Algebra & Number Theory

Volume 2
2008
No. 1

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mathematical sciences publishers

# $R$-equivalence on three-dimensional tori and zero-cycles

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We prove that the natural map $T(F)/R \to A_0(X)$, where $T$ is an algebraic torus over a field $F$ of dimension at most 3, $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset and $A_0(X)$ is the group of classes of zero-dimensional cycles on $X$ of degree zero, is an isomorphism. In particular, the group $A_0(X)$ is finite if $F$ is finitely generated over the prime subfield, over the complex field, or over a $p$-adic field.

Let $T$ be an algebraic torus over a field $F$ and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Let $A_0(X)$ be the subgroup of the Chow group $\mathrm{CH}_0(X)$ of classes of zero-dimensional cycles on $X$ consisting of classes of degree zero. The map $T(F) \to A_0(X)$ taking a rational point $t$ in $T(F)$ to $[t]-[1]$ factors through the $R$-equivalence on $T(F)$ (see Section 2C):

$$\varphi : T(F)/R \to A_0(X).$$

One can ask the following questions:

1. Is $\varphi$ a homomorphism?
2. Is $\varphi$ an isomorphism?

Note that $\varphi$ is a homomorphism if and only if $[ts]-[t]=[s]-[1]$ for any two rational points $s, t \in T(F)$. If the translation action of $T$ on itself extends to an action on $X$, the latter means that the natural action of $T(F)$ on $A_0(X)$ is trivial.

In the present paper we prove that $\varphi$ is an isomorphism for all algebraic tori of dimension at most 3 (Theorem 4.4). All tori of dimension 1 and 2 are rational [Voskresenskiĭ 1998, § 4.9], therefore, $\varphi$ is an isomorphism of trivial groups. Birational classification of 3-dimensional tori was given in [Kunyavskiĭ 1987].

We use the following notation in the paper:

The word "*variety*" will mean a separated scheme of finite type over a field.

$F$ is a field.

$F_{\mathrm{sep}}$ is a separable closure of $F$.

*MSC2000:* 19E15.

*Keywords:* algebraic tori, $R$-equivalence, $K$-cohomology, zero-dimensional cycle.

The work has been supported by the NSF grant DMS #0652316.

$\Gamma$ is the Galois group of $F_{\mathrm{sep}}/F$.
$X_L := X \times_F \operatorname{Spec} L$ for a scheme $X$ over $F$ and a field extension $L/F$.
$X_{\mathrm{sep}}$ is $X \times_F \operatorname{Spec} F_{\mathrm{sep}}$.
$T^*$ is the character group of an algebraic torus $T$ over $F_{\mathrm{sep}}$ with $\Gamma$-action.
$T_* = \operatorname{Hom}(T^*, \mathbb{Z})$ is the cocharacter group of a torus $T$.
$T^\circ$ is the dual torus, $(T^\circ)^* = T_*$.
$K_*(X)$ is Quillen's $K$-group of a scheme $X$.
$H^*(X, K_*)$ is the $K$-cohomology group.
$\mathrm{CH}^i(X)$ is the Chow groups of cycles of codimension $i$ on $X$.
$\mathrm{CH}_i(X)$ is the Chow groups of cycles of dimension $i$ on $X$.
*Fields*$/F$ is the category of field extensions of $F$.
*Ab* is the category of abelian groups.
*Sets* is the category of sets.
$\mathbb{G}_m = \mathbb{G}_{m,F}$.

## 1. Preliminaries

**1A.** ***R-equivalence.*** Let $F$ be a field. For a field extension $L/F$, we write $H_L$ for the semilocal ring of all rational functions $f(t)/g(t) \in L(t)$ such that $g(0)$ and $g(1)$ are nonzero. Let $A$ be a functor from the category of semisimple commutative $F$-algebras to the category *Sets*. If $i = 0$ or 1, we have a map $A(H_L) \to A(L)$, $a \mapsto a(i)$, induced by the $L$-algebra homomorphism $H_L \to L$ taking a function $h$ to $h(i)$.

Two points $a_0, a_1 \in A(L)$ are called *strictly $R$-equivalent* if there is an $a \in A(H_L)$ with $a(0) = a_0$ and $a(1) = a_1$. The strict $R$-equivalence generates an equivalence relation $R$ on $A(L)$, called the *$R$-equivalence relation*. The set of $R$-equivalence classes is denoted by $A(L)/R$.

**Example 1.1.** A scheme $X$ over $F$ defines the functor

$$X(A) := \operatorname{Mor}_F(\operatorname{Spec} A, X).$$

The notion of $R$-equivalence in $X(L)$ is classical and was introduced in [Manin 1986, Ch. 2, § 4]. If $G$ is an algebraic group over $F$, then $G(L)/R = G(L)/RG(L)$, where $RG(L)$ is the subgroup of $G(L)$ consisting of all elements that are $R$-equivalent to the identity.

**Example 1.2.** Let $G$ be an algebraic group over $F$. We can define the functor taking a commutative $F$-algebra $A$ to the set of isomorphism classes $H^1_{\text{ét}}(A, G)$ of $G$-torsors over $\operatorname{Spec} A$.

**Example 1.3.** Let $1 \to S \to P \to T \to 1$ be an exact sequence of algebraic tori over $F$ with $P$ a quasitrivial torus, that is, $P \simeq R_{K/F}(\mathbb{G}_{m,K})$ for an étale $F$-algebra

$K$. As $H^1_{ét}(A, P) = H^1_{ét}(A \otimes_F K, \mathbb{G}_m) = 0$ for any semilocal commutative $F$-algebra $A$ by Shapiro–Faddeev Lemma and Grothendieck's Hilbert Theorem 90, the sequence

$$P(A) \to T(A) \to H^1_{ét}(A, S) \to 0$$

is exact. Since $P$ is an open subset in the affine space of $K$, we have $P(L)/R = 1$ for any field extension $L/F$. Hence the image of $P(L) \to T(L)$ consists of $R$-trivial elements in $T(L)$ and therefore,

$$T(L)/R \simeq H^1(L, S)/R.$$

If in addition $S$ is a flasque torus (see [Voskresenskiĭ 1998, § 4.6]) then by [Colliot-Thélène and Sansuc 1977, Th. 2],

$$T(L)/R \simeq H^1(L, S).$$

**1B. *Category of Chow motives.*** Let $CM(F)$ be the category of Chow motives over $F$ (see [Manin 1968]). Recall that $CM(F)$ is an additive category with objects formal finite direct sums $\coprod_k (X_k, i_k)$ (called *Chow motives*) where $X_k$ are smooth proper varieties over $F$ and $i_k \in \mathbb{Z}$. For a smooth proper variety $X$ we write $M(X)(i)$ for the object $(X, i)$ of $CM(F)$ and shortly $M(X)$ for $M(X)(0)$. If $M(X)$ and $M(Y)$ are objects in $CM(F)$ and $X$ is irreducible of dimension $d$ then

$$\mathrm{Mor}_{CM(F)}(M(X)(i), M(Y)(j)) = \mathrm{CH}_{d+i-j}(X \times Y).$$

We have the functor from the category $SP(F)$ of smooth proper varieties over $F$ to $CM(F)$ taking a variety $X$ to $M(X)$ and a morphism $f : X \to Y$ to the cycle of the graph of $f$.

