k)} \longrightarrow \prod_{v \in S} A(\mathbb{F}_v)$$

are injective.

Proof. Let $\hat{Q}$ be in the kernel. If $\hat{Q} \neq 0$, then by Lemma 3.8, there is a set of places v of positive density such that the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is nonzero, contradicting the assumptions. So $\hat{Q} = 0$, and the map is injective. □

For applications, it is useful to remove in Proposition 3.6 the requirement that all points of Z have to be defined over k.

Theorem 3.11. *Let $Z \subset A$ be a finite subscheme of an abelian variety A over k. Let $\hat{P} \in \widehat{\mathrm{Sel}}(k, A)$ be such that $P_v \in Z(k_v)$ for a set of places v of k of density* 1. *Then $\hat{P}$ is in the image of $Z(k)$ in $\widehat{\mathrm{Sel}}(k, A)$.*

Proof. Let K/k be a finite extension such that $Z(K) = Z(\bar{k})$. By Proposition 3.6, we have that the image of $\hat{P}$ in $A(\mathbb{A}_K)_\bullet$ is in $Z(K)$. Since $\hat{P}$ is k-rational, this implies that the image of $\hat{P}$ in $A(\mathbb{A}_K)_\bullet$ is in $Z(k)$. Now the canonical map

$A(\mathbb{A}_k)_\bullet \to A(\mathbb{A}_K)_\bullet$ is injective except possibly at some of the infinite places, so $P_v \in Z(k)$ for all but finitely many places. Now, replacing Z by $Z(k)$ and applying Proposition 3.6 again (this time over k), we find that $\hat{P} \in Z(k)$, as claimed. □

We have seen that for zero-dimensional subvarieties $Z \subset A$, we have

$$Z(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = Z(k),$$

or even more generally,

$$Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) = Z(k)$$

(writing intersections for simplicity). One can ask if this is valid more generally for subvarieties $X \subset A$, that do not contain the translate of an abelian subvariety of positive dimension.

Question 3.12. Is there such a thing as an "Adelic Mordell–Lang Conjecture"?

A possible statement is as follows. Let A/k be an abelian variety and $X \subset A$ a subvariety not containing the translate of a nontrivial subabelian variety of A. Then there is a finite subscheme $Z \subset X$ such that

$$X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) \subset Z(\mathbb{A}_k)_\bullet \,.$$

If this holds, Theorem 3.11 above implies that

$$X(k) \subset X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) \subset Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) = Z(k) \subset X(k)$$

and therefore $X(k) = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A)$. In the notation introduced in Section 5 below and by the discussion in Section 6, this implies

$$X(k) \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) = X(k)\,,$$

and so X is excellent with respect to the abelian coverings (and hence "very good").

Remark 3.13. Note that the Adelic Mordell–Lang Conjecture formulated above is true when k is a global function field, A is ordinary, and X is not defined over k^p (where p is the characteristic of k); see [Voloch 1991]. (The result is also implicit in [Hrushovski 1996].)

4. Torsors and twists

We now introduce torsors (under finite étale group schemes) and twists, and describe various constructions that can be done with these objects.

Let X be a smooth projective (reduced, but not necessarily geometrically connected) variety over k.

We will consider the following category $\mathscr{C}ov(X)$. Its objects are X-torsors Y under G (see for example [Skorobogatov 2001] for definitions), where G is a finite étale group scheme over k. More concretely, the data consist of a k-morphism $\mu : Y \times G \to Y$ describing a right action of G on Y, together with a finite étale k-morphism $\pi : Y \to X$ such that the following diagram is cartesian, *id est*, identifies $Y \times G$ with the fiber product $Y \times_X Y$,

$$\begin{CD} Y \times G @>{\mu}>> Y \\ @V{\mathrm{pr}_1}VV @VV{\pi}V \\ Y @>{\pi}>> X. \end{CD}$$

We will usually just write (Y, G) for such an object, with the maps μ and π being understood. Morphisms $(Y', G') \to (Y, G)$ in $\mathscr{C}ov(X)$ are given by a pair of maps, as k-morphisms of (group) schemes, $\phi : Y' \to Y$ and $\gamma : G' \to G$ such that the obvious diagram

$$\begin{CD} Y' \times G' @>{\mu'}>> Y' @>{\pi'}>> X \\ @V{\phi\times\gamma}VV @VV{\phi}V @| \\ Y \times G @>{\mu}>> Y @>{\pi}>> X \end{CD}$$

commutes. Note that γ is uniquely determined by ϕ: if $y' \in Y'$, $g' \in G'$, there is a unique $g \in G$ such that $\phi(y') \cdot g = \phi(y' \cdot g')$, so we must have $\gamma(g') = g$.

We will denote by $\mathscr{S}ol(X)$ and $\mathscr{A}b(X)$ the full subcategories of $\mathscr{C}ov(X)$ whose objects are the torsors (Y, G) such that G is solvable or abelian, respectively.

If $X' \to X$ is a k-morphism of (smooth projective) varieties, then we can pull back X-torsors under G to obtain X'-torsors under G. This defines covariant functors $\mathscr{C}ov(X) \to \mathscr{C}ov(X')$, $\mathscr{S}ol(X) \to \mathscr{S}ol(X')$ and $\mathscr{A}b(X) \to \mathscr{A}b(X')$.

The following constructions are described for $\mathscr{C}ov(X)$, but they are similarly valid for $\mathscr{S}ol(X)$ and $\mathscr{A}b(X)$.

If (Y_1, G_1), $(Y_2, G_2) \in \mathscr{C}ov(X)$ are two X-torsors, then we can construct their fiber product $(Y, G) \in \mathscr{C}ov(X)$, where $Y = Y_1 \times_X Y_2$ and $G = G_1 \times G_2$. More generally, if $(Y_1, G_1) \to (Y, G)$ and $(Y_2, G_2) \to (Y, G)$ are two morphisms in $\mathscr{C}ov(X)$, there is a fiber product $(Z, H) \in \mathscr{C}ov(X)$, where $Z = Y_1 \times_Y Y_2$ and $H = G_1 \times_G G_2$.

If $(Y, G) \in \mathscr{C}ov(X)$ is an X-torsor, where now everything is over K with a finite extension K/k, then we can apply restriction of scalars to obtain

$$(R_{K/k}Y, R_{K/k}G) \in \mathscr{C}ov(R_{K/k}X).$$

If $(Y, G) \in \mathscr{C}ov(X)$ is an X-torsor and ξ is a cohomology class in $H^1(k, G)$, then we can construct the *twist* (Y_ξ, G_ξ) of (Y, G) by ξ. Here G_ξ is the inner form of G corresponding to ξ (compare, for example, [Skorobogatov 2001, pp. 12, 20]). We will denote the structure maps by μ_ξ and π_ξ. Usually, $H^1(k, G)$ is just a pointed set with distinguished element corresponding to the given torsor; if the torsor is abelian, $H^1(k, G)$ is a group, and $G_\xi = G$ for all $\xi \in H^1(k, G)$.

If $(\phi, \gamma) : (Y', G') \to (Y, G)$ is a morphism and $\xi \in H^1(k, G')$, then we get an induced morphism

$$(Y'_\xi, G'_\xi) \to (Y_{\gamma_*\xi}, G_{\gamma_*\xi})$$

where γ_* is the induced map $H^1(k, G') \to H^1(k, G)$. Similarly, twists are compatible with pull-backs, fiber products and restriction of scalars.

Twists are transitive in the following sense. If $(Y, G) \in \mathscr{C}ov(X)$ is an X-torsor and $\xi \in H^1(k, G)$, $\eta \in H^1(k, G_\xi)$, then there is a $\zeta \in H^1(k, G)$ such that $((Y_\xi)_\eta, (G_\xi)_\eta) \cong (Y_\zeta, G_\zeta)$. Conversely, if ξ and ζ are given, then there is an $\eta \in H^1(k, G_\xi)$ such that the relation above holds.

The following observation does not hold in general for $\mathscr{S}ol(X)$ and $\mathscr{A}b(X)$. If $Y \xrightarrow{\pi} X$ is any finite étale morphism, then there is some $(\tilde{Y}, G) \in \mathscr{C}ov(X)$ such that $\tilde{\pi} : \tilde{Y} \to X$ factors through π. Also, if we have $(Y, G) \in \mathscr{C}ov(X)$ and $(Z, H) \in \mathscr{C}ov(Y)$, then there is some $(\tilde{Z}, \Gamma) \in \mathscr{C}ov(X)$ such that $\tilde{Z}$ maps to Z over X and such that the induced map $\tilde{Z} \to Y$ gives rise to a Y-torsor $(\tilde{Z}, \tilde{H}) \in \mathscr{C}ov(Y)$. This last statement is also valid with $\mathscr{S}ol(X)$ and $\mathscr{S}ol(Y)$ in place of $\mathscr{C}ov(X)$ and $\mathscr{C}ov(Y)$ (since extensions of solvable groups are solvable).

5. Finite descent conditions

In this section, we use torsors and their twists, as described in the previous section, in order to obtain obstructions against rational points. The use of torsors under finite abelian group schemes is classical; it is what is behind the usual descent procedures on elliptic curves or abelian varieties (and so one can claim that they go all the way back to Fermat). The nonabelian case was first studied by Harari and Skorobogatov [2002]; see also [Harari 2000].

The following theorem (going back to Chevalley and Weil [1932]) summarizes the standard facts about descent via torsors. Compare also [Harari and Skorobogatov 2002, Lemma 4.1] and [Skorobogatov 2001, pp. 105, 106].

Theorem 5.1. *Let $(Y, G) \in \mathscr{C}ov(X)$ be a torsor, where X is a smooth projective k-variety. Then*

(1) $X(k) = \coprod_{\xi \in H^1(k,G)} \pi_\xi(Y_\xi(k))$;

(2) *the* (Y, G)-Selmer set

$$\mathrm{Sel}^{(Y,G)}(k, X) = \{\xi \in H^1(k, G) : Y_\xi(\mathbb{A}_k)_\bullet \neq \varnothing\}$$

is finite: there are only finitely many twists (Y_ξ, G_ξ) such that Y_ξ has points everywhere locally.

At least in principle, the Selmer set in the second statement can be determined explicitly, and the union in the first statement can be restricted to this finite set.

The idea behind the following considerations is to see how much information one can get out of the various torsors regarding the image of $X(k)$ in $X(\mathbb{A}_k)_\bullet$. Compare Definition 4.2 in [Harari and Skorobogatov 2002] and Definition 5.3.1 in Skorobogatov's book Skorobogatov [2001].

Definition 5.2. Let $(Y, G) \in \mathscr{C}ov(X)$ be an X-torsor. We say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives* (Y, G), if it lifts to a point in $Y_\xi(\mathbb{A}_k)_\bullet$ for some twist (Y_ξ, G_ξ) of (Y, G).

There is a cohomological description of this property. An X-torsor under G is given by an element of $H^1_{\text{ét}}(X, G)$. Pull-back through the map $\operatorname{Spec} k \to X$ corresponding to a point in $X(k)$ gives a map

$$X(k) \longrightarrow H^1(k, G)\,.$$

Note that it is not necessary to refer to nonabelian étale cohomology here: the map $X(k) \to H^1(k, G)$ induced by a torsor (Y, G) simply arises by associating to a point $P \in X(k)$ its fiber $\pi^{-1}(P) \subset Y$, which is a k-torsor under G and therefore corresponds to an element of $H^1(k, G)$.

We get a similar map on adelic points,

$$X(\mathbb{A}_k)_\bullet \longrightarrow \prod_v H^1(k_v, G)\,.$$

There is the canonical restriction map

$$H^1(k, G) \longrightarrow \prod_v H^1(k_v, G)\,,$$

and the various maps piece together to give a commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & H^1(k, G) \\ \downarrow & & \downarrow \\ X(\mathbb{A}_k)_\bullet & \longrightarrow & \prod_v H^1(k_v, G). \end{array}$$

A point $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if its image in $\prod_v H^1(k_v, G)$ is in the image of the global set $H^1(k, G)$. The (Y, G)-Selmer set is then the preimage in $H^1(k, G)$ of the image of $X(\mathbb{A}_k)_\bullet$; this is completely analogous to the definition of a Selmer group in case X is an abelian variety A, and $G = A[n]$ is the n-torsion subgroup of A.

Here are some basic properties.

Lemma 5.3. (1) *If $(\phi, \gamma) : (Y', G') \to (Y, G)$ is a morphism in $\mathscr{C}ov(X)$, and if $P \in X(\mathbb{A}_k)_\bullet$ survives (Y', G'), then P also survives (Y, G).*

(2) *If $(Y', G) \in \mathscr{C}ov(X')$ is the pull-back of $(Y, G) \in \mathscr{C}ov(X)$ under a morphism $\psi : X' \to X$, then $P \in X'(\mathbb{A}_k)_\bullet$ survives (Y', G) if and only if $\psi(P)$ survives (Y, G).*

(3) *If $(Y_1, G_1), (Y_2, G_2) \in \mathscr{C}ov(X)$ have fiber product (Y, G), then $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if P survives both (Y_1, G_1) and (Y_2, G_2).*

(4) *Let X be over K, where K/k is a finite extension, and let $(Y, G) \in \mathscr{C}ov(X)$ be an X-torsor. Then $P \in (R_{K/k}X)(\mathbb{A}_k)_\bullet$ survives $(R_{K/k}Y, R_{K/k}G)$ if and only if its image in $X(\mathbb{A}_K)_\bullet$ survives (Y, G).*

(5) *If $(Y, G) \in \mathscr{C}ov(X)$ and $\xi \in H^1(k, G)$, then $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if P survives (Y_ξ, G_ξ).*

Proof. (1) By assumption, there are $\xi \in H^1(k, G')$ and $Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ such that $\pi'_\xi(Q) = P$. We have the morphism $\phi_\xi : Y'_\xi \to Y_{\gamma_*\xi}$ over X; hence $\pi_{\gamma_*\xi}(\phi_\xi(Q)) = \pi'_\xi(Q) = P$, whence P survives (Y, G).

(2) Assume that P survives (Y', G). There are $\xi \in H^1(k, G)$ and $Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ such that $\pi'_\xi(Q) = P$. There is a morphism $\Psi_\xi : Y'_\xi \to Y_\xi$ over ψ, and hence we have that $\pi_\xi(\Psi_\xi(Q)) = \psi(P)$, so $\psi(P)$ survives (Y, G). Conversely, assume that $\psi(P)$ survives (Y, G). Then there are $\xi \in H^1(k, G)$ and $Q \in Y_\xi(\mathbb{A}_k)_\bullet$ such that $\pi_\xi(Q) = \psi(P)$. The twist (Y'_ξ, G_ξ) is the pull-back of (Y_ξ, G_ξ) under ψ; in particular, $Y'_\xi = Y_\xi \times_X X'$, and so there is $Q' \in Y'_\xi(\mathbb{A}_k)_\bullet$ mapping to Q in Y_ξ and to P in X'. Hence P survives (Y', G).

(3) We have obvious morphisms $(Y, G) \to (Y_i, G_i)$. So by part (1), if P survives (Y, G), then it also survives (Y_1, G_1) and (Y_2, G_2). Now assume that P survives both (Y_1, G_1) and (Y_2, G_2). Then there are $\xi_1 \in H^1(k, G_1)$ and $\xi_2 \in H^1(k, G_2)$ and points $Q_1 \in Y_{1,\xi_1}(\mathbb{A}_k)_\bullet$, $Q_2 \in Y_{2,\xi_2}(\mathbb{A}_k)_\bullet$ such that $\pi_{1,\xi_1}(Q_1) = P$ and $\pi_{2,\xi_2}(Q_2) = P$. Consider $\xi = (\xi_1, \xi_2) \in H^1(k, G) = H^1(k, G_1) \times H^1(k, G_2)$. We have that $Y_\xi = Y_{1,\xi_1} \times_X Y_{2,\xi_2}$; hence there is $Q \in Y_\xi(\mathbb{A}_k)_\bullet$ mapping to Q_1 and Q_2 under the canonical maps $Y_\xi \to Y_{i,\xi_i}$ $(i = 1, 2)$, and to P under $\pi_\xi : Y_\xi \to X$. Hence P survives (Y, G).

(4) We have $H^1(k, R_{K/k}G) = H^1(K, G)$, and the corresponding twists are compatible. For any ξ in this set, we have $R_{K/k}Y_\xi = (R_{K/k}Y)_\xi$, and the adelic points $(R_{K/k}Y_\xi)(\mathbb{A}_k)_\bullet$ and $Y_\xi(\mathbb{A}_K)_\bullet$ are identified. The claim follows.

(5) This comes from the fact that every twist of (Y, G) is also a twist of (Y_ξ, G_ξ) and vice versa. □

By the Descent Theorem 5.1, it is clear that (the image in $X(\mathbb{A}_k)_\bullet$ of) a rational point $P \in X(k)$ survives every torsor. Therefore it makes sense to study the set of adelic points that survive every torsor (or a suitable subclass of torsors) in order to obtain information on the location of the rational points within the adelic points. Note that the set of points in $X(\mathbb{A}_k)_\bullet$ surviving a given torsor is closed — it is a finite union of images of compact sets $Y_\xi(\mathbb{A}_k)_\bullet$ under continuous maps.

We are led to the following definitions.

Definition 5.4. Let X be a smooth projective variety over k.

(1) $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathscr{C}ov(X)\}$.

(2) $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathscr{S}ol(X)\}$.

(3) $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathscr{A}b(X)\}$.

(The "f" in the superscripts stands for "finite", since we are dealing with torsors under finite group schemes only.)

By the remark made before the definition above, we have

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet .$$

Here, $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

Recall the "evaluation map" for $P \in X(\mathbb{A}_k)_\bullet$ and G a finite étale k-group scheme,

$$\mathrm{ev}_{P,G} : H^1_{\text{ét}}(X, G) \longrightarrow \prod_v H^1(k_v, G)$$

(the set on the left can be considered as the set of isomorphism classes of X-torsors under G) and the restriction map

$$\mathrm{res}_G : H^1(k, G) \longrightarrow \prod_v H^1(k_v, G) .$$

In these terms, we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcap_G \{P \in X(\mathbb{A}_k)_\bullet : \mathrm{im}(\mathrm{ev}_{P,G}) \subset \mathrm{im}(\mathrm{res}_G)\} ,$$

where G runs through all finite étale k-group schemes. We obtain $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in a similar way, by restricting G to solvable or abelian group schemes.

In the definition above, we can restrict to (Y, G) with Y connected (over k) if X is connected: if we have (Y, G) with Y not connected, then let Y_0 by a connected component of Y, and let $G_0 \subset G$ be the stabilizer of this component. Then (Y_0, G_0) is again a torsor of the same kind as (Y, G), and we have a morphism $(Y_0, G_0) \to (Y, G)$. Hence, by Lemma 5.3, (1), if P survives (Y_0, G_0), then it also survives (Y, G).

However, we cannot restrict to geometrically connected torsors when X is geometrically connected. The reason is that there can be obstructions coming from the fact that a suitable geometrically connected torsor does not exist.

Lemma 5.5. *Assume that X is geometrically connected. If there is a torsor $(Y, G) \in \mathcal{C}ov(X)$ such that Y and all twists Y_ξ are k-connected, but not geometrically connected, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$. The analogous statement holds for the solvable and abelian versions.*

Proof. If Y_ξ is connected, but not geometrically connected, then $Y_\xi(\mathbb{A}_k)_\bullet = \emptyset$ (this is because the finite scheme $\pi_0(Y_\xi)$ is irreducible and therefore satisfies the Hasse Principle, compare the proof of Proposition 5.12). Hence no point in $X(\mathbb{A}_k)_\bullet$ survives (Y, G). □

Let us briefly discuss how this relates to the geometric fundamental group of X over $\bar{k}$, assuming X to be geometrically connected. In the following, we write $\bar{X} = X \times_k \bar{k}$ and so forth, for the base-change of X to a variety over $\bar{k}$. Every torsor $(Y, G) \in \mathcal{C}ov(X)$ ($\mathcal{S}ol(X)$ or $\mathcal{A}b(X)$, respectively) gives rise to a covering $\bar{Y} \to \bar{X}$ that is Galois with (solvable or abelian) Galois group $G(\bar{k})$. The stabilizer Γ of a connected component of $\bar{Y}$ is then a finite quotient of the geometric fundamental group $\pi_1(\bar{X})$. If we fix an embedding $k \to \mathbb{C}$, then $\pi_1(\bar{X})$ is the profinite completion of the topological fundamental group $\pi_1(X(\mathbb{C}))$, so Γ is also a finite quotient of $\pi_1(X(\mathbb{C}))$. If Γ is trivial, then $\pi_0(Y)$ is a k-torsor under G, and (Y, G) is the pull-back of $(\pi_0(Y), G)$ under the structure morphism $X \to \operatorname{Spec} k$. We call such a torsor *trivial*. Note that all points in $X(\mathbb{A}_k)_\bullet$ survive a trivial torsor (since their image in $(\operatorname{Spec} k)(\mathbb{A}_k)_\bullet = (\operatorname{Spec} k)(k) = \{\text{pt}\}$ survives everything); therefore trivial torsors do not give information.

Conversely, given a finite quotient Γ of $\pi_1(\bar{X})$ or of $\pi_1(X(\mathbb{C}))$, there is a corresponding covering $\bar{Y} \to \bar{X}$ that will be defined over some finite extension K of k. Let $\pi : Y \to X_K$ be the covering over K; it is a torsor under a K-group scheme G such that $G(\bar{k}) = \Gamma$. We now construct a torsor $(Z, R_{K/k}G) \in \mathcal{C}ov(X)$ that over K factors through π. By restriction of scalars, we obtain $(R_{K/k}Y, R_{K/k}G) \in \mathcal{C}ov(R_{K/k}X_K)$. We pull back via the canonical morphism $X \to R_{K/k}X_K$ to obtain $(Z, R_{K/k}G) \in \mathcal{C}ov(X)$. Over K, we have the following diagram

$$\begin{tikzcd}
Z_K \arrow[r] \arrow[d, "(R_{K/k}G)_K"] & (R_{K/k}Y)_K \arrow[r, "\text{can}"] \arrow[d, "(R_{K/k}G)_K"] & Y \arrow[d, "G"] \\
X_K \arrow[r, "\text{can}"] & (R_{K/k}X_K)_K \arrow[r, "\text{can}"] & X_K .
\end{tikzcd}$$

(Here the right hand horizontal maps come from the identity morphism $W \to W$ of a K-variety W, under the identification of $\operatorname{Mor}_k(V, R_{K/k}W)$ with $\operatorname{Mor}_K(V_K, W)$, taking $V = R_{K/k}W$; for $W = Y$ and $W = X_K$, respectively.) The composition of the lower horizontal maps is the identity morphism; hence $(Z_K, (R_{K/k}G)_K) \in \mathscr{C}ov(X_K)$ maps to (Y, G). Note that the torsor we construct is in $\mathscr{S}ol(X)$ (respectively, $\mathscr{A}b(X)$) when Γ is solvable (respectively, abelian).

Lemma 5.6. *Let X be geometrically connected, $(Y, G), (Y', G') \in \mathscr{C}ov(X)$ such that Y is geometrically connected and such that $(\bar{Y}, \bar{G})$ maps to $(\bar{Y}', \bar{G}')$ as torsors of $\bar{X}$. Then there is a twist (Y'_ξ, G'_ξ) of (Y', G') such that (Y, G) maps to (Y'_ξ, G'_ξ).*

Proof. Let $(\phi, \gamma) : (\bar{Y}, \bar{G}) \to (\bar{Y}', \bar{G}')$ be the given morphism. Note that by assumption, the covering maps $\pi : Y \to X$ and $\pi' : Y' \to X$ are defined over k. For $\sigma \in \mathscr{G}_k$, this implies that $({}^\sigma\phi, {}^\sigma\gamma)$ is also a morphism $(\bar{Y}, \bar{G}) \to (\bar{Y}', \bar{G}')$. We can then consider the composite morphism

$$\bar{Y} \xrightarrow{(\phi, {}^\sigma\phi)} \bar{Y}' \times_{\bar{X}} \bar{Y}' \xrightarrow{\cong} \bar{Y}' \times \bar{G}' \xrightarrow{\mathrm{pr}_2} \bar{G}' .$$

Since $\bar{Y}$ is connected and $\bar{G}'$ is discrete, this morphism must be constant. Let $\xi_\sigma \in G'(\bar{k})$ be its image. It can then be checked that $\xi = (\xi_\sigma)_{\sigma \in \mathscr{G}_k}$ is a G'-valued cocycle and that after twisting (Y', G') by ξ, the morphism ϕ becomes defined over k; since γ is uniquely determined by ϕ, the same is true for γ. □

We still assume X to be geometrically connected. Let us call a family of torsors $(Y_i, G_i) \in \mathscr{C}ov(X)$ ($\mathscr{S}ol(X)$ or $\mathscr{A}b(X)$, respectively) with Y_i geometrically connected a *cofinal family of coverings of X* (respectively, of *solvable* or *abelian coverings of X*) if for every (respectively, every solvable or abelian) connected $(\bar{Y}, \bar{G}) \in \mathscr{C}ov(\bar{X})$ (respectively, $\mathscr{S}ol(\bar{X})$ or $\mathscr{A}b(\bar{X})$), there is a torsor (Y_i, G_i) such that $(\bar{Y}_i, \bar{G}_i)$ maps to $(\bar{Y}, \bar{G})$. We then have the following.

Lemma 5.7. *Let X be geometrically connected.*

(1) *If $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$, then there is a cofinal family of coverings of X. A similar statement holds for $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and solvable coverings, and for $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and abelian coverings.*

(2) *If $(Y_i, G_i)_i$ is a cofinal family of coverings of X, then $P \in X(\mathbb{A}_k)_\bullet$ is in $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ if and only if P survives every (Y_i, G_i). Similarly for the solvable and abelian variants.*

Proof.

(1) Let $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, and let $\bar{Y} \to \bar{X}$ be a finite étale Galois covering with Galois group Γ. Then by the discussion before Lemma 5.6, there is a torsor $(Z, G) \in \mathscr{C}ov(X)$, which we can assume to be k-connected, such that $(\bar{Z}, \bar{G})$ maps to $(\bar{Y}, \Gamma)$. Without loss of generality (after perhaps twisting (Z, G)), we can assume that (Z, G) lifts P. This implies that Z is geometrically connected (compare Lemma 5.5). So if we take all torsors (Z, G) obtained in this way, we obtain a cofinal family of coverings of X. The proof in the solvable and abelian cases is analogous.

(2) The "only if" part is clear. So assume that P survives all (Y_i, G_i), and let $(Z, \Gamma) \in \mathscr{C}ov(X)$ be arbitrary. Let $\bar{Z}_0$ be a connected component of $\bar{Z}$, and let $\bar{\Gamma}_0$ be the stabilizer of $\bar{Z}_0$. Then there is some (Y_i, G_i) such that $(\bar{Y}_i, \bar{G}_i) \to (\bar{Z}_0, \bar{\Gamma}_0) \to (\bar{Z}, \bar{\Gamma})$; hence by Lemma 5.6, there is a twist (Z_ξ, Γ_ξ) such that (Y_i, G_i) maps to it. Since P survives (Y_i, G_i) by assumption, it also survives (Z_ξ, Γ_ξ) and therefore (Z, Γ), by Lemma 5.3. The proof in the solvable and abelian cases is again analogous.

□

Lemma 5.8. *Let X be geometrically connected.*

(1) *If $\pi_1(\bar{X})$ is trivial (that is, X is simply connected), then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$.*

(2) *If the abelianization $\pi_1(\bar{X})^{\text{ab}}$ is trivial, then $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet$.*

(3) *If $\pi_1(\bar{X})$ is abelian (respectively, solvable), then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ (respectively, $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$).*

Proof.

(1) In this case, all torsors are trivial and are therefore survived by all points in $X(\mathbb{A}_k)_\bullet$.

(2) Here the same holds for all abelian torsors.

(3) We always have $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. So let $P \in X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$; then by Lemma 5.7, (1), there is a cofinal family (Y_i, G_i) of abelian coverings of X, and since $\pi_1(\bar{X})$ is abelian, this is also a cofinal family of coverings without restriction. By part (2) of the same lemma, it suffices to check that P survives all (Y_i, G_i), which we know to be true, in order to conclude that $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. Similarly for the solvable variant.

□

We now list some fairly elementary properties of the sets $X(\mathbb{A}_k)_\bullet^{\text{f-ab/f-sol/f-cov}}$.

Proposition 5.9. *If $X' \xrightarrow{\psi} X$ is a morphism, then $\psi(X'(\mathbb{A}_k)_\bullet^{\text{f-cov}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. Similarly for the solvable and abelian variants.*

Proof. Let $P \in X'(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, and let $(Y, G) \in \mathscr{C}ov(X)$ be an X-torsor. By assumption, P survives the pull-back (Y', G) of (Y, G) under ψ, so by Lemma 5.3, part (2), $\psi(P)$ survives (Y, G). Since (Y, G) is arbitrary, $\psi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. The same proof works for the solvable and abelian variants. □

Lemma 5.10. *Let $Z = \operatorname{Spec} k \amalg \operatorname{Spec} k = \{P_1, P_2\}$. Then*

$$\{P_1, P_2\} = Z(k) = Z(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proof. Let $Q \in Z(\mathbb{A}_k)_\bullet$ and assume that $Q \notin Z(k)$. We have to show that $Q \notin Z(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. By assumption, there are places v and w of k such that $Q_v = P_1$ and

$Q_w = P_2$. We will consider torsors under $G = \mathbb{Z}/2\mathbb{Z}$. Pick some $\alpha \in k^\times$ such that $\alpha \notin (k_v^\times)^2$ and $\alpha \notin (k_w^\times)^2$. Let $Y = \operatorname{Spec} k(\sqrt{\alpha}) \amalg (\operatorname{Spec} k \amalg \operatorname{Spec} k)$; then $(Y, G) \in \mathscr{A}b(Z)$ in an obvious way. We want to show that no twist (Y_ξ, G) for $\xi \in H^1(k, G) = k^\times/(k^\times)^2$ lifts Q. Such a twist is of one of the following forms:

$$(Y_\xi, G) = \operatorname{Spec} k(\sqrt{\alpha}) \amalg (\operatorname{Spec} k \amalg \operatorname{Spec} k),$$
$$(Y_\xi, G) = (\operatorname{Spec} k \amalg \operatorname{Spec} k) \amalg \operatorname{Spec} k(\sqrt{\alpha}),$$
$$(Y_\xi, G) = \operatorname{Spec} k(\sqrt{\beta}) \amalg \operatorname{Spec} k(\sqrt{\gamma}),$$

where in the last case, β and γ are independent in $k^\times/(k^\times)^2$. In the first two cases, Q does not lift, since in the first case, the first component does not lift Q_v, and in the second case, the second component does not lift Q_w (by our choice of α). In the third case, there is a set of places of k of density $1/4$ that are inert in both $k(\sqrt{\beta})$ and $k(\sqrt{\gamma})$, so $Y_\xi(\mathbb{A}_k)_\bullet = \emptyset$. In particular, Q does not lift to any of these twists. □

Proposition 5.11. *If $X = X_1 \amalg X_2 \amalg \cdots \amalg X_n$ is a disjoint union, then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \coprod_{j=1}^{n} X_j(\mathbb{A}_k)_\bullet^{\text{f-cov}},$$

and similarly for the solvable and abelian variants.

Proof. It is sufficient to consider the case $n = 2$. We have maps $X_1 \to X$ and $X_2 \to X$, so by Proposition 5.9, $X_1(\mathbb{A}_k)_\bullet^{\text{f-cov}} \amalg X_2(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (same for the solvable and abelian variants). For the reverse inclusion, consider the morphism $X \to \operatorname{Spec} k \amalg \operatorname{Spec} k = Z$ mapping X_1 to the first point and X_2 to the second point. If $Q \in X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, then its image is in $Z(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$ (by Proposition 5.9 again and Lemma 5.10). This means that $Q \in X_1(\mathbb{A}_k)_\bullet \amalg X_2(\mathbb{A}_k)_\bullet$. The claim then follows easily. □

Proposition 5.12. *If Z is a (reduced) finite scheme, then $Z(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$.*

Proof. By Proposition 5.11, it suffices to prove this when $Z = \operatorname{Spec} K$ is connected. But in this case, it is known that Z satisfies the Hasse Principle. On the other hand, if $Z(k) \neq \emptyset$, then $Z = \operatorname{Spec} k$ and $Z(\mathbb{A}_k)_\bullet$ has just one point, so $Z(k) = Z(\mathbb{A}_k)_\bullet$.

(The statement that $\operatorname{Spec} K$ as a k-scheme satisfies the Hasse Principle comes down to the following fact.

Fact. If a group G acts transitively on a finite set X such that every $g \in G$ fixes at least one element of X, then $\#X = 1$.

To see this, let $n = \#X$ and assume (without loss of generality) that $G \subset S_n$. The stabilizer G_x of $x \in X$ is a subgroup of index n in G. By assumption, $G = \bigcup_{x \in X} G_x$, so $G \setminus \{1\} = \bigcup_{x \in X} (G_x \setminus \{1\})$. Counting elements now gives $\#G - 1 \le n(\#G/n - 1) = \#G - n$, which implies $n = 1$.) ☐

Remark 5.13. Note that the Hasse Principle does not hold in general for finite schemes. A typical counterexample is given by the $\mathbb{Q}$-scheme

$$\operatorname{Spec} \mathbb{Q}(\sqrt{13}) \amalg \operatorname{Spec} \mathbb{Q}(\sqrt{17}) \amalg \operatorname{Spec} \mathbb{Q}(\sqrt{13 \cdot 17})\,.$$

Proposition 5.14. *We have*

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}\,.$$

Similarly for the solvable and abelian variants.

Proof. Proposition 5.9 implies that

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}$$

(and similarly for the solvable and abelian variants).

For the other direction, we can assume that X and Y are k-connected, compare Proposition 5.11. If X (say) is not geometrically connected, then $X(\mathbb{A}_k)_\bullet = \varnothing$, and hence $(X \times Y)(\mathbb{A}_k)_\bullet = \varnothing$ as well, and the statement is trivially true. So we can assume that X and Y are geometrically connected.

We now use the fact that $\pi_1(\bar{X} \times \bar{Y}) = \pi_1(\bar{X}) \times \pi_1(\bar{Y})$. Let $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and $Q \in Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. By Lemma 5.7, (1), there are cofinal families of coverings (V_i, G_i) of X and (W_j, H_j) of Y, which we can assume to lift P, respectively, Q. Then the products $(V_i \times W_j, G_i \times H_j)$ form a cofinal family of coverings of $X \times Y$, and it is clear that they lift (P, Q). By Lemma 5.7, (2), this implies that $(P, Q) \in (X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}}$.

The solvable and abelian variants are proved similarly, using the corresponding product property of the maximal abelian and solvable quotients of the geometric fundamental group. ☐

Proposition 5.15. *If K/k is a finite extension and X is a K-variety, then*

$$(R_{K/k}X)(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$$

(*under the canonical identification* $(R_{K/k}X)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_K)_\bullet$), *and similarly for the solvable and abelian variants.*

Proof. Let $P \in (R_{K/k}X)(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, and let $(Y, G) \in \mathscr{C}ov(X)$. By assumption, P survives $(R_{K/k}Y, R_{K/k}G) \in \mathscr{C}ov(R_{K/k}X)$, so by Lemma 5.3, part (4), P also survives (Y, G). Since (Y, G) was arbitrary, $P \in X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$, so the left hand side is contained in the right hand side.

For the proof of the reverse inclusion, we can reduce to the case that X is K-connected, by Proposition 5.11. If X is K-connected, but not geometrically connected, then $(R_{K/k}X)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_K)_\bullet = \varnothing$, and there is nothing to prove. So we can assume that X is geometrically connected. Take $P \in X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$. Then by Lemma 5.7, there is a cofinal family (Y_i, G_i) of coverings of X. We show that $(R_{K/k}Y_i, R_{K/k}G_i)$ is then a cofinal family of coverings of $R_{K/k}X$. Indeed, it is known that $\overline{R_{K/k}X} \cong \bar{X}^{[K:k]}$ (with the factors coming from the various embeddings of K into $\bar{k}$), so $\pi_1(\overline{R_{K/k}X}) \cong \pi_1(\bar{X})^{[K:k]}$. This easily implies the claim. Now, viewing P as an element of $(R_{K/k}X)(\mathbb{A}_k)_\bullet$, we see by Lemma 5.3 that P survives every $(R_{K/k}Y_i, R_{K/k}G_i)$, and hence $P \in (R_{K/k}X)(\mathbb{A}_k)_\bullet^{\text{f-cov}}$.

The same proof works for the solvable and abelian variants. □

Proposition 5.16. *If K/k is a finite extension, then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet \cap X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$$

and similarly for the solvable and abelian variants. Note that the intersection is to be interpreted as the pullback of $X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$ under the canonical map $X(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_K)_\bullet$, which may not be injective at the infinite places.

Proof. We have a morphism $X \to R_{K/k}X_K$, inducing the canonical map

$$X(\mathbb{A}_k)_\bullet \longrightarrow (R_{K/k}X_K)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_K)_\bullet\,.$$

The claim now follows from combining Propositions 5.9 and 5.15. □

We also have an analogue of the Descent Theorem 5.1.

Proposition 5.17. *Let $(Y, G) \in \mathcal{C}ov(X)$ be an X-torsor. Then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcup \pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet^{\text{f-cov}})\,,$$

where the union is extended over all twists (Y_ξ, G_ξ) of (Y, G), or equivalently, over the finite set of twists with points everywhere locally. A similar statement holds for the solvable variant, when G is solvable.

Proof. Note first that by Proposition 5.9, the right hand side is a subset of the left hand side.

For the reverse inclusion, take $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. To ease notation, we will suppress the group schemes when denoting torsors in the following. Let $Y_1, \dots, Y_s \in \mathcal{C}ov(X)$ (or $\mathcal{S}ol(X)$) be the finitely many twists of Y such that P lifts.

Define $\tau(j) \subset \{1, \dots, s\}$ to be the set of indices i such that for every X-torsor Z mapping to Y_j (or an X-torsor Z over Y_j for short), there is a twist Z_ξ that lifts P and induces a twist of Y_j that is isomorphic to Y_i. We make a number of claims about this function.

(i) *$\tau(j)$ is nonempty.* To see this, note first that for any given Z, the corresponding set (call it $\tau(Z)$) is nonempty, since by assumption P must lift to some twist of Z, and this twist induces a twist of Y_j to which P also lifts, and hence this twist must be one of the Y_i. Second, if Z maps to Z' (as X-torsors over Y_j), we have

$\tau(Z) \subset \tau(Z')$. Third, for every pair of X-torsors Z and Z' over Y_j, their relative fiber product $Z \times_{Y_j} Z'$ maps to both of them. Taking these together, we see that $\tau(j)$ is a filtered intersection of nonempty subsets of a finite set and hence nonempty.

(ii) *If $i \in \tau(j)$, then $\tau(i) \subset \tau(j)$.* Let $h \in \tau(i)$, and let Z be an X-torsor over Y_j. By definition of $\tau(j)$, there is a twist Z_ξ of Z lifting P and inducing the twist Y_i of Y_j. Now by definition of $\tau(i)$, there is a twist $(Z_\xi)_\eta$ of Z_ξ lifting P and inducing the twist Y_h of Y_i. By transitivity of twists, this means that we have a twist of Z lifting P and inducing the twist Y_h of Y_j. Since Z was arbitrary, this shows that $h \in \tau(j)$.

(iii) *For some j, we have $j \in \tau(j)$.* Indeed, selecting for each j some $\sigma(j) \in \tau(j)$ (this is possible by (i)), the map σ will have a cycle: $\sigma^m(j) = j$ for some $m \geq 1$ and j. Then by (ii), it follows that $j \in \tau(j)$.

For this specific value of j, we have therefore proved that every X-torsor Z over Y_j has a twist that lifts P and induces the trivial twist of Y_j. This means in particular that this twist is also a twist of Z as a Y_j-torsor.

Now assume that P does not lift to $Y_j(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (or $Y_j(\mathbb{A}_k)_\bullet^{\text{f-sol}}$). Since the preimages of P in $Y_j(\mathbb{A}_k)_\bullet$ form a compact set and since surviving a torsor is a closed condition, we can find a Y_j-torsor V that is not survived by any of the preimages of P. We can then find an X-torsor Z mapping to V, staying in $\mathscr{S}ol$ when working in that category. (Note that this step does not work for $\mathscr{A}b$, since extensions of abelian groups need not be abelian again.) But by what we have just proved, Z has a twist as a Y_j-torsor that lifts a preimage of P, a contradiction. Hence our assumption that P does not lift to $Y_j(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (or $Y_j(\mathbb{A}_k)_\bullet^{\text{f-sol}}$) must be false. $\square$

Remark 5.18. The analogous statement for $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and G abelian is not true in general: it would follow that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$, but Skorobogatov (see [Skorobogatov 2001, § 8] or [Skorobogatov 1999]) has a celebrated example of a surface X such that

$$\emptyset = X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

In fact, there is an abelian covering $\pi : Y \to X$ such that $\bigcup_\xi \pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet^{\text{f-ab}}) = \emptyset$, which therefore gives a counterexample to the abelian version of the statement.

Skorobogatov shows that the "Brauer set" $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ is nonempty. In a later paper, Harari and Skorobogatov [2002, § 5.1] show that there exists an obstruction coming from a nilpotent, nonabelian covering (arising from an abelian covering of Y). The latter means that $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \emptyset$, whereas the former implies that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \emptyset$, since $X(\mathbb{A}_k)_\bullet^{\text{Br}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$; see Section 7 below. The interest in this result comes from the fact that it is the first example known of a variety where there is no Brauer–Manin obstruction, yet there are no rational points.

6. Finite descent conditions and rational points

The ultimate goal behind considering the sets cut out in the adelic points by the various covering conditions is to obtain information on the rational points. There is a three-by-three matrix of natural statements relating these sets; see the diagram below. Here, $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

$$\begin{tikzcd}
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(k)} \arrow[r, Rightarrow] & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \overline{X(k)}} \arrow[r, Rightarrow] & \boxed{X(k) = \varnothing \iff X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \varnothing} \\
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(k)} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \overline{X(k)}} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{X(k) = \varnothing \iff X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \varnothing} \arrow[u, Rightarrow] \\
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \overline{X(k)}} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{X(k) = \varnothing \iff X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing} \arrow[u, Rightarrow]
\end{tikzcd} \tag{6–1}$$

We have the implications shown. If $X(k)$ is finite, then we obviously have $X(k) = \overline{X(k)}$, and the corresponding statements in the left and middle columns are equivalent. In particular, this is the case when X is a curve of genus at least 2.

Let us discuss these statements. The ones in the middle column are perhaps the most natural ones, whereas the ones in the left column are better suited for proofs (as we will see below). The statements in the right column can be considered as variants of the Hasse Principle; in some sense they state that the Hasse Principle will eventually hold if one allows oneself to replace X by finite étale coverings. Note that the weakest of the nine statements (the one in the upper right corner), if valid for a class of varieties, would imply that there is an effective procedure to decide whether there are k-rational points on a variety X within that class or not: at least in principle, we can list all the X-torsors and for each torsor compute the finite set of twists with points everywhere locally. If this set is empty, we know that $X(k) = \varnothing$. In order to obtain the torsors, we can for example enumerate all finite extensions of the function field of X (assuming that X is geometrically connected, say) and check whether such an extension corresponds to an étale covering of X that is a torsor under a finite group scheme. On the other hand, we can search for k-rational points on X at the same time, and as soon as we find one such point, we know that $X(k) \neq \varnothing$. The statement $X(k) = \varnothing \iff X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \varnothing$ guarantees that one of the two events must occur. (Note that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ can be written as a filtered intersection of compact subsets of $X(\mathbb{A}_k)_\bullet$, each coming from one specific torsor, so if $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \varnothing$, then already one of these conditions will provide an obstruction.)

For X of dimension at least two, none of these statements can be expected to hold in general. For example, a rational surface X has trivial geometric fundamental group, and so $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$. On the other hand, there are examples known of such surfaces that violate the Hasse principle, so we have $\varnothing = X(k) \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$. The first example (a smooth cubic surface) was given by Swinnerton-Dyer [1962]. There are also examples among smooth diagonal cubic surfaces, see [Cassels and Guy 1966], and in [Colliot-Thélène et al. 1980], an infinite family of rational surfaces violating the Hasse principle is given.

Let us give names to the properties in the left two columns in the diagram (6–1) above.

Definition 6.1. Let X be a smooth projective k-variety. We call X

(1) *good with respect to all coverings* or simply *good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,

(2) *good with respect to solvable coverings* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$,

(3) *good with respect to abelian coverings* or *very good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$,

(4) *excellent with respect to all coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,

(5) *excellent with respect to solvable coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$,

(6) *excellent with respect to abelian coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Now let us look at curves in more detail. When C is a curve of genus 0, then it satisfies the Hasse Principle, so

$$C(\mathbb{A}_k)_\bullet = \varnothing \iff C(k) = \varnothing\,,$$

and then all the intermediate sets are equal and empty. On the other hand, when $C(k) \neq \varnothing$, then $C \cong \mathbb{P}^1$, and $C(k)$ is dense in $C(\mathbb{A}_k)_\bullet$, so

$$\overline{C(k)} = C(\mathbb{A}_k)_\bullet^{\text{f-cov}} = C(\mathbb{A}_k)_\bullet^{\text{f-sol}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet\,.$$

So curves of genus 0 are always very good.

Now consider the case of a genus 1 curve. If A is an elliptic curve, or more generally, an abelian variety, then $\pi_1(\bar{A})$ is abelian, so by Lemma 5.8 we have

$$A(\mathbb{A}_k)_\bullet^{\text{f-cov}} = A(\mathbb{A}_k)_\bullet^{\text{f-sol}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}\,.$$

Furthermore, among the abelian coverings, we can restrict to the multiplication-by-n maps $A \xrightarrow{n} A$. (In the terminology used earlier, these coverings are a cofinal family.) This shows that

$$A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \widehat{\mathrm{Sel}}(k, A)\,.$$

Since the cokernel of the canonical map

$$\overline{A(k)} \cong \widehat{A(k)} \longrightarrow \widehat{\mathrm{Sel}}(k, A)$$

is the Tate module of $\text{Ш}(k, A)$, we get the following.

Corollary 6.2. (1) *A is very good if and only if $\text{Ш}(k, A)_{\text{div}} = 0$.*

(2) *A is excellent with respect to abelian coverings if and only if $A(k)$ is finite and $\text{Ш}(k, A)_{\text{div}} = 0$.*

See [Wang 1996] for a discussion of the situation when one works with $A(\mathbb{A}_k)$ instead of $A(\mathbb{A}_k)_\bullet$. Note that Wang's discussion is in the context of the Brauer–Manin obstruction, which is closely related to the "finite abelian" obstruction considered here, as discussed in Section 7 below.

Corollary 6.3. *If $A/\mathbb{Q}$ is a modular abelian variety of analytic rank zero, then A is excellent with respect to abelian coverings. In particular, if $E/\mathbb{Q}$ is an elliptic curve of analytic rank zero, then E is excellent with respect to abelian coverings.*

Proof. In [Kolyvagin 1988; Kolyvagin and Logachëv 1989], it is proved that $A(\mathbb{Q})$ and $\text{Ш}(\mathbb{Q}, A)$ are both finite. Corollary 6.2 then implies that $A(\mathbb{A}_\mathbb{Q})_\bullet^{\text{f-ab}} = A(\mathbb{Q})$.

For elliptic curves $E/\mathbb{Q}$, Wiles [1995], Taylor and Wiles [1995], and Breuil, Conrad, Diamond and Taylor [Breuil et al. 2001] have proved that E is modular, and so the first assertion applies. □

Now let X be a principal homogeneous space for the abelian variety A. If $X(\mathbb{A}_k)_\bullet = \varnothing$, then all statements in (6–1) are trivially true. So assume $X(\mathbb{A}_k)_\bullet \neq \varnothing$, and let $\xi \in \text{Ш}(k, A)$ denote the element corresponding to X. By Lemma 5.8, we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}},$$

and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing$ if and only if $\xi \notin \text{Ш}(k, A)_{\text{div}}$. So for $\xi \neq 0$, X is very good if and only if $\xi \notin \text{Ш}(k, A)_{\text{div}}$ (since $X(k) = \varnothing$ in this case).

For curves C of genus 2 or higher, we always have that $C(k)$ is finite, and so the statements in the left and middle columns in (6–1) are equivalent. In this case, we can characterize the set $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in a different way.

Theorem 6.4. *Let C be a smooth projective geometrically connected curve over k. Let $A = \mathrm{Alb}_C^0$ be its Albanese variety, and let $V = \mathrm{Alb}_C^1$ be the torsor under A that parametrizes classes of zero-cycles of degree 1 on C. Then there is a canonical map $\phi : C \to V$, and we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

Of course, since C is a curve, A is the same as the Jacobian variety $\mathrm{Jac}_C = \mathrm{Pic}_C^0$, and V is its torsor Pic_C^1, parametrizing divisor classes of degree 1 on C.

Proof. We know by Proposition 5.9 that $\phi(C(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. It therefore suffices to prove that $\phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

By [Serre 1988, § VI.2], all (connected) finite abelian unramified coverings of $\bar{C} = C \times_k \bar{k}$ are obtained through pull-back from isogenies into $\bar{V} \cong \bar{A}$. From this, we can deduce that the induced homomorphism $\phi^* : H^1_{\text{ét}}(\bar{V}, \bar{G}) \to H^1_{\text{ét}}(\bar{C}, \bar{G})$ is an isomorphism for all finite abelian k-group schemes G. Since the map ϕ is defined over k, we obtain an isomorphism as k-Galois modules. The spectral sequence associated to the composition of functors $H^0(k, H^0_{\text{ét}}(\bar{V}, -)) = H^0_{\text{ét}}(V, -)$ (and similarly for C) gives a diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & H^1(k, G) \arrow[r] \arrow[d, equal] & H^1_{\text{ét}}(V, G) \arrow[r] \arrow[d, "\phi^*"] & H^0(k, H^1_{\text{ét}}(\bar{V}, \bar{G})) \arrow[r] \arrow[d, "\phi^*", "\cong"'] & H^2(k, G) \arrow[d, equal] \\
0 \arrow[r] & H^1(k, G) \arrow[r] & H^1_{\text{ét}}(C, G) \arrow[r] & H^0(k, H^1_{\text{ét}}(\bar{C}, \bar{G})) \arrow[r] & H^2(k, G).
\end{tikzcd}$$

By the 5-lemma, $\phi^* : H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(C, G)$ is an isomorphism.

