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Complexes of injective kG-modules

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Let G be a finite group and k be a field of characteristic p. We investigate the homotopy category $\mathsf{K}(\mathsf{Inj}kG)$ of the category $\mathsf{C}(\mathsf{Inj}kG)$ of complexes of injective (= projective) kG-modules. If G is a p-group, this category is equivalent to the derived category $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ of the cochains on the classifying space; if G is not a p-group, it has better properties than this derived category. The ordinary tensor product in $\mathsf{K}(\mathsf{Inj}kG)$ with diagonal G-action corresponds to the E_∞ tensor product on $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$.

We show that $\mathsf{K}(\mathsf{Inj}kG)$ can be regarded as a slight enlargement of the stable module category $\mathsf{StMod}kG$. It has better formal properties inasmuch as the ordinary cohomology ring $H^*(G,k)$ is better behaved than the Tate cohomology ring $\hat{H}^*(G,k)$.

It is also better than the derived category $\mathsf{D}(\mathsf{Mod}kG)$, because the compact objects in $\mathsf{K}(\mathsf{Inj}kG)$ form a copy of the bounded derived category $\mathsf{D}^b(\mathsf{mod}kG)$, whereas the compact objects in $\mathsf{D}(\mathsf{Mod}kG)$ consist of just the perfect complexes.

Finally, we develop the theory of support varieties and homotopy colimits in $\mathsf{K}(\mathsf{Inj}kG)$.

1. Introduction

Let k be a field and G a finite group. The purpose of this paper is to develop the properties of $\mathsf{K}(\mathsf{Inj}\,kG)$, the homotopy category of complexes of injective kG-modules.

For any ring Λ, we write $\mathsf{C}(\mathsf{Inj}\,\Lambda)$ for the category whose objects are the chain complexes of injective Λ-modules and whose arrows are the degree zero morphisms of chain complexes. We write $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ for the category with the same objects, but where the maps are the homotopy classes of degree zero maps of chain complexes. We write $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,\Lambda)$ for the full subcategory whose objects are the acyclic chain complexes of injective Λ-modules.

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We investigate a recollement relating $\mathsf{K}(\mathsf{Inj}\,kG)$ to the stable module category $\mathsf{StMod}\,kG$ and the derived category $\mathsf{D}(\mathsf{Mod}\,kG)$:

$$\mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG) \underset{\underset{-\otimes_k tk}{\longleftarrow}}{\overset{\overset{\mathrm{Hom}_k(tk,-)}{\longleftarrow}}{\longrightarrow}} \mathsf{K}(\mathsf{Inj}\,kG) \underset{\underset{-\otimes_k pk}{\longleftarrow}}{\overset{\overset{\mathrm{Hom}_k(pk,-)}{\longleftarrow}}{\longrightarrow}} \mathsf{D}(\mathsf{Mod}\,kG).$$

For notation, we write pk, ik and tk for a projective resolution, injective resolution and Tate resolution of k as a kG-module respectively. The compact objects in these categories are

$$\mathsf{stmod}\,kG \longleftarrow \mathsf{D}^b(\mathsf{mod}\ kG) \longleftarrow \mathsf{D}^b(\mathsf{proj}\,kG).$$

This means that $\mathsf{K}(\mathsf{Inj}\,kG)$ can be regarded as the appropriate "big" category for $\mathsf{D}^b(\mathsf{mod}\ kG)$, whereas $\mathsf{D}(\mathsf{Mod}\,kG)$ has too few compact objects for this purpose. In this sense, $\mathsf{K}(\mathsf{Inj}\,kG)$ is a nicer category to work in than $\mathsf{D}(\mathsf{Mod}\,kG)$.

From the point of view of algebraic topology, what $\mathsf{K}(\mathsf{Inj}\,kG)$ does for us is provide an algebraic replacement for the derived category of the differential graded algebra of singular cochains on the classifying space, $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$. Namely, if G is a p-group there is an equivalence of categories

$$\mathsf{K}(\mathsf{Inj}\,kG) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)).$$

We prove that the tensor product over k of complexes in $\mathsf{K}(\mathsf{Inj}\,kG)$ corresponds under this equivalence to the left derived tensor product over $C^*(BG;k)$ coming from the fact that the latter is E_∞, or "commutative up to all higher homotopies" (see Theorem 7.8 and the remarks after Theorem 4.1).

If G is not a p-group, then there is more than one simple kG-module, and the only one $C^*(BG;k)$ "sees" is the trivial kG-module. In this sense, $\mathsf{K}(\mathsf{Inj}\,kG)$ is nicer to work in than $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$, even though it is not necessarily equivalent to it. Writing ik for an injective resolution of the trivial module, what we obtain in general is an equivalence between $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ and the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ generated by ik.

In the work of Dwyer, Greenlees and Iyengar [2006], a close relationship was established between $\mathsf{D}(\mathsf{Mod}\,kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$. For a general finite group, the relationship between $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ is much closer, and provides some sort of context for understanding what is going on in [Dwyer et al. 2006]. Traces of arguments from that paper can be seen from time to time in this paper.

We develop the theory of support varieties for objects in $\mathsf{K}(\mathsf{Inj}\,kG)$, extending the theory developed by Benson, Carlson and Rickard [1996]. The extra information not included in $\mathsf{StMod}\,kG$ is reflected in the fact that the maximal ideal $\mathfrak{m}$ of positive degree elements in $H^*(G,k)$ becomes relevant in the variety theory. Thus

$\mathsf{K}(\mathsf{Inj}\, kG)$ can be regarded as a slight enlargement of $\mathsf{StMod}(kG)$ in which one more prime ideal $\mathfrak{m}$ of the cohomology ring is reflected. We also construct objects with injective cohomology, extending the work of Benson and Krause [2002]; the theory in $\mathsf{K}(\mathsf{Inj}\, kG)$ is easier than in $\mathsf{StMod}\, kG$ because one does not have to compare ordinary and Tate cohomology.

Homotopy colimits in $\mathsf{K}(\mathsf{Inj}\, kG)$ are harder to deal with than in $\mathsf{StMod}\, kG$ or than in $\mathsf{D}(\mathsf{Mod}\, kG)$, so we conclude with a section describing how the theory works in this case. The main theorem here is that localizing subcategories of $\mathsf{K}(\mathsf{Inj}\, kG)$ are closed under filtered colimits in $\mathsf{C}(\mathsf{Inj}\, kG)$, in spite of the fact that the compact objects in $\mathsf{K}(\mathsf{Inj}\, kG)$ do not lift to compact objects in $\mathsf{C}(\mathsf{Inj}\, kG)$.

2. $\mathsf{K}(\mathsf{Inj}\, kG)$ is compactly generated

Let Λ be a Noetherian ring. We consider the category $\mathsf{Mod}\,\Lambda$ of Λ-modules and denote by $\mathsf{mod}\,\Lambda$ the full subcategory which is formed by all finitely generated modules. The injective Λ-modules form a subcategory $\mathsf{Inj}\,\Lambda$ of $\mathsf{Mod}\,\Lambda$ that is closed under taking arbitrary coproducts. This implies that $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is a triangulated category which admits arbitrary coproducts.

We need to recall some definitions. Let T be a triangulated category with arbitrary coproducts. An object X of T is called *compact* if the functor $\mathrm{Hom}_{\mathsf{T}}(X, -)$ into the category of abelian groups preserves all coproducts. We denote by T^c the full subcategory which is formed by all compact objects of T and observe that T^c is a thick subcategory. The triangulated category T is *compactly generated* if the isomorphism classes of objects of T^c form a set and if T coincides with its smallest triangulated subcategory containing T^c and closed under all coproducts.

Well known examples of compactly generated triangulated categories include the stable module category $\mathsf{StMod}\,\Lambda$ provided that Λ is self-injective, and the derived category $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ for any ring Λ. For references, see [Happel 1988] and [Verdier 1996]. Note that the inclusion functors $\mathsf{stmod}\,\Lambda \to \mathsf{StMod}\,\Lambda$ and $\mathsf{proj}\,\Lambda \to \mathsf{Mod}\,\Lambda$ induce equivalences

$$\mathsf{stmod}\,\Lambda \xrightarrow{\sim} (\mathsf{StMod}\,\Lambda)^c \quad \text{and} \quad \mathsf{D}^b(\mathsf{proj}\,\Lambda) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\,\Lambda)^c.$$

Proposition 2.1. *The triangulated category* $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ *is compactly generated. Let* $\mathsf{K}^c(\mathsf{Inj}\,\Lambda)$ *denote the full subcategory which is formed by all compact objects. Then the canonical functor* $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Lambda)$ *induces an equivalence*

$$\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\,\Lambda).$$

Proof. See [Krause 2005, Proposition 2.3]. □

Remark 2.2. The canonical functor $Q\colon \mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Lambda)$ has a right adjoint sending a complex X to its semiinjective resolution $i\,X$ (the definition can be found just before Corollary 6.2). This right adjoint induces an equivalence

$$\mathsf{D}^b(\mathsf{mod}\,\Lambda) \xrightarrow{\sim} \mathsf{K}^c(\mathsf{Inj}\,\Lambda)$$

which is a quasiinverse for the equivalence $\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \to \mathsf{D}^b(\mathsf{mod}\,\Lambda)$ induced by Q. For details of this construction we refer to Section 6.

3. $\mathsf{K}(\mathsf{Inj}\,kG)$ is a derived category

Given two chain complexes X and Y in $\mathsf{Mod}\,\Lambda$, we define the chain complex $\mathrm{Hom}_\Lambda(X, Y)$. The n-th component is

$$\prod_{p\in\mathbb{Z}} \mathrm{Hom}_\Lambda(X_p, Y_{n+p})$$

and the differential is defined so that

$$(d(f))(x) = d(f(x)) - (-1)^{|f|} f(d(x)).$$

Note that

$$H_n \mathrm{Hom}_\Lambda(X, Y) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(X, Y[n]).$$

Composition of maps gives

$$\mathrm{End}_\Lambda(X) = \mathrm{Hom}_\Lambda(X, X)$$

the structure of a differential graded algebra (DG algebra), over which $\mathrm{Hom}_\Lambda(X, Y)$ is a differential graded module (DG module).