We write $\mathbb{Z}(i)$ for $M(\mathrm{Spec}\, F)(i)$. A motive is called *split* if it is isomorphic to a motive of the form $\coprod_{i=1}^r \mathbb{Z}(d_i)$.

The functor taking an $X$ to the $K$-cohomology groups $H^*(X, K_*)$ (see [Quillen 1973]) from the category $SP(F)$ to the category of (bigraded) abelian groups factors through the category $CM(F)$ as follows. Let $\alpha \in \mathrm{CH}(X \times Y)$ be a morphism $M(X)(i) \to M(Y)(j)$ in $CM(F)$. Then the functor takes $\alpha$ to the homomorphism $H^*(X, K_*) \to H^*(Y, K_*)$ defined by $\beta \mapsto (p_2)_*(\alpha \cdot p_1^*(\beta))$ where $p_1^*$ and $(p_2)_*$ are the pull-back and the push-forward homomorphisms for the first and the second projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ respectively.

Recall that $H^p(X, K_p) = \mathrm{CH}^p(X)$ for a smooth $X$ and every $p \geq 0$ by [Quillen 1973, § 7, Prop. 5.14].

**Lemma 1.4.** *Let $M$ be a split motive. Then the product map*

$$\mathrm{CH}^p(M) \otimes K_q(F) \to H^p(M, K_{p+q})$$

*is an isomorphism.*

*Proof.* The statement is obviously true for the motive $M = \mathbb{Z}(i)$. □

Let $X$ be a smooth proper irreducible variety over $F$. The push-forward homomorphism

$$\deg : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} F) = \mathbb{Z}$$

with respect to the the structure morphism $X \to \operatorname{Spec} F$ is called the *degree homomorphism*. For every $i \geq 0$, we have the intersection pairing

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}_p(X) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta). \tag{1}$$

**Proposition 1.5.** *Let $X$ be a smooth proper irreducible variety over $F$. Then the Chow motive of $X$ is split if and only if*

(i) *the Chow group* $\mathrm{CH}(X)$ *is free abelian of finite rank and the map*

$$\mathrm{CH}(X) \to \mathrm{CH}(X_L)$$

*is an isomorphism for every field extension $L/F$ and*

(ii) *the pairing* (1) *is a perfect duality for every $p$.*

*Proof.* Suppose that the motive of $X$ is split. Mutually inverse isomorphisms between $M(X)$ and a split motive $\coprod_{i=1}^{r} \mathbb{Z}(d_i)$ are given by two $r$-tuples of elements $u_i \in \mathrm{CH}_{d_i}(X)$ and $v_i \in \mathrm{CH}^{d_i}(X)$ such that the tuple $u$ (and also $v$) form a $\mathbb{Z}$-basis of $\mathrm{CH}(X)$ and $\deg(u_i v_j) = \delta_{ij}$ over any field extension of $F$.

Conversely, suppose that (i) and (ii) hold. Choose dual bases $u_i$ and $v_j$ of $\mathrm{CH}(X)$. They define morphisms $\alpha$ and $\beta$ from a split motive $N$ to $M(X)$ and back respectively so that $\beta \circ \alpha$ is the identity of $N$. By Yoneda Lemma, it suffices to prove that for every variety $Y$ over $F$ the morphism

$$u \otimes 1_Y : \mathrm{CH}\big(N \otimes M(Y)\big) \to \mathrm{CH}(X \times Y)$$

is an isomorphism. The injectivity follows from the fact that $\beta \circ \alpha = \mathrm{id}$. The surjectivity follows by induction on the dimension of $Y$ using the localization and the fact that the map $u \otimes 1_Y$ is an isomorphism if $Y$ is the spectrum of a field extension of $F$. □

**1C.** ***K-theory, K-cohomology and the Brown–Gersten–Quillen spectral sequence.*** Let $X$ be a smooth variety over $F$. Let $K_*(X)^{(i)}$ denote the $i$-th term of the topological filtration on $K_*(X)$. Consider the Brown–Gersten–Quillen (BGQ) spectral sequence (see [Quillen 1973, § 7, Th. 5.4])

$$E_2^{p,q} = H^p(X, K_{-q}) \Rightarrow K_{-p-q}(X) \tag{2}$$

converging to the $K$-groups of $X$ with the topological filtration. The $K$-cohomology groups $H^*(X, K_*)$ can be computed via Gersten complexes [Quillen 1973, § 7.5].

We have $E_2^{p,q} = 0$ if $p < 0$ or $p + q > 0$, or $p > \dim X$ and $E_2^{p,-p} = \mathrm{CH}^p(X)$. The $E_2$-term is as follows.

$$\begin{array}{ccccc}
\mathrm{CH}^0(X) & 0 & & & \\
H^0(X, K_1) \searrow & \mathrm{CH}^1(X) & 0 & & \\
& H^1(X, K_2) \searrow & \mathrm{CH}^2(X) & 0 & \\
& & H^2(X, K_3) & \mathrm{CH}^3(X) &
\end{array}$$

If in addition $X$ is geometrically irreducible proper, we have $H^0(X, K_1) = F^\times$. The composition of the pull-back homomorphism $F^\times = K_1(F) \to K_1(X)$ for the structure morphism of $X$ with the edge homomorphism $K_1(X) \to H^0(X, K_1)$ is the identity. Hence all the differentials starting at $E_*^{0,-1}$ are trivial. If in addition $\dim X = 3$, the spectral sequence yields an exact sequence

$$K_1(X)^{(1)} \to H^1(X, K_2) \to \mathrm{CH}^3(X) \xrightarrow{g} K_0(X), \tag{3}$$

where $g$ is the edge homomorphism.

## 2. Zero cycles on toric models

**2A.** ***K-theory of toric models.*** Let $T$ be an algebraic torus over a field $F$. Let $X$ be a geometrically irreducible variety containing $T$ as an open subset. We say that $X$ is a *toric model of* $T$ if the translation action of $T$ on itself extends to an action on $X$. Every torus admits a smooth proper toric model [Brylinski 1979; Colliot-Thélène et al. 2005].

Let $X$ be a smooth proper toric model of $T$. It follows from [Klyachko 1982, Prop. 3, Cor. 2] that $X_{\mathrm{sep}}$ satisfies the conditions (i) and (ii) of Proposition 1.5. Thus by Proposition 1.5, we have:

**Proposition 2.1.** *Let $X$ be a smooth proper toric model of $T$. Then the Chow motive of $X_{\mathrm{sep}}$ is split.*

The proposition and Lemma 1.4 yield:

**Corollary 2.2.** *Let $X$ be a smooth proper toric model of an algebraic torus $T$. Then the product map*

$$\mathrm{CH}^p(X_{\mathrm{sep}}) \otimes K_q(F_{\mathrm{sep}}) \to H^p(X_{\mathrm{sep}}, K_{p+q})$$

*is an isomorphism.*

The absolute Galois group $\Gamma$ acts naturally on $K_0(X_{\mathrm{sep}})$ leaving each term $K_0(X_{\mathrm{sep}})^{(i)}$ invariant.