Let $P \in C(\mathbb{A}_k)_\bullet$ such that $\phi(P) \in V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and let $(Y, G) \in \mathscr{A}b(C)$. Then by the above, there is $(W, G) \in \mathscr{A}b(V)$ such that Y is the pull-back of W. By assumption, $\phi(P)$ survives (W, G); without loss of generality, (W, G) already lifts $\phi(P)$. (G is abelian, hence equal to all its inner forms.) Then (Y, G) lifts P, so P survives (Y, G). Since (Y, G) was arbitrary, $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. □

Remark 6.5. The result in the preceding theorem will hold more generally for smooth projective geometrically connected varieties X instead of curves C, provided all finite étale abelian coverings of $\bar{X}$ can be obtained as pullbacks of isogenies into the Albanese variety of X. For this, it is necessary and sufficient that the (geometric) Néron–Severi group of X is torsion-free; see [Serre 1988, VI.20].

For arbitrary varieties X, we can define a set $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$ consisting of the adelic points on X surviving all torsors that are pull-backs of V-torsors (where V is the k-torsor under A that receives a canonical map ϕ from X), and then the result above will hold in the form

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}})\,.$$

We trivially have $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\text{Alb}}$.

In particular, we get that $X(\mathbb{A}_k)_\bullet^{\text{Alb}} = X(\mathbb{A}_k)_\bullet$ if X has trivial Albanese variety. For example, this is the case for all complete intersections of dimension at least 2 in some projective space. (By Exercise III.5.5 in [Hartshorne 1977], $H^1(\bar{X}, \mathbb{O}) = 0$ in this case, so the Picard variety and therefore also its dual $\mathrm{Alb}^0(X)$ are trivial.) If in addition NS_X is torsion-free, then $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet$ as well.

Corollary 6.6. *Let C be a smooth projective geometrically connected curve over k. Let A be its Albanese* (*or Jacobian*) *variety, and let $V = \mathrm{Alb}^1_C = \mathrm{Pic}^1_C$ as above.*

(1) *If $C(\mathbb{A}_k)_\bullet = \varnothing$, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k) = \varnothing$.*

(2) *If* $C(\mathbb{A}_k)_\bullet \neq \varnothing$ *and* $V(k) \neq \varnothing$ (*so that* C *has a* k*-rational divisor class of degree* 1), *then there is a* k*-defined embedding* $\phi : C \hookrightarrow A$, *and we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(\widehat{\mathrm{Sel}}(k, A)) .$$

If $Ш(k, A)_{\mathrm{div}} = 0$, *we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(\overline{A(k)}) .$$

(3) *If* $C(\mathbb{A}_k)_\bullet \neq \varnothing$ *and* $V(k) = \varnothing$, *then, using the canonical map* $\phi : C \to V$, *we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}) .$$

Let $\xi \in Ш(k, A)$ *be the element corresponding to* V. *By assumption,* $\xi \neq 0$. *Then if* $\xi \notin Ш(k, A)_{\mathrm{div}}$ (*and so in particular when* $Ш(k, A)_{\mathrm{div}} = 0$), *we have* $C(k) = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing$.

Similar statements are true for more general X *in place of* C, *with* $X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}}$ *in place of* $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Proof. This follows immediately from Theorem 6.4, taking into account the descriptions of $A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and $V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in Corollary 6.2 and the text following it. □

Let X be a smooth projective geometrically connected k-variety, let A be its Albanese variety, and denote by V the k-torsor under A such that there is a canonical map $\phi : X \to V$. (V corresponds to the cocycle class of $\sigma \mapsto [P^\sigma - P] \in A(\bar{k})$ for any point $P \in X(\bar{k})$.) If $V(k) \neq \varnothing$, then V is the trivial torsor, and there is an n-covering of V, that is, a V-torsor under $A[n]$. So the nonexistence of an n-covering of V is an obstruction against rational points on V and therefore on X.

If an n-covering of V exists, we can pull it back to a torsor $(Y, A[n]) \in \mathcal{A}b(X)$, and we will say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives the* n*-covering of* X if it survives $(Y, A[n])$. If there is no n-covering, then by definition no point in $X(\mathbb{A}_k)_\bullet$ survives the n-covering of X. If we denote the set of adelic points surviving the n-covering of X by $X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$, then we have

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = \bigcap_{n \geq 1} X(\mathbb{A}_k)_\bullet^{n\text{-ab}} .$$

In particular, for a curve C, we get

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \bigcap_{n \geq 1} C(\mathbb{A}_k)_\bullet^{n\text{-ab}} .$$

7. Relation with the Brauer–Manin obstruction

In this section, we study the relationship between the finite covering obstructions introduced in Section 5 and the Brauer–Manin obstruction. This latter obstruction was introduced by Manin [1971] in 1970 in order to provide a unified framework to explain violations of the Hasse Principle.

The idea is as follows. Let X be, as usual, a smooth projective geometrically connected k-variety. We then have the (cohomological) Brauer group

$$\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)\,.$$

If K/k is any field extension and $P \in X(K)$ is a K-point of X, then the corresponding morphism $\operatorname{Spec} K \to X$ induces a homomorphism $\phi_P : \mathrm{Br}(X) \to \mathrm{Br}(K)$. If $K = k_v$ is a completion of k, then there is a canonical injective homomorphism

$$\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

(which is an isomorphism when v is a finite place). In this way, we can set up a pairing

$$X(\mathbb{A}_k)_\bullet \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \quad ((P_v), b) \longmapsto \langle (P_v), b\rangle_{Br} = \sum_v \mathrm{inv}_v(\phi_{P_v}(b))\,.$$

By a fundamental result of Class Field Theory, k-rational points on X pair trivially with all elements of $\mathrm{Br}(X)$. This implies that

$$\overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \{P \in X(\mathbb{A}_k)_\bullet : \langle P, b\rangle_{\mathrm{Br}} = 0 \text{ for all } b \in \mathrm{Br}(X)\}\,.$$

The set $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ is called the *Brauer set* of X. If it is empty, one says that there is a *Brauer–Manin obstruction* against rational points on X. More generally, if $B \subset \mathrm{Br}(X)$ is a subgroup (or subset), we can define $X(\mathbb{A}_k)_\bullet^B$ in a similar way as the subset of points in $X(\mathbb{A}_k)_\bullet$ that pair trivially with all $b \in B$.

The main result of this section is that for a curve C, we have

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}\,,$$

see Corollary 7.3 below. This implies that all the results we have deduced or will deduce about finite abelian descent obstructions on curves also apply to the Brauer–Manin obstruction.

We first recall that the (algebraic) Brauer–Manin obstruction is at least as strong as the obstruction coming from finite abelian descent. For a more precise statement, see [Harari and Skorobogatov 2002, Thm. 4.9]. We define

$$\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X \times_k \bar{k})) \subset \mathrm{Br}(X)$$

and set $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1(X)}$.

Theorem 7.1. *For any smooth projective geometrically connected variety X, we have*

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proof. The main theorem of descent theory of Colliot-Thélène and Sansuc [1987], as extended by Skorobogatov (see [Skorobogatov 1999] and [Skorobogatov 2001, Thm. 6.1.1]), states that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1}$ is equal to the set obtained from descent obstructions with respect to torsors under k-groups G of multiplicative type, which includes all finite abelian k-groups. This proves the second inclusion. The first one follows from the definitions. □

It is known that (see [Skorobogatov 2001, Cor. 2.3.9]; use that $H^3(k, \bar{k}^\times) = 0$)

$$\frac{\mathrm{Br}_1(X)}{\mathrm{Br}_0(X)} \cong H^1(k, \mathrm{Pic}_X),$$

where $\mathrm{Br}_0(X)$ denotes the image of $\mathrm{Br}(k)$ in $\mathrm{Br}(X)$. We also have the canonical map $H^1(k, \mathrm{Pic}^0_X) \to H^1(k, \mathrm{Pic}_X)$. Define $\mathrm{Br}_{1/2}(X)$ to be the subgroup of $\mathrm{Br}_1(X)$ that maps into the image of $H^1(k, \mathrm{Pic}^0_X)$ in $H^1(k, \mathrm{Pic}_X)$. (Manin [1971] calls it $\mathrm{Br}_1'(X)$.) In addition, for $n \geq 1$, let $\mathrm{Br}_{1/2,n}(X)$ be the subgroup of $\mathrm{Br}_1(X)$ that maps into the image of $H^1(k, \mathrm{Pic}^0_X)[n]$. Then

$$\mathrm{Br}_{1/2}(X) = \bigcup_{n \geq 1} \mathrm{Br}_{1/2,n}(X), \quad \text{and} \quad X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}} = \bigcap_{n \geq 1} X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}.$$

Recall the definition of $X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}}$ from Remark 6.5 and the fact that

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = \bigcap_{n \geq 1} X(\mathbb{A}_k)_\bullet^{n\text{-ab}}.$$

Theorem 7.2. *Let X be a smooth projective geometrically connected variety, and let $n \geq 1$. Then*

$$X(\mathbb{A}_k)_\bullet^{n\text{-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}.$$

In particular,

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

Proof. Given the first statement, the second statement is clear.

The first statement follows from Theorem 7.5 below. However, since our proof of the inclusion given here is fairly simple, we include it.

So consider $P \in X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$ and $b \in \mathrm{Br}_{1/2,n}(X)$. We have to show that $\langle b, P\rangle_{\mathrm{Br}}$ vanishes, where $\langle \cdot, \cdot \rangle_{\mathrm{Br}}$ is the Brauer pairing between $X(\mathbb{A}_k)_\bullet$ and $\mathrm{Br}(X)$.

Let b' be the image of b in

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong H^1(k, \mathrm{Pic}_X),$$

and let $b'' \in H^1(k, \mathrm{Pic}^0_X)[n]$ be an element mapping to b' (which exists because $b \in \mathrm{Br}_{1/2,n}(X)$).

Let A be the Albanese variety of X, and let V be the k-torsor under A that has a canonical map $\phi : X \to V$. Then we have $\mathrm{Pic}^0_X \cong \mathrm{Pic}^0_A \cong \mathrm{Pic}^0_V$. Since $P \in X(\mathbb{A}_k)^{n\text{-ab}}_\bullet \xrightarrow{\phi} V(\mathbb{A}_k)^{n\text{-ab}}_\bullet$, the latter is nonempty, and hence V admits a torsor of the form $(W, A[n])$.

Since P maps into $V(\mathbb{A}_k)^{n\text{-ab}}_\bullet$, there is some twist of $(W, A[n])$ such that $\phi(P)$ lifts to it. Without loss of generality, $(W, A[n])$ is already this twist, so there is $Q' \in W(\mathbb{A}_k)_\bullet$ such that $\pi'(Q') = \phi(P)$, where $\pi' : W \to V$ is the covering map associated to $(W, A[n])$.

Let $(Y, A[n]) \in \mathscr{A}b(X)$ be the pull-back of $(W, A[n])$ to X. Then there is some $Q \in Y(\mathbb{A}_k)_\bullet$ such that $\pi(Q) = P$. Now the left hand diagram below induces the one on the right, where the rightmost vertical map is the multiplication by n.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & W \arrow[d, "\pi'"] \\ X \arrow[r, "\phi"] & V \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pic}_Y & \mathrm{Pic}^0_Y \arrow[l] & \mathrm{Pic}^0_W \arrow[l] \arrow[r, equal] & \mathrm{Pic}^0_A \\ \mathrm{Pic}_X \arrow[u, "\pi^*"] & \mathrm{Pic}^0_X \arrow[l] \arrow[u, "\pi^*"] & \mathrm{Pic}^0_V \arrow[l, "\cong"'] \arrow[u, "\pi'^*"] \arrow[r, equal] & \mathrm{Pic}^0_A \arrow[u, "\cdot n"'] \end{tikzcd}$$

Chasing b'' around the diagram on the right, after applying $H^1(k, -)$ to it, we see that $\pi^*(b') = 0$ in $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$. Finally, we have

$$\langle b, P\rangle_{\mathrm{Br}} = \langle b', \pi(Q)\rangle_{\mathrm{Br}} = \langle \pi^*(b'), Q\rangle_{\mathrm{Br}} = 0\,.$$

□

So we have the chain of inclusions

$$X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \subset X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet \subset X(\mathbb{A}_k)^{\text{f-ab}}_\bullet \subset X(\mathbb{A}_k)^{\mathrm{Alb}}_\bullet \subset X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet\,.$$

It is then natural to ask to what extent one might have equality in this chain of inclusions. We certainly get something when $\mathrm{Br}_{1/2}(X)$ already equals $\mathrm{Br}_1(X)$ or even $\mathrm{Br}(X)$.

Corollary 7.3. *If X is a smooth projective geometrically connected variety such that the canonical map $H^1(k, \mathrm{Pic}^0_X) \to H^1(k, \mathrm{Pic}_X)$ is surjective, then*

$$X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet = X(\mathbb{A}_k)^{\text{f-ab}}_\bullet = X(\mathbb{A}_k)^{\mathrm{Alb}}_\bullet\,.$$

In particular, if C is a curve, then $C(\mathbb{A}_k)^{\mathrm{Br}}_\bullet = C(\mathbb{A}_k)^{\text{f-ab}}_\bullet$.

Proof. In this case, $\mathrm{Br}_{1/2}(X) = \mathrm{Br}_1(X)$, and so the result follows from the two preceding theorems.

When $X = C$ is a curve, then we know that $\mathrm{Br}(C \times_k \bar{k})$ is trivial (Tsen's Theorem); also $H^1(k, \mathrm{Pic}^0_C)$ surjects onto $H^1(k, \mathrm{Pic}_C)$, since the Néron–Severi group of C is $\mathbb{Z}$ with trivial Galois action, and $H^1(k,\mathbb{Z}) = 0$. Hence $\mathrm{Br}(C) = \mathrm{Br}_{1/2}(C)$, and the assertion follows. □

The result of Corollary 7.3 means that we can replace $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ by $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ everywhere. For example, from Corollary 6.6, we obtain the following.

Corollary 7.4. *Let C be a smooth projective geometrically connected curve over k, and let A be its Albanese (or Jacobian) variety. Assume that $\text{Ш}(k, A)_{\mathrm{div}} = 0$.*

(1) *If C has a k-rational divisor class of degree 1 inducing a k-defined embedding $C \hookrightarrow A$, then*
$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \phi^{-1}(\overline{A(k)}),$$
where ϕ denotes the induced map $C(\mathbb{A}_k)_\bullet \to A(\mathbb{A}_k)_\bullet$.

(2) *If C has no k-rational divisor class of degree 1, then $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \varnothing$.*

These results can be found in Scharaschkin's thesis Scharaschkin [1999]. Our approach provides an alternative proof, and the more precise version in Corollary 6.6 shows how to extend the result to the case when the Shafarevich–Tate group of the Jacobian is not necessarily assumed to have trivial divisible subgroup.

We can strengthen Theorem 7.2.

Theorem 7.5. *Let X be a smooth projective geometrically connected variety. Then*
$$X(\mathbb{A}_k)_\bullet^{n\text{-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}$$
for all $n \geq 1$. In particular,
$$X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

Proof. This follows from the descent theory of Colliot-Thélène and Sansuc. Let $M = \mathrm{Pic}^0_X[n]$, and let $\lambda : M \to \mathrm{Pic}_X$ be the inclusion. Then the n-coverings of X are exactly the torsors of type λ in the language of the theory; compare for example [Skorobogatov 2001]. (Note that the dual of M is $A[n]$, where A is the Albanese variety of X.) We have $\mathrm{Br}_\lambda = \mathrm{Br}_{1/2,n}$, and the result then follows from Thm. 6.1.2,(a) in [Skorobogatov 2001]. □

Remark 7.6. Since $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$, it is natural to ask whether there might be a subgroup $B \subset \mathrm{Br}_1(X)$ such that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{B}$. As Joost van Hamel pointed out to me, a natural candidate for B is the subgroup mapping to the image of $H^1(k,\mathrm{Pic}^\tau_X)$ in $H^1(k,\mathrm{Pic}_X)$, where Pic^τ_X is the saturation of Pic^0_X in Pic_X, that is, the subgroup of elements mapping into the torsion subgroup of the Néron–Severi group NS_X. It is tempting to denote this B by $\mathrm{Br}_{2/3}$, but perhaps Br_τ is the better choice. Note that $\mathrm{Br}_\tau = \mathrm{Br}_{1/2}$ when NS_X is torsion free, in which case we have $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$.

Corollary 7.7. *If C/k is a curve that has a rational divisor class of degree* 1, *then*

$$C(\mathbb{A}_k)_\bullet^{n\text{-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}[n]}\,.$$

In words, the information coming from n-torsion in the Brauer group is exactly the information obtained by an n-descent on C.

Proof. Under the given assumptions, $H^1(k, \operatorname{Pic}_C^0) = H^1(k, \operatorname{Pic}_C) = \operatorname{Br}(C)/\operatorname{Br}(k)$, and $\operatorname{Br}(k)$ is a direct summand of $\operatorname{Br}(C)$. Therefore, the images of $\operatorname{Br}_{1/2,n}(C)$ and of $\operatorname{Br}(C)[n]$ in $\operatorname{Br}(C)/\operatorname{Br}_0(C)$ agree, and the claim follows. □

Corollary 7.8. *If X is a smooth projective geometrically connected variety such that the Néron–Severi group of X (over $\bar{k}$) is torsion-free, then there is a finite field extension K/k such that*

$$X(\mathbb{A}_K)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_K)_\bullet^{\text{f-ab}}\,.$$

Proof. We have an exact sequence

$$H^1(k, \operatorname{Pic}_X^0) \longrightarrow H^1(k, \operatorname{Pic}_X) \longrightarrow H^1(k, \mathrm{NS}_X)\,.$$

Since NS_X is a finitely generated abelian group, the Galois action on it factors through a finite quotient $\operatorname{Gal}(K/k)$ of the absolute Galois group of k. Then

$$H^1(K, \mathrm{NS}_X) = \operatorname{Hom}(G_K, \mathbb{Z}^r) = 0,$$

and the claim follows from Theorem 7.2. □

Note that it is not true in general that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ (even when the Néron–Severi group of X over $\bar{k}$ is torsion-free). For example, a smooth cubic surface X in $\mathbb{P}^3$ has $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$ (since it has trivial geometric fundamental group), but may well have $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = \varnothing$, even though there are points everywhere locally. See [Colliot-Thélène et al. 1987a], where the algebraic Brauer–Manin obstruction is computed for all smooth diagonal cubic surfaces

$$X:\ a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0$$

with integral coefficients $0 < a_i < 100$, thereby verifying that it is the only obstruction against rational points on X (and thus providing convincing experimental evidence that this may be true for smooth cubic surfaces in general). This computation produces a list of 245 such surfaces with points everywhere locally, but no rational points, since $X(\mathbb{A}_\mathbb{Q})_\bullet^{\mathrm{Br}_1} = \varnothing$.

It is perhaps worth mentioning that our condition that $H^1(k, \operatorname{Pic}_X^0)$ surjects onto $H^1(k, \operatorname{Pic}_X)$, which leads to the identification of the "algebraic Brauer–Manin obstruction" and the "finite abelian descent obstruction", is in some sense orthogonal to the situation studied (quite successfully) in [Colliot-Thélène and Sansuc 1987; Colliot-Thélène et al. 1980; 1987b], where it is assumed that Pic_X is torsion-free (and therefore Pic^0_X is trivial), and so there can be a Brauer–Manin obstruction only when our condition fails. There is then no finite abelian descent obstruction, and one has to look at torsors under tori instead.

In general, we have a diagram of inclusions:

$$\begin{tikzcd}[column sep=small] & & X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \arrow[r, phantom, "\subset"] & X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet \arrow[dr, hook] & & & \\ X(k) \arrow[r, phantom, "\subset"] & \overline{X(k)} \arrow[ur, hook] \arrow[dr, hook] & & & X(\mathbb{A}_k)^{\text{f-ab}}_\bullet \arrow[r, phantom, "\subset"] & X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet \arrow[r, phantom, "\subset"] & X(\mathbb{A}_k)_\bullet\,. \\ & & X(\mathbb{A}_k)^{\text{f-cov}}_\bullet \arrow[r, phantom, "\subset"] & X(\mathbb{A}_k)^{\text{f-sol}}_\bullet \arrow[ur, hook] & & & \end{tikzcd}$$

We expect that every inclusion can be strict. We discuss them in turn.

(1) $X = \mathbb{P}^1$ has $X(k) \subsetneq \overline{X(k)} = X(\mathbb{A}_k)_\bullet$.

(2) Skorobogatov's famous example (see [Skorobogatov 1999; Harari and Skorobogatov 2002]) has $X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \neq \varnothing$, but $X(\mathbb{A}_k)^{\text{f-sol}}_\bullet = \varnothing$, showing that $\overline{X(k)} \subsetneq X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet$ and $X(\mathbb{A}_k)^{\text{f-sol}}_\bullet \subsetneq X(\mathbb{A}_k)^{\text{f-ab}}_\bullet$ are both possible.

(3) As mentioned above, Colliot-Thélène et al. [1987a] have examples such that $X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet = \varnothing$, but $X(\mathbb{A}_k)^{\text{f-cov}}_\bullet = X(\mathbb{A}_k)_\bullet$. This shows that $\overline{X(k)} \subsetneq X(\mathbb{A}_k)^{\text{f-cov}}_\bullet$ and $X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet \subsetneq X(\mathbb{A}_k)^{\text{f-ab}}_\bullet$ are both possible.

(4) [Harari 1996] has examples, where there is a "transcendental", but no "algebraic" Brauer–Manin obstruction, which means that $X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet = \varnothing$, but $X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet \neq \varnothing$. Hence we can have $X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \subsetneq X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet$.

(5) If we take a finite nonabelian simple group for $\pi_1(\bar{X})$ in Cor. 6.1 in [Harari 2000], then the proof of this result shows that $X(\mathbb{A}_k)^{\text{f-cov}}_\bullet \subsetneq X(\mathbb{A}_k)_\bullet$. On the other hand, $X(\mathbb{A}_k)^{\text{f-sol}}_\bullet = X(\mathbb{A}_k)_\bullet$, since there are only trivial torsors in $\mathcal{S}ol(X)$; compare Lemma 5.8.

(6) A construction using Enriques surfaces like that in [Harari and Skorobogatov 2005] should produce an example such that $X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet = X(\mathbb{A}_k)^{\mathrm{Alb}}_\bullet = X(\mathbb{A}_k)_\bullet$, since the Albanese variety is trivial, but $X(\mathbb{A}_k)^{\text{f-ab}}_\bullet \subsetneq X(\mathbb{A}_k)_\bullet$, since there is a nontrivial abelian covering.

(7) Finally, in Section 8 below, we will see many examples of curves X that have $X(k) = X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet \subsetneq X(\mathbb{A}_k)_\bullet$.

A new obstruction? For curves, we expect the interesting part of the diagram of inclusions above to collapse: $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet$; see the discussion in Section 9 below. For higher-dimensional varieties, this is far from true; see the discussion above. So one could consider a new obstruction obtained from a combination of the Brauer–Manin and the finite descent obstructions, as follows. Define

$$X(\mathbb{A}_k)^{\text{f-cov},\mathrm{Br}}_\bullet = \bigcap_{(Y,G)\in\mathcal{C}ov(X)} \; \bigcup_{\xi\in H^1(k,G)} \pi_\xi\bigl(Y_\xi(\mathbb{A}_k)^{\mathrm{Br}}_\bullet\bigr)\,.$$

(This is similar in spirit to the "refinement of the Manin obstruction" introduced in [Skorobogatov 1999].)

It would be interesting to find out how strong this obstruction is and whether it is strictly weaker than the obstruction obtained from *all* torsors under (not necessarily finite or abelian) k-group schemes. Note that the latter is at least as strong as the Brauer–Manin obstruction by [Harari and Skorobogatov 2002, Thm. 4.10] (see also Prop. 5.3.4 in [Skorobogatov 2001]), at least if one assumes that all elements of $\mathrm{Br}(X)$ are represented by Azumaya algebras over X.

8. Finite descent conditions on curves

Let us now prove some general properties of the notions, introduced in Section 6 above, of being excellent with respect to all, solvable, or abelian coverings in the case of curves. In the following, C, D, and so forth, will be (smooth projective geometrically connected) curves over k. We will use ι to denote an embedding of C into its Jacobian (if it exists). Also, if $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \varnothing$ (and therefore $C(k) = \varnothing$, too), we say that *the absence of rational points is explained by the Brauer–Manin obstruction.* Note that by Corollary 7.3, $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, which implies that the absence of rational points is explained by the Brauer–Manin obstruction when C is excellent with respect to abelian coverings and $C(k) = \varnothing$. We will use this observation below without explicit mention.

Corollary 8.1. *Let C/k be a curve of genus at least* 1, *with Jacobian J. Assume that $\text{Ш}(k, J)_{\mathrm{div}} = 0$ and that $J(k)$ is finite. Then C is excellent with respect to abelian coverings. If $C(k) = \varnothing$, then the absence of rational points is explained by the Brauer–Manin obstruction.*

Proof. By Corollary 6.6, under the assumption on $\text{Ш}(k, J)$, either $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing$, and there is nothing to prove, or else

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\overline{J(k)}) = \iota^{-1}(J(k)) = C(k)\,.$$

∎

The following result shows that the statement we would like to have, namely that $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$, holds for finite subschemes of a curve.

Theorem 8.2. *Let C/k be a curve of genus at least* 1, *and let $Z \subset C$ be a finite subscheme. Then the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$ meets $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in $Z(k)$. More generally, if $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ is such that $P_v \in Z(k_v)$ for a set of places v of k of density* 1, *then $P \in Z(k)$.*

Proof. Let K/k be a finite extension such that C has a rational divisor class of degree 1 over K. By Corollary 6.6, we have that

$$C(\mathbb{A}_K)_\bullet^{\text{f-ab}} = \iota^{-1}(\widehat{\mathrm{Sel}}(K, J))\,,$$

where $\iota : C(\mathbb{A}_K)_\bullet \to J(\mathbb{A}_K)_\bullet$ is the map induced by an embedding $C \hookrightarrow J$ over K. Now we apply Theorem 3.11 to the image of Z in J. We find that $\iota(P) \in \widehat{\mathrm{Sel}}(K, J)$ and so $\iota(P) \in \iota(Z(K))$. Since ι is injective (even at the infinite places!), we find that the image of P in $C(\mathbb{A}_K)_\bullet$ is in (the image of) $Z(k)$. Now if $Z(k)$ is empty, this gives a contradiction and proves the claim in this case. Otherwise, $C(k) \supset Z(k)$ is nonempty, and we can take $K = k$ above, which gives the statement directly. □

The following results show that the "excellence properties" behave nicely.

Proposition 8.3. *Let K/k be a finite extension, and let C/k be a curve of genus at least 1. If C_K is excellent with respect to all, solvable, or abelian coverings, then so is C.*

Proof. By Proposition 5.16, we have

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset C(\mathbb{A}_k)_\bullet \cap C(\mathbb{A}_K)_\bullet^{\text{f-cov}} = C(\mathbb{A}_k)_\bullet \cap C(K) = C(k)\,.$$

Similarly for $C(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. Strictly speaking, this means that $C(k)$ and $C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ have the same image in $C(\mathbb{A}_K)_\bullet$. Now, since $C(K)$ has to be finite in order to equal $C(\mathbb{A}_K)_\bullet^{\text{f-cov}}$, $C(k)$ is also finite, and we can apply Theorem 8.2 to $Z = C(k) \subset C$ and the set of finite places of k. □

Proposition 8.4. *Let $(D, G) \in \mathscr{C}ov(C)$ (or $\mathscr{S}ol(C)$). If all twists D_ξ of (D, G) are excellent with respect to all (respectively, solvable) coverings, then C is excellent with respect to all (respectively, solvable) coverings.*

Proof. By Theorem 5.1, $C(k) = \coprod_\xi \pi_\xi(D_\xi(k))$. Now, by Proposition 5.17,

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcup_\xi \pi_\xi(D_\xi(\mathbb{A}_k)_\bullet^{\text{f-cov}}) = \bigcup_\xi \pi_\xi(D_\xi(k)) = C(k)\,.$$

If G is solvable, the same proof shows the statement for $C(\mathbb{A}_k)_\bullet^{\text{f-sol}}$. □

Proposition 8.5. *Let $C \xrightarrow{\phi} X$ be a nonconstant morphism over k from the curve C into a variety X. If X is excellent with respect to all, solvable, or abelian coverings, then so is C. In particular, if $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)$ and $C(k) = \emptyset$, then the absence of rational points on C is explained by the Brauer–Manin obstruction.*

Proof. First assume that C is of genus zero. Then either $C(\mathbb{A}_k)_\bullet = \emptyset$, and there is nothing to prove, or else $C(k)$ is dense in $C(\mathbb{A}_k)_\bullet$, implying that $X(k) \subsetneq \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and thus contradicting the assumption.

Now assume that C is of genus at least 1. Let $P \in C(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}}$. Then by Proposition 5.9, $\phi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}} = X(k)$. Let $Z \subset C$ be the preimage (subscheme) of $\phi(P) \in X(k)$ in C. This is finite, since ϕ is nonconstant. Then we have that P is in the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$. Now apply Theorem 8.2 to conclude that $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}} \cap Z(\mathbb{A}_k)_\bullet = Z(k) \subset C(k)$. □

As an application, we have the following.

Theorem 8.6. *Let $C \to A$ be a nonconstant morphism over k of a curve C into an abelian variety A. Assume that $Ш(k, A)_{\mathrm{div}} = 0$ and that $A(k)$ is finite.* (*For example, this is the case when $k = \mathbb{Q}$ and A is modular of analytic rank zero.*) *Then C is excellent with respect to abelian coverings. In particular, if $C(k) = \varnothing$, then the absence of rational points on C is explained by the Brauer–Manin obstruction.*

Proof. By Corollary 6.2, we have $A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = A(k)$. Now by Proposition 8.5, the claim follows. □

This generalizes a result proved by Siksek [2004] under additional assumptions on the Galois action on the fibers of ϕ above k-rational points of A, in the case that $C(k)$ is empty. A similar observation was made independently by Colliot-Thélène [2004]. Note that both previous results are in the context of the Brauer–Manin obstruction.

Examples 8.7. We can use Theorem 8.6 to produce many examples of curves C over $\mathbb{Q}$ that are excellent with respect to abelian coverings. Concretely, let us look at the curves $C_a : y^2 = x^6 + a$, where a is a nonzero integer. C_a maps to the two elliptic curves $E_a : y^2 = x^3 + a$ and E_{a^2} (the latter by sending (x, y) to $(a/x^2, ay/x^3)$). So whenever one of these elliptic curves has (analytic) rank zero, we know that C_a is excellent with respect to abelian coverings. For example, this is the case for all a such that $|a| \leq 20$, with the exception of $a = -15, -13, -11, 3, 10, 11, 15, 17$. Note that $C_a(\mathbb{Q})$ is always nonempty (there are two rational points at infinity).

We can even show a whole class of interesting curves to be excellent with respect to abelian coverings.

Corollary 8.8. *If $C/\mathbb{Q}$ is one of the modular curves $X_0(N)$, $X_1(N)$, $X(N)$ and such that the genus of C is positive, then C is excellent with respect to abelian coverings.*

Proof. By a result of Mazur [1977], every Jacobian $J_0(p)$ of $X_0(p)$, where $p = 11$ or $p \geq 17$ is prime, has a nontrivial factor of analytic rank zero. Also, if $M \mid N$, then there are nonconstant morphisms $X_1(N) \to X_0(N) \to X_0(M)$. Hence the assertion is true for all $X_0(N)$ and $X_1(N)$ such that N is divisible by one of the primes in Mazur's result. For the other minimal N such that $X_0(N)$ (respectively, $X_1(N)$) is of positive genus, William Stein's tables [$\geq$ 2007] prove that there is a factor of $J_0(N)$ (respectively, $J_1(N)$) of analytic rank zero. So we get the result for all $X_0(N)$ and $X_1(N)$ of positive genus. Finally, $X(N)$ maps to $X_0(N^2)$, and so we obtain the result also for $X(N)$ (except in the genus zero cases $N = 1, 2, 3, 4, 5$). □

For another example, involving high-genus Shimura curves, see [Skorobogatov 2005].

Remark 8.9. There is some relation with the "Section Conjecture" from Grothendieck's anabelian geometry [Grothendieck 1997]. Let C/k be a smooth projective geometrically connected curve of genus ≥ 2. One can prove that if C has the "section property", then C is excellent with respect to all coverings, which in turn implies that C has the "birational section property". See [Koenigsmann 2005] for definitions. For example, all the curves $X_0(N)$, $X_1(N)$ and $X(N)$ have the birational section property if they are of higher genus.

9. Discussion

In the preceding section, we have seen that we can construct many examples of higher-genus curves that are excellent with respect to abelian coverings. This leads us to the following conjecture.

Conjecture 9.1 (Main Conjecture). *If C is a smooth projective geometrically connected curve over a number field k, then C is very good.*

By what we have seen, for curves of genus 1, this is equivalent to saying that the divisible subgroup of $Ш(k, E)$ is trivial, for every elliptic curve E over k. For curves C of higher genus, the statement means that C is excellent with respect to abelian coverings. We recall that our conjecture would follow in this case from the "Adelic Mordell–Lang Conjecture" formulated in Question 3.12.

Remark 9.2. When k is a global function field of characteristic p, then the Main Conjecture holds when $J = \mathrm{Jac}_C$ has no isotrivial factor and $J(k^{\mathrm{sep}})[p^\infty]$ is finite. See recent work by Poonen and Voloch [2006].

If the Main Conjecture holds for C and $C(k)$ is empty, then (as previously discussed) we can find a torsor that has no twists with points everywhere locally and thus *prove* that $C(k)$ is empty. The validity of the conjecture (even just in case $C(k)$ is empty) therefore implies that *we can algorithmically decide whether a given smooth projective geometrically connected curve over a number field k has rational points or not.*

In Section 7 above, we have shown that for a curve C, we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}},$$

where on the right hand side, we have the *Brauer subset* of $C(\mathbb{A}_k)_\bullet$, that is, the subset cut out by conditions coming from the Brauer group of C. We say that, if $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \varnothing$, there is a *Brauer–Manin obstruction* against rational points on C. A corollary of our Main Conjecture is that the Brauer–Manin obstruction is the only obstruction against rational points on curves over number fields (which means that $C(k) = \varnothing$ implies $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \varnothing$). To our knowledge, before this work (and Poonen's heuristic, see his conjecture below, which was influenced by the discussions we had at the IHP in Paris in Fall 2004) nobody gave a conjecturally positive answer to the question, first formulated on page 133 in [Skorobogatov 2001], whether the Brauer–Manin obstruction might be the only obstruction against rational points on curves. No likely counterexample is known, but there is an ever-growing list of examples, for which the failure of the Hasse Principle could be explained by the Brauer–Manin obstruction; see the discussion below (which does not pretend to be exhaustive) or also Skorobogatov's recent paper Skorobogatov [2005] on Shimura curves.

Let v be a place of k. Under a *local condition at* v on a rational point $P \in C$, we understand the requirement that the image of P in $C(k_v)$ is contained in a specified closed and open ("clopen") subset of $C(k_v)$. If v is an infinite place, this just means that we require P to be on some specified connected component(s) of $C(k_v)$; for finite places, we can take something like a "residue class". With this notion, the Main Conjecture above is equivalent to the following statement.

Let C/k be a curve as above. Specify local conditions at finitely many places of k and assume that there is no point in $C(k)$ satisfying these conditions. Then there is some $n \geq 1$ such that no point in $\prod_v X_v \subset C(\mathbb{A}_k)_\bullet$ survives the n-covering of C, where X_v is the set specified by the local condition at those places where a condition is specified, and $X_v = C(k_v)$ (or $\pi_0(C(k_v))$) otherwise.

This says that the "finite abelian" obstruction (equivalently, the Brauer–Manin obstruction) is the only obstruction against weak approximation in $C(\mathbb{A}_k)_\bullet$.

We see that the conjecture implies that we can decide if a given finite collection of local conditions can be satisfied by a rational point. Now the question is how practical it might be to actually do this in concrete cases. For certain classes of curves and specific values of n, it may be possible to explicitly and efficiently find the relevant twists. For example, this can be done for hyperelliptic curves and $n = 2$; compare [Bruin and Stoll 2007a]. However, for general curves and/or general n, this approach is likely to be infeasible.

On the other hand, assume that we can find $J(k)$ explicitly, where J, as usual, is the Jacobian of C. This is the case (at least in principle) when $Ш(k, J)_{\text{div}} = 0$. Then we can approximate $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ more and more precisely by looking at the images of $C(\mathbb{A}_k)_\bullet$ and of $J(k)$ in $\prod_{v \in S} J(k_v)/NJ(k_v)$ for increasing N and finite sets S of places of k. If $C(k)$ is empty and the Main Conjecture holds, then for some choice of S and N, the two images will not intersect, giving an explicit proof that $C(k) = \varnothing$. An approach like this was proposed (and carried out for some twists of the Fermat quartic) by Scharaschkin [1999]. See [Flynn 2004] for an implementation of this method and [Bruin and Stoll 2007b] for improvements. In [Poonen et al. 2007], this procedure is used to rule out rational points satisfying certain local conditions on a genus 3 curve whose Jacobian has Mordell–Weil rank 3.

In order to test the conjecture, Nils Bruin and the author conducted an experiment; see [Bruin and Stoll 2006]. We considered all genus 2 curves over $\mathbb{Q}$ of the form

$$y^2 = f_6 x^6 + f_5 x^5 + \cdots + f_1 x + f_0 \tag{9–1}$$

with coefficients $f_0, \ldots, f_6 \in \{-3, -2, \ldots, 3\}$. For each isomorphism class of curves thus obtained, we attempted to decide if there are rational points or not. On about 140,000 of these roughly 200,000 curves (up to isomorphism), we found a (fairly) small rational point. Of the remaining about 60,000, about half failed to have local points at some place. On the remaining about 30,000 curves, we performed a 2-descent and found that for all but 1,492 curves C, $C(\mathbb{A}_{\mathbb{Q}})_\bullet^{2\text{-ab}} = \varnothing$, proving that $C(\mathbb{Q}) = \varnothing$ as well. For the 1,492 curves that were left over, we found generators of the Mordell–Weil group (assuming the Birch and Swinnerton-Dyer Conjecture for a small number of them) and then did a computation along the lines sketched above. This turned out to be successful for *all* curves, proving that none of them has a rational point. The conclusion is that the Main Conjecture holds for curves C as in (9–1) if $C(\mathbb{Q}) = \varnothing$, assuming $\text{Ш}(\mathbb{Q}, J)_{\text{div}} = 0$ for the Jacobian J if C is one of the 1,492 curves mentioned, and assuming in addition the Birch and Swinnerton-Dyer Conjecture if C is one of 42 specific curves.

At least in case $C(k)$ is empty, there are heuristic arguments due to Poonen [2006] that suggest that an even stronger form of our conjecture might be true.

Conjecture 9.3 (Poonen). *Let C be a smooth projective geometrically connected curve of genus ≥ 2 over a number field k, and assume that $C(k) = \varnothing$. Assume further that C has a rational divisor class of degree 1, and let $\iota : C \to J$ be the induced embedding of C into its Jacobian J. Then there is a finite set S of finite places of good reduction for C such that the image of $J(k)$ in $\prod_{v \in S} J(\mathbb{F}_v)$ does not meet $\prod_{v \in S} \iota(C(\mathbb{F}_v))$.*

Note that under the assumption $\text{Ш}(k, J)_{\text{div}} = 0$, we must have a rational divisor (class) of degree 1 on C whenever $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \varnothing$, compare Corollary 6.6, so the condition above is not an essential restriction.

Let us for a moment assume that Poonen's Conjecture holds and that all abelian varieties A/k satisfy $\text{Ш}(k, A)_{\text{div}} = 0$. Then for all curves C/k of higher genus, $C(k) = \varnothing$ implies $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing$. If we apply this observation to coverings of C, then we find that C must be excellent with respect to solvable coverings. The argument goes like this. Let $P \in C(\mathbb{A}_k)_\bullet^{\text{f-sol}}$, and assume $P \notin C(k)$. There are only finitely many rational points on C, and hence there is an n such that P lifts to a different n-covering D of C than all the rational points. (Take n such that $P - Q$ is not divisible by n in $J(\mathbb{A}_k)_\bullet$, for all $Q \in C(k)$, where J is the Jacobian of C.) In particular, $D(k)$ must be empty. But then, by Poonen's Conjecture, we have

$D(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \varnothing$, so P cannot lift to D either. This contradiction shows that P must be a rational point.

In particular, this would imply that all higher-genus curves have the "birational section property"; compare Remark 8.9.

A more extensive and detailed discussion of these conjectures, their relations to other conjectures, and evidence for them will be published elsewhere.

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Del Pezzo surfaces and representation theory

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To Yuri Ivanovich Manin on his seventieth birthday

The connection between del Pezzo surfaces and root systems goes back to Coxeter and Du Val, and was given modern treatment by Manin in his seminal book *Cubic forms*. Batyrev conjectured that a universal torsor on a del Pezzo surface can be embedded into a certain projective homogeneous space of the semisimple group with the same root system, equivariantly with respect to the maximal torus action. Computational proofs of this conjecture based on the structure of the Cox ring have been given recently by Popov and Derenthal. We give a new proof of Batyrev's conjecture using an inductive process, interpreting the blowing-up of a point on a del Pezzo surface in terms of representations of Lie algebras corresponding to Hermitian symmetric pairs.

Introduction

Del Pezzo surfaces, classically defined as smooth surfaces of degree d in the projective space $\mathbb{P}^d$, $d \geq 3$, are among the most studied and best understood algebraic varieties. Over an algebraically closed ground field such a surface is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ or the projective plane $\mathbb{P}^2$ with $r = 9 - d$ points in general positions blown up. In this definition, d can be any integer between 1 and 9. Despite the apparent simplicity the enumerative geometry of these surfaces displays amazing symmetries and puzzling coincidences. The 27 lines on a smooth cubic surface were discovered by Cayley and Salmon, and the symmetries of their configurations were explored by Schoutte, Coxeter and Du Val. Manin [1986] gave a modern exposition of this subject, with many geometric and arithmetic applications. He showed that the Picard group of a del Pezzo surface X of degree $d = 9 - r$, where $d \leq 6$, contains a root system R_r of rank r in such a way that the automorphism group of the incidence graph of the exceptional curves on X is the Weyl group $\mathrm{W}(\mathrm{R}_r)$. These root systems are embedded into one another: $\mathrm{R}_8 = \mathrm{E}_8$, and as r decreases one chops one by one the nodes off the long end of the Dynkin diagram of E_8, until the diagram

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becomes disconnected. Let α_r be the simple root of R_r corresponding to the node which must be removed from the Dynkin diagram of R_r in order to obtain that of R_{r-1}; let ω_r be the fundamental weight dual to α_r. For $r = 4, 5, 6, 7$ the number of exceptional curves on X is $|\mathrm{W}(\mathrm{R}_r)/\mathrm{W}(\mathrm{R}_{r-1})| = 10, 16, 27, 56$, respectively, and this is also the dimension of the irreducible minuscule representation $V(\omega_r)$ of the Lie algebra $\mathfrak{g}_r$ of type R_r with the highest weight ω_r. It is tempting to try to recover the Lie algebra directly from a del Pezzo surface, but one has to bear in mind that the del Pezzo surfaces of degree $d \leq 5$ depend on $10-2d$ moduli, so the Lie algebra should somehow take into account all del Pezzo surfaces of given degree; see [Manivel 2006], and also [Friedman and Morgan 2002; Leung 2000].

Universal torsors were introduced by Colliot-Thélène and Sansuc in the 1970's in a seemingly unrelated line of research; see [Colliot-Thélène and Sansuc 1987] or [Skorobogatov 2001]. If X is a smooth projective variety over a field k, then an X-torsor under a torus T is a pair (Y, f), where Y is a variety over k with a free action of T, and f is an affine morphism $Y \to X$ whose fibres are the orbits of T. An X-torsor is universal if all invertible regular functions on Y are constant, and the Picard group of Y is trivial (see Section 1 for details). Then T is isomorphic to the Néron–Severi torus of X, that is, the algebraic torus dual to the Picard lattice of X over an algebraic closure of k. In the work of Colliot-Thélène, Sansuc, Swinnerton-Dyer, Salberger and the second named author (see the references in [Skorobogatov 2001]) the birational geometry of universal torsors on del Pezzo surfaces of degrees 3 and 4 played a crucial role in gaining some understanding of the rational points on these surfaces over number fields, for example, the Hasse principle, weak approximation, the Brauer–Manin obstruction, and R-equivalence. The work of Batyrev, Tschinkel, Peyre, Salberger, Hassett, de la Bretèche, Heath-Brown, Browning and others on the Manin–Batyrev conjecture on the number of rational points of bounded height, highlighted the importance of explicitly describing universal torsors as algebraic varieties, and not merely their birational structure. However, in the most interesting cases such as those of (smooth) del Pezzo surfaces of degrees 3 and 4, the explicit equations of universal torsors turned out to be quite complicated to write down.

Around 1990, Victor Batyrev told one of us (Skorobogatov) about his conjecture relating universal torsors on del Pezzo surfaces to certain projective homogeneous spaces. Let G_r be a simply connected semisimple group of type R_r. We fix a maximal torus $H_r \subset G_r$, and a basis of simple roots in the character group of H_r. Let $P_r \subset G_r$ be the maximal parabolic subgroup defined by the root α_r (the stabilizer of the line spanned by the highest weight vector of $V(\omega_r)$). Batyrev conjectured that a universal torsor $\mathcal{T}$ on a del Pezzo surface X of degree $d = 9 - r$ over an algebraically closed field can be embedded into the affine cone $(G_r/P_r)_a \subset V(\omega_r)$ over G_r/P_r, equivariantly with respect to the action of the Néron–Severi torus T_r

of X, identified with an extension of H_r by the scalar matrices $\mathbf{G}_m$. Moreover, the exceptional curves on X should be the images of the weight hyperplane sections of $\mathcal{T}$, that is, the intersections of $\mathcal{T}$ with the H_r-invariant hyperplanes in $V(\omega_r)$. Inspired by these ideas, one of us showed in [Skorobogatov 1993] that the set of stable points of the affine cone over the Grassmannian $G(3, 5)$ with respect to the action of the diagonal torus of SL(5), is a universal torsor over the del Pezzo surface of degree 5 which is the geometric invariant theory (GIT) quotient by this action. Batyrev's line of attack on the general case of his conjecture uses the Cox ring of X, which can be interpreted as the ring of regular functions on a universal torsor over X. Indeed, Batyrev and Popov [2004] (see also [Derenthal 2006]) found the generators and the relations of the Cox ring, which enabled Popov in his thesis [2001] in the case $d = 4$ and Derenthal [2007] in the cases $d = 3$ and $d = 2$ to prove Batyrev's conjecture by identifying the generators with the weights of $V(\omega_r)$, and comparing the relations with the well-known equations of G_r/P_r. The proofs of [Popov 2001] and [Derenthal 2007] are based on a substantial amount of calculation which grows rapidly with r, and do not seem to give much insight into why things work this way.

Here we prove Batyrev's conjecture for del Pezzo surfaces of degrees 2 to 4 using a totally different approach, the representation theory of Lie algebras. We start with the known case of a del Pezzo surface of degree 5. (Alternatively, one could start with the simpler though somewhat irregular case of degree 6; see [Batyrev and Popov 2004].) Let $\mathfrak{p}_r$ be the Lie algebra of $P_r \subset G_r$. We build an inductive process based on the fact that the pair (R_r, α_r) for $r = 4, 5, 6, 7$ is a Hermitian symmetric pair, that is, the complementary nilpotent algebra of $\mathfrak{p}_r$ in $\mathfrak{g}_r$ is commutative. We show that $V(\omega_r)$, as a $\mathfrak{g}_{r-1}$-module, has a direct factor isomorphic to $V(\omega_{r-1})$, and that the restriction of the projection $V(\omega_r) \to V(\omega_{r-1})$ to a certain open subset $U \subset (G_r/P_r)_a$ is the composition of a $\mathbf{G}_m$-torsor and a morphism inverse to the blowing-up of $V(\omega_{r-1}) \setminus \{0\}$ at $(G_{r-1}/P_{r-1})_a \setminus \{0\}$ (see Corollary 4.2). Now we can explain the main idea of our proof. Suppose that a universal torsor $\mathcal{T}$ over a del Pezzo surface X of degree $9-(r-1)$ is T_{r-1}-equivariantly embedded into the affine cone $(G_{r-1}/P_{r-1})_a \subset V(\omega_{r-1})$. Let M be a point on X outside of the exceptional curves, and $\mathrm{Bl}_M(X)$ the blowing-up of X at M. The space $V(\omega_{r-1})$ is a direct sum of 1-dimensional weight spaces of H_{r-1}, so the torus consisting of the diagonal matrices with respect to a weight basis of $V(\omega_{r-1})$ does not depend on the choice of this basis. We show how to choose an element t_M of this torus so that the translation $t_M^{-1}(G_{r-1}/P_{r-1})_a$ intersects $\mathcal{T}$ exactly in the fibre of $f : \mathcal{T} \to X$ over M. Then the closure of the inverse image of $t_M(\mathcal{T} \setminus f^{-1}(M))$ in U is a universal torsor over $\mathrm{Bl}_M(X)$. This yields a T_r-equivariant embedding of this universal torsor into $(G_r/P_r)_a$. We then show that the image of this embedding is contained in the open subset of stable points with a free action of the Néron–Severi torus, so $\mathrm{Bl}_M(X)$ embeds into the corresponding quotient.

Here is the structure of the paper. In Section 1 we recall equivalent definitions and some basic properties of universal torsors. In Section 2 we prove that the left action of a maximal torus of G on G/P, where P is a maximal parabolic subgroup of a semisimple algebraic group G, turns the set of stable points with free action of the maximal torus into a universal torsor on an open subset of the GIT quotient of G/P by this action (with an explicit list of exceptions, see Theorem 2.7 for the precise statement). In Section 3 we recall the necessary background from the representation theory of Lie algebras. The implications for the structure of the projection of $(G_r/P_r)_a$ to $V(\omega_{r-1})$ are studied in Section 4. In Section 5 we list some well-known properties of del Pezzo surfaces. Our main result, Theorem 6.1, is stated and proved in Section 6.

