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Complexes of injective kG-modules

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Let G be a finite group and k be a field of characteristic p. We investigate the homotopy category $\mathsf{K}(\mathsf{Inj}kG)$ of the category $\mathsf{C}(\mathsf{Inj}kG)$ of complexes of injective (= projective) kG-modules. If G is a p-group, this category is equivalent to the derived category $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ of the cochains on the classifying space; if G is not a p-group, it has better properties than this derived category. The ordinary tensor product in $\mathsf{K}(\mathsf{Inj}kG)$ with diagonal G-action corresponds to the E_∞ tensor product on $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$.

We show that $\mathsf{K}(\mathsf{Inj}kG)$ can be regarded as a slight enlargement of the stable module category $\mathsf{StMod}kG$. It has better formal properties inasmuch as the ordinary cohomology ring $H^*(G,k)$ is better behaved than the Tate cohomology ring $\hat{H}^*(G,k)$.

It is also better than the derived category $\mathsf{D}(\mathsf{Mod}kG)$, because the compact objects in $\mathsf{K}(\mathsf{Inj}kG)$ form a copy of the bounded derived category $\mathsf{D}^b(\mathsf{mod}kG)$, whereas the compact objects in $\mathsf{D}(\mathsf{Mod}kG)$ consist of just the perfect complexes.

Finally, we develop the theory of support varieties and homotopy colimits in $\mathsf{K}(\mathsf{Inj}kG)$.

1. Introduction

Let k be a field and G a finite group. The purpose of this paper is to develop the properties of $\mathsf{K}(\mathsf{Inj}\,kG)$, the homotopy category of complexes of injective kG-modules.

For any ring Λ, we write $\mathsf{C}(\mathsf{Inj}\,\Lambda)$ for the category whose objects are the chain complexes of injective Λ-modules and whose arrows are the degree zero morphisms of chain complexes. We write $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ for the category with the same objects, but where the maps are the homotopy classes of degree zero maps of chain complexes. We write $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,\Lambda)$ for the full subcategory whose objects are the acyclic chain complexes of injective Λ-modules.

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We investigate a recollement relating $\mathsf{K}(\mathsf{Inj}\, kG)$ to the stable module category $\mathsf{StMod}\, kG$ and the derived category $\mathsf{D}(\mathsf{Mod}\, kG)$:

$$\mathsf{StMod}\, kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \;\begin{smallmatrix} \xleftarrow{\mathrm{Hom}_k(tk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k tk]{} \end{smallmatrix}\; \mathsf{K}(\mathsf{Inj}\, kG) \;\begin{smallmatrix} \xleftarrow{\mathrm{Hom}_k(pk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k pk]{} \end{smallmatrix}\; \mathsf{D}(\mathsf{Mod}\, kG).$$

For notation, we write pk, ik and tk for a projective resolution, injective resolution and Tate resolution of k as a kG-module respectively. The compact objects in these categories are

$$\mathsf{stmod}\, kG \longleftarrow \mathsf{D}^b(\mathsf{mod}\ kG) \longleftarrow \mathsf{D}^b(\mathsf{proj}\, kG).$$

This means that $\mathsf{K}(\mathsf{Inj}\, kG)$ can be regarded as the appropriate “big” category for $\mathsf{D}^b(\mathsf{mod}\ kG)$, whereas $\mathsf{D}(\mathsf{Mod}\, kG)$ has too few compact objects for this purpose. In this sense, $\mathsf{K}(\mathsf{Inj}\, kG)$ is a nicer category to work in than $\mathsf{D}(\mathsf{Mod}\, kG)$.

From the point of view of algebraic topology, what $\mathsf{K}(\mathsf{Inj}\, kG)$ does for us is provide an algebraic replacement for the derived category of the differential graded algebra of singular cochains on the classifying space, $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$. Namely, if G is a p-group there is an equivalence of categories

$$\mathsf{K}(\mathsf{Inj}\, kG) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

We prove that the tensor product over k of complexes in $\mathsf{K}(\mathsf{Inj}\, kG)$ corresponds under this equivalence to the left derived tensor product over $C^*(BG; k)$ coming from the fact that the latter is E_∞, or “commutative up to all higher homotopies” (see Theorem 7.8 and the remarks after Theorem 4.1).

If G is not a p-group, then there is more than one simple kG-module, and the only one $C^*(BG; k)$ “sees” is the trivial kG-module. In this sense, $\mathsf{K}(\mathsf{Inj}\, kG)$ is nicer to work in than $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$, even though it is not necessarily equivalent to it. Writing ik for an injective resolution of the trivial module, what we obtain in general is an equivalence between $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$ and the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by ik.

In the work of Dwyer, Greenlees and Iyengar [2006], a close relationship was established between $\mathsf{D}(\mathsf{Mod}\, kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$. For a general finite group, the relationship between $\mathsf{K}(\mathsf{Inj}\, kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$ is much closer, and provides some sort of context for understanding what is going on in [Dwyer et al. 2006]. Traces of arguments from that paper can be seen from time to time in this paper.

We develop the theory of support varieties for objects in $\mathsf{K}(\mathsf{Inj}\, kG)$, extending the theory developed by Benson, Carlson and Rickard [1996]. The extra information not included in $\mathsf{StMod}\, kG$ is reflected in the fact that the maximal ideal $\mathfrak{m}$ of positive degree elements in $H^*(G, k)$ becomes relevant in the variety theory. Thus

$\mathsf{K}(\mathsf{Inj}\,kG)$ can be regarded as a slight enlargement of $\mathsf{StMod}(kG)$ in which one more prime ideal $\mathfrak{m}$ of the cohomology ring is reflected. We also construct objects with injective cohomology, extending the work of Benson and Krause [2002]; the theory in $\mathsf{K}(\mathsf{Inj}\,kG)$ is easier than in $\mathsf{StMod}\,kG$ because one does not have to compare ordinary and Tate cohomology.

Homotopy colimits in $\mathsf{K}(\mathsf{Inj}\,kG)$ are harder to deal with than in $\mathsf{StMod}\,kG$ or than in $\mathsf{D}(\mathsf{Mod}\,kG)$, so we conclude with a section describing how the theory works in this case. The main theorem here is that localizing subcategories of $\mathsf{K}(\mathsf{Inj}\,kG)$ are closed under filtered colimits in $\mathsf{C}(\mathsf{Inj}\,kG)$, in spite of the fact that the compact objects in $\mathsf{K}(\mathsf{Inj}\,kG)$ do not lift to compact objects in $\mathsf{C}(\mathsf{Inj}\,kG)$.

2. $\mathsf{K}(\mathsf{Inj}\,kG)$ is compactly generated

Let Λ be a Noetherian ring. We consider the category $\mathsf{Mod}\,\Lambda$ of Λ-modules and denote by $\mathsf{mod}\,\Lambda$ the full subcategory which is formed by all finitely generated modules. The injective Λ-modules form a subcategory $\mathsf{Inj}\,\Lambda$ of $\mathsf{Mod}\,\Lambda$ that is closed under taking arbitrary coproducts. This implies that $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is a triangulated category which admits arbitrary coproducts.

We need to recall some definitions. Let T be a triangulated category with arbitrary coproducts. An object X of T is called *compact* if the functor $\mathrm{Hom}_{\mathsf{T}}(X,-)$ into the category of abelian groups preserves all coproducts. We denote by T^c the full subcategory which is formed by all compact objects of T and observe that T^c is a thick subcategory. The triangulated category T is *compactly generated* if the isomorphism classes of objects of T^c form a set and if T coincides with its smallest triangulated subcategory containing T^c and closed under all coproducts.

Well known examples of compactly generated triangulated categories include the stable module category $\mathsf{StMod}\,\Lambda$ provided that Λ is self-injective, and the derived category $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ for any ring Λ. For references, see [Happel 1988] and [Verdier 1996]. Note that the inclusion functors $\mathsf{stmod}\,\Lambda \to \mathsf{StMod}\,\Lambda$ and $\mathsf{proj}\,\Lambda \to \mathsf{Mod}\,\Lambda$ induce equivalences

$$\mathsf{stmod}\,\Lambda \xrightarrow{\sim} (\mathsf{StMod}\,\Lambda)^c \quad \text{and} \quad \mathsf{D}^b(\mathsf{proj}\,\Lambda) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\,\Lambda)^c.$$

Proposition 2.1. *The triangulated category* $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ *is compactly generated. Let* $\mathsf{K}^c(\mathsf{Inj}\,\Lambda)$ *denote the full subcategory which is formed by all compact objects. Then the canonical functor* $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Lambda)$ *induces an equivalence*

$$\mathsf{K}^c(\mathsf{Inj}\,\Lambda) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\,\Lambda).$$

Proof. See [Krause 2005, Proposition 2.3]. □

Remark 2.2. The canonical functor $Q\colon \mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Lambda)$ has a right adjoint sending a complex X to its semiinjective resolution iX (the definition can be found just before Corollary 6.2). This right adjoint induces an equivalence

$$\mathsf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\simeq} \mathsf{K}^c(\operatorname{Inj}\Lambda)$$

which is a quasiinverse for the equivalence $\mathsf{K}^c(\operatorname{Inj}\Lambda) \to \mathsf{D}^b(\operatorname{mod}\Lambda)$ induced by Q. For details of this construction we refer to Section 6.

3. $\mathsf{K}(\operatorname{Inj} kG)$ is a derived category

Given two chain complexes X and Y in $\operatorname{Mod}\Lambda$, we define the chain complex $\operatorname{Hom}_\Lambda(X, Y)$. The n-th component is

$$\prod_{p\in\mathbb{Z}} \operatorname{Hom}_\Lambda(X_p, Y_{n+p})$$

and the differential is defined so that

$$(d(f))(x) = d(f(x)) - (-1)^{|f|} f(d(x)).$$

Note that

$$H_n \operatorname{Hom}_\Lambda(X, Y) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{Mod}\Lambda)}(X, Y[n]).$$

Composition of maps gives

$$\operatorname{End}_\Lambda(X) = \operatorname{Hom}_\Lambda(X, X)$$

the structure of a differential graded algebra (DG algebra), over which $\operatorname{Hom}_\Lambda(X, Y)$ is a differential graded module (DG module).

Given a DG algebra Γ, we denote by $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ the derived category of DG Γ-modules. The objects in this category are DG Γ-modules. The arrows are homotopy classes of degree zero morphisms of DG modules, with the quasiisomorphisms (maps that induce an isomorphism on homology) inverted. So for example if Γ is a ring, regarded as a DG algebra concentrated in degree zero with zero differential, then a DG Γ-module is a complex of modules, and we recover the usual definition of the derived category of a ring. See [Keller 1994] for further details.

Proposition 3.1. *Let C be an object of $\mathsf{K}^c(\operatorname{Inj}\Lambda) \simeq \mathsf{D}^b(\operatorname{mod}\Lambda)$ and let $\Gamma = \operatorname{End}_\Lambda C$. Denote by $\mathbb{C}$ the smallest full triangulated subcategory of $\mathsf{K}(\operatorname{Inj}\Lambda)$ closed under all coproducts and containing C. Then the functor*

$$\operatorname{Hom}_\Lambda(C, -)\colon\ \mathsf{K}(\operatorname{Inj}\Lambda) \longrightarrow \mathsf{D}_{\mathrm{dg}}(\Gamma)$$

induces an equivalence $\mathbb{C} \xrightarrow{\simeq} \mathsf{D}_{\mathrm{dg}}(\Gamma)$ of triangulated categories.