The following theorem was proven in [Merkurjev and Panin 1997].

**Theorem 2.3.** *Let $X$ be a smooth proper toric model of an algebraic torus of dimension $d$ over $F$. Then*

(1) $K_0(X_{\mathrm{sep}})$ *is a direct summand of a permutation* $\Gamma$*-module*;

(2) *the subgroup* $K_0(X_{\mathrm{sep}})^{(d)}$ *is infinite cyclic generated by the class of a rational point of* $X$;

(3) *the natural map* $K_i(X) \to K_i(X_{\mathrm{sep}})^{\Gamma}$ *is an isomorphism for* $i \le 1$;

(4) *the product map* $K_0(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^{\times} \to K_1(X_{\mathrm{sep}})$ *is an isomorphism.*

**Corollary 2.4.** *Let $X$ be a smooth proper toric model of a torus of dimension $d$ over $F$. We have the following natural isomorphisms*:

(1) $K_i(X)^{(1)} \xrightarrow{\sim} \big(K_i(X_{\mathrm{sep}})^{(1)}\big)^{\Gamma}$ *for* $i \le 1$.

(2) $K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^{\times} \xrightarrow{\sim} K_1(X_{\mathrm{sep}})^{(1)}$.

*Proof.* (1): The group $K_i(X)^{(1)}$ is the kernel of the restriction to the generic point $K_i(X) \to K_i F(X)$. The image of this map is equal to $H^0(X, K_i) = K_i(F)$ for $i = 0, 1$. Statement (1) follows from Theorem 2.3(3) applied to the exact sequence

$$0 \to (K_i(X_{\mathrm{sep}})^{(1)})^{\Gamma} \to K_i(X_{\mathrm{sep}})^{\Gamma} \to K_i(F_{\mathrm{sep}})^{\Gamma}$$

for $i = 0, 1$.

(2): Tensoring with $F_{\mathrm{sep}}^{\times}$ the split exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}}) \to \mathbb{Z} \to 0$$

we get (2) by Theorem 2.3(4). □

**Corollary 2.5.** *Let $X$ be a smooth proper toric model of a torus of dimension $d$ over $F$. Then*

(1) $K_0(X_{\mathrm{sep}})^{(1)}$ *is a direct summand of a permutation* $\Gamma$*-module.*

(2) $K_0(X_{\mathrm{sep}})^{(d)}$ *is a direct summand of the* $\Gamma$*-module* $K_0(X_{\mathrm{sep}})$.

*Proof.* (1): We have the canonical decomposition of $\Gamma$-modules via the structure sheaf $\mathbb{O}_X$:

$$K_0(X_{\mathrm{sep}}) = K_0(X_{\mathrm{sep}})^{(1)} \oplus \mathbb{Z} \cdot 1.$$

Hence $K_0(X_{\mathrm{sep}})^{(1)}$ is a direct summand of a permutation $\Gamma$-module by Theorem 2.3(1).

(2): For a rational point $x \in X(F)$, the composition of the push-forward homomorphism $K_0(F_{\mathrm{sep}}) = K_0(F_{\mathrm{sep}}(x)) \to K_0(X_{\mathrm{sep}})$ with the push-forward map $p_* : K_0(X_{\mathrm{sep}}) \to K_0(F_{\mathrm{sep}})$ induced by the structure morphism $p$ of $X_{\mathrm{sep}}$ is the identity. It follows from Theorem 2.3(2) that the inclusion

$$K_0(X_{\mathrm{sep}})^{(d)} \to K_0(X_{\mathrm{sep}})$$

is split by $p_*$ as a homomorphism of $\Gamma$-modules. □

We shall need the following property of $K$-cohomology groups of smooth proper toric models.

**Proposition 2.6.** *Let $X$ be a smooth proper toric model of a torus of dimension $d$ over $F$. Then the natural morphism $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)^\Gamma$ is an isomorphism.*

*Proof.* As $X$ is geometrically rational and has a rational point, the statement follows from [Colliot-Thélène and Raskind 1985, Prop. 4.3] (if $\mathrm{char}(F) = 0$) and [Kahn 1996, Th. 1(a)] or [Garibaldi et al. 2003, Th. 8.9] (in general). □

**2B.** ***The group $A_0(X)$ of 3-dimensional toric models.*** Let $T$ be an algebraic torus and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Let $P$ and $S$ be algebraic tori over $F$ such that $P^*$ is the permutation $\Gamma$-module with $\mathbb{Z}$-basis the set of irreducible components of $(X \setminus T)_{\mathrm{sep}}$ and $S^* = \mathrm{CH}^1(X_{\mathrm{sep}})$. We have natural $\Gamma$-homomorphisms $T^* \to P^*$ taking a character $\chi$ to $\mathrm{div}(\chi)$ (we consider $\chi$ as a rational function on $X_{\mathrm{sep}}$) and $P^* \to S^*$ taking a component of $(X \setminus T)_{\mathrm{sep}}$ to its class in the Chow group. The sequence

$$0 \to T^* \to P^* \to S^* \to 0 \tag{4}$$

is a flasque resolution of $T^*$ (see [Colliot-Thélène and Sansuc 1977, Prop. 6], [Voskresenskiĭ 1998, § 4.6]). Thus we have an exact sequence of algebraic tori

$$1 \to S \to P \to T \to 1, \tag{5}$$

a flasque resolution of $T$.

By [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3),

$$T(L)/R \simeq H^1(L, S) \tag{6}$$

for any field extension $L/F$.

The spectral sequence (2) for $X_{\mathrm{sep}}$ yields isomorphisms of $\Gamma$-modules

$$K_0(X_{\mathrm{sep}})^{(1/2)} \simeq \mathrm{CH}^1(X_{\mathrm{sep}}) = S^*$$

and

$$K_0(X_{\mathrm{sep}})^{(2/3)} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}).$$

Let $T$ be a 3-dimensional torus and $X$ a smooth proper toric model of $T$. By [Klyachko 1982, Prop. 3, Cor. 2], the pairing

$$\mathrm{CH}^1(X_{\mathrm{sep}}) \otimes \mathrm{CH}^2(X_{\mathrm{sep}}) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta)$$

is a perfect duality of $\Gamma$-lattices. It follows that $\mathrm{CH}^2(X_{\mathrm{sep}}) \simeq S_*$. Thus, the exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(2)} \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}})^{(1/2)} \to 0$$

yields an exact sequence of algebraic tori

$$1 \to S' \xrightarrow{\tau} Q \to S^\circ \to 1 \tag{7}$$

with $S'_* = K_0(X_{\mathrm{sep}})^{(2)}$ and $Q_* = K_0(X_{\mathrm{sep}})^{(1)}$ a direct summand of a permutation $\Gamma$-module by Corollary 2.5(1). By Theorem 2.3(2) and Corollary 2.5(2), we have isomorphisms of $\Gamma$-modules

$$S'_* = K_0(X_{\mathrm{sep}})^{(2)} \simeq K_0(X_{\mathrm{sep}})^{(2/3)} \oplus \mathbb{Z} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}) \oplus \mathbb{Z} \simeq S_* \oplus \mathbb{Z}.$$