1. Universal torsors

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$. Let X be a geometrically integral variety over k. We write $\overline{X}$ for $X \times_k \bar{k}$. We denote by $\bar{k}[X]$ the $\bar{k}$-algebra of regular functions on $\overline{X}$, and by $\bar{k}[X]^*$ the group of its invertible elements.

Let T be an algebraic k-torus, that is, an algebraic group such that $\overline{T} \simeq \mathbf{G}_m^n$ for some n. Let $\hat{T} \simeq \mathbf{Z}^n$ be the group of characters of T. The Galois group $\Gamma = \mathrm{Gal}\,(\bar{k}/k)$ naturally acts on $\hat{T}$.

For generalities on torsors the reader is referred to [Skorobogatov 2001]. An *X-torsor* under T is a pair $(\mathcal{T}, f)$, where $\mathcal{T}$ is a k-variety with an action of T, and $f : \mathcal{T} \to X$ is a morphism such that locally in étale topology $\mathcal{T} \to X$ is T-equivariantly isomorphic to $X \times_k T$. The following lemma is well known.

Lemma 1.1. *Suppose that a k-torus T acts on a k-variety Y with trivial stabilizers, and $g : Y \to X$ is an affine morphism of k-varieties, whose fibres are orbits of T. Then $g : Y \to X$ is a torsor under T.*

Proof. The property of g to be a torsor can be checked locally on X. Let U be an open affine subset of X. Since g is affine, $g^{-1}(U)$ is also affine [Hartshorne 1977, II, 5, Exercise 5.17]. Since the stabilizers of all $\bar{k}$-points of $g^{-1}(U)$ are trivial, by a corollary of Luna's étale slice theorem [Mumford et al. 1994, p. 153] the natural map $g^{-1}(U) \to U$ is a torsor under T. The lemma follows. □

Colliot-Thélène and Sansuc associated to a torsor $f : \mathcal{T} \to X$ under a torus T the exact sequence of Γ-modules [Colliot-Thélène and Sansuc 1987, 2.1.1]

$$1 \to \bar{k}[X]^*/\bar{k}^* \to \bar{k}[\mathcal{T}]^*/\bar{k}^* \to \hat{T} \to \mathrm{Pic}\,\overline{X} \to \mathrm{Pic}\,\overline{\mathcal{T}} \to 0. \qquad (1)$$

Here the second and fifth arrows are induced by f. The fourth arrow is called the *type* of $\mathcal{T} \to X$. To define it, consider the natural pairing compatible with the action of the Galois group Γ,

$$\cup : \ \mathrm{H}^1(\bar{X}, T) \times \hat{T} \to \mathrm{H}^1(\bar{X}, \mathbf{G}_m) = \operatorname{Pic} \bar{X},$$

where the cohomology groups are in étale or Zariski topology. The type sends $\chi \in \hat{T}$ to $[\overline{\mathcal{T}}] \cup \chi$, where $[\overline{\mathcal{T}}] \in \mathrm{H}^1(\bar{X}, T)$ is the class of the torsor $\overline{\mathcal{T}} \to \bar{X}$. A torsor $\mathcal{T} \to X$ is called *universal* if its type is an isomorphism. If X is projective, Equation (1) gives the following characterization of the universal torsors: an X-torsor under a torus is universal if and only if $\operatorname{Pic} \overline{\mathcal{T}} = 0$ and $\bar{k}[\mathcal{T}]^* = \bar{k}^*$, that is, $\overline{\mathcal{T}}$ has no nonconstant invertible regular functions.

We now give an equivalent definition of type which does not involve cohomology. Let $K = \bar{k}(X)$ be the function field of $\bar{X}$, and $\mathcal{T}_K$ the generic fibre of $\overline{\mathcal{T}} \to \bar{X}$. By Hilbert's Theorem 90, the K-torsor $\mathcal{T}_K$ is trivial, that is, is isomorphic to $T_K = T \times_k K$. By Rosenlicht's lemma we have an isomorphism of Γ-modules

$$K[\mathcal{T}_K]^*/K^* = K[T_K]^*/K^* = \hat{T}.$$

This isomorphism associates to a character $\chi \in \hat{T}$ a rational function $\phi \in \bar{k}(\mathcal{T})^*$ such that $\phi(tx) = \chi(t)\phi(x)$; the function ϕ is well defined up to an element of $K^* = \bar{k}(X)^*$. The divisor of ϕ on $\overline{\mathcal{T}}$ does not meet the generic fibre $\mathcal{T}_K$, and hence comes from a divisor on $\bar{X}$ defined up to a principal divisor. We obtain a well-defined class $\tau(\chi)$ in $\operatorname{Pic} \bar{X}$.

Lemma 1.2. *The map $\tau : \hat{T} \to \operatorname{Pic} \bar{X}$ coincides with the type of $f : \mathcal{T} \to X$ up to sign.*

Proof. According to [Skorobogatov 2001, Lemma 2.3.1 (ii)], the type associates to χ the subsheaf $\mathcal{O}_\chi$ of χ-semiinvariants of the sheaf $f_*(\mathcal{O}_{\mathcal{T}})$. The function ϕ is a rational section of $\mathcal{O}_\chi$; hence, the class of its divisor represents $\mathcal{O}_\chi \in \operatorname{Pic} \bar{X}$. □

For the sake of completeness we note that if $f : \mathcal{T} \to X$ is a universal torsor, then the group of divisors on $\bar{X}$ is naturally identified with $K[\mathcal{T}_K]^*/\bar{k}^*$; this identifies the semigroup of effective divisors on $\bar{X}$ with $(K[\mathcal{T}_K]^* \cap \bar{k}[\mathcal{T}])/\bar{k}^*$.

We have

$$\bar{k}[\mathcal{T}] = \bigoplus_{\chi \in \hat{T}} \bar{k}[\mathcal{T}]_\chi,$$

where $\bar{k}[\mathcal{T}]_\chi$ is the set of regular functions ϕ on $\mathcal{T}$, satisfying $\phi(tx) = \chi(t)\phi(x)$ for any t in T. We also define $\bar{k}(\mathcal{T})_\chi$ as the set of rational functions on $\mathcal{T}$, satisfying the same condition. Since $\bar{k}(\mathcal{T})_\chi$ is the group of rational sections of the sheaf $\mathcal{O}_\chi$, we have $\bar{k}[\mathcal{T}]_\chi = \mathrm{H}^0(\bar{X}, \mathcal{O}_\chi)$. Hence if the sheaf $\mathcal{O}_\chi$ defines a morphism

$$X \to \mathbb{P}(\mathrm{H}^0(\bar{X}, \mathcal{O}_\chi)^*),$$

we obtain a commutative diagram

$$\begin{array}{ccc}\mathcal{T} & \longrightarrow & \bar{k}[\mathcal{T}]^*_\chi \setminus \{0\} = \mathrm{H}^0(\bar{X}, \mathcal{O}_\chi)^* \setminus \{0\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}(\mathrm{H}^0(\bar{X}, \mathcal{O}_\chi)^*).\end{array} \tag{2}$$

Here the asterisk denotes the dual vector space.

2. G/P and the torus quotient

Let G be a split simple simply connected algebraic group over k, with a split maximal torus $H \subset G$; in this case the root system R of G relative to H is irreducible. Write $\hat{H}$ for the character group of H. We use the standard notation $Q(\mathrm{R})$ for the lattice generated by the simple roots, then $P(\mathrm{R}) = \hat{H}$ is the dual lattice generated by the fundamental weights. We denote the Weyl group by $\mathrm{W} = \mathrm{W}(\mathrm{R})$.

Let $G \to \mathrm{GL}(V)$ be an irreducible representation of G with a fundamental highest weight $\omega \in \hat{H}$. Let $v \in V$ be a highest weight vector. The stabilizer of the line kv is a maximal parabolic subgroup $P \subset G$. The homogeneous space G/P is thus a smooth projective subvariety of $\mathbb{P}(V)$, which is indeed the only closed orbit of G in $\mathbb{P}(V)$. We write $\hat{P}$ (respectively, $\hat{G}$) for the character group of P (respectively, of G). Let $\varepsilon : \hat{P} \to \operatorname{Pic} G/P$ be the map associating to the character $\chi \in \hat{P}$ the G/P-torsor under $\mathbf{G}_m$ defined as the quotient of $G \times \mathbf{G}_m$ by P, where $p \in P$ sends (g, t) to $(gp^{-1}, \chi(p)t)$. This map fits into the exact sequence

$$0 \to \hat{G} \to \hat{P} \to \operatorname{Pic} G/P \to \operatorname{Pic} G \to 0.$$

Since G is semisimple and simply connected we have $\hat{G} = \operatorname{Pic} G = 0$ so ε is an isomorphism; see, for example, [Popov 1974]. Since $\hat{P}$ is the subgroup of $\hat{H}$ generated by ω, we see that $\operatorname{Pic} G/P$ is generated by the hyperplane section class. This fact implies the following elementary statement from projective geometry.

Lemma 2.1. *Let L_1 and L_2 be distinct hyperplanes in the projective space $\mathbb{P}(V)$. Then $(G/P) \cap L_1 \cap L_2$ has codimension 2 in G/P.*

Proof. Since $\operatorname{Pic} G/P$ is generated by the class of a hyperplane section, for any hyperplane $L \subset \mathbb{P}(V)$ the closed subset $(G/P) \cap L$ is irreducible of codimension 1, and the intersection has multiplicity 1. If the codimension of $(G/P) \cap L_1 \cap L_2$ in G/P is 1, we have $(G/P) \cap L_1 \cap L_2 = (G/P) \cap L$ for any L in the linear family spanned by L_1 and L_2. Choosing L passing through a point of G/P not contained in L_1, we deduce a contradiction. $\square$

By the irreducibility of V the center $Z(G)$ acts diagonally on V, and hence it acts trivially on $\mathbb{P}(V)$. For a $\bar{k}$-point $x \in \mathbb{P}(V)$ we denote the stabilizer of x in H by $\mathrm{St}_H(x)$. We now show that for x in a dense open subset of G/P we have

$\mathrm{St}_H(x) = Z(G)$, and determine the points such that $\mathrm{St}_H(x)$ is strictly bigger than $Z(G)$.

Proposition 2.2. *Let x be a $\bar{k}$-point of G/P, and let K_x be the connected component of the centralizer of* $\mathrm{St}_H(x)$ *in G. Then we have the following properties.*

(i) *K_x is a reductive subgroup of G, $H \subset K_x$;*

(ii) *$x \in K_x wv = K_x/(wPw^{-1} \cap K_x)$ for some $w \in$* W;

(iii) $Z(K_x) = \mathrm{St}_H(x)$;

(iv) $\mathrm{St}_H(x)$ *is finite if and only if K_x is semisimple, in which case the ranks of K_x and G are equal.*

Proof. If $\mathrm{St}_H(x) = Z(G)$, then $K_x = G$, and all the statements are clearly true. Assume that $\mathrm{St}_H(x)$ is bigger than $Z(G)$, then K_x is a closed subgroup of G, $K_x \neq G$.

Let $\mathfrak{k}_x$ be the Lie algebra of K_x; explicitly $\mathfrak{k}_x \subset \mathfrak{g}$ is the fixed set of $\mathrm{Ad}(\mathrm{St}_H(x))$. Since $\mathfrak{k}_x$ contains the Cartan subalgebra $\mathfrak{h}$, it has a root decomposition

$$\mathfrak{k}_x = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha,$$

where $S \subset \mathrm{R}$. Let $\exp_\alpha \in \hat{H}$ be the multiplicative character defined by the root $\alpha \in \mathrm{R}$. The space $\mathfrak{g}_\alpha$ consists of $y \in \mathfrak{g}$ such that $\mathrm{Ad}(h)y = \exp_\alpha(h)y$ for all $h \in H$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{k}_x$ if and only if $\mathrm{St}_H(x) \subset H$ is in the kernel of $\exp_\alpha$. Therefore $S = -S$, so $\mathfrak{k}_x$ is reductive, and hence so is K_x.

The fixed points of H in G/P come from the points wv, where $w \in \mathrm{W}$. One of these, say $x_0 = wv$, is contained in the closure of the orbit Hx. The stabilizer of x_0 in G is the parabolic subgroup wPw^{-1}. To prove (ii) we need to show that x belongs to the K_x-orbit of x_0. Let $N \subset G$ be the unipotent subgroup complementary to wPw^{-1}, such that the corresponding Lie algebras satisfy $\mathfrak{g} = \mathfrak{n} \bigoplus w\mathfrak{p}w^{-1}$. Then $N \cap wPw^{-1} = \{1\}$, and the N-orbit of the line kx_0 is the open Schubert cell $Nx_0 \subset G/wPw^{-1} \simeq G/P$. The intersection of this open Schubert cell with Hx is a nonempty open subset of Hx; thus there is a $\bar{k}$-point $x_1 \in Hx \cap Nx_0$. We can write $x_1 = u.x_0$ for some $u \in N$. The complement to the union of connected components of the centralizer of $\mathrm{St}_H(x)$ other than K_x, is an open neighborhood of 1 in G. We choose x_1 in such a way that u belongs to this open set. Since $H \subset K_x$, the points x and x_1 are in the same K_x-orbit, so it is enough to show that $x_1 \in K_x x_0$. Any $t \in \mathrm{St}_H(x)$ fixes both x_1 and x_0; thus $x_1 = u.x_0 = t^{-1}ut.x_0$. Therefore, $u^{-1}t^{-1}ut$ fixes x_0; hence $u^{-1}t^{-1}ut \in wPw^{-1}$. On the other hand, H normalizes N; thus $t^{-1}ut \in N$, implying $u^{-1}t^{-1}ut \in N$. Since the intersection of wPw^{-1} and N is $\{1\}$, we see that u and t commute. By the choice of x_1 we see that u is in the connected component of 1 of the centralizer of $\mathrm{St}_H(x)$, that is, $u \in K_x$. This completes the proof of (ii).

The center of K_x is contained in every maximal torus, in particular, in H. Any element of $Z(K_x)$ fixes x, since $x \in K_x/(wPw^{-1} \cap K_x)$, so $Z(K_x) \subset \mathrm{St}_H(x)$. On the other hand, every element of $\mathrm{St}_H(x)$ commutes with K_x by the definition of K_x. But $\mathrm{St}_H(x) \subset H \subset K_x$; hence $\mathrm{St}_H(x) \subset Z(K_x)$. This proves (iii).

The rank of the semisimple part of K_x equals the rank of G if and only if $Z(K_x)$ is finite. If $Z(K_x)$ is finite, then K_x is semisimple by definition. Thus (iv) follows from (iii). $\square$

Fix a weight basis in V, that is, a basis in which H is diagonal. The weight of a coordinate is the character of H by which H acts on it. Denote by Λ the set of weights of H in V, and by $\mathrm{wt}(x)$ the set of weights of $x \in G/P$, that is, the weights of the nonvanishing coordinates of x.

Corollary 2.3. *Assume that* R *is simply laced. Then the codimension of the set of* $\bar{k}$*-points* $x \in G/P$ *such that* $\mathrm{St}_H(x)$ *is finite, and* $\mathrm{St}_H(x) \neq Z(G)$, *is at least* 2.

Proof. By Proposition 2.2 and W-invariance it is sufficient to show that the codimension of Kv in Gv is at least 2 for any proper connected semisimple subgroup $K \subset G$ containing H. The set of such subgroups is clearly finite.

For any $x \in G/P$ the property $\mathrm{wt}(x) = \Lambda$ implies $\mathrm{St}_H(x) = Z(G)$. Let $V' \subset V$ be the irreducible representation of K generated by v. Denote by Λ' the set of weights of V', and write $V = V' \oplus U$, where U is another K-invariant subspace. First, we claim that $\Lambda' \neq \Lambda$ because otherwise one can find $x \in \mathbb{P}(Kv)$ such that $\mathrm{wt}(x) = \Lambda$, and $\mathrm{St}_H(x) = Z(G) = Z(K)$ would imply $K = G$. In particular, $U \neq 0$. If $\dim U > 1$, then the codimension of $Kv \subset Gv \cap V'$ is at least 2 by Lemma 2.1.

If $\dim U = 1$, then U is a trivial representation of K and 0 is not a weight of V'. But then U is invariant under the action of the Weyl group W. Therefore wKw^{-1} acts trivially on U for any $w \in \mathrm{W}$. If $a \in \mathrm{R}$ is a root of K, then $w(a)$ is a root of wKw^{-1}. But in the simply laced case W acts transitively on R; hence, the subgroups wKw^{-1}, $w \in \mathrm{W}$, generate the whole group G. Thus, U is G-invariant, but that contradicts the irreducibility of V. $\square$

Recall that a $\bar{k}$-point $x \in V$ is called *stable* for the action of H if the orbit Hx is closed, and the stabilizer of x in H is finite [Mumford et al. 1994, p. 194]. We always consider the stability with respect to the action of H, and drop the reference to H when it causes no confusion.

For a subset $M \subset \hat{H}$ we write $\mathrm{Conv}(M)$ for the convex hull of M in the vector space $\hat{H} \otimes \mathbb{R}$. It is well known that $\mathrm{Conv}(\Lambda) = \mathrm{Conv}(\mathrm{W}\omega)$ [Gel'fand and Serganova 1987; Flaschka and Haine 1991]; see [Dabrowski 1996, Proposition 2.2 (i)] for a short proof. The Hilbert–Mumford numerical criterion of stability says that x is stable if and only if 0 belongs to the interior of $\mathrm{Conv}(\mathrm{wt}(x))$ [Dolgachev 2003, Theorem 9.2].

In the following statement and thereafter the numeration of the nodes of Dynkin diagrams, simple roots and fundamental weights follows the conventions of [Bourbaki 1981].

Proposition 2.4. *Assume that the pair* (R, ω) *is not in the following list*:

$$(\mathrm{R}_r, \omega_1),\ (\mathrm{A}_r, \omega_r),\ (\mathrm{A}_3, \omega_2),\ (\mathrm{B}_2, \omega_2),\ (\mathrm{C}_2, \omega_2),\ (\mathrm{D}_4, \omega_3),\ (\mathrm{D}_4, \omega_4), \tag{3}$$

where R_r *is* A_r, B_r, C_r, *or* D_r. *Let x be a point of $V \otimes_k \bar{k}$ such that no two elements of* $\mathrm{W}\omega \setminus \mathrm{wt}(x)$ *are adjacent vertices of* $\mathrm{Conv}(\mathrm{W}\omega)$. *Then x is stable. In particular, the set of unstable points of G/P has codimension at least* 2.

Proof. Since $\sum_{w \in \mathrm{W}} w\omega = 0$, the point 0 is contained in the interior of

$$\mathrm{Conv}(\mathrm{W}\omega) = \mathrm{Conv}(\Lambda)$$

in $\hat{H} \otimes \mathbb{R}$. Thus if all the coordinates of x with weights in $\mathrm{W}\omega$ are nonzero, then x is stable.

Now assume that exactly one such coordinate of x is zero; because of the action of W it is no loss of generality to assume that it corresponds to ω. The dimension of the corresponding eigenspace is 1, so to check that x is stable it is enough to show that 0 lies in the interior of $\mathrm{Conv}(\mathrm{W}\omega \setminus \{\omega\})$. The vertices of $\mathrm{Conv}(\mathrm{W}\omega)$ adjacent to ω are $\omega - w\alpha$, where α is the root dual to ω, for all w in the stabilizer of ω in W [Flaschka and Haine 1991, Lemma 3 and Cor. 2]. All these are contained in the hyperplane $L = 0$, where

$$L(y) = (y, \omega) - (\omega^2) + (\omega, \alpha) = (y, \omega) - (\omega^2) + \tfrac{1}{2}(\alpha^2).$$

We have $L(\omega) > 0$. Thus 0 belongs to the interior of $\mathrm{Conv}(\mathrm{W}\omega \setminus \{\omega\})$ if and only if ω and 0 are separated by this hyperplane, that is, if and only if $L(0) < 0$. Therefore, we need to check the condition

$$(\omega^2) > \tfrac{1}{2}(\alpha^2).$$

Note that the numbers $2(\omega^2)/(\alpha^2)$, for all possible fundamental weights, are the diagonal elements of the inverse Cartan matrix of R. A routine verification using the tables of [Bourbaki 1981] or [Onishchik and Vinberg 1990] shows that this inequality is satisfied for the pairs (R, ω) not in the list (3).

Finally, let $\mathrm{W}\omega \setminus \mathrm{wt}(x) = \{\lambda_1, \dots, \lambda_n\}$. By assumption $\lambda_1, \dots, \lambda_n$ correspond to pairwise nonadjacent vertices of $\mathrm{Conv}(\mathrm{W}\omega)$. Thus

$$\mathrm{Conv}(\mathrm{W}\omega \setminus \{\lambda_1, \dots, \lambda_n\}) = \bigcap_{i=1}^{n} \mathrm{Conv}(\mathrm{W}\omega \setminus \{\lambda_i\}).$$

Since 0 is in the interior of each convex hull on the right hand side, it is also in the interior of $\mathrm{Conv}(\mathrm{wt}(x))$.

The last statement is an application of Lemma 2.1. $\square$

Definition 2.5. Let $T \subset \mathrm{GL}(V)$ be the torus generated by the image of H in $\mathrm{GL}(V)$ and the scalar matrices $\mathbf{G}_m \subset \mathrm{GL}(V)$. We write $(G/P)_a$ for the affine cone over G/P in V, and $(G/P)_a^{sf}$ for the open subset of stable points with trivial stabilizers in T.

By the irreducibility of V, the stabilizer of $x \in V \otimes_k \bar{k}$, $v \neq 0$, in T is trivial if and only if $\mathrm{St}_H(pr(x)) = Z(G)$, where $pr(x)$ is the image of x in $\mathbb{P}(V)$.

Lemma 2.6. *There exist a smooth quasiprojective variety Y and an affine morphism $f : (G/P)_a^{sf} \to Y$ which is a torsor with structure group T with respect to its natural left action on G/P.*

Proof. By geometric invariant theory there exist a quasiprojective variety Y and an affine morphism $f : (G/P)_a^{sf} \to Y$ such that every fibre of f is an orbit of T [Mumford et al. 1994, Theorem 1.10 (iii)]. Since the stabilizers of all $\bar{k}$-points of $(G/P)_a^{sf}$ are trivial, Lemma 1.1 implies that $f : (G/P)_a^{sf} \to Y$ is a torsor under T. The smoothness of Y follows from the smoothness of $(G/P)_a$, since a torsor is locally trivial in étale topology. $\square$

Theorem 2.7. *Assume that the root system* R *is simply laced, and the pair* (R, ω) *is not in the list* (3). *Then the only invertible regular functions on* $(G/P)_a^{sf}$ *are constants, so* $f : (G/P)_a^{sf} \to Y$ *is a universal torsor.*

Proof. By Lemma 2.6 we need to show that $\mathrm{Pic}\,\overline{\mathcal{T}} = 0$ and $\bar{k}[\mathcal{T}]^* = \bar{k}^*$ where we write $\mathcal{T} = (G/P)_a^{sf}$ (see Section 1). The Picard group of $(G/P)_a$ is trivial since that of G/P is generated by the class of a hyperplane section. Thus it suffices to show that the complement to $(G/P)_a^{sf}$ in $(G/P)_a$ has codimension at least 2. The set of unstable points has codimension at least 2, by Proposition 2.4. The closed subset of its complement consisting of the stable points with nontrivial (finite) stabilizers in T also has codimension at least 2, as follows from Corollary 2.3. $\square$

3. Hermitian symmetric pairs

Let $\mathfrak{g}$ be a semisimple Lie algebra over the field k with Chevalley basis $\{H_\beta, X_\gamma\}$, where γ is a root of R, and $H_\beta = [X_\beta, X_{-\beta}]$, where β is a simple root of R.

A simple root α of $\mathfrak{g}$ defines a $\mathbf{Z}$-grading on $\mathfrak{g}$ in the following way. We set $\deg(X_\alpha) = 1$, $\deg(X_{-\alpha}) = -1$, $\deg(X_{\pm\beta}) = 0$ for all other simple roots $\beta \neq \alpha$, and $\deg(H_\beta) = 0$ for all simple roots β. Then

$$\mathfrak{g} = \bigoplus_{i=-l(\alpha)}^{l(\alpha)} \mathfrak{g}_i, \tag{4}$$

where $l(\alpha)$ is the *label* of α, that is, the coefficient of α in the decomposition of the maximal root as a linear combination of the simple roots. The Lie algebra $\mathfrak{p}=\bigoplus_{i\geq 0}\mathfrak{g}_i$ is the parabolic subalgebra defined by α, and $\mathfrak{n}=\bigoplus_{i<0}\mathfrak{g}_i$ is the complementary nilpotent algebra. The center of the Lie algebra $\mathfrak{g}_0$ is one-dimensional, so $\mathfrak{g}_0=Z(\mathfrak{g}_0)\oplus\mathfrak{g}'$, where $\mathfrak{g}'$ is the semisimple Lie algebra whose Dynkin diagram is that of $\mathfrak{g}$ with the node corresponding to α removed.

It is clear from (4) that $l(\alpha)=1$ if and only if $[\mathfrak{n},\mathfrak{n}]=0$. The following terminology has its origin in the theory of symmetric spaces; see [Helgason 2001, Chapter VIII].

Definition 3.1. The pair (R,α) is a Hermitian symmetric pair if $l(\alpha)=1$, or, equivalently, if $\mathfrak{n}$ is a commutative Lie algebra.

If R is simply laced, then (R,α) is a Hermitian symmetric pair if and only if $\mathrm{R}=\mathrm{A}_n$, or if it is one of the following pairs: (D_n,α_i), where $i=1$, $n-1$ or n, (E_6,α_1), (E_6,α_6), and (E_7,α_7).

We now assume that $\mathfrak{n}$ is commutative. Our next goal is to explore the implications of this assumption for the restriction of the $\mathfrak{g}$-module V to the semisimple subalgebra $\mathfrak{g}'$. We write $\mathrm{U}(\mathfrak{l})$ for the universal enveloping algebra of the Lie algebra $\mathfrak{l}$, and $S(W)$ for the symmetric algebra of the vector space W. Since $\mathfrak{n}$ is commutative we have $\mathrm{U}(\mathfrak{n})=S(\mathfrak{n})$.

The line kv is a 1-dimensional $\mathfrak{p}$-submodule of V; hence the $\mathfrak{g}$-module V is the quotient of the induced module $\mathrm{U}(\mathfrak{g})\otimes_{\mathrm{U}(\mathfrak{p})}kv$ by the submodule generated by $X_{-\alpha}^2v$. (This follows from the construction of V as the quotient of the Verma module by the submodule generated by $X_{-\beta}v$ for the simple roots $\beta\neq\alpha$, and $X_{-\alpha}^2v$.) By the Poincaré–Birkhoff–Witt theorem we have $\mathrm{U}(\mathfrak{g})=\mathrm{U}(\mathfrak{p})\otimes_k\mathrm{U}(\mathfrak{n})$. The line kv is a trivial $\mathfrak{g}'$-module. Therefore, the $\mathfrak{g}'$-module $\mathrm{U}(\mathfrak{g})\otimes_{\mathrm{U}(\mathfrak{p})}kv$ is isomorphic to $\mathrm{U}(\mathfrak{n})=S(\mathfrak{n})$, so the finite dimensional vector space V inherits the $\mathbf{Z}_{\leq 0}$-graded commutative k-algebra structure from $S(\mathfrak{n})$, $V=\bigoplus_{n\leq 0}V^n$. We turn this grading into a $\mathbf{Z}_{\geq 0}$-grading by setting $V_n=V^{-n}$. Since $\mathfrak{g}'$ has grading 0, the direct sum $V=\bigoplus_{n\geq 0}V_n$ is the direct sum of $\mathfrak{g}'$-modules, and we can write

$$V=k\oplus\mathfrak{n}\oplus\left(S^{\geq 2}(\mathfrak{n})/S(\mathfrak{n})\mathrm{U}(\mathfrak{g}')X_{-\alpha}^2\right),$$

where $k=V_0$, $\mathfrak{n}=V_1$. Note that $1\in V_0$ is a highest weight vector; it generates V as a $S(\mathfrak{n})$-module.

Lemma 3.2. *Let* (R,α) *be a Hermitian symmetric pair. Then the adjoint representation of* $\mathfrak{g}'$ *on* $V_1=\mathfrak{n}=\mathfrak{g}_{-1}$ *is the irreducible representation such that* $X_{-\alpha}$ *is a highest weight vector. If* R *is simply laced, then the highest weight* ω' *of* V_1 *is the sum of the fundamental weights corresponding to the nodes of the Dynkin diagram of* R *adjacent to the node* α.

Proof. We have $[X_\beta, X_{-\alpha}] = 0$ for all simple roots $\beta \neq \alpha$, so $X_{-\alpha}$ is annihilated by the positive roots of $\mathfrak{g}'$. Every root of $\mathfrak{n}$ is the sum of $-\alpha$ and a root of $\mathfrak{g}'$, so $\mathfrak{n}$ is generated by $X_{-\alpha}$ as a $\mathfrak{g}'$-module. The computation of the weight of $X_{-\alpha}$ is immediate from the defining relations among the elements of the Chevalley basis. □

We have the exponential map

$$\exp : \mathfrak{n} \to S(\mathfrak{n}), \quad \exp(u) = 1 + u + \frac{1}{2}u^2 + \frac{1}{3!}u^3 + \cdots.$$

Let G be the simply connected semisimple algebraic k-group with Lie algebra $\mathfrak{g}$, $P \subset G$ the parabolic subgroup with Lie algebra $\mathfrak{p}$, and N the unipotent k-group with Lie algebra $\mathfrak{n}$. By the Chevalley construction of the Lie group from its Lie algebra, N acts on V by the rule $1+x \mapsto \exp(x)$. Recall that the open Schubert cell of $G/P \subset \mathbb{P}(V)$ is the N-orbit of the highest weight vector, and hence is identified with $\exp(\mathfrak{n})$. (In particular, $\dim G/P = \dim V_1$.) Thus $\exp(x)$ is a polynomial G'-equivariant map, where G' is the simply connected semisimple k-group with Lie algebra $\mathfrak{g}'$

$$\exp : V_1 \to (G/P)_a \subset V = \bigoplus_{n\geq 0} V_n.$$

Let $p : V_1 = \mathfrak{n} \to V_2$ be the degree 2 graded component of $\exp(x)$.

Lemma 3.3. *Let G' be the simply connected semisimple k-group with the Lie algebra $\mathfrak{g}'$, and $P' \subset G'$ the parabolic subgroup which is the stabilizer of the line spanned by the highest weight vector $X_{-\alpha} \in \mathfrak{n}$. The restriction of* $\exp(x)$ *to $(G'/P')_a$ coincides with* $(1, \mathrm{id}, 0, 0, \ldots)$. *We have $(G'/P')_a = p^{-1}(0)$, and the ideal of $(G'/P')_a$ is generated by the coordinates of $p(x)$.*

Proof. It is clear that every graded component of $\exp(x)$ of degree at least 2 sends the orbit $(G'/P')_a$ of the highest weight vector $X_{-\alpha}$ to 0. Indeed, $X_{-\alpha}^m$ is in the kernel of the natural map $S^m(\mathfrak{n}) \to V_m$, for $m \geq 2$. To prove the second statement let us observe that the symmetric square $S^2(\mathfrak{n})$ decomposes as the direct sum of V_2 and the $\mathfrak{g}'$-submodule generated by $X_{-\alpha}^2$, which is the irreducible representation $V(2\omega')$ with highest weight $2\omega'$. It is well known from [Lancaster and Towber 1979, proof of Theorem 1.1] or [Batyrev and Popov 2004, Proposition 4.2] that the orbit of the highest weight vector is the intersection of the second Veronese embedding with $V(2\omega')$. This completes the proof. □

Consider the following series of root systems,

$$\mathrm{A}_4 \subset \mathrm{D}_5 \subset \mathrm{E}_6 \subset \mathrm{E}_7. \tag{5}$$

Let (R, α) be one of the Hermitian symmetric pairs

$$(\mathrm{A}_4, \alpha_3), \quad (\mathrm{D}_5, \alpha_5), \quad (\mathrm{E}_6, \alpha_6), \quad (\mathrm{E}_7, \alpha_7), \tag{6}$$

where the roots are numbered as in [Bourbaki 1981]. By Lemma 3.2 the pair (G', P') is defined by (R', α') which is the previous pair to (R, α) in (6). In other words, P' corresponds to the only node of the smaller diagram adjacent to α. (If G is of type A_4, then G' is of type $\mathrm{A}_1 \times \mathrm{A}_2$, $G'/P' \simeq \mathbb{P}^1 \times \mathbb{P}^2$, but we shall not have to consider this case.)

We note that the fundamental weight ω dual to α is minuscule, that is, the weights of V are $\mathrm{W}\omega$, and $\mathrm{W}v$ is a basis of V; see [Bourbaki 1981, VIII.7.3]. We also note that the G-module V defined by ω is faithful (this follows from the fact that ω generates $P(\mathrm{R})/Q(\mathrm{R})$, which can be checked from the tables). Thus the faithful representation of G in V defines a faithful representation of G', and this implies that $G' \subset G$ (in fact, G' is the Levi subgroup of P).

Let us identify the graded components of V in various cases. Let $d_r = \dim V$. We have

$$d_4 = 10,\ d_5 = 16,\ d_6 = 27,\ d_7 = 56.$$

The details given below show that for $r = 4, 5, 6$ the graded components of $\exp(x)$ of degree at least 3 are zero.

Let $\mathrm{R} = \mathrm{A}_4$. Then $G = \mathrm{SL}(5)$, and G/P is the Grassmannian $G(2, 5)$. Denote by E_n the standard n-dimensional representation of $\mathrm{SL}(n)$. We have $V = \Lambda^2(E_5)$, $\dim V = 10 = 1+6+3$. The group $G' = \mathrm{SL}(2) \times \mathrm{SL}(3)$ is embedded into $\mathrm{SL}(5)$ in the obvious way, and the graded factors of V are $V_1 = E_2 \otimes E_3$, $V_2 = \Lambda^2(E_3) \cong E_3^*$. The map $p : V_1 \to V_2$ sends x to the $\Lambda^2(E_3)$-component of

$$x \wedge x \in \Lambda^2(E_5) = \Lambda^2(E_2) \oplus (E_2 \otimes E_3) \oplus \Lambda^2(E_3).$$

Let $\mathrm{R} = \mathrm{D}_5$. Then V is a spinor representation of $G = \mathrm{Spin}(10)$ of dimension $16 = 1 + 10 + 5$, and G/P is the isotropic Grassmannian (one of two families of maximal isotropic subspaces of the nondegenerate quadric of rank 10), and $\dim G/P = 10$. The graded components are $V_1 = \Lambda^2(E_5)$ and $V_2 = \Lambda^4(E_5) \cong E_5^*$. The map $p : V_1 \to V_2$ sends x to $x \wedge x$.

Let $\mathrm{R} = \mathrm{E}_6$. Then $\dim V = 27 = 1 + 16 + 10$, V_1 is the spinor representation of $\mathrm{Spin}(10)$ as above, and V_2 is the standard 10-dimensional representation of $\mathrm{SO}(10)$. We have $\dim G/P = 16$.

Let $\mathrm{R} = \mathrm{E}_7$. Then $\dim V = 56 = 1 + 27 + 27 + 1$, V_1 is the 27-dimensional representation of the group of type E_6 considered above, $V_2 = (V_1)^*$, and $V_3 = k$ is the trivial 1-dimensional representation. (The graded components of degree at least 4 are zero.) We have $\dim G/P = 27$. We define $q : V_1 = \mathfrak{n} \to V_3 = k$ as the degree 3 graded component of $\exp(x)$. This is a E_6-invariant cubic form in 27 variables. The 27 weight coordinates of $p(x)$ are partial derivatives of $q(x)$. This identifies the space G/P of type E_6 with the singular locus of the cubic hypersurface $q(x) = 0$.

Define a symmetric bilinear form $p(x, y)$ on V_1 with values in V_2 by the formula $p(x+y) = p(x) + 2p(x, y) + p(y)$. Then $\exp(x+y) = \exp(x)\exp(y)$ implies that

$$2p(x, y) = x \cdot y \tag{7}$$

is the product of $x \in V_1$ and $y \in V_1$ in the commutative k-algebra V.

We have a decomposition of $S^2(V_1)$ as the direct sum of V_2 and the representation with highest weight $2\omega'$ (see the proof of Lemma 3.3). In the notation of [Bourbaki 1981] the representation V_2 is irreducible with highest weight ω_1; in particular, it is minuscule. Thus the eigenspaces for the action of the maximal torus $H' = H \cap G'$ are 1-dimensional, so on V_2, in the same way as on V_1, we have weight coordinates well defined up to a multiplicative constant. The coordinates $p_\lambda(x, y)$ of $p(x, y)$ are symmetric bilinear forms of degree 2 with values in k. We can write

$$p_\lambda(x, y) = \sum_{\lambda=\mu+\nu} p_{\mu\nu} x_\mu y_\nu, \tag{8}$$

where μ and ν are weights of V_1, $p_{\mu\nu} \in k$, and x_μ is a nonzero linear form on the weight μ subspace $(V_1)_\mu \subset V_1$ (and similarly for y_ν). One checks that for $r = 4, 5, 6, 7$ the ranks of the quadratic forms $p_\lambda(x)$ are 4, 6, 8, 10, respectively. If $r = 7$ we associate to the cubic form

$$q(x) = \sum_{\mu+\nu+\xi=0} q_{\mu\nu\xi} x_\mu x_\nu x_\xi$$

the symmetric trilinear form

$$q(x, y, z) = \sum_{\mu+\nu+\xi=0} q_{\mu\nu\xi} x_\mu y_\nu z_\xi .$$

In this case the weights of V_2 are the negatives of the weights of V_1. Moreover, $p_{-\mu}(x) = \partial q(x)/\partial x_\mu$, so

$$\begin{aligned} 3q(x, y, z) &= \sum_{\mu} p_{-\mu}(x, y) z_\mu, \\ p_{-\mu}(x, y) &= \sum_{-\mu=\nu+\xi} 3q_{\mu\nu\xi} x_\nu y_\xi . \end{aligned} \tag{9}$$

For future reference we note that if $p_\lambda(x, y) = 0$ for all λ, then $q(x, y, y) = 0$. It follows from $\exp(x+y) = \exp(x)\exp(y)$ that

$$3q(x, x, y) = p(x) \cdot y \tag{10}$$

is the product of $p(x) \in V_2$ and $y \in V_1$ in the commutative k-algebra V.

4. G/P and blowing-up

Let $\pi : (G/P)_a \to V_1$ be the restriction to $(G/P)_a$ of the natural projection

$$V = k \oplus V_1 \oplus V_2 \oplus V_3 \to V_1.$$

We have $\exp(x) = (1, x, p(x), q(x))$; hence $\pi \circ \exp = \mathrm{id}$. Here and in what follows we write our formulae for the case $r = 7$, with the convention that if $r < 7$ the last coordinate must be discarded.

We now describe the fibres of π.

Lemma 4.1. *Let $g_t = (t, 1, t^{-1}, t^{-2})$, $t \in \bar{k}^*$. For $x \in V_1 \otimes_k \bar{k}$ we have the following statements.*

(a) *If $x \notin (G'/P')_a$, then $\pi^{-1}(x) = \{g_t \cdot \exp(x) \mid t \in \bar{k}^*\}$.*

(b) *If $x \in (G'/P')_a \setminus \{0\}$, then*

$$\pi^{-1}(x) = \big\{(t, x, 0, 0) \mid t \in \bar{k}^*\big\} \cup \big\{(0, x, 2p_\lambda(x, u), 3q(x, u, u)) \mid u \in V_1 \otimes_k \bar{k}\big\}.$$

Proof. Recall that the torus T is generated by the maximal torus $H \subset G$ and the scalar matrices (t, t, t, t), $t \in \bar{k}^*$. Let $h \in \mathfrak{h}$ be an element of the Lie algebra of H such that $\beta(h) = 0$ for all simple roots β of G, $\beta \neq \alpha$, and $\alpha(h) = 1$. The 1-parameter subgroup $\mathbf{G}_m \subset H$ whose tangent vector at the identity is h, acts on V as $(t^m, t^{m-1}, t^{m-2}, t^{m-3})$, where $m = \omega(h)$, and ω is the fundamental weight dual to α. Hence $g_t \in T$ for any $t \in \bar{k}^*$.

Every $\bar{k}$-point $y = (y_0, y_1, y_2, y_3)$ of the closed set $(G/P)_a$ satisfies the equations

$$y_0 y_2 = p(y_1), \quad y_0^2 y_3 = q(y_1), \tag{11}$$

since these are satisfied on the affine cone over $\exp(V_1)$ which is dense in $(G/P)_a$. Therefore, if π sends a $\bar{k}$-point y of $(G/P)_a$ to $x = y_1$, and $y_0 \neq 0$, we can write $y = g_t \cdot (1, x, p(x), q(x)) = g_t \cdot \exp(x)$ for $t = y_0 \in \bar{k}^*$. All such points are in $(G/P)_a$ since the action of T preserves $(G/P)_a$, and $\exp(V_1) \subset (G/P)_a$. If $y_0 = 0$ we see from (11) and Lemma 3.3 that $x \in (G'/P')_a$. This proves (a).

To prove (b), assume $x \in (G'/P')_a$, $x \neq 0$. If $y_0 \neq 0$, then $y = (t, x, 0, 0)$, by (11).

We need some preparations for the case $y_0 = 0$. Recall that V_0 is identified with k by the choice of a highest weight vector $v \in V_0$, and V_1 is identified with $\mathfrak{n}$. Consider $\mathfrak{g}_1 = \mathfrak{n}^-$, the opposite nilpotent algebra of $\mathfrak{n}$. Any nonzero element $X \in \mathfrak{g}_1$ sends V_i to V_{i-1} because of the grading. Hence we can write

$$\exp(Xt)(y_0, y_1, y_2, y_3) = \big(y_0 + s(y_1, X)t + z_1 t^2 + z_2 t^3,\ y_1 + u_1 t + u_2 t^2,\ y_2 + wt,\ y_3\big),$$

where $z_1,\ z_2 \in k$, $u_1,\ u_2 \in V_1$, $w \in V_2$, and $s(y_1, X) \in k$ is defined by

$$s(y_1, X)v = X y_1 v = [X, y_1]v.$$

For any nonzero $y_1 \in \mathfrak{n} \otimes_k \bar{k} = V_1 \otimes_k \bar{k}$ one can find $X \in \mathfrak{g}_1 \otimes_k \bar{k}$ such that $s(y_1, X) = 1$. Otherwise $\mathfrak{g}_1 y_1 v = 0$, and so $y_1 v$ is a highest vector of the $\mathfrak{g}$-module $V \otimes_k \bar{k}$, which is not a multiple of v. This contradicts the irreducibility of $V \otimes_k \bar{k}$. Fix such an element $X \in \mathfrak{g}_1 \otimes_k \bar{k}$.

Now let $y_0 = 0$. Then

$$g_{t^{-1}} \exp(Xt)(0, y_1, y_2, y_3) = (1 + z_1 t + z_2 t^2, y_1 + u_1 t + u_2 t^2, y_2 t + w t^2, y_3 t^2)$$

is a $\bar{k}[t]$-point of $(G/P)_a$, and hence its coordinates satisfy (11) identically in t. Equating to 0 the coefficient at t in the first equation in (11) we obtain $y_2 = 2p(y_1, u)$, where $u = u_1$. Equating to 0 the coefficient at t^2 in the second equation, and using that $q(y_1, y_1, v) = 0$ for all $v \in V_1$ according to (9), we obtain $y_3 = 3q(y_1, u, u)$.

To complete the proof of (b) we need to show that for any $\bar{k}$-point $x \in (G'/P')_a$ and any $u \in V_1 \otimes_k \bar{k}$ the point $\big(0, x, 2p_\lambda(x, u), 3q(x, u, u)\big)$ is contained in $(G/P)_a$. We note that

$$\big(0, x, 2p_\lambda(x, u), 3q(x, u, u)\big) = \exp(u) \cdot (0, x, 0, 0),$$

as immediately follows from (7) and (10). Since $\exp(u)$ is in the unipotent group $N \subset G$ it is enough to show that $(0, x, 0, 0)$ is in $(G/P)_a$. Clearly $(1, x, 0, 0) = \exp(x)$ is in $(G/P)_a$. Choosing $X \in \mathfrak{g}_1 \otimes_k \bar{k}$ as above such that $s(x, X) = -1$ we obtain $\exp(X)(1, x, 0, 0) = (0, x, 0, 0)$. $\square$

Corollary 4.2. *Let $U \subset (G/P)_a$ be the complement to the intersection of $(G/P)_a$ with $(V_0 \oplus V_1) \cup (V_2 \oplus V_3)$. The restriction of π to U is a morphism $U \to V_1 \setminus \{0\}$, which is the composition of a torsor under the torus $\mathbf{G}_m = \{g_t \mid t \in \bar{k}^*\}$, and the morphism inverse to the blowing-up of $V_1 \setminus \{0\}$ at $(G'/P')_a \setminus \{0\}$.*

Proof. The set U is covered by the open subsets $U_0 : y_0 \neq 0$, and $U_\lambda : y_\lambda \neq 0$, where y_λ are the weight coordinates in V_2. Indeed, if $y_0 = y_\lambda = 0$ for all λ, then we are in case (b) of Lemma 4.1, but $p_\lambda(x, u) = 0$ for all λ implies $q(x, u, u) = 0$, and such points are not in U. Each of these open subsets is $\mathbf{G}_m$-equivariantly isomorphic to the direct product of $\mathbf{G}_m$ and the closed subvariety of $(G/P)_a$ given by $y_i = 1$ with trivial $\mathbf{G}_m$-action. Gluing them together we obtain the quotient $\tilde{U}$.

The equations (11) show that $\pi^{-1}(0) \cap U = \emptyset$; thus π projects U to $V_1 \setminus \{0\}$. The action of $\mathbf{G}_m$ preserves the fibres, hence π factors through a morphism $\tilde{U} \to V_1 \setminus \{0\}$. It is an isomorphism outside $(G'/P')_a$, whereas the inverse image of $(G'/P')_a \setminus \{0\}$ is the projectivisation of the normal bundle to $(G'/P')_a \setminus \{0\}$ in $V_1 \setminus \{0\}$, by Lemma 4.1 (b). It is not hard to prove (and is well known to experts) that this implies that $\tilde{U}$ is the blowing-up of $V_1 \setminus \{0\}$ at $(G'/P')_a \setminus \{0\}$. $\square$

5. Del Pezzo surfaces

For the geometry of exceptional curves on del Pezzo surfaces the reader is referred to [Manin 1986, Chapter IV]; see also [Friedman and Morgan 2002, Section 5]. Let $M_1, \dots, M_r$, $4 \le r \le 7$, be k-points in general position in the projective plane $\mathbb{P}^2$, which says that no three points are on a line and no six on a conic. The blowing-up X of $\mathbb{P}^2$ in $M_1, \dots, M_r$ is called a *split del Pezzo surface* of degree $d = 9 - r$. The surface X contains exactly d_r *exceptional curves*, that is, smooth rational curves with self-intersection -1. For $r \le 6$ the exceptional curves on X arise in one of these ways: the inverse images of the M_i; the proper transforms of the lines through M_i and M_j, $i \ne j$; the proper transforms of the conics through five of the M_i. For $r = 7$ one also has the proper transforms of singular cubics passing through all 7 points with a double point at some M_i. The intersection index defines an integral bilinear form $(\,.\,)$ on Pic X. The opposite of the canonical class $-K_X$ is an ample divisor, $(K_X^2) = d$. The Picard group Pic $\overline{X}$ = Pic X is generated by the classes of exceptional curves (the complement to the union of these curves is an open subset of $\mathbb{A}^2$). The triple (Pic X, K_X, $(\,.\,)$) coincides, up to isomorphism, with the triple $(N_r, K_r, (\,.\,))$ defined as [Manin 1986, Theorem 23.9]

$$N_r = \bigoplus_{i=0}^{r} \mathbf{Z}\ell_i, \quad K_r = -3\ell_0 + \sum_{i=1}^{r} \ell_i, \quad (\ell_0^2) = 1, \ (\ell_i^2) = -1, \ i \ge 1, \ (\ell_i.\ell_j) = 0, \ i \ne j.$$

Moreover, the exceptional curves are identified with the elements $\ell \in N_r$ such that $(\ell^2) = (\ell.K_r) = -1$, which are called the *exceptional classes* [Manin 1986, Theorem 23.8]. By definition, a geometrically integral conic on X is a smooth rational curve with self-intersection 0. By the Riemann–Roch theorem each conic belongs to a 1-dimensional pencil of curves which are fibres of a morphism $X \to \mathbb{P}^1$, called a *conic bundle*. We refer to the fibres of such a morphism as *conics*. In particular, through every point of X passes exactly one conic of a given pencil. The classes of conic bundles can be characterized by the properties $(c^2) = 0$, $(c.K_r) = -2$.

Let $K_r^\perp$ be the orthogonal complement to K_r in N_r. The elements $\alpha \in K_r^\perp$ such that $(\alpha^2) = -2$ form a root system R in the vector space $K_r^\perp \otimes \mathbb{R} \simeq \mathbb{R}^r$ with the negative definite scalar product $(\,.\,)$. In fact, R is a root system of rank r in the series (5). Moreover, the lattice $K_r^\perp$ is generated by roots so $K_r^\perp \simeq Q(\mathrm{R})$. For example, we can choose

$$\beta_1 = -\ell_1 + \ell_2, \ \dots, \ \beta_{r-1} = -\ell_{r-1} + \ell_r, \ \beta_r = -\ell_0 + (\ell_1 + \ell_2 + \ell_3)$$

as a basis of simple roots of R. The relation to our standard numeration, which follows [Bourbaki 1981], is $\alpha_r = \beta_{r-1}$, $\alpha_1 = \beta_1$.

The Weyl group W = W(R) generated by the reflections in the roots, is the automorphism group of the triple $(N_r, K_r, (\,.\,))$. It operates transitively on the set of exceptional curves, and also on the set of conic bundle classes; see, for example, [Friedman and Morgan 2002, Lemma 5.3]. Let

$$P(\mathrm{R}) = \{n \in K_r^{\perp} \otimes \mathbb{R} : \ (n.m) \in \mathbf{Z} \text{ for any } m \in Q(\mathrm{R})\}$$

be the lattice dual to $Q(\mathrm{R})$; we have $Q(\mathrm{R}) \subset P(\mathrm{R})$. The image of the map

$$N_r \to N_r \otimes \mathbb{R} = \mathbb{R}K_r \oplus (K_r^{\perp} \otimes \mathbb{R})$$

is contained in the orthogonal direct sum $\frac{1}{d}\mathbf{Z}K_r \oplus P(\mathrm{R})$ as a subgroup of index d.