Proof. We begin by defining $\operatorname{Hom}_\Lambda(C,-)$ as a functor from $\mathsf{C}(\operatorname{Inj}\Lambda)$ to the category of differential graded Γ-modules. This functor is exact, and the composite to $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ takes homotopic maps to the same place. So we obtain a well defined exact functor from $\mathsf{K}(\operatorname{Inj}\Lambda)$ to $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ (compare [Keller 1994, §4.3, bottom of p. 77]). To see that it preserves coproducts, fix a family of objects X_i in $\mathsf{K}(\operatorname{Inj}\Lambda)$. Then we have for every $n \in \mathbb{Z}$

$$\begin{aligned} H_n \coprod_i \operatorname{Hom}_\Lambda(C, X_i) \cong \coprod_i H_n \operatorname{Hom}_\Lambda(C, X_i) \cong \coprod_i \operatorname{Hom}_{\mathsf{K}(\operatorname{Inj}\Lambda)}(C, X_i[n]) \\ \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{Inj}\Lambda)}(C, \coprod_i X_i[n]) \cong H_n \operatorname{Hom}_\Lambda(C, \coprod_i X_i[n]) \end{aligned}$$

since C is compact in $\mathsf{K}(\operatorname{Inj}\Lambda)$. Thus the canonical map

$$\coprod_i \operatorname{Hom}_\Lambda(C, X_i) \longrightarrow \operatorname{Hom}_\Lambda(C, \coprod_i X_i)$$

is an isomorphism. Furthermore, the functor induces bijections

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{Inj}\Lambda)}(C, C[n]) \cong H_n \operatorname{Hom}_\Lambda(C, C) \cong \operatorname{Hom}_{\mathsf{D}_{\mathrm{dg}}(\Gamma)}(\Gamma, \Gamma[n]).$$

Thus the class D of objects in $\mathsf{K}(\operatorname{Inj}\Lambda)$ such that the induced map

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{Inj}\Lambda)}(X, Y) \longrightarrow \operatorname{Hom}_{\mathsf{D}_{\mathrm{dg}}(\Gamma)}(\operatorname{Hom}_\Lambda(C, X), \operatorname{Hom}_\Lambda(C, Y))$$

is bijective for all X, Y in D contains $\mathbb{C}$. The functor $\operatorname{Hom}_\Lambda(C,-)$ is, up to isomorphism, surjective on objects since the image contains the free Γ-module Γ which generates $\mathsf{D}_{\mathrm{dg}}(\Gamma)$. □

Remark 3.2. (1) The functor $\operatorname{Hom}_\Lambda(C,-)$ admits left and right adjoints. This is a consequence of Brown representability (see [Neeman 2001]) because the functor preserves (co)products. Thus $\operatorname{Hom}_\Lambda(C,-)$ induces a recollement of the form

$$\operatorname{Ker}\operatorname{Hom}_\Lambda(C,-) \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow}\; \mathsf{K}(\operatorname{Inj}\Lambda) \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow}\; \mathsf{D}_{\mathrm{dg}}(\Gamma).$$

Here, $\operatorname{Ker}\operatorname{Hom}_\Lambda(C,-)$ denotes the full subcategory of $\mathsf{K}(\operatorname{Inj}\Lambda)$ formed by all objects X with $\operatorname{Hom}_\Lambda(C,X) = 0$. The functors between $\operatorname{Ker}\operatorname{Hom}_\Lambda(C,-)$ and $\mathsf{K}(\operatorname{Inj}\Lambda)$ are the inclusion together with its left and right adjoints.

(2) If the object C generates $\mathsf{K}^c(\operatorname{Inj}\Lambda)$, that is, there is no proper thick subcategory containing C, then $\mathbb{C} = \mathsf{K}(\operatorname{Inj}\Lambda)$ and the functor $\operatorname{Hom}_\Lambda(C,-)$ is an equivalence.

In the case where G is a finite p-group, one choice for the compact generator C of Proposition 3.1 is ik, an injective resolution of k. For a more general finite group, we may take the sum of the injective resolutions of the simple modules. We write $\mathscr{E}_G$ for the differential graded algebra $\operatorname{End}_{kG}(ik)$ whether or not G is a p-group.

4. The Rothenberg–Steenrod construction

We now relate the category $\mathsf{K}(\mathsf{Inj}\, kG)$ to the classifying space BG. For general background references on classifying spaces of groups, see for example [Benson 1991; Brown 1982]. The basic link between $\mathsf{K}(\mathsf{Inj}\, kG)$ and the derived category of $C^*(BG; k)$ is achieved through the Rothenberg–Steenrod construction [Rothenberg and Steenrod 1965], which we now make precise. For any path-connected space X, this construction gives a quasiisomorphism of differential graded algebras from the derived endomorphisms of k over the chains on the loop space and the cochains on X:

$$\mathbb{R}\mathrm{End}_{C_*(\Omega X;k)}(k) \simeq C^*(X; k).$$

In the case where X is the classifying space BG, ΩX is equivalent to G, and $C_*(\Omega X; k)$ is equivalent as a differential graded algebra to the group algebra kG in degree zero. So in this case the left hand side is just $\mathscr{E}_G = \mathrm{End}_{kG}(ik)$, and therefore we obtain

$$\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

The purpose of this section is to investigate this equivalence algebraically.

We begin by remarking that $\mathrm{End}_{kG}(pk)$ and $\mathrm{End}_{kG}(ik)$ are quasiisomorphic differential graded algebras. To see this, choose a quasiisomorphism $pk \to ik$. Then we have quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \to \mathrm{Hom}_{kG}(pk, ik) \leftarrow \mathrm{End}_{kG}(ik).$$

The middle object is not a differential graded algebra, but the pullback of this pair of maps

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{End}_{kG}(ik) \arrow[d] \\ \mathrm{End}_{kG}(pk) \arrow[r] & \mathrm{Hom}_{kG}(pk, ik) \end{tikzcd}$$

is a differential graded algebra

$$X = \mathrm{End}_{kG}(pk) \times_{\mathrm{Hom}_{kG}(pk,ik)} \mathrm{End}_{kG}(ik)$$

that comes with quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \overset{\simeq}{\leftarrow} X \overset{\simeq}{\rightarrow} \mathrm{End}_{kG}(ik).$$

Thus we obtain equivalences of derived categories

$$\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(X) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik)).$$

Similarly, if $p'k$ is another projective resolution there is a comparison map $pk \to p'k$, and hence there are homomorphisms

$$\mathrm{End}_{kG}(pk) \to \mathrm{Hom}_{kG}(pk, p'k) \leftarrow \mathrm{End}_{kG}(p'k).$$

The pullback of this pair of maps is a differential graded algebra

$$Y = \mathrm{End}_{kG}(pk) \times_{\mathrm{Hom}_{kG}(pk,p'k)} \mathrm{End}_{kG}(p'k)$$

that comes with quasiisomorphisms

$$\mathrm{End}_{kG}(pk) \xleftarrow{\simeq} Y \xrightarrow{\simeq} \mathrm{End}_{kG}(p'k).$$

Thus we obtain equivalences of derived categories

$$\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(Y) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(p'k)).$$

It follows that $\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk))$ is, up to natural equivalence, independent of choice of projective resolution, and is also equivalent to $\mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik))$.

The augmentation map $\varepsilon\colon pk \to k$ gives a quasiisomorphism of complexes

$$\mathrm{End}_{kG}(pk) \simeq \mathrm{Hom}_{kG}(pk, k).$$

Suppose that pk is a resolution supporting a strictly coassociative and counital diagonal $\Delta\colon pk \to pk \otimes_k pk$, meaning that the following diagrams commute:

$$\begin{tikzcd} pk \arrow[r, "\Delta"] \arrow[d, "\Delta"] & pk \otimes_k pk \arrow[d, "1\otimes\Delta"] \\ pk \otimes_k pk \arrow[r, "\Delta\otimes 1"] & pk \otimes_k pk \otimes_k pk \end{tikzcd}, \qquad \begin{tikzcd} & pk \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ pk \otimes_k k & pk \otimes_k pk \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & k \otimes_k pk \end{tikzcd}.$$

This happens, for example, when pk is the bar resolution, and when pk is equal to the singular cochains on EG. Then there is a multiplication on $\mathrm{Hom}_{kG}(pk, k)$ given as follows. If $\alpha, \beta\colon pk \to k$ then $\alpha.\beta$ is given by the composite

$$pk \xrightarrow{\Delta} pk \otimes_k pk \xrightarrow{\alpha\otimes\beta} k \otimes_k k \xrightarrow{\cong} k.$$

The fact that Δ is coassociative and counital implies that this multiplication is associative and unital.

We claim that there is a quasiisomorphism of differential graded algebras

$$\mathrm{Hom}_{kG}(pk, k) \to \mathrm{End}_{kG}(pk)$$

given by sending $\alpha\colon pk \to k$ to the map $\tilde{\alpha}\colon pk \to pk$ given by the composite $pk \xrightarrow{\Delta} pk \otimes_k pk \xrightarrow{1\otimes\alpha} pk \otimes_k k \xrightarrow{\cong} pk$. Since Δ is counital, we have $\varepsilon \circ \tilde{\alpha} = \alpha$, so that $\alpha \mapsto \tilde{\alpha}$ is a quasiisomorphism. The commutative diagram

$$\begin{tikzcd}
pk \arrow[r, "\tilde{\alpha}\circ\tilde{\beta}"] \arrow[d, equal] & pk \arrow[r, "\varepsilon"] & k \arrow[d, equal] \\
pk \arrow[r, "\tilde{\beta}"] \arrow[d, "\Delta"'] & pk \arrow[r, "\alpha"] \arrow[u, "\tilde{\alpha}"'] & k \\
pk \otimes_k pk \arrow[r, "1\otimes\beta"] \arrow[rr, bend right, "\alpha\otimes\beta"'] & pk \otimes_k k \arrow[r, "\alpha\otimes 1"] \arrow[u, "\cong"'] & k \otimes_k k \arrow[u, "\cong"']
\end{tikzcd}$$

shows that the map $\alpha \mapsto \tilde{\alpha}$ preserves multiplication.

Using this, we see that we have quasiisomorphisms of differential graded algebras

$$\begin{aligned}
\mathrm{End}_{kG}(pk) \simeq \mathrm{Hom}_{kG}(pk, k) &\simeq \mathrm{Hom}_{kG}(C_*(EG; k), k) \\
&\cong \mathrm{Hom}_k(C_*(BG; k), k) \cong C^*(BG; k).
\end{aligned}$$

Now suppose that H is a subgroup of G. Then EG can be used as a model for EH. In particular, $C_*(EG; k)$ is another model of pk in $\mathsf{K}(\mathsf{Inj}\, kH)$ with a strictly coassociative and counital diagonal map. Restricting resolutions for G to the subgroup H gives us resolutions for H, so we have a restriction map of differential graded algebras $\mathrm{res}_{G,H} \colon \mathscr{E}_G \to \mathscr{E}_H$. We also have a restriction map $\mathrm{res}_{G,H} \colon C^*(BG; k) \to C^*(BH; k)$. Naturality of the Rothenberg–Steenrod construction gives us the following theorem.

Theorem 4.1. *There are equivalences of categories*

$$\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) = \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(ik)) \simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kG}(pk)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

The equivalence $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$ is natural, in the sense that if H is a subgroup of G then the square

$$\begin{tikzcd}
\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}^*_{G,H}"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH; k)) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))
\end{tikzcd}$$

commutes up to natural isomorphism. □

Next, we discuss the tensor product $- \otimes_{C^*(BG;k)} -$ on $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$. It is convenient at this stage to be able to pass back and forth between differential graded algebras and S-algebras (S here is the sphere spectrum). The point of this formalism is to have a category of spectra with a smash product that is commutative and associative up to coherent natural isomorphism, and not just up to homotopy. In the 1990s, several sets of authors produced such categories. We will work with the formalism of S-algebras introduced by Elmendorf, Kříž, Mandell and May [1997].

We make use of [Shipley 2007] to translate between the language of S-algebras and the language of differential graded algebras. Shipley shows that if R is a discrete commutative ring, with associated Eilenberg–Mac Lane spectrum HR, then the model categories of differential graded R-algebras and S-algebras over HR are Quillen equivalent. In particular, their homotopy categories are equivalent as triangulated categories. It would be possible to work directly in the category of E_∞ differential graded algebras, but we would need to be working over an E_∞ operad such as the surjection operad of McClure and Smith [2003] and then transfer to an E_∞ operad satisfying the Hopkins lemma in [Elmendorf et al. 1997]. Alternatively, one could work directly with the formalism of Hovey, Shipley and Smith [Hovey et al. 2000] and use the algebraic analogue of symmetric spectra. Further comments on the relationships between E_∞ algebras and singular cochains on spaces can be found in [Mandell 2001].

In any case, the upshot of the discussion is that if X and Y are objects in $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ then so is the left derived tensor product $X \otimes_{C^*(BG;k)} Y$. This tensor product is symmetric monoidal, so there are coherent natural isomorphisms

$$X \otimes_{C^*(BG;k)} Y \cong Y \otimes_{C^*(BG;k)} X,$$
$$(X \otimes_{C^*(BG;k)} Y) \otimes_{C^*(BG;k)} Z \cong X \otimes_{C^*(BG;k)} (Y \otimes_{C^*(BG;k)} Z).$$

In the case where G is a p-group, we can compare $\mathsf{D}_{\mathrm{dg}}(\Gamma)$ with $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ as in the following theorem. If G is not a p-group, then $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ is not equivalent to the whole of $\mathsf{K}(\mathsf{Inj}\,kG)$, but just the part generated by ik. This is because there is more than one simple kG-module, and $C^*(BG;k)$ only "sees" what is generated by the trivial module; in particular, nonprincipal blocks of kG are invisible to $C^*(BG;k)$.