Hence $S' \simeq S \times \mathbb{G}_m$ is a flasque torus. Let $\widetilde{Q}$ be a torus such that $Q \times \widetilde{Q}$ is a quasi-split torus. Then the exact sequence

$$1 \to S' \times \widetilde{Q} \xrightarrow{\tau \times 1_{\widetilde{Q}}} Q \times \widetilde{Q} \to S^\circ \to 1$$

is a flasque resolution of $S^\circ$. By [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3) and (6), we have

$$S^\circ(L)/R \simeq H^1(L, S' \times \widetilde{Q}) \simeq H^1(L, S') \simeq H^1(L, S) \simeq T(L)/R \tag{8}$$

for any field extension $L/F$, and hence it follows from (7) that

$$\mathrm{Coker}(Q(F) \to S^\circ(F)) = S^\circ(F)/R. \tag{9}$$

As $K_0(X)$ injects into $K_0(X_{\mathrm{sep}})$ and $K_0(X_{\mathrm{sep}})^{(3)}$ is infinite cyclic group generated by the class of a rational point by Theorem 2.3, the kernel of the homomorphism $g$ in (3) coincides with the kernel of the composition

$$\mathrm{CH}^3(X) \to \mathrm{CH}^3(X_{\mathrm{sep}}) \to K_0(X_{\mathrm{sep}})^{(3)} \simeq \mathbb{Z},$$

which is the degree map. Recall that we write $A_0(X)$ for the kernel of $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$. We then have

$$\mathrm{Ker}(g) = A_0(X). \tag{10}$$

The group $A_0(X)$ is 2-torsion, by [Merkurjev and Panin 1997, Cor. 5.11(4)].

By Corollary 2.4, we have isomorphisms

$$K_1(X)^{(1)} \simeq \big(K_1(X_{\mathrm{sep}})^{(1)}\big)^\Gamma \simeq \big(K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^\times\big)^\Gamma = (Q_* \otimes F_{\mathrm{sep}}^\times)^\Gamma = Q(F). \tag{11}$$

It follows from Corollary 2.2 and Proposition 2.6 that

$$H^1(X, K_2) \simeq H^1(X_{\mathrm{sep}}, K_2)^\Gamma \simeq \big(\mathrm{CH}^1(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^\times\big)^\Gamma = (S^* \otimes F_{\mathrm{sep}}^\times)^\Gamma = S^\circ(F). \tag{12}$$

**Remark 2.7.** The referee has pointed out that using results from [Colliot-Thélène and Raskind 1985] one can deduce that $\mathrm{CH}^1(X) \otimes F^\times \simeq H^1(X, K_2)$ for a smooth projective rational variety $X$ over an algebraically closed field $F$ of characteristic zero.

Under the identifications (11) and (12), and the fact that the BGQ spectral sequence is compatible with products [Gillet 1981, § 7], the map $K_1(X)^{(1)} \to H^1(X, K_2)$ in (3) coincides with the homomorphism $Q(F) \to S^\circ(F)$ given by (7). It follows from (3), (9) and (10) that

$$\begin{aligned} S^\circ(F)/R &= \mathrm{Coker}(Q(F) \to S^\circ(F)) \\ &\simeq \mathrm{Coker}\big(K_1(X)^{(1)} \to H^1(X, K_2)\big) \simeq \mathrm{Ker}(g) = A_0(F). \end{aligned} \tag{13}$$

By (8), there are natural isomorphisms

$$T(F)/R \simeq S^\circ(F)/R \simeq A_0(X). \tag{14}$$

Similarly, over any field extension $L/F$ we have an isomorphism

$$\boxed{\rho_L : T(L)/R \simeq A_0(X_L).} \tag{15}$$

We shall view $\rho$ as an isomorphism of functors $L \mapsto T(L)/R$ and $L \mapsto A_0(X_L)$ from *Fields*/$F$ to *Ab*.

The following remark was suggested by J.-L. Colliot-Thélène.

**Remark 2.8.** The isomorphism (14) yields finiteness of $A_0(X)$ in all cases when $T(F)/R$ is known to be finite, that is, $F$ a finitely generated over the prime subfield, over the complex field, over a $p$-adic field (see [Colliot-Thélène and Sansuc 1977, Th. 1 and Prop. 14] and [Colliot-Thélène et al. 2004, Th. 3.4]).

**2C. *The map* $\varphi_L : T(L)/R \to A_0(X_L)$.** Let $T$ be an algebraic torus over $F$, $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset, and $L/F$ a field extension. By [Colliot-Thélène and Sansuc 1977, Prop. 12, Cor.], the map

$$\boxed{\varphi_L : T(L)/R \to A_0(X_L)} \tag{16}$$

taking the $R$-equivalence class of an $L$-point $t \in T(L)$ to the class of the zero cycle $[t] - [1]$, is well defined. We view $\varphi$ as a morphism of functors from *Fields*/$F$ to *Sets*.

**Proposition 2.9.** *The map $\varphi_L$ does not depend (up to canonical isomorphism) on the choice of $X$.*

*Proof.* We may assume that $L = F$. Let $X$ and $X'$ be two smooth proper geometrically irreducible varieties containing $T$ as an open subset. The closure of the graph of a birational isomorphism between $X$ and $X'$ that is identical on $T$ yields morphisms between the motives $M(X)$ and $M(X')$ in $\mathcal{CM}(F)$. These morphisms induce mutually inverse isomorphisms between $A_0(X)$ and $A_0(X')$ [Fulton 1984, 16.1.11]. □

Let $X$ be a smooth proper toric model of $T$. Consider the flasque resolution (5). The $S$-torsor $P_L$ over $T_L$ can be extended to an $S$-torsor $q : U \to X_L$ (see [Colliot-Thélène and Sansuc 1977, Prop. 9] or [Merkurjev and Panin 1997, Prop. 5.4]). For any point $x \in X_L$, the fiber $U_x$ of $q$ over $x$ is an $S$-torsor over $\operatorname{Spec} L(x)$. Denote by $[U_x]$ its class in $H^1(L(x), S)$. By [Colliot-Thélène and Sansuc 1977, Prop. 12], the map

$$\psi_L : \mathrm{CH}_0(X_L) \to H^1(L, S) = T(L)/R, \tag{17}$$

taking the class $[x]$ of a closed point $x \in X_L$ to $N_{L(x)/L}([U_x])$ extends to a well defined group homomorphism. The composition $\psi|_{A_0(X_L)} \circ \varphi$ is the identity. It follows that the map $\varphi_L$ is injective.

## 3. Functors from *Fields*/$F$ to *Sets*

We consider functors from the category *Fields*/$F$ to the category *Sets*.