Lemma 5.1. *Let $\alpha = \beta_{r-1} \in \mathrm{R}$ be the simple root such that (R, α) is one of the pairs in (6), and let $\omega \in P(\mathrm{R})$ be the dual fundamental weight, $(\alpha.\omega) = -1$.*

(i) *The exceptional classes in N_r are $-\frac{1}{d}K_r + w\omega$, for all $w \in \mathrm{W}$.*

(ii) *Two distinct exceptional curves intersect in X if and only if the corresponding weights are not adjacent vertices of the convex hull* $\mathrm{Conv}(\mathrm{W}\omega)$.

(iii) *Let ω_1 be the fundamental weight dual to the root β_1. The conic bundle classes in N_r are $-\frac{2}{d}K_r + w\omega_1$, for all $w \in \mathrm{W}$.*

Note that since W acts transitively on the set of bases, the choice of a basis of simple roots is not important for the conclusion of this lemma.

Proof. (i) and (iii) The image of the exceptional class ℓ_r in $P(\mathrm{R})$ is the fundamental weight $\omega = \omega_{r-1}$, and the image of the conic bundle class $\ell_0 - \ell_1$ is the fundamental weight ω_1. The statement now follows from the transitivity of action of W on these classes. See [Friedman and Morgan 2002, Lemma 5.2].

(ii) By the transitivity of W on the exceptional classes it is enough to check this for the classes $-\frac{1}{d}K_r + \omega$ and $-\frac{1}{d}K_r + x$, where $x = w\omega$ for some $w \in \mathrm{W}$. The intersection index

$$\Big(-\frac{1}{d}K_r + x . -\frac{1}{d}K_r + \omega\Big) = \frac{1}{d} + (x.\omega) \tag{12}$$

equals $-L(x)$ in the notation of the proof of Proposition 2.4 (with the opposite sign of the scalar product). In the simply laced case this proof shows that $L(x) = 1$ when $x = \omega$, $L(x) = 0$ if x is a vertex of the convex hull $\mathrm{Conv}(\mathrm{W}\omega)$ adjacent to ω, and $L(x) < 0$ for all other $x \in \mathrm{W}\omega$. □

We observe that for any conic bundle class x there exists a conic bundle class y such that $(x.y) = 1$. Indeed, by the transitivity of W on conic bundle classes we can assume that $x = \ell_0 - \ell_1$. For $y = \ell_0 - \ell_2$ we have $(x.y) = 1$.

6. Main theorem

We recall our notation.

- (R, α) is the pair in (6) such that R has rank r;

- G is the simply connected semisimple group with a split maximal torus H and a maximal parabolic subgroup $P \supset H$, such that (G, P) is defined by the pair (R, α);
- V is the fundamental representation of G such that P is the stabilizer of the line spanned by a highest weight vector (this representation is faithful);
- $T \subset \mathrm{GL}(V)$ is the torus generated by the image of H in $\mathrm{GL}(V)$, and the scalar matrices;
- Y is the geometric quotient of $(G/P)_a^{sf} \subset (G/P)_a$ with respect to the natural left action of T;
- the morphism $f : (G/P)_a^{sf} \to Y$ is a universal torsor (see Theorem 2.7).

Let $\Lambda \subset \hat{H}$ be the set of weights of H in V, and let $V_\lambda \subset V$ be the subspace of weight λ, so that $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$. In our case $\dim V_\lambda = 1$ (since V is minuscule; see Section 3). Let $\pi_\lambda : V \to V_\lambda$ be the natural projections, and let $L_\lambda = \pi_\lambda^{-1}(0)$ be the weight coordinate hyperplanes. For a subset $A \subset V$ we write $A^\times$ for the set of points of A outside $\cup_{\lambda \in \Lambda} L_\lambda$. For a subset $B \subset Y$ we write $B^\times$ for $f(f^{-1}(B)^\times)$.

We now state our main theorem, whose proof occupies the rest of the paper.

Theorem 6.1. *For $r = 4, 5, 6$ or 7 let $M_1, \dots, M_r$ be k-points in general position in $\mathbb{P}^2$ (no three on a line, no six on a conic). Let X be the blowing-up of $\mathbb{P}^2$ in $M_1, \dots, M_r$. There exists an embedding $X \hookrightarrow Y$ such that $X \setminus X^\times$ is the union of exceptional curves on X. For such an embedding $f^{-1}(X) \to X$ is a universal torsor.*

We write $S^n_\chi(V)$ for the H-eigenspace of $S^n(V)$ of weight $\chi \in \hat{H}$, and $S^n_\chi(V)^*$ for the dual space of functions. Let $I(\mathcal{T}) \subset k[V] = S(V^*)$ be the ideal of $\mathcal{T}$. We shall prove the following statement from which the main theorem will follow.

There exists an embedding of a universal torsor $\mathcal{T}$ over X into $(G/P)_a^{sf} \subset V$ such that the restriction of f to $\mathcal{T}$ is the structure morphism $\mathcal{T} \to X$, and $f(\mathcal{T}^\times)$ is the complement to the union of exceptional curves on X. Moreover, for $r < 7$ the ideal $I(\mathcal{T}^\times) \subset k[V^\times]$ is generated by the graded components of degree 2 and weight $w\omega_1$, for all $w \in \mathrm{W}$.

The last statement will be used in the case $r = 7$, and can be ignored by the reader interested in the cases $r = 5$ and $r = 6$ only. Recall that ω_1 is the highest weight of a nontrivial irreducible $\mathfrak{g}$-module of least dimension.

Proof. The proof is by induction on r starting from $r = 4$. In this case Y is a del Pezzo surface of degree 5, G/P is the Grassmannian variety $G(3, 5) \simeq G(2, 5)$, and $G(3, 5)^{sf} = G(3, 5)^s$ is a universal torsor over Y; see [Skorobogatov 1993] or [Skorobogatov 2001, Lemma 3.1.6]. It is well known that the ideal of $G(3, 5)_a \subset V$

is generated by the (quadratic) Plücker relations, and it is easy to see that their weights are of the form $w\omega_1$, so our statement is true in this case.

Suppose we know the statement for $r-1 \geq 4$. This means that we are given the following data.

- (R', α') is the "previous" pair to (R, α) in (6);
- $\mathrm{W}' = \mathrm{W}(\mathrm{R}')$ is the Weyl group;
- G' and P' are defined by (R', α'), so that $(G'/P')_a \subset V_1$ (see Section 3);
- $H' = H \cap G'$, so that R' is the root system of G' with respect to H';
- $T' \subset \mathrm{GL}(V_1)$ is the torus generated by the image of H' in $\mathrm{GL}(V_1)$ and the scalars (T' is also the image of H in $\mathrm{GL}(V_1)$);
- x_μ is a nonzero linear form on the weight μ subspace of V_1;
- Y' is the quotient of $(G'/P')_a^{sf}$ by T';
- $f' : (G'/P')_a^{sf} \to Y'$ is a universal torsor;
- X' is the blowing-up of $\mathbb{P}^2$ in $M_1, \ldots, M_{r-1}$ (it is a del Pezzo surface of degree $d' = 8-r$);
- there exists an embedding $X' \hookrightarrow Y'$ satisfying the conditions of the theorem, in particular,
- $\mathcal{T}' = f'^{-1}(X') \to X'$ is a universal torsor.

The general position assumption implies that M_r does not belong to the exceptional curves of X'. Thus, by Hilbert's Theorem 90, we can find a k-point $x_0 \in \mathcal{T}'^{\times}$ such that $f'(x_0) = M_r$.

Let $\tau : \hat{T}' \to \operatorname{Pic} X'$ be the map defined in Section 1; up to the sign, τ coincides with the type of the torsor $f' : \mathcal{T}' \to X'$ (Lemma 1.2). Since the torsor $f' : \mathcal{T}' \to X'$ is universal, τ is an isomorphism of $\hat{T}' = K[\mathcal{T}'_K]^*/K^*$ and $\operatorname{Pic} X'$ as abelian groups. To account for the duality between vectors and linear forms on V_1 we identify these groups by the isomorphism $-\tau$. Recall that the Weyl group W' acts on $\hat{T}'$ via the normalizer of H' in G', permuting the weights of V_1. By induction assumption $-\tau$ sends these weights bijectively onto the exceptional classes in $\operatorname{Pic} X'$. If we transport the action of W' from $\hat{T}'$ to $\operatorname{Pic} X'$ using $-\tau$, then the action of W' so obtained preserves the intersection index of exceptional curves; see (12). Thus $-\tau$ is a homomorphism of W'-modules, where W' acts on $\operatorname{Pic} X'$ as the automorphism group of the triple $(N_{r-1}, K_{r-1}, (\,.\,))$. In particular, $-\tau$ identifies the W'-(co)invariants on both sides (isomorphic to $\mathbf{Z}$). This implies that if χ is a weight of T' in $S^n(V_1)$, then the restriction of χ to the scalar matrices $\mathbf{G}_m \subset T'$ coincides with the intersection index of $-\tau(\chi)$ and $-K_{X'}$, that is,

$$(\tau(\chi).K_{X'}) = n \tag{13}$$

(the sign is uniquely determined by the fact that effective divisors intersect positively with $-K_{X'}$). The isomorphism $-\tau$ also identifies the quotients by the W$'$-invariants, that is, $P(\mathrm{R}')$ and $\hat{H}'$. We fix these identifications from now on.

For $\phi(x) \in S^n_\chi(V_1)^*$, $\chi \in \hat{T}'$, let $C_\phi \subset X'$ be the image of the intersection of $\mathcal{T}'$ with the T'-invariant hypersurface $\phi(x) = 0$. If $C_\phi \neq X'$, then the class $[C_\phi]$ in $\operatorname{Pic} X'$ is $-\tau(\chi)$, and (13) can be written as

$$([C_\phi].(-K_{X'})) = n. \tag{14}$$

We have (see the end of Section 1 for the first equality)

$$\mathrm{H}^0(X', \mathcal{O}_{-\chi}) = k[\mathcal{T}']_{-\chi} = S^n_\chi(V_1)^*/I(\mathcal{T}') \cap S^n_\chi(V_1)^*. \tag{15}$$

Apart from the weights of V_1 which correspond to the exceptional curves, the following two cases will be particularly relevant. For $n = 2$ let λ be a weight of T' in V_2. The restriction of λ to H' is $w\omega_1 \in \hat{H}' = P(\mathrm{R}')$, where $w \in \mathrm{W}'$ (see the end of Section 3). If $\phi \in S^2_\lambda(V_1)^*$ is such that $C_\phi \neq X'$, then by (14) we see that $[C_\phi] = -\frac{2}{d'}K_{X'} + w\omega_1$, so C_ϕ is a conic on X' by Lemma 5.1 (iii). The Riemann–Roch theorem implies that $\dim \mathrm{H}^0(X', \mathcal{O}_{-\lambda}) = 2$, where $\mathcal{O}_{-\lambda} = \mathcal{O}(C_\phi)$ is the invertible sheaf associated to C_ϕ. Thus $I(\mathcal{T}') \cap S^2_\lambda(V_1)^*$ has codimension 2 in $S^2_\lambda(V_1)^*$. Note that by Lemma 3.3 we have $p_\lambda(x) \in I(\mathcal{T}') \cap S^2_\lambda(V_1)^*$. For $r = 7$ and $n = 3$ the space V_3 is a trivial 1-dimensional representation of G', hence of weight $0 \in \hat{H}'$. Thus for $\phi \in S^3_0(V_1)^*$ we have $[C_\phi] = -K_{X'}$, by (14). If $C_\phi \neq X'$, then C_ϕ is a plane section of the cubic surface $X' \subset \mathbb{P}^3$. The vector space $\mathrm{H}^0\big(X', \mathcal{O}(C_\phi)\big) = \mathrm{H}^0\big(X', \mathcal{O}(-K_{X'})\big)$ has dimension 4; thus $I_0 = I(\mathcal{T}') \cap S^3_0(V_1)^*$ has codimension 4 in $S^3_0(V_1)^*$. It is clear that $q(x) \in I_0$; see, for example, (10).

The following proposition is a crucial technical step in the proof of our main theorem.

Proposition 6.2. *There exists a nonempty open subset $\Omega(x_0) \subset (G'/P')^\times_a$ such that for any $y_0 \in \Omega(x_0)$ we have $p_\lambda(x_0^{-1}y_0x) \notin I(\mathcal{T}') \cap S^2_\lambda(V_1)^*$ for all weights λ of V_2, and $q(x_0^{-1}y_0x) \notin I_0$ if $r = 7$.*

Proof. We begin with pointing out the following useful fact. Let Ver_λ be the composition of the second Veronese embedding $V_1 \to S^2(V_1)$ with the projection of $S^2(V_1)$ to its direct summand $S^2_\lambda(V_1)$. By Lemma 3.3, $p_\lambda(x) = 0$ is the only quadratic equation of G'/P' of weight λ; thus $\mathrm{Ver}_\lambda((G'/P')_a)$ spans a codimension 1 subspace of $S^2_\lambda(V_1)$, namely, the zero set of the linear form $p_\lambda(x) \in S^2_\lambda(V_1)^*$.

Next, we claim that the quadratic forms $p_\lambda(x_0^{-1}y_0x)$, $y_0 \in (G'/P')^\times_a$, span a codimension 1 subspace of $S^2_\lambda(V_1)^*$. Using (8) we write

$$p_\lambda(x_0^{-1}y_0x) = \sum_{\lambda=\mu+\nu} p_{\mu\nu}\frac{y_{0\mu}y_{0\nu}}{x_{0\mu}x_{0\nu}}x_\mu x_\nu.$$

Suppose that for some coefficients $c_{\mu\nu}$ we have a linear relation

$$\sum_{\lambda=\mu+\nu} c_{\mu\nu} p_{\mu\nu} \frac{y_{0\mu} y_{0\nu}}{x_{0\mu} x_{0\nu}} = 0.$$

This can be read as a relation with coefficients $c_{\mu\nu} p_{\mu\nu} x_{0\mu}^{-1} x_{0\nu}^{-1}$ satisfied by all the vectors $(y_{0\mu} y_{0\nu})$, where $y_0 \in (G'/P')_a^\times$ and $\mu+\nu=\lambda$. The set of these vectors is precisely $\mathrm{Ver}_\lambda((G'/P')_a^\times)$. The linear span of $\mathrm{Ver}_\lambda((G'/P')_a^\times)$ is the same as the linear span of $\mathrm{Ver}_\lambda((G'/P')_a)$. By the argument in the beginning of the proof, up to a multiplicative constant there is only one linear relation satisfied by the elements of $\mathrm{Ver}_\lambda((G'/P')_a)$, namely the one with coefficients $p_{\mu\nu}$. Therefore, $c_{\mu\nu} = x_{0\mu} x_{0\nu}$ is uniquely determined up to a multiplicative constant. This proves our claim. Note that the linear span under discussion is thus the space of forms vanishing at x_0.

It follows that the set of $\bar{k}$-points $y \in (G'/P')_a^\times$ such that $p_\lambda(x_0^{-1} y x)$ belongs to the codimension 2 subspace $I(\mathcal{T}') \cap S_\lambda^2(V_1)^*$, is a proper closed subset of $(G'/P')_a^\times$. For $r<7$ we define $\Omega(x_0)$ as the complement to the union of these closed subsets for all weights λ of V_2.

For the rest of the proof we let $r=7$. Let

$$\mathrm{Ver}_0^3 : V_1 \to S_0^3(V_1)$$

be the composition of the natural map $V_1 \to S^3(V_1)$ with the projection $S^3(V_1) \to S_0^3(V_1)$. The map Ver_0^3 sends $x=(x_\mu)$ to the vector $(x_\mu x_\nu x_\xi)$, for all μ, ν, ξ such that $\mu+\nu+\xi=0$. If we write the invariant cubic form (defined up to a scalar multiple) as

$$q(x) = \sum_{\mu+\nu+\xi=0} q_{\mu\nu\xi} x_\mu x_\nu x_\xi,$$

then it is well known that all the coefficients $q_{\mu\nu\xi}$ are nonzero; see, for example, [Faulkner 2001]. Recall that the singular locus of the cubic hypersurface $q(x)=0$ is $(G'/P')_a$.

Let $L_{x_0} \subset S_0^3(V_1)^*$ be the subspace of forms such that all their (first order) partial derivatives vanish at x_0. We claim that L_{x_0} coincides with the linear span of the forms $q(x_0^{-1} y_0 u)$, where y_0 ranges over $(G'/P')_a$.

Let us prove this claim. The partial derivatives of $q(x)$ vanish on $(G'/P')_a$; hence $q(x_0^{-1} y_0 u) \in L_{x_0}$ for any $y_0 \in (G'/P')_a$. Thus the linear span of the forms $q(x_0^{-1} y_0 u)$, where $y_0 \in (G'/P')_a$, is contained in L_{x_0}. We now prove that these spaces have the same dimension.

Let $f(x) = \sum_{\mu+\nu+\xi=0} f_{\mu\nu\xi} x_\mu x_\nu x_\xi$ be a form in L_{x_0}. The partial derivative with respect to x_ξ is

$$3 \sum_{\mu+\nu=-\xi} f_{\mu\nu\xi} x_\mu x_\nu.$$

It vanishes at $x_0 \in V_1^\times$ if and only if

$$x_\xi \sum_{\mu+\nu=-\xi} f_{\mu\nu\xi} x_\mu x_\nu = \sum_{\mu+\nu=-\xi} q_{\mu\nu\xi}^{-1} f_{\mu\nu\xi} \cdot q_{\mu\nu\xi} x_\mu x_\nu x_\xi$$

does. Hence $(q_{\mu\nu\xi}^{-1} L_{x_0})^\perp$ is spanned by the 27 vectors $(q_{\mu\nu\xi} x_{0\mu} x_{0\nu} x_{0\xi})$, where ξ is fixed, and μ, ν are arbitrary. Since the coordinates of x_0 are not zero, this space has the same dimension as the space $M \subset S_0^3(V_1)$ spanned by the 27 vectors $(q_{\mu\nu\xi})$, where ξ is fixed, and μ, ν are arbitrary weights satisfying $\mu+\nu+\xi=0$. The fact that the ideal of $(G'/P')_a$ is generated by the partial derivatives of $q(x)$, implies that $M^\perp$ is the linear span of $\mathrm{Ver}_0^3((G'/P')_a)$. We conclude that $\dim L_{x_0}$ equals the dimension of this linear span. Since all the coefficients $q_{\mu\nu\xi}$ are nonzero, the forms $q(x_0^{-1} y_0 u)$, where $y_0 \in (G'/P')_a$, span the space of the same dimension. This proves our claim.

We complete the proof of the proposition in the case $r=7$. A cubic form $f \in S_0^3(V_1)^*$ is in L_{x_0} if and only if $f(x)=0$ is singular at $x_0 \in V_1^\times$. This is the case if and only if the corresponding hyperplane $H_f \subset S_0^3(V_1)$ contains the tangent space Φ to $\mathrm{Ver}_0^3(V_1)$ at the point $m = \mathrm{Ver}_0^3(x_0)$. We have a commutative diagram (compare (15) and (2))

$$\begin{tikzcd}
X' \arrow[d] & \mathcal{T}' \arrow[l] \arrow[r, hook] \arrow[d] & V_1 \arrow[d] \\
\mathbb{P}\big(\mathrm{H}^0(X', \mathcal{O}(-K_{X'}))^*\big) & \mathrm{H}^0\big(X', \mathcal{O}(-K_{X'})\big)^* \setminus \{0\} \arrow[l] \arrow[r, hook] & S_0^3(V_1)
\end{tikzcd}$$

where the left-hand vertical map is the anticanonical embedding of X', and the other two are Ver_0^3. The image of $\mathcal{T}'$ in the 4-dimensional vector space

$$\mathrm{H}^0\big(X', \mathcal{O}(-K_{X'})\big)^* = \big(k[\mathcal{T}'] \cap S_0^3(V_1)^*\big)^* = (S_0^3(V_1)^*/I_0)^* \simeq \mathbb{A}^4 \subset S_0^3(V_1)$$

is the affine cone X'_a (without 0) over the cubic surface $X' \subset \mathbb{P}^3$.

By the induction assumption $I(\mathcal{T}'^\times)$ is generated by its graded components I_λ of degree 2 and weight λ, for all weights λ of V_2. The weights of V_1 are the negatives of the weights of V_2, so $x_{-\lambda} I_\lambda$ has degree 3 and weight 0. Since the coordinates $x_{-\lambda}$ are invertible on $\mathcal{T}'^\times$, the ideal $I(\mathcal{T}'^\times)$ is generated by its graded component of degree 3 and weight 0. Hence locally in the neighborhood $\mathcal{T}'^\times$ of x_0 the ideal $I(\mathcal{T}')$ is generated by I_0, that is, by the equations of $\mathbb{A}^4$ in $S_0^3(V_1)$.

This implies that the tangent space $T_{X'_a,m} \subset \mathbb{A}^4$ is $\Phi \cap \mathbb{A}^4$. Thus for any f in a dense open subset of L_{x_0} we have $H_f \cap \mathbb{A}^4 = T_{X'_a,m}$. Since $X' \subset \mathbb{P}^3$ is a smooth cubic surface, $X'_a \setminus T_{X'_a,m}$ is dense and open in X'_a. Therefore, for the general $f \in L_{x_0}$ we have $X'_a \cap H_f \neq X'_a$, so $f \notin I_0$. Now the above claim implies the statement of the proposition. □

Corollary 6.3. *For any k-point $y_0 \in \Omega(x_0)$ and any weight λ of V_2 the closed subset of $\mathcal{T}'$ given by $p_\lambda(x_0^{-1}y_0x) = 0$ is the preimage $f'^{-1}(C_\lambda)$ of a geometrically integral k-conic $C_\lambda \subset X'$ passing through M_r. For $r = 7$ the closed subset of $\mathcal{T}'$ given by $q(x_0^{-1}y_0x) = 0$, for any $y_0 \in \Omega(x_0)$, is the preimage $f'^{-1}(Q)$ of a geometrically integral cubic k-curve Q with a double point at M_r (the intersection of the cubic surface X' with its tangent plane at M_r).*

Proof. To check that $M_r \in C_\lambda$, set $x = x_0$; then $p_\lambda(x_0^{-1}y_0x) = p_\lambda(y_0) = 0$ by Lemma 3.3 since $y_0 \in (G'/P')_a$. If the conic C_λ is not geometrically integral, then its components must have intersection index 1 with $-K_{X'}$, so there are two of them. It is well known that a curve on X' has such a property if and only if it is an exceptional curve. However, M_r does not belong to the exceptional curves of X'. Thus C_λ is geometrically integral.

If $r = 7$, by substituting $x = x_0$ one shows as before that Q contains M_7 (the cubic form q vanishes on G'/P'). Since the $p_\lambda(x)$ are partial derivatives of $q(x)$, and $M_7 \in C_\lambda$, we see that Q has a double point at M_7. If Q is not geometrically integral, then it is the union of a geometrically integral conic and an exceptional curve, or the union of three exceptional curves. In each of these cases the singular point $M_7 \subset Q$ will have to lie on an exceptional curve, and this is a contradiction. □

Corollary 6.4. *For any $y_0 \in \Omega(x_0)$ the scheme-theoretic intersection of $x_0^{-1}y_0\mathcal{T}'$ and $(G'/P')_a$ is the orbit $T'y_0$.*

Proof. By Lemma 3.3 the ideal of $(G'/P')_a$ is generated by $p_\lambda(x)$, for all weights λ of V_2. As was remarked at the end of Section 5, there exist weights λ and ν such that the intersection index of C_λ and C_ν on X' is 1, that is, M_r is the scheme-theoretic intersection $C_\lambda \cap C_\nu$. Thus the orbit $T'y_0$ is the closed subscheme of $x_0^{-1}y_0\mathcal{T}'$ given by $p_\lambda(x) = p_\nu(x) = 0$, and our statement follows. □

Let $\sigma : X = \mathrm{Bl}_{M_r}(X') \to X'$ be the morphism inverse to the blowing-up of M_r. Then σ induces an isomorphism of $X \setminus \sigma^{-1}(M_r)$ with $X' \setminus M_r$, and $\sigma^{-1}(M_r) \cong \mathbb{P}^1$. The proper transform of a curve $D \subset X'$ is defined as the closure of $\sigma^{-1}(D \setminus M_r)$ in X. The comparison of intersection indices on X' and X shows that the proper transforms of the conics C_λ and the singular cubic Q (for $r = 7$) are exceptional curves on X. By comparing the numbers we see that these curves together with $\sigma^{-1}(M_r)$ and the inverse images of the exceptional curves on X' give the full set of exceptional curves on X.

End of proof of Theorem 6.1 Consider the open set $U \subset (G/P)_a$ and the morphism $\pi : U \to V_1 \setminus \{0\}$; see Corollary 4.2. Choose any $y_0 \in \Omega(x_0)$, and define $\mathcal{T} \subset U$ as the “proper transform” of $x_0^{-1}y_0\mathcal{T}'$ with respect to π. Explicitly, $\mathcal{T} \subset U$

is defined as the Zariski closure of

$$\pi^{-1}\big(x_0^{-1}y_0\mathcal{T}'\setminus(G'/P')_a\big)=\pi^{-1}(x_0^{-1}y_0\mathcal{T}'\setminus T'y_0),$$

where the equality is due to Corollary 6.4. The torus T' acts on $\mathcal{T}'$, and π is T'-equivariant; hence T' acts on $\mathcal{T}$. But $\mathbf{G}_m=\{g_t\}$ (see Lemma 4.1) also acts on $\mathcal{T}$. The torus T is generated by T' and $\mathbf{G}_m=\{g_t\}$, so T acts on $\mathcal{T}$.

Corollaries 4.2 and 6.4 imply that the restriction of π to $\mathcal{T}$ is the composition of a torsor under $\mathbf{G}_m=\{g_t\}$ and the morphism $\mathrm{Bl}_{y_0T'}(x_0^{-1}y_0\mathcal{T}')\to x_0^{-1}y_0\mathcal{T}'$ inverse to the blowing-up of the orbit $T'y_0$ in $x_0^{-1}y_0\mathcal{T}'$. The blowing-up of $T'y_0$ in $x_0^{-1}y_0\mathcal{T}'$ is naturally isomorphic to the pullback $\mathcal{T}'\times_{X'}X$ of the torsor $\mathcal{T}'\to X'$ to X. This can be summarized in the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r] & \mathcal{T}'\times_{X'}X \arrow[r] \arrow[d] & X \arrow[d, "\sigma"] \\ & \mathcal{T}' \arrow[r] & X' \end{tikzcd} \tag{16}$$

where the horizontal arrows are torsors under tori, and the vertical arrows are contractions. The composed morphism $f:\mathcal{T}\to X$ is a composition of two torsors under tori, and hence is an affine morphism whose fibres are orbits of T. Therefore $\mathcal{T}$ is an X-torsor under T, by Lemma 1.1. We obtain a T-equivariant embedding $\mathcal{T}\hookrightarrow(G/P)_a$.

For $r<7$ we note that $I(\mathcal{T}^\times)\subset k[V^\times]$ is generated by $I(x_0^{-1}y_0\mathcal{T}'^\times)$ and the equations of $(G/P)_a$; moreover, for each weight $w\omega_1$, $w\in\mathrm{W}$, there is exactly one quadratic equation, by Lemma 3.3. The restriction of $\omega_1\in\hat{H}=P(\mathrm{R})$ to H' is again the weight $\omega_1\in\hat{H}'=P(\mathrm{R}')$. By the induction assumption $I(\mathcal{T}'^\times)$ is generated by its graded components of degree 2 of such weights; hence the same is true for $I(\mathcal{T}^\times)$.

It remains to prove that $\mathcal{T}\subset(G/P)_a^{sf}$, and that the torsor $f:\mathcal{T}\to X$ is universal. The action of T on $\mathcal{T}$ is free; we show that every point of $\mathcal{T}$ is stable. We claim that f sends the weight hyperplane sections of $\mathcal{T}$ to the exceptional curves on X. By the results of Section 4 this follows from the induction assumption for the weights of V_1, and from Corollary 6.3 for the weights of $V_2\oplus V_3$. Corollary 6.4 implies that the highest weight hyperplane $x_\omega=0$ corresponds to $\sigma^{-1}(M_r)$. By Lemma 5.1 (ii) the set of exceptional curves of X is identified with the set $\mathrm{W}\omega$ in such a way that two distinct exceptional curves intersect in X if and only if the corresponding weights are not adjacent vertices of the convex hull $\mathrm{Conv}(\mathrm{W}\omega)$. Now Proposition 2.4 implies that $\mathcal{T}\subset(G/P)_a^{sf}$. We thus obtain an embedding $X\hookrightarrow Y$.

The pull-back of the torsor $(G/P)_a^{sf}\to Y$ to X gives rise to the following commutative diagram, where the horizontal arrows represent the types of corresponding torsors

$$\begin{tikzcd} \hat{T} \arrow[r, "\sim"] \arrow[d, equal] & \operatorname{Pic} Y \arrow[d] \\ \hat{T} \arrow[r] & \operatorname{Pic} X. \end{tikzcd}$$

The upper horizontal arrow is an isomorphism since the torsor $(G/P)_a^{sf} \to Y$ is universal, by Theorem 2.7. Since the exceptional curves on X are cut by divisors on Y, the restriction map $\operatorname{Pic} Y \to \operatorname{Pic} X$ is surjective. However, the ranks of $\operatorname{Pic} Y$ and $\operatorname{Pic} X$ are equal, so this map is an isomorphism. Now it follows from the diagram that the type of the torsor $f : \mathcal{T} \to X$ is an isomorphism, so this torsor is universal as well. The theorem is proved. □

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The zeta function of monomial deformations of Fermat hypersurfaces

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This paper intends to give a mathematical explanation for results on the zeta function of some families of varieties recently obtained in the context of mirror symmetry. In the process we obtain concrete and explicit examples for some results recently used in algorithms to count points on smooth hypersurfaces in $\mathbb{P}^n$.

In particular, we extend the monomial-motive correspondence of Kadir and Yui and we give explicit solutions to the p-adic Picard–Fuchs equation associated with monomial deformations of Fermat hypersurfaces.

As a byproduct we obtain Poincaré duality for the rigid cohomology of certain singular affine varieties.

1. Introduction

One of the families under consideration in this paper is the famous one-parameter family (Dwork family) of quintic threefolds $X_{\bar{\lambda}} \subset \mathbb{P}^4_{\mathbb{F}_q}$ given by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \bar{\lambda} x_0 x_1 x_2 x_3 x_4 = 0, \tag{1}$$

where $\bar{\lambda} \in \mathbb{F}_q$ is a parameter. Candelas et al. [2003] observed that the zeta function of this variety can be written as

$$\frac{R_1(t,\bar{\lambda}) R_2(t,\bar{\lambda})^{20} R_3(t,\bar{\lambda})^{30}}{(1-t)(1-qt)(1-q^2t)(1-q^3t)},$$

where the R_i are of degree 4. Candelas et al. gave expressions in $\bar{\lambda}$ for the zeroes of the R_i: to explain this, note that we can lift this family to a family over the ring $\mathbb{Z}_q$ of Witt vector over $\mathbb{F}_q$. This enables us to consider this family as a family in $\mathbb{P}^4$ over the field of fractions $\mathbb{Q}_q$ of $\mathbb{Z}_q$. Assume that $\bar{\lambda} \in \mathbb{F}_q$ is chosen such that $X_{\bar{\lambda}}$ is smooth. Denote by λ the Teichmüller lift of $\bar{\lambda}$. Specifically, Candelas et al. show

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that the zeroes of the zeta function of $X_{\bar{\lambda}}$ can be expressed in certain solutions of the p-adic Picard–Fuchs equation (associated with the family $X_{\bar{\lambda}}$) evaluated at λ.

This fact was proved in a more general context, but less explicitly, by N. Katz [1968]. His description of the zeta function in terms of the Picard–Fuchs equation is exploited by Lauder [2004] in order to give an algorithm to count points on smooth hypersurfaces in $\mathbb{P}^n$.

Some other families are investigated by Kadir [2004]. She obtained similar results. From this, one might conjecture that various factors of the zeta function are enumerated by so-called (admissible) monomial types modulo certain equivalence relations. We come back to this in Section 1.3.

Kadir and Yui [2006] noticed that monomial types are occurring in the study of several objects related to (1), for example in the Picard–Fuchs equation or in the enumeration of the factors of the zeta function. In the case $\lambda = 0$, they also appear in the enumeration of the Jacobi sums needed to compute the number of points of the variety at $\lambda = 0$. They proved a certain correspondence between these monomial types for Fermat varieties. Our aim is to present a different view on the above mentioned phenomena.

We should mention that N. Katz [2007] and Rojas-Leon and Wan [2007] studied the zeta function of families similar to (1) by using (ℓ-adic) hypergeometric sheaves. We recommend [Katz 2007] for a discussion on previous results on the Dwork family.

The main object of study in this paper are families $X_{\bar{\lambda}}/\mathbb{F}_q$ defined by the vanishing of polynomials of the form

$$F_{\bar{\lambda}} := \sum_{i=0}^{n} x_i^{d_i} + \bar{\lambda} \prod_i x_i^{a_i} \tag{2}$$

in a weighted projective space $\mathbb{P} := \mathbb{P}(w_0, \ldots, w_n)$, with $w_i d_i = d$ for all i, the a_i are nonnegative and $\sum w_i a_i = d$; moreover, we assume that $\gcd(q, d) = 1$. Such families will be called one-parameter monomial deformations of a Fermat hypersurface. For the rest of the introduction fix such a weighted projective space, and such a one-parameter deformation of a Fermat hypersurface. Let $\boldsymbol{a}$ denote the vector $(w_0 a_0, w_1 a_1, \ldots, w_n a_n) \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$. We call $\boldsymbol{a}$ the *deformation vector*.

The main technical result of this paper implies that the p-adic Picard–Fuchs equation associated with such a family is a generalized hypergeometric differential equation. We refer to Sections 1.2 and 5 for more on this.

Let $U_{\bar{\lambda}} := \mathbb{P} \setminus X_{\bar{\lambda}}$. Since

$$Z(X_{\bar{\lambda}}, t) Z(U_{\bar{\lambda}}, t) = Z(\mathbb{P}, t),$$

the value of $Z(X_{\bar{\lambda}}, t)$ is uniquely determined by $Z(U_{\bar{\lambda}}, t)$. Hence from now on we will only discuss how to calculate $Z(U_{\bar{\lambda}}, t)$.

1.1. ***Choice of the cohomology theory.*** The Lefschetz fixed point formula allows us to prove statements on the zeta function by considering the action of geometric Frobenius on certain cohomology groups. Very often one uses étale cohomology. This is particularly useful when one wants to compare results in characteristic $p > 0$ with results in characteristic 0, or if one wants to consider Galois-representations on certain ℓ-adic vector spaces.

However, for our purposes it seems more natural to use p-adic cohomology theories instead. One can represent cohomology classes of a variety over a finite field $\mathbb{F}_q$ by differential forms with coefficients in $\mathbb{Q}_q$. This allows us to perform several (basic) analytic tricks when computing with cohomology classes.

To be more precise, let λ be a lift of $\overline{\lambda}$ to $\mathbb{Q}_q$, let F_λ be a lift of $F_{\overline{\lambda}}$ and $U_{\overline{\lambda}}$. Since U_λ is affine, we can define Monsky–Washnitzer groups cohomology (see Section 3) $H^i(U_\lambda, \mathbb{Q}_q)$. The elements in $H^i(U_\lambda, \mathbb{Q}_q)$ are differential forms with $\mathbb{Q}_q$-coefficients. There is a lift Frob_q of the Frobenius acting on these groups.

To illustrate how explicitly one can compute with Monsky–Washnitzer cohomology, we proceed to produce a basis for $H^i(U_\lambda)$. Let

$$\Omega := \Big(\prod_j x_j\Big) \sum (-1)^i w_i \frac{dx_0}{x_0} \wedge \frac{dx_1}{x_1} \wedge \cdots \wedge \widehat{\frac{dx_i}{x_i}} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

Proposition 1.1. *Let $X_{\overline{\lambda}}$ be quasismooth. Then the cohomology groups $H^i(U_\lambda, \mathbb{Q}_q)$ are zero except for $i = 0, n$. The group $H^0(U_\lambda, \mathbb{Q}_q)$ is one-dimensional and Frobenius acts trivially on it. The following set is a basis for $H^n(U_\lambda, \mathbb{Q}_q)$:*

$$\left\{ \frac{\prod_{i=0}^n x_i^{k_i}}{(F_\lambda)^t} \Omega : 0 \le k_i < d_i - 1\ \forall i,\ \sum_i w_i(k_i + 1) = td \right\}.$$

This basis will be called the *standard basis*. We are not aware of a proper reference for this standard fact in our context. We prove this proposition in Section 3. Proposition 1.1 is a combination of Theorem 3.8 and Proposition 3.16.

The proof is based on the fact that for quasismooth $X_{\overline{\lambda}}$, de Rham cohomology of U_λ with $\mathbb{Q}_p$ coefficients is isomorphic to the Monsky–Washnitzer cohomology of $U_{\overline{\lambda}}$ [Baldassarri and Chiarellotto 1994]. By a theorem of Steenbrink [1977] we have the isomorphism

$$H^n_{\mathrm{dR}}(U_\lambda) \cong \bigoplus_{t>0} H^0(\Omega^n(tX_\lambda))/dH^0(\Omega^{n-1}((t-1)X_\lambda)).$$

The vector space on the right-hand side is very well understood.

However, if $X_{\overline{\lambda}}$ is not quasismooth then the dimension of the right-hand side depends on the choice of the lift λ. If we choose λ in such a way that X_λ is not quasismooth then the right-hand side is infinite-dimensional. In that case one needs to add more relations to get an isomorphism with $H^n(U_\lambda)$. Which relations one needs to add is not very well understood.

A vector

$$\boldsymbol{k} := (\overline{w_0(k_0+1)}, \dots, \overline{w_n(k_n+1)}) \in \prod_i (w_i\mathbb{Z}/d\mathbb{Z})$$

is called an admissible monomial type if for all i, we have $k_i \not\equiv -1 \bmod d_i$ and $\sum \overline{w_i(k_i+1)} \equiv 0 \bmod d$. Fix an admissible monomial type $\boldsymbol{k}$. Take elements $k_i \in \mathbb{Z}$ satisfying $0 \le k_i \le d_i - 2$ and $k_i \equiv \bar{k}_i \bmod d_i$. Then with $\boldsymbol{k}$ we associate the standard basis vector

$$\omega_{\boldsymbol{k}} := \frac{\prod x_i^{k_i}}{(F_\lambda)^t}\Omega.$$

Remark 1.2. The results mentioned in Section 3 imply that

$$Z(U_{\bar{\lambda}}, t) = \frac{\Big(\det\Big(I - q^n(\mathrm{Frob}_q^*)^{-1}t \mid H^n(U_\lambda, \mathbb{Q}_q)\Big)\Big)^{(-1)^{n+1}}}{(1-q^n t)}.$$

From here on we formulate our results in terms of the characteristic polynomial of $q^n(\mathrm{Frob}_q^*)^{-1}$ on $H^n(U_\lambda, \mathbb{Q}_q)$, rather than in terms of $Z(U_{\bar{\lambda}}, t)$.

1.2. ***Deformation behavior.*** We produce a solution to the p-adic Picard–Fuchs equation that turns out to give us a description of the dependence of λ of the action of Frobenius on $H^n(U_\lambda)$, where λ is in the p-adic unit disc.

Following [Katz 1968], we consider the commutative diagram

$$\begin{tikzcd}
H^n(U_{\lambda^q}) \arrow[r, "\mathrm{Frob}_q^*"] \arrow[d, "A(\lambda^q)"'] & H^n(U_\lambda) \arrow[d, "A(\lambda)"'] \\
H^n(U_0) \arrow[r, "\mathrm{Frob}_q^*"] & H^n(U_0)
\end{tikzcd}$$

where λ is on a small p-adic disc around the origin, and A is a solution to the Picard–Fuchs equation associated with the family X_λ. Using p-adic analytic continuation we can extend $A(\lambda)^{-1}\,\mathrm{Frob}_{q,0}^*\,A(\lambda^q)$ to the closed unit disc, although $A(\lambda)$ itself cannot be extended to the p-adic unit disc.

Let $\lambda_0 \in \mathbb{Q}_q$ be the Teichmüller lift of some element $\bar{\lambda}_0 \in \mathbb{F}_q$. Then $\lambda_0^q = \lambda_0$, hence the above diagram implies that the action of Frob_q on $H^n(U_{\lambda_0})$ can be recovered from the p-adic analytic continuation of $A(\lambda)^{-1}\,\mathrm{Frob}_{q,0}^*\,A(\lambda^q)$. Therefore, to determine the zeta function of $X_{\bar{\lambda}_0}$ we need to know the Frobenius action in the Fermat-case (see 1.3) and compute the correct solution of the Picard–Fuchs equation.

We describe the action of $A(\lambda)$ on the standard basis. We call two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ *strongly equivalent* if and only if there is a j_0 such that $\boldsymbol{k}-\boldsymbol{m}=j_0\boldsymbol{a}$, where $\boldsymbol{a}$ is the deformation vector (see above).

Theorem 1.3. *Let $\boldsymbol{k}$ be an admissible monomial type. Write $A(\lambda)\,\omega_{\boldsymbol{k}}=\sum c_{\boldsymbol{m}}(\lambda)\,\omega_{\boldsymbol{m}}$, where the sum is taken over all admissible monomial types. Then $c_{\boldsymbol{m}}(\lambda)$ is nonzero only if $\boldsymbol{k}$ and $\boldsymbol{m}$ are strongly equivalent. If this is the case then $c_{\boldsymbol{m}}(\lambda)$ is of the form $c_0\lambda^{j_0}F(\alpha_i;\,\beta_j;\,\lambda^d c_1^d)$, with F a p-adic generalized hypergeometric function with parameters α_i, β_j and $j_0\in\{0,1,\dots,d-1\}$ is chosen such that $\boldsymbol{k}-\boldsymbol{m}=\overline{j}_0\boldsymbol{a}$.*

Explicit formulas for the α_i, β_j, c_0 and c_1 are given in Lemma 5.1 and Proposition 5.3. See Section 5 for a proof of Theorem 1.3.

In our proof we exploit the fact that there is a straightforward way of computing in groups like $H^n(U_\lambda)$, relying on the fact that this group is a quotient of a module of differentials over a *power series* ring. This allows us to perform some easy analytic operations that would be impossible in a module of differentials over a polynomial ring.

1.3. ***Factorization of the zeta function.*** We call the case $\lambda=0$ the Fermat case. One can show that Frob_q^* on $H^n(U_0,\mathbb{Q}_q)$ sends the standard basis vector $\omega_{\boldsymbol{k}}$ to a constant $c_{\boldsymbol{k},q}$ times the standard basis vector $\omega_{\overline{q}\boldsymbol{k}}$. Hence, if $q\equiv 1 \bmod d$ then the standard basis is a basis of eigenvectors for Frob_q^*. In this case Theorem 1.3 tells us that for every admissible monomial type $\boldsymbol{k}$ the operator $\mathrm{Frob}_{q,\lambda}$ fixes the subspace spanned by the $\omega_{\boldsymbol{m}}$, where $\boldsymbol{m}$ is strongly equivalent to $\boldsymbol{k}$.

The general case is slightly different, for this we introduce another equivalence relation: we call two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ *weakly equivalent* if $j_0\in\mathbb{Z}/d\mathbb{Z}$ and invertible $s,t\in(\mathbb{Z}/d\mathbb{Z})^*$ exists such that $s\boldsymbol{k}+t\boldsymbol{m}=j_0\boldsymbol{a}$.

Theorem 1.4. *Let $\boldsymbol{k}$ be an admissible monomial type. Write*

$$\mathrm{Frob}_{q,\lambda}\,\omega_{\boldsymbol{k}}=\sum c_{\boldsymbol{m}}(\lambda)\,\omega_{\boldsymbol{m}},$$

where the sum is taken over all admissible monomial types. Then $c_{\boldsymbol{m}}(\lambda)$ is nonzero only if $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent.

This is a weak form of Theorem 6.4. Theorem 1.4 implies that the zeta function of U_λ can be factored (as a rational function with $\mathbb{Q}(\lambda)$-coefficients) in such a way that each factor corresponds to a weak-equivalence class. If one only considers the zeta function over fields containing all d-th roots of unity, then there is a factorization of the zeta function of U_λ such that each factor corresponds to a strong-equivalence class.

Explicitly determining the constants $c_{\boldsymbol{k},q}$ is actually very hard. In some cases it is known that the eigenvalues of Frob_q^* correspond to the Fourier coefficients of a modular form. For example if $n=2$, $w=(1,1,1)$ and $d=3$, then X_0 is the $j=0$

elliptic curve $x_0^3+x_1^3+x_2^3$. Also the case $n=3$, $w=(1,1,1,1)$, $d=4$ and the case $n=5$, $w=(1,1,1,1)$, $d=3$ are known to correspond to modular forms; see [Hulek and Kloosterman 2007; Shioda and Inose 1977].

A more general result on $c_{\boldsymbol{k},q}$ is due to Weil: Assume that $\mathbb{F}_q \supset \mathbb{F}_p(\zeta_d)$. Let χ be the d-th power residue symbol. Let $\boldsymbol{k}$ be an admissible monomial type. Let $\boldsymbol{k}_i$ be the i-th entry of $\boldsymbol{k}$, i.e., $w_i(k_i+1)$. Then

$$J_{\boldsymbol{k},q} := (-1)^{n+1} \sum_{(v_1,\dots,v_n)\in\mathbb{F}_q^n : \sum_i v_i=-1} \chi(v_1)^{\boldsymbol{k}_1}\chi(v_2)^{\boldsymbol{k}_2}\dots\chi(v_n)^{\boldsymbol{k}_n}.$$

The following theorem coincides with Corollary 6.9.

Theorem 1.5. *Assume q is chosen such that $\mathbb{F}_q \supset \mathbb{F}_p(\zeta_d)$. Let $\boldsymbol{k}$ be an admissible monomial type. Let S be the set of monomial types that are weakly equivalent to $\boldsymbol{k}$. Then the sets $\{q^{n-1}/c_{\boldsymbol{m},q} : m \in S\}$ and $\{J_{\boldsymbol{m},q} : m \in S\}$ coincide.*

1.4. ***Monomial-motive correspondence.*** We call $\boldsymbol{b} \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$ an admissible automorphism type if $\boldsymbol{b}=(w_0b_0, w_1b_1,\dots,w_nb_n) \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$ is such that

$$\sum w_ib_ia_i \equiv 0 \bmod d.$$

Define $\sigma_{\boldsymbol{b}}$ to be the automorphism

$$[x_0 : x_1 : \dots : x_n] \mapsto [\zeta_d^{w_0b_0}x_0 : \zeta_d^{w_1b_1}x_1 : \dots : \zeta_d^{w_nb_n}x_n].$$

We call two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ *distinguishable by automorphisms* if there exists an admissible automorphism type $\boldsymbol{b} \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$ such that

$$\sigma_{\boldsymbol{b}}\left(\prod x_i^{k_i}\right) = \prod x_i^{k_i} \quad \text{and} \quad \sigma_{\boldsymbol{b}}\left(\prod x_i^{m_i}\right) \neq \prod x_i^{m_i}.$$

Theorem 1.6. *Two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent if and only if $\boldsymbol{k}$ and $\boldsymbol{m}$ are not distinguishable by automorphisms.*

This result enables us to give a different proof for the monomial-motive correspondence of Kadir and Yui [2006], and to generalize it as follows: fix an admissible monomial type $\boldsymbol{k}$. Let $G_{\boldsymbol{k}}$ be the group of automorphisms of the form $\sigma_{\boldsymbol{b}}$ that fix $\omega_{\boldsymbol{k}}$. Then the subspace of $H^n(U)$ fixed by $G_{\boldsymbol{k}}$ is the spanned by the $\omega_{\boldsymbol{m}}$ such that $\boldsymbol{m}$ is weakly equivalent to $\boldsymbol{k}$. This can be also extended to the level of motives, i.e., we find a submotive $\mathfrak{h}(U_\lambda/G_{\boldsymbol{k}})$ of the (Chow-)motive $\mathfrak{h}(U_\lambda)$. Moreover, we obtain that

$$\mathfrak{h}(U_\lambda) = \bigoplus_{[\boldsymbol{k}]} \mathfrak{h}(U_\lambda/G_{\boldsymbol{k}}),$$

where we sum over all the weak-equivalence classes.

Kadir and Yui decompose $\mathfrak{h}(U_\lambda/G_{\boldsymbol{k}})$ further. To explain this, we need to change our context, and consider our family X_λ over the field $\mathbb{Q}$ of rational numbers. Then the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ acts nontrivially on $G_{\boldsymbol{k}}$, and this enables us to find correspondences in $CH^n(U_\lambda/G_{\boldsymbol{k}} \times U_\lambda/G_{\boldsymbol{k}})$ that decompose $\mathfrak{h}(U_\lambda/G_{\boldsymbol{k}})$ into smaller motives. It is easy to see that each such motive corresponds to a strong-equivalence class of monomial types. This correspondence between admissible monomial types and submotives of $\mathfrak{h}^n(U_\lambda)$ is called by Kadir and Yui *monomial-motive correspondence*. They also relate monomial types with the Picard–Fuchs equation. For this issue we refer to Section 1.2.

Kadir and Yui [2006] could only prove their monomial-motive correspondence if X_λ is a Calabi–Yau hypersurface of dimension 3 and $\lambda = 0$. The above discussion extends this correspondence to any quasismooth member of a one-parameter monomial deformation of a Fermat hypersurface in a weighted projective space, for any degree d such that $w_i|d$ for all i and provided that the characteristic does not divide d.

Kadir and Yui prove the monomial-motive correspondence using Jacobi sums. We take a more direct approach using subgroups of the automorphism group.

This paper is organized as follows: in Section 2 we fix some notation and list some standard definitions. In Section 3 we discuss Monsky–Washnitzer cohomology groups and recall some of the properties of these groups. In Section 4 we recall Katz' result on the deformation of the zeta function of a hypersurface in $\mathbb{P}^n$. In Section 5 we make Katz' result explicit. In Section 6 we discuss the Frobenius action on the cohomology of a Fermat hypersurface and prove some results on the structure of the zeta function of a monomial deformation of a Fermat hypersurface.