Given a DG algebra Γ, we denote by $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ the derived category of DG Γ-modules. The objects in this category are DG Γ-modules. The arrows are homotopy classes of degree zero morphisms of DG modules, with the quasiisomorphisms (maps that induce an isomorphism on homology) inverted. So for example if Γ is a ring, regarded as a DG algebra concentrated in degree zero with zero differential, then a DG Γ-module is a complex of modules, and we recover the usual definition of the derived category of a ring. See [Keller 1994] for further details.

Proposition 3.1. *Let C be an object of $\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \simeq \mathsf{D}^b(\mathsf{mod}\,\Lambda)$ and let $\Gamma = \mathrm{End}_\Lambda C$. Denote by $\mathbb{C}$ the smallest full triangulated subcategory of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ closed under all coproducts and containing C. Then the functor*

$$\mathrm{Hom}_\Lambda(C, -)\colon\ \mathsf{K}(\mathsf{Inj}\,\Lambda) \longrightarrow \mathsf{D}_{\mathrm{dg}}(\Gamma)$$

induces an equivalence $\mathbb{C} \xrightarrow{\sim} \mathsf{D}_{\mathrm{dg}}(\Gamma)$ of triangulated categories.

Proof. We begin by defining $\mathrm{Hom}_\Lambda(C,-)$ as a functor from $\mathsf{C}(\mathsf{Inj}\,\Lambda)$ to the category of differential graded Γ-modules. This functor is exact, and the composite to $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ takes homotopic maps to the same place. So we obtain a well defined exact functor from $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ to $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ (compare [Keller 1994, §4.3, bottom of p. 77]). To see that it preserves coproducts, fix a family of objects X_i in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$. Then we have for every $n \in \mathbb{Z}$

$$H_n \coprod_i \mathrm{Hom}_\Lambda(C, X_i) \cong \coprod_i H_n \mathrm{Hom}_\Lambda(C, X_i) \cong \coprod_i \mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(C, X_i[n])$$
$$\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(C, \coprod_i X_i[n]) \cong H_n \mathrm{Hom}_\Lambda(C, \coprod_i X_i[n])$$

since C is compact in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$. Thus the canonical map

$$\coprod_i \mathrm{Hom}_\Lambda(C, X_i) \longrightarrow \mathrm{Hom}_\Lambda(C, \coprod_i X_i)$$

is an isomorphism. Furthermore, the functor induces bijections

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(C, C[n]) \cong H_n \mathrm{Hom}_\Lambda(C, C) \cong \mathrm{Hom}_{\mathsf{D}_{\mathrm{dg}}(\Gamma)}(\Gamma, \Gamma[n]).$$

Thus the class D of objects in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ such that the induced map

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{D}_{\mathrm{dg}}(\Gamma)}(\mathrm{Hom}_\Lambda(C, X), \mathrm{Hom}_\Lambda(C, Y))$$

is bijective for all X, Y in D contains $\mathbb{C}$. The functor $\mathrm{Hom}_\Lambda(C,-)$ is, up to isomorphism, surjective on objects since the image contains the free Γ-module Γ which generates $\mathsf{D}_{\mathrm{dg}}(\Gamma)$. □

Remark 3.2. (1) The functor $\mathrm{Hom}_\Lambda(C,-)$ admits left and right adjoints. This is a consequence of Brown representability (see [Neeman 2001]) because the functor preserves (co)products. Thus $\mathrm{Hom}_\Lambda(C,-)$ induces a recollement of the form

$$\mathrm{Ker}\,\mathrm{Hom}_\Lambda(C,-) \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; \mathsf{K}(\mathsf{Inj}\,\Lambda) \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; \mathsf{D}_{\mathrm{dg}}(\Gamma).$$

Here, $\mathrm{Ker}\,\mathrm{Hom}_\Lambda(C,-)$ denotes the full subcategory of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ formed by all objects X with $\mathrm{Hom}_\Lambda(C, X) = 0$. The functors between $\mathrm{Ker}\,\mathrm{Hom}_\Lambda(C,-)$ and $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ are the inclusion together with its left and right adjoints.

(2) If the object C generates $\mathsf{K}^c(\mathsf{Inj}\,\Lambda)$, that is, there is no proper thick subcategory containing C, then $\mathbb{C} = \mathsf{K}(\mathsf{Inj}\,\Lambda)$ and the functor $\mathrm{Hom}_\Lambda(C,-)$ is an equivalence.

In the case where G is a finite p-group, one choice for the compact generator C of Proposition 3.1 is ik, an injective resolution of k. For a more general finite group, we may take the sum of the injective resolutions of the simple modules. We write $\mathscr{E}_G$ for the differential graded algebra $\mathrm{End}_{kG}(ik)$ whether or not G is a p-group.

4. The Rothenberg–Steenrod construction

We now relate the category $\mathsf{K}(\mathsf{Inj}\, kG)$ to the classifying space BG. For general background references on classifying spaces of groups, see for example [Benson 1991; Brown 1982]. The basic link between $\mathsf{K}(\mathsf{Inj}\, kG)$ and the derived category of $C^*(BG;k)$ is achieved through the Rothenberg–Steenrod construction [Rothenberg and Steenrod 1965], which we now make precise. For any path-connected space X, this construction gives a quasiisomorphism of differential graded algebras from the derived endomorphisms of k over the chains on the loop space and the cochains on X:

$$\mathbb{R}\mathrm{End}_{C_*(\Omega X;k)}(k) \simeq C^*(X;k).$$

In the case where X is the classifying space BG, ΩX is equivalent to G, and $C_*(\Omega X;k)$ is equivalent as a differential graded algebra to the group algebra kG in degree zero. So in this case the left hand side is just $\mathscr{E}_G = \mathrm{End}_{kG}(ik)$, and therefore we obtain

$$\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)).$$

The purpose of this section is to investigate this equivalence algebraically.

We begin by remarking that $\mathrm{End}_{kG}(pk)$ and $\mathrm{End}_{kG}(ik)$ are quasiisomorphic differential graded algebras. To see this, choose a quasiisomorphism $pk \to ik$. Then we have quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \to \mathrm{Hom}_{kG}(pk, ik) \leftarrow \mathrm{End}_{kG}(ik).$$

The middle object is not a differential graded algebra, but the pullback of this pair of maps

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{End}_{kG}(ik) \arrow[d] \\ \mathrm{End}_{kG}(pk) \arrow[r] & \mathrm{Hom}_{kG}(pk, ik) \end{tikzcd}$$

is a differential graded algebra

$$X = \mathrm{End}_{kG}(pk) \times_{\mathrm{Hom}_{kG}(pk,ik)} \mathrm{End}_{kG}(ik)$$

that comes with quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \xleftarrow{\simeq} X \xrightarrow{\simeq} \mathrm{End}_{kG}(ik).$$

Thus we obtain equivalences of derived categories

$$\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(X) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik)).$$

Similarly, if $p'k$ is another projective resolution there is a comparison map $pk \to p'k$, and hence there are homomorphisms

$$\mathrm{End}_{kG}(pk) \to \mathrm{Hom}_{kG}(pk,\, p'k) \leftarrow \mathrm{End}_{kG}(p'k).$$

The pullback of this pair of maps is a differential graded algebra

$$Y = \mathrm{End}_{kG}(pk) \times_{\mathrm{Hom}_{kG}(pk,p'k)} \mathrm{End}_{kG}(p'k)$$

that comes with quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \xleftarrow{\simeq} Y \xrightarrow{\simeq} \mathrm{End}_{kG}(p'k).$$

Thus we obtain equivalences of derived categories

$$\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(Y) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(p'k)).$$

It follows that $\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk))$ is, up to natural equivalence, independent of choice of projective resolution, and is also equivalent to $\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik))$.

The augmentation map $\varepsilon\colon pk \to k$ gives a quasiisomorphism of complexes

$$\mathrm{End}_{kG}(pk) \simeq \mathrm{Hom}_{kG}(pk, k).$$

Suppose that pk is a resolution supporting a strictly coassociative and counital diagonal $\Delta\colon pk \to pk \otimes_k pk$, meaning that the following diagrams commute:

$$\begin{tikzcd} pk \arrow[r, "\Delta"] \arrow[d, "\Delta"] & pk \otimes_k pk \arrow[d, "1\otimes\Delta"] \\ pk \otimes_k pk \arrow[r, "\Delta\otimes 1"] & pk \otimes_k pk \otimes_k pk \end{tikzcd} , \qquad \begin{tikzcd} & pk \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ pk \otimes_k k & pk \otimes_k pk \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & k \otimes_k pk \end{tikzcd} .$$

This happens, for example, when pk is the bar resolution, and when pk is equal to the singular cochains on EG. Then there is a multiplication on $\mathrm{Hom}_{kG}(pk, k)$ given as follows. If $\alpha, \beta\colon pk \to k$ then $\alpha.\beta$ is given by the composite

$$pk \xrightarrow{\Delta} pk \otimes_k pk \xrightarrow{\alpha\otimes\beta} k \otimes_k k \xrightarrow{\cong} k.$$

The fact that Δ is coassociative and counital implies that this multiplication is associative and unital.

We claim that there is a quasiisomorphism of differential graded algebras

$$\mathrm{Hom}_{kG}(pk, k) \to \mathrm{End}_{kG}(pk)$$

given by sending $\alpha\colon pk \to k$ to the map $\tilde{\alpha}\colon pk \to pk$ given by the composite $pk \xrightarrow{\Delta} pk \otimes_k pk \xrightarrow{1\otimes\alpha} pk \otimes_k k \xrightarrow{\cong} pk$. Since Δ is counital, we have $\varepsilon \circ \tilde{\alpha} = \alpha$, so that $\alpha \mapsto \tilde{\alpha}$ is a quasiisomorphism. The commutative diagram

$$\begin{tikzcd}
pk \arrow[r, "\tilde{\alpha}\circ\tilde{\beta}"] \arrow[d, equal] & pk \arrow[r, "\varepsilon"] & k \arrow[d, equal] \\
pk \arrow[r, "\tilde{\beta}"] \arrow[d, "\Delta"'] & pk \arrow[u, "\tilde{\alpha}"'] \arrow[r, "\alpha"] & k \\
pk \otimes_k pk \arrow[r, "1\otimes\beta"] \arrow[rr, bend right, "\alpha\otimes\beta"'] & pk \otimes_k k \arrow[u, "\cong"'] \arrow[r, "\alpha\otimes 1"] & k \otimes_k k \arrow[u, "\cong"']
\end{tikzcd}$$

shows that the map $\alpha \mapsto \tilde{\alpha}$ preserves multiplication.