All functors we are considering take values in *Ab*, but some of the morphisms between such functors (namely, $\varphi$) may not be given by group homomorphisms.

In this section, we study compatibility properties for morphisms between functors with respect to norm and specialization maps.

**3A.** ***Functors with norm maps.*** Let $A$ : *Fields*/$F \to$ *Sets* be a functor. We say that $A$ is a *functor with norms* if for any finite field extension $E/F$, there is given a *norm map* $N_{E/F} : A(E) \to A(F)$.

**Example 3.1.** Let $T$ be an algebraic torus over $F$ and $E/F$ a finite field extension. There is an obvious norm map

$$N_{E/F} : T(E) = H^0(E, T_* \otimes E^\times_{\mathrm{sep}}) \to H^0(F, T_* \otimes F^\times_{\mathrm{sep}}) = T(F).$$

Thus the functor $L \mapsto T(L)$ is equipped with norms. Similarly, the functors $L \mapsto T(L)/R$, $L \mapsto H^1(L, T)$, and $L \mapsto A_0(X_L)$ also have norms.

A morphism $\alpha : A \to B$ of functors with norms from *Fields*$/F$ to *Sets* *commutes with norms* if for any field extension $E/F$, the diagram

$$\begin{array}{ccc} A(E) & \xrightarrow{\alpha_E} & B(E) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ A(F) & \xrightarrow{\alpha_F} & B(F) \end{array}$$

is commutative.

**Example 3.2.** Let $T$ be a torus of dimension 3. The sequence (5) yields an isomorphism of functors $T(L)/R \xrightarrow{\sim} H^1(L, S)$ that commutes with norms. It follows that the isomorphism $T(L)/R \simeq S^\circ(L)/R$ in (8) commutes with norms.

**Example 3.3.** Let $T$ be an arbitrary torus and $1 \to S \to P \to T \to 1$ a flasque resolution. Let $\mathrm{End}_F(S) = \mathrm{Hom}_\Gamma(S^*, S^*)$ be the endomorphism ring of $S$. For a field extension $L/F$, the group $T(L)/R = H^1(L, S)$ has a natural structure of an $\mathrm{End}_F(S)$-module. For any $\alpha \in \mathrm{End}_F(S)$, the endomorphism of the functor $L \mapsto T(L)/R$ taking a $t$ to $\alpha(t)$ commutes with norms.

**Proposition 3.4.** *Let $T$ be an algebraic torus over $F$ and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Then the morphism $\psi$ in* (17) *commutes with norms.*

*Proof.* Let $E/F$ be a finite field extension, $x \in X_E$ a closed point and $x'$ the image of $x$ under the natural morphism $X_E \to X$. We have $N_{E/F}([x]) = m[x']$ in $\mathrm{CH}_0(X)$, where $m = [E(x) : F(x')]$. The torsor $U_x$ in the definition of $\psi$ is the restriction of $U_{x'}$ to $E(x)$. By [Fulton 1984, Example 1.7.4], we have

$$N_{E(x)/F(x')}\big([U_{x'}]_{E(x)}\big) = m[U_{x'}].$$

Hence

$$\begin{aligned} N_{E/F}\big(\psi_E([x])\big) = N_{E(x)/F}([U_x]) &= N_{F(x')/F} N_{E(x)/F(x')}([U_{x'}]_{E(x)}) \\ &= m N_{F(x')/F}([U_{x'}]) = \psi_F\big(N_{E/F}([x])\big). \quad \square \end{aligned}$$

**Proposition 3.5.** *Let $T$ be an algebraic torus over $F$ and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Then the map $\varphi_F : T(F)/R \to A_0(X)$ in* (16) *is an isomorphism of groups if and only if the morphism $\varphi$ commutes with norms.*

*Proof.* Suppose that $\varphi$ commutes with norms. We show that $\varphi$ is surjective. Every closed point in $X$ is rationally equivalent to a zero-divisor with support in $T$. Let $x \in T$ be a closed point of degree $n$. It is sufficient to prove that $[x] - n[1]$ belongs to the image of $\varphi_F$. Let $E = F(x)$ and $x' \in T_E$ the canonical rational point over $x$. We have $\varphi_E(x') = [x'] - [1]$ and as $\varphi$ commutes with norms,

$$[x] - n[1] = N_{E/F}([x'] - [1]) = N_{E/F} \circ \varphi_E(x') = \varphi_F(N_{E/F}(x')).$$

Thus, $\varphi$ is a bijection. The inverse map given by (17) is a group homomorphism. Hence $\varphi$ is a group isomorphism.

Conversely, if $\varphi$ is an isomorphism, then $\varphi$ commutes with norms as $\psi$ does by Proposition 3.4. □

**Proposition 3.6.** *Let $T$ be an algebraic torus of dimension* 3 *over $F$ and $X$ a smooth proper toric model of $T$. Then the morphism of functors $\rho$ in* (15) *commutes with norms.*

*Proof.* By Example 3.2, it suffices to prove that the morphism $S^\circ(L)/R \to A_0(X_L)$ given by (13) commutes with norms. Let $E/F$ be a finite field extension. The statement follows from the commutativity of the diagram

$$\begin{array}{ccccc} S^\circ(E)/R & \longrightarrow & H^1(X_E, K_2) & \longrightarrow & \mathrm{CH}^3(X_E) \\ \big\downarrow N_{E/F} & & \big\downarrow N_{E/F} & & \big\downarrow N_{E/F} \\ S^\circ(F)/R & \longrightarrow & H^1(X, K_2) & \longrightarrow & \mathrm{CH}^3(X). \end{array}$$

The exact direct image functor $f_*$ takes the category $M^p(X_E)$ of coherent sheaves on $X_E$ supported in codimension at least $p$ to $M^p(X)$. Therefore, $f_*$ yields a map of the BGQ spectral sequences for $X_E$ and $X$. Hence the right square of the diagram is commutative.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, it suffices to prove commutativity of the left square in the split case. The left square coincides with

$$\begin{array}{ccc} S^* \otimes E^\times & \longrightarrow & H^1(X_E, K_2) \\ \big\downarrow 1\otimes N_{E/F} & & \big\downarrow N_{E/F} \\ S^* \otimes F^\times & \longrightarrow & H^1(X, K_2), \end{array}$$

where the horizontal maps are product maps after the identification of $S^*$ with $\mathrm{CH}^1(X)$. The commutativity follows from the projection formula in $K$-cohomology [Rost 1996, § 14.5]. □

**3B.** ***Functors with specializations.*** Let $A$ : *Fields*/$F$ → *Sets* be a functor. We say that $A$ is a *functor with specializations* if for any DVR (discrete valuation ring) over $F$ of geometric type (a localization of an $F$-algebra of finite type) with quotient field $L$ and residue field $K$ there is given a map $s_A : A(L) \to A(F)$ called a *specialization map*.