2. Notation

Fix once and for all :

- a prime p (the characteristic) and a positive integer r,
- an integer n (the dimension of the ambient space),
- a vector $(w_0, w_1, \dots, w_n) \in \mathbb{Z}^{n+1}$ such that none of the w_i is divisible by p.
- an integer d divisible by all the w_i and p does not divide d.

Set $q = p^r$ and $d_i := d/w_i$. Let $\mathbb{Q}_q$ denote the unique unramified extension of degree r of $\mathbb{Q}_p$. Let $w := \sum w_i$ denote the total weight. Let $\mathbb{P}_{\mathbb{F}_q} := \mathbb{P}^n_{\mathbb{F}_q}(w_0, \dots, w_n)$ be the associated weighted projective space over the finite field $\mathbb{F}_q$.

Definition 2.1. A *monomial type* $\boldsymbol{m} = (\overline{m}_0, \dots, \overline{m}_n)$ is an element of $\prod_i w_i\mathbb{Z}/d\mathbb{Z}$ such that $\sum \overline{m}_i = 0$ in $\mathbb{Z}/d\mathbb{Z}$. Choose representatives $m_i \in \mathbb{Z}$ of $\overline{m}_i$ such that $0 \le m_i < d$. The *relative degree* of $\boldsymbol{m}$ is $\sum m_i/d$.

Fix once and for all a monomial type $\boldsymbol{a}$ of relative degree 1, with at least 2 nonzero entries. We call $\boldsymbol{a}$ the *deformation vector*. Let a_i be integers such that

$0 \le a_i < d_i$ and $\boldsymbol{a} \equiv (\overline{w}_0\overline{a}_0, \dots, \overline{w}_n\overline{a}_n)$. Set

$$F_\lambda := \sum x_i^{d_i} + \lambda \prod x_j^{a_j}.$$

Let $F := F_0$. If $\overline{\lambda} \in \mathbb{F}_q$, denote by $X_{\overline{\lambda}}$ the zero set of $F_{\overline{\lambda}}$ in $\mathbb{P}$. If $\lambda \in \mathbb{Q}_q$, denote by X_λ the zero set of F_λ. Let $U_{\overline{\lambda}}$ be the complement $\mathbb{P} \setminus X_{\overline{\lambda}}$. Let U_λ be the complement $\mathbb{P} \setminus X_\lambda$.

Let $\Omega := \prod_i x_i \sum_j (-1)^j w_j \frac{dx_0}{x_0} \wedge \dots \wedge \widehat{\frac{dx_j}{x_j}} \wedge \frac{dx_n}{x_n}$.

Definition 2.2. A monomial type $\boldsymbol{k}$ is called *admissible* if there exist integers k_i, for $i = 0, \dots, n$, such that $0 \le k_i \le d_i - 2$ and $\boldsymbol{k} = (w_0(k_0+1), \dots, w_n(k_n+1))$. Let t be the relative degree of $\boldsymbol{k}$. With $\boldsymbol{k}$ we associate the differential form

$$\omega_{\boldsymbol{k}} := \frac{\prod x_i^{k_i}}{F_\lambda^t} \Omega.$$

Denote by $(a)_m$ the Pochhammer symbol $a(a+1)\dots(a+m-1)$.

Definition 2.3. Let $\pi : \mathbb{P}^n \to \mathbb{P}$ be the natural quotient map sending x_i to $x_i^{w_i}$. Let $G := \times \boldsymbol{\mu}_{w_i}/\Delta$ be the Galois group associated with this quotient. We call π the *standard quotient map* and G the *group associated with π*.

3. Monsky–Washnitzer cohomology

We will not define rigid cohomology in complete detail, but give a simplified presentation for the case of quasismooth hypersurfaces. For a good introduction to the theory of rigid cohomology we refer to [Berthelot 1983; 1997b].

Since U_λ is affine, we can write $U_\lambda = \operatorname{Spec} R_\lambda$, with

$$R_\lambda = \mathbb{Q}_q[\lambda, Y_0, \dots, Y_m]/(G_{1,\lambda}, \dots, G_{k,\lambda}).$$

Definition 3.1. Fix λ_0 in the closed p-adic unit disc and set

$$R_{\lambda_0}^\dagger = \frac{\{H \in \mathbb{Q}_q[[Y_0, \dots, Y_m]] : \text{the radius of convergence of } H \text{ is at least } r > 1\}}{(G_{1,\lambda_0}, \dots, G_{k,\lambda_0})}.$$

Then $R_{\lambda_0}^\dagger$ is called the *overconvergent completion* (or weak completion) of R_{λ_0}.

If π is the standard quotient map, G is its associated group (Definition 2.3), and $S := S_{\lambda_0}^\dagger$ is the overconvergent completion of the coordinate ring of $\mathbb{P}^n \setminus \pi^{-1}(X_{\lambda_0})$, there is on the module of differential forms Ω_S^i a natural G-action. Set $\Omega_R^i = (\Omega_S^i)^G$. The *$i$-th Monsky–Washnitzer cohomology group* $H^i(U_{\lambda_0}, \mathbb{Q}_q)$ is the i-th cohomology group of the complex $\Omega_R^\bullet$.

Notation 3.2. Let $X \subset \mathbb{P}$ be a quasiprojective variety. Denote by $H_{\text{rig}}^i(X)$ the *i-th rigid cohomology group* of X and by $H_{\text{rig},c}^i(X)$ the *i-th rigid cohomology group with compact support* of X, as defined in [Berthelot 1983].

There exists a second, equivalent, definition of $H^i(U_{\lambda_0}, \mathbb{Q}_q)$. This goes as follows: since $U_{\lambda_0,\mathrm{sing}}$ is affine, there is a ring S such that $U_{\lambda_0,\mathrm{sing}} = \operatorname{Spec} S$. Let $S^\dagger$ be an overconvergent completion of S. Let $\iota : \operatorname{Spec} R^\dagger_{\lambda_0} \setminus \operatorname{Spec} S^\dagger \to \operatorname{Spec} R^\dagger_{\lambda_0}$ be the inclusion. Let $\Omega^i_{\operatorname{Spec} R^\dagger_{\lambda_0}}$ be the sheaf $\iota_*\Omega_{\operatorname{Spec} R^\dagger_{\lambda_0}\setminus \operatorname{Spec} S^\dagger}$. Then define the Monsky–Washnitzer cohomology groups $H^i(U_{\lambda_0}, \mathbb{Q}_q)$ as the cohomology groups of the complex obtained by taking global sections. The proof that these two definitions are equivalent is very similar to [Dolgachev 1982, 2.2.4].

Definition 3.3. Let R be a ring over $\mathbb{Z}_q$. Let π be the maximal ideal of $\mathbb{Z}_q$. A *lift of Frobenius* is a ring homomorphism $\mathrm{Frob}_q^* : R \to R$ whose reduction modulo π,

$$\mathrm{Frob}_q^* \bmod \pi : R \otimes_{\mathbb{Z}_q} \mathbb{F}_q \to R \otimes_{\mathbb{Z}_q} \mathbb{F}_q,$$

is well-defined and equals $x \mapsto x^q$.

Fix a lift of Frobenius Frob_q^* to $R^\dagger_{\lambda_0}$, such that $\mathrm{Frob}_q^*(\lambda) = \lambda^q$. By abuse of notation we denote by Frob_q^* also the induced morphism on $H^i(U_{\lambda_0}, \mathbb{Q}_q)$.

Proposition 3.4. *There is a natural isomorphism*

$$H^i_{\mathrm{rig}}(U_{\bar{\lambda}_0}, \mathbb{Q}_q) \cong H^i(U_{\lambda_0}, \mathbb{Q}_q)$$

which is compatible with the action of Frobenius.

Proof. Similar to the proof of [Berthelot 1997b, Proposition 1.10]. □

Definition 3.5. Let K be a field. Let $G \in K[x_0, \dots, x_n]$ be a weighted homogeneous polynomial (with weights $(w_0, \dots, w_n)$). Let Y be the hypersurface $G = 0$ in $\mathbb{P}$. Then Y is said to be *quasismooth* if the affine cone $\operatorname{Spec} K[X_0, \dots, X_n]/G$ is smooth or has exactly one singular point, namely $(0, 0, \dots, 0)$.

Remark 3.6. If $\mathbb{P} = \mathbb{P}^n$ then a hypersurface $X \subset \mathbb{P}$ is quasismooth if and only if it is smooth.

An easy calculation shows:

Lemma 3.7. *Let $I = \{i \in \{0, 1, \dots, n\} : a_i \not\equiv 0 \bmod p\}$. Let $g = \gcd_{i\in I}(a_i w_i)$ and $d' := d/g$. If there is a nonzero a_i such that $a_i \equiv 0 \bmod p$ then X_λ is quasismooth for all λ. Otherwise, $X_{\bar{\lambda}}$ is quasismooth if and only if*

$$\bar{\lambda}^{d'} \neq \frac{(-1)^{d'}\bar{d}^{d'}}{\prod_{i\in I}(\bar{a}_i \bar{w}_i)^{a_i w_i/g}}.$$

Proof. Consider the partial derivative of F with respect to x_j. If $\bar{a}_j = 0$ then this derivative equals $x_j^{d_j-1}$ and vanishes if and only if $x_j = 0$.

Suppose there is a j such that $a_j \neq 0$ and $x_j = 0$. Then for all for all $k \neq j$ we have

$$0 = \frac{\partial F_{\bar{\lambda}}}{\partial x_k} = \bar{d}_k x_k^{d_k-1} + \bar{a}_k \bar{\lambda} \frac{\prod x_i^{a_i}}{x_k} = \bar{d}_k x_k^{d_k-1}.$$

This implies that all the x_k would vanish. Hence if $X_{\bar{\lambda}}$ is singular at $(x_0 : \cdots : x_n)$ then $x_j = 0$ if $\bar{a}_j = 0$ and $x_j \neq 0$ if $a_j \neq 0$. If there is a j such that p divides a nonzero a_j then $X_{\bar{\lambda}}$ is quasismooth.

Suppose now that p does not divide any of the positive a_i.

Suppose $\bar{a}_j \neq 0$. Consider now the derivative with respect to x_j:

$$\frac{\partial F_{\bar{\lambda}}}{\partial x_j} = \bar{d}_j x_j^{d_j-1} + \overline{a_j \lambda} \frac{\prod x_i^{a_i}}{x_j}.$$

This derivative vanishes if and only if

$$\bar{\lambda} \prod x_i^{a_i} = -\frac{\bar{d}_j}{\bar{a}_j} x_j^{d_j}.$$

In particular,

$$-\frac{\bar{d}_j}{\bar{a}_j} x_j^{d_j} = -\frac{\bar{d}_k}{\bar{a}_k} x_k^{d_k} \quad \text{for } j, k \in I.$$

Fix d_j-th roots α_j of $\bar{d}_j/\bar{a}_j$. Let ζ be a primitive d'-th root of unity. A solution of the above set of equations is of the form

$$x_j = \frac{\gamma^{w_j}}{\alpha_j} \zeta^{k_j w_j} \quad \text{for some } \gamma, k_j.$$

Substituting gives

$$\bar{\lambda} \prod_{i \in I} \frac{\zeta^{k_i w_i a_i}}{\alpha^{a_i}} = -1,$$

which is equivalent with

$$\bar{\lambda}^{d'} = (-1)^{d'} \prod \alpha^{a_j d'} = \frac{(-1)^{d'} \bar{d}^{d'}}{\prod_{i \in I} (\overline{a_i w_i})^{a_i w_i / g}}. \qquad \square$$

Let $X \subset \mathbb{P}$ be a hypersurface. Let $U = \mathbb{P} \setminus X$. Recall that we have a Gysin-type exact sequence (see [Berthelot 1983, Section 3])

$$\cdots \to H^{i-1}_{\text{rig},c}(X, \mathbb{Q}_q) \to H^i_{\text{rig},c}(U, \mathbb{Q}_q) \to H^i_{\text{rig},c}(\mathbb{P}, \mathbb{Q}_q) \to H^i_{\text{rig},c}(X, \mathbb{Q}_q) \to \cdots \quad (3)$$

Theorem 3.8. *Let $\lambda_0 \in \mathbb{Z}_q$ be such that $X_{\bar{\lambda}_0}$ is quasismooth. Then the groups $H^i(U_{\lambda_0}, \mathbb{Q}_q)$ are zero except for $i = 0, n$.*

Proof. Set $X = X_{\lambda_0}$ and $U = U_{\lambda_0}$. Consider first the case $\mathbb{P} = \mathbb{P}^n$. From Remark 3.6 it follows that U and X are smooth. Since $\Omega^i_R = 0$ for $i > n$ we have $H^i(U, \mathbb{Q}_q) = 0$ for $i > n$. Proposition 3.4 implies that $H^i_{\text{rig}}(U, \mathbb{Q}_q)$ is trivial for $i > n$. Using Poincaré duality [Berthelot 1997a] it follows that

$$H^i_{\text{rig},c}(U, \mathbb{Q}_q) = 0 \text{ for } i < n.$$

From (3) it follows that

$$H^i_{\text{rig},c}(X, \mathbb{Q}_q) \cong H^i_{\text{rig},c}(\mathbb{P}, \mathbb{Q}_q) \text{ for } i < n-1.$$

Using Poincaré duality, it follows that

$$H^i_{\text{rig}}(X, \mathbb{Q}_q) \cong H^i_{\text{rig}}(\mathbb{P}, \mathbb{Q}_q) \text{ for } n-1 < i < 2n.$$

Using that X is compact, it follows that

$$H^i_{\text{rig},c}(X, \mathbb{Q}_q) = H^i_{\text{rig}}(X, \mathbb{Q}_q) \cong H^i_{\text{rig}}(\mathbb{P}, \mathbb{Q}_q) = H^i_{\text{rig},c}(\mathbb{P}, \mathbb{Q}_q) \text{ for } n-1 < i < 2n.$$

Using the sequence (3) again, we obtain that $H^i_{\text{rig},c}(U, \mathbb{Q}_q) = 0$ for $i \notin \{n, 2n\}$. Applying Poincaré duality yields $H^i_{\text{rig}}(U, \mathbb{Q}_q) = H^i(U, \mathbb{Q}_q) = 0$ for $i \notin \{0, \dim U\}$.

The general case can be deduced from this as follows: consider the standard quotient map $\pi : \mathbb{P}^n \to \mathbb{P}$ sending x_i to $x_i^{w_i}$. Let Y be $\pi^{-1}(X)$. Let G be the group associated with π. From Lemma 3.7 it follows that X is quasismooth if and only if Y is smooth. Let V be the complement of Y in $\mathbb{P}^n$. Then from the above it follows that $H^i(V) = 0$ except for $i = 0, n$. In particular, $d\Omega_V^{j-1,\dagger} = \Omega_V^{j,\dagger}$. One easily shows that $(d\Omega_V^{j-1,\dagger})^G = d((\Omega_V^{j-1,\dagger})^G)$. This implies that

$$H^j(U) = \frac{(\Omega_V^{j,\dagger,cl})^G}{(d\Omega_V^{j-1,\dagger})^G} = 0 \quad \text{if } j \neq 0, n. \qquad \square$$

Remark 3.9. One might try to prove the vanishing of $H^i(U)$ for the complement of an arbitrary quasismooth hypersurface along the lines of the above proof. This fails if the following happens: Let $H_1, \dots, H_j$ be the coordinate hyperplanes corresponding to coordinates with weight $w_i > 1$. Suppose there is a subset of $\{H_i\}$ such that $X \cap H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$ is not quasismooth. Then $\pi^{-1}(X)$ is singular, so the strategy of the above proof does not apply. Conversely, if $\pi^{-1}(X)$ is singular then such a set of coordinate hyperplanes exists.

Theorem 3.10 (Poincaré duality for $H^i(U_\lambda, \mathbb{Q}_q)$). *Let $\bar{\lambda}_0 \in \mathbb{F}_q$ be such that $X_{\bar{\lambda}_0}$ is quasismooth. There is a nondegenerate pairing*

$$H^i_{\text{rig},c}(U_{\bar{\lambda}_0}, \mathbb{Q}_q) \times H^{2n-i}_{\text{rig}}(U_{\bar{\lambda}_0}, \mathbb{Q}_q) \to H^{2n}_{\text{rig},c}(U_{\bar{\lambda}_0}, \mathbb{Q}_q)$$

respecting the Frobenius action.

Proof. Set $X = X_{\lambda_0}$ and $U = U_{\lambda_0}$. Consider first the case $\mathbb{P} = \mathbb{P}^n$. Then from Lemma 3.7 it follows that U and X are smooth. The main theorem of [Berthelot 1997a] asserts the existence of such pairings.

The general case can be obtained as follows: consider the standard quotient map $\pi : \mathbb{P}^n \to \mathbb{P}$ sending x_i to $x_i^{w_i}$. Let Y be $\pi^{-1}(X)$. Let G be the group associated with π. From Lemma 3.7 it follows that X is quasismooth if and only if Y is smooth. Let V be the complement of Y in $\mathbb{P}^n$. Since Poincaré duality is G-equivariant, one obtains a pairing

$$H^i_{\mathrm{rig}}(V, \mathbb{Q}_q)^G \times H^{2n-i}_{\mathrm{rig},c}(V, \mathbb{Q}_q)^G \to H^{2n}_{\mathrm{rig},c}(V, \mathbb{Q}_q).$$

Using the isomorphism $(\Omega^k_V)^G \cong \Omega^k_U$, we obtain isomorphisms

$$H^i_{\mathrm{rig}}(V, \mathbb{Q}_q)^G \cong H^i_{\mathrm{rig}}(U, \mathbb{Q}_q) \quad \text{and} \quad H^{2n-i}_{\mathrm{rig},c}(V, \mathbb{Q}_q)^G \cong H^{2n-i}_{\mathrm{rig},c}(U, \mathbb{Q}_q).$$

This yields the proof. □

Theorem 3.11 (Lefschetz trace formula). *Let $\bar{\lambda}_0 \in \mathbb{F}_q$ be such that $X_{\bar{\lambda}_0}$ is quasismooth. Then*

$$\sum_i (-1)^i \operatorname{trace}((q^n(\mathrm{Frob}^*)^{-1}) | H^i(U_{\lambda_0})) = \#U_{\bar{\lambda}_0}(\mathbb{F}_q).$$

Proof. Combine the Lefschetz trace formula for rigid cohomology with compact support [Étesse and Le Stum 1993, théorème I] with Poincaré duality (Theorem 3.10) and Proposition 3.4. □

Proposition 3.12. *The group $H^0(U_\lambda, \mathbb{Q}_q)$ is one-dimensional, and Frobenius acts trivially on $H^0(U_\lambda, \mathbb{Q}_q)$.*

Proof. Straightforward. □

Let $H^n_{\mathrm{dR}}(U_\lambda, \mathbb{Q}_q)$ denote the algebraic de Rham cohomology of U_λ.

Theorem 3.13 (Baldassarri and Chiarellotto). *Suppose λ is chosen such that $X_{\bar{\lambda}}$ is quasismooth. Then the natural map*

$$H^n_{\mathrm{dR}}(U_\lambda, \mathbb{Q}_q) \to H^n(U_\lambda, \mathbb{Q}_q)$$

is an isomorphism.

Proof. Consider first the case $\mathbb{P} = \mathbb{P}^n$. Then this is precisely the main theorem of [Baldassarri and Chiarellotto 1994].

The general case can be obtained as follows: consider the standard quotient map $\pi : \mathbb{P}^n \to \mathbb{P}$ sending x_i to $x_i^{w_i}$. Let Y_λ be $\pi^{-1}(X_\lambda)$. Let G be the group associated with π. From Lemma 3.7 it follows that X_λ is quasismooth if and only if Y_λ is smooth. Let V_λ be the complement of Y_λ in $\mathbb{P}^n$. Then we have an isomorphism

$$H^n_{\mathrm{dR}}(V_\lambda, \mathbb{Q}_q) \to H^n(V_\lambda, \mathbb{Q}_q).$$

There is a natural G-action on both groups and it is easy to see that this isomorphism is G-equivariant. Moreover, using [Dolgachev 1982, Lemma 2.2.2] we obtain that π induces isomorphisms $H^n_{\mathrm{dR}}(V_\lambda)^G \cong H^n_{\mathrm{dR}}(U_\lambda)$ and $H^n(V_\lambda)^G \cong H^n(U_\lambda)$; hence the natural map

$$H^n_{\mathrm{dR}}(U_\lambda, \mathbb{Q}_q) \to H^n(U_\lambda, \mathbb{Q}_q)$$

is an isomorphism. □

Let G be the defining equation of a quasismooth hypersurface $Y \subset \mathbb{P}$. Let $V := \mathbb{P} \setminus Y$. Similar to the case of ordinary projective space, the algebraic de Rham cohomology of V can be computed using the complex $C^p_k = \Omega^p((k+p)Y)$. I.e., the hypercohomology group $\mathbb{H}^n(\mathbb{P}, C^\bullet_k)$ equals $H^0(\mathbb{P}, C^n_k)/dH^0(\mathbb{P}, C^{n-1}_k)$ and

$$H^n_{\mathrm{dR}}(V) = \bigoplus_k H^0(\mathbb{P}, C^n_k)/dH^0(\mathbb{P}, C^{n-1}_k).$$

(A proof of this equality can be obtained as follows. After fixing an embedding $\mathbb{Q}_q \hookrightarrow \mathbb{C}$ and tensoring both sides with $\mathbb{C}$, we obtain that it suffices to prove this result over $\mathbb{C}$. This is precisely the main result of [Steenbrink 1977].)

More explicitly, the vector space $H^n(V, \mathbb{Q}_q)$ can be identified with the quotient of the infinite-dimensional vector space spanned by

$$\frac{H}{G^t}\Omega$$

with $\deg(H) = t\deg(G) - \sum w_i$, by the relations

$$\frac{(t-1)HG_x - GH_x}{G^t}\Omega,$$

where the subscript x means the partial derivative with respect to a coordinate x on $\mathbb{P}$.

If $G = F$ (the polynomial whose zero-set is the Fermat hypersurface) then this formula reads as

$$\frac{(t-1)d_i H x_i^{d_i-1}}{F^t}\Omega = \frac{H_{x_i}}{F^{t-1}}\Omega$$

in $H^n(U)$. This motivates the following definition:

Definition 3.14. Let $\omega \in \Omega^n(U_0)$ be a form of the type

$$\frac{H}{F^t}\Omega$$

with H a monomial. Let x_i be a coordinate of $\mathbb{P}$ such that $x_i^{d_i-1}$ divides H. Then the *reduction of ω with respect to x_i* is the form

$$\frac{\frac{\partial}{\partial x_i}\left(\frac{H}{x_i^{d_i-1}}\right)}{(t-1)d_i F^{t-1}}\Omega.$$

The complete reduction $\operatorname{red}\omega := H'/F^s\Omega$ of ω is the form obtained by successively reducing with respect to the coordinates x_i of $\mathbb{P}$, such that for all i the exponent of x_i in H' is at most $d_i - 2$.

Note that ω and the reduction with respect to x_i of ω represent the same class in $H^n(U_0, \mathbb{Q}_q)$, and that the complete reduction of ω cannot be further reduced.

Definition 3.15. Let $P^\bullet$ be the pole order filtration on $H^n(U_\lambda)$, that is $\omega \in P^t$ if $\omega = \frac{G}{F_\lambda^t}\Omega$ for some $G \in \mathbb{Q}_q[x_0, \dots x_n]$.

Let $\boldsymbol{k}$ be an admissible monomial type. Recall that we can associate a differential form $\omega_{\boldsymbol{k}}$ with it. By definition $\omega_{\boldsymbol{k}}$ lies in P^t, where t is the relative degree of $\boldsymbol{k}$.

Proposition 3.16. *Let λ be such that X_λ is quasismooth. Then the set*

$$\{\omega_{\boldsymbol{k}} : \boldsymbol{k} \text{ an admissible monomial type}\}$$

is a basis for $H^n(U_\lambda, \mathbb{Q}_q)$.

Proof. The above discussion implies the statement for $\lambda = 0$.

We start by proving that for every integer t the set

$$\{w_{\boldsymbol{k}} : \boldsymbol{k} \text{ an admissible monomial type of relative degree } t\}$$

is linearly independent in P^t/P^{t-1}.

The relations in P^t/P^{t-1} are generated by (cf. the discussion before Definition 3.14)

$$\frac{x_i^{d_i-1}\prod x_j^{k_j}}{F_\lambda^t}\Omega = \frac{-\lambda a_i}{d_i}\frac{\prod x_j^{k_j+a_j}}{x_i F_\lambda^t}\Omega.$$

Suppose i is chosen such that $a_i \neq 0$. Let

$$\sigma_i(G) := \frac{-d_i x_i^{d_i}}{\lambda a_i \prod_j x_j^{a_j}} G.$$

If G is a monomial of degree $td - \sum w_i$ such that all the exponents of the x_j are at least $a_j - \delta_{i,j}$, then

$$\frac{G}{F_\lambda^t}\Omega \equiv \frac{\sigma_i(G)}{F_\lambda^t}\Omega \bmod P^{t-1}.$$

Note that σ_i is defined if the exponent of x_j is at least $a_j - \delta_{i,j} d_i$, but σ_i corresponds to a relation in P^t only if the exponent x_j is at least $a_j - \delta_{i,j}$. Similarly, if the exponent of x_i in G is at least $d_i - 1$, then

$$\frac{G}{F_\lambda^t}\Omega \equiv \frac{\sigma_i^{-1}(G)}{F_\lambda^t}\Omega \bmod P^{t-1}.$$

Take a nontrivial expression $\sum b_{\boldsymbol{k}} \omega_{\boldsymbol{k}}$ that is zero modulo P^{t-1}. Since the σ_i generate the relations, and the σ_i map monomials to monomials, there exists two distinct admissible monomial types $\boldsymbol{k}, \boldsymbol{m}$ of relative degree t and a sequence of σ_i and σ_j^{-1} such that

$$\tau(\omega_{\boldsymbol{k}}) := \sigma_{i_s}^{\epsilon_s} \dots \sigma_{i_1}^{\epsilon_1}(\omega_{\boldsymbol{k}}) = c_0 \omega_{\boldsymbol{m}},$$

with

- $c_0 \in \mathbb{Q}_q$,
- $\epsilon_j \in \{\pm 1\}$,
- $a_{i_j} \neq 0$ for all j,
- for all j such that $\epsilon_j = 1$ and for all k, the exponent of x_k in $\sigma_{i_{j-1}}^{\epsilon_{j-1}} \dots \sigma_{i_1}^{\epsilon_1}(\omega_{\boldsymbol{k}})$ is at least $a_k - \delta_{i_j,k}$, and
- for all j such that $\epsilon_j = -1$, the exponent of x_{i_j} in $\sigma_{i_{j-1}}^{\epsilon_{j-1}} \dots \sigma_{i_1}^{\epsilon_1}(\omega_{\boldsymbol{k}})$ is at least $d_{i_j} - 1$.

We will prove below that given such a τ, we can always shorten the length of this expression by 2, and that this expression cannot consist of one σ_i. Hence the only possibility for τ is to be the identity and $k_i = m_i$ for all i, a contradiction.

We claim that $\epsilon_1 = 1$ and $\epsilon_s = -1$. If ϵ_1 were -1, then in order to apply σ_{i_1} we would need that the exponent x_{i_1} in $\omega_{\boldsymbol{k}}$ is at least $d_{i_1} - 1$, contradicting that $\omega_{\boldsymbol{k}}$ is associated with an admissible monomial type. Similarly, if $\epsilon_s = 1$ we obtain that the exponent of x_{i_s} in $\omega_{\boldsymbol{m}}$ is at least $d_{i_s} - 1$, contradicting that $\omega_{\boldsymbol{m}}$ is associated with an admissible monomial type.

Let j be the smallest integer such that $\epsilon_j = -1$. This implies that the exponent of x_{i_j} in $\sigma_{i_{j-1}} \dots \sigma_{i_1}(\prod x_i^{k_i})$ is at least $d_{i_j} - 1$, hence at least for one of the $j' < j$ we have $i_j = i_{j'}$. Let j' be the largest integer smaller than j such that $i_j = i_{j'}$.

Note that the σ_i commute as operators on $\mathbb{Q}_q(x_0, \dots, x_n)$. Hence, if we consider the σ_i as operators on $\mathbb{Q}_q(x_0, \dots x_n)$ then we have the identities

$$\begin{aligned}\sigma_{i_j}^{-1}\sigma_{i_{j-1}} \dots \sigma_{i_{j'}}\sigma_{i_{j'-1}} \dots \sigma_{i_1}\left(\prod x_i^{k_i}\right) &= \sigma_{i_j}^{-1}\sigma_{i_{j'}}\sigma_{i_{j-1}} \dots \sigma_{i_{j'+1}}\sigma_{i_{j'-1}} \dots \sigma_{i_1}\left(\prod x_i^{k_i}\right) \\ &= \sigma_{i_{j-1}} \dots \sigma_{i_{j'+1}}\sigma_{i_{j'-1}} \dots \sigma_{i_1}\left(\prod x_i^{k_i}\right).\end{aligned}$$

We need to show that the latter expression corresponds to a series of relations in P^t/P^{t-1}, i.e., we need to show that for each j'' such that $j' < j'' < j$, if

$$\sigma_{i_{j''}} \dots \sigma_{i_{j'+1}} \sigma_{i_{j'-1}} \dots \sigma_{i_1} \big(\prod x_i^{k_i}\big) = c \prod x_r^{e_r}$$

with $c \in \mathbb{Q}_q$ then $e_r \geq a_r - \delta_{r, i_{j''+1}}$ for all r.

Suppose that $r \neq i_j$. Since

$$\sigma_{i_{j''}} \dots \sigma_{i_1} \big(\prod x_i^{k_i}\big) = c' \prod x_r^{e'_r}$$

with $c' \in \mathbb{Q}_q$ and $e'_r \geq a_r - \delta_{r, i_{j''+1}}$ and σ_{i_j} lowers the exponent of x_r by a_r we obtain $e'_r = e_r - a_r$, whence $e_r \geq a_r - \delta_{r, i_{j''+1}}$.

Suppose that $r = i_j$. Since

$$\sigma_{i_{j-1}} \dots \sigma_{i_1} \big(\prod x_i^{k_i}\big) = c'' \prod x_r^{e''_r}$$

with $c'' \in \mathbb{Q}_q$ and $e''_r \geq d_r - 1$, it follows that

$$\sigma_{i_{j-1}} \dots \sigma_{i_{j'+1}} \sigma_{i_{j'-1}} \dots \sigma_{i_1} \big(\prod x_i^{k_i}\big) = c''' \prod x_r^{e'''_r}$$

with $c''' \in \mathbb{Q}_q$ and $e'''_r \geq 0$. Since the σ_{i_k} for $j'' < k < j'$ lower the exponent of x_r by a_r we obtain $e_r = e'''_r + (j - j'')a_r \geq a_r$.

We need to show that

$$\{\omega_{\boldsymbol{k}} : \boldsymbol{k} \text{ an admissible monomial type}\}$$

spans $H^n(U_\lambda, \mathbb{Q}_q)$. If $\lambda = 0$ then this follows from the discussion before this proposition. If $\lambda \neq 0$ and all the weights equal 1 then [Katz 1968, Theorem 1.10] shows that $\dim H^{n-1}(X_\lambda, \mathbb{Q}_q)$ is independent of λ. Using (3) we obtain that $\dim H^n(U_\lambda, \mathbb{Q}_q)$ is independent of λ. The general case follows from this case by applying the standard quotient map and [Dolgachev 1982, Lemma 2.2.2]. □

4. Deformation theory

Assume for the moment that $\mathbb{P} = \mathbb{P}^n$. Following N. Katz, consider the commutative diagram

$$\begin{array}{ccc} H^n(U_{\lambda^q}) & \xrightarrow{\mathrm{Frob}^*_{q,\lambda}} & H^n(U_\lambda) \\ {\scriptstyle A(\lambda^q)}\downarrow & & \downarrow{\scriptstyle A(\lambda)} \\ H^n(U_0) & \xrightarrow{\mathrm{Frob}^*_{q,0}} & H^n(U_0), \end{array}$$

where $\mathrm{Frob}_{q,\lambda}$ is the Frobenius acting on the complete family. Since it maps the fiber over 0 to the fiber over 0 this map can be restricted to U_0. Katz studied the differential equation associated to $A(\lambda)$. He remarked in a note that $A(\lambda)$ is actually the solution of the Picard–Fuchs equation.

We first give a way of computing a map $B(\lambda)$ such that

$$\mathrm{Frob}_{q,0}^{*} B(\lambda^q) = B(\lambda)\,\mathrm{Frob}_{q,\lambda}$$

on a small neighborhood of 0. This matrix $B(\lambda)$ is enough to deduce $\mathrm{Frob}_{q,\lambda}^{*}$ from $\mathrm{Frob}_{q,0}^{*}$.

Fix a basis

$$\frac{G_i}{F_\lambda^t}\Omega$$

for $H^n(U_\lambda)$ and write

$$\frac{G_i}{F_\lambda^t}\Omega = \sum_{j=0}^{\infty} \binom{j+t-1}{j} \frac{G_i(F-F_\lambda)^j}{F^{j+t}}\Omega. \tag{4}$$

Since $F - F_\lambda$ is the product of λ with a polynomial with integral coefficients, the above power series in the x_i converges on a small disc. By choosing λ sufficiently small, we obtain an overconvergent power series in the x_i, hence $\frac{G_i}{F_\lambda^t}$ defines an element of $H^n(U_0)$. Let $B(\lambda) : H^n(U_\lambda) \to H^n(U_0)$ be the analytic continuation of the operator mapping

$$\frac{G_i}{F_\lambda^t}\Omega$$

to the complete reduction of (4) in $H^n(U_0)$.

In this way we obtain a local expansion of the matrix $B(\lambda)$ around $\lambda = 0$. In the following section we will make this more explicit.

Proposition 4.1 (Katz). *We have* $B(\lambda)\,\mathrm{Frob}_{q,\lambda}^{*} = \mathrm{Frob}_{q,0}^{*} B(\lambda^q)$ *and* $B(\lambda) = A(\lambda)$.

Proof. The case $\mathbb{P} = \mathbb{P}^n$ is a combination of [Katz 1968, Lemma 2.10, Lemma 2.13, Theorem 2.14]. The general case is a formal consequence of the special case by Lemma 3.7, Proposition 4.1 and the definition of $H^n(U_\lambda, \mathbb{Q}_q)$ in terms of the standard quotient map $\pi : \mathbb{P}^n \to \mathbb{P}$. □

Remark 4.2. Proposition 4.1 is particularly interesting in the case when we specialize to $\lambda = \lambda_0$ where λ_0 is the Teichmüller lift of some element $\bar{\lambda}_0$. Then $\lambda_0^q = \lambda_0$, hence $\mathrm{Frob}_{q,\lambda_0}^{*}$ is a lift of Frobenius on $H^n(U_{\lambda_0}, \mathbb{Q}_q)$. Using Theorem 3.11 and Theorem 3.8 we obtain that

$$Z(U_{\bar{\lambda}_0}, t) = \lim_{\lambda \to \lambda_0} \frac{\Bigl(\det\bigl(I - tq^n \mid A(\lambda^q)^{-1}(\mathrm{Frob}_{q,0}^{*})^{-1}A(\lambda)\bigr)\Bigr)^{(-1)^{n+1}}}{1 - q^n t}.$$

5. Actual computation of the deformation matrix

In order to compute the matrix $A(\lambda)$ we need to reduce the right hand side of (4) in $H^n(U_0)$. We start with a very useful lemma.

Lemma 5.1. *Fix nonnegative integers b_i such that $\sum b_i w_i + w = td$ for some integer t. The complete reduction of $\omega := \frac{\prod x_i^{b_i}}{F^t}\Omega$ equals*

$$\frac{\prod_i ((c_i+1)w_i/d)_{q_i}}{(s)_{t-s}} \frac{\prod x_i^{c_i}}{F^s}\Omega,$$

where $0 \le c_i < d_i$ and q_i, s are integers such that $b_i = q_i d_i + c_i$, and $sd = \sum c_i w_i + w$, i.e., $t - s = \sum q_i$, provided that $c_i \neq d_i - 1$ for all i. If for one of the i we have $c_i = d_i - 1$ then ω reduces to zero in $H(U_0, \mathbb{Q}_q)$.

Proof.

The reduction with respect to x_0 of

$$\frac{x_0^{b_0}\prod_{i=1}^n x_i^{b_i}}{F^t}\Omega$$

(cf. Definition 3.14) equals

$$\frac{x_0^{b_0-d_0}((b_0+1)-d_0)\prod x_i^{b_i}}{(t-1)d_0F^{t-1}}\Omega = \frac{x_0^{b_0-d_0}((b_0+1)w_0-d)\prod x_i^{b_i}}{(t-1)dF^{t-1}}\Omega$$

(provided $b_0 \ge d_0$). After reducing q_i times with respect to x_i for $i = 0, \dots, n$, we obtain that ω reduces to

$$\frac{(s-1)!\prod_i \left(\prod_{j=0}^{q_i-1}((c_i+1)w_i + jd)\right)\prod x_i^{c_i}}{(t-1)!d^{t-s}F^s}\Omega.$$

This in turn equals

$$\tau := \frac{(s-1)!d^{\sum q_i}\prod((c_i+1)w_i/d)_{q_i}\prod x_i^{c_i}}{(t-1)!d^{t-s}F^s}\Omega.$$

If none of the c_i equals $d_i - 1$ then this is a complete reduction. Using $\sum q_i = t - s$ the first formula follows.

If $c_i = d_i - 1$ then we can write τ as $(F_{x_i}G/F^s)\Omega$, where G does not contain the variable x_i. The reduction of this form is a constant times

$$\frac{G_{x_i}}{F^{s-1}}\Omega.$$

Since $G_{x_i} = 0$, this reduction is zero. □

Fix an admissible monomial type $\boldsymbol{k} = (w_0(k_0+1), \dots, w_n(k_n+1)) \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$ of relative degree t. We want to calculate the reduction of

$$\frac{\prod x_i^{k_i}}{(F + \lambda \prod x_i^{a_i})^t}\Omega$$

in $H^n(U_0)$. In order to find a power series expression, we assume that λ is sufficiently small, then by (4) this form equals

$$\frac{\prod x_i^{k_i}}{F^t}\frac{1}{\left(1-\frac{(-\lambda)\prod x_i^{a_i}}{F}\right)^t}\Omega=\sum_j \binom{t+j-1}{j}\frac{\prod x_i^{k_i+a_i j}}{F^{t+j}}(-\lambda)^j\Omega. \tag{5}$$

Note that at most d distinct monomials occur in the reduction of the form.

Definition 5.2. Let r, s be nonnegative integers, let $\alpha_i \in \mathbb{Q}_q$, for $i \in \{1, 2, \ldots, r\}$, let $\beta_j \in \mathbb{Q}_q \setminus \mathbb{Z}_{<0}$ for $j \in \{1, 2, \ldots, s\}$. We define the *(generalized) hypergeometric function*

$$_rF_s\left(\begin{matrix}\alpha_1\ \alpha_2\ \ldots\ \alpha_r\\ \beta_1\ \beta_2\ \ldots\ \beta_s\end{matrix};z\right)$$

to be

$$\sum_{k=0}^{\infty} b_j z^j,$$

with $b_0 = 1$, and

$$\frac{b_{j+1}}{b_j}=\frac{(j+\alpha_1)\ldots(j+\alpha_r)}{(j+\beta_1)\ldots(j+\beta_s)(j+1)},$$

for all positive integers j.

Let d_i' be the order of a_i mod d_i in $\mathbb{Z}/d_i\mathbb{Z}$. Let d' be the least common multiple of all the d_i'. Set $b_i = a_i d'/d_i$. In the following proposition and its proof we identify elements in $a \in \mathbb{Z}/m\mathbb{Z}$ with their representative $\tilde{a} \in \mathbb{Z}$ such that $0 \le \tilde{a} \le m-1$.

Proposition 5.3. *Let $\boldsymbol{k}$ be an admissible monomial type. Let t be the relative degree of $\boldsymbol{k}$. Write $A(\lambda)\omega_{\boldsymbol{k}} = \sum c_{\boldsymbol{m}}(\lambda)\omega_{\boldsymbol{m}}$, where the sum is taken over all admissible monomial types. Then $c_{\boldsymbol{m}}(\lambda)$ is nonzero only if there is a $j_0 \in \mathbb{Z}$ with $0 \le j_0 \le d'-1$ and such that $\boldsymbol{m}-\boldsymbol{k} = \overline{j}_0\boldsymbol{a}$. If this is the case then*

$$\frac{c_{\boldsymbol{m}}(\lambda)}{\operatorname{red}\dfrac{\prod x_i^{a_i j_0+k_i}}{F^{t+j_0}}\Omega}$$

equals

$$\binom{t+j_0-1}{j_0}(-\lambda)^{j_0}{}_{d'}F_{d'-1}\left(\begin{matrix} & & \alpha_{i,s} & \\ \frac{j_0+1}{d'} & \frac{j_0+2}{d'} & \ldots\ \widehat{1}\ \ldots & \frac{j_0+d'}{d'}\end{matrix};\prod_{i:a_i\neq 0}\left(\frac{a_i}{d_i}\right)^{b_i}(-\lambda)^{d'}\right),$$

with

$$\alpha_{i,s}=\frac{(s-1)d_i+1+a_i j_0+k_i}{a_i d'}, s=1,\ldots,b_i; i=0,\ldots n.$$

This proposition almost gives a complete reduction of the form $\mathrm{Frob}_{q,\lambda}(\omega_{\boldsymbol{k}})$, in the sense that $c_{\boldsymbol{k}}(\lambda)$ is described as the product of a hypergeometric function and the reduction of a rational function in the x_i multiplied by Ω. The latter form can be easily reduced using Lemma 5.1.

Proof. It suffices to compute explicitly a complete reduction of $\omega := \frac{\prod x_i^{k_i}}{F_\lambda^t}\Omega$ in $H^n(U_0)$. We can write ω as

$$\sum_j \binom{t+j-1}{j} \frac{\prod x_i^{k_i+a_i j}}{F^{t+j}}(-\lambda)^j\Omega.$$

Set $c_{t,j} := \binom{t+j-1}{j}$. Since each reduction step decreases the exponent of x_i by d_i, we split this sum as follows: write

$$\omega = \sum_{j_0=0}^{d'-1}\sum_j c_{t,j_0+d'j}\frac{\prod x_i^{k_i+a_i(j_0+d'j)}}{F^{t+j_0+d'j}}(-\lambda)^{j_0+d'j}\Omega.$$

For $0 \leq j_0 \leq d'-1$ set

$$\omega_{j_0} := \sum_j c_{t,j_0+d'j}\frac{\prod x_i^{k_i+a_i(j_0+d'j)}}{F^{t+j_0+d'j}}(-\lambda)^{j_0+d'j}\Omega.$$

From Lemma 5.1 it follows that if $k_i+a_i(j_0+d'j) \equiv -1 \bmod d_i$ for some i, then ω_{j_0} reduces to zero. Otherwise, we claim that the reduction of ω_j is a generalized hypergeometric function. In order to prove this and to calculate the parameters, we need to show that

$$\frac{c_{t,j_0+d'j+d'} \operatorname{red} \dfrac{\prod x_i^{k_i+a_i(j_0+d'j)+a_i d'}}{F^{t+j_0+d'j+d'}}\Omega}{c_{t,j_0+d'j} \operatorname{red} \dfrac{\prod x_i^{k_i+a_i(j_0+d'j)}}{F^{t+j_0+d'j}}\Omega} \tag{6}$$

is a rational function in j. If we reduce with respect to x_i then the exponent of x_i is lowered by d_i. So if we reduce the numerator $b_i = a_i d'/d_i$ times with respect to x_i, then the exponent of x_i in the numerator and denominator coincide. Now

$$\operatorname{red} \frac{\prod x_i^{k_i+a_i(j_0+d'j)+a_i d'}}{F^{t+j_0+d'j+d'}}\Omega$$

equals

$$\frac{\prod_i \prod_{s=1}^{b_i}(k_i+a_i(j_0+d'j)+(s-1)d_i+1)}{(t+j_0+d'j)_{\sum b_i}\prod_i d_i^{b_i}} \operatorname{red} \frac{\prod x_i^{k_i+a_i(j_0+d'j)}}{F^{t+j_0+d'j}}\Omega$$

and

$$\frac{c_{t,\,j_0+d'j+d'}}{c_{t,\,j_0+d'j}} = \frac{(t+j_0+d'j)_{d'}}{(j_0+d'j+1)_{d'}}.$$

Putting this together we conclude that (6) equals

$$\frac{\prod_i \prod_{s=1}^{b_i}(k_i + a_i(j_0+d'j)+(s-1)d_i+1)}{(j_0+d'j+1)_{d'}\prod_i d_i^{b_i}}.$$

This equals

$$\frac{\prod_{i:a_i\neq 0}(a_i d')^{b_i}}{(d')^{d'}\prod_{i:a_i\neq 0} d_i^{b_i}}\;\frac{\prod_i \prod_{s=1}^{b_i}\Big(j+\frac{(s-1)d_i+1+a_i j_0+k_i}{a_i d'}\Big)}{\prod_{s=1}^{d'}\Big(j+\frac{j_0+s}{d'}\Big)}.$$

Since $\sum b_i = d'$ the first factor simplifies to

$$\prod_{i:a_i\neq 0}\left(\frac{a_i}{d_i}\right)^{b_i}.$$

From the second factor we can read off the hypergeometric parameters.

Since the first summand of ω_{j_0} equals

$$(-\lambda)^{j_0} c_{t,\,j_0}\frac{\prod x_i^{a_i j_0+k_i}}{F^{t+j_0}}\Omega,$$

by collecting everything together, we obtain that

$$\frac{\operatorname{red}\omega_j}{c_{t,\,j_0}(-\lambda)^{j_0}}$$

equals

$${}_{d'}F_{d'-1}\left(\begin{matrix} & & \alpha_{i,s} & \\ \frac{j_0+1}{d'} & \frac{j_0+2}{d'} & \ldots\ \widehat{1}\ \ldots & \frac{j_0+d'}{d'}\end{matrix}\;;\;(-\lambda)^{d'}\prod_{i:a_i\neq 0}\left(\frac{a_i}{d_i}\right)^{b_i}\right)\operatorname{red}\frac{\prod x_i^{a_i j_0+k_i}}{F^{t+j_0}}\Omega,$$

as desired. □

Example 5.4. Consider the family $X^3+Y^3+Z^3+\lambda XYZ$. Then we obtain the following matrix $A(\lambda)$ (with respect to the basis $\{\omega_{(1,1,1)},\omega_{(2,2,2)}\}$)

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac13 & \frac13\\ & \frac23\end{matrix}\;;\;\frac{-\lambda^3}{27}\right) & \frac{\lambda^2}{54}\,{}_2F_1\left(\begin{matrix}\frac43 & \frac43\\ & \frac53\end{matrix}\;;\;\frac{-\lambda^3}{27}\right)\\ -\lambda\,{}_2F_1\left(\begin{matrix}\frac23 & \frac23\\ & \frac43\end{matrix}\;;\;\frac{-\lambda^3}{27}\right) & {}_2F_1\left(\begin{matrix}\frac23 & \frac23\\ & \frac13\end{matrix}\;;\;\frac{-\lambda^3}{27}\right)\end{pmatrix}.$$

Example 5.5. Another famous example is $C_\lambda : X^4+Y^4+Z^4+\lambda X^2Y^2$.

Note that $d'=2$, $d_1'=d_2'=2$, $d_3'=0$, $b_1=b_2=1$, $b_3=0$.

One easily obtains

$$A(\lambda)\omega_{(2,1,1)} = {}_1F_0\left(\begin{matrix}\frac{1}{4}\\ -\end{matrix};\frac{\lambda^2}{16}\right)\omega_{(2,1,1)},\quad A(\lambda)\omega_{(1,2,1)} = {}_1F_0\left(\begin{matrix}\frac{1}{4}\\ -\end{matrix};\frac{\lambda^2}{16}\right)\omega_{(1,2,1)},$$

$$A(\lambda)\omega_{(2,3,3)} = {}_1F_0\left(\begin{matrix}\frac{3}{4}\\ -\end{matrix};\frac{\lambda^2}{16}\right)\omega_{(2,3,3)},\quad A(\lambda)\omega_{(3,2,3)} = {}_1F_0\left(\begin{matrix}\frac{3}{4}\\ -\end{matrix};\frac{\lambda^2}{16}\right)\omega_{(3,2,3)}.$$

$A(\lambda)$ acts as follows on the basis $\{\omega_{(1,1,2)},\omega_{(3,3,2)}\}$

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac{1}{4}\,\frac{1}{4}\\ \frac{1}{2}\end{matrix};\frac{\lambda^2}{16}\right) & \frac{\lambda^2}{16}\,{}_2F_1\left(\begin{matrix}\frac{5}{4}\,\frac{5}{4}\\ \frac{3}{2}\end{matrix};\frac{\lambda^2}{16}\right) \\ -\lambda\,{}_2F_1\left(\begin{matrix}\frac{3}{4}\,\frac{3}{4}\\ \frac{3}{2}\end{matrix};\frac{\lambda^2}{16}\right) & {}_2F_1\left(\begin{matrix}\frac{3}{4}\,\frac{3}{4}\\ \frac{1}{2}\end{matrix};\frac{\lambda^2}{16}\right)\end{pmatrix}.$$

It is classically known that the Jacobian of C_λ is isogenous to the product of two elliptic curves with j-invariant 1728 and one elliptic curve E_λ whose j-invariant depends properly on λ. This factor can also be obtained from the above information:

When we restrict $A(\lambda)$ to the subspace spanned by $\omega_{(1,1,2),(3,2,2)}$, we find the same operator as the operator $A'(\lambda)$ associated with the family $E_\lambda : X^4+Y^4+Z^2+\lambda X^2Y^2$ considered in $\mathbb{P}(1,1,2)$. One easily shows that this is a family of elliptic curves, with j-invariant depending on λ. The curve E_0 has an automorphism of order 4 with fixed points, hence $j(E_0)=1728$.