Theorem 4.2. *Let G be a finite group. Then we have functors*

$$\mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\mathrm{Hom}_{kG}(ik,-)} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$$

whose composite we denote by Φ. If G is a finite p-group, this gives an equivalence of categories

$$\Phi\colon \mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\simeq} \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)).$$

Proof. This is proved by combining Proposition 3.1 and Theorem 4.1. As remarked above, in the case of a p-group, we can take ik as a generator for $\mathsf{K}^c(\mathsf{Inj}\,kG)$, so that the differential graded algebra Γ of Proposition 3.1 is equal to $\mathscr{E}_G$. $\square$

Remark 4.3. An explicit right adjoint $\Psi\colon \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)) \to \mathsf{K}(\mathsf{Inj}\,kG)$ to Φ is described just before Lemma 7.4; see also Remark 3.2. The functor Ψ satisfies $\Phi\circ\Psi \simeq \mathrm{Id}_{\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))}$.

5. $\mathsf{K}(\mathsf{Inj}\,kG)$ is a tensor category

If G_1 and G_2 are groups then there is a natural isomorphism of group algebras $k(G_1\times G_2)\cong kG_1\otimes_k kG_2$. Taking the tensor product of complexes gives an *external tensor product*

$$\mathsf{C}(\mathsf{Mod}\,kG_1)\times\mathsf{C}(\mathsf{Mod}\,kG_2)\to\mathsf{C}(\mathsf{Mod}\,k(G_1\times G_2))$$

and hence also

$$\mathsf{K}(\mathsf{Mod}\,kG_1)\times\mathsf{K}(\mathsf{Mod}\,kG_2)\to\mathsf{K}(\mathsf{Mod}\,k(G_1\times G_2)).$$

If $G=G_1=G_2$, then restricting the external tensor product via the diagonal embedding of G in $G\times G$ defines an *internal tensor product*

$$\mathsf{C}(\mathsf{Mod}\,kG)\times\mathsf{C}(\mathsf{Mod}\,kG)\to\mathsf{C}(\mathsf{Mod}\,kG)$$

which induces

$$\mathsf{K}(\mathsf{Mod}\,kG)\times\mathsf{K}(\mathsf{Mod}\,kG)\to\mathsf{K}(\mathsf{Mod}\,kG).$$

Similar arguments show that $\mathrm{Hom}_k(-,-)$ induces internal products on the categories $\mathsf{C}(\mathsf{Mod}\,kG)$ and $\mathsf{K}(\mathsf{Mod}\,kG)$. Note that we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,kG)}(X\otimes_k Y, Z)\cong\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,kG)}(X,\mathrm{Hom}_k(Y,Z)) \tag{5.1}$$

for all X, Y, Z in $\mathsf{K}(\mathsf{Mod}\,kG)$.

The subcategories $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG)$ inherit tensor products from the category $\mathsf{K}(\mathsf{Mod}\,kG)$ because they are tensor ideals. This follows from the next lemma.

Lemma 5.2. *Let X, Y be complexes of kG-modules.*

(i) *If X is a complex of injective kG-modules, then $X\otimes_k Y$ and* $\mathrm{Hom}_k(X,Y)$ *are complexes of injective kG-modules.*

(ii) *If X is an acyclic complex, then $X\otimes_k Y$ and* $\mathrm{Hom}_k(X,Y)$ *are acyclic complexes.*

Proof. The first assertion is clear since $M\otimes_k N$ and $\mathrm{Hom}_k(M,N)$ are injective for any pair of kG-modules M, N provided that one of them is injective. The second assertion follows from the fact that the tensor product and Hom are computed over k. $\square$

Proposition 5.3. *The unit for the tensor product on* $\mathsf{K}(\mathsf{Inj}\, kG)$ *is the injective resolution* ik *of the trivial representation* k.

Proof. For any object X in $\mathsf{K}(\mathsf{Inj}\, kG)$, the map of complexes $k \to ik$ induces the following chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik \otimes_k X, -) &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik, \mathrm{Hom}_k(X, -)) \\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k, \mathrm{Hom}_k(X, -)) \\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k \otimes_k X, -).\end{aligned}$$

Here we use (5.1) and that $k \to ik$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(ik, Y) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, kG)}(k, Y)$$

for all Y in $\mathsf{K}(\mathsf{Inj}\, kG)$ by [Krause 2005, Lemma 2.1]. Thus the map of complexes $k \otimes_k X \to ik \otimes_k X$ is an isomorphism in $\mathsf{K}(\mathsf{Inj}\, kG)$. □

Definition 5.4. If X is an object in $\mathsf{K}(\mathsf{Inj}\, kG)$, we define

$$H^*(G, X) = \mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, X)$$

where the n-th component is $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, X[n])$. This is a graded module for the cohomology ring $H^*(G, k) = \mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\, kG)}(ik, ik)$.

6. A recollement for $\mathsf{K}(\mathsf{Inj}\, kG)$

Let Λ be a Noetherian ring. We have seen that $\mathsf{K}(\mathsf{Inj}\, \Lambda)$ is compactly generated and this fact has some interesting consequences. For instance, any exact functor $\mathsf{K}(\mathsf{Inj}\, \Lambda) \to \mathsf{T}$ into a triangulated category T admits a right adjoint if it preserves coproducts and a left adjoint if it preserves products. We apply this consequence of Brown representability (see [Neeman 2001]) to the canonical functor

$$Q \colon \mathsf{K}(\mathsf{Inj}\, \Lambda) \xrightarrow{\text{inc}} \mathsf{K}(\mathsf{Mod}\, \Lambda) \xrightarrow{\text{can}} \mathsf{D}(\mathsf{Mod}\, \Lambda)$$

and obtain the following result [Krause 2005, Corollary 4.3].

Proposition 6.1. *The pair of canonical functors*

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, \Lambda) \xrightarrow{I} \mathsf{K}(\mathsf{Inj}\, \Lambda) \xrightarrow{Q} \mathsf{D}(\mathsf{Mod}\, \Lambda)$$

induces a recollement

$$\begin{tikzcd} \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, \Lambda) \arrow[r, "I"] & \mathsf{K}(\mathsf{Inj}\, \Lambda) \arrow[l, shift right=2ex, "I_\rho"'] \arrow[l, shift left=2ex, "I_\lambda"] \arrow[r, "Q"] & \mathsf{D}(\mathsf{Mod}\, \Lambda). \arrow[l, shift right=2ex, "Q_\rho"'] \arrow[l, shift left=2ex, "Q_\lambda"] \end{tikzcd}$$

More precisely, the functors I and Q admit left adjoints I_λ and Q_λ as well as right adjoints I_ρ and Q_ρ such that the following adjunction morphisms

$$I_\lambda \circ I \xrightarrow{\cong} \mathrm{Id}_{\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,\Lambda)} \xrightarrow{\cong} I_\rho \circ I \quad \text{and} \quad Q \circ Q_\rho \xrightarrow{\cong} \mathrm{Id}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)} \xrightarrow{\cong} Q \circ Q_\lambda$$

are isomorphisms.

Recall from [Avramov et al. 2003] (see also [Spaltenstein 1988]) that for any differential graded algebra Γ, a DG Γ-module X is said to be *semiprojective* if $\mathrm{Hom}_\Gamma(X, -)$ carries surjective quasiisomorphisms to surjective quasiisomorphisms. Similarly, X is *semiinjective* if $\mathrm{Hom}_\Gamma(-, X)$ carries injective quasiisomorphisms to surjective quasiisomorphisms. A *semiprojective resolution* of a DG Γ-module X is a quasiisomorphism $pX \to X$ with pX semiprojective, and a *semiinjective resolution* of X is a quasiisomorphism $X \to iX$ with iX semiinjective. If Γ is a ring, these definitions are applied by regarding Γ as a DG algebra concentrated in degree zero, so that a DG module is just a complex of Γ-modules.

Note that the recollement provides two embeddings of $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ into $\mathsf{K}(\mathsf{Inj}\,\Lambda)$. The more familiar one is the fully faithful functor $Q_\rho\colon \mathsf{D}(\mathsf{Mod}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$ which sends a complex X of Λ-modules to a semiinjective resolution iX. The less familiar embedding is the fully faithful functor $Q_\lambda\colon \mathsf{D}(\mathsf{Mod}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$ which identifies $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ with the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ generated by $i\Lambda$. If Λ is self-injective, then Q_λ sends a complex X of Λ-modules to a semiprojective resolution pX.

We summarize this discussion as follows.

Corollary 6.2. *Let Λ be a Noetherian ring, and let X be a complex of injective Λ-modules. Then the following are equivalent.* (*i*) *X is semiinjective.* (*ii*) *$X \cong Q_\rho Y$ for some Y in* $\mathsf{D}(\mathsf{Mod}\,\Lambda)$. (*iii*) $I_\rho X \cong 0$.

If Λ is selfinjective (*so that projective and injective Λ-modules coincide*), *then the following are equivalent.* (*i*) *X is semiprojective.* (*ii*) *$X \cong Q_\lambda Y$ for some Y in* $\mathsf{D}(\mathsf{Mod}\,\Lambda)$. (*iii*) $I_\lambda X \cong 0$.

In the case where $\Lambda = kG$, we have $\mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG)$, and the adjoints in the recollement take the form

$$\mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG) \begin{smallmatrix} \xleftarrow{\mathrm{Hom}_k(tk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k tk]{} \end{smallmatrix} \mathsf{K}(\mathsf{Inj}\,kG) \begin{smallmatrix} \xleftarrow{\mathrm{Hom}_k(pk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k pk]{} \end{smallmatrix} \mathsf{D}(\mathsf{Mod}\,kG). \quad (6.3)$$

Here, we write ik for a semiinjective resolution, pk for a semiprojective resolution, and tk for a Tate resolution of the trivial kG-module k. Note that these resolutions fit into an exact triangle

$$pk \longrightarrow ik \longrightarrow tk \longrightarrow pk[1]$$

in $\mathsf{K}(\mathsf{Inj}\,kG)$. This triangle induces for each object X of $\mathsf{K}(\mathsf{Inj}\,kG)$ the following exact triangles:

$$X \otimes_k pk \longrightarrow X \otimes_k ik \longrightarrow X \otimes_k tk \longrightarrow X \otimes_k pk[1],$$
$$\mathrm{Hom}_k(tk, X) \longrightarrow \mathrm{Hom}_k(ik, X) \longrightarrow \mathrm{Hom}_k(pk, X) \longrightarrow \mathrm{Hom}_k(tk[-1], X).$$

The first two maps in each triangle are the obvious adjunction morphisms which are induced by the recollement. This becomes clear once we observe that the canonical map $k \to ik$ induces isomorphisms

$$X = X \otimes_k k \xrightarrow{\sim} X \otimes_k ik \quad \text{and} \quad \mathrm{Hom}_k(ik, X) \xrightarrow{\sim} \mathrm{Hom}_k(k, X) = X$$

(see Proposition 5.3). Thus $\mathsf{K}(\mathsf{Inj}\,kG)$ is a sort of intermediary between $\mathsf{StMod}\,kG$ and $\mathsf{D}(\mathsf{Mod}\,kG)$, and in some ways is better behaved than either of them. The problem with $\mathsf{StMod}\,kG$ is that the graded endomorphisms of the trivial module form a usually non-Noetherian ring (the Tate cohomology ring). The problem with $\mathsf{D}(\mathsf{Mod}\,kG)$, on the other hand, is that k is usually not a compact object.

The compact objects in the three categories in the recollement give the perhaps more familiar sequence of categories and functors

$$\mathsf{stmod}\,kG \longleftarrow \mathsf{D}^b(\mathsf{mod}\;kG) \longleftarrow \mathsf{D}^b(\mathsf{proj}\,kG).$$

Note that only the left adjoints in the recollement preserve compact objects.

7. The dictionary between $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$

Let G be a finite group. Then by Theorem 4.1 we have functors

$$\mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\mathrm{Hom}_{kG}(ik,-)} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$$

(where $\mathscr{E}_G = \mathrm{End}_{kG}(ik)$), which in the case of a p-group give an equivalence of triangulated categories

$$\Phi\colon \mathsf{K}(\mathsf{Inj}\,kG) \to \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).$$

In this section, we investigate the functor Φ further, and we develop a dictionary for translating between $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{D}_{\mathrm{dg}}(C^*(BG; k))$.