**Example 3.7.** Let $O$ be a DVR over $F$ with quotient field $L$ and residue field $K$

and $X$ a variety over $F$. The specialization homomorphism

$$s : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(X_K)$$

is defined as follows. Let $\alpha \in \mathrm{CH}_0(X_L)$. As the restriction map $\mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_L)$ is surjective, we can choose $\alpha' \in \mathrm{CH}_1(X_O)$ such that $\alpha'_L = \alpha$. Then set $s(\alpha) = i^*(\alpha')$, the image of $\alpha'$ under the Gysin homomorphism $i^* : \mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_K)$, where $i : X_K \to X_O$ is the regular closed embedding of codimension one [Fulton 1984, § 2.6]. The map $s$ is well defined as $i^* \circ i_* = 0$ for the principal divisor $X_K$ in $X_O$ by [Fulton 1984, Prop. 2.6(c)].

**Example 3.8.** (see [Gille 2004, Prop. 2.2]) Let $T$ be a torus over $F$ and $O$ a DVR over $F$ with quotient field $L$ and residue field $K$. Let $1 \to S \to P \to T \to 1$ be a flasque resolution of $T$. The homomorphism

$$H^1_{\text{ét}}(O, S) \to H^1(L, S)$$

is an isomorphism by [Colliot-Thélène and Sansuc 1987, Cor. 4.2]. The composition

$$s : T(L)/R \simeq H^1(L, S) \simeq H^1_{\text{ét}}(O, S) \to H^1(K, S) \simeq T(K)/R$$

is called the *specialization homomorphism with respect to $O$*. One can easily see that the specialization homomorphism does not depend on the choice of a flasque resolution of $T$. It follows from the triviality of $H^1_{\text{ét}}(O, P)$ that the composition $T(O) \to T(L) \to T(L)/R$ is surjective.

$$\begin{array}{ccccccccc}
T(L)/R & \twoheadleftarrow & T(L) & \longleftarrow & T(O) & \longrightarrow & T(K) & \twoheadrightarrow & T(K)/R \\
\downarrow \wr & & & & \downarrow & & & & \downarrow \wr \\
H^1(L, S) & & \xleftarrow{\quad\sim\quad} & & H^1_{\text{ét}}(O, S) & & \longrightarrow & & H^1(K, S)
\end{array}$$

Let $p \in T(L)/R$ and $q \in T(O)$ be a lift of $p$. Then it readily follows from the definition that $s(p)$ is the image of $q$ under the composition $T(O) \to T(K) \to T(K)/R$.

**Lemma 3.9.** *Let $T$ be an algebraic torus over $F$. Let $t, t' \in T$ be two points such that $t$ belongs to the closure of $t'$ and the local ring $O_{t',t}$ is a DVR. Let $s : T(F(t'))/R \to T(F(t))/R$ be the specialization homomorphism with respect to $O_{t',t}$. Then $s(t') = t$.*

*Proof.* In the ring $A := F[T]$ let $P$ and $P'$ be the prime ideals of $y$ and $y'$ respectively. Then $O$ is the ring $A_P/P'A_P$. Let $\tilde{t} \in T(O) = \mathrm{Mor}(\mathrm{Spec}\, O, T)$ be the point given by the natural homomorphism of $A \to O$. Then the images of $\tilde{t}$

under the maps $T(O) \to T(F(t))$ and $T(O) \to T(F(t'))$ coincide with $y$ and $y'$ respectively. The statement follows now from Example 3.8. □

Let $\theta : A \to B$ be a morphism of functors from *Fields*/$F$ to *Sets* with specializations (for example, the functors $L \mapsto T(L)/R$ or $L \mapsto \mathrm{CH}_0(X_L)$). We say that $\theta$ *commutes with specializations* if for every DVR as above, the diagram

$$\begin{array}{ccc} A(L) & \xrightarrow{\theta_L} & B(L) \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ A(K) & \xrightarrow{\theta_K} & B(K) \end{array}$$

is commutative.

**Proposition 3.10.** *Let $T$ be an algebraic torus over $F$ and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Then the morphism $\varphi$ in* (16) *commutes with specializations.*

*Proof.* Let $O$ be a DVR over $F$ with quotient field $L$ and residue field $K$. For an $O$-point $p$ of $T$ let $[p]$ denote the class of its graph in $\mathrm{CH}_1(X_O)$. Consider the diagram

$$\begin{array}{ccccc} T(K) & \longleftarrow & T(O) & \longrightarrow & T(L) \\ {\scriptstyle \varphi_K}\downarrow & & {\scriptstyle \varphi_O}\downarrow & & {\scriptstyle \varphi_L}\downarrow \\ \mathrm{CH}_0(X_K) & \longleftarrow & \mathrm{CH}_1(X_O) & \longrightarrow & \mathrm{CH}_0(X_L) \end{array}$$

where $\varphi_O(p) = [p] - [1]$ and the bottom maps are the pull-back homomorphisms. The statement follows from the commutativity property of the diagram. To prove commutativity let $E$ be either $K$ or $L$ and $f : \operatorname{Spec} E \to \operatorname{Spec} O$, $g : X_E \to X_O$ the natural morphisms. Let $p \in T(O)$ be a point and $q \in T(E)$ its image. We view $p$ and $q$ as morphisms $p : \operatorname{Spec} O \to X_O$ and $q : \operatorname{Spec} E \to X_E$. By [Fulton 1984, Th. 6.2(a)], the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(\operatorname{Spec} O) & \xrightarrow{f^*} & \mathrm{CH}_0(\operatorname{Spec} E) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \mathrm{CH}_1(X_O) & \xrightarrow{g^*} & \mathrm{CH}_0(X_E) \end{array}$$

is commutative. It follows that $[q] = q_*(1_E) = q_* f^*(1_O) = g^* p_*(1_O) = g^*([p])$ and the result follows. □

**Proposition 3.11.** *Let $T$ be an algebraic torus over $F$ and $\theta, \theta' : T(?)/R \to B$ two morphisms of functors commuting with specializations. Suppose that $\theta_{F(T)}$ and $\theta'_{F(T)}$ coincide at the generic point of $T$. Then $\theta = \theta'$.*

*Proof.* Let $p : \operatorname{Spec} L \to T$ be a point of $T$ over a field extension $L$ over $F$. We need to prove that $\theta_L(p) = \theta'_L(p)$. Let $t \in T$ be the point in the image of $p$. We view $t$ as a point of $T$ over the residue field $F(t)$. As $F(t) \subset L$ and $p$ is the image of $t$ under the map $T(F(t)) \to T(L)$, it suffices to show that $\theta_{F(t)}(t) = \theta'_{F(t)}(t)$.