In the next section we prove that if $q\equiv 1 \bmod 4$ then all the $\omega_{\boldsymbol{k}}$ are eigenvectors for $\mathrm{Frob}_{q,0}^*$, let $c_{\boldsymbol{k},q}$ be the corresponding eigenvalue. Then, for $\boldsymbol{k}=(2,1,1)$,

$$\mathrm{Frob}_{q,\lambda}^*\,\omega_{\boldsymbol{k}} = A(\lambda)^{-1}\,\mathrm{Frob}_{q,0}^*\,A(\lambda^q)\omega_{\boldsymbol{k}} = \frac{{}_1F_0\left(\begin{matrix}\frac{1}{4}\\ -\end{matrix};\frac{\lambda^{2q}}{16}\right)}{{}_1F_0\left(\begin{matrix}\frac{1}{4}\\ -\end{matrix};\frac{\lambda^2}{16}\right)}c_{\boldsymbol{k},q}\omega_{\boldsymbol{k}}.$$

One easily shows that the factor in front of $c_{\boldsymbol{k},q}$ is a fourth root of unity, which implies that we have twisted the Frobenius action on $\omega_{\boldsymbol{k}}$ by a quartic character. Something similar happens when $\boldsymbol{k}\in\{(1,2,1),(2,3,3),(3,2,3)\}$. This implies that on a 4-dimensional subspace V_λ of $H^1(X_\lambda,\mathbb{Q}_q)$ the Frobenius action is a quartic twist of the Frobenius action on $V_0\subset H^1(X_0,\mathbb{Q}_q)$. The curve X_0 has the automorphism $[X,Y,Z]\mapsto[Z,X,Y]$. From this we obtain that the action of Frobenius on V_0 is isomorphic to two copies of the Frobenius action on E_0.

Example 5.6. Consider now the quintic threefold $X_0^5+X_1^5+X_2^5+X_3^5+X_4^5+\lambda X_0X_1X_2X_3X_4$. This family is studied for example by Candelas, de la Ossa, and

Rodriguez-Villegas [Candelas et al. 2003]. We discuss another aspect of this family in Example 6.11.

One can distinguish between the following five types of subspaces:

We start with $V_1 = \operatorname{span}\{\omega_{(1,1,1,1,1)}, \omega_{(2,2,2,2,2)}, \omega_{(3,3,3,3,3)}, \omega_{(4,4,4,4,4)}\}$. The corresponding matrix is

$$\begin{pmatrix} \Phi\binom{1}{2\,3\,4} & \frac{\lambda^4}{2^3\cdot 3\cdot 5^5}\Phi\binom{6}{7\,8\,9} & \frac{-\lambda^3}{2^2\cdot 3\cdot 5^5}\Phi\binom{6}{4\,7\,8} & \frac{\lambda^2}{2^2\cdot 3\cdot 5^5}\Phi\binom{6}{3\,4\,7} \\ -\lambda\,\Phi\binom{2}{3\,4\,6} & \Phi\binom{2}{1\,3\,4} & \frac{2\lambda^4}{3\cdot 5^5}\Phi\binom{7}{6\,8\,9} & \frac{-8\lambda^3}{3^3\cdot 5^5}\Phi\binom{7}{4\,6\,8} \\ \lambda^2\,\Phi\binom{3}{4\,6\,7} & -2\lambda\,\Phi\binom{3}{2\,4\,6} & \Phi\binom{3}{1\,2\,4} & \frac{27\lambda^4}{2^3\cdot 5^5}\Phi\binom{8}{6\,7\,9} \\ -\lambda^3\,\Phi\binom{4}{6\,7\,8} & 3\lambda^2\,\Phi\binom{4}{3\,6\,7} & -3\lambda\,\Phi\binom{4}{2\,3\,6} & \Phi\binom{4}{1\,2\,3} \end{pmatrix},$$

where we used the shorthand

$$\Phi\begin{pmatrix} a \\ b\;c\;d \end{pmatrix} := {}_4F_3\left(\begin{matrix} \frac{a}{5}\;\frac{a}{5}\;\frac{a}{5}\;\frac{a}{5} \\ \frac{b}{5}\;\frac{c}{5}\;\frac{d}{5} \end{matrix}\;;\;\frac{-\lambda^5}{5^5}\right).$$

The other four spaces are less interesting: on $V_2 = \operatorname{span}\{\omega_{(1,1,1,3,4)}, \omega_{(4,4,4,1,2)}\}$, $A(\lambda)$ acts as

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac15\;\frac15\\ \frac25\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & \frac{\lambda^2}{500}\,{}_2F_1\left(\begin{matrix}\frac65\;\frac65\\ \frac75\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \\ -4\lambda^3\,{}_2F_1\left(\begin{matrix}\frac45\;\frac45\\ \frac85\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & {}_2F_1\left(\begin{matrix}\frac45\;\frac45\\ \frac35\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \end{pmatrix}.$$

On $V_3 = \operatorname{span}\{\omega_{(2,2,2,1,3)}, \omega_{(3,3,3,2,4)}\}$, $A(\lambda)$ acts as

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac25\;\frac25\\ \frac45\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & \frac{\lambda^4}{6250}\,{}_2F_1\left(\begin{matrix}\frac75\;\frac75\\ \frac95\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \\ -2\lambda\,{}_2F_1\left(\begin{matrix}\frac35\;\frac35\\ \frac65\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & {}_2F_1\left(\begin{matrix}\frac35\;\frac35\\ \frac15\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \end{pmatrix}.$$

On $V_4 = \operatorname{span}\{\omega_{(1,1,2,2,4)}, \omega_{(3,3,4,4,1)}\}$, $A(\lambda)$ acts as

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac15\;\frac25\\ \frac35\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & \frac{-\lambda^3}{1875}\,{}_2F_1\left(\begin{matrix}\frac65\;\frac75\\ \frac85\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \\ \frac{\lambda^2}{5}\,{}_2F_1\left(\begin{matrix}\frac35\;\frac45\\ \frac75\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) & {}_2F_1\left(\begin{matrix}\frac35\;\frac45\\ \frac25\end{matrix}\;;\;\frac{-\lambda^5}{3125}\right) \end{pmatrix}.$$

On $V_5 = \text{span}\{\omega_{(3,3,1,1,2)}, \omega_{(4,4,2,2,3)}\}$, $A(\lambda)$ acts as

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac{1}{5}\ \frac{3}{5}\\ \frac{4}{5}\end{matrix};\frac{-\lambda^5}{3125}\right) & \frac{3\lambda^4}{25000}\,{}_2F_1\left(\begin{matrix}\frac{6}{5}\ \frac{8}{5}\\ \frac{9}{5}\end{matrix};\frac{-\lambda^5}{3125}\right) \\ -2\lambda\,{}_2F_1\left(\begin{matrix}\frac{2}{5}\ \frac{4}{5}\\ \frac{6}{5}\end{matrix};\frac{-\lambda^5}{3125}\right) & {}_2F_1\left(\begin{matrix}\frac{2}{5}\ \frac{4}{5}\\ \frac{1}{5}\end{matrix};\frac{-\lambda^5}{3125}\right) \end{pmatrix}.$$

Example 5.7. The final example is the family $X^4 + Y^4 + Z^4 + W^4 + \lambda XYZW$. This is a family of $K3$-surfaces. This family is also studied in [Dwork 1969, pp. 73–77].

Considered over a number field, every smooth member of this family has geometric Picard number 19 or 20. This implies that when we consider this family over a finite field, then every smooth member has geometric Picard number at least 20. From the Tate conjecture (which is proven in this case [Nygaard and Ogus 1985] if $p \geq 5$) it follows that every smooth member has Picard number 20 or 22. This implies that at least 19 of the eigenvalues of $\text{Frob}^*_{q,\lambda}$ on $H^3(U_\lambda)$ are of the form $q\zeta$, with ζ a root of unity. We will indicate how one can obtain this result from the methods described in this section.

First we calculate the operator $A(\lambda)$. We obtain that

$$A(\lambda)\,\omega_{(1,2,2,3)} = {}_1F_0\left(\begin{matrix}\frac{1}{2}\\ -\end{matrix};\frac{\lambda^4}{256}\right)\omega_{(1,2,2,3)}.$$

The operator $A(\lambda)$ leaves the space spanned by $\omega_{(1,1,3,3)}$ and $\omega_{(3,3,1,1)}$ invariant. Its action is as follows:

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\frac{\lambda^4}{256}\right) & \frac{\lambda^2}{32}\,{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\frac{\lambda^4}{256}\right) \\ \frac{\lambda^2}{32}\,{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\frac{\lambda^4}{256}\right) & {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\frac{\lambda^4}{192}\right) \end{pmatrix}.$$

One easily computes that

$${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\frac{\lambda^4}{256}\right) \pm \frac{\lambda^2}{32}\,{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\frac{\lambda^4}{256}\right) = {}_1F_0\left(\begin{matrix}\frac{1}{2}\\ -\end{matrix};\frac{\pm\lambda^2}{16}\right)$$

hence

$$A(\lambda)\,\omega_{(1,1,3,3)} \pm \omega_{(3,3,1,1)} = {}_1F_0\left(\begin{matrix}\frac{1}{2}\\ -\end{matrix};\frac{\pm\lambda^2}{16}\right)\omega_{(1,1,3,3)} \pm \omega_{(3,3,1,1)}.$$

As explained in the previous example, this implies that if $q \equiv 1 \bmod 4$ then $\text{Frob}^*_{\lambda,q}$ restricted to the subspace generated by the $\omega_{(1,2,2,3)}$, $\omega_{(1,1,3,3)}$ and all the coordinate permutations of these forms, is a (quartic) twist of $\text{Frob}^*_{0,q}$. Using Jacobi sums one can show that the $\text{Frob}^*_{0,q}$ restricted to this subspace has only eigenvalues of the form $q\zeta$, with ζ a root of unity. This yields 18 eigenvalues of $\mathrm{Frob}_{\lambda,q}$ of this form. Since the number of eigenvalues of $\mathrm{Frob}^*_{\lambda,q}$ that are not of this form is even, and the complementary subspace has dimension 3, there is a nineteenth eigenvalue of the form $q\zeta$.

The final subspace under consideration is $\mathrm{span}\{\omega_{(1,1,1,1)}, \omega_{(2,2,2,2)}, \omega_{(3,3,3,3)}\}$. We obtain the following matrix with respect to this basis:

$$\begin{pmatrix} {}_3F_2\left(\begin{matrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) & \frac{-\lambda^3}{1536}\, {}_3F_2\left(\begin{matrix} \frac{5}{4} & \frac{5}{4} & \frac{5}{4} \\ \frac{3}{2} & \frac{7}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) & \frac{\lambda^2}{1024}\, {}_3F_2\left(\begin{matrix} \frac{5}{4} & \frac{5}{4} & \frac{5}{4} \\ \frac{3}{4} & \frac{3}{2} & \end{matrix}; \frac{\lambda^4}{256}\right) \\ -\lambda\, {}_3F_2\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) & {}_3F_2\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) & \frac{-\lambda^3}{192}\, {}_3F_2\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{4} & \frac{7}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) \\ \lambda^2\, {}_3F_2\left(\begin{matrix} \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{5}{4} & \frac{3}{2} & \end{matrix}; \frac{\lambda^4}{256}\right) & -2\lambda\, {}_3F_2\left(\begin{matrix} \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{5}{4} & \end{matrix}; \frac{\lambda^4}{256}\right) & {}_3F_2\left(\begin{matrix} \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \end{matrix}; \frac{\lambda^4}{256}\right) \end{pmatrix}$$

6. Fermat hypersurfaces and equivalence relations

In the previous sections it is shown how to calculate the deformation matrix $A(\lambda)$. In this section we discuss the Frobenius action on the central fiber.

Lemma 6.1. *Let $\boldsymbol{k}$ be an admissible monomial type. Let $\boldsymbol{m} = \overline{q}\boldsymbol{k}$. We have* $\mathrm{Frob}_{0,q}\, \omega_{\boldsymbol{k}} = c_{\boldsymbol{k},q}\omega_{\boldsymbol{m}}$ *for some $c_{\boldsymbol{k},q}$.*

Proof. Take as a lift of Frobenius the morphism $x_i \mapsto x_i^q$. Then

$$\mathrm{Frob}^*_{q,0}(\omega) = \frac{x_i^{qk_i+q-1}}{F(x_i^q)^t}\Omega = \sum_{j=0} c_{t,j} \frac{x_i^{qk_i}(F^q - F(x_i^q))^j}{F^{qj+t}}.$$

One can easily show that any exponent of x_i in this sum is congruent to $qk_i + q - 1 \bmod d_i$. Hence there is only one monomial type $\boldsymbol{m}$ occurring in the reduction, namely $\overline{q}\boldsymbol{k}$. □

Remark 6.2. Suppose $q \equiv 1 \bmod d$. It is well-known that the eigenvalues of Frobenius on $H^n(U)$ are of the form $q^{n-1}/J_{\boldsymbol{k},q}$, where $J_{\boldsymbol{k},q}$ is a so-called Jacobi sum. Note that the assumption on q implies that $\overline{q}\boldsymbol{k} = \boldsymbol{k}$. So the set of Jacobi sums coincides with the set of $c_{\boldsymbol{k},q}$ (cf. the Introduction). A stronger result will be proved in the sequel.

Definition 6.3. Two monomial types are called *strongly equivalent* if and only if their difference is a multiple of the deformation vector. Two monomial types are called *weakly equivalent* if and only if there exists nonzero multiples of both monomial types that differ by the deformation vector.

The characteristic polynomial of Frobenius on the cohomology can be factorized in factors corresponding to the weak-equivalence classes of monomial types:

Theorem 6.4. *Let $\boldsymbol{k}$ be an admissible monomial type. Let S be the set of monomial types that are weakly equivalent to $\boldsymbol{k}$ and let S' be the set of monomial types that are strongly equivalent to $\boldsymbol{k}$. Then*

$$\mathrm{Frob}_{\lambda,q}\,\omega_{\boldsymbol{k}} = \sum_{\boldsymbol{m}\in S} c'_{\boldsymbol{m},q}\omega_{\boldsymbol{m}}$$

for some $c'_{\boldsymbol{m},q}\in\mathbb{Q}_q$. In particular, the characteristic polynomial $P(T)$ of Frobenius on $H^n(U)$ can be factored as $P(T)=\prod_{[\boldsymbol{k}]}P_{[\boldsymbol{k}]}(T)$, where the product is taken over all weak-equivalence classes, and $P_{[\boldsymbol{k}]}(T)$ is an element of $\mathbb{Q}_q[T]$ of degree equal to the number of distinct admissible monomial types in the weak-equivalence class $[\boldsymbol{k}]$.

If, moreover, $q\equiv 1 \bmod d$ then

$$\mathrm{Frob}_{\lambda,q}\,\omega_{\boldsymbol{k}} = \sum_{\boldsymbol{m}\in S'} c'_{\boldsymbol{m},q}\omega_{\boldsymbol{m}}$$

for some $c'_{\boldsymbol{m},q}\in\mathbb{Q}_q$. In particular, the characteristic polynomial $P(T)$ of Frobenius on $H^n(U)$ can be factored as $P(T)=\prod_{[\boldsymbol{k}]}P_{[\boldsymbol{k}]}(T)$, where the product is taken over all strong-equivalence classes, and $P_{[\boldsymbol{k}]}(T)$ is an element of $\mathbb{Q}_q[T]$ of degree equal to the number of distinct admissible monomial types in the strong-equivalence class $[\boldsymbol{k}]$.

Proof. Since $\mathrm{Frob}_{\lambda,q} = A(\lambda)^{-1}\,\mathrm{Frob}_{\lambda,0}\,A(\lambda^q)$, it suffices to prove that all these three operators leave the subspace $\mathrm{span}_{m\in S}(\omega_{\boldsymbol{m}})$ (if $q\not\equiv 1 \bmod d$) or the subspace $\mathrm{span}_{m\in S'}(\omega_{\boldsymbol{m}})$ (if $q\equiv 1 \bmod d$) invariant. For $A(\lambda)^{-1}$ and $A(\lambda^q)$ this follows from Proposition 5.3. For $\mathrm{Frob}_{\lambda,0}$ this follows from Lemma 6.1. □

Remark 6.5. In Corollary 6.10 we show that the factorization mentioned above gives factors which are polynomials with $\mathbb{Q}$-coefficients rather then with $\mathbb{Q}_q$-coefficients.

It remains to show that weak equivalence is the same relation as "indistinguishable by automorphisms".

Definition 6.6. We call $\boldsymbol{b}\in(\mathbb{Z}/d\mathbb{Z})^{n+1}$ an *admissible automorphism type* if $\boldsymbol{b}$ can be written as $(w_0b_0, w_1b_1,\dots,w_nb_n)\in(\mathbb{Z}/d\mathbb{Z})^{n+1}$, such that $\sum w_ib_ia_i\equiv 0 \bmod d$. Define $\sigma_{\boldsymbol{b}}$ to be the automorphism

$$[x_0:x_1:\cdots:x_n]\mapsto[\zeta_d^{w_0b_0}x_0:\zeta_d^{w_1b_1}x_1:\cdots:\zeta_d^{w_nb_n}x_n].$$

We call two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ *distinguishable by automorphisms* if there exists an admissible automorphism type $\boldsymbol{b}\in(\mathbb{Z}/d\mathbb{Z})^{n+1}$ such that

$$\sigma_{\boldsymbol{b}}\left(\prod x_i^{k_i}\right)=\prod x_i^{k_i}\quad\text{and}\quad\sigma_{\boldsymbol{b}}\left(\prod x_i^{m_i}\right)\neq\prod x_i^{m_i}.$$

Theorem 6.7. *Two monomial types $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent if and only if $\boldsymbol{k}$ and $\boldsymbol{m}$ are not distinguishable by automorphisms.*

Proof. One easily sees that $\sigma_{(b_i)}$ fixes $\omega_{\boldsymbol{k}}$ if and only if $\prod x_i^{k_i+1}$ is fixed by $\sigma_{(b_i)}$. This in turn is equivalent with

$$\sum b_i(k_i+1)w_i \equiv 0 \bmod d,$$

and similarly for $\boldsymbol{m}$.

('$\Rightarrow$'). Suppose $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent. Then we have a relation

$$s\boldsymbol{k} + t\boldsymbol{m} = r\boldsymbol{a},$$

with $s, t \in (\mathbb{Z}/d\mathbb{Z})^*$. It suffices to show that if $\boldsymbol{b}$ is an admissible automorphism type then

$$\sum b_i(k_i+1)w_i \equiv 0 \bmod d \iff \sum b_i(m_i+1)w_i \equiv 0 \bmod d.$$

Since $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent we have

$$s\sum b_i w_i(k_i+1) + t\sum b_i w_i(m_i+1) \equiv r\sum b_i a_i w_i \equiv 0 \bmod d.$$

Hence

$$s\sum b_i w_i(m_i+1) \equiv -t\sum b_i w_i(k_i+1) \bmod d.$$

Since s and t are invertible, the above claim follows.

('$\Leftarrow$'). Suppose $\boldsymbol{k}$ and $\boldsymbol{m}$ are not distinguishable by automorphisms. Take $\boldsymbol{b}$ such that $\sigma_{\boldsymbol{b}}(\omega_{\boldsymbol{k}}) = \omega_{\boldsymbol{k}}$ and $\sigma_{\boldsymbol{b}}(\omega_{\boldsymbol{m}}) \neq \omega_{\boldsymbol{m}}$ Hence

$$\sum b_i(k_i+1)w_i \equiv 0 \bmod d,$$

and

$$\sum b_i(m_i+1)w_i \not\equiv 0 \bmod d.$$

Suppose $\boldsymbol{k}$ and $\boldsymbol{m}$ are weakly equivalent, i.e., we have a relation

$$s\boldsymbol{k} + t\boldsymbol{m} = r\boldsymbol{a}$$

where s and t are invertible in $\mathbb{Z}/d\mathbb{Z}$. Then

$$s\sum b_i w_i(k_i+1) + t\sum b_i q_i(m_i+1) - r\sum b_i a_i q_i \equiv 0 \bmod d.$$

Since the first and third summand are zero, the same holds for the second summand. Contradicting that it should be nonzero. So we cannot have a relation

$$s\boldsymbol{k} + t\boldsymbol{m} = r\boldsymbol{a}.$$

Hence $\boldsymbol{k}$ and $\boldsymbol{m}$ are not weakly equivalent. $\square$

Definition 6.8. Assume that $q \equiv 1 \bmod d$ (i.e., $\mathbb{F}_q \supset \mathbb{F}_p(\zeta_d)$). Let χ be the d-th power residue symbol. Let $\boldsymbol{k}$ be an admissible monomial type. Let $\boldsymbol{k}_i$ be the i-th entry of $\boldsymbol{k}$, i.e., $w_i(k_i+1)$. Then the Jacobi-sum associated with $\boldsymbol{k}$ is defined as

$$J_{\boldsymbol{k},q} := (-1)^{n+1} \sum_{(v_1,\dots,v_n)\in\mathbb{F}_q^n : \sum_i v_i = -1} \chi(v_1)^{\boldsymbol{k}_1} \chi(v_2)^{\boldsymbol{k}_2} \dots \chi(v_n)^{\boldsymbol{k}_n}.$$

Corollary 6.9. *Assume $q \equiv 1 \bmod d$. Let $\boldsymbol{k}$ be an admissible monomial type. Let S be the set of monomial types that cannot be distinguished by automorphisms from $\boldsymbol{k}$. Then the sets $S_1 := \{q^n/c_{\boldsymbol{m},q} : m \in S\}$ and $S_2 := \{J_{\boldsymbol{m},q} : m \in S\}$ coincide.*

Proof. Let $G \subset \prod \mathbb{Z}/d_i\mathbb{Z}$ be the group of automorphisms that fixes $\omega_{\boldsymbol{k}}$. Then X_0/G is a Fermat variety in a different weighted projective space $\mathbb{P}'$. It is well-known that the eigenvalues of Frobenius on the primitive part of $H^{n-1}_{\mathrm{rig},c}(X_0/G)$ are Jacobi-sums appearing in S_2; see [Gouvêa and Yui 1995], for example.

The group $H^n(U/G)$ is canonically isomorphic with the subspace of $H^n(U)$ generated by the forms $\omega_{\boldsymbol{m}}$, where $\boldsymbol{m} \in S$ (this follows from [Dolgachev 1982, Lemma 2.2.2]). This implies that all the $q^n/c_{\boldsymbol{m},q}$ with $\boldsymbol{m} \in S$ are eigenvalues of Frob on $H^{n-1}_{\mathrm{rig},c}(X_0/G)$. Hence $S_1 = S_2$. □

Corollary 6.10. *Let $\bar{\lambda} \in \mathbb{F}_q$. Let $P(t)$ be the characteristic polynomial of* $\mathrm{Frob}_{\bar{\lambda}}$ *on $H^n(U_\lambda, \mathbb{Q}_q)$. Then*

$$P(t) = \prod_{[\boldsymbol{k}]} P_{[\boldsymbol{k}]}(t),$$

where the product is taken over all weak-equivalence classes of admissible monomial types. Let $\boldsymbol{k}$ be an admissible polynomial type. Then $P_{[\boldsymbol{k}]}(t)$ is an element of $\mathbb{Q}[t]$ and its degree equals the number of admissible monomial types that are weakly equivalent with $\boldsymbol{k}$.

Proof. Fix for the moment a monomial type $\boldsymbol{k}$. Let $G_{\boldsymbol{k}} \subset \prod \mathbb{Z}/d_i\mathbb{Z}$ be the group of automorphisms that fixes $\omega_{\boldsymbol{k}}$. Then $X_0/G_{\boldsymbol{k}}$ is a Fermat variety in a different weighted projective space $\mathbb{P}'$ and $H^n(U_0/G_{\boldsymbol{k}}, \mathbb{Q}_q)$ is canonically isomorphic with the subspace of $H^n(U_0, \mathbb{Q}_q)$ generated by the form $\omega_{\boldsymbol{m}}$, where $\boldsymbol{m}$ is weakly equivalent with $\boldsymbol{k}$. This enables us to write

$$H^n(U) = \bigoplus_{[\boldsymbol{k}]} H^n(U/G_{[\boldsymbol{k}]}).$$

For every weak-equivalence class of monomial types, set $P_{[\boldsymbol{k}]}(t) \in \mathbb{Q}[t]$ to be the characteristic polynomial of Frobenius acting on $H^n(U/G_{[\boldsymbol{k}]})$. Then $P(t) = \prod P_{[\boldsymbol{k}]}(t)$, we have $P_{[\boldsymbol{k}]}(t) \in \mathbb{Q}[t]$ and

$$\deg(P_{[\boldsymbol{k}]}(t)) = \dim H^n(U/G_{[\boldsymbol{k}]}) = \#\{\boldsymbol{m} : \boldsymbol{k} \text{ and } \boldsymbol{m} \text{ are weakly equivalent}\},$$

which finishes the proof. □

Example 6.11. Consider the case of the quintic threefold in $\mathbb{P}^4$, with deformation vector $a = (1, 1, 1, 1, 1)$. Up to interchanging coordinates we have the following five strong equivalence classes:

(1) $[0, 0, 0, 0, 0], [1, 1, 1, 1, 1], [2, 2, 2, 2, 2], [3, 3, 3, 3, 3]$.

(2) $[0, 0, 0, 2, 3], [3, 3, 3, 0, 1]$. (20 Permutations possible)

(3) $[1, 1, 1, 0, 2], [2, 2, 2, 1, 3]$. (20 Permutations possible)

(4) $[0, 0, 1, 1, 3], [2, 2, 3, 3, 0]$. (30 Permutations possible)

(5) $[2, 2, 0, 0, 1], [3, 3, 1, 1, 2]$. (30 Permutations possible)

The classes (2) and (3) form one weak-equivalence class, the same holds for (4) and (5). Over an arbitrary finite field we obtain three distinguishable factors of the zeta function, all three of degree 4. One factor is occurring with multiplicity 30, one factor is occurring with multiplicity 20, and one factor is occurring with multiplicity one. This is in agreement with [Candelas et al. 2003].

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Dual graded graphs for Kac–Moody algebras

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Motivated by affine Schubert calculus, we construct a family of dual graded graphs (Γ_s, Γ_w) for an arbitrary Kac–Moody algebra $\mathfrak{g}$. The graded graphs have the Weyl group W of $\mathfrak{g}\mathfrak{e}\mathfrak{h}$ as vertex set and are labeled versions of the strong and weak orders of W respectively. Using a construction of Lusztig for quivers with an admissible automorphism, we define folded insertion for a Kac–Moody algebra and obtain Sagan–Worley shifted insertion from Robinson–Schensted insertion as a special case. Drawing on work of Proctor and Stembridge, we analyze the induced subgraphs of (Γ_s, Γ_w) which are distributive posets.

1. Introduction

The Robinson–Schensted correspondence is perhaps the most important algorithm in algebraic combinatorics. It exhibits a bijection between permutations and pairs of standard Young tableaux of the same shape. Stanley [1988] investigated the class of differential posets (also studied in [Fomin 1986]). Fomin [1994] studied the more general notion of a dual graded graph to formalize local conditions which guarantee the existence of a Robinson–Schensted style algorithm.

In this article, we construct a family of dual graded graphs (Γ_s, Γ_w) associated to each Kac–Moody algebra $\mathfrak{g}$. These graded graphs have as vertex set the Weyl group W of $\mathfrak{g}$. The pair $(\Gamma_s, \Gamma_w) = (\Gamma_s(\Lambda), \Gamma_w(K))$ depends on a pair (Λ, K) where Λ is a dominant integral weight and K is a "positive integral" element of the center $Z(\mathfrak{g})$. In every case Γ_w is obtained by labeling the left weak order of W and Γ_s is obtained by labeling the strong Bruhat order of W.

These labelings are motivated by the Schubert calculus for homogeneous spaces associated to the Kac–Moody group G with Lie algebra $\mathfrak{g}$. For $w \in W$, let $\xi^w \in H^*(G/B)$ denote the cohomology Schubert classes of the flag manifold of G. If $\Lambda = \Lambda_i$ is the i-th fundamental weight, then an edge $v \lessdot w$ in $\Gamma_s(\Lambda)$ is labeled with

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the coefficient of ξ^w in the product $\xi^{s_i}\xi^v$, also called a Chevalley coefficient. When $\mathfrak{g}$ is of affine type and $K = K_{\mathrm{can}}$ is the canonical central element, the analogous statement holds (Proposition 2.17) for $\Gamma_w(K)$ with the homology Schubert classes $\xi_w \in H_*(\mathrm{Gr})$ of the affine Grassmannian corresponding to $\mathfrak{g}$ replacing the cohomology classes. Thus the combinatorics of these graphs encode computations in Schubert calculus, and the duality of the graded graphs (Γ_s, Γ_w) is a combinatorial skeleton of the duality between cohomology and homology of homogeneous spaces of G.

In the case of the affine Grassmannian, the dual graded graph structure arises from the pair of dual graded Hopf algebras given by $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$: one may define the down operator by the action of the homology class ξ_{s_0} on the Schubert basis of $H^*(\mathrm{Gr})$ and the up operator by multiplication by ξ^{s_i} for any fixed simple reflection s_i. It is a general phenomenon that pairs of dual graded combinatorial Hopf algebras yield dual graded graphs; we shall pursue this in a separate publication [Lam and Shimozono $\geq$ 2007].

Chains in the graded graphs (Γ_s, Γ_w), which we call strong and weak tableaux, are natural generalizations of standard Young tableaux. To go one speculative step further, we believe that the generating functions of an appropriate semistandard notion of strong and weak tableaux would give polynomials which *represent* certain homology and cohomology Schubert classes, in particular for homogeneous spaces corresponding to maximal parabolics, generalizing Schur functions, Schur Q-functions and the like. While this statement is vague in general, it can be made much more precise when $\mathfrak{g}$ is of affine type, and has already been achieved in one case.

In the case that $\mathfrak{g}$ is of the affine type $A_{n-1}^{(1)}$ our construction recovers the dual graded graphs that were implicitly studied in our joint work with Lapointe and Morse [2006]. The weak and strong tableaux in [Lam et al. 2006] are semistandard generalizations of the corresponding objects here; in the same work, an affine insertion algorithm was explicitly constructed for semistandard weak and strong tableaux, and from [Lam 2006; Lam et al. 2006] we know that the corresponding generating functions do indeed represent Schubert classes of the affine Grassmannian of type A. In the limit $n \to \infty$ of the $A_{n-1}^{(1)}$ case, our construction reproduces Young's lattice, which is the self-dual graded graph that gives rise to the Robinson–Schensted algorithm.

Having constructed the Kac–Moody dual graded graphs we study two further aspects of these graphs in detail.

The first aspect is motivated by the relation between the Robinson–Schensted insertion and Sagan–Worley shifted insertion. Using Lusztig's construction [1993] which associates to each symmetrizable generalized Cartan matrix A, a symmetric generalized Cartan matrix B equipped with an admissible automorphism π, we show that any dual graded graph of the form (Γ_s^A, Γ_w^A) for $\mathfrak{g}(A)$ can be realized in terms of one of the form (Γ_s^B, Γ_w^B) for $\mathfrak{g}(B)$. In particular, for any affine algebra, any of the dual graded graphs (Γ_s, Γ_w) can be realized using a dual graded graph for a simply-laced affine algebra. In particular we obtain a Schensted bijection for type $C_n^{(1)}$, using the insertion algorithm of [Lam et al. 2006] for type $A_{2n-1}^{(1)}$. As n goes to infinity, the type $C_n^{(1)}$ insertion converges to Sagan–Worley insertion [Sagan 1987; Worley 1984]. As a related result, we define a notion of mixed insertion for dual graded graphs equipped with a pair of automorphisms. This generalizes Haiman's variants of Schensted insertion known as left-right, mixed, and doubly dual insertion [Haiman 1989].

The second aspect we investigate are the induced subgraphs of the pair (Γ_s, Γ_w) which are distributive lattices when considered as posets. These are precisely the conditions under which one may describe our strong and weak tableaux by "filling cells with numbers" as in a usual standard Young tableau. Here we draw on [Proctor 1984; 1999; Stembridge 1996], which classify the parabolic quotients of Weyl groups of simple Lie algebras whose left weak orders (or equivalently Bruhat orders) are distributive lattices. We sharpen these results slightly to show that in these cases, the distributivity is compatible with the edge labels of the graphs (Γ_s, Γ_w); see Section 6B. These distributive parabolic quotients have also appeared recently in the geometric work [Thomas and Yong 2006]. They show that in these cases one may use the *jeu-de-taquin* to calculate Schubert structure constants of the cohomology of (co)minuscule flag varieties. We do not recover this result, but we note that their notion of standard tableau, fits into our framework as strong (or weak) tableaux for the distributive parabolic quotients, with the edge labels forgotten.

2. Dual graded graphs for Kac–Moody algebras

2A. ***Dual graded graphs.*** We recall Fomin's notion of dual graded graphs [Fomin 1994]. A *graded graph* is a directed graph

$$\Gamma = (V, E, h, m)$$

with vertex set V and set of directed edges $E \subset V^2$, together with a grading function $h : V \to \mathbb{Z}_{\geq 0}$, such that every directed edge $(v, w) \in E$ satisfies $h(w) = h(v)+1$ and has a multiplicity $m(v, w) \in \mathbb{Z}_{\geq 0}$. Forgetting the edge labels m, Γ may be regarded as the Hasse diagram of a graded poset. We shall interpret $m(v, w)$ as making Γ into a directed multigraph in which there are $m(v, w)$ distinct edges from v to w.

Γ is *locally finite* if, for every $v \in V$, there are finitely many $w \in V$ such that $(v, w) \in E$ and finitely many $u \in V$ such that $(u, v) \in E$; we shall assume this condition without further mention. For a graded graph $\Gamma = (V, E, h, m)$ define the

$\mathbb{Z}$-linear down and up operators $D, U : \mathbb{Z}V \to \mathbb{Z}V$ on the free abelian group $\mathbb{Z}V$ of formal $\mathbb{Z}$-linear combinations of vertices, by

$$U_\Gamma(v) = \sum_{(v,w)\in E} m(v,w)w \quad \text{and} \quad D_\Gamma(w) = \sum_{(v,w)\in E} m(v,w)v.$$

A pair of graded graphs (Γ, Γ') is *dual* if Γ and Γ' have the same vertex sets and grading function but possibly different edge sets and edge multiplicities, such that

$$D_{\Gamma'} U_\Gamma - U_\Gamma D_{\Gamma'} = r\,\mathrm{Id} \tag{1}$$

as $\mathbb{Z}$-linear operators on $\mathbb{Z}V$, for some fixed $r \in \mathbb{Z}_{>0}$. We call r the *differential coefficient*. When $\Gamma = \Gamma'$ and all the edges have multiplicity one, we obtain the r-differential posets of [Stanley 1988].

Remark 2.1. The duality property implies that V is infinite.

Example 2.2. Let $\Gamma = \mathbb{Y}$ be Young's lattice, with $(\lambda, \mu) \in E$ if the diagram of the partition μ is obtained from that of λ by adding a single cell (in which case we say that the cell is λ*-addable* and μ*-removable*), all edge multiplicities are 1, and $h(\lambda) = |\lambda|$ is the number of cells in the diagram of λ. Then $(\mathbb{Y}, \mathbb{Y})$ is a pair of dual graded graphs with differential coefficient 1.

2B. *The labeled Kac–Moody weak and strong orders.* In this section a new family of dual graded graphs is introduced.

Let I be a set of Dynkin nodes and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix (GCM), that is, one with integer entries which satisfies $a_{ii} = 2$ for all $i \in I$, and for all $i \neq j$, $a_{ij} \leq 0$ and $a_{ij} < 0$ if and only if $a_{ji} < 0$. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the Kac–Moody algebra over $\mathbb{C}$ associated to A [Kac 1990], $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra, and $\mathfrak{h}^*$ the dual. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ be the simple roots, $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ the simple coroots, and $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ the fundamental weights, with $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ where $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ is the natural pairing. We assume that the simple roots are linearly independent and the dimension of $\mathfrak{h}$ is chosen to be minimal. Let W be the Weyl group of $\mathfrak{g}$: it has generators s_i for $i \in I$ and relations $s_i^2 = 1$ for $i \in I$ and $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $i \neq j$, where m_{ij} is $2, 3, 4, 6$ or ∞ according as $a_{ij}a_{ji}$ is $0, 1, 2, 3$ or > 3. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function on W. Let $\Delta_{\mathrm{re}} = W \cdot \{\alpha_i \mid i \in I\}$ be the set of real roots and $\Delta_{\mathrm{re}}^+ = \Delta_{\mathrm{re}} \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\,\alpha_i$ the positive real roots. The associated coroot $\alpha^\vee$ of $\alpha \in \Delta_{\mathrm{re}}^+$ is defined by $\alpha^\vee = u\alpha_i^\vee$, where $u \in W$ and $i \in I$ are such that $\alpha = u\alpha_i$. For $\alpha \in \Delta_{\mathrm{re}}$ let $s_\alpha = us_iu^{-1}$ denote the reflection associated to α. The *strong order* (or Bruhat order) $\leq$ on W is defined by the cover relations $w \lessdot ws_\alpha$ whenever $\ell(ws_\alpha) = \ell(w) + 1$ for some $\alpha \in \Delta_{\mathrm{re}}^+$ and $w \in W$. The *left weak order* $(W, \preceq)$ is the subposet of $(W, \leq)$ generated by the cover relations $w \prec s_i w$ whenever $\ell(s_i w) = \ell(w) + 1$ for some $i \in I$ and $w \in W$. The *left descent set* of v is defined by $\mathrm{Des}(v) = \{i \in I \mid s_i v \prec v\}$.

Given Λ in the set P^+ of dominant integral weights, let $\Gamma_s(\Lambda)$ be the graded graph with vertex set W and edges $(v, w) \in W^2$ such that $v \lessdot w$, with multiplicity $m_\Lambda(v, w) = \langle \alpha^\vee, \Lambda\rangle$, where $\alpha \in \Delta^+_{\mathrm{re}}$ is such that $w = vs_\alpha$. Let $i \in I$ and $u \in W$ be such that $\alpha = u\alpha_i$. Then

$$m_\Lambda(v, w) = \langle u\alpha_i^\vee, \Lambda\rangle = \langle \alpha_i^\vee, u^{-1}\Lambda\rangle. \tag{2}$$

Let $Z^+ = Z^+(\mathfrak{g}(A)) = Z(\mathfrak{g}(A)) \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$, where $Z(\mathfrak{g}(A))$ is the center of $\mathfrak{g}(A)$. If $K \in Z^+$, writing $K = \sum_{i\in I} k_i\alpha_i^\vee$, the vector $(k_i)_{i\in I}$ defines a linear dependence amongst the rows of A.

Given $K \in Z^+$, let $\Gamma_w(K)$ be the graded graph with vertex set W and edges $(v, w) \in W^2$ such that $v \prec w = s_i v$, with multiplicity

$$n_K(v, w) = k_i = \langle K, \Lambda_i\rangle. \tag{3}$$

Both $\Gamma_s(\Lambda)$ and $\Gamma_w(K)$ are graded by the length function.

Theorem 2.3. *Let $(\Lambda, K) \in P^+ \times Z^+$. Then $(\Gamma_s(\Lambda), \Gamma_w(K))$ is a pair of dual graded graphs with differential coefficient $r = \langle K, \Lambda\rangle$.*

Proof. Let $U = U_{\Gamma_s(\Lambda)}$ and $D = D_{\Gamma_w(K)}$. The coefficient of $u \neq v$ in $(DU - UD)v$ is given by

$$\sum_{\substack{(i,\alpha)\in I\times\Delta^+_{\mathrm{re}} \\ v\lessdot vs_\alpha \\ u = s_i vs_\alpha \prec vs_\alpha}} k_i\langle\alpha^\vee, \Lambda\rangle - \sum_{\substack{(i,\alpha)\in I\times\Delta^+_{\mathrm{re}} \\ s_i v\prec v \\ s_i v \lessdot s_i vs_\alpha = u}} k_i\langle\alpha^\vee, \Lambda\rangle.$$

This quantity is zero because the indexing sets of both sums coincide, by two versions of [Humphreys 1990, Lemma 5.11].

For every $i \in I$ and $v \in W$, either $v \prec s_i v$ or $s_i v \prec v$ is a covering relation. It follows that the coefficient of v in $(DU - UD)v$ is

$$\begin{aligned}\sum_{i\in I\setminus \mathrm{Des}(v)} k_i\langle v^{-1}\alpha_i^\vee, \Lambda\rangle - \sum_{i\in \mathrm{Des}(v)} k_i\langle (s_i v)^{-1}\alpha_i^\vee, \Lambda\rangle \qquad&\\ = \sum_{i\in I} k_i\langle v^{-1}\alpha_i^\vee, \Lambda\rangle = \sum_{i\in I} k_i\langle \alpha_i^\vee, v\Lambda\rangle&\\ = \langle K, v\Lambda\rangle = \langle v^{-1}K, \Lambda\rangle = \langle K, \Lambda\rangle.&\end{aligned}$$

We have used the W-invariance of $\langle\cdot,\cdot\rangle$ and K. $\square$

Remark 2.4. For $i \in I$ and $v \in W$, the multiplicity of the edge (v, vs_α) in $\Gamma_s(\Lambda_i)$, is the Chevalley multiplicity, given by the coefficient of ξ^{vs_α} in the product $\xi^{s_i}\xi^v$, where $\xi^v \in H^*(G/B)$ is the Schubert cohomology class for the flag manifold G/B associated with the Kac–Moody algebra $\mathfrak{g}$ [Kostant and Kumar 1986].

In Proposition 2.17 we will relate the multiplicities of the weak graph $\Gamma_w(K)$ with the homology multiplication of the affine Grassmannian, in the case that $\mathfrak{g}(A)$ is of untwisted affine type.

2C. *Tableaux and enumeration.* Let $\Gamma = (V, E, h, m)$ be a graded graph and $v, w \in V$. A Γ-*tableau* T of shape v/w is a directed path

$$T = (w = v_0 \xrightarrow{m_1} v_1 \xrightarrow{m_2} \cdots \xrightarrow{m_k} v_k = v)$$

from w to v in Γ, where m_i is an element taken from a set of $m(v_{i-1}, v_i)$ possible markings of the edge (v_{i-1}, v_i). In the multigraph interpretation, m_i indicates which of the $m(v_{i-1}, v_i)$ edges going from v_{i-1} to v_i, is traversed by the path.

Let $v = \mathrm{out}(T)$ and $w = \mathrm{in}(T)$ the outer and inner shapes of T respectively, and denote by $\mathcal{T}(\Gamma)$ the set of Γ-tableaux, and $\mathcal{T}(\Gamma, v/w)$ the subset of those of shape v/w.

Example 2.5. For $\mathbb{Y}$ as in Example 2.2, $\mathbb{Y}$-tableaux are standard Young tableaux.

If Γ has a unique minimum element $\hat{0}$, we say T has shape v if $\mathrm{in}(T) = \hat{0}$.

Theorem 2.6 [Fomin 1994]. *Let (Γ, Γ') be a pair of dual graded graphs with differential coefficient r. Then*

$$r^n\, n! = \sum_{\substack{v \in V \\ h(v)=n}} f_\Gamma^v f_{\Gamma'}^v \tag{4}$$

where $f_\Gamma^v = |\mathcal{T}(\Gamma, v)|$ and $f_{\Gamma'}^v = |\mathcal{T}(\Gamma', v)|$.

Example 2.7. Let $\Gamma = \Gamma' = \mathbb{Y}$. Then by Examples 2.2 and 2.5, Equation (4) is the well known identity $n! = \sum_\lambda f_\lambda^2$, where λ ranges over the partitions of n and f_λ is the number of standard Young tableaux of shape λ.

The graphs $\Gamma_w(K)$ and $\Gamma_s(\Lambda)$ both have minimum element $\mathrm{id} \in W$. We call $\Gamma_w(K)$-tableaux *(standard) K-weak tableaux* and $\Gamma_s(\Lambda)$-tableaux *(standard) Λ-strong tableaux.*

Corollary 2.8. *Let $\mathfrak{g}$ be a Kac–Moody algebra and $(\Lambda, K) \in P^+ \times Z^+$. Then for each $n \in \mathbb{Z}_{\geq 0}$ we have*

$$r^n\, n! = \sum_{\substack{w \in W \\ \ell(w)=n}} f_{\mathrm{weak}}^w f_{\mathrm{strong}}^w \tag{5}$$

where $r = \langle K\,,\, \Lambda\rangle$, f_{weak}^w is the number of K-weak tableaux of shape w and f_{strong}^w is the number of Λ-strong tableaux of shape w.

In Section 3 standard Young tableaux are realized as special cases of both K-weak and Λ-strong tableaux using affine algebras of type $A^{(1)}$.

2D. ***From dual graded graphs to Schensted bijections.*** A *differential bijection* for the pair of dual graded graphs (Γ, Γ') is one that exhibits the equality (1); we give a precise definition below. By [Fomin 1995], a differential bijection induces a *Schensted bijection* (see (8)) which describes the enumerative identity (4).

We recall Fomin's theory in more detail. Given $r \in \mathbb{Z}_{\geq 0}$, let $S(r)$ be a set of cardinality r. Sometimes we may write $S(r)$ for a particular set of cardinality r. Let (Γ, Γ') be a pair of dual graded graphs with $\Gamma = (V, E, h, m)$ and $\Gamma' = (V, E', h, m')$ and differential coefficient r. Given $x, y \in V$, define $\mathcal{UD}_{xy} = \{(z, m, m') \in V \times \mathbb{Z}^2_{>0} \mid (z, y) \in E, (z, x) \in E', m \leq m(z, y), m' \leq m'(z, x)\}$ and let $\mathcal{DU}_{xy} = \{(w, M, M') \in V \times \mathbb{Z}^2_{>0} \mid (x, w) \in E, (y, w) \in E', M \leq m(x, w), M' \leq m'(y, w)\}$. $\mathcal{UD}_{xy}$ represents the set of marked paths going down one step in Γ' from x to some $z \in V$ and then up one step in Γ from z to y. Similarly, $\mathcal{DU}_{xy}$ represents the set of marked paths going up one step in Γ from x to some $w \in V$, and then down one step in Γ' from w to y. To cancel the off-diagonal terms in (1), for every $(x, y) \in V^2$ with $x \neq y$, there must be a bijection

$$\Phi_{xy} : \mathcal{UD}_{xy} \to \mathcal{DU}_{xy}, \tag{6}$$

and to obtain agreement of diagonal terms in (1), for each $x \in V$ there must be a bijection

$$\Phi_x : S(r) \sqcup \mathcal{UD}_x \to \mathcal{DU}_x \tag{7}$$

where $\mathcal{DU}_x = \mathcal{DU}_{xx}$ and $\mathcal{UD}_x = \mathcal{UD}_{xx}$. By definition, a *differential bijection* for (Γ, Γ'), is a collection $\Phi = (\Phi_{xy}; \Phi_x)$ of such bijections Φ_{xy} and Φ_x.

Example 2.9. Let $\Gamma = \mathbb{Y}$ with dual graded graph structure on $(\mathbb{Y}, \mathbb{Y})$ as in Example 2.2. For $\lambda \in \mathbb{Y}$, since all edges have multiplicity 1, $\mathcal{DU}_\lambda$ is in bijection with λ-addable corner cells and $\mathcal{UD}_\lambda$ is in bijection with λ-removable corner cells. The λ-addable and λ-removable corner cells of λ are interleaved. Let Φ_λ send the unique element of the set $S(1)$, to the λ-addable corner cell in the first row of λ, and send a λ-removable corner to the nearest λ-addable corner with higher row index. For $\lambda \neq \mu$ the sets $\mathcal{DU}_{\lambda\mu}$ and $\mathcal{UD}_{\lambda\mu}$ have the same cardinality, which is either 0 or 1, so there is no choice for the definition of $\Phi_{\lambda\mu}$. This defines a differential bijection Φ for $(\mathbb{Y}, \mathbb{Y})$.

Let $P_n(r)$ be the set of r-colored permutations of n elements. We realize $\sigma \in P_n(r)$ as an $n \times n$ monomial matrix (one with exactly one nonzero element in each row and in each column, whose nonzero entries must be taken from a set $S(r)$ of cardinality r such that $0 \notin S(r)$).

We assume that Γ and Γ' have a common minimum element $\hat{0}$ such that $h(\hat{0}) = 0$.

A *Schensted bijection* I for (Γ, Γ') is a family of bijections (for all $n \in \mathbb{Z}_{\geq 0}$)

$$\begin{aligned} P_n(r) &\to \bigsqcup_{\substack{v \in V \\ h(v)=n}} \mathcal{T}(\Gamma, v) \times \mathcal{T}(\Gamma', v) \\ \sigma &\mapsto (P, Q). \end{aligned} \tag{8}$$

We fix a differential bijection Φ for (Γ, Γ') and define the induced Schensted bijection I_Φ.

Given $\sigma \in P_n(r)$, we shall define a directed graph G with vertices $G_{ij} \in V$ for $0 \leq i, j \leq n$, which is depicted matrix-style. It shall have the property that (1) adjacent vertices $G_{i,j-1}, G_{ij}$ in a row, are either equal, or they form an edge $(G_{i,j-1}, G_{ij}) \in E$ with marking $m \in S(m(G_{i,j-1}, G_{ij}))$, and (2) adjacent vertices $G_{i-1,j}, G_{ij}$ in a column, are either equal, or they form an edge $(G_{i-1,j}, G_{ij}) \in E'$ with marking $m' \in S(m'(G_{i-1,j}, G_{ij}))$. Moreover, (3) $G_{i-1,j} \neq G_{ij}$ (resp. $G_{i,j-1} \neq G_{ij}$) if and only if the unique p such that $\sigma_{pj} \neq 0$ (resp. q such that $\sigma_{iq} \neq 0$) satisfies $p \leq i$ (resp. $q \leq j$). In particular (ignoring the equalities), for each i, the i-th row $G_{i\bullet}$ is a Γ-tableau and for each j, the j-th column $G_{\bullet j}$ is a Γ'-tableau. For the sake of uniform language we shall always imagine that there is a marked edge $G_{i,j-1} \to G_{ij}$ and $G_{i-1,j} \to G_{ij}$, but when the vertices coincide the marked edge degenerates.

G is defined inductively as follows. The north and west edges $G_{0\bullet}$ and $G_{\bullet 0}$ of G are initialized to the empty tableau: $G_{i0} = G_{0j} = \hat{0}$ for all $0 \leq i, j \leq n$. To define the rest of G, it suffices to give a *local rule*, which, given the marked edges

$$G_{i-1,j-1} \xrightarrow{m} G_{i-1,j} \quad \text{and} \quad G_{i-1,j-1} \xrightarrow{m'} G_{i,j-1},$$

and the value σ_{ij}, determines $G_{ij} \in V$ with markings $M \in S(m(G_{i,j-1}, G_{ij}))$ and $M' \in S(m'(G_{i-1,j}, G_{ij}))$.