First we deal with external tensor products. Now if R_1 and R_2 are commutative S-algebras over k, then $R_1 \otimes_k R_2$ is also a commutative S-algebra over k by VII.1.6 of [Elmendorf et al. 1997]. If X and Y are spaces then the Eilenberg–Zilber map gives an equivalence between $C^*(X; k) \otimes_k C^*(Y; k)$ and $C^*(X \times Y; k)$ as S-algebras over k. If $\delta\colon X \to X \times X$ is the diagonal map, then the composite

$$C^*(X; k) \otimes_k C^*(X; k) \simeq C^*(X \times X; k) \xrightarrow{\delta^*} C^*(X; k)$$

is the multiplication map, and is a map of commutative S-algebras over k.

In particular, if $X = BG_1$ and $Y = BG_2$ then $X \times Y = B(G_1 \times G_2)$, and we get the equivalence of $C^*(BG_1; k) \otimes_k C^*(BG_2; k)$ with $C^*(B(G_1 \times G_2); k)$. This means that if X and Y are modules over $C^*(BG_1; k)$ and $C^*(BG_2; k)$ respectively, we have an *external tensor product* $X \otimes_k Y$ as a module over $C^*(B(G_1 \times G_2); k)$.

If $\Delta\colon G \to G \times G$ is the diagonal map, then the composite

$$\begin{aligned} &C^*(BG; k) \otimes_k C^*(BG; k) \simeq C^*(BG \times BG; k) \\ &\qquad\qquad = C^*(B(G \times G); k) \xrightarrow{B\Delta^*} C^*(BG; k) \end{aligned}$$

is the multiplication map on $C^*(BG; k)$.

Theorem 7.1. *The functor Φ takes the external tensor product over k discussed in Section 5 to the external tensor product described above.*

Proof. If ik_{G_1} and ik_{G_2} are injective resolutions of k for G_1 and G_2, then the external tensor product $ik_{G_1} \otimes_k ik_{G_2}$ is an injective resolution of k for $G_1 \times G_2$. We have a commutative diagram

$$\begin{tikzcd} \mathsf{K}(\mathsf{Inj}\, kG_1) \times \mathsf{K}(\mathsf{Inj}\, kG_2) \arrow[r, "\mathrm{Hom}_{kG_1}(ik_{G_1},-)\times\mathrm{Hom}_{kG_2}(ik_{G_2},-)"] \arrow[d, "\otimes_k"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1}) \times \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_2}) \arrow[d, "\otimes_k"] \\ \mathsf{K}(\mathsf{Inj}\, k(G_1 \times G_2)) \arrow[r, "\mathrm{Hom}_{k(G_1\times G_2)}(ik_{G_1}\otimes_k ik_{G_2},-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1} \otimes_k \mathscr{E}_{G_2}) \end{tikzcd}$$

We combine this with the commutative diagram

$$\begin{tikzcd} \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1}) \times \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_2}) \arrow[r, "\simeq"] \arrow[d, "\otimes_k"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG_1, k) \times \mathsf{D}_{\mathrm{dg}}(C^*(BG_2; k)) \arrow[d, "\otimes_k"] \\ \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_{G_1} \otimes_k \mathscr{E}_{G_2}) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG_1; k) \otimes_k C^*(BG_2; k)) \end{tikzcd}$$

and the equivalence

$$\mathsf{D}_{\mathrm{dg}}(C^*(BG_1; k) \otimes_k C^*(BG_2; k)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(B(G_1 \times G_2); k))$$

to prove the theorem. □

Next we deal with subgroups.

Lemma 7.2. *If H is a subgroup of G, the following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\, kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] \arrow[d, "\mathrm{ind}_{H,G}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{K}(\mathsf{Inj}\, kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)
\end{tikzcd}$$

Proof. This follows from the Frobenius reciprocity (or Eckmann–Shapiro) isomorphism

$$\mathrm{Hom}_{kG}(ik, \mathrm{ind}_{H,G}(X)) \cong \mathrm{Hom}_{kH}(ik, X). \qquad \square$$

Theorem 7.3. *The functor Φ takes induction from kH-modules to kG-modules to restriction from $C^*(BH; k)$-modules to $C^*(BG; k)$-modules.*

Proof. By Theorem 4.1 and Lemma 7.2, the following diagram commutes up to natural isomorphisms:

$$\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\, kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] \arrow[d, "\mathrm{ind}_{H,G}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}^*_{G,H}"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH; k)) \arrow[d, "\mathrm{res}^*_{G,H}"] \\
\mathsf{K}(\mathsf{Inj}\, kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG; k)).
\end{tikzcd} \qquad \square$$

The corresponding statement for restriction from $\mathsf{K}(\mathsf{Inj}\, kG)$ to $\mathsf{K}(\mathsf{Inj}\, kH)$ requires more preparation. We begin by defining a functor

$$- \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik \colon \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \to \mathsf{K}(\mathsf{Inj}\, kG)$$

as the left adjoint of $\mathrm{Hom}_{kG}(ik, -)$. The existence of such a left adjoint follows from Brown's representability theorem (see [Neeman 2001]) since $\mathrm{Hom}_{kG}(ik, -)$ preserves products. Alternatively, we construct this functor explicitly by tensoring over $\mathscr{E}_G$ a semiprojective resolution (for the definition, see Section 6) of the given differential graded $\mathscr{E}_G$-module with ik. It is clear from the construction that

$$- \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$$

identifies $\mathscr{E}_G$ with ik.

Lemma 7.4. *Let X be an object in $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$. Then the natural map*

$$X \to \mathrm{Hom}_{kG}(ik, X \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik)$$

is an isomorphism in $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$.

Proof. This is obviously true for $X = \mathscr{E}_G$. The functor on the right preserves triangles and direct sums in the variable X because ik is compact. So the assertion is true for any object in the localizing subcategory generated by $\mathscr{E}_G$, which is all of $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$. $\square$

Remark 7.5. The functor $-\overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$ identifies $\mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G)$ with the localizing subcategory $\mathrm{Loc}(ik)$ of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by ik. In particular, for each object Y in $\mathsf{K}(\mathsf{Inj}\, kG)$, the natural map

$$\eta_Y \colon \operatorname{Hom}_{kG}(ik, Y) \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik \to Y$$

is the best left approximation of Y by objects in $\mathrm{Loc}(ik)$. More precisely, the object

$$\operatorname{Hom}_{kG}(ik, Y) \overset{\mathbb{L}}{\otimes}_{\mathscr{E}_G} ik$$

belongs to $\mathrm{Loc}(ik)$ and the induced map $\operatorname{Hom}_{\mathsf{K}(\mathsf{Inj}\, kG)}(X, \eta_Y)$ is bijective for all X in $\mathrm{Loc}(ik)$.

Lemma 7.6. *Suppose we have given a diagram of functors*

$$\begin{tikzcd} \mathsf{S} \arrow[r, "H"] \arrow[d, "F"] & \mathsf{T} \arrow[d, "G"] \\ \mathsf{S}' \arrow[r, "H'"] & \mathsf{T}' \end{tikzcd}$$

which is commutative up to isomorphism such that F, G, H' admit right adjoints F_ρ, G_ρ, H'_ρ, and H admits a left adjoint H_λ. Suppose in addition that

$$\mathrm{Id}_{\mathsf{T}} \cong H \circ H_\lambda \quad \textit{and} \quad H'_\rho \circ H' \cong \mathrm{Id}_{\mathsf{S}'}.$$

Then the diagram of functors:

$$\begin{tikzcd} \mathsf{S}' \arrow[r, "H'"] \arrow[d, "F_\rho"] & \mathsf{T}' \arrow[d, "G_\rho"] \\ \mathsf{S} \arrow[r, "H"] & \mathsf{T} \end{tikzcd}$$

commutes up to isomorphism.

Proof. We have

$$G \cong G \circ H \circ H_\lambda \cong H' \circ F \circ H_\lambda.$$

Taking right adjoints, we obtain

$$G_\rho \cong H \circ F_\rho \circ H'_\rho$$

and this implies

$$G_\rho \circ H' \cong H \circ F_\rho \circ H'_\rho \circ H' \cong H \circ F_\rho. \qquad \square$$

Theorem 7.7. *Let G be a finite p-group and let H be a subgroup of G. Then the functor Φ takes restriction from kG-modules to kH-modules to coinduction from $C^*(BG;k)$-modules to $C^*(BH;k)$-modules.*

Proof. We claim that the diagram

$$\begin{tikzcd}
\mathsf{K}(\mathsf{Inj}\,kG) \arrow[r, "\mathrm{Hom}_{kG}(ik,-)"] \arrow[d, "\mathrm{res}_{G,H}"'] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_G) \arrow[r, "\simeq"] \arrow[d, "\mathbb{R}\mathrm{Hom}_{\mathscr{E}_G}(\mathscr{E}_H,-)"] & \mathsf{D}_{\mathrm{dg}}(C^*(BG;k)) \arrow[d, "\mathbb{R}\mathrm{Hom}_{C^*(BG;k)}(C^*(BH;k),-)"] \\
\mathsf{K}(\mathsf{Inj}\,kH) \arrow[r, "\mathrm{Hom}_{kH}(ik,-)"] & \mathsf{D}_{\mathrm{dg}}(\mathscr{E}_H) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{dg}}(C^*(BH;k))
\end{tikzcd}$$

commutes. For the right-hand square this is clear. For the left hand square, this follows from Lemma 7.2, 7.4 and 7.6. The assumption on G to be a p-group is needed for $\mathrm{Hom}_{kG}(ik,-)$ to be an equivalence. □

Theorem 7.8. *Let G be a finite p-group. Then the functor Φ takes the internal tensor product with diagonal G-action to the E_∞ tensor product discussed at the end of Section 4.*

Proof. The internal tensor product in $\mathsf{K}(\mathsf{Inj}\,kG)$ is given by external tensor product to $\mathsf{K}(\mathsf{Inj}\,k(G\times G))$ followed by restriction to the diagonal copy of G. Using Theorems 7.1 and 7.7, we see that

$$\begin{aligned}
\mathrm{Hom}_{kG}(ik,(X\otimes_k Y){\downarrow}_G^{G\times G}) &\cong \mathbb{R}\mathrm{Hom}_{\mathscr{E}_{G\times G}}(\mathscr{E}_G, \mathrm{Hom}_{k(G\times G)}(ik, X\otimes_k Y)) \\
&\cong \mathbb{R}\mathrm{Hom}_{\mathscr{E}_G\otimes_k\mathscr{E}_G}(\mathscr{E}_G, \mathrm{Hom}_{kG}(ik,X)\otimes_k \mathrm{Hom}_{kG}(ik,Y)).
\end{aligned}$$

Applying the equivalence with $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ to the latter, we obtain

$$\mathbb{R}\mathrm{Hom}_{C^*(BG;k)\otimes_k C^*(BG;k)}(C^*(BG;k), \Phi(X)\otimes_k\Phi(Y))$$

which is isomorphic to

$$\Phi(X)\otimes_{C^*(BG;k)}\Phi(Y)$$

with the E_∞ tensor product. □

Theorems 7.3 and 7.7 above can be thought of as saying that the roles of restriction and (co)induction are reversed by the equivalence. So it makes sense that the roles of the trivial representation and the regular representation should also be reversed.

It is easy to see that ik in $\mathsf{K}(\mathsf{Inj}\,kG)$ corresponds to the regular representation of $C^*(BG;k)$, and that the regular representation kG corresponds to the trivial representation k of $C^*(BG;k)$.

We summarize all this information in Table 1.

$\mathsf{K}(\mathsf{Inj}\, kG)$	$\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$
external $-\otimes_k -$	external $-\otimes_k -$
internal $-\otimes_k -$ diagonal G-action	$-\otimes_{C^*(BG;k)} -$ E_∞ tensor product
Induction from H to G	Restriction via $C^*(BG;k) \to C^*(BH;k)$
Restriction from G to H	Coinduction $\mathrm{Hom}_{C^*(BG;k)}(C^*(BH;k), -)$
ik	$C^*(BG;k)$
kG	k

Table 1

8. $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is a derived invariant

The classical Morita theory for derived categories [Rickard 1989; 1991] can be extended to complexes of injective modules as follows.