We prove this by induction on $\operatorname{codim}(t)$. By assumption, the statement holds if $t$ is the generic point. Otherwise let $t' \in T$ be a point such that $t$ is a direct specialization of $t'$. Then the local ring $O_{t',t}$ is a DVR with quotient field $F(t')$ and residue field $F(t)$. As $\theta$ and $\theta'$ commute with specializations, it follows from Lemma 3.9 that

$$\begin{aligned}\theta_{F(t)}(t) = \theta_{F(t)}(s(t')) &= s_B\big(\theta_{F(t')}(t')\big) \\ &= s_B\big(\theta'_{F(t')}(t')\big) = \theta'_{F(t)}(s(t')) = \theta'_{F(t)}(t).\end{aligned}$$

□

**Proposition 3.12.** *Let $T$ be an algebraic torus of dimension* 3 *over $F$ and $X$ a smooth proper toric model of $T$. Then the morphism of functors $\rho$ in* (15) *commutes with specializations.*

*Proof.* Let $O$ be a DVR over $F$ of geometric type with quotient field $L$ and residue field $K$. The diagram

$$\begin{array}{ccccc} H^1(X_K, K_2) & \longleftarrow & H^1(X_O, K_2) & \longrightarrow & H^1(X_L, K_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CH}^3(X_K) & \longleftarrow & \mathrm{CH}^3(X_O) & \longrightarrow & \mathrm{CH}^3(X_L) \end{array}$$

where the middle vertical map is the differential in the $E_2$-term of the BGQ spectral sequence (2) for $X_O$. The right square is commutative since the morphism $X_L \to X_O$ is flat [Quillen 1973, § 7, Th. 5.4].

The pull-back homomorphism $f^*$ for the morphism $f : X_K \to X_O$ in $K$-theory is defined as follows (see [Quillen 1973, § 7.2.5]). Let $\pi \in O$ be a prime element and $M(X_O, f)$ the full subcategory of the category $M(X_O)$ of coherent sheaves on $X_O$ consisting of sheaves $G$ with $\pi$ a nonzero-divisor in $G$. Then $f^*$ is the composition of the inverse of the isomorphism induced by the inclusion functor

$$\alpha : M(X_O, f) \to M(X_O)$$

on $K$-groups and the map induced by the restriction

$$\beta : M(X_O, f) \to M(X_K)$$

of the unverse image functor $M(X_O) \to M(X_K)$. Note that functors $\alpha$ and $\beta$ take sheaves supported in codimension $p$ into $M^p(X_O)$ and $M^p(X_K)$ respectively. Hence $f$ induces a pull-back map of the BGQ spectral sequences for $X_O$ and $X_K$. It follows that the left square of the diagram is commutative too.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, we may consider the split situation. In the diagram

$$\begin{array}{ccccc} S^\circ(K) & \longleftarrow & S^\circ(O) & \longrightarrow & S^\circ(L) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X_K, K_2) & \longleftarrow & H^1(X_O, K_2) & \longrightarrow & H^1(X_L, K_2) \end{array}$$

the vertical maps are the product maps. The commutativity follows from the projection formula in $K$-cohomology [Rost 1996, § 14.5].

Finally, it follows from the definition that the isomorphism $T(L)/R \xrightarrow{\sim} S^\circ(L)/R$ of functors in (15) commutes with specializations. □

## 4. Main theorem

Let $T$ be a torus over $F$ and $1 \to S \to P \to T \to 1$ a flasque resolution.

**4A.** ***The group $T(F(T))/R$.*** Tensoring the exact sequence

$$0 \to F_{\mathrm{sep}}^\times \oplus T^* \to F_{\mathrm{sep}}(T)^\times \to \mathrm{Div}(T_{\mathrm{sep}}) \to 0$$

with $S_*$ and applying Galois cohomology yields a surjective homomorphism

$$H^1(F, S) \oplus H^1(F, S_* \otimes T^*) \to H^1(F(T), S)$$

since $H^1\big(F, S_* \otimes \mathrm{Div}(T_{\mathrm{sep}})\big) = 0$ as $S$ is flasque.

Tensoring (4) with $S_*$ yields a surjective homomorphism

$$\mathrm{End}_F(S) = H^0(F, S_* \otimes S^*) \to H^1(F, S_* \otimes T^*)$$

as $H^1(F, S_* \otimes P^*) = 0$. Combining these two surjections we get another surjective homomorphism

$$(T(F)/R) \oplus \mathrm{End}_F(S) \to T(F(T))/R.$$

Note that the group $T(L)/R = H^1(L, S)$ is a left module over the ring $\mathrm{End}_F(S)$ for any field extension $L/F$. The image of an element $\alpha \in \mathrm{End}_F(S)$ in $T(F(T))/R$ is equal to $\alpha(\xi)$ (up to sign), where $\xi$ is the generic point of $T$.

We have proven:

**Proposition 4.1.** *Every element of the group $T(F(T))/R$ is of the form $t \cdot \alpha(\xi)$ where $t \in T(F)/R$ and $\alpha \in \mathrm{End}_F(S)$.*

Now assume that $\dim T = 3$ and $X$ is a smooth proper toric model of $T$.

**Corollary 4.2.** *There is an $\alpha \in \mathrm{End}_F(S)$ such that the composition $\rho^{-1} \circ \varphi$ takes every $t \in T(L)/R$ over a field extension $L/F$ to $\alpha(t)$.*

*Proof.* By Propositions 3.10, 3.11 and 3.12, it is sufficient to prove the statement in the case when $t$ is the generic point $\xi$ of $T$. By Proposition 4.1, $(\rho^{-1} \circ \varphi)(\xi) = t \cdot \alpha(\xi)$ for some $\alpha \in \mathrm{End}_F(S)$ and $t \in T(F)/R$. As $(\rho^{-1} \circ \varphi)(1) = 1$, specializing at 1, we get $t = 1$. □

Example 3.3 then yields:

**Corollary 4.3.** *The composition $\rho^{-1} \circ \varphi$ commutes with norms.*

### 4B. *Main theorem.*

**Theorem 4.4.** *Let $T$ be an algebraic torus of dimension 3 and $X$ a smooth proper geometrically irreducible variety over $F$ containing $T$ as an open subset. Then the map $\varphi : T(F)/R \to A_0(X)$ is an isomorphism.*

*Proof.* In view of Proposition 2.9, we may assume that $X$ is a smooth proper toric model of $T$. By Proposition 3.6 and Corollary 4.3, $\varphi$ commutes with norms. It follows from Proposition 3.5 that $\varphi$ is an isomorphism. □

**Remark 4.5.** The following is an alternative proof of Theorem 4.4. It avoids the machinery of Section 3, but it is based on deep, albeit classical, arithmetic-geometric result. We may assume that the field $F$ is finitely generated over the prime subfield. By [Colliot-Thélène and Sansuc 1977, Th. 1], the group $T(F)/R$ is finite. It follows from (15) that $A_0(X)$ is also finite of the same order. As $\varphi$ is injective, it is a bijection. Therefore, $\varphi$ is an isomorphism of groups as we have a homomorphism of groups $\psi$ with $\psi \circ \varphi = \mathrm{id}$.

The statement of the following theorem (but not the proof) does not involve a toric model.

**Theorem 4.6.** *Let $T$ be an algebraic torus of dimension 3. Then there is a natural isomorphism $T(F)/R \simeq H^1(F, T^\circ)/R$.*

*Proof.* The sequence dual to (5)

$$1 \to T^\circ \to P^\circ \to S^\circ \to 1$$

and [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3) yield an isomorphism

$$S^\circ(F)/R \simeq H^1(F, T^\circ)/R.$$

On the other hand, by (8), $S^\circ(F)/R \simeq H^1(F, S) \simeq T(F)/R$. □

In the following examples we give two applications of Theorem 4.6.