This is depicted below. Use z, y, x, w to denote $G_{i-1,j-1}, G_{i-1,j}, G_{i,j-1}, G_{ij}$ for convenience and write $c = \sigma_{ij}$. In later examples we shall indicate $\sigma_{ij} = 1$ by the symbol $\otimes$ and $\sigma_{ij} = 0$ by a blank.

$$\begin{tikzcd} G_{i-1,j-1} \arrow[r, "m"] \arrow[d, "m'"'] \arrow[dr, phantom, "\sigma_{ij}"] & G_{i-1,j} \arrow[d, dashed, "M'"] \\ G_{i,j-1} \arrow[r, dashed, "M"'] & G_{ij} \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, "m"] \arrow[d, "m'"'] \arrow[dr, phantom, "c"] & y \arrow[d, dashed, "M'"] \\ x \arrow[r, dashed, "M"'] & w \end{tikzcd} \tag{9}$$

The local rule is defined using Φ.

(1) If $z = x = y$:
  (a) If $c = 0$, set $w = z$.
  (b) If $c \neq 0$, let $\Phi_x(c) = (w, M, M')$.

(2) If $z \neq x = y$ then let $\Phi_x(z, m, m') = (w, M, M')$.

(3) If $z = y$ and $z \neq x$ then let $w = x$ and $M' = m'$.

(4) If $z = x$ and $z \neq y$ then let $w = y$ and $M = m$.

(5) If z, x, y are all distinct, then let $(w, M, M') = \Phi_{xy}(z, m, m')$.

This uniquely determines G [Fomin 1995]. Its south edge $G_{n\bullet}$ is a Γ-tableau P and its east edge $G_{\bullet n}$ is a Γ'-tableau Q, both of a common shape $v = G_{nn} \in V$ with $h(v) = n$. This well-defines a map I_Φ as in (8).

For the inverse of I_Φ, let $v \in V$ be such that $h(v) = n$, and let $(P, Q) \in \mathcal{T}(\Gamma, v) \times \mathcal{T}(\Gamma', v)$. To recover $\sigma \in P_n(r)$, we initialize the south and east edges of G to P and Q respectively. Then for each i, j and two by two subgraph as above, we apply the inverse of the above local rule. Given labeled edges

$$x \xrightarrow{M} w \quad \text{and} \quad y \xrightarrow{M'} w,$$

it determines $z \in V$ and marked edges

$$z \xrightarrow{m} y \quad \text{and} \quad z \xrightarrow{m'} x$$

and a value $c \in \{0\} \sqcup S(r)$, such that $c \neq 0$ if and only if $z = x = y \neq w$ and $\Phi_z^{-1}(w, M, M') = c$. The inverse local rule is defined as follows.

(1) If $x = y$:

- (a) If $w = x$, let $z = x$.
- (b) If $w \neq x$:
  - (i) If $c := \Phi_x^{-1}(w, M, M') \in S(r)$: let $z = x$.
  - (ii) Otherwise $\Phi_x^{-1}(w, M, M') = (z, m, m')$.

(2) If $w = x \neq y$, let $z = y$ and $m' = M'$.

(3) If $w = y \neq x$, let $z = x$ and $m = M$.

(4) If x, y, w are all distinct, let $(z, m, m') = \Phi_{xy}^{-1}(w, M, M')$.

In all cases but (1)(b)(i) let $c = 0$. Using the inverse local rule the rest of G is defined [Fomin 1995] and one obtains a well-defined element $\sigma \in P_n(r)$.

Theorem 2.10 [Fomin 1995]. *Let* (Γ, Γ') *be a dual graded graph with differential coefficient* r. *Then for any differential bijection* Φ *for* (Γ, Γ'), *the above construction defines a Schensted bijection* I_Φ *of the form* (8).

We call I_Φ is the *Schensted bijection* induced by the differential bijection Φ.

Example 2.11. The differential bijection Φ of Example 2.9 induces Schensted's row insertion bijection [1961].

Remark 2.12. For the Kac–Moody dual graded graphs $(\Gamma_w(K), \Gamma_s(\Lambda))$ there is a natural choice for the off-diagonal part of the differential bijection. If $v \neq w$ with $v, w \in W$ then Φ_{vw} is essentially obtained from [Humphreys 1990, Lemma 5.11], just as in the proof of Theorem 2.3. For $i \in \mathrm{Des}(v)$, the marked down-then-up path

$$v \xrightarrow{m'} s_i v \xrightarrow{m} s_i v s_\alpha = w$$

maps to the marked up-then-down path

$$v \xrightarrow{m} v s_\alpha \xrightarrow{m'} s_i v s_\alpha = w.$$

Here m and m' denote edge markings, which in either case are selected from sets of size $m_\Lambda(s_i v, s_i v s_\alpha) = \langle \alpha^\vee , \Lambda\rangle = m_\Lambda(v, v s_\alpha)$ and $n_K(\Lambda_i)$ respectively.

Currently we are not aware of a general rule for Φ_v which exhibits the coefficient of v in $(DU - UD)v$ as $\langle K , \Lambda\rangle$. In Section 3 we shall give a special case where the bijection Φ_v has been constructed explicitly.

2E. ***Automorphisms and mixed insertion.*** This section is a natural synthesis of the ideas of [Fomin 1994] and [Haiman 1989] which does not seem to have been written down before. We believe this construction is particularly interesting for Kac–Moody dual graded graphs (see also Section 4).

Let (Γ, Γ') be a pair of dual graded graphs with $\Gamma = (V, E, m, h)$ and $\Gamma = (V, E', m', h)$. Say that a permutation $\tau : V \to V$ is an automorphism of (Γ, Γ') if (1) $h \circ \tau = h$, (2) $(x, y) \in E$ if and only if $(\tau(x), \tau(y)) \in E$, and in this case, $m(x, y) = m(\tau(x), \tau(y))$, and (3) $(x, y) \in E'$ if and only if $(\tau(x), \tau(y)) \in E'$, and in this case, $m'(x, y) = m'(\tau(x), \tau(y))$.

Given a differential bijection Φ for (Γ, Γ'), we define its twist Φ^τ by τ as follows. For every $x, y \in V$ there are natural bijections $\tau : \mathcal{UD}_{xy} \to \mathcal{UD}_{\tau(x)\tau(y)}$ given by $(z, m, m') \mapsto (\tau(z), m, m')$ and $\tau : \mathcal{DU}_{xy} \to \mathcal{DU}_{\tau(x)\tau(y)}$ given by $(w, M, M') \mapsto (\tau(w), M, M')$. Let $\tau : S(r) \to S(r)$ be the identity permutation. Then define $\Phi^\tau_{xy} = \tau^{-1} \circ \Phi_{\tau(x)\tau(y)} \circ \tau$. It is easy to verify that Φ^τ is also a differential bijection for (Γ, Γ').

Example 2.13. Let $\Gamma = \Gamma' = \mathbb{Y}$ and $\mathrm{tr} : \mathbb{Y} \to \mathbb{Y}$ the automorphism of (Γ, Γ) that transposes partition diagrams. Let Φ be the differential bijection in Example 2.9. Then $\mathrm{I}_{\Phi^{\mathrm{tr}}}$ is Schensted's column insertion bijection [Schensted 1961].

For the sequel we assume that τ has finite order κ. A Γ-tableau whose edges have an auxiliary marking parameter $p \in S(\kappa) = \{0, 1, \ldots, \kappa - 1\}$ is called a τ-*mixed* Γ-tableau. Let $\mathcal{T}_\tau(\Gamma)$ be the set of τ-mixed Γ-tableaux. Suppose τ' is an automorphism of (Γ, Γ') of order κ'. Let $(\Gamma, \Gamma'; \tau, \tau')$ denote the pair of dual graded graphs given by Γ and Γ' except that Γ-edges (resp. Γ'-edges) are labeled by (m, p) with $p \in S(\kappa)$ (resp. (m', p') with $p' \in S(\kappa'))$ and m (resp. m') is a usual edge label for Γ (resp. Γ'). This multiplies the number of markings for each edge of Γ (resp. Γ') by κ (resp. κ'). The differential coefficient of $(\Gamma, \Gamma'; \tau, \tau')$ is $r\kappa\kappa'$ where r is the differential coefficient of (Γ, Γ').

Let Φ be a differential bijection for (Γ, Γ'). Then there is an obvious differential bijection (also denoted Φ) for $(\Gamma, \Gamma'; \tau, \tau')$, defined by a trivial scaling by κ in Γ and by κ' in Γ'.

We define another bijection

$$P_n(\kappa\kappa' r) \to \bigsqcup_{\substack{x\in V\\ h(x)=n}} \mathcal{T}_\tau(\Gamma, x) \times \mathcal{T}_{\tau'}(\Gamma', x), \tag{10}$$

called (τ, τ')-*mixed insertion* by modifying the process in which we construct the matrix G_{ij} from the colored permutation σ. Instead of using the same differential bijection Φ to compute each G_{ij}, we use twists of Φ by automorphisms that depend on (i, j).

Let $\sigma \in P_n(\kappa\kappa' r)$. Regard σ as a monomial matrix in which each nonzero entry has three labels $(c, p, p') \in S(r) \times S(\kappa) \times S(\kappa')$, where $S(\kappa) = \{0, 1, \dots, \kappa - 1\}$ and $S(\kappa') = \{0, 1, \dots, \kappa' - 1\}$. Each horizontal (resp. vertical) edge is marked by a pair (m, p) (resp. (m', p')) where m (resp. m') is the usual marking and $p \in S(\kappa)$ (resp. $p' \in S(\kappa')$). Let $(z, y, x, w) = (G_{i-1,j-1}, G_{i-1,j}, G_{i,j-1}, G_{ij})$ and suppose that

$$z \xrightarrow{(m,p)} y \quad \text{and} \quad z \xrightarrow{(m',p')} x$$

are given, where it is understood that if $z = y$ (resp $z = x$) then (m, p) (resp. (m', p')) need not be specified. Then G_{ij} is determined as before, except that instead of using Φ_{xy} we use the twist

$$\Phi_{xy}^{\tau^k(\tau')^{k'}},$$

where k and k' are as follows. Let (c, p, p') be the nonzero entry of σ in the i-th row, say, σ_{il}. We set $k' = p'$. Separately, let (c, p, p') be the nonzero entry of σ in the j-th column, say, σ_{qj}. We set $k = p$. Note that in the case $q > i$ (resp. $l > j$) the bijection Φ_{xy} is not used in the local rule so the value of k (resp. k') does not affect the algorithm.

In other words, if $\sigma_{ij} = (c, p, p')$ is a nonzero entry, then $(\tau')^{p'}$ acts everywhere to the right in the i-th row and all vertical edges to the right (those of the form $G_{i-1,l} \to G_{il}$ for $l \geq j$) are given the auxiliary marking p', and τ^p acts everywhere below in the j-th column, and all horizontal edges below (those of the form $G_{l,j-1} \to G_{lj}$ for $l \geq i$) are given the auxiliary marking p. The output is the pair $(P, Q) \in \mathcal{T}_\tau(\Gamma, v) \times \mathcal{T}_{\tau'}(\Gamma', v)$ where $v = G_{nn}$ and P and Q are obtained from the south and east edges of G respectively.

Proposition 2.14. (τ, τ')-*mixed insertion gives a well-defined bijection* (10).

Example 2.15. In the context of Example 2.13, (τ, τ')-mixed insertion specializes to the following kinds of insertion algorithms, the first from [Schensted 1961] and the other three from [Haiman 1989].

(1) $(\tau, \tau') = (\mathrm{id}, \mathrm{id})$: Schensted row insertion

(2) $(\tau, \tau') = (\mathrm{id}, \mathrm{tr})$: left-right insertion

(3) $(\tau, \tau') = (\mathrm{tr}, \mathrm{id})$: mixed insertion

(4) $(\tau, \tau') = (\mathrm{tr}, \mathrm{tr})$: doubly mixed insertion

2F. ***Restriction to parabolics.*** Let $J \subset I$. We say that a weight Λ is supported on J if $\Lambda = \sum_{j\in J} a_j \Lambda_j$. The Kac–Moody dual graded graphs $(\Gamma_s(\Lambda), \Gamma_w(K))$ are compatible with restriction to parabolics. Let $W_J \subset W$ be the parabolic subgroup generated by $\{s_j \mid j \in J\}$ and let W^J be the set of minimal length coset representatives in W/W_J. Note that W^J inherits weak and strong orders from W by restriction.

Proposition 2.16. *Fix $J \subset I$. If Λ is supported on $I\backslash J$ then the restriction of $(\Gamma_s(\Lambda), \Gamma_w(K))$ to W^J is a pair of dual graded graphs with differential coefficient $\langle K , \Lambda\rangle$.*

Proof. Suppose $v \prec w$ is a weak cover. If $w \in W^J$ then $v \in W^J$ since $W^J \subset W$ is a lower order ideal for $\preceq$.

Suppose $w \lessdot v$ and $w \in W^J$ and $v \notin W^J$. Since v has a reduced expression ending in s_j for some $j \in J$ and w is obtained from this reduced expression by omitting a simple generator, we conclude that $v = ws_j$. But Λ is supported on $I\backslash J$, so $\langle \alpha_j^\vee , \Lambda\rangle = 0$.

Combining these two facts we see that the proof of Theorem 2.3 restricts to W^J. □

We shall use the following notation for maximal parabolic subgroups of W. For $i \in I$ we shall write W^i for W^J where $J = I\backslash\{i\}$. We denote by $(\Gamma_s(\Lambda), \Gamma_w(K))^i = (\Gamma_s^i(\Lambda), \Gamma_w^i(K))$, the dual graded graph given by restricting $(\Gamma_s(\Lambda), \Gamma_w(K))$ to W^i.

2G. ***The affine case.*** If the GCM A is of finite type, then $Z^+ = \{0\}$ and all of the edges of $\Gamma_w(K)$ are labeled 0.

In this section let A be of untwisted affine type. Let $0 \in I$ be the distinguished Kac 0 node and $J = I \setminus \{0\}$. Then W is the affine Weyl group, $W_J = W_{\mathrm{fin}}$ is the finite Weyl group, and we write $W^0 = W^J$. By Proposition 2.16 the restriction of the Kac–Moody dual graded graph to W^0, is a dual graded graph. In this case the weak graph $\Gamma_w(K)$ has an interpretation involving the Schubert calculus of the homology of the affine Grassmannian, and the duality is a combinatorial expression of the pairing between the homology and cohomology of the affine Grassmannian.

For affine algebras, $Z^+ = \mathbb{Z}_{\geq 0}K$ where $K = K_{\text{can}} = \sum_{i\in I} k_i\alpha_i^\vee$ is the canonical central element; the vector $(k_i)_{i\in I}$ is the unique linear dependence of the rows of A given by positive relatively prime integers [Kac 1990]. In this case, since the labels of $\Gamma_w(K)$ are linear in K, without loss of generality we shall only work with K_{can} and define $\Gamma_w := \Gamma_w(K_{\text{can}})$. The edge labels of Γ_w are related to the homology multiplication in affine Grassmannians, as follows.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an untwisted affine algebra. Let $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{G}}$ denote the affine Grassmannian of the simple Lie group G whose Lie algebra $\mathfrak{g}_{\text{fin}}$ is the canonical simple Lie subalgebra of the affine algebra $\mathfrak{g}$. For $w \in W^0$ we let $\xi_w \in H_*(\mathrm{Gr})$ denote the corresponding *homology* Schubert class. Recall the constants $n(w, v)$ from (3).

Proposition 2.17. *Let $\xi_0 = \xi_{s_0}$ be the Schubert class indexed by the unique simple generator $s_0 \notin W_{\text{fin}}$. Then for every $w \in W^0$, we have in $H_*(\mathrm{Gr})$ the identity*

$$\xi_0\, \xi_w = \sum_v n(w, v)\, \xi_v$$

where $v \in W^0$ runs over the weak covers $w \prec v$ of w.

Proof. We rely on the results of [Lam 2006] which in turn are based on unpublished work of Peterson. Let $S = \mathrm{Sym}((\mathfrak{h}^*_{\mathbb{Z}})_{\text{fin}}) = H^{T_{\text{fin}}}(\mathrm{pt})$ denote the symmetric algebra in the weights of the $\mathfrak{g}_{\text{fin}}$ and $\phi_0 : S \to \mathbb{Z}$ denote the evaluation at 0. Let $\mathbb{A}_0$ denote the affine nilCoxeter algebra corresponding to W. As a free $\mathbb{Z}$-module $\mathbb{A}_0$ is spanned by elements $\{A_w \mid w \in W\}$ with multiplication given by

$$A_w\, A_v = \begin{cases} A_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv) \\ 0 & \text{otherwise.} \end{cases}$$

The affine nilHecke algebra $\mathbb{A}$ is the $\mathbb{Z}$-algebra generated by $\mathbb{A}_0$ and S with the additional relation [Lam 2006, Lemma 3.1]

$$A_w\, \lambda = (w \cdot \lambda) A_w + \sum_{w\, r_\alpha \lessdot w} \langle \lambda\,, \alpha^\vee \rangle A_{w\, r_\alpha}, \tag{11}$$

where α is always taken to be a positive root of W.

Now let

$$\mathbb{B} = \{a \in \mathbb{A}_0 \mid \phi_0(as) = \phi_0(s)a \text{ for any } s \in S\} \subset \mathbb{A}_0$$

denote the affine Fomin–Stanley subalgebra, where $\phi_0 : \mathbb{A} \to \mathbb{A}_0$ is given by $\phi_0(\sum_w a_w\, A_w) = \sum_w \phi_0(a_w) A_w$. Let $j_0 : H_*(\mathrm{Gr}_G) \to \mathbb{B}$ denote the ring isomorphism [Lam 2006, Theorem 5.5] from the homology of Gr_G to the affine Fomin–Stanley algebra $\mathbb{B}$. We first show that $j_0(\xi_0) = \sum_{i\in I} k_i A_i$, where A_i are the generators of the nilCoxeter algebra and $K_{\text{can}} = \sum_i k_i \alpha_i^\vee$.

By [Lam 2006, Proposition 5.4], the element $j_0(\xi_0)$ is characterized by having unique Grassmannian term A_0, and the property that it lies in $\mathbb{B}$. Since $k_0 = 1$, the unique Grassmannian term property is immediate. Using (11), we calculate that

$$\phi_0(a\alpha_j) = \sum_{i \in I} k_i \langle \alpha_i^\vee , \alpha_j \rangle = \langle K_{\mathrm{can}} , \alpha_j \rangle = 0.$$

Since the α_j span $(\mathfrak{h}^*_{\mathbb{Z}})_{\mathrm{fin}}$ over $\mathbb{Q}$, we deduce that $\phi_0(as) = \phi_0(s)a$ for any $s \in S$, and thus $j_0(\xi_0) = \sum_{i \in I} k_i A_i$. (This was first pointed out to us by Alex Postnikov.)

By [Lam 2006, Lemma 4.3, Theorem 5.5], we thus have

$$\xi_0\, \xi_w = j(\xi_0) \cdot \xi_w = \Big(\sum_{i \in I} k_i A_i\Big) \cdot \xi_w = \sum_{w \prec s_i w} k_i\, \xi_{s_i w}.$$

where we have used the action of $\mathbb{A}_0$ on $H_*(\mathrm{Gr}_G)$ given by

$$A_i \cdot \xi_w = \begin{cases} \xi_{s_i w} & \text{if } s_i w > w, \\ 0 & \text{otherwise;} \end{cases}$$

see [Lam 2006, (3.2)]. Recalling the definition $n(w, s_i w) = k_i$ of the weak graph Γ_w from (3) this completes the proof. $\square$

3. Affine type A and LLMS insertion

For this section let $\mathfrak{g}(A)$ be the affine algebra of type $A^{(1)}_{n-1}$. In this case the combinatorics of the pair of dual graded graphs $(\Gamma_s(\Lambda_i), \Gamma_w)$ was studied extensively in [Lam et al. 2006] and was one of the main motivations of the current work. The affine insertion algorithm of [Lam et al. 2006] (which we shall call LLMS insertion) furnishes an explicit differential bijection for $(\Gamma_s(\Lambda_i), \Gamma_w)$. LLMS insertion involves nontrivial extensions of the notion of tableaux to semistandard weak and strong tableaux, and proves Pieri rules (formulae for certain Schubert structure constants) in the homology $H_*(\mathrm{Gr})$ and cohomology $H^*(\mathrm{Gr})$ of the affine Grassmannian of $\mathrm{SL}(n, \mathbb{C})$ [Lam et al. 2006].

For type $A^{(1)}_{n-1}$ the coefficients of the canonical central element K are all 1. Therefore the weak graph Γ_w has all edge multiplicities equal to 1. Using the rotational symmetry of the Dynkin diagram $A^{(1)}_{n-1}$, we may assume that $\Lambda = \Lambda_0$ and for brevity we write Γ_s for $\Gamma_s(\Lambda_0)$.

Let $I = \{0, 1, \dots, n-1\}$ and let the Cartan matrix be defined by $a_{i,i+1} = a_{i+1,i} = -1$ for all i, with indices taken modulo n, $a_{ii} = 2$ for all $i \in I$, and $a_{ij} = 0$ otherwise. As in Section 2B the Weyl group is defined by $m_{i,i+1} = 3$ and $m_{ij} = 2$ for $|i - j| \geq 2$.

3A. ***Affine permutations.*** We use the following explicit realization of the affine symmetric group $W = \widetilde{S}_n$. A bijection $w : \mathbb{Z} \to \mathbb{Z}$ is an affine permutation with period n if $w(i+n) = w(i)$ for each $i \in \mathbb{Z}$ and $\sum_{i=1}^{n} (w(i) - i) = 0$. The set of affine permutations with period n form a group isomorphic to $\widetilde{S}_n$, with multiplication given by function composition. The reflections t_{ij} in $\widetilde{S}_n$ are indexed by a pair of integers (i, j) satisfying $i < j$ and $i \neq j \mod n$. Suppose $v \lessdot vt_{ij} = w$ is a cover in $\widetilde{S}_n$. Then the edge (v, w) in Γ_s has multiplicity equal to $\#\{k \in \mathbb{Z} \mid v(i) \leq k < v(j) \text{ and } k = 0 \mod n\}$ [Lam et al. 2006].

3B. ***Action of $\widetilde{S}_n$ on partitions.*** Given a partition λ, one may associate a bi-infinite binary word $p(\lambda) = p = \cdots p_{-1}p_0p_1\cdots$ called its *edge sequence*. The edge sequence $p(\lambda)$ traces the border of the (French) diagram of λ, going from northwest to southeast, such that every letter 0 (resp. 1) represents a south (resp. east) step, and some cell in the i-th diagonal is touched by the steps p_{i-1} and p_i. Here the cell (i, j) lies in row i (where row indices increase from south to north), column j (where column indices increase from west to east), and diagonal $j - i$.

The affine symmetric group $\widetilde{S}_n$ acts on partitions, since elements of $\widetilde{S}_n$ are certain permutations $\mathbb{Z} \to \mathbb{Z}$ and partitions can be identified with their edge sequences, which are certain functions $\mathbb{Z} \to \{0, 1\}$. Then for $i \in \mathbb{Z}/n\mathbb{Z}$, $s_i\lambda$ is obtained by removing from λ every λ-removable cell of residue i, and adding to λ every λ-addable cell of residue i. Here the residue of a box (i, j) is the diagonal index $j - i$ taken modulo n.

3C. ***Cores and affine Grassmannian permutations.*** Using the language of cores, we shall describe the combinatorics of the dual graded graph $(\Gamma_s, \Gamma_w)^0$ afforded by Proposition 2.16.

An *n-ribbon* is a skew partition diagram λ/μ (the difference of the diagrams of the partitions λ and μ) consisting of n rookwise connected cells, all with distinct residues. We say that this ribbon is *λ-removable* and *μ-addable*. An *n-core* is a partition that admits no removable n-ribbon. Since the removal of an n-ribbon is the same thing as exchanging bits $p_i = 0$ and $p_{i+n} = 1$ in the edge sequence for some i, it follows that λ is a core if and only if for every i, the sequence $p^{(i)}(\lambda) := \cdots p_{i-2n}p_{i-n}p_ip_{i+n}p_{i+2n}\cdots$ consisting of the subsequence of bits indexed by $i \mod n$, has the form $\cdots 1111100000 \cdots$. We denote the set of n-cores by $\mathscr{C}_n$.

Proposition 3.1 [Lam et al. 2006; Misra and Miwa 1990]. *The map $w \mapsto w \cdot \varnothing$ is a bijection $c : \widetilde{S}_n^0 \to \mathscr{C}_n$. Moreover, for $v, w \in \widetilde{S}_n^0$, we have $v \leq w$ if and only if $c(v) \subseteq c(w)$, and if $v \lessdot w$ then $c(w)/c(v)$ is a disjoint union of translates of some ribbon R, and the number of components of $c(w)/c(v)$ is equal to the multiplicity $m(v, w)$ in Γ_s.*

We say that $\mu \in \mathscr{C}_n$ covers $\lambda \in \mathscr{C}_n$ if $c^{-1}(\mu) \gtrdot c^{-1}(\lambda)$. Thus a standard strong tableau in Γ_s is a sequence $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \mu$ such that λ^i covers λ^{i-1} and one of the components of λ^i/λ^{i-1} has been marked.

It is easy to show that a core cannot have both an addable and a removable cell of the same residue. Thus, in the special case that $v \lessdot s_i v = w$ for some $i \in I$, $c(w)$ is obtained from $c(v)$ by adding all $c(v)$-addable cells of residue i, and the ribbon R of Proposition 3.1 must be a single box. In this case we say that $c(v) \subset c(w)$ is a weak cover.

3D. ***LLMS insertion.*** In [Lam et al. 2006], for the affine symmetric group $\widetilde{S}_n$, semistandard analogues of weak and strong tableaux were defined (for all of $\widetilde{S}_n$, not just $\widetilde{S}_n^0$), and an RSK correspondence was given between a certain set of biwords or matrix words, and pairs of tableaux, one semistandard weak and the other semistandard strong. Let us consider the following restriction of this bijection. We first restrict to "standard" tableau pairs, that is, the case in which the tableaux are weak and strong tableaux as defined in Section 2C. Next we take the parabolic restriction from $\widetilde{S}_n$ to $\widetilde{S}_n^0$. Let us denote the restricted bijection by I_{LLMS}.

Let Φ be the differential bijection for $(\Gamma_s, \Gamma_w)^0$ such that $\mathrm{I}_\Phi = I_{\mathrm{LLMS}}$. We describe it explicitly.

For $u, v \in \widetilde{S}_n^0$ with $u \neq v$, the off-diagonal part Φ_{uv} of Φ coincides with the natural definition given in Remark 2.12. The diagonal part Φ_v for $v \in \widetilde{S}_n^0$ is specified as follows. Let $\lambda = c(v)$ be the n-core corresponding to v. If $\lambda \subset \mu$ is a weak cover then μ/λ consists of all the λ-addable cells of λ which have a fixed residue. As μ varies over all the weak covers of λ we obtain all the λ-addable cells in this way. Thus there is a natural identification of the set $\mathcal{D}\mathcal{U}_v$ with the set of λ-addable cells. Similarly $\mathcal{U}\mathcal{D}_v$ may be identified with the set of λ-removable cells. This given, we may use the differential bijection denoted Φ_λ in Example 2.9 for Young's lattice. This defines a differential bijection Φ for $(\Gamma_s, \Gamma_w)^0$.

Example 3.2. Figure 1 shows the calculation of I_{LLMS} of the permutation $\sigma = 412635$ (written here in one-line notation; it corresponds to the permutation matrix with ones located positions $(i, \sigma(i))$ for $1 \le i \le 6$) for $\widetilde{S}_3$. The symbols $\otimes$ encode σ as described above Equation (9). Each arrow indicates a marked strong cover; the subscript 2 indicates that the marked component is the second from the southeast, and no subscript means the marked component is the southeastmost. Stars in the P tableau indicate the marked components.

Remark 3.3. As n goes to infinity, $(\Gamma_s, \Gamma_w)^0$ converges to the dual graded graph $(\mathbb{Y}, \mathbb{Y})$ of Example 2.2 and LLMS insertion converges to Schensted row insertion [Lam et al. 2006] because the respective differential bijections coincide in the limit.

4. Folding

An automorphism of the GCM $B = (b_{ij} \mid i, j \in J)$ is a permutation π of J such that $b_{\pi(i)\pi(j)} = b_{ij}$ for all $i, j \in J$. The automorphism π is *admissible* if $b_{ij} = 0$ for all i and j in the same π-orbit.

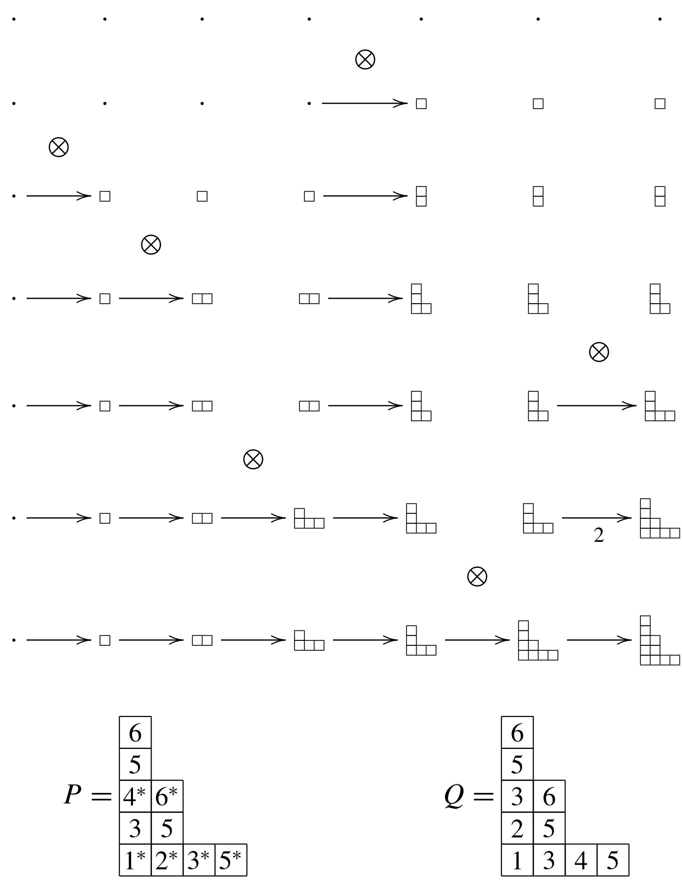


Figure 1. Growth diagram for LLMS insertion of 412635 for $\widetilde{S}_3$.

A GCM $A = (a_{ij} \mid i, j \in I)$ is *symmetric* if it is a symmetric matrix. It is *symmetrizable* if there are positive integers o_i for $i \in I$, such that DA is symmetric, where D is the diagonal matrix with diagonal entries o_i.

Lusztig [1993] showed that every symmetrizable GCM A can be constructed from a symmetric GCM B that is equipped with an admissible automorphism π. We call this construction *folding*.

For A and B related in this manner, we show that the structure of every dual graded graph of the form $(\Gamma_s^A(\Lambda_{i'}), \Gamma_s^A(K))$ for $\mathfrak{g}(A)$, is encoded by some dual graded graph for $\mathfrak{g}(B)$. Thus the combinatorics for $\mathfrak{g}(A)$ is reduced to that of $\mathfrak{g}(B)$. In particular, for any affine algebra $\mathfrak{g}(A)$ there is a simply-laced affine algebra $\mathfrak{g}(B)$ related by folding, so that all affine Schensted bijections can be realized using only the simply-laced affine algebras.

4A. ***Folding data.*** Let $B = (b_{ij} \mid i, j \in J)$ be a GCM and π an admissible automorphism of B. Let I be a set which indexes the π-orbits of J; we write $O_i \subset J$ for the π-orbit indexed by $i \in I$. Let $o_i = |O_i|$. It is easy to show that the matrix $A = (a_{i'i} \mid i', i \in I)$ defined by

$$a_{i'i} = \frac{o_{i'}}{o_i} \sum_{j \in O_i} b_{j'j} \qquad \text{for any } j' \in O_{i'}, \tag{12}$$

is a well-defined GCM. We say that A is obtained from (B, π) by folding.

Proposition 4.1 [Lusztig 1993, Proposition 14.1.2]. *Given any symmetrizable GCM A, there is a symmetric GCM B with admissible automorphism π, such that A is obtained from (B, π) by folding. In particular, if A is of affine type then B can be taken to be of simply-laced affine type.*

For the Kac–Moody algebras $\mathfrak{g}(A)$ and $\mathfrak{g}(B)$, we denote their weight lattices by P_A and P_B, their coweight lattices by $P_A^\vee$ and $P_B^\vee$ and their coroot lattices by $Q_A^\vee$ and $Q_B^\vee$. For simplicity, we let $\Lambda_i, \alpha_i, \alpha_i^\vee$ be the fundamental weights, simple roots, and simple coroots for $\mathfrak{g}(A)$ and write $\omega_i, \beta_i, \beta_i^\vee$ for the corresponding data for $\mathfrak{g}(B)$.

Let $P'_A = P_A/(\bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee)^0$ be the weight lattice of A modulo the annihilator of the coroots $\{\alpha_i^\vee\}$. Similarly define P'_B. Note that

$$\alpha_i = \sum_{i' \in I} a_{i'i}\, \Lambda_{i'} \quad \text{and} \quad \beta_j = \sum_{j' \in J} b_{j'j}\, \omega_{j'}, \tag{13}$$

where α_i, Λ_i and β_j, ω_j also denote their respective images inside P'_A and P'_B. Set $\kappa = \mathrm{lcm}_{i \in I}(o_i)$, and define $\psi : P'_A \to P'_B$ by

$$\psi(\Lambda_i) = \frac{\kappa}{o_i}\, \widetilde{\omega}_i, \tag{14}$$

where $\widetilde{\omega}_i = \sum_{j \in O_i} \omega_j$ and $\widetilde{\beta}_i = \sum_{j \in O_i} \beta_j$ for $i \in I$. We have

$$\begin{aligned} \psi(\alpha_i) &= \sum_{i'} a_{i'i}\, \psi(\Lambda_{i'}) = \sum_{i'} a_{i'i} \frac{\kappa}{o_{i'}} \sum_{j' \in O_{i'}} \omega_{j'} \\ &= \frac{\kappa}{o_i} \sum_{i'} \sum_{j \in O_i} \sum_{j' \in O_{i'}} b_{j'j}\omega_{j'} = \frac{\kappa}{o_i} \widetilde{\beta}_i, \end{aligned} \tag{15}$$

by (13) and (12). Define $\varphi : Q_A^\vee \to Q_B^\vee$ by

$$\varphi(\alpha_i^\vee) = \widetilde{\beta}_i^\vee = \sum_{j \in O_i} \beta_j. \tag{16}$$

4B. ***Weyl groups.*** For $i \in I$ define

$$f(s_i^A) = \prod_{j \in O_i} s_j^B \in W_B. \tag{17}$$

Since π is admissible the reflections $\{s_j^B \mid j \in O_i\}$ commute with each other so that the product in (17) is independent of the order of its factors.

Steinberg [1968] showed that there is an embedding of Weyl groups $W_A \to W_B$. By [Nanba 2005] it respects the Bruhat order.

Theorem 4.2.

(1) [Steinberg 1968] *There is an injective group homomorphism $f : W_A \to W_B$ defined by (17), whose image is the subgroup W_B^π of π-fixed elements in W_B, where π acts on W_B by $\pi(s_i) = s_{\pi(i)}$.*

(2) [Nanba 2005, Proposition 3.3 and Theorem 1.2] *$v \le w$ in W_A if and only if $f(v) \le f(w)$ in W_B. Moreover, if $w = s_{i_1} \cdots s_{i_N}$ is a reduced decomposition in W_A then $f(w) = f(s_{i_1}) f(s_{i_2}) \cdots f(s_{i_N})$ is a length-additive factorization in W_B.*

Corollary 4.3.

(1) *π acts on $\Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$.*

(2) *Suppose $v \lessdot w = v s_\alpha$ in W_A for $\alpha \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(A))$. Then there is a unique π-orbit $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ such that the reflections $\{s_\gamma \mid \gamma \in O\}$ commute, $f(s_\alpha) = \prod_{\gamma \in O} s_\gamma$, and there is an isomorphism of the boolean lattice of subsets O' of O with the interval $[f(v), f(w)]$ in $(W_B, \le)$ given by $O' \mapsto f(v) \prod_{\gamma \in O'} s_\gamma$. We call O the orbit associated with the cover $v \lessdot w$.*

(3) *Let $w \in W_A$ and $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be such that $f(w) s_\gamma \lessdot f(w)$ in W_B, and let $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be the π-orbit of γ. Then there is a covering relation $v \lessdot w$ of w in W_A of which O is the associated orbit.*

(4) *Let $v \in W_A$, $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ be such that $f(v) \lessdot f(v) s_\gamma$, and $O \subset \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$ the π-orbit of γ. Then there is a covering relation $v \lessdot w$ in W_A of which O is the associated cover.*

Proof. For (1), let $\gamma \in \Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$, with $\gamma = \tilde{u}\beta_j$ for some $\tilde{u} \in W_B$ and $j \in J$. Then $\pi(\gamma) = \pi(\tilde{u})\beta_{\pi(j)} \in \Delta_{\mathrm{re}}(\mathfrak{g}(B))$ and π clearly preserves the set of positive roots, so that π acts on $\Delta_{\mathrm{re}}^+(\mathfrak{g}(B))$.

For (2), there is a unique length-additive factorization $w = u_1 s_i u_2$ in W_A such that $v = u_1 u_2$. We have $\alpha = u\alpha_i$ and $s_\alpha = u s_i u^{-1}$ where $u = u_2^{-1}$. Define $O = \{f(u)\beta_j \mid j \in O_i\}$; since $f(u)$ is π-invariant, we have $\pi(f(u)\beta_j) = \pi(f(u))\beta_{\pi(j)} = f(u)\beta_{\pi(j)} \in O(\alpha)$, so that O is a π-orbit. For $j \in O_i$ we have the relation $s_{f(u)\beta_j} = f(u) s_j f(u)^{-1}$, so the reflections $\{s_\gamma \mid \gamma \in O\}$ commute, being conjugate to commuting reflections $\{s_j \mid j \in O_i\}$. Since f is a homomorphism we have $f(s_\alpha) = f(u)\big(\prod_{j\in O_i} s_j\big) f(u)^{-1} = \prod_{j\in O_i} f(u)s_j f(u)^{-1} = \prod_{\gamma\in O(\alpha)} s_\gamma$.

By Theorem 4.2 $f(w) = f(u_1)(\prod_{j\in O_i} s_j) f(u_2)$ and $f(v) = f(u_1)f(u_2)$ are length-additive factorizations. It follows that there is an isomorphism of the boolean lattice of subsets $S \subset O_i$ with $[f(v), f(w)]$ where $S \mapsto f(u_1)\big(\prod_{j\in S} s_j\big) f(u_2)$. The desired isomorphism is given by sending $S \to O'$ where $O' = \{f(u)\beta_j \mid j \in S\}$.

For (3), let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition. Then the image $f(w) = f(s_{i_1}) \cdots f(s_{i_k})$ is length-additive by Theorem 4.2. Therefore the cover $f(w)s_\gamma \lessdot f(w)$ is obtained by removing some unique reflection in $f(s_{i_r})$ for some unique r. Let $u_1 = s_{i_1} \cdots s_{i_{r-1}}$, $i = i_r$, and $u^{-1} = u_2 = s_{i_{r+1}} \cdots s_{i_k}$. Letting $\alpha = u\alpha_i \in \Delta^+_{\mathrm{re}}(\mathfrak{g}(A))$ we find that the π-orbit O of γ is the orbit associated with the cover $v \lessdot w$ where $v = u_1u_2$.

The proof of (4) is similar. □

The following result is proved similarly.

Corollary 4.4. *Let $v \prec s_iv$ in W_A for some $i \in I$. Then there is a poset isomorphism from the boolean lattice of subsets O' of O_i, to the interval $[f(v), f(s_iv)]$ of $(W_B, \preceq)$ given by $O' \mapsto \big(\prod_{j\in O'} s_j\big) f(v)$.*

4C. *Pairings.* The action of W_B on P_B descends to P'_B, and similarly for W_A.

Theorem 4.5.

(1) *For all $\alpha^\vee \in Q^\vee_A$ and $\lambda \in P'_A$ we have*

$$\langle \varphi(\alpha^\vee), \psi(\lambda)\rangle = \kappa\,\langle \alpha^\vee, \lambda\rangle. \tag{18}$$

(2) *For all $\Lambda \in P'_A$ and $w \in W_A$ we have*

$$\psi(w\Lambda) = f(w)\psi(\Lambda). \tag{19}$$

(3) *Let $w \in W_A$, $\Lambda \in P'_A$ and $i \in I$. Then*

$$\langle \varphi(\alpha^\vee), f(w)\psi(\Lambda)\rangle = \kappa\langle \alpha^\vee, w\Lambda\rangle \tag{20}$$

and in particular we have

$$\langle \widetilde{\beta}_i^\vee, f(w)\psi(\Lambda)\rangle = \kappa\langle \alpha_i^\vee, w\Lambda\rangle. \tag{21}$$

(4) *The map φ sends $Q^\vee_A \cap Z(\mathfrak{g}(A))$ into $Z(\mathfrak{g}(B))$.*

Proof. By linearity it suffices to check (18) for $\alpha^\vee = \alpha_i^\vee$ and $\lambda = \Lambda_k$ for $i, k \in I$. We have

$$\langle \varphi(\alpha_i^\vee), \psi(\Lambda_k)\rangle = \frac{\kappa}{o_k}\langle \widetilde{\beta}_i^\vee, \widetilde{\omega}_k\rangle = \frac{\kappa}{o_k} o_i\delta_{ik} = \kappa\delta_{ik} = \kappa\langle \alpha_i^\vee, \Lambda_k\rangle.$$

This implies (18).

It suffices to prove (19) for $w = s^A_{i'}$ and $\Lambda = \Lambda_i$. For $i = i'$ we have

$$\psi(s_i\Lambda_i) = \psi(\Lambda_i - \alpha_i) = \frac{\kappa}{o_i}(\widetilde{\omega}_i - \widetilde{\beta}_i)$$

by (14) and (15). Since π is admissible, $\big(\prod_{j\in O_i} s^B_j\big)\widetilde{\omega}_i = \widetilde{\omega}_i - \widetilde{\beta}_i$, giving the required result. The case that $i \neq i'$ is even easier.

Equation (20) follows immediately from (19) and (18) with $\lambda = w\Lambda$. Equation (21) is implied by (20) with $\alpha = \alpha_i$ and (16).

Let $K \in Q^\vee_A$. For any $i \in I$ and $j \in O_i$, by (18), (15), and the π-invariance of $\varphi(K)$, we have

$$\kappa\langle\varphi(K)\,,\beta_j\rangle = \frac{\kappa}{o_i}\langle\varphi(K)\,,\widetilde{\beta}_i\rangle = \kappa\langle K\,,\alpha_i\rangle. \tag{22}$$

It follows that φ sends $Z(\mathfrak{g}(A))$ into $Z(\mathfrak{g}(B))$. □

Remark 4.6. Let A be a GCM of affine type, obtained as in [Lusztig 1993] by folding (B,π). Then the canonical central elements and null roots are related by $\varphi(K^A) = K^B$ and $\psi(\delta^A) = r^\vee\delta^B$ where $r^\vee$ is the "twist" of the dual affine root system $X^{(r^\vee)}_N$ to that of A in the nomenclature of [Kac 1990].

4D. *Folding and insertion.* Let $A = (a_{ij} \mid i, j \in I)$ be a GCM with associated Kac–Moody algebra $\mathfrak{g}(A)$, $i' \in I$, and $K \in Z^+(\mathfrak{g}(A))$. Suppose A is obtained by folding the GCM $B = (b_{ij} \mid i, j \in J)$ with admissible automorphism π. Choose $j' \in O_{i'}$. We shall construct the dual graded graph $(\Gamma^A_s(\Lambda_{i'}), \Gamma^A_w(K))$ from the dual graded graph $(\Gamma^B_s(\omega_{j'}), \Gamma^B_w(\varphi(K)))$. The construction only requires the subset $W^\pi_B \subset W_B$ of π-invariant vertices, and the edges incident to them, grouped according to their π-orbits.

Remark 4.7. The choice of $j' \in O_{i'}$ is immaterial; if one chooses another element of $O_{i'}$ then the resulting type B structures are transported to each other by a power of the automorphism π.

Proposition 4.8. *Let $v \lessdot w$ in W_A with $\alpha \in \Delta^+_{\mathrm{re}}(\mathfrak{g}(A))$ such that $w = vs_\alpha$ and let $O \subset \Delta^+_{\mathrm{re}}(\mathfrak{g}(B))$ be the associated orbit of the cover $v \lessdot w$, defined in Corollary 4.3. Then the following sets have the same cardinality.*

(1) *Marked edges $v \xrightarrow{m} w$ in $\Gamma^A_s(\Lambda_{i'})$.*

(2) *The disjoint union over $\gamma \in O$ of the sets $\{f(w)s_\gamma \xrightarrow{M} f(w)\}$ of marked edges going into $f(w)$ in the interval $[f(v), f(w)]$ in $\Gamma^B_s(\omega_{j'})$.*

(3) *The disjoint union over $\gamma \in O$ of the sets $\{f(v) \xrightarrow{M} f(v)s_\gamma\}$ of marked edges coming out of $f(v)$ in the interval $[f(v), f(w)]$ in $\Gamma^B_s(\omega_{j'})$.*

Proof. By Corollary 4.3 the last two sets are in bijection. Let $u \in W_A$ and $i \in I$ be such that $\alpha = u\alpha_i$. Then $O = \{f(u)\beta_j \mid j \in O_i\}$. Using Theorem 4.5 and the π-invariance of $\sum_{\gamma \in O} \gamma^\vee$, we have

$$\begin{aligned}
\sum_{\gamma \in O} m^B_{\omega_{j'}}(f(w)s_\gamma, f(w)) &= \sum_{\gamma \in O} \langle \gamma^\vee, \omega_{j'} \rangle = \frac{1}{o_{i'}} \sum_{\gamma \in O} \langle \gamma^\vee, \widetilde{\omega}_{i'} \rangle \qquad (23)\\
&= \frac{1}{\kappa} \sum_{\gamma \in O} \langle \gamma^\vee, \psi(\Lambda_{i'}) \rangle = \frac{1}{\kappa} \sum_{j \in O_i} \langle f(u)\beta_j^\vee, \psi(\Lambda_{i'}) \rangle \\
&= \frac{1}{\kappa} \langle \widetilde{\beta}_i^\vee, f(u)^{-1}\psi(\Lambda_{i'}) \rangle \\
&= \langle \alpha_i^\vee, u^{-1}\Lambda_{i'} \rangle = \langle \alpha^\vee, \Lambda_{i'} \rangle = m^A_{\Lambda_{i'}}(ws_\alpha, w). \qquad \square
\end{aligned}$$

By definition the graded graph $\Gamma^B_s(\omega_{j'})^\pi$ has vertex set $f(W_A) = W^\pi_B \subset W_B$, grading function $h(f(v)) = \ell(v)$ for $v \in W_A$, and for every cover $v \lessdot w$ in W_A, an edge from $f(v)$ to $f(w)$ whose multiplicity is the common number in Proposition 4.8. It is completely specified by the π-invariant elements of $\Gamma^B_s(\omega_{j'})$ and their incident edges.

Corollary 4.9. *The graded graphs $\Gamma^A_s(\Lambda_{i'})$ and $\Gamma^B_s(\omega_{j'})^\pi$ are isomorphic.*

Proof. By Theorem 4.2 the map $f : W_A \to W^\pi_B$ is a grade- and edge-preserving bijection. The edge multiplicities agree by Proposition 4.8. $\square$

We call $\Gamma^B_s(\omega_{j'})^\tau$ the *folded strong graph* and its tableaux *folded strong tableaux*.

Proposition 4.10. *Let $v \in W_A$ and $i \in I \setminus \mathrm{Des}(v)$ so that $v \prec w = s_i v$. Fix any $j \in O_i$. Then the following sets have the same cardinality.*

(1) *The marked edges $v \xrightarrow{m'} w$ in $\Gamma^A_w(K)$.*

(2) *The marked edges $f(v) \xrightarrow{M'} s_j f(v)$ in $\Gamma^B_w(\varphi(K))$.*

(3) *The marked edges $s_j f(w) \xrightarrow{M'} f(w)$ in $\Gamma^B_w(\varphi(K))$.*

Proof. By a proof similar to the one for (22) and recalling the definition (3) we have

$$n^A_K(v, s_i v) = \langle K, \Lambda_i \rangle = \langle \varphi(K), \omega_j \rangle = n^B_{\varphi(K)}(f(v), s_j f(v)). \qquad (24)$$

This proves the proposition. $\square$

By definition the graded graph $\Gamma^B_w(\varphi(K))^\pi$ has vertex set $f(W_A) = W^\pi_B$, grading function $h(f(w)) = \ell(w)$, and for each $v \in W_A$ and $i \in I \setminus \mathrm{Des}(v)$, an edge from $f(v)$ to $f(s_i v)$ whose multiplicity is the common multiplicity in Proposition 4.10.

Corollary 4.11. *The graded graphs $\Gamma^A_w(K)$ and $\Gamma^B_w(\varphi(K))^\pi$ are isomorphic.*

We call $\Gamma_w^B(\varphi(K))^\pi$ the *folded weak graph*. Its tableaux are called *folded weak tableaux*.

Corollary 4.12. *$(\Gamma_s^B(\omega_{j'})^\pi, \Gamma_w^B(\varphi(K))^\pi)$ and $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$ are isomorphic dual graded graphs.*

Proof. This follows immediately from Corollaries 4.9 and 4.11. □

Using a differential bijection Φ^B for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K)))$, we construct a differential bijection for the folded dual graded graph $(\Gamma_s^B(\omega_{j'})^\pi, \Gamma_w^B(\varphi(K))^\pi)$, which, by the identifications given in Propositions 4.8 and 4.10, yields a differential bijection Φ^A for $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.

By Remark 2.12 the off-diagonal part Φ_{vw}^A for $v \neq w$ in W_A, has already been specified.