Proposition 8.1. *Let Λ and Γ be Noetherian algebras over a commutative ring k. Suppose Λ and Γ are projective as k-modules. Then the following are equivalent.*

(i) *Λ and Γ are derived equivalent, that is, there exists a tilting complex T over Λ such that the endomorphism ring $\mathrm{End}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(T)$ is isomorphic to Γ.*

(ii) *There exists an exact equivalence $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Gamma)$.*

(iii) *There exists an exact equivalence $\mathsf{D}^b(\mathsf{mod}\,\Lambda) \to \mathsf{D}^b(\mathsf{mod}\,\Gamma)$.*

Proof. (i) $\Rightarrow$ (ii): In [Rickard 1991], it is shown that Λ and Γ admit a standard derived equivalence. Thus there is a bounded complex P of Γ-Λ-bimodules which in each degree is finitely generated projective over Λ and over Γ. The functor $\mathrm{Hom}_\Lambda(P,-)$ sends complexes of injective Λ-modules to complexes of injective Γ-modules and semiinjective complexes to semiinjective complexes. The last assertion follows from the isomorphism

$$\mathrm{Hom}_\Gamma(A, \mathrm{Hom}_\Lambda(P, X)) \cong \mathrm{Hom}_\Lambda(A \otimes_\Gamma P, X).$$

Thus $\mathrm{Hom}_\Lambda(P,-)$ induces the commutative diagram of exact functors

$$\begin{tikzcd}
\mathsf{D}(\mathsf{Mod}\,\Lambda) \arrow[r, "\mathrm{Hom}_\Lambda(P,-)"] \arrow[d, "(Q_\Lambda)_\rho"] & \mathsf{D}(\mathsf{Mod}\,\Gamma) \arrow[d, "(Q_\Gamma)_\rho"] \\
\mathsf{K}(\mathsf{Inj}\,\Lambda) \arrow[r, "\mathrm{Hom}_\Lambda(P,-)"] & \mathsf{K}(\mathsf{Inj}\,\Gamma)
\end{tikzcd}$$

because we know from Corollary 6.2 that the right adjoint functors $(Q_\Lambda)_\rho$ and $(Q_\Gamma)_\rho$ identify the derived categories with the full subcategories formed by all semiinjective complexes. By our assumption, the functor $\mathsf{D}(\operatorname{Mod}\Lambda) \to \mathsf{D}(\operatorname{Mod}\Gamma)$ is an equivalence inducing an equivalence $\mathsf{D}^b(\operatorname{mod}\Lambda) \to \mathsf{D}^b(\operatorname{mod}\Gamma)$. Now we apply Proposition 2.1 as follows. The commutativity of the diagram implies that $\operatorname{Hom}_\Lambda(P, -)$ induces an equivalence $\mathsf{K}^c(\operatorname{Inj}\Lambda) \to \mathsf{K}^c(\operatorname{Inj}\Gamma)$. Then a standard dévissage argument shows that $\operatorname{Hom}_\Lambda(P, -)$ induces an equivalence $\mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{K}(\operatorname{Inj}\Gamma)$ since $\mathsf{K}(\operatorname{Inj}\Lambda)$ is compactly generated and the functor preserves all coproducts.

(ii) $\Rightarrow$ (iii): An exact equivalence $\mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{K}(\operatorname{Inj}\Gamma)$ induces an exact equivalence $\mathsf{K}^c(\operatorname{Inj}\Lambda) \to \mathsf{K}^c(\operatorname{Inj}\Gamma)$ and therefore, again by Proposition 2.1, an exact equivalence $\mathsf{D}^b(\operatorname{mod}\Lambda) \to \mathsf{D}^b(\operatorname{mod}\Gamma)$.

(iii) $\Rightarrow$ (i): Let $F\colon \mathsf{D}^b(\operatorname{mod}\Gamma) \to \mathsf{D}^b(\operatorname{mod}\Lambda)$ be an exact equivalence. Then $T = F\Gamma$ is a tilting complex with $\operatorname{End}_{\mathsf{D}^b(\operatorname{mod}\Lambda)}(T) \cong \Gamma$. $\square$

9. Bousfield localization

We recall briefly some basic facts about Bousfield localization. Let T be triangulated with arbitrary coproducts. We fix a full triangulated subcategory S of T which is *localizing* in the sense that S is closed under taking all coproducts. Then we have a sequence

$$\mathsf{S} \xrightarrow{I} \mathsf{T} \xrightarrow{Q} \mathsf{T}/\mathsf{S}$$

of canonical functors and observe that I has a right adjoint I_ρ if and only if Q has a right adjoint Q_ρ. In this case we call the sequence a *localization sequence*. Following [Rickard 1997], we write $E_\mathsf{S} = I \circ I_\rho$ and $F_\mathsf{S} = Q_\rho \circ Q$. Note that E_S and F_S are idempotent functors.

Let us collect the basic facts of such a localization sequence.

Lemma 9.1. *A localization sequence $\mathsf{S} \xrightarrow{I} \mathsf{T} \xrightarrow{Q} \mathsf{T}/\mathsf{S}$ has the following properties.*

(i) *The functor Q_ρ is fully faithful and identifies T/S with the full subcategory*

$$\mathsf{S}^\perp = \{Y \in \mathsf{T} \mid \operatorname{Hom}_\mathsf{T}(X, Y) = 0 \textit{ for all } X \in \mathsf{S}\}.$$

(ii) *We have*

$$\mathsf{S} = \{X \in \mathsf{T} \mid \operatorname{Hom}_\mathsf{T}(X, Y) = 0 \textit{ for all } Y \in \mathsf{S}^\perp\}.$$

(iii) *For each object X of T, there exists up to isomorphism a unique exact triangle*

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

with $X' \in \mathsf{S}$ and $X'' \in \mathsf{S}^\perp$.

(iv) *For each object X of T, the adjunction morphisms $E_{\mathsf{S}}X \to X$ and $X \to F_{\mathsf{S}}X$ fit into an exact triangle*

$$E_{\mathsf{S}}X \longrightarrow X \longrightarrow F_{\mathsf{S}}X \longrightarrow E_{\mathsf{S}}X[1].$$

There is a finite variant of Bousfield localization for compactly generated triangulated categories which Rickard [1997] introduced into representation theory. Here we use the tensor product $\otimes_k$ which is defined on $\mathsf{K}(\mathsf{Inj}\, kG)$.

Let S_0 be a class of compact objects of $\mathsf{K}(\mathsf{Inj}\, kG)$ and denote by $\mathsf{S} = \mathrm{Loc}(\mathsf{S}_0)$ the localizing subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ which is generated by S_0. Then the sequence

$$\mathsf{S} \xrightarrow{I} \mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow{Q} \mathsf{K}(\mathsf{Inj}\, kG)/\mathsf{S}$$

of canonical functors is a localization sequence. Moreover, S is compactly generated and the subcategory S^c of compact objects equals the thick subcategory $\mathrm{Thick}(\mathsf{S}_0)$ of $\mathsf{K}^c(\mathsf{Inj}\, kG)$ which is generated by S_0.

Now suppose that S_0 is a *thick tensor ideal* of $\mathsf{K}^c(\mathsf{Inj}\, kG)$. Thus S_0 is by definition a thick subcategory and a *tensor ideal*, that is, $X \otimes_k Y$ belongs to S_0 for all X in S_0 and Y in $\mathsf{K}^c(\mathsf{Inj}\, kG)$. Then $\mathsf{S} = \mathrm{Loc}(\mathsf{S}_0)$ is a localizing tensor ideal and therefore the exact triangle

$$E_{\mathsf{S}}ik \longrightarrow ik \longrightarrow F_{\mathsf{S}}ik \longrightarrow E_{\mathsf{S}}ik[1]$$

induces for each X in $\mathsf{K}(\mathsf{Inj}\, kG)$ an exact triangle

$$X \otimes_k E_{\mathsf{S}}ik \longrightarrow X \otimes_k ik \longrightarrow X \otimes_k F_{\mathsf{S}}ik \longrightarrow X \otimes_k E_{\mathsf{S}}ik[1]$$

which is isomorphic to

$$E_{\mathsf{S}}X \longrightarrow X \longrightarrow F_{\mathsf{S}}X \longrightarrow E_{\mathsf{S}}X[1].$$

10. Varieties

In this section, we indicate how the theory of support for kG-modules from [Benson et al. 1996] may be modified to work in $\mathsf{K}(\mathsf{Inj}\, kG)$.

Let $H^*(G,k)$ be the cohomology ring of G, and denote by $\mathrm{Spec}^* H^*(G,k)$ the set of homogeneous prime ideals of $H^*(G,k)$. We consider the Zariski topology on $\mathrm{Spec}^* H^*(G,k)$, that is, a subset of $\mathrm{Spec}^* H^*(G,k)$ is *Zariski closed* if it is of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}^* H^*(G,k) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some homogeneous ideal $\mathfrak{a}$ of $H^*(G,k)$. We write $\mathfrak{m} = H^+(G,k)$ for the unique maximal ideal of $H^*(G,k)$ and obtain the *projective variety*

$$\mathrm{Proj}\, H^*(G,k) = \mathrm{Spec}^* H^*(G,k) \setminus V(H^+(G,k)) = \mathrm{Spec}^* H^*(G,k) \setminus \{\mathfrak{m}\}.$$

Now fix a specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec}^* H^*(G,k)$, that is, $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \mathcal{V}$ imply $\mathfrak{q} \in \mathcal{V}$. We obtain the localizing tensor ideal

$$\mathsf{S}_{\mathcal{V}} = \operatorname{Loc}\big(\{X \in \mathsf{K}^c(\mathsf{Inj}\, kG) \mid H^*(G,X)_{\mathfrak{q}} = 0 \text{ for all } \mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \setminus \mathcal{V}\}\big),$$

of $\mathsf{K}(\mathsf{Inj}\, kG)$. To simplify our notation, we write

$$E_{\mathcal{V}} = E_{\mathsf{S}_{\mathcal{V}}} \quad \text{and} \quad F_{\mathcal{V}} = F_{\mathsf{S}_{\mathcal{V}}}.$$

Now fix $\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k)$ and let

$$\mathcal{V}_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \mid \mathfrak{p} \subseteq \mathfrak{q}\} \quad \text{and} \quad \mathcal{W}_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec}^* H^*(G,k) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

Note that $\mathcal{W}_{\mathfrak{p}} \setminus \mathcal{V}_{\mathfrak{p}} = \{\mathfrak{p}\}$. We define

$$\kappa_{\mathfrak{p}} = (F_{\mathcal{W}_{\mathfrak{p}}} \circ E_{\mathcal{V}_{\mathfrak{p}}})ik \cong (E_{\mathcal{V}_{\mathfrak{p}}} \circ F_{\mathcal{W}_{\mathfrak{p}}})ik.$$

For example, one computes

$$\kappa_{\mathfrak{m}} = (E_{\mathcal{V}_{\mathfrak{m}}} \circ F_{\mathcal{W}_{\mathfrak{m}}})ik = E_{\mathcal{V}_{\mathfrak{m}}}ik = pk.$$

Given X in $\mathsf{K}(\mathsf{Inj}\, kG)$, we have

$$X \otimes_k \kappa_{\mathfrak{p}} \cong (F_{\mathcal{W}_{\mathfrak{p}}} \circ E_{\mathcal{V}_{\mathfrak{p}}})X \cong (E_{\mathcal{V}_{\mathfrak{p}}} \circ F_{\mathcal{W}_{\mathfrak{p}}})X$$

and the *variety* of X is by definition

$$\mathcal{V}_G(X) = \{\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k) \mid X \otimes_k \kappa_{\mathfrak{p}} \neq 0\}.$$

Lemma 10.1. *Let* $\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k)$. *Then* $\mathcal{V}_G(\kappa_{\mathfrak{p}}) = \{\mathfrak{p}\}$.

Proof. The proof is essentially the same as the proof of Lemma 10.4 of [Benson et al. 1996]. □

Lemma 10.2. *A complex X in* $\mathsf{K}(\mathsf{Inj}\, kG)$ *is acyclic if and only if* $\mathcal{V}_G(X)$ *is contained in* $\operatorname{Proj} H^*(G,k)$.

Proof. A complex X is acyclic if and only if $X \otimes_k pk = 0$ if and only if

$$\mathcal{V}_G(X) \subseteq \operatorname{Proj} H^*(G,k).$$ □

It follows that $\kappa_{\mathfrak{p}}$ is in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \cong \mathsf{StMod}\, kG$ unless $\mathfrak{p} = \mathfrak{m}$, and that these modules agree with the modules κ_V introduced in [Benson et al. 1996].

11. Objects with injective cohomology

Modules over kG with injective cohomology were studied in [Benson and Krause 2002]. In this section, we indicate how this works in $\mathsf{K}(\mathsf{Inj}\, kG)$. The theory is actually easier than in $\mathsf{StMod}\, kG$, because it does not involve a discussion of injective modules over the non-Noetherian Tate cohomology ring.