Using this, we see that we have quasiisomorphisms of differential graded algebras

$$\begin{aligned}\mathrm{End}_{kG}(pk) &\simeq \mathrm{Hom}_{kG}(pk, k) \simeq \mathrm{Hom}_{kG}(C_*(EG; k), k)\\ &\cong \mathrm{Hom}_k(C_*(BG; k), k) \cong C^*(BG; k).\end{aligned}$$

Now suppose that H is a subgroup of G. Then EG can be used as a model for EH. In particular, $C_*(EG; k)$ is another model of pk in $\mathsf{K}(\mathsf{Inj}\, kH)$ with a strictly coassociative and counital diagonal map. Restricting resolutions for G to the subgroup H gives us resolutions for H, so we have a restriction map of differential graded algebras $\mathrm{res}_{G,H}\colon \mathscr{E}_G \to \mathscr{E}_H$. We also have a restriction map $\mathrm{res}_{G,H}\colon C^*(BG; k) \to C^*(BH; k)$. Naturality of the Rothenberg–Steenrod construction gives us the following theorem.

Theorem 4.1. *There are equivalences of categories*

$$\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) = \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik)) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

The equivalence $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$ is natural, in the sense that if H is a subgroup of G then the square

$$\begin{tikzcd}
\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}^*_{G,H}"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH; k)) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))
\end{tikzcd}$$

commutes up to natural isomorphism. □

Next, we discuss the tensor product $- \otimes_{C^*(BG;k)} -$ on $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$. It is convenient at this stage to be able to pass back and forth between differential graded algebras and S-algebras (S here is the sphere spectrum). The point of this formalism is to have a category of spectra with a smash product that is commutative and associative up to coherent natural isomorphism, and not just up to homotopy. In the 1990s, several sets of authors produced such categories. We will work with the formalism of S-algebras introduced by Elmendorf, Kříž, Mandell and May [1997].

We make use of [Shipley 2007] to translate between the language of S-algebras and the language of differential graded algebras. Shipley shows that if R is a discrete commutative ring, with associated Eilenberg–Mac Lane spectrum HR, then the model categories of differential graded R-algebras and S-algebras over HR are Quillen equivalent. In particular, their homotopy categories are equivalent as triangulated categories. It would be possible to work directly in the category of E_∞ differential graded algebras, but we would need to be working over an E_∞ operad such as the surjection operad of McClure and Smith [2003] and then transfer to an E_∞ operad satisfying the Hopkins lemma in [Elmendorf et al. 1997]. Alternatively, one could work directly with the formalism of Hovey, Shipley and Smith [Hovey et al. 2000] and use the algebraic analogue of symmetric spectra. Further comments on the relationships between E_∞ algebras and singular cochains on spaces can be found in [Mandell 2001].

In any case, the upshot of the discussion is that if X and Y are objects in $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ then so is the left derived tensor product $X \otimes_{C^*(BG;k)} Y$. This tensor product is symmetric monoidal, so there are coherent natural isomorphisms

$$X \otimes_{C^*(BG;k)} Y \cong Y \otimes_{C^*(BG;k)} X,$$
$$(X \otimes_{C^*(BG;k)} Y) \otimes_{C^*(BG;k)} Z \cong X \otimes_{C^*(BG;k)} (Y \otimes_{C^*(BG;k)} Z).$$

In the case where G is a p-group, we can compare $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ with $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ as in the following theorem. If G is not a p-group, then $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ is not equivalent to the whole of $\mathsf{K}(\mathsf{Inj}\, kG)$, but just the part generated by ik. This is because there is more than one simple kG-module, and $C^*(BG;k)$ only "sees" what is generated by the trivial module; in particular, nonprincipal blocks of kG are invisible to $C^*(BG;k)$.

Theorem 4.2. *Let G be a finite group. Then we have functors*

$$\mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow{\mathrm{Hom}_{kG}(ik,-)} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$$

whose composite we denote by Φ. If G is a finite p-group, this gives an equivalence of categories

$$\Phi\colon \mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow{\simeq} \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)).$$

Proof. This is proved by combining Proposition 3.1 and Theorem 4.1. As remarked above, in the case of a p-group, we can take ik as a generator for $\mathsf{K}^c(\mathsf{Inj}\, kG)$, so that the differential graded algebra Γ of Proposition 3.1 is equal to $\mathscr{E}_G$. □

Remark 4.3. An explicit right adjoint $\Psi\colon \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)) \to \mathsf{K}(\mathsf{Inj}\, kG)$ to Φ is described just before Lemma 7.4; see also Remark 3.2. The functor Ψ satisfies $\Phi\circ\Psi \simeq \mathrm{Id}_{\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))}$.

5. $\mathsf{K}(\mathsf{Inj}\, kG)$ is a tensor category

If G_1 and G_2 are groups then there is a natural isomorphism of group algebras $k(G_1\times G_2)\cong kG_1\otimes_k kG_2$. Taking the tensor product of complexes gives an *external tensor product*

$$\mathsf{C}(\mathsf{Mod}\, kG_1)\times \mathsf{C}(\mathsf{Mod}\, kG_2)\to \mathsf{C}(\mathsf{Mod}\, k(G_1\times G_2))$$

and hence also

$$\mathsf{K}(\mathsf{Mod}\, kG_1)\times \mathsf{K}(\mathsf{Mod}\, kG_2)\to \mathsf{K}(\mathsf{Mod}\, k(G_1\times G_2)).$$

If $G=G_1=G_2$, then restricting the external tensor product via the diagonal embedding of G in $G\times G$ defines an *internal tensor product*

$$\mathsf{C}(\mathsf{Mod}\, kG)\times \mathsf{C}(\mathsf{Mod}\, kG)\to \mathsf{C}(\mathsf{Mod}\, kG)$$

which induces

$$\mathsf{K}(\mathsf{Mod}\, kG)\times \mathsf{K}(\mathsf{Mod}\, kG)\to \mathsf{K}(\mathsf{Mod}\, kG).$$

Similar arguments show that $\mathrm{Hom}_k(-,-)$ induces internal products on the categories $\mathsf{C}(\mathsf{Mod}\, kG)$ and $\mathsf{K}(\mathsf{Mod}\, kG)$. Note that we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(X\otimes_k Y, Z)\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(X, \mathrm{Hom}_k(Y,Z)) \tag{5.1}$$

for all X, Y, Z in $\mathsf{K}(\mathsf{Mod}\, kG)$.

The subcategories $\mathsf{K}(\mathsf{Inj}\, kG)$ and $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG)$ inherit tensor products from the category $\mathsf{K}(\mathsf{Mod}\, kG)$ because they are tensor ideals. This follows from the next lemma.

Lemma 5.2. *Let X, Y be complexes of kG-modules.*

(i) *If X is a complex of injective kG-modules, then $X\otimes_k Y$ and* $\mathrm{Hom}_k(X,Y)$ *are complexes of injective kG-modules.*

(ii) *If X is an acyclic complex, then $X\otimes_k Y$ and* $\mathrm{Hom}_k(X,Y)$ *are acyclic complexes.*

Proof. The first assertion is clear since $M\otimes_k N$ and $\mathrm{Hom}_k(M,N)$ are injective for any pair of kG-modules M, N provided that one of them is injective. The second assertion follows from the fact that the tensor product and Hom are computed over k. $\square$

Proposition 5.3. *The unit for the tensor product on* $\mathsf{K}(\mathsf{Inj}\, kG)$ *is the injective resolution* ik *of the trivial representation* k.

Proof. For any object X in $\mathsf{K}(\mathsf{Inj}\, kG)$, the map of complexes $k \to ik$ induces the following chain of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik \otimes_k X, -) &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik, \mathrm{Hom}_k(X, -)) \\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k, \mathrm{Hom}_k(X, -)) \\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k \otimes_k X, -). \end{aligned}$$

Here we use (5.1) and that $k \to ik$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik, Y) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k, Y)$$

for all Y in $\mathsf{K}(\mathsf{Inj}\, kG)$ by [Krause 2005, Lemma 2.1]. Thus the map of complexes $k \otimes_k X \to ik \otimes_k X$ is an isomorphism in $\mathsf{K}(\mathsf{Inj}\, kG)$. ☐

Definition 5.4. If X is an object in $\mathsf{K}(\mathsf{Inj}\, kG)$, we define

$$H^*(G, X) = \mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, X)$$

where the n-th component is $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, X[n])$. This is a graded module for the cohomology ring $H^*(G, k) = \mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, ik)$.

6. A recollement for $\mathsf{K}(\mathsf{Inj}\, kG)$

Let Λ be a Noetherian ring. We have seen that $\mathsf{K}(\mathsf{Inj}\, \Lambda)$ is compactly generated and this fact has some interesting consequences. For instance, any exact functor $\mathsf{K}(\mathsf{Inj}\, \Lambda) \to \mathsf{T}$ into a triangulated category T admits a right adjoint if it preserves coproducts and a left adjoint if it preserves products. We apply this consequence of Brown representability (see [Neeman 2001]) to the canonical functor

$$Q \colon \mathsf{K}(\mathsf{Inj}\, \Lambda) \xrightarrow{\text{inc}} \mathsf{K}(\mathsf{Mod}\, \Lambda) \xrightarrow{\text{can}} \mathsf{D}(\mathsf{Mod}\, \Lambda)$$

and obtain the following result [Krause 2005, Corollary 4.3].