For the diagonal terms, since $\mathrm{Des}(f(v)) = \bigsqcup_{i \in \mathrm{Des}(v)} O_i$, by (23) in the special case of a cover of the form $s_i v = v s_\alpha \gtrdot v$ with $\alpha = v^{-1}\alpha_i \in \Delta_{\mathrm{re}}^+$ and (24) we have

$$\begin{aligned}\sum_{i \in I \setminus \mathrm{Des}(v)} & m_{\Lambda_{i'}}^A(v, s_i v)\, n_K^A(v, s_i v) \\ &= \sum_{j \in J \setminus \mathrm{Des}(f(v))} m_{\omega_{j'}}^B(f(v), s_j f(v))\, n_{\varphi(K)}^B(f(v), s_j f(v)). \quad (25)\end{aligned}$$

For $i \in \mathrm{Des}(v)$ and $j \in O_i$, we have $v s_\alpha = s_i v \lessdot v$ where $\alpha = v^{-1}\alpha_i \in -\Delta_{\mathrm{re}}^+$, and an analogous computation yields

$$\begin{aligned}\sum_{i \in \mathrm{Des}(v)} & m_{\Lambda_{i'}}^A(s_i v, v) n_K^A(s_i v, v) \\ &= \sum_{j \in \mathrm{Des}(f(v))} m_{\omega_{j'}}^B(s_j f(v), f(v))\, n_{\varphi(K)}^B(s_j f(v), f(v)). \quad (26)\end{aligned}$$

Using the bijections of Propositions 4.8 and 4.10, we obtain bijections $\mathfrak{DU}_v^A \to \mathfrak{DU}_{f(v)}^B$ and $\mathfrak{UD}_v^A \to \mathfrak{UD}_{f(v)}^B$. Under these identifications we obtain a differential bijection Φ^A for $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.

Proposition 4.13. *Let the GCM $A = (a_{ij} \mid i, j \in I)$ be obtained by folding from the GCM $B = (b_{ij} \mid i, j \in J)$ with admissible automorphism π. Then for any $i' \in I$ and $j' \in O_{i'}$, a differential bijection Φ^B for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K))$ restricts to a differential bijection Φ^A for the pair of dual graded graphs $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))$.*

We shall give an extensive example in Section 5.

Remark 4.14. It is possible to axiomatize conditions for an arbitrary pair of dual graded graphs (Γ, Γ') and an automorphism π of (Γ, Γ') to give rise to a folded insertion in the manner we have described for Kac–Moody graded graphs. The key properties needed are abstract graph-theoretic formulations of Theorem 4.2 and Corollary 4.3. Since we have no interesting examples that do not come from Kac–Moody dual graded graphs, we will not make this precise.

5. Affine type C combinatorics

In this section we consider folded insertion for the affine root system $C_n^{(1)}$. Folding works for the entire Weyl group. However here we shall restrict our discussion to the explicit description of folded insertion for the maximal parabolic quotient of the affine Weyl group given by the dual graded graphs $(\Gamma_s(\Lambda_{i'}), \Gamma_w(K))^{i'}$ using $2n$-cores, where $i' \in I$ is any Dynkin node. In the limit $n \to \infty$ one obtains a new Schensted bijection for every integer i, which for $i = 0$ coincides with "standard" Sagan–Worley insertion into shifted tableaux.

Let $I = \{0, 1, \dots, n\}$ be the Dynkin node set and (a_{ij}) the GCM, with $a_{ii} = 2$ for $i \in I$, $a_{i,i+1} = a_{i+1,i} = -1$ for $1 \le i \le n-2$, $a_{01} = a_{n,n-1} = -1$, $a_{10} = a_{n-1,n} = -2$, and other entries zero. Using the recipe in Section 2B, the Weyl group W_n has generators s_i for $i \in I$ satisfying $s_i^2 = 1$ for $i \in I$ and $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $i \neq j$, where $m_{01} = m_{n-1,n} = 4$, $m_{i,i+1} = 3$ for $1 \le i \le n-2$, and $m_{ij} = 0$ for $|i - j| \ge 2$.

5A. *Folding for $C_n^{(1)}$.* Let $A = C_n^{(1)}$ and $B = A_{2n-1}^{(1)}$ denote the two GCMs. We use the notation of Section 4.

Let π be the admissible automorphism of B given by $j \mapsto 2n - j$ where indices are taken modulo $2n$. We index the π-orbits by $O_0 = \{0\}$, $O_n = \{n\}$, and $O_i = \{i, 2n - i\}$ for $i \in I \setminus \{0, n\}$. It is easy to check that A is obtained from (B, π) by folding.

Let K be the canonical central element for $C_n^{(1)}$. Let $i' \in I$ and $j' \in O_{i'}$. We define folded insertion for the dual graded graph $(\Gamma_s^A(\Lambda_{i'}), \Gamma_w^A(K))^{i'}$, realized by LLMS insertion for $(\Gamma_s^B(\omega_{j'}), \Gamma_w^B(\varphi(K)))$. We call this induced folded insertion the "LLMS insertion for $W_n^{i'}$" (even though it also depends on j').

5B. *$2n$-cores.* As before, fix $i' \in I$ and $j' \in O_{i'}$. The elements of the parabolic quotient $W_n^{i'}$ may be realized by $2n$-cores as follows.

By Proposition 3.1 there is a bijection $c : \widetilde{S}_{2n}^0 \to \mathscr{C}_{2n}$. Using a rotational automorphism of the Dynkin diagram of type $A_{2n-1}^{(1)}$, for any $k \in J$ one may define the k-action of $\widetilde{S}_{2n}$ on $\mathscr{C}_{2n}$, denoted $w \cdot_k \lambda$, which is the same as before except that the diagonal of the cell (i, j) is $j - i + k$. Since the stabilizer of $\varnothing$ under the k-action of $\widetilde{S}_{2n}$ on $\mathscr{C}_{2n}$ is $(\widetilde{S}_{2n})_{J \setminus \{k\}}$, there is a bijection $c_k : \widetilde{S}_{2n}^k \to \widetilde{S}_{2n}/(\widetilde{S}_{2n})_{J\setminus\{k\}} \to \mathscr{C}_{2n}$ defined by $c_k(w) = w \cdot_k \varnothing$.

Define the map $sc_{i'} : W_n \to \mathscr{C}_{2n}$ by $w \mapsto f(w) \cdot_{j'} \varnothing$, where $f : W_n \to \widetilde{S}_{2n}$ is the Weyl group homomorphism of Section 4. Note that $f(W_n^{i'}) \subset \widetilde{S}_{2n}^{J \setminus O_{i'}}$.

Denote by $\mathscr{C}_{2n}^{j'}$ the image of $sc_{i'}$.

The following result is the $C_n^{(1)}$-analogue of (part of) Proposition 3.1.

Proposition 5.1. *The map $sc_{i'}$ restricts to a bijection $W_n^{i'} \to \mathscr{C}_{2n}^{j'}$. For $v, w \in W_n^{i'}$ we have $v \le w$ if and only if $sc_{i'}(v) \subset sc_{i'}(w)$.*

Proof. The stabilizer of $W = W_n$ acting on $\varnothing$ is equal to $W_{I-\{i'\}}$, so the first statement is immediate. Let $v, w \in W_n^{i'}$. The following are equivalent: (1) $v \leq w$; (2) $f(v) \leq f(w)$; (3) $f(v)(\widetilde{S}_{2n})_{J\setminus\{j\}} \leq f(w)(\widetilde{S}_{2n})_{J\setminus\{j\}}$ for all $j \in O_{i'}$; (4) $f(v)(\widetilde{S}_{2n})_{J\setminus\{j'\}} \leq f(w)(\widetilde{S}_{2n})_{J\setminus\{j'\}}$; (5) $sc_{i'}(v) \subset sc_{i'}(w)$. (1) and (2) are equivalent by Theorem 4.2. (2) and (3) are equivalent by Proposition 5.2 below applied to the data $f(v)$, $f(w)$, and $J \setminus O_{i'}$ in $\widetilde{S}_{2n}$. Since $f(v)$ and $f(w)$ are π-invariant, (3) and (4) are equivalent, because $j' \in O_{i'}$ and the condition for j is invariant as j runs over a π-orbit. (4) and (5) are equivalent by Proposition 3.1. □

For a Coxeter group W and a parabolic subgroup W_J, the strong (Bruhat) order denoted $\leq$ on the quotient W/W_J is the partial order naturally induced from the strong order on W^J. The following result is due to [Deodhar 1977].

Proposition 5.2. *Let W be a Coxeter group with simple generators indexed by P and let $Q \subset P$. Suppose $x, y \in W^Q$. Then $x \leq y$ if and only if $xW_{Q'} \leq yW_{Q'}$ for every maximal parabolic subgroup $W_{Q'} \supset W_Q$.*

We describe $\mathscr{C}_{2n}^0$ explicitly in Section 5D, together with an explicit description of LLMS insertion for W_n^0. It would be interesting to obtain an explicit description of $\mathscr{C}_{2n}^{j'}$ for arbitrary j'. The explicit description of the Chevalley coefficients in a manner similar to Proposition 3.1, and of LLMS insertion for $W_n^{i'}$ appears to be rather subtle.

5C. ***Large rank limit of folded LLMS insertion.*** We now consider the limit of LLMS insertion for $W_n^{i'}$ as n goes to ∞, in such a way that the nodes near 0 in $A_{2n-1}^{(1)}$ are stable; for this purpose we label these nodes $\dots, -2, -1, 0, 1, 2, \dots$.

Let $A_{\pm\infty}$ be the Kac–Moody algebra[1] whose Dynkin diagram has vertex set $J_\infty = \mathbb{Z}$, with Cartan matrix (b_{ij}) such that $b_{ii} = 2$ and $b_{i,i+1} = b_{i+1,i} = -1$, and $b_{ij} = 0$ otherwise. Let $S_{\pm\infty}$ be its Weyl group: it has generators s_j for $j \in J_\infty$, with relations $s_j^2 = 1$, $(s_j s_{j+1})^3 = 1$, and $(s_i s_j)^2 = 1$ for $|i - j| \geq 2$. Then $S_{\pm\infty}$ acts on partitions: $s_j \cdot \lambda$ is obtained from λ by adding the unique λ-addable cell in diagonal j if it exists, and removing the unique λ-removable cell in diagonal j if it exists (remembering the shift in diagonal index by j'). Then $S_{\pm\infty}\varnothing = \mathbb{Y}$ is the set of all partitions and there is a bijection $c_{j'} : S_{\pm\infty}^{j'} \cong \mathbb{Y}$.

Let C_∞ be the Kac–Moody algebra with Dynkin node set $I_\infty = \mathbb{Z}_{\geq 0}$ and Cartan matrix a_{ij} with $a_{ii} = 2$ for $i \in I_\infty$, $a_{i,i+1} = a_{i+1,i} = -1$ for $i \in I_\infty \setminus \{0\}$, $a_{01} = -1$ and $a_{10} = -2$. Then its Weyl group W_∞ has generators s_i for $i \in I_\infty$ with relations $s_i^2 = 1$, $(s_0 s_1)^4 = 1$, $(s_i s_{i+1})^3 = 1$ for $i \in I_\infty \setminus \{0\}$, and $(s_i s_j)^2 = 1$ for $|i - j| \geq 2$. As before, there is an injective homomorphism $f : W_\infty \to S_{\pm\infty}$ given by $f(s_0) = s_0$ and $f(s_i) = s_i s_{-i}$ for $i > 0$.

[1] We allow infinite Dynkin diagrams in a formal manner.

Then W_∞ acts on partitions via f. Define $\mathbb{Y}^{i'} = W_\infty \cdot \varnothing$. The limit of Proposition 5.1 gives a bijection $sc_{i'} : W_\infty^{i'} \cong \mathbb{Y}^{i'}$. The strong and weak orders on $\mathscr{C}_{2n} \cong \widetilde{S}_{2n}^{j'}$ both converge to Young's lattice $\mathbb{Y}$. The weak order on $\mathscr{C}_{2n}^{j'} \cong W_n^{i'}$ converges to the weak order on $\mathbb{Y}^{i'} \cong W_\infty^{i'}$, in which a cover $\lambda \subset s_i\lambda$ adds cells in diagonals i and $-i$ if $i > 0$ or just the cell in diagonal 0 if $i = 0$.

Proposition 5.3. *Suppose $\lambda \subset \mu$ is a strong cover in $\mathbb{Y}^{i'} \cong W_\infty^{i'}$ with $sc_{i'}^{-1}(\lambda) = w \lessdot ws_\alpha = sc_{i'}^{-1}(\mu)$. If s_α is conjugate to s_0 then the Chevalley coefficient $\langle \alpha^\vee , \Lambda_{i'} \rangle$ is equal to 1 and μ/λ has a single connected component which is necessarily a ribbon. Otherwise suppose $f(s_\alpha) = s_\beta s_{\beta'}$. Then the Chevalley coefficient $\langle \alpha^\vee , \Lambda_{i'} \rangle$ is equal 1 or 2 depending on whether one or both of $sc(ws_\beta)$ and $sc(ws_{\beta'})$ strictly contain λ. Furthermore, each strict containment has a single connected component which is a ribbon.*

Proof. That the skew partitions in question contain a single connected component which equals a ribbon follows from the fact that they are obtained by the action of a reflection on a partition, which always changes the shape by a ribbon. This follows from the edge sequence discussion in Section 3.

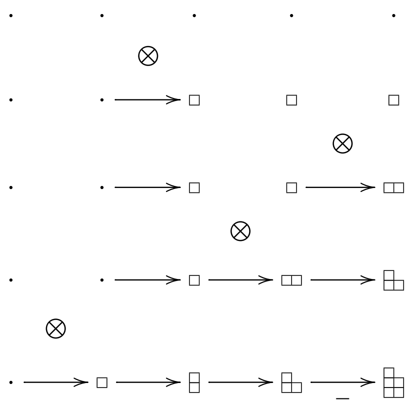
Suppose s_α is conjugate to s_0 and $f(s_\alpha) = s_\beta$. Every strong cover in $\mathbb{Y}$ has Chevalley coefficient equal to 1 or 0 (since a single box is added). Now write $f(w) = xy$ so that $f(ws_\alpha) = xs_0y$ in $S_{\pm\infty}$. The Chevalley coefficient $\langle \alpha^\vee , \Lambda_{i'} \rangle$ is equal to the Chevalley coefficient $\langle \beta^\vee , \omega_{j'} \rangle$ which is equal to the Chevalley coefficient of the cover $y \lessdot s_0 y$ in $S_{\pm\infty}$ (with respect to $\omega_{j'}$). Since $\mu \neq \lambda$, we must have $y \cdot \varnothing \subsetneq (s_0 y) \cdot \varnothing$. Thus the required Chevalley coefficient must be nonzero, and hence equal to 1. We have used the calculation $\langle y^{-1}\beta_0^\vee , \omega_{j'} \rangle = \langle \beta_0^\vee , y\omega_{j'} \rangle = \langle \beta_0^\vee , y'\omega_{j'} \rangle$, where $y' = c_{j'}^{-1}(y \cdot \varnothing)$ is the parabolic component" of y.

The proof for $f(s_\alpha) = s_\beta s_{\beta'}$ follows in a similar manner. The only delicate issue is to show that if $\langle \beta^\vee , \omega_{j'} \rangle = 1$ then $\lambda \subset sc(ws_\beta)$ is a strict inclusion. But $\langle \beta^\vee , \omega_{j'} \rangle = 1$ implies that $(s_\beta \cdot \varnothing) \neq \varnothing$ so that $(ws_\beta \cdot \varnothing) \neq (w \cdot \varnothing)$. □

Figure 3 shows the case of a domino appearing in the strong tableau P, corresponding to the strong cover $s_2s_0s_1 \lessdot s_2s_1s_0s_1$ in W_∞^1.

In the case the Chevalley coefficient described in Proposition 5.3 is equal to 2, the difference μ/λ is a union of two ribbons, since s_β and $s_{\beta'}$ commute. We have not shown that these ribbons do not touch, so the difference μ/λ can potentially be written as the union of the two ribbons in two ways, corresponding to the left action of $s_{f(w)\beta}s_{f(w)\beta'}$ and $s_{f(w)\beta'}s_{f(w)\beta}$. To obtain a strong tableau in $\mathbb{Y}^{i'}$, we must mark a ribbon for each strong cover which consists of two ribbons.

For the differential bijection, we note that for $\lambda \in \mathbb{Y}^{i'}$, once again, $\mathscr{UD}_\lambda$ is in natural bijection with the set of λ-addable corners and $\mathscr{DU}_\lambda$ is in natural bijection with the set of λ-removable corners; in this context the corners are grouped by diagonals of the form $\pm i$ for various i. Using the differential bijection in Example 2.9,



$$P = \begin{array}{|c|c}\hline 4' & \\\hline 2 & \multicolumn{1}{|c|}{4} \\\hline 1 & \multicolumn{1}{|c|}{3} \\\hline\end{array} \qquad Q = \begin{array}{|c|c}\hline 4 & \\\hline 3 & \multicolumn{1}{|c|}{4} \\\hline 1 & \multicolumn{1}{|c|}{2} \\\hline\end{array}$$

Figure 2. Folded insertion of $\sigma = 2431$ for $i' = j' = 1$.

we obtain a folded insertion for the limit $\mathbb{Y}^{i'}$ of $\mathscr{C}_{2n}^{j'}$. It defines a bijection from permutations $P_n(1)$ to pairs (P, Q) where P and Q are n-step strong and weak tableaux with respect to $\mathbb{Y}^{i'}$.

For example, we let $i' = j' = 1$ and compute the folded insertion of $\sigma = 2431$; see the graph G in Figure 2. For the meaning of $\otimes$ see (9) and Example 3.2. The arrows represent strong covers, and an arrow is labeled with $-$ if the strong cover adds two nonadjacent cells and the marked cell is in a more negative diagonal. The unique arrow labeled with $-$, corresponds to the entry $4'$ in P. Strictly speaking we should mark one ribbon for each number used, but when there is no choice we have omitted the marking.

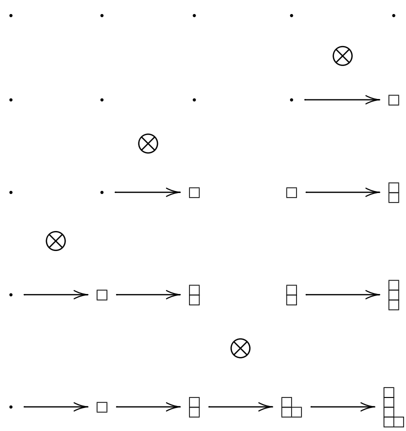
Again, with $i' = j' = 1$ we compute the folded insertion of 4213; see the graph G in Figure 3. Note that there is a unique strong cover that is not a weak cover, corresponding to the domino in P containing 4s.

5D. ***Sagan–Worley insertion.*** We now consider the important case that $i' = 0$. We must have $j' = 0$. The following result is straightforward.

Lemma 5.4. *The set $\mathscr{C}_{2n}^0$ is the subset of $\mathscr{C}_{2n}$ of elements fixed by the transpose* tr.

Using the fact that $f(W_n^0) \subset \widetilde{S}_{2n}^0$, the next result is an easy consequence of Propositions 4.13 and 3.1. A similar statement holds for $i' = n$.

Proposition 5.5. *Suppose $\lambda \subset \mu$ is a strong cover in $\mathscr{C}_{2n}^0 \cong W_n^0$ with $sc_0^{-1}(\mu) = sc_0^{-1}(\lambda)s_\alpha$. Then the Chevalley coefficient $\langle \alpha^\vee, \Lambda_0 \rangle$ is equal to the total number of*



$$P = \begin{array}{|c|c}\cline{1-1} 4 & \\ \cline{1-1} 4 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad Q = \begin{array}{|c|c}\cline{1-1} 4 & \\ \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Figure 3. Folded insertion of $\sigma = 4213$ for $i' = j' = 1$.

components of ν/λ for a strong cover $\lambda \subset \nu$ in $\mathcal{C}_{2n} \cong \widetilde{S}^0_{2n}$ satisfying $\lambda \subset \nu \subset \mu$. In particular, if $\lambda \subset \mu$ is also a left weak cover in $\mathcal{C}^0_{2n} \cong W^0_n$ with $\mu = s_i\lambda$ for $i \in I$, then $\langle \alpha^\vee, \Lambda_0 \rangle$ is equal to the number of λ-addable corner cells of residue i or $-i$ modulo $2n$.

Thus for $i' = 0$, a *folded strong tableau* of shape $\lambda \in \mathcal{C}^0_{2n}$ is a sequence of strong covers in $\mathcal{C}^0_{2n} \cong W^0_n$ from $\varnothing$ to λ, such that every cover has a marked component. A *folded weak tableau* of shape $\lambda \in \mathcal{C}^0_{2n}$ is a sequence of weak covers in $\mathcal{C}^0_{2n} \cong W^0_n$ going from $\varnothing$ to λ; no marking is necessary. With these explicit descriptions we have the following:

Corollary 5.6. *LLMS insertion induces a bijection from the set of permutations $P_n(1)$ to pairs (P, Q) of tableaux of the same shape $\lambda \in \mathcal{C}^0_{2n}$, where $\ell(sc_0^{-1}(\lambda)) = n$, P is a folded strong tableau and Q a folded weak tableau.*

Again we now consider the $n \to \infty$ limit. In this case the limit of $\mathcal{C}^0_{2n}$ is the set $\mathbb{Y}^0$ of partitions fixed under the transpose. The strong and weak orders both converge to the same order on $\mathbb{Y}^0$. Since added cells are in transpose-symmetric positions, when marking a strong cover of the form $\lambda \lessdot s_i\lambda$, one must mark either the added cell in diagonal i or $-i$ if $i > 0$.

Folded weak tableaux Q are in obvious bijection with standard shifted tableaux Q^*, given by taking only the part on one side of the diagonal. A similarly obvious bijection exists from folded strong tableaux to standard shifted tableaux in which off-diagonal entries may or may not have a mark; we choose the bijection so that a mark in a shifted tableau P^* indicates that the corresponding cell with negative diagonal index is marked in the folded strong tableau P.

Thus we have the correct kinds of tableaux to compare with Sagan–Worley insertion.

Example 5.7. Let $n = 7$ and $\sigma = 2673541$. See Example 3.2 for the way to interpret σ and the symbols $\otimes$. We draw the graph G_{ij} for the folded insertion of σ. We draw arrows to represent strong covers and place an asterisk on an edge if the marked cell is on a negative diagonal. P and Q are the folded strong and weak tableaux respectively. P^* and Q^* are the shifted tableaux corresponding to P and Q.

By reformulating Sagan–Worley insertion using Fomin's setup, we obtain the following theorem. Note the exchange of P^* and Q^*.

Theorem 5.8. *Let σ map to (P, Q) under folded insertion. Then the pair (Q^*, P^*) of shifted tableaux, is the image of σ^{-1} under Sagan–Worley insertion.*

Remark 5.9. In [1989, Proposition 6.2], Haiman relates Sagan–Worley shifted insertion with left-right insertion. It is natural to ask whether one can connect Sections 2E and 4D in a similar manner. Unfortunately, a straightforward generalization of Haiman's result does not appear to be possible. For example when π has order 2, one would need to relate the left-right (or mixed) insertion of a colored permutation on $2r$ letters with folded insertion of a permutation on r letters. Length considerations show that this can be done only if each orbit of π on J has order 2, which nearly never happens in our setup.

6. Distributive parabolic quotients

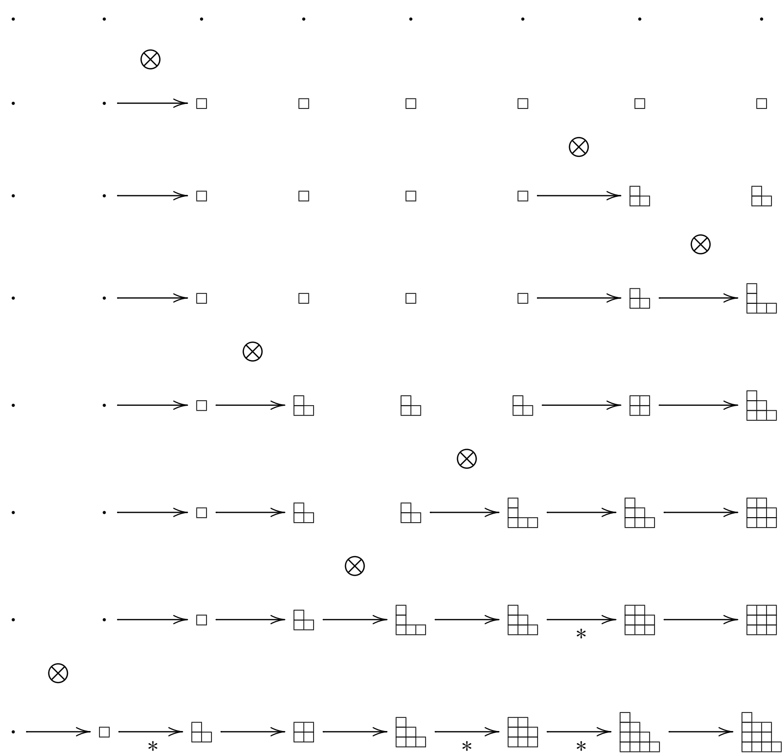
6A. *Proctor's classification.* Let W be a finite irreducible Weyl group with simple generators $\{s_i \mid i \in I\}$ and set of reflections T. Recall the notations W_J and W^J from before Proposition 2.16. We have

$$W^J = \{w \in W \mid w < ws_i \text{ for any } i \in J\}.$$

Proctor [1984] classified the cases when W^J is a distributive lattice under the weak order. In all such cases, Stembridge [1996] showed that the weak and strong orders agree on W^J and that W_J is a maximal parabolic subgroup of W, that is, $J = I \setminus \{i\}$ for some $i \in I$. We call such $W^J := W^i$ *distributive parabolic quotients*.

Theorem 6.1 [Proctor 1984]. *The distributive parabolic quotients are*:

(1) $W \simeq A_n$; $J = I \setminus \{i\}$ *for any* $i \in I$.



$$P = \begin{array}{llll} 6' & & & \\ 4 & 5' & 7 & \\ 2' & 3 & 5 & \\ 1 & 2 & 4' & 6 \end{array} \qquad Q = \begin{array}{llll} 7 & & & \\ 3 & 5 & 6 & \\ 2 & 4 & 5 & \\ 1 & 2 & 3 & 7 \end{array}$$

$$P^* = \begin{array}{llll} & & 7 & \\ & 3 & 5' & \\ 1 & 2' & 4 & 6' \end{array} \qquad Q^* = \begin{array}{llll} & & 6 & \\ & 4 & 5 & \\ 1 & 2 & 3 & 7 \end{array}$$

Figure 4. Growth diagram illustrating Theorem 5.8.

(2) $W \simeq B_n$; $W_J \simeq B_{n-1}$ *or* $W_J \simeq A_{n-1}$.

(3) $W \simeq D_n$; $W_J \simeq D_{n-1}$ *or* $W_J \simeq A_{n-1}$.

(4) $W \simeq G_2$; $J = I \setminus \{i\}$ *for any* $i \in I$.

In [Stembridge 1996], it is shown that these cases are also exactly the parabolic quotients W^J of Weyl groups such that every element $w \in W^J$ is fully commutative, that is, every two reduced decompositions of w can be obtained from each other using just the relations of the form $s_i s_j = s_j s_i$ for $i, j \in I$.

6B. ***Distributive labeled posets.*** We need a slightly more precise form of the results of Proctor and Stembridge. If Q is a finite poset we let $J(Q)$ denote the poset of (lower) order ideals of Q. The poset $J(Q)$ is a distributive lattice and the fundamental theorem of finite distributive posets [Stanley 1999] says that the correspondence $Q \mapsto J(Q)$ is a bijection between finite posets and finite distributive lattices. Suppose P is a finite poset and $\omega : \{x \lessdot y\} \to A$ is a labeling of the edges of the Hasse diagram of P with elements of some set A. We call (P, ω) an edge-labeled poset. We say that (P, ω) is a *distributively labeled lattice* if

(1) $P = J(Q)$ is a distributive lattice; and

(2) there is a vertex (element) labeling $\pi : Q \to A$ such that

$$\omega(I \setminus \{q\} \lessdot I) = \pi(q)$$

for any $I \in J(Q)$ and q maximal in I.

If W is a Weyl group, we may label the edges of the Hasse diagram of the weak order $(W, \prec)$ with simple reflections: the cover $w \prec s_i w$ is labeled with s_i. We denote the resulting edge-labeled poset by W_{weak}. Similarly define W_{strong} to be the strong order where $w \lessdot wt$ is labeled with $t \in T$. These labeled posets restrict to give labeled posets W^i_{weak} and W^i_{strong}. Note that each cover relation in W^i under either order is itself a cover relation in W. Thus W^i_{weak} and W^i_{strong} are induced subgraphs of W_{weak} and W_{strong}.

Theorem 6.2. *Suppose W^i is a distributive parabolic quotient. Then the strong and weak orders on W^i coincide. In particular W^i_{weak} and W^i_{strong} are distributively labeled lattices.*

Stembridge [1996, Theorem 2.2] proved that W^i_{weak} is a distributively labeled lattice. For the sake of completeness we give a self-contained proof of Theorem 6.2.

6C. ***Cominuscule parabolic quotients.*** Let Φ be an irreducible finite root system and W be its Weyl group. Let $\Phi = \Phi^+ \sqcup \Phi^-$ denote the decomposition of the roots into the disjoint subsets of positive and negative roots. Let $\theta = \sum_{i \in I} a_i \alpha_i$ denote the highest root of Φ. We say that $i \in I$ is *cominuscule* if $a_i = 1$.

It can be checked case-by-case using Theorem 6.1 that the distributive parabolic quotients W^i correspond to cominuscule nodes $i \in I$ except in the cases $W = G_2$, $W^i = B_n/A_{n-1}$ or $W^i = C_n/C_{n-1}$. In the latter two cases, one may use the isomorphic quotients given by their duals C_n/A_{n-1} and B_n/B_{n-1}, which are cominuscule.

For now we suppose that a cominuscule node $i \in I$ has been fixed. If α and β are two roots, we say $\alpha \geq \beta$ if $\alpha - \beta$ is a sum of positive roots. Recall that θ is the unique maximal root under this order. Let $\Phi^{(i)}$ denote the poset of positive roots which lie above α_i. Clearly $\theta \in \Phi^{(i)}$. The inversion set of $w \in W$ is defined by

$$\mathrm{Inv}(w) = \{\alpha \in \Phi^+ \mid w\alpha < 0\}.$$

Lemma 6.3. *Suppose $\alpha, \beta \in \Phi^{(i)}$. Write $s_\alpha\beta = \beta + k\alpha$ where $k = -\langle \alpha^\vee, \beta\rangle$. Then $k \in \{0, -1, -2\}$ and the following facts hold:*

(1) *If α and β are incomparable then $s_\alpha\beta = \beta$.*

(2) *If $\alpha > \beta$ then $s_\alpha\beta$ is equal to one of the following: (i) β; (ii) $-\gamma$ where $\gamma \in \Phi^+ \setminus \Phi^{(i)}$; or (iii) $-\gamma$ where $\gamma > \alpha$.*

Proof. To obtain the bounds on k we observe that for all roots $\gamma \in \Phi$, $-\theta \le \gamma \le \theta$, so that the coefficient of α_i in γ, lies between the corresponding coefficients in $-\theta$ and θ, which are -1 and 1 by the assumption that i is cominuscule.

Suppose that α and β are incomparable. Then $\beta - \alpha$ is neither positive nor negative and hence not a root. Since the roots in Φ occur in strings, we must have $k = 0$.

If $\alpha > \beta$, the three cases correspond to $k = 0$, $k = -1$, and $k = -2$. □

For our results on distributive parabolic quotients, we require the following result, which is a slight strengthening of [Thomas and Yong 2006, Proposition 2.1, Lemma 2.2]. We include a self-contained proof, part of which is the same as the proof of [Thomas and Yong 2006, Proposition 2.1]. In particular we prove directly that the edge labeled poset W^i_{strong} defined in Section 6B is a distributively labeled lattice.

Proposition 6.4. *The map $w \longmapsto \mathrm{Inv}(w)$ defines an isomorphism of posets $\mathrm{Inv}|_{W^i} : (W^i, \le) \to J(\Phi^{(i)})$. Moreover, if $u \lessdot w$ for $u, w \in W^i$, then writing $w = us_\alpha$ for $\alpha \in \Phi^+$, we have $\alpha \in \Phi^{(i)}$ and $\mathrm{Inv}(w) = \mathrm{Inv}(u) \sqcup \{\alpha\}$.*

Proof. Let $w \in W^i$. First we show that $\mathrm{Inv}(w) \subset \Phi^{(i)}$. Suppose that $\gamma \in \mathrm{Inv}(w) \setminus \Phi^{(i)}$. If $\gamma = \alpha_k$ where $k \neq i$ this means $ws_k < w$ which contradicts the assumption that $w \in W^i$. Otherwise $\gamma = \delta + \rho$ where $\delta, \rho \in \Phi^+ \setminus \Phi^{(i)}$. Since $w\gamma < 0$ we have $w\delta < 0$ or $w\rho < 0$ so the same argument applies. Repeating we obtain a contradiction.

Now we show that $\mathrm{Inv}(w) \in J(\Phi^{(i)})$. Suppose $\alpha \in \mathrm{Inv}(w)$ and $\beta < \alpha$. Then $\gamma = \alpha - \beta \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i in γ is zero. Since $\mathrm{Inv}(w) \subset \Phi^{(i)}$, we have $\gamma \notin \mathrm{Inv}(w)$, that is, $w\alpha - w\beta = w\gamma > 0$. Since $w\alpha < 0$ this shows that $w\beta < 0$ as desired. Thus $\mathrm{Inv}|_{W^i}$ is well-defined.

Next we show that $\mathrm{Inv}|_{W^i}$ sends covers to covers. Let $u \lessdot w$ with $u, w \in W^i$ and $\alpha \in \Phi^+$ such that $w = us_\alpha$. Then $0 > w\alpha = -u\alpha$ so $\alpha \in \mathrm{Inv}(w) \setminus \mathrm{Inv}(u)$. For all $\beta \in \mathrm{Inv}(u)$, since $\mathrm{Inv}(u) \in J(\Phi^{(i)})$, $\alpha > \beta$ or $\alpha > \beta$. Either way we have $w\beta = us_\alpha\beta < 0$, since by Lemma 6.3, $s_\alpha\beta$ is either equal to β or $-\gamma$ for

$\gamma \in \Phi^+ \setminus \mathrm{Inv}(u)$. That is, $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$. Since $|\mathrm{Inv}(w)| = |\mathrm{Inv}(u)| + 1$ it follows that $\mathrm{Inv}(w) = \mathrm{Inv}(u) \sqcup \{\alpha\}$, so that $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$ is a covering relation in $J(\Phi^{(i)})$.

Next we show that every covering relation in $J(\Phi^{(i)})$ is the image of a covering relation in W^i, and in particular, that $\mathrm{Inv}|_{W^i}$ is onto. An arbitrary covering relation in $J(\Phi^{(i)})$ is given by $S \setminus \{\alpha\} \subset S$ where $S \in J(\Phi^{(i)})$ and α is maximal in S.

By induction there is a $u \in W^i$ such that $\mathrm{Inv}(u) = S \setminus \{\alpha\}$. Let $w = us_\alpha$. It suffices to show that

$$\mathrm{Inv}(w) = S \quad \text{and} \quad w \in W^i.$$

The second claim follows from the first since none of the α_k for $k \neq i$ lie in $\mathrm{Inv}(w)$. For the first claim, since $\alpha \in \Phi^{(i)} \setminus \mathrm{Inv}(u)$, we may argue as before to show that $S = \mathrm{Inv}(u) \sqcup \{\alpha\} \subset \mathrm{Inv}(w)$.

For the opposite inclusion, suppose $\beta \in \Phi^+ \setminus S$. We must show that $w\beta > 0$. Write $s_\alpha\beta = \beta + k\alpha$ for $k \in \mathbb{Z}$. If $k = 0$ then we are done as before. If $k > 0$ then $s_\alpha\beta > \alpha$, so that $s_\alpha\beta \in \Phi^+ \setminus S$ since S is an order ideal. But then $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. So we may assume that $k < 0$.

Suppose first that $\beta \in \Phi^{(i)}$. We may assume that α and β are comparable by Lemma 6.3. Since S is an order ideal we have $\beta > \alpha$. If $k = -1$ then $s_\alpha\beta = \beta - \alpha \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i is 1 in both α and β. In particular $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. Otherwise $k = -2$. Then $s_\alpha\beta = \beta - 2\alpha < 0$. We have $0 < \beta - \alpha < \alpha$ and $-s_\alpha\beta = 2\alpha - \beta = \alpha - (\beta - \alpha) < \alpha$. Since S is an order ideal it follows that $-s_\alpha\beta \in \mathrm{Inv}(u)$ and $w\beta = us_\alpha\beta > 0$ as desired.

Otherwise $\beta \in \Phi^+ \setminus \Phi^{(i)}$. Since i is cominuscule we have $k \in \{-1, 0, 1\}$. We assume $k = -1$ as the other cases were already done. Then $s_\alpha\beta = \beta - \alpha < 0$ since its coefficient of α_i is -1. Moreover $\alpha - \beta \in \Phi^{(i)}$. Since $\alpha > \alpha - \beta$ and S is an order ideal, it follows that $\alpha - \beta \in \mathrm{Inv}(u)$. Therefore $w\beta = us_\alpha\beta > 0$ as desired.

We have shown that every cover in $J(\Phi^{(i)})$ is the image under $\mathrm{Inv}|_{W^i}$ of a cover in $(W^i, \leq)$.

The bijectivity of $\mathrm{Inv}|_{W^i}$ follows by induction and the explicit description of the image of a cover under $\mathrm{Inv}|_{W^i}$. □

Proof of Theorem 6.2. For the case $W = G_2$, both labeled posets W^i_{weak} and W^i_{strong} are chains, so the result follows immediately. Thus we may assume that W^i is a cominuscule parabolic quotient.

For W^i_{strong} the result follows from Proposition 6.4. We label the vertices of $\Phi^{(i)}$ by reflections, defining $\pi : \Phi^{(i)} \to T$ by $\pi(\alpha) = s_\alpha$. Each cover $w \lessdot ws_\alpha$ in W^i_{strong} corresponds to adding $\alpha \in \Phi^{(i)}$ to $\mathrm{Inv}(w)$. Thus the edge label of $w \lessdot ws_\alpha$ agrees with the vertex label $\pi(\alpha) = s_\alpha$.

For the weak order W^i_{weak} let us consider two covers $w \lessdot ws_\alpha = s_\beta w$ and $v \lessdot vs_\alpha = s_{\beta'} v$ which have the same label s_α in W^i_{strong}. We claim that $s_\beta = s_{\beta'} = s_k$ for some $k \in I$. The elements w and v differ by right multiplication by some s_γ's where

$\gamma \in \Phi^{(i)}$ is incomparable with α; this is accomplished by passing between w or v to the element $u \in W^i$ such that $\mathrm{Inv}(u) = \mathrm{Inv}(w) \cap \mathrm{Inv}(v)$. By Lemma 6.3 these s_γ's commute with s_α, and so $w\alpha = v\alpha$. This gives us a map $f : \Phi^{(i)} \to \Phi^+$ defined by $f(\alpha) = \beta = w\alpha$, which does not depend on $w \in W^i$ as long as $w \lessdot ws_\alpha$.

To show that $f(\alpha)$ is simple for each $\alpha \in \Phi^{(i)}$, consider a reduced word $ws_\alpha = s_{k_1}s_{k_2}\cdots s_{k_l}$. We know that $w^{(r)} = s_{k_r}\cdots s_{k_l} \in W^i$ and that $\mathrm{Inv}(w^{(r)})$ differs from $\mathrm{Inv}(w^{(r+1)})$ by some root in $\Phi^{(i)}$ since $w^{(r+1)} \lessdot w^{(r)}$. For some value $r = r^*$, this root is α and by the well-definedness just proved $f(\alpha) = \alpha_{k_{r^*}}$, since $w^{(r^*)} = w^{(r^*+1)}s_\alpha$. This shows that the strong order and weak order on W^i coincide, and that W^i_{weak} is isomorphic to the poset of order ideals of $\Phi^{(i)}$ where $\Phi^{(i)}$ is labeled with $\pi(\alpha) = f(\alpha)$. □

7. Distributive subgraphs of Kac–Moody graded graphs

In this section we apply Theorem 6.2 to the dual graded graphs constructed in Section 2.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody algebra associated to the generalized Cartan matrix A and let W be its Weyl group. Let $W_{\mathrm{fin}} \subset W$ be a finite parabolic subgroup corresponding to some index set $I' \subset I$. Now suppose that W_{fin} has a distributive parabolic quotient as in Theorem 6.1 corresponding to $J = I' \setminus \{i\} \subset I'$. We let $W^J \subset W_{\mathrm{fin}}$ denote the distributive parabolic quotient (we use W^J instead of W^i in this section since W^J is not a maximal parabolic quotient of W, but of W_{fin}).

Now let $(\Lambda, K) \in P^+ \times Z^+$ and $(\Gamma_s(\Lambda), \Gamma_w(K))$ be the pair of dual graded graphs constructed in Section 2. By restricting to the set of vertices $W^J \subset W_{\mathrm{fin}} \subset W$ we obtain the induced pair of graded graphs $(\Gamma_s(\Lambda), \Gamma_w(K))^J$. These graded graphs are not dual (see Remark 2.1) but they still have rich combinatorics.

The distributive lattice $(W^J, \leq)$ has two edge labelings. Recall that in W^J_{strong}, the edge $v \lessdot w = vs_\alpha$ is labeled either by the reflection s_α, while in the strong Kac–Moody subgraph $\Gamma^J_s(\Lambda)$, the edge $v \lessdot w = vs_\alpha$ is labeled by the integer $\langle \alpha^\vee, \Lambda\rangle$. Similarly the distributive lattice $(W^J, \preceq)$ has two edge labelings; in W^J_{weak}, the edge $v \prec s_j v$ is labeled by the simple reflection s_j, while in the weak Kac–Moody subgraph $\Gamma^J_w(K)$, the edge $v \prec s_j v$ is labeled by the integer $\langle K, \Lambda_j\rangle$. The following result is an immediate consequence of Theorem 6.2.

Theorem 7.1. *The induced graded subgraphs $\Gamma^J_s(\Lambda)$ and $\Gamma^J_w(K)$ are distributively labeled lattices.*

Thus $\Gamma^J_s(\Lambda)$ (resp. $\Gamma^J_w(K)$) can be thought of as the poset of order ideals in some integer labeled poset P^J (resp. Q^J). The Λ-strong and K-weak tableaux can be thought of as linear extensions of P^J and Q^J with additional markings.

In the rest of the paper, we give examples of the posets P^J and Q^J and relate them to classically understood tableaux. In each case we let $\mathfrak{g}$ be of untwisted affine

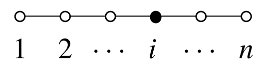
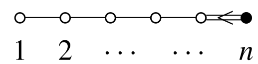
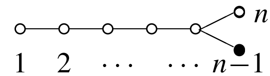
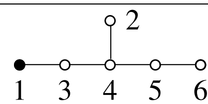
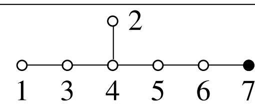
Root system	Dynkin Diagram
A_n	
$C_n, n \geq 3$	
$D_n, n \geq 4$	
E_6	
E_7	

Figure 5. Some cominuscule parabolic quotients.

type, $I_{\text{fin}} = I \setminus \{0\}$ and $J = I \setminus \{i\}$ for a fixed node $i \in I_{\text{fin}}$ to be specified. We use the canonical central element $K_{\text{can}} = \sum_{i \in I} a_i^\vee \alpha_i^\vee$ for K and Λ_i for the dominant weight. In this case P^J and Q^J are both labelings of the poset $\Phi^{(i)} \subset \Phi^+$ for the simple Lie algebra $\mathfrak{g}_{\text{fin}}$ whose Dynkin diagram is the subdiagram of that of $\mathfrak{g}$ given by removing the 0 node. These examples, with the exception of G_2, can be viewed as providing some additional data for the posets $\Phi^{(i)}$, whose unlabeled versions were given explicitly in [Thomas and Yong 2006]. As in that reference, we rotate the labeled Hasse diagrams clockwise by 45 degrees so that the minimal element is in the southwest corner. In the following, we let V^J_{weak}, V^J_{strong} denote the vertex-labeled posets such that $W^J_{\text{weak}} = J(V^J_{\text{weak}})$ and $W^J_{\text{strong}} = J(V^J_{\text{strong}})$.

7A. ***Type*** $\boldsymbol{A_n^{(1)}}$**.** Let $i \in I_{\text{fin}}$ be arbitrary. The poset $\Phi^{(i)}$ consists of elements $\alpha_{p,q} = \alpha_p + \cdots + \alpha_q$ for $1 \leq p \leq i \leq q \leq n$. The weak labeling of $\Phi^{(i)}$ is given by $\alpha_{p,q} \mapsto s_{p+q-i}$. For example, for $n = 7$ and $i = 3$ and abbreviating $\alpha_{p,q}$ by pq and s_j by j, the labelings of $\Phi^{(i)}$ by positive roots and simple reflections are given by

$V^J_{\text{strong}} =$

13	14	15	16	17
23	24	25	26	27
33	34	35	36	37

$V^J_{\text{weak}} =$

1	2	3	4	5
2	3	4	5	6
3	4	5	6	7

All labelings in P^J and Q^J are given by the constant 1. The resulting strong and weak tableaux are usual standard tableaux.

7B. ***Type*** $\boldsymbol{C_n^{(1)}}$**.** Let $i = n$. Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = 2e_n$ where e_i is the i-th standard basis element of the weight lattice $\mathbb{Z}^n$. Then $\Phi^{(n)}$ consists of the roots $\alpha_{i,j} = e_i + e_j$ for $1 \le i \le j \le n$. We have $a_i^\vee = 1$ for all i. For $n = 4$ we have

$$V^J_{\text{strong}} = \begin{array}{|c|c|c|c|} \hline 14 & 13 & 12 & 11 \\ \hline 24 & 23 & 22 & \\ \hline 34 & 33 & & \\ \hline 44 & & & \\ \hline \end{array} \qquad V^J_{\text{weak}} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & \\ \hline 3 & 4 & & \\ \hline 4 & & & \\ \hline \end{array} \qquad P^J = \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 1 \\ \hline 2 & 2 & 1 & \\ \hline 2 & 1 & & \\ \hline 1 & & & \\ \hline \end{array} \qquad Q^J = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & \\ \hline 1 & 1 & & \\ \hline 1 & & & \\ \hline \end{array}$$

The strong tableaux are shifted standard tableaux with two kinds of markings on off-diagonal entries; these are the standard recording tableaux for shifted insertion [Sagan 1987]. The weak tableaux are standard shifted tableaux.

7C. ***Type*** $\boldsymbol{D_n^{(1)}}$**.** Let $i = n$. Letting $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = e_{n-1} + e_n$, the roots of $\Phi^{(n)}$ are given by $\alpha_{p,q} = e_p + e_q$ for $1 \le p < q \le n$. We have $a_j^\vee = 1$ for $j \in \{0, 1, n-1, n\}$ and $a_j^\vee = 2$ otherwise. For $n = 5$ we give the labelings of $\Phi^{(n)}$ below. Note the 1 in the upper left corner of Q^J.

$$V^J_{\text{strong}} = \begin{array}{|c|c|c|c|} \hline 15 & 14 & 13 & 12 \\ \hline 25 & 24 & 23 & \\ \hline 35 & 34 & & \\ \hline 45 & & & \\ \hline \end{array} \qquad V^J_{\text{weak}} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 & \\ \hline 3 & 4 & & \\ \hline 5 & & & \\ \hline \end{array} \qquad P^J = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & \\ \hline 1 & 1 & & \\ \hline 1 & & & \\ \hline \end{array} \qquad Q^J = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 1 \\ \hline 2 & 2 & 1 & \\ \hline 2 & 1 & & \\ \hline 1 & & & \\ \hline \end{array}$$

7D. ***Type E.*** The computations in this section were made using Stembridge's Coxeter/Weyl package [Stembridge 2004]. In both of the following cases, P^J has all labels 1.

For $E_6^{(1)}$ and $i = 1$ with the Dynkin labeling in Figure 5, we have

$$V^J_{\text{weak}} = \begin{array}{cccccccc} & & & 1 & 3 & 4 & 5 & 6 \\ & & & 3 & 4 & 2 & & \\ & & 2 & 4 & 5 & & & \\ 1 & 3 & 4 & 5 & 6 & & & \end{array} \qquad Q^J = \begin{array}{cccccccc} & & & 1 & 2 & 3 & 2 & 1 \\ & & & 2 & 3 & 2 & & \\ & & 2 & 3 & 2 & & & \\ 1 & 2 & 3 & 2 & 1 & & & \end{array}$$

For $E_7^{(1)}$ and $i = 7$ with the Dynkin labeling in Figure 5, we have

$$V^J_{\text{weak}} = \begin{array}{ccccccccc} & & & & & & & & 7 \\ & & & & & & & & 6 \\ & & & & & & & & 5 \\ & & & & & & & 2 & 4 \\ & & & & 7 & 6 & 5 & 4 & 3 \\ & & & & 6 & 5 & 4 & 3 & 1 \\ & & & & 5 & 4 & 2 & \square & \\ & & & 2 & 4 & 3 & & & \\ 7 & 6 & 5 & 4 & 3 & 1 & & & \end{array} \qquad Q^J = \begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & & & & & 2 \\ & & & & & & & & 3 \\ & & & & & & & 2 & 4 \\ & & & & 1 & 2 & 3 & 4 & 3 \\ & & & & 2 & 3 & 4 & 3 & 2 \\ & & & & 3 & 4 & 2 & & \\ & & & 2 & 4 & 3 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & & & \end{array}$$

7E. ***Type*** $\boldsymbol{G_2^{(1)}}$**.** This case does *not* correspond to a cominiscule root. Pick $i = 1$ and let α_1, α_2 be the two simple roots, so that the highest root is $3\alpha_1 + 2\alpha_2$. Then

$a_1^\vee = 1$ and $a_2^\vee = 2$. Abbreviating the reflection $s_{p\alpha_1+q\alpha_2}$ by pq, we have:

$$V^J_{\text{strong}} = \boxed{1}\boxed{31}\boxed{21}\boxed{32}\boxed{11} \qquad P^J = \boxed{1}\boxed{3}\boxed{2}\boxed{3}\boxed{1}$$

$$V^J_{\text{weak}} = \boxed{1}\boxed{2}\boxed{1}\boxed{2}\boxed{1} \qquad Q^J = \boxed{1}\boxed{2}\boxed{1}\boxed{2}\boxed{1}$$

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