Let I be an injective $H^*(G,k)$-module. Then the functor from $\mathsf{K}(\mathsf{Inj}\,kG)$ to the category of abelian groups which takes an object X to

$$\mathrm{Hom}_{H^*(G,k)}(H^*(G,X),I)$$

takes triangles to exact sequences and coproducts to products. So by Brown representability (see [Neeman 2001]) there is an object $T(I)$ in $\mathsf{K}(\mathsf{Inj}\,kG)$ satisfying

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,kG)}(X,T(I)) \cong \mathrm{Hom}_{H^*(G,k)}(H^*(G,X),I).$$

The assignment $I \mapsto T(I)$ extends via Yoneda's lemma to a functor

$$T\colon\ \mathsf{Inj}\,H^*(G,k) \to \mathsf{K}(\mathsf{Inj}\,kG).$$

A dimension shifting argument (see [Benson and Krause 2002, §3]) shows that we obtain an isomorphism of graded $H^*(G,k)$-modules

$$\mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,kG)}(X,T(I)) \cong \mathrm{Hom}^*_{H^*(G,k)}(H^*(G,X),I).$$

In particular, setting $X = ik$, we see that $H^*(G,T(I)) \cong I$ for all I in $\mathsf{Inj}\,H^*(G,k)$, and setting $X = T(I')$ we see that

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,kG)}(T(I'),T(I)) \cong \mathrm{Hom}_{H^*(G,k)}(I',I),$$

so that the functor T is fully faithful. Thus, if $\mathfrak{p} \in \mathrm{Spec}^*\, H^*(G,k)$ and $I_\mathfrak{p}$ is the injective hull of $H^*(G,k)/\mathfrak{p}$, we have

$$\mathrm{End}^*_{\mathsf{K}(\mathsf{Inj}\,kG)}(T(I_\mathfrak{p})) \cong H^*(G,k)^{\wedge}_{\mathfrak{p}} = \varprojlim_n H^*(G,k)_\mathfrak{p}/\mathfrak{p}^n_\mathfrak{p},$$

using [Matlis 1958].

Proposition 11.1. *Let $I_\mathfrak{m} = H_*(G,k)$, the graded dual of $H^*(G,k)$. This is the injective hull of the trivial $H^*(G,k)$-module $k = H^*(G,k)/\mathfrak{m}$ where $\mathfrak{m} = H^+(G,k)$ is the maximal ideal generated by the positive degree elements. Then $T(I_\mathfrak{m}) \cong pk$, the projective resolution of k.*

Proof. The proof is essentially the same as the proof of Lemma 3.1 of [Benson and Krause 2002]. □

Proposition 11.2. *Let H be a subgroup of G, and write T_G and T_H for the functor T with respect to kG and kH respectively. If I is an injective $H^*(G,k)$-module, we have*

$$T_G(I){\downarrow_H} \cong T_H\big(\mathrm{Hom}^*_{H^*(G,k)}(H^*(H,k),I)\big).$$

Proof. The proof is essentially the same as the proof of Proposition 7.1 of [Benson and Krause 2002]. □

Proposition 11.3. *Let $\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k)$. Then $\mathcal{V}_G(T(I_{\mathfrak{p}})) = \{\mathfrak{p}\}$.*

Proof. The proof is essentially the same as the proof of Theorem 7.3 of [Benson and Krause 2002]. □

It follows that $T(I_{\mathfrak{p}})$ is in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \cong \mathsf{StMod}\, kG$ unless $\mathfrak{p} = \mathfrak{m}$, and that these objects agree with the objects of the same name constructed in [Benson and Krause 2002].

Theorem 11.4. *Let $\mathfrak{p}$ be a homogeneous prime ideal in $H^*(G,k)$, and let d be the Krull dimension of $H^*(G,k)/\mathfrak{p}$. Then*

$$\kappa_{\mathfrak{p}} \cong T(I_{\mathfrak{p}}[d]).$$

Proof. If $d > 0$ then both objects are in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \cong \mathsf{StMod}\, kG$ and the theorem is proved in [Benson 2008; Benson and Greenlees 2008]. If $d = 0$ then $\mathfrak{p} = \mathfrak{m}$ and both sides are isomorphic to the projective resolution pk. □

12. Chouinard and Dade

In this section we describe the analogues in $\mathsf{K}(\mathsf{Inj}\, kG)$ of the theorem of Chouinard [1976] and of Benson, Carlson and Rickard's version [1996] of the lemma from [Dade 1978].

Theorem 12.1. *Let G be a finite group and k a field of characteristic p. An object in $\mathsf{K}(\mathsf{Inj}\, kG)$ is semiinjective, respectively semiprojective, respectively zero, if and only if its restriction to every elementary abelian p-subgroup of G is semiinjective, respectively semiprojective, respectively zero.*

Proof. It follows from the recollement (6.3) that an object X in $\mathsf{K}(\mathsf{Inj}\, kG)$ is semiinjective, respectively semiprojective, if and only if $\operatorname{Hom}_k(tk, X) = 0$, respectively $X \otimes_k tk = 0$. By Chouinard's theorem [1976] in $\mathsf{StMod}\, kG$, this is true if and only if the restriction of $\operatorname{Hom}_k(tk, X)$, respectively $X \otimes_k tk$ to each elementary abelian p-subgroup E of G is zero. This is equivalent to the statement that the restriction of X to each such E is semiinjective, respectively semiprojective.

If an object X in $\mathsf{K}(\mathsf{Inj}\, kG)$ restricts to zero on every elementary abelian p-subgroup then it is acyclic, so it is in $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \simeq \mathsf{StMod}(kG)$. So we can apply Chouinard's theorem in $\mathsf{StMod}(kG)$ to deduce that $X \cong 0$. □

Now if $E = \langle g_1, \ldots, g_r \rangle$ is an elementary abelian group of rank r, we write X_i for the element $g_i - 1 \in J(kE)$, the radical of the group algebra. If K is an extension field of k, and $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a nonzero point in affine space $\mathbb{A}^r(K)$, then

$$X_\lambda = \lambda_1 X_1 + \cdots + \lambda_r X_r$$

is an element of $J(KE)$ satisfying $X_\lambda^p = 0$, and $\langle 1 + X_\lambda \rangle$ is a cyclic subgroup of order p in the group algebra KE. It is called a *cyclic shifted subgroup* of E over K.

Theorem 12.2. *An object in* $\mathsf{K}(\mathsf{Inj}\, kE)$ *is semiinjective, respectively semiprojective, respectively zero if and only if for all extension fields K of k and all cyclic shifted subgroups of E over K the restriction is semiinjective, respectively semiprojective, respectively zero.*

Proof. The proof follows the same lines as that of Theorem 12.1, using the version of Dade's lemma in [Benson et al. 1996, Theorem 5.2] instead of Chouinard's theorem. We also need to observe that

$$K \otimes_k \mathrm{Hom}_k(tk, X) \cong \mathrm{Hom}_K(tK, K \otimes_k X),$$
$$K \otimes_k (X \otimes_k tk) \cong (K \otimes_k X) \otimes_K tK. \qquad \square$$

Remark 12.3. As in [Benson et al. 1996], it suffices to check the hypothesis for K the algebraic closure of an extension of k of transcendence degree $r - 1$.

13. Homotopy colimits and localizing subcategories

The goal of this section is to show that in the stable module category

$$\mathsf{StMod}\, kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG),$$

the homotopy category of complexes of injectives $\mathsf{K}(\mathsf{Inj}\, kG)$ and the derived category $\mathsf{D}(\mathsf{Mod}\, kG)$, localizing subcategories are closed under taking filtered colimits in the corresponding category of chain complexes and chain homomorphisms. This amounts to filling in the details of arguments of Bousfield and Kan [1972] and Bökstedt and Neeman [1993] for the sake of easy access.

Let $\mathbb{C}$ denote one of the categories $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG)$, $\mathsf{D}(\mathsf{Mod}\, kG)$, $\mathsf{K}(\mathsf{Inj}\, kG)$ (the arguments work in other situations, but it seems difficult to make precise the conditions on $\mathbb{C}$). Let I be a small category, and let $\phi \colon \mathsf{I} \to \mathbb{C}$ be a covariant functor. Then we call ϕ an I-*diagram* in $\mathbb{C}$. We define the *homotopy colimit* of the diagram ϕ to be the total complex of the double complex formed from finite chains of maps in I in the following manner:

$$\cdots \xrightarrow{d_3} \bigoplus_{i \to j \to k} \phi(i) \xrightarrow{d_2} \bigoplus_{i \to j} \phi(i) \xrightarrow{d_1} \bigoplus_{i} \phi(i)\,. \tag{13.1}$$

We regard this as a complex of objects in $\mathbb{C}$, where the differentials are alternating sums over deleted objects in the chain in the usual way. So for example d_1 takes the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j$ via $\phi(\alpha)$ to $\phi(j)$ minus the identity to $\phi(i)$; while d_2 takes the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ via $\phi(\alpha)$ to the copy of $\phi(j)$ indexed by $j \xrightarrow{\beta} k$ minus the identity to the copy of $\phi(i)$ indexed by $i \xrightarrow{\beta\circ\alpha} k$ plus the identity to the copy of $\phi(i)$ indexed by $i \xrightarrow{\alpha} j$. It is easy to see that $d_j \circ d_{j+1} = 0$. Note that the cokernel of d_1 is

$$\varinjlim_{\mathsf{I}} \phi.$$

We write

$$\underset{\mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi \quad \text{or} \quad \underset{i \in \mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi(i)$$

for the homotopy colimit.

We say that I is a *right filter* if it is a small category satisfying

(i) given objects x and y in I, there exists an object z in I and arrows $x \to z$ and $y \to z$, and

(ii) given objects x and y in I and arrows $f, g \colon x \to y$, there exists an object z in I and an arrow $\alpha \colon y \to z$ such that $\alpha \circ f = \alpha \circ g$.

For example, I could be a poset in which every pair of elements has an upper bound. If I is a right filter, then an I-diagram $\phi \colon \mathsf{I} \to \mathbb{C}$ is called a *filtered system* in $\mathbb{C}$. We assume that every filtered system in $\mathbb{C}$ has a colimit, which we write as

$$\varinjlim_{\mathsf{I}} \phi \quad \text{or} \quad \varinjlim_{i \in \mathsf{I}} \phi(i).$$

Whether I is a filtered system or a more general small category, there is an obvious map $\underset{\mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi \to \varinjlim_{\mathsf{I}} \phi$.

Lemma 13.2 [Bousfield and Kan 1972]. *Let ϕ be an I-diagram in $\mathbb{C}$. Then*

$$\underset{\mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi \to \varinjlim_{\mathsf{I}} \phi$$

is an equivalence.

Proof. In the case where I has a terminal object, say ℓ, there is a homotopy on the complex (13.1) sending the copy of $\phi(i)$ indexed by $i \to \cdots \to j$ to the copy in one degree higher indexed by $i \to \cdots \to j \to \ell$. This is a homotopy from the identity to the projection onto the subcomplex consisting of the single copy of $\phi(\ell)$ in degree zero. This proves that the map from the homotopy colimit to the colimit is an equivalence (that is, passes down to an isomorphism in the corresponding homotopy category) in this case.

The homotopy colimit can be written as a colimit of homotopy colimits over smaller diagrams, so we have

$$\underset{\mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi = \varinjlim_{\ell \in I} \underset{\mathsf{I}/\ell}{\underrightarrow{\mathrm{hocolim}}}\, \phi \to \varinjlim_{\ell \in I} \varinjlim_{\mathsf{I}/\ell} \phi = \varinjlim_{\mathsf{I}} \phi.$$

Since I/ℓ has a terminal object,

$$\underrightarrow{\operatorname{hocolim}}_{\mathsf{I}/\ell}\,\phi \to \underrightarrow{\operatorname{colim}}_{\mathsf{I}/\ell}\,\phi$$

is an equivalence, and it remains to prove that a colimit of equivalences is an equivalence. This is where the mild assumptions on the category $\mathbb{C}$ come in. Bousfield and Kan were working in the homotopy category of simplicial sets, where equivalences are detected by maps from spheres, and any such map to the filtered colimit factors through some term in the filtered system.

For a countable filtered system, we can argue as follows. If there is no terminal object, then we may choose a cofinal subsystem consisting of a countable sequence of objects and maps

$$\phi(0) \xrightarrow{\alpha_0} \phi(1) \xrightarrow{\alpha_1} \phi(2) \xrightarrow{\alpha_2} \cdots .$$

Then the colimit fits into a triangle

$$\bigoplus_n \phi(n) \xrightarrow{1-\alpha} \bigoplus_n \phi(n) \to \underrightarrow{\operatorname{colim}}_n\,\phi(n).$$

It follows that a colimit of equivalences is an equivalence in this case. So it is only for uncountable filtered systems that there is any problem.