**Example 4.7.** Let $L/F$ be a degree 4 separable field extension and $T$ the norm 1 torus for $L/F$, that is,

$$T = \mathrm{Ker}\big(R_{L/F}(\mathbb{G}_{m,L}) \xrightarrow{N_{L/F}} \mathbb{G}_m\big).$$

Then $T^\circ = R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$ and

$$H^1(F, T^\circ) = \mathrm{Br}(L/F),$$

the relative Brauer group of the extension $L/F$. Thus by Theorem 4.6, we have a canonical isomorphism

$$\mathrm{Br}(L/F)/R \simeq T(F)/R.$$

The case of a biquadratic extension $L/F$ was considered in [Tignol 1981, p. 427].

**Example 4.8.** Let $L$ and $K$ be finite separable field extensions of a field $F$ and set $M := K \otimes_F L$. Let $T$ be the kernel of the norm homomorphism

$$N_{M/L} : R_{M/F}(\mathbb{G}_{m,M})/R_{K/F}(\mathbb{G}_{m,K}) \to R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m.$$

We have

$$T(F) = \{x \in M^\times \quad \text{such that} \quad N_{M/L}(x) \in F^\times\}/K^\times.$$

The dual torus $T^\circ$ is the kernel of the norm homomorphism

$$N_{M/K} : R_{M/F}(\mathbb{G}_{m,M})/R_{L/F}(\mathbb{G}_{m,L}) \to R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_m.$$

We have an exact sequence

$$K^\times \to H^1(F, T^\circ) \to \mathrm{Br}(M/L) \to \mathrm{Br}(K/F).$$

Now suppose that $[K : F] = 2$ and $[L : F] = 4$. Then $T$ is a 3-dimensional torus and the last homomorphism in the exact sequence is isomorphic to the norm map

$$N_{L/F} : L^\times/N_{M/L}(M^\times) \to F^\times/N_{K/F}(K^\times).$$

Let $U$ be the subtorus of $R_{L/F}(\mathbb{G}_{m,L}) \times R_{K/F}(\mathbb{G}_{m,K})$ consisting of all pairs $(l, k)$ with $N_{L/F}(l) = N_{K/F}(k)$. It follows that

$$T(F)/R \simeq H^1(F, T^\circ)/R \simeq U(F)/R.$$

This isomorphism was known when $L/F$ is a biquadratic extension (see [Shapiro et al. 1982, Cor. 1.13] and [Gille 1997, Prop. 3]).

## 5. Chow group of a 3-dimensional torus

Let $T$ be an algebraic torus over a field $F$ and $X$ a smooth proper geometrically irreducible variety containing $T$ as an open subset. Set $Z = X \setminus T$.

**Lemma 5.1.** (see [Colliot-Thélène and Sansuc 1977, Lemme 12], [Voskresenskiĭ 1998, Prop. 17.3] and [Gille 2004, Prop. 1.1]) *The torus $T$ is isotropic if and only if $Z(F) \neq \varnothing$.*

*Proof.* Suppose $T$ is isotropic. Then $T$ contains a subgroup isomorphic to $\mathbb{G}_m$. The embedding of $\mathbb{G}_m$ into $T$ extends to a regular morphism $f : \mathbb{P}^1 \to X$. Then $f(0)$ or $f(\infty)$ is a rational point of $Z$.

Conversely, suppose $Z$ has a rational point $z$. Since $z$ is regular on $X$, there is a geometric valuation $v$ of $F(X)$ dominating $z$ with residue field $F = F(z)$. Suppose that $T$ is anisotropic. Then there is a proper geometrically irreducible variety $X'$ containing $T$ as an open subset such that $X' \setminus T$ has no rational points (see [Colliot-Thélène and Sansuc 1977, Lemme 12], [Voskresenskiĭ 1998, Prop. 17.3]). But $v$ dominates a rational point on $X' \setminus T$, a contradiction. □

Write $i_T$ (respectively $n_Z$) for the greatest common divisor of the integers $[L : F]$ for all finite field extensions $L/F$ such that $T$ is isotropic over $L$ (respectively $Z(L) \neq \varnothing$).

**Corollary 5.2.** *The number $i_T$ coincides with $n_Z$. In particular, the integer $n_Z$ does not depend on the smooth proper geometrically irreducible variety $X$ containing $T$ as an open subset.*

**Proposition 5.3.** *The order of the class* $[1]$ *in* $\mathrm{CH}_0(T)$ *is equal to* $i_T$.

*Proof.* If $T$ is isotropic, there is a subgroup $H$ of $T$ isomorphic to $\mathbb{G}_m$. As $\mathrm{CH}_0(\mathbb{G}_m) = 0$, we have $[1] = 0$ in $\mathrm{CH}_0(H)$ and therefore in $\mathrm{CH}_0(T)$. In the general case, let $L$ be a finite field extension such that $T_L$ is isotropic. By the first part of the proof, $[1]$ is trivial in $\mathrm{CH}_0(T_L)$; hence applying the norm map for the extension $L/F$ yields $[L : F] \cdot [1] = 0$ in $\mathrm{CH}_0(T)$. Therefore, $i_T \cdot [1] = 0$.

Now let $m \cdot [1] = 0$ in $\mathrm{CH}_0(T)$ for some integer $m$. Hence the cycle $m \cdot [1]$ in $\mathrm{CH}_0(X)$ belongs to the image of the push-forward map $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$ [Fulton 1984, Prop. 1.8]. In particular, there is a zero-cycle on $Z$ of degree $m$, hence $i_F = n_Z$ divides $m$. □

Consider the map

$$\alpha_T : T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$$

taking a pair $(t, k)$ to the cycle $[t] + (k - 1) \cdot [1]$.

**Theorem 5.4.** *Let $T$ be a torus of dimension at most* 3. *Then the map $\alpha_T$ : $T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$ is an isomorphism.*

*Proof.* The Chow group $\mathrm{CH}_0(T)$ is the factor group of $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z} \cdot [1]$ by the image of $\mathrm{CH}_0(Z)$. Let $z \in Z$ be a closed point. By Lemma 5.1, the torus $T_{F(z)}$ is isotropic and hence is stably birational to a 2-dimensional torus. Therefore, $T_{F(z)}$ is rational, $A_0(X_{F(z)}) = 0$ and the image of the class of $z$ in $A_0(X) \oplus \mathbb{Z} \cdot [1]$ is equal to $0 \oplus \deg(z) \cdot [1]$. Hence $\mathrm{CH}_0(T)$ is isomorphic to $A_0(X) \oplus \mathbb{Z}/i_T\mathbb{Z}$. The result follows from Theorem 4.4. □

## Acknowledgments

I would like to thank J.-L. Colliot-Thélène for useful discussions.

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Communicated by Raman Parimala
Received 2007-06-28 Revised 2007-10-23 Accepted 2007-11-20

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