Proposition 6.1. *The pair of canonical functors*

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, \Lambda) \xrightarrow{I} \mathsf{K}(\mathsf{Inj}\, \Lambda) \xrightarrow{Q} \mathsf{D}(\mathsf{Mod}\, \Lambda)$$

induces a recollement

$$\begin{tikzcd} \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, \Lambda) \arrow[r] & \mathsf{K}(\mathsf{Inj}\, \Lambda) \arrow[l, shift right=2ex, "I_\rho"'] \arrow[l, shift left=2ex, "I_\lambda"] \arrow[r] & \mathsf{D}(\mathsf{Mod}\, \Lambda). \arrow[l, shift right=2ex, "Q_\rho"'] \arrow[l, shift left=2ex, "Q_\lambda"] \end{tikzcd}$$

More precisely, the functors I and Q admit left adjoints I_λ and Q_λ as well as right adjoints I_ρ and Q_ρ such that the following adjunction morphisms

$$I_\lambda \circ I \xrightarrow{\sim} \mathrm{Id}_{\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,\Lambda)} \xrightarrow{\sim} I_\rho \circ I \quad \text{and} \quad Q \circ Q_\rho \xrightarrow{\sim} \mathrm{Id}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)} \xrightarrow{\sim} Q \circ Q_\lambda$$

are isomorphisms.

Recall from [Avramov et al. 2003] (see also [Spaltenstein 1988]) that for any differential graded algebra Γ, a DG Γ-module X is said to be *semiprojective* if $\mathrm{Hom}_\Gamma(X,-)$ carries surjective quasiisomorphisms to surjective quasiisomorphisms. Similarly, X is *semiinjective* if $\mathrm{Hom}_\Gamma(-,X)$ carries injective quasiisomorphisms to surjective quasiisomorphisms. A *semiprojective resolution* of a DG Γ-module X is a quasiisomorphism $pX \to X$ with pX semiprojective, and a *semiinjective resolution* of X is a quasiisomorphism $X \to iX$ with iX semiinjective. If Γ is a ring, these definitions are applied by regarding Γ as a DG algebra concentrated in degree zero, so that a DG module is just a complex of Γ-modules.

Note that the recollement provides two embeddings of $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ into $\mathsf{K}(\mathsf{Inj}\,\Lambda)$. The more familiar one is the fully faithful functor $Q_\rho\colon \mathsf{D}(\mathsf{Mod}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$ which sends a complex X of Λ-modules to a semiinjective resolution iX. The less familiar embedding is the fully faithful functor $Q_\lambda\colon \mathsf{D}(\mathsf{Mod}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$ which identifies $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ with the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ generated by $i\Lambda$. If Λ is self-injective, then Q_λ sends a complex X of Λ-modules to a semiprojective resolution pX.

We summarize this discussion as follows.

Corollary 6.2. *Let Λ be a Noetherian ring, and let X be a complex of injective Λ-modules. Then the following are equivalent.* (*i*) *X is semiinjective.* (*ii*) *$X \cong Q_\rho Y$ for some Y in* $\mathsf{D}(\mathsf{Mod}\,\Lambda)$. (*iii*) *$I_\rho X \cong 0$.*

If Λ is selfinjective (*so that projective and injective Λ-modules coincide*), *then the following are equivalent.* (*i*) *X is semiprojective.* (*ii*) *$X \cong Q_\lambda Y$ for some Y in* $\mathsf{D}(\mathsf{Mod}\,\Lambda)$. (*iii*) *$I_\lambda X \cong 0$.*

In the case where $\Lambda = kG$, we have $\mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG)$, and the adjoints in the recollement take the form

$$\begin{tikzcd} \mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG) \arrow[r] & \mathsf{K}(\mathsf{Inj}\,kG) \arrow[l, bend right, "\mathrm{Hom}_k(tk,-)"'] \arrow[l, bend left, "-\otimes_k tk"] \arrow[r] & \mathsf{D}(\mathsf{Mod}\,kG). \arrow[l, bend right, "\mathrm{Hom}_k(pk,-)"'] \arrow[l, bend left, "-\otimes_k pk"] \end{tikzcd} \tag{6.3}$$

Here, we write ik for a semiinjective resolution, pk for a semiprojective resolution, and tk for a Tate resolution of the trivial kG-module k. Note that these resolutions fit into an exact triangle

$$pk \longrightarrow ik \longrightarrow tk \longrightarrow pk[1]$$

in $\mathsf{K}(\mathsf{Inj}\, kG)$. This triangle induces for each object X of $\mathsf{K}(\mathsf{Inj}\, kG)$ the following exact triangles:

$$X \otimes_k pk \longrightarrow X \otimes_k ik \longrightarrow X \otimes_k tk \longrightarrow X \otimes_k pk[1],$$
$$\mathrm{Hom}_k(tk, X) \longrightarrow \mathrm{Hom}_k(ik, X) \longrightarrow \mathrm{Hom}_k(pk, X) \longrightarrow \mathrm{Hom}_k(tk[-1], X).$$

The first two maps in each triangle are the obvious adjunction morphisms which are induced by the recollement. This becomes clear once we observe that the canonical map $k \to ik$ induces isomorphisms

$$X = X \otimes_k k \xrightarrow{\sim} X \otimes_k ik \quad \text{and} \quad \mathrm{Hom}_k(ik, X) \xrightarrow{\sim} \mathrm{Hom}_k(k, X) = X$$

(see Proposition 5.3). Thus $\mathsf{K}(\mathsf{Inj}\, kG)$ is a sort of intermediary between $\mathsf{StMod}\, kG$ and $\mathsf{D}(\mathsf{Mod}\, kG)$, and in some ways is better behaved than either of them. The problem with $\mathsf{StMod}\, kG$ is that the graded endomorphisms of the trivial module form a usually non-Noetherian ring (the Tate cohomology ring). The problem with $\mathsf{D}(\mathsf{Mod}\, kG)$, on the other hand, is that k is usually not a compact object.

The compact objects in the three categories in the recollement give the perhaps more familiar sequence of categories and functors

$$\mathsf{stmod}\, kG \longleftarrow \mathsf{D}^b(\mathsf{mod}\ kG) \longleftarrow \mathsf{D}^b(\mathsf{proj}\, kG).$$

Note that only the left adjoints in the recollement preserve compact objects.

7. The dictionary between $\mathsf{K}(\mathsf{Inj}\, kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$

Let G be a finite group. Then by Theorem 4.1 we have functors

$$\mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow{\mathrm{Hom}_{kG}(ik,-)} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$$

(where $\mathscr{E}_G = \mathrm{End}_{kG}(ik)$), which in the case of a p-group give an equivalence of triangulated categories

$$\Phi\colon \mathsf{K}(\mathsf{Inj}\, kG) \to \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

In this section, we investigate the functor Φ further, and we develop a dictionary for translating between $\mathsf{K}(\mathsf{Inj}\, kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$.

First we deal with external tensor products. Now if R_1 and R_2 are commutative S-algebras over k, then $R_1 \otimes_k R_2$ is also a commutative S-algebra over k by VII.1.6 of [Elmendorf et al. 1997]. If X and Y are spaces then the Eilenberg–Zilber map gives an equivalence between $C^*(X; k) \otimes_k C^*(Y; k)$ and $C^*(X \times Y; k)$ as S-algebras over k. If $\delta\colon X \to X \times X$ is the diagonal map, then the composite

$$C^*(X; k) \otimes_k C^*(X; k) \simeq C^*(X \times X; k) \xrightarrow{\delta^*} C^*(X; k)$$

is the multiplication map, and is a map of commutative S-algebras over k.

In particular, if $X = BG_1$ and $Y = BG_2$ then $X \times Y = B(G_1 \times G_2)$, and we get the equivalence of $C^*(BG_1;k) \otimes_k C^*(BG_2;k)$ with $C^*(B(G_1 \times G_2);k)$. This means that if X and Y are modules over $C^*(BG_1;k)$ and $C^*(BG_2;k)$ respectively, we have an *external tensor product* $X \otimes_k Y$ as a module over $C^*(B(G_1 \times G_2);k)$.

If $\Delta\colon G \to G \times G$ is the diagonal map, then the composite

$$\begin{aligned} &C^*(BG;k) \otimes_k C^*(BG;k) \simeq C^*(BG \times BG;k) \\ &\qquad\qquad = C^*(B(G \times G);k) \xrightarrow{B\Delta^*} C^*(BG;k) \end{aligned}$$

is the multiplication map on $C^*(BG;k)$.

Theorem 7.1. *The functor Φ takes the external tensor product over k discussed in Section 5 to the external tensor product described above.*

Proof. If ik_{G_1} and ik_{G_2} are injective resolutions of k for G_1 and G_2, then the external tensor product $ik_{G_1} \otimes_k ik_{G_2}$ is an injective resolution of k for $G_1 \times G_2$. We have a commutative diagram

$$\begin{tikzcd} \mathsf{K}(\mathsf{Inj}\, kG_1) \times \mathsf{K}(\mathsf{Inj}\, kG_2) \arrow[r, "\mathrm{Hom}_{kG_1}(ik_{G_1},-)\times\mathrm{Hom}_{kG_2}(ik_{G_2},-)"] \arrow[d, "\otimes_k"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1}) \times \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_2}) \arrow[d, "\otimes_k"] \\ \mathsf{K}(\mathsf{Inj}\, k(G_1 \times G_2)) \arrow[r, "\mathrm{Hom}_{k(G_1\times G_2)}(ik_{G_1}\otimes_k ik_{G_2},-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1} \otimes_k \mathscr{E}_{G_2}) \end{tikzcd}$$

We combine this with the commutative diagram

$$\begin{tikzcd} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1}) \times \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_2}) \arrow[r, "\simeq"] \arrow[d, "\otimes_k"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG_1,k) \times \mathsf{D}_{\mathrm{dg}}(C^*(BG_2;k)) \arrow[d, "\otimes_k"] \\ \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1} \otimes_k \mathscr{E}_{G_2}) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG_1;k) \otimes_k C^*(BG_2;k)) \end{tikzcd}$$

and the equivalence

$$\mathsf{D}_{\mathrm{dg}}(C^*(BG_1;k) \otimes_k C^*(BG_2;k)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(B(G_1 \times G_2);k))$$

to prove the theorem. $\square$

Next we deal with subgroups.