In the category $\mathsf{StMod}\,kG \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\,kG)$, equivalences are detected by maps from the modules $\Omega^n S$ for $n \in \mathbb{Z}$ with S simple, in the sense that for a map $f\colon M \to N$, if for all $n \in \mathbb{Z}$ and S simple

$$f_*\colon \underline{\operatorname{Hom}}_{kG}(\Omega^n S, M) \to \underline{\operatorname{Hom}}_{kG}(\Omega^n S, N)$$

is an isomorphism, then f is an equivalence. So the argument of Bousfield and Kan works here: any map from $\Omega^n S$ to a filtered colimit factors through some term in the filtered system.

The same argument works in $\mathsf{D}(\mathsf{Mod}\,kG)$, where equivalences are detected by maps from perfect complexes, and any map from a perfect complex to a filtered colimit factors through some object in the system.

For the category $\mathsf{K}(\mathsf{Inj}\,kG)$, we pass to $\mathsf{K}(\mathsf{Mod}\,kG)$ and use the fact that for each simple kG-module S the injective resolution $S \to iS$ induces an isomorphism

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(iS, X) \cong \operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(S, X)$$

for all X in $\mathsf{K}(\mathsf{Inj}\,kG)$ by [Krause 2005, Lemma 2.1]. In $\mathsf{K}(\mathsf{Mod}\,kG)$ any map from S to a filtered colimit factors through some object in the system since S is finitely presented. Thus equivalences in $\mathsf{K}(\mathsf{Inj}\,kG)$ are detected by maps from the injective resolutions iS of simple kG-modules S. □

Theorem 13.3. *Let L be a localizing subcategory of $\mathbb{C}$. Then L is closed under taking filtered colimits in the underlying category of chain complexes.*

Proof. According to Lemma 13.2, it suffices to show that the homotopy colimit is in L.

For $n \geq 0$, write $X(n)$ for the total complex of the truncation of the sequence (13.1) consisting of just the last $n+1$ objects and the maps $d_n, \dots, d_1$. Since each $\phi(i)$ is in L and L is closed under direct sums, each of the terms in (13.1) is in L, and so by the induction on n, $X(n)$ is in L.

There are inclusions $\alpha_n \colon X(n) \to X(n+1)$, and we have a short exact sequence of complexes

$$0 \to \bigoplus_n X(n) \xrightarrow{1-\alpha} \bigoplus_n X(n) \to \varinjlim_n X(n) \to 0.$$

The corresponding triangle shows that

$$\underset{\mathsf{I}}{\underrightarrow{\mathrm{hocolim}}}\, \phi = \underset{n}{\underrightarrow{\mathrm{colim}}}\, X(n)$$

is in L. □

14. $\mathsf{K}(\mathsf{Inj}\, kE)$ for an elementary abelian 2-group E

Let

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/2)^r$$

be an elementary abelian 2-group of rank r, and let k be a field of characteristic two. Let

$$H^*(E, k) = k[x_1, \dots, x_r].$$

where the polynomial generators $x_1, \dots, x_r$ have degree one. The purpose of this section is to give an equivalence of triangulated categories

$$\mathsf{K}(\mathsf{Inj}\, kE) \simeq \mathsf{D}_{\mathrm{dg}}(k[x_1, \dots, x_r]).$$

This can be viewed as a version of Bernšteĭn–Gel′fand–Gel′fand duality [1978], and is also related to a construction of Carlsson [1983].

First we discuss the cyclic group of order two. The discussion begins with the observation that the reduced bar construction on a cyclic group of order two is the minimal resolution. The Alexander–Whitney map on the reduced bar construction is strictly associative, and so it follows that the minimal resolution supports a strictly associative comultiplication. Applying $\mathrm{Hom}_{k(\mathbb{Z}/2)}(-, k)$ to the reduced bar construction gives a differential graded algebra quasiisomorphic to cochains on $B(\mathbb{Z}/2)$. From this, it follows that we have a quasiisomorphism of differential graded algebras

$$C^*(B(\mathbb{Z}/2); k) \simeq H^*(\mathbb{Z}/2, k)$$

where the right hand side is regarded as a differential graded algebra with zero differential. A differential graded algebra is said to be formal if it is quasiisomorphic to its cohomology. The statement above says that $C^*(B(\mathbb{Z}/2);k)$ is formal.

Using the Künneth theorem and the Eilenberg–Zilber theorem, it follows that $C^*(BE;k)$ is also formal, since we have quasiisomorphisms

$$\begin{aligned} C^*(BE;k) &\simeq C^*(B(\mathbb{Z}/2);k) \otimes_k \cdots \otimes_k C^*(B(\mathbb{Z}/2);k) \\ &\simeq H^*(\mathbb{Z}/2,k) \otimes_k \cdots \otimes_k H^*(\mathbb{Z}/2,k) \\ &\cong H^*(E,k) = k[x_1,\dots,x_r]. \end{aligned}$$

Thus we have equivalences of categories

$$\mathsf{D}_{\mathrm{dg}}(C^*(BE;k)) \simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]). \tag{14.1}$$

Theorem 14.2. *Let E be an elementary abelian 2-group and k a field of characteristic two. Then there is an equivalence of triangulated categories*

$$\mathsf{K}(\mathsf{Inj}\, kE) \simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]).$$

Proof. This follows by combining the equivalences

$$\begin{aligned} \mathsf{K}(\mathsf{Inj}\, kE) &\simeq \mathsf{D}_{\mathrm{dg}}(\mathrm{End}_{kE}(ik)) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BE;k)) \\ &\simeq \mathsf{D}_{\mathrm{dg}}(H^*(E,k)) = \mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r]) \end{aligned}$$

coming from Proposition 3.1, Theorem 4.1 and Equation (14.1). □

Remark 14.3. The curious reader may wonder whether these equivalences are monoidal, and if so, why this does not imply that the Steenrod operations on $H^*(BE;k)$ are trivial. The point here is that there are in fact many inequivalent E_∞ structures on the formal differential graded algebra $k[x_1,\dots,x_r]$. There is a trivial one which would make the Steenrod operations act trivially, but this is not the one coming from $C^*(BE;k)$. If E' is a subgroup of the group of units of kE of augmentation one, inducing an isomorphism $kE' \cong kE$, then this gives another, usually inequivalent E_∞ structure on $k[x_1,\dots,x_r]$. There is another one coming from viewing kE as a restricted universal enveloping algebra. The fact that these E_∞ structures are inequivalent can be seen by examining the corresponding tensor products of kE-modules. So the point is that the equivalences in the theorem *are* monoidal, but the monoidal structure on $\mathsf{D}_{\mathrm{dg}}(k[x_1,\dots,x_r])$ is *not* the one coming from the derived tensor product over this graded commutative ring.

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References

[Avramov et al. 2003] L. L. Avramov, H.-B. Foxby, and S. Halperin, "Differential graded homological algebra", preprint, 2003.

[Benson 1991] D. J. Benson, *Representations and cohomology. II*, Cambridge Studies in Advanced Mathematics **31**, Cambridge University Press, Cambridge, 1991. MR 93g:20099 Zbl 0731.20001

[Benson 2008] D. J. Benson, "Idempotent kG-modules with injective cohomology", *J. Pure Appl. Alebra* (2008). to appear.

[Benson and Greenlees 2008] D. J. Benson and J. P. C. Greenlees, "Localization and duality in topology and modular representation theory", *J. Pure Appl. Algebra* (2008). to appear.

[Benson and Krause 2002] D. Benson and H. Krause, "Pure injectives and the spectrum of the cohomology ring of a finite group", *J. Reine Angew. Math.* **542** (2002), 23–51. MR 2002m:20086 Zbl 0987.20026

[Benson et al. 1996] D. J. Benson, J. F. Carlson, and J. Rickard, "Complexity and varieties for infinitely generated modules. II", *Math. Proc. Cambridge Philos. Soc.* **120**:4 (1996), 597–615. MR 97f:20008 Zbl 0888.20003

[Bernšteĭn et al. 1978] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, "Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra", *Funktsional. Anal. i Prilozhen.* **12**:3 (1978), 66–67. MR 80c:14010a

[Bökstedt and Neeman 1993] M. Bökstedt and A. Neeman, "Homotopy limits in triangulated categories", *Compositio Math.* **86**:2 (1993), 209–234. MR 94f:18008 Zbl 0802.18008

[Bousfield and Kan 1972] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer, Berlin, 1972. MR 51 #1825 Zbl 0259.55004

[Brown 1982] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, New York, 1982. MR 83k:20002 Zbl 0584.20036

[Carlsson 1983] G. Carlsson, "On the homology of finite free $(\mathbf{Z}/2)^n$-complexes", *Invent. Math.* **74**:1 (1983), 139–147. MR 85h:57045 Zbl 0526.57024

[Chouinard 1976] L. G. Chouinard, "Projectivity and relative projectivity over group rings", *J. Pure Appl. Algebra* **7**:3 (1976), 287–302. MR 53 #5769 Zbl 0327.20020

[Dade 1978] E. Dade, "Endo-permutation modules over p-groups. II", *Ann. of Math.* (2) **108**:2 (1978), 317–346. MR 80a:13008b Zbl 0404.16003

[Dwyer et al. 2006] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar, "Duality in algebra and topology", *Adv. Math.* **200**:2 (2006), 357–402. MR 2006k:55017 Zbl 05015044

[Elmendorf et al. 1997] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs **47**, American Mathematical Society, Providence, RI, 1997. MR 97h:55006 Zbl 0894.55001

[Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, Cambridge, 1988. MR 89e:16035 Zbl 0635.16017

[Hovey et al. 2000] M. Hovey, B. Shipley, and J. Smith, "Symmetric spectra", *J. Amer. Math. Soc.* **13**:1 (2000), 149–208. MR 2000h:55016 Zbl 0931.55006

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR 95e:18010 Zbl 0799.18007

[Krause 2005] H. Krause, "The stable derived category of a Noetherian scheme", *Compos. Math.* **141**:5 (2005), 1128–1162. MR 2006e:18019 Zbl 02212628

[Mandell 2001] M. A. Mandell, "E_∞ algebras and p-adic homotopy theory", *Topology* **40**:1 (2001), 43–94. MR 2001m:55025 Zbl 0974.55004

[Matlis 1958] E. Matlis, "Injective modules over Noetherian rings", *Pacific J. Math.* **8** (1958), 511–528. MR 20 #5800 Zbl 0084.26601

[McClure and Smith 2003] J. E. McClure and J. H. Smith, "Multivariable cochain operations and little n-cubes", *J. Amer. Math. Soc.* **16**:3 (2003), 681–704. MR 2004c:55021 Zbl 1014.18005

[Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ, 2001. MR 2001k:18010 Zbl 0974.18008

[Rickard 1989] J. Rickard, "Derived categories and stable equivalence", *J. Pure Appl. Algebra* **61**:3 (1989), 303–317. MR 91a:16004 Zbl 0685.16016

[Rickard 1991] J. Rickard, "Derived equivalences as derived functors", *J. London Math. Soc.* (2) **43**:1 (1991), 37–48. MR 92b:16043 Zbl 0683.16030

[Rickard 1997] J. Rickard, "Idempotent modules in the stable category", *J. London Math. Soc.* (2) **56**:1 (1997), 149–170. MR 98d:20058 Zbl 0910.20034

[Rothenberg and Steenrod 1965] M. Rothenberg and N. E. Steenrod, "The cohomology of classifying spaces of H-spaces", *Bull. Amer. Math. Soc.* **71** (1965), 872–875. MR 34 #8405 Zbl 0132.19201

[Shipley 2007] B. Shipley, "$H\mathbb{Z}$-algebra spectra are differential graded algebras", *Amer. J. Math.* **129**:2 (2007), 351–379. MR 2306038 Zbl 1120.55007

[Spaltenstein 1988] N. Spaltenstein, "Resolutions of unbounded complexes", *Compositio Math.* **65**:2 (1988), 121–154. MR 89m:18013 Zbl 0636.18006

[Verdier 1996] J.-L. Verdier, "Des catégories dérivées des catégories abéliennes", *Astérisque* 239 (1996), xii+253 pp. (1997). MR 98c:18007 Zbl 0882.18010

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Joint moments of derivatives of characteristic polynomials

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We investigate the joint moments of the $2k$-th power of the characteristic polynomial of random unitary matrices with the $2h$-th power of the derivative of this same polynomial. We prove that for a fixed h, the moments are given by rational functions of k, up to a well-known factor that already arises when $h = 0$.