Lemma 7.2. *If H is a subgroup of G, the following diagram commutes up to natural isomorphism:*

$$
\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\,kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] \arrow[d, "\mathrm{ind}_{H,G}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{K}(\mathsf{Inj}\,kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)
\end{tikzcd}
$$

Proof. This follows from the Frobenius reciprocity (or Eckmann–Shapiro) isomorphism

$$\mathrm{Hom}_{kG}(ik, \mathrm{ind}_{H,G}(X)) \cong \mathrm{Hom}_{kH}(ik, X). \qquad \square$$

Theorem 7.3. *The functor Φ takes induction from kH-modules to kG-modules to restriction from $C^*(BH; k)$-modules to $C^*(BG; k)$-modules.*

Proof. By Theorem 4.1 and Lemma 7.2, the following diagram commutes up to natural isomorphisms:

$$
\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\,kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] \arrow[d, "\mathrm{ind}_{H,G}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}^*_{G,H}"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH; k)) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{K}(\mathsf{Inj}\,kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).
\end{tikzcd}
\qquad \square
$$

The corresponding statement for restriction from $\mathsf{K}(\mathsf{Inj}\,kG)$ to $\mathsf{K}(\mathsf{Inj}\,kH)$ requires more preparation. We begin by defining a functor

$$- \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik \colon \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \to \mathsf{K}(\mathsf{Inj}\,kG)$$

as the left adjoint of $\mathrm{Hom}_{kG}(ik, -)$. The existence of such a left adjoint follows from Brown's representability theorem (see [Neeman 2001]) since $\mathrm{Hom}_{kG}(ik, -)$ preserves products. Alternatively, we construct this functor explicitly by tensoring over $\mathscr{E}_G$ a semiprojective resolution (for the definition, see Section 6) of the given differential graded $\mathscr{E}_G$-module with ik. It is clear from the construction that

$$- \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$$

identifies $\mathscr{E}_G$ with ik.

Lemma 7.4. *Let X be an object in $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$. Then the natural map*

$$X \to \mathrm{Hom}_{kG}(ik, X \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik)$$

is an isomorphism in $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$.

Proof. This is obviously true for $X = \mathscr{E}_G$. The functor on the right preserves triangles and direct sums in the variable X because ik is compact. So the assertion is true for any object in the localizing subcategory generated by $\mathscr{E}_G$, which is all of $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$. $\square$

Remark 7.5. The functor $-\overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$ identifies $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$ with the localizing subcategory $\mathrm{Loc}(ik)$ of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by ik. In particular, for each object Y in $\mathsf{K}(\mathsf{Inj}\, kG)$, the natural map

$$\eta_Y\colon\ \mathrm{Hom}_{kG}(ik, Y) \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik \to Y$$

is the best left approximation of Y by objects in $\mathrm{Loc}(ik)$. More precisely, the object

$$\mathrm{Hom}_{kG}(ik, Y) \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$$

belongs to $\mathrm{Loc}(ik)$ and the induced map $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(X, \eta_Y)$ is bijective for all X in $\mathrm{Loc}(ik)$.

Lemma 7.6. *Suppose we have given a diagram of functors*

$$\begin{tikzcd} \mathsf{S} \arrow[r, "H"] \arrow[d, "F"] & \mathsf{T} \arrow[d, "G"] \\ \mathsf{S}' \arrow[r, "H'"] & \mathsf{T}' \end{tikzcd}$$

which is commutative up to isomorphism such that F, G, H' admit right adjoints F_ρ, G_ρ, H'_ρ, and H admits a left adjoint H_λ. Suppose in addition that

$$\mathrm{Id}_{\mathsf{T}} \cong H \circ H_\lambda \quad \textit{and} \quad H'_\rho \circ H' \cong \mathrm{Id}_{\mathsf{S}'}.$$

Then the diagram of functors:

$$\begin{tikzcd} \mathsf{S}' \arrow[r, "H'"] \arrow[d, "F_\rho"] & \mathsf{T}' \arrow[d, "G_\rho"] \\ \mathsf{S} \arrow[r, "H"] & \mathsf{T} \end{tikzcd}$$

commutes up to isomorphism.

Proof. We have

$$G \cong G \circ H \circ H_\lambda \cong H' \circ F \circ H_\lambda.$$

Taking right adjoints, we obtain

$$G_\rho \cong H \circ F_\rho \circ H'_\rho$$

and this implies

$$G_\rho \circ H' \cong H \circ F_\rho \circ H'_\rho \circ H' \cong H \circ F_\rho. \qquad \square$$

Theorem 7.7. *Let G be a finite p-group and let H be a subgroup of G. Then the functor Φ takes restriction from kG-modules to kH-modules to coinduction from $C^*(BG;k)$-modules to $C^*(BH;k)$-modules.*

Proof. We claim that the diagram

$$\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\,kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] \arrow[d, "\mathrm{res}_{G,H}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] \arrow[d, "\mathbb{R}\mathrm{Hom}_{\mathscr{E}_G}(\mathscr{E}_H,-)"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)) \arrow[d, "\mathbb{R}\mathrm{Hom}_{C^*(BG;k)}(C^*(BH;k),-)"] \\
\mathsf{K}(\mathsf{Inj}\,kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH;k))
\end{tikzcd}$$

commutes. For the right-hand square this is clear. For the left hand square, this follows from Lemma 7.2, 7.4 and 7.6. The assumption on G to be a p-group is needed for $\mathrm{Hom}_{kG}(ik,-)$ to be an equivalence. □

Theorem 7.8. *Let G be a finite p-group. Then the functor Φ takes the internal tensor product with diagonal G-action to the E_∞ tensor product discussed at the end of Section 4.*

Proof. The internal tensor product in $\mathsf{K}(\mathsf{Inj}\,kG)$ is given by external tensor product to $\mathsf{K}(\mathsf{Inj}\,k(G\times G))$ followed by restriction to the diagonal copy of G. Using Theorems 7.1 and 7.7, we see that

$$\begin{aligned}
\mathrm{Hom}_{kG}(ik,(X\otimes_k Y){\downarrow}_G^{G\times G}) &\cong \mathbb{R}\mathrm{Hom}_{\mathscr{E}_{G\times G}}(\mathscr{E}_G, \mathrm{Hom}_{k(G\times G)}(ik, X\otimes_k Y)) \\
&\cong \mathbb{R}\mathrm{Hom}_{\mathscr{E}_G\otimes_k\mathscr{E}_G}(\mathscr{E}_G, \mathrm{Hom}_{kG}(ik,X)\otimes_k \mathrm{Hom}_{kG}(ik,Y)).
\end{aligned}$$

Applying the equivalence with $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ to the latter, we obtain

$$\mathbb{R}\mathrm{Hom}_{C^*(BG;k)\otimes_k C^*(BG;k)}(C^*(BG;k), \Phi(X)\otimes_k\Phi(Y))$$

which is isomorphic to

$$\Phi(X)\otimes_{C^*(BG;k)}\Phi(Y)$$

with the E_∞ tensor product. □

Theorems 7.3 and 7.7 above can be thought of as saying that the roles of restriction and (co)induction are reversed by the equivalence. So it makes sense that the roles of the trivial representation and the regular representation should also be reversed.

It is easy to see that ik in $\mathsf{K}(\mathsf{Inj}\,kG)$ corresponds to the regular representation of $C^*(BG;k)$, and that the regular representation kG corresponds to the trivial representation k of $C^*(BG;k)$.

We summarize all this information in Table 1.

$\mathsf{K}(\mathsf{Inj}\, kG)$	$\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$
external $-\otimes_k -$	external $-\otimes_k -$
internal $-\otimes_k -$ diagonal G-action	$-\otimes_{C^*(BG;k)} -$ E_∞ tensor product
Induction from H to G	Restriction via $C^*(BG;k) \to C^*(BH;k)$
Restriction from G to H	Coinduction $\mathrm{Hom}_{C^*(BG;k)}(C^*(BH;k), -)$
ik	$C^*(BG;k)$
kG	k

Table 1

8. $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is a derived invariant

The classical Morita theory for derived categories [Rickard 1989; 1991] can be extended to complexes of injective modules as follows.

Proposition 8.1. *Let Λ and Γ be Noetherian algebras over a commutative ring k. Suppose Λ and Γ are projective as k-modules. Then the following are equivalent.*

(i) *Λ and Γ are derived equivalent, that is, there exists a tilting complex T over Λ such that the endomorphism ring $\mathrm{End}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(T)$ is isomorphic to Γ.*

(ii) *There exists an exact equivalence $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Gamma)$.*

(iii) *There exists an exact equivalence $\mathsf{D}^b(\mathsf{mod}\,\Lambda) \to \mathsf{D}^b(\mathsf{mod}\,\Gamma)$.*

Proof. (i) $\Rightarrow$ (ii): In [Rickard 1991], it is shown that Λ and Γ admit a standard derived equivalence. Thus there is a bounded complex P of Γ-Λ-bimodules which in each degree is finitely generated projective over Λ and over Γ. The functor $\mathrm{Hom}_\Lambda(P, -)$ sends complexes of injective Λ-modules to complexes of injective Γ-modules and semiinjective complexes to semiinjective complexes. The last assertion follows from the isomorphism

$$\mathrm{Hom}_\Gamma(A, \mathrm{Hom}_\Lambda(P, X)) \cong \mathrm{Hom}_\Lambda(A \otimes_\Gamma P, X).$$

Thus $\mathrm{Hom}_\Lambda(P, -)$ induces the commutative diagram of exact functors

$$\begin{tikzcd}
\mathsf{D}(\mathsf{Mod}\,\Lambda) \arrow[r, "\mathrm{Hom}_\Lambda(P,-)"] \arrow[d, "(Q_\Lambda)_\rho"] & \mathsf{D}(\mathsf{Mod}\,\Gamma) \arrow[d, "(Q_\Gamma)_\rho"] \\
\mathsf{K}(\mathsf{Inj}\,\Lambda) \arrow[r, "\mathrm{Hom}_\Lambda(P,-)"] & \mathsf{K}(\mathsf{Inj}\,\Gamma)
\end{tikzcd}$$

because we know from Corollary 6.2 that the right adjoint functors $(Q_\Lambda)_\rho$ and $(Q_\Gamma)_\rho$ identify the derived categories with the full subcategories formed by all semiinjective complexes. By our assumption, the functor $\mathsf{D}(\mathsf{Mod}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Gamma)$ is an equivalence inducing an equivalence $\mathsf{D}^b(\mathsf{mod}\,\Lambda) \to \mathsf{D}^b(\mathsf{mod}\,\Gamma)$. Now we apply Proposition 2.1 as follows. The commutativity of the diagram implies that $\mathrm{Hom}_\Lambda(P, -)$ induces an equivalence $\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \to \mathsf{K}^c(\mathsf{Inj}\,\Gamma)$. Then a standard dévissage argument shows that $\mathrm{Hom}_\Lambda(P, -)$ induces an equivalence $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Gamma)$ since $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is compactly generated and the functor preserves all coproducts.

(ii) $\Rightarrow$ (iii): An exact equivalence $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Gamma)$ induces an exact equivalence $\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \to \mathsf{K}^c(\mathsf{Inj}\,\Gamma)$ and therefore, again by Proposition 2.1, an exact equivalence $\mathsf{D}^b(\mathsf{mod}\,\Lambda) \to \mathsf{D}^b(\mathsf{mod}\,\Gamma)$.

(iii) $\Rightarrow$ (i): Let $F\colon \mathsf{D}^b(\mathsf{mod}\,\Gamma) \to \mathsf{D}^b(\mathsf{mod}\,\Lambda)$ be an exact equivalence. Then $T = F\Gamma$ is a tilting complex with $\mathrm{End}_{\mathsf{D}^b(\mathsf{mod}\,\Lambda)}(T) \cong \Gamma$. $\square$

9. Bousfield localization

We recall briefly some basic facts about Bousfield localization. Let T be triangulated with arbitrary coproducts. We fix a full triangulated subcategory S of T which is *localizing* in the sense that S is closed under taking all coproducts. Then we have a sequence

$$\mathsf{S} \xrightarrow{I} \mathsf{T} \xrightarrow{Q} \mathsf{T}/\mathsf{S}$$

of canonical functors and observe that I has a right adjoint I_ρ if and only if Q has a right adjoint Q_ρ. In this case we call the sequence a *localization sequence*. Following [Rickard 1997], we write $E_\mathsf{S} = I \circ I_\rho$ and $F_\mathsf{S} = Q_\rho \circ Q$. Note that E_S and F_S are idempotent functors.

Let us collect the basic facts of such a localization sequence.

Lemma 9.1. *A localization sequence* $\mathsf{S} \xrightarrow{I} \mathsf{T} \xrightarrow{Q} \mathsf{T}/\mathsf{S}$ *has the following properties.*

(i) *The functor* Q_ρ *is fully faithful and identifies* T/S *with the full subcategory*

$$\mathsf{S}^\perp = \{Y \in \mathsf{T} \mid \mathrm{Hom}_\mathsf{T}(X, Y) = 0 \textit{ for all } X \in \mathsf{S}\}.$$

(ii) *We have*

$$\mathsf{S} = \{X \in \mathsf{T} \mid \mathrm{Hom}_\mathsf{T}(X, Y) = 0 \textit{ for all } Y \in \mathsf{S}^\perp\}.$$

(iii) *For each object* X *of* T, *there exists up to isomorphism a unique exact triangle*

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

with $X' \in \mathsf{S}$ *and* $X'' \in \mathsf{S}^\perp$.

(iv) *For each object X of T, the adjunction morphisms $E_{\mathsf{S}}X \to X$ and $X \to F_{\mathsf{S}}X$ fit into an exact triangle*

$$E_{\mathsf{S}}X \longrightarrow X \longrightarrow F_{\mathsf{S}}X \longrightarrow E_{\mathsf{S}}X[1].$$

There is a finite variant of Bousfield localization for compactly generated triangulated categories which Rickard [1997] introduced into representation theory. Here we use the tensor product $\otimes_k$ which is defined on $\mathsf{K}(\mathsf{Inj}\, kG)$.

Let S_0 be a class of compact objects of $\mathsf{K}(\mathsf{Inj}\, kG)$ and denote by $\mathsf{S} = \mathrm{Loc}(\mathsf{S}_0)$ the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ which is generated by S_0. Then the sequence

$$\mathsf{S} \xrightarrow{I} \mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow{Q} \mathsf{K}(\mathsf{Inj}\, kG)/\mathsf{S}$$

of canonical functors is a localization sequence. Moreover, S is compactly generated and the subcategory S^c of compact objects equals the thick subcategory $\mathrm{Thick}(\mathsf{S}_0)$ of $\mathsf{K}^c(\mathsf{Inj}\, kG)$ which is generated by S_0.

Now suppose that S_0 is a *thick tensor ideal* of $\mathsf{K}^c(\mathsf{Inj}\, kG)$. Thus S_0 is by definition a thick subcategory and a *tensor ideal*, that is, $X \otimes_k Y$ belongs to S_0 for all X in S_0 and Y in $\mathsf{K}^c(\mathsf{Inj}\, kG)$. Then $\mathsf{S} = \mathrm{Loc}(\mathsf{S}_0)$ is a localizing tensor ideal and therefore the exact triangle

$$E_{\mathsf{S}}ik \longrightarrow ik \longrightarrow F_{\mathsf{S}}ik \longrightarrow E_{\mathsf{S}}ik[1]$$

induces for each X in $\mathsf{K}(\mathsf{Inj}\, kG)$ an exact triangle

$$X \otimes_k E_{\mathsf{S}}ik \longrightarrow X \otimes_k ik \longrightarrow X \otimes_k F_{\mathsf{S}}ik \longrightarrow X \otimes_k E_{\mathsf{S}}ik[1]$$

which is isomorphic to

$$E_{\mathsf{S}}X \longrightarrow X \longrightarrow F_{\mathsf{S}}X \longrightarrow E_{\mathsf{S}}X[1].$$

10. Varieties

In this section, we indicate how the theory of support for kG-modules from [Benson et al. 1996] may be modified to work in $\mathsf{K}(\mathsf{Inj}\, kG)$.

Let $H^*(G,k)$ be the cohomology ring of G, and denote by $\mathrm{Spec}^* H^*(G,k)$ the set of homogeneous prime ideals of $H^*(G,k)$. We consider the Zariski topology on $\mathrm{Spec}^* H^*(G,k)$, that is, a subset of $\mathrm{Spec}^* H^*(G,k)$ is *Zariski closed* if it is of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}^* H^*(G,k) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some homogeneous ideal $\mathfrak{a}$ of $H^*(G,k)$. We write $\mathfrak{m} = H^+(G,k)$ for the unique maximal ideal of $H^*(G,k)$ and obtain the *projective variety*

$$\mathrm{Proj}\, H^*(G,k) = \mathrm{Spec}^* H^*(G,k) \setminus V(H^+(G,k)) = \mathrm{Spec}^* H^*(G,k) \setminus \{\mathfrak{m}\}.$$

Now fix a specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec}^* H^*(G,k)$, that is, $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \mathcal{V}$ imply $\mathfrak{q} \in \mathcal{V}$. We obtain the localizing tensor ideal

$$\mathsf{S}_{\mathcal{V}} = \operatorname{Loc}\big(\{X \in \mathsf{K}^c(\operatorname{Inj} kG) \mid H^*(G,X)_{\mathfrak{q}} = 0 \text{ for all } \mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \setminus \mathcal{V}\}\big),$$

of $\mathsf{K}(\operatorname{Inj} kG)$. To simplify our notation, we write

$$E_{\mathcal{V}} = E_{\mathsf{S}_{\mathcal{V}}} \quad \text{and} \quad F_{\mathcal{V}} = F_{\mathsf{S}_{\mathcal{V}}}.$$

Now fix $\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k)$ and let

$$\mathcal{V}_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \mid \mathfrak{p} \subseteq \mathfrak{q}\} \quad \text{and} \quad \mathcal{W}_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

Note that $\mathcal{W}_{\mathfrak{p}} \setminus \mathcal{V}_{\mathfrak{p}} = \{\mathfrak{p}\}$. We define

$$\kappa_{\mathfrak{p}} = (F_{\mathcal{W}_{\mathfrak{p}}} \circ E_{\mathcal{V}_{\mathfrak{p}}})ik \cong (E_{\mathcal{V}_{\mathfrak{p}}} \circ F_{\mathcal{W}_{\mathfrak{p}}})ik.$$

For example, one computes

$$\kappa_{\mathfrak{m}} = (E_{\mathcal{V}_{\mathfrak{m}}} \circ F_{\mathcal{W}_{\mathfrak{m}}})ik = E_{\mathcal{V}_{\mathfrak{m}}} ik = pk.$$

Given X in $\mathsf{K}(\operatorname{Inj} kG)$, we have

$$X \otimes_k \kappa_{\mathfrak{p}} \cong (F_{\mathcal{W}_{\mathfrak{p}}} \circ E_{\mathcal{V}_{\mathfrak{p}}})X \cong (E_{\mathcal{V}_{\mathfrak{p}}} \circ F_{\mathcal{W}_{\mathfrak{p}}})X$$

and the *variety* of X is by definition

$$\mathcal{V}_G(X) = \{\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k) \mid X \otimes_k \kappa_{\mathfrak{p}} \neq 0\}.$$

Lemma 10.1. *Let* $\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k)$. *Then* $\mathcal{V}_G(\kappa_{\mathfrak{p}}) = \{\mathfrak{p}\}$.

Proof. The proof is essentially the same as the proof of Lemma 10.4 of [Benson et al. 1996]. □

Lemma 10.2. *A complex X in* $\mathsf{K}(\operatorname{Inj} kG)$ *is acyclic if and only if* $\mathcal{V}_G(X)$ *is contained in* $\operatorname{Proj} H^*(G,k)$.

Proof. A complex X is acyclic if and only if $X \otimes_k pk = 0$ if and only if

$$\mathcal{V}_G(X) \subseteq \operatorname{Proj} H^*(G,k). \qquad \square$$

It follows that $\kappa_{\mathfrak{p}}$ is in $\mathsf{K}_{\mathrm{ac}}(\operatorname{Inj} kG) \cong \operatorname{StMod} kG$ unless $\mathfrak{p} = \mathfrak{m}$, and that these modules agree with the modules κ_V introduced in [Benson et al. 1996].

11. Objects with injective cohomology

Modules over kG with injective cohomology were studied in [Benson and Krause 2002]. In this section, we indicate how this works in $\mathsf{K}(\operatorname{Inj} kG)$. The theory is actually easier than in $\operatorname{StMod} kG$, because it does not involve a discussion of injective modules over the non-Noetherian Tate cohomology ring.

Let I be an injective $H^*(G,k)$-module. Then the functor from $\mathsf{K}(\mathsf{Inj}\, kG)$ to the category of abelian groups which takes an object X to

$$\mathrm{Hom}_{H^*(G,k)}(H^*(G,X), I)$$

takes triangles to exact sequences and coproducts to products. So by Brown representability (see [Neeman 2001]) there is an object $T(I)$ in $\mathsf{K}(\mathsf{Inj}\, kG)$ satisfying

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(X, T(I)) \cong \mathrm{Hom}_{H^*(G,k)}(H^*(G,X), I).$$

The assignment $I \mapsto T(I)$ extends via Yoneda's lemma to a functor

$$T\colon\ \mathsf{Inj}\, H^*(G,k) \to \mathsf{K}(\mathsf{Inj}\, kG).$$

A dimension shifting argument (see [Benson and Krause 2002, §3]) shows that we obtain an isomorphism of graded $H^*(G,k)$-modules

$$\mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(X, T(I)) \cong \mathrm{Hom}^*_{H^*(G,k)}(H^*(G,X), I).$$

In particular, setting $X = ik$, we see that $H^*(G, T(I)) \cong I$ for all I in $\mathsf{Inj}\, H^*(G,k)$, and setting $X = T(I')$ we see that

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(T(I'), T(I)) \cong \mathrm{Hom}_{H^*(G,k)}(I', I),$$

so that the functor T is fully faithful. Thus, if $\mathfrak{p} \in \mathrm{Spec}^* H^*(G,k)$ and $I_\mathfrak{p}$ is the injective hull of $H^*(G,k)/\mathfrak{p}$, we have

$$\mathrm{End}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(T(I_\mathfrak{p})) \cong H^*(G,k)^\wedge_\mathfrak{p} = \varprojlim_n H^*(G,k)_\mathfrak{p}/\mathfrak{p}^n_\mathfrak{p},$$

using [Matlis 1958].

Proposition 11.1. *Let $I_\mathfrak{m} = H_*(G,k)$, the graded dual of $H^*(G,k)$. This is the injective hull of the trivial $H^*(G,k)$-module $k = H^*(G,k)/\mathfrak{m}$ where $\mathfrak{m} = H^+(G,k)$ is the maximal ideal generated by the positive degree elements. Then $T(I_\mathfrak{m}) \cong pk$, the projective resolution of k.*

Proof. The proof is essentially the same as the proof of Lemma 3.1 of [Benson and Krause 2002]. □

Proposition 11.2. *Let H be a subgroup of G, and write T_G and T_H for the functor T with respect to kG and kH respectively. If I is an injective $H^*(G,k)$-module, we have*

$$T_G(I){\downarrow_H} \cong T_H\big(\mathrm{Hom}^*_{H^*(G,k)}(H^*(H,k), I)\big).$$

Proof. The proof is essentially the same as the proof of Proposition 7.1 of [Benson and Krause 2002]. □

Proposition 11.3. *Let* $\mathfrak{p} \in \operatorname{Spec}^* H^*(G, k)$. *Then* $\mathcal{V}_G(T(I_\mathfrak{p})) = \{\mathfrak{p}\}$.

Proof. The proof is essentially the same as the proof of Theorem 7.3 of [Benson and Krause 2002]. □

It follows that $T(I_\mathfrak{p})$ is in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \cong \mathsf{StMod}\, kG$ unless $\mathfrak{p} = \mathfrak{m}$, and that these objects agree with the objects of the same name constructed in [Benson and Krause 2002].

Theorem 11.4. *Let* $\mathfrak{p}$ *be a homogeneous prime ideal in* $H^*(G, k)$, *and let* d *be the Krull dimension of* $H^*(G, k)/\mathfrak{p}$. *Then*

$$\kappa_\mathfrak{p} \cong T(I_\mathfrak{p}[d]).$$

Proof. If $d > 0$ then both objects are in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \cong \mathsf{StMod}\, kG$ and the theorem is proved in [Benson 2008; Benson and Greenlees 2008]. If $d = 0$ then $\mathfrak{p} = \mathfrak{m}$ and both sides are isomorphic to the projective resolution pk. □

12. Chouinard and Dade

In this section we describe the analogues in $\mathsf{K}(\mathsf{Inj}\, kG)$ of the theorem of Chouinard [1976] and of Benson, Carlson and Rickard's version [1996] of the lemma from [Dade 1978].

Theorem 12.1. *Let* G *be a finite group and* k *a field of characteristic* p. *An object in* $\mathsf{K}(\mathsf{Inj}\, kG)$ *is semiinjective*, *respectively semiprojective*, *respectively zero*, *if and only if its restriction to every elementary abelian* p*-subgroup of* G *is semiinjective*, *respectively semiprojective*, *respectively zero.*

Proof. It follows from the recollement (6.3) that an object X in $\mathsf{K}(\mathsf{Inj}\, kG)$ is semiinjective, respectively semiprojective, if and only if $\operatorname{Hom}_k(tk, X) = 0$, respectively $X \otimes_k tk = 0$. By Chouinard's theorem [1976] in $\mathsf{StMod}\, kG$, this is true if and only if the restriction of $\operatorname{Hom}_k(tk, X)$, respectively $X \otimes_k tk$ to each elementary abelian p-subgroup E of G is zero. This is equivalent to the statement that the restriction of X to each such E is semiinjective, respectively semiprojective.

If an object X in $\mathsf{K}(\mathsf{Inj}\, kG)$ restricts to zero on every elementary abelian p-subgroup then it is acyclic, so it is in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \simeq \mathsf{StMod}(kG)$. So we can apply Chouinard's theorem in $\mathsf{StMod}(kG)$ to deduce that $X \cong 0$. □

Now if $E = \langle g_1, \ldots, g_r \rangle$ is an elementary abelian group of rank r, we write X_i for the element $g_i - 1 \in J(kE)$, the radical of the group algebra. If K is an extension field of k, and $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a nonzero point in affine space $\mathbb{A}^r(K)$, then

$$X_\lambda = \lambda_1 X_1 + \cdots + \lambda_r X_r$$

is an element of $J(KE)$ satisfying $X_\lambda^p = 0$, and $\langle 1 + X_\lambda \rangle$ is a cyclic subgroup of order p in the group algebra KE. It is called a *cyclic shifted subgroup* of E over K.

Theorem 12.2. *An object in* $\mathsf{K}(\mathsf{Inj}\, kE)$ *is semiinjective, respectively semiprojective, respectively zero if and only if for all extension fields K of k and all cyclic shifted subgroups of E over K the restriction is semiinjective, respectively semiprojective, respectively zero.*

Proof. The proof follows the same lines as that of Theorem 12.1, using the version of Dade's lemma in [Benson et al. 1996, Theorem 5.2] instead of Chouinard's theorem. We also need to observe that

$$K \otimes_k \operatorname{Hom}_k(tk, X) \cong \operatorname{Hom}_K(tK, K \otimes_k X),$$
$$K \otimes_k (X \otimes_k tk) \cong (K \otimes_k X) \otimes_K tK. \qquad \square$$

Remark 12.3. As in [Benson et al. 1996], it suffices to check the hypothesis for K the algebraic closure of an extension of k of transcendence degree $r - 1$.

13. Homotopy colimits and localizing subcategories

The goal of this section is to show that in the stable module category

$$\mathsf{StMod}\, kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG),$$

the homotopy category of complexes of injectives $\mathsf{K}(\mathsf{Inj}\, kG)$ and the derived category $\mathsf{D}(\mathsf{Mod}\, kG)$, localizing subcategories are closed under taking filtered colimits in the corresponding category of chain complexes and chain homomorphisms. This amounts to filling in the details of arguments of Bousfield and Kan [1972] and Bökstedt and Neeman [1993] for the sake of easy access.

Let $\mathbb{C}$ denote one of the categories $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG)$, $\mathsf{D}(\mathsf{Mod}\, kG)$, $\mathsf{K}(\mathsf{Inj}\, kG)$ (the arguments work in other situations, but it seems difficult to make precise the conditions on $\mathbb{C}$). Let I be a small category, and let $\phi\colon \mathsf{I} \to \mathbb{C}$ be a covariant functor. Then we call ϕ an I-*diagram* in $\mathbb{C}$. We define the *homotopy colimit* of the diagram ϕ to be the total complex of the double complex formed from finite chains of maps in I in the following manner:

$$\cdots \xrightarrow{d_3} \bigoplus_{i \to j \to k} \phi(i) \xrightarrow{d_2} \bigoplus_{i \to j} \phi(i) \xrightarrow{d_1} \bigoplus_{i} \phi(i)\,. \tag{13.1}$$

We regard this as a complex of objects in $\mathbb{C}$, where the differentials are alternating sums over deleted objects in the chain in the usual way. So for example d_1 takes the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j$ via $\phi(\alpha)$ to $\phi(j)$ minus the identity to $\phi(i)$; while d_2 takes the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ via $\phi(\alpha)$ to the copy of $\phi(j)$ indexed by $j \xrightarrow{\beta} k$ minus the identity to the copy of $\phi(i)$ indexed by $i \xrightarrow{\beta\circ\alpha} k$ plus the identity to the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j$. It is easy to see that $d_j \circ d_{j+1} = 0$. Note that the cokernel of d_1 is

$$\varinjlim_{\mathsf{I}} \phi.$$

We write

$$\operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathsf{I}}} \phi \quad \text{or} \quad \operatorname*{hocolim}_{\substack{\longrightarrow \\ i \in \mathsf{I}}} \phi(i)$$

for the homotopy colimit.

We say that I is a *right filter* if it is a small category satisfying

(i) given objects x and y in I, there exists an object z in I and arrows $x \to z$ and $y \to z$, and

(ii) given objects x and y in I and arrows $f, g \colon x \to y$, there exists an object z in I and an arrow $\alpha \colon y \to z$ such that $\alpha \circ f = \alpha \circ g$.

For example, I could be a poset in which every pair of elements has an upper bound. If I is a right filter, then an I-diagram $\phi \colon \mathsf{I} \to \mathbb{C}$ is called a *filtered system* in $\mathbb{C}$. We assume that every filtered system in $\mathbb{C}$ has a colimit, which we write as

$$\varinjlim_{\mathsf{I}} \phi \quad \text{or} \quad \varinjlim_{i \in \mathsf{I}} \phi(i).$$

Whether I is a filtered system or a more general small category, there is an obvious map $\operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathsf{I}}} \phi \to \varinjlim_{\mathsf{I}} \phi$.

Lemma 13.2 [Bousfield and Kan 1972]. *Let ϕ be an I-diagram in $\mathbb{C}$. Then*

$$\operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathsf{I}}} \phi \to \varinjlim_{\mathsf{I}} \phi$$

is an equivalence.

Proof. In the case where I has a terminal object, say ℓ, there is a homotopy on the complex (13.1) sending the copy of $\phi(i)$ indexed by $i \to \cdots \to j$ to the copy in one degree higher indexed by $i \to \cdots \to j \to \ell$. This is a homotopy from the identity to the projection onto the subcomplex consisting of the single copy of $\phi(\ell)$ in degree zero. This proves that the map from the homotopy colimit to the colimit is an equivalence (that is, passes down to an isomorphism in the corresponding homotopy category) in this case.

The homotopy colimit can be written as a colimit of homotopy colimits over smaller diagrams, so we have

$$\operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathsf{I}}} \phi = \varinjlim_{\ell \in I} \operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathsf{I}/\ell}} \phi \to \varinjlim_{\ell \in I} \varinjlim_{\mathsf{I}/\ell} \phi = \varinjlim_{\mathsf{I}} \phi.$$

Since I/ℓ has a terminal object,

$$\underrightarrow{\mathrm{hocolim}}_{\mathsf{I}/\ell}\, \phi \to \underrightarrow{\mathrm{colim}}_{\mathsf{I}/\ell}\, \phi$$

is an equivalence, and it remains to prove that a colimit of equivalences is an equivalence. This is where the mild assumptions on the category $\mathbb{C}$ come in. Bousfield and Kan were working in the homotopy category of simplicial sets, where equivalences are detected by maps from spheres, and any such map to the filtered colimit factors through some term in the filtered system.

For a countable filtered system, we can argue as follows. If there is no terminal object, then we may choose a cofinal subsystem consisting of a countable sequence of objects and maps

$$\phi(0) \xrightarrow{\alpha_0} \phi(1) \xrightarrow{\alpha_1} \phi(2) \xrightarrow{\alpha_2} \cdots .$$

Then the colimit fits into a triangle

$$\bigoplus_n \phi(n) \xrightarrow{1-\alpha} \bigoplus_n \phi(n) \to \underrightarrow{\mathrm{colim}}_n\, \phi(n).$$

It follows that a colimit of equivalences is an equivalence in this case. So it is only for uncountable filtered systems that there is any problem.

In the category $\mathsf{StMod}\, kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG)$, equivalences are detected by maps from the modules $\Omega^n S$ for $n \in \mathbb{Z}$ with S simple, in the sense that for a map $f\colon M \to N$, if for all $n \in \mathbb{Z}$ and S simple

$$f_*\colon \underline{\mathrm{Hom}}_{kG}(\Omega^n S, M) \to \underline{\mathrm{Hom}}_{kG}(\Omega^n S, N)$$

is an isomorphism, then f is an equivalence. So the argument of Bousfield and Kan works here: any map from $\Omega^n S$ to a filtered colimit factors through some term in the filtered system.

The same argument works in $\mathsf{D}(\mathsf{Mod}\, kG)$, where equivalences are detected by maps from perfect complexes, and any map from a perfect complex to a filtered colimit factors through some object in the system.

For the category $\mathsf{K}(\mathsf{Inj}\, kG)$, we pass to $\mathsf{K}(\mathsf{Mod}\, kG)$ and use the fact that for each simple kG-module S the injective resolution $S \to iS$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(iS, X) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(S, X)$$

for all X in $\mathsf{K}(\mathsf{Inj}\, kG)$ by [Krause 2005, Lemma 2.1]. In $\mathsf{K}(\mathsf{Mod}\, kG)$ any map from S to a filtered colimit factors through some object in the system since S is finitely presented. Thus equivalences in $\mathsf{K}(\mathsf{Inj}\, kG)$ are detected by maps from the injective resolutions iS of simple kG-modules S. □

Theorem 13.3. *Let L be a localizing subcategory of $\mathbb{C}$. Then L is closed under taking filtered colimits in the underlying category of chain complexes.*

Proof. According to Lemma 13.2, it suffices to show that the homotopy colimit is in L.

For $n \geq 0$, write $X(n)$ for the total complex of the truncation of the sequence (13.1) consisting of just the last $n+1$ objects and the maps $d_n, \dots, d_1$. Since each $\phi(i)$ is in L and L is closed under direct sums, each of the terms in (13.1) is in L, and so by the induction on n, $X(n)$ is in L.

There are inclusions $\alpha_n \colon X(n) \to X(n+1)$, and we have a short exact sequence of complexes

$$0 \to \bigoplus_n X(n) \xrightarrow{1-\alpha} \bigoplus_n X(n) \to \varinjlim_n X(n) \to 0.$$

The corresponding triangle shows that

$$\underrightarrow{\mathrm{hocolim}}_{\mathsf{I}}\, \phi = \varinjlim_n X(n)$$

is in L. □

14. $\mathsf{K}(\mathsf{Inj}\, kE)$ for an elementary abelian 2-group E

Let

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/2)^r$$

be an elementary abelian 2-group of rank r, and let k be a field of characteristic two. Let

$$H^*(E,k) = k[x_1, \dots, x_r].$$

where the polynomial generators $x_1, \dots, x_r$ have degree one. The purpose of this section is to give an equivalence of triangulated categories

$$\mathsf{K}(\mathsf{Inj}\, kE) \simeq \mathsf{D}_{\mathrm{dg}}(k[x_1, \dots, x_r]).$$

This can be viewed as a version of Bernšteĭn–Gel′fand–Gel′fand duality [1978], and is also related to a construction of Carlsson [1983].

First we discuss the cyclic group of order two. The discussion begins with the observation that the reduced bar construction on a cyclic group of order two is the minimal resolution. The Alexander–Whitney map on the reduced bar construction is strictly associative, and so it follows that the minimal resolution supports a strictly associative comultiplication. Applying $\mathrm{Hom}_{k(\mathbb{Z}/2)}(-,k)$ to the reduced bar construction gives a differential graded algebra quasiisomorphic to cochains on $B(\mathbb{Z}/2)$. From this, it follows that we have a quasiisomorphism of differential graded algebras

$$C^*(B(\mathbb{Z}/2);k) \simeq H^*(\mathbb{Z}/2,k)$$

where the right hand side is regarded as a differential graded algebra with zero differential. A differential graded algebra is said to be formal if it is quasiisomorphic to its cohomology. The statement above says that $C^*(B(\mathbb{Z}/2);k)$ is formal.

Using the Künneth theorem and the Eilenberg–Zilber theorem, it follows that $C^*(BE;k)$ is also formal, since we have quasiisomorphisms

$$\begin{aligned} C^*(BE;k) &\simeq C^*(B(\mathbb{Z}/2);k) \otimes_k \cdots \otimes_k C^*(B(\mathbb{Z}/2);k) \\ &\simeq H^*(\mathbb{Z}/2,k) \otimes_k \cdots \otimes_k H^*(\mathbb{Z}/2,k) \\ &\cong H^*(E,k) = k[x_1,\dots,x_r]. \end{aligned}$$

Thus we have equivalences of categories

$$\mathsf{D}_{\mathrm{dg}}(C^*(BE;k)) \simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]). \tag{14.1}$$

Theorem 14.2. *Let E be an elementary abelian 2-group and k a field of characteristic two. Then there is an equivalence of triangulated categories*

$$\mathsf{K}(\mathsf{Inj}\,kE) \simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]).$$

Proof. This follows by combining the equivalences

$$\begin{aligned} \mathsf{K}(\mathsf{Inj}\,kE) &\simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kE}(ik)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BE;k)) \\ &\simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]) \end{aligned}$$

coming from Proposition 3.1, Theorem 4.1 and Equation (14.1). □

Remark 14.3. The curious reader may wonder whether these equivalences are monoidal, and if so, why this does not imply that the Steenrod operations on $H^*(BE;k)$ are trivial. The point here is that there are in fact many inequivalent E_∞ structures on the formal differential graded algebra $k[x_1,\dots,x_r]$. There is a trivial one which would make the Steenrod operations act trivially, but this is not the one coming from $C^*(BE;k)$. If E' is a subgroup of the group of units of kE of augmentation one, inducing an isomorphism $kE' \cong kE$, then this gives another, usually inequivalent E_∞ structure on $k[x_1,\dots,x_r]$. There is another one coming from viewing kE as a restricted universal enveloping algebra. The fact that these E_∞ structures are inequivalent can be seen by examining the corresponding tensor products of kE-modules. So the point is that the equivalences in the theorem *are* monoidal, but the monoidal structure on $\mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r])$ is *not* the one coming from the derived tensor product over this graded commutative ring.

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