We fully describe the denominator in those rational functions (this had already been done by Hughes experimentally), and define the numerators through various formulas, mostly sums over partitions.

We also use this to formulate conjectures on joint moments of the zeta function and its derivatives, or even the same questions for the Hardy function, if we use a "real" version of characteristic polynomials.

Our methods should easily be applied to other similar problems, for instance with higher derivatives of characteristic polynomials.

More data and computer programs are available as expanded content.

1. Introduction

Our central object of study is the characteristic polynomial

$$Z_U(\theta) := \prod_{j=1}^{N} (1 - e^{\mathrm{i}(\theta_j - \theta)})$$

of an $N \times N$ random unitary matrix U with eigenangles θ_j, and specifically the *joint moments* of the powers of this polynomial and its derivative. Our results, which we state forthwith, are motivated by questions in number theory (page 33), and obtain by techniques from representation theory and algebraic combinatorics (see Section 1).

We define

$$V_U(\theta) := e^{\mathrm{i}N(\theta+\pi)/2} e^{-\mathrm{i}\sum_{j=1}^{N} \theta_j/2} Z_U(\theta). \tag{1}$$

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It is easily checked that for real θ, $V_U(\theta)$ is real and $|V_U(\theta)| = |Z_U(\theta)|$.

In this paper, we will investigate the averages (with respect to Haar measure)

$$|\mathcal{M}|_N(2k, r) := \left\langle |Z_U(0)|^{2k} \left| \frac{Z'_U(0)}{Z_U(0)} \right|^r \right\rangle_{\mathrm{U}(N)},$$

$$(\mathcal{M})_N(2k, r) := \left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)},$$

$$|\mathcal{V}|_N(2k, r) := \left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^r \right\rangle_{\mathrm{U}(N)}$$

and their asymptotics

$$|\mathcal{M}|(2k, r) := \lim_{N\to\infty} |\mathcal{M}|_N(2k, r)/N^{k^2+r},$$

$$(\mathcal{M})(2k, r) := \lim_{N\to\infty} (\mathcal{M})_N(2k, r)/N^{k^2+r},$$

$$|\mathcal{V}|(2k, r) := \lim_{N\to\infty} |\mathcal{V}|_N(2k, r)/N^{k^2+r}.$$

As they involve both the characteristic polynomials and their derivatives, we call these averages joint moments. It is easy to show (by expanding the Haar measure explicitly) that the joint moments at finite N only make sense when $2k - r > -1$. For the asymptotics, the normalization by N^{k^2+r} is due to Hughes [2005] (and proved in this paper anyway).

This and related problems have been looked at by Conrey et al. [2006], Hughes [2001; 2005], Hughes et al. [2000], Forrester and Witte [2006a] and Mezzadri [2003]. However, much mystery remains, in particular for the dependency in r when $r \in \mathbb{R} \setminus \mathbb{N}$.

While $r \in \mathbb{R} \setminus \mathbb{N}$ remains out of reach, we offer in this paper an alternative approach that uncovers some of the structure in those averages.

Theorem 1.1. *For $r \in \mathbb{N}$ and $k \in \mathbb{C}$, the moments $(\mathcal{M})(2k, r)$ are essentially given by rational functions, that is, as meromorphic functions of k we have*

$$(\mathcal{M})(2k, r) = \left(-\frac{\mathrm{i}}{2}\right)^r \frac{G(k+1)^2}{G(2k+1)} \frac{X_r(2k)}{Y_r(2k)}, \tag{2}$$

where X_r and Y_r are even monic polynomials with integer coefficients and with $\deg X_r = \deg Y_r$ *and G is the Barnes G-function [Hughes et al. 2000, Appendix].*

Moreover

$$Y_r(u) = \prod_{\substack{1\le a\le r-1\\ a\ odd}} (u^2 - a^2)^{\alpha_a(r)},$$

with the $\alpha_a(\cdot)$ *given by*

$$\alpha_a(r) = \left\lfloor \frac{-a+\sqrt{a^2+4r}}{2} \right\rfloor.$$

We derive from this a similar result (Theorem 6.1, page 55) for $|\mathcal{M}|(2k, 2h)$ and $|\mathcal{V}|(2k, 2h)$ (for h an integer). Finally, we have explicit expressions for $(\mathcal{M})(2k, r)$ given in Theorem 5.11, page 53 and Theorem 8.2, page 62 which allow us to compute the $X_r(u)$s, as given in Table 2, page 56, and additional data (available in Section 7).

Motivation. Ever since the works by Keating and Snaith [2000a; 2000b], the Riemann ζ-function can be (conjecturally but quantitatively) better understood through the modeling by characteristic polynomials of unitary matrices. The classical example concerns moments. Let

$$g(k) := \frac{G(k+1)^2}{G(2k+1)},$$
$$a(k) := \prod_{p \text{ prime}} \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}.$$

Then one can prove (fairly immediately, using the Selberg integral) that

$$|M|(2k, 0) = g(k), \tag{3}$$

which according to the Keating–Snaith philosophy leads to the following conjecture (for $k > -1/2$):

$$\frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k} \mathrm{d}t \sim_T g(k)\, a(k) \left(\log \frac{T}{2\pi}\right)^{k^2}. \tag{4}$$

The main point is thus that $a(k)$ is obtained by looking at primes, while $g(k)$ is guessed at from the random matrix side.

Observe also that Equations (3) and then (4) can be analytically continued in k.

Many of the authors cited above have now shown that this philosophy should be extended to the derivatives of characteristic polynomials.

In particular, $|\mathcal{M}|(2k, r)$ should show up as the RMT factor of[1]

$$\mathcal{I}(2k, r) := \lim_{T\to\infty} T^{-1}\left(\log \frac{T}{2\pi}\right)^{-k^2-r} \int_0^T \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k-r} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}t)\right|^{r} \mathrm{d}t,$$

[1]It is a conjecture of Hall [2004] and Hughes [2001] that this is the appropriate normalization with respect to T.

k	h	$a(k)$	$\lvert\mathcal{M}\rvert(2k,2h)$	$\mathcal{I}(2k,2h)$	
1	1	1	$\left(\frac{1}{2^2}\right)\frac{1^2}{1}\frac{(2\times 1)^2}{(2^2-1^2)^1}$	$\frac{1}{3}$	[Ingham 1928]
2	1	$\frac{6}{\pi^2}$	$\left(\frac{1}{2^2}\right)\frac{1^2}{12}\frac{(2\times 2)^2}{(4^2-1^2)^1}$	$\frac{2}{15\pi^2}$	[Conrey 1988]
2	2	$\frac{6}{\pi^2}$	$\left(\frac{1}{2^4}\right)\frac{1^2}{12}\frac{(2\times 2)^4-8(2\times 2)^2-6}{(4^2-1^2)^1(4^2-3^2)^1}$	$\frac{61}{1680\pi^2}$	[Conrey 1988]
k	h	$a(k)$	$\lvert\mathcal{V}\rvert(2k,2h)$	$\mathcal{J}(2k,2h)$	
1	1	1	$\left(\frac{2!}{1!2^3}\right)\frac{1^2}{1}\frac{1}{(2^2-1^2)^1}$	$\frac{1}{12}$	[Ingham 1928]; see also [Hughes 2005]
2	1	$\frac{6}{\pi^2}$	$\left(\frac{2!}{1!2^3}\right)\frac{1^2}{12}\frac{1}{(4^2-1^2)^1}$	$\frac{1}{120\pi^2}$	[Conrey 1988]; see also [Hall 2002b]
2	2	$\frac{6}{\pi^2}$	$\left(\frac{4!}{2!2^6}\right)\frac{1^2}{12}\frac{1}{(4^2-1^2)^1(4^2-3^2)^1}$	$\frac{1}{1120\pi^2}$	[Conrey 1988]; see also [Hall 2002b]

Table 1. Summary of results on $\mathcal{I}(2k,2h)$ and $\mathcal{J}(2k,2h)$ when $h \neq 0$. The values for $\lvert\mathcal{M}\rvert(2k,2h)$ and $\lvert\mathcal{V}\rvert(2k,2h)$ are as obtained from Theorem 6.1. The fifth column equals the product of the third and the fourth. The last column gives the source where the result in the fifth column was first published.

and similarly $\lvert\mathcal{V}\rvert(2k,r)$ is needed for

$$\mathcal{J}(2k,r) := \lim_{T\to\infty} T^{-1}\left(\log\frac{T}{2\pi}\right)^{-k^2-r}\int_0^T \left|\mathfrak{Z}(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k-r}\left|\mathfrak{Z}'(\tfrac{1}{2}+\mathrm{i}t)\right|^{r}\mathrm{d}t,$$

where $\mathfrak{Z}$ is Hardy's function (the relationship of $\mathfrak{Z}$ to ζ is analogous to the relationship of V_U to Z_U, that is, when $t\in\mathbb{R}$, $\mathfrak{Z}(\frac{1}{2}+\mathrm{i}t)\in\mathbb{R}$ and $\pm\mathfrak{Z}(\frac{1}{2}+\mathrm{i}t) = \left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|$). More precisely, it is expected that

$$\mathcal{I}(2k,r) = a(k)\ \lvert\mathcal{M}\rvert(2k,r) \quad \text{and} \quad \mathcal{J}(2k,r) = a(k)\ \lvert\mathcal{V}\rvert(2k,r).$$

Thus Theorems 1.1 and 6.1 give us a conjectural handle on the moments of ζ and $\mathfrak{Z}$.

One can compute some small cases (for integer k and r) and show that they agree with previous Number Theory (proved) results. This had already been done before and is repeated in Table 1.

However, while Keating and Snaith obtained a full conjecture for $\mathcal{I}(2k,0)$ and $\mathcal{J}(2k,0)$ by computing $\lvert\mathcal{M}\rvert(2k,0)$ and $\lvert\mathcal{V}\rvert(2k,0)$, for the case of joint moments this goal remains elusive. All the available formulas for $|\mathcal{M}|(2k,r)$ or $|\mathcal{V}|(2k,r)$ are rather inadequate. In particular, those formulas are limited to $r := 2h$ (h an integer), they are hard to compute for large values of k and h, they obscure some of the structure in the results, and finally they cannot be analytically continued in h.

Analytic continuation would be important, because Conrey and Ghosh [1989] have proved (assuming the Riemann Hypothesis) that

$$\mathcal{J}(2,1) = \frac{e^2-5}{4\pi}$$

and hence effectively conjectured[2]

$$|\mathcal{V}|(2,1) = \frac{e^2-5}{4\pi}$$

as well since $a(1) = 1$. In order to get this, we would need to have a sufficiently nice formula for $|\mathcal{V}|(2k,2h)$ that would allow for the analytic continuation in h. We have simply been unable to do this but have no doubt that our results should be helpful for that goal (see the connection with Noumi's work below).

On the other hand, the formulas obtained in Theorem 5.11, page 53 allow for much more effective computation than possible before, and we can compute longer tables for the different moments (see Section 7).

This numerical data is useful as well, as Hall has devised (around 2002) a method that uses $\mathcal{J}(2k,2h)$ for all $0 \le h \le k$ to produce a lower bound $\Lambda(k)$ on

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1}-t_n}{2\pi/\log t_n},$$

where the t_n is the n-th positive real zero of $\zeta(1/2+\mathrm{i}t)$. It is probably good to insist that this method does not depend on the Riemann Hypothesis, but only on values for moments! At the time of writing [Hall 2004], Hall only had the information he needed for k up to 2 (conjecturally, up to 6). In Section 7, we present our conjectural data for $\mathcal{J}(2k,2h)$ as a direct function of k for h up to 15 (also available online at [Dehaye 2007a] for h up to 30). For a fixed h, various conjectural formulas are also given in this paper for $\mathcal{J}(2k,2h)$ as a function of k. This, combined with Hall's method, should lead to more (conjectural) lower bounds on Λ. It is widely believed that $\Lambda = \infty$ so potentially we could also see if Hall's method has any hope to reach that, assuming only information on the $\mathcal{J}(2k,2h)$, but not on the Riemann Hypothesis. In other words, it would also inform us on the relationship between moment conjectures, the Riemann Hypothesis and the conjecture $\Lambda = \infty$. We leave this to a further paper.

[2] This is completely backwards from the usual flow of conjectures *from* random matrix theory *to* number theory, and possibly an unique instance of a reversal of this type.

Finally, Noumi [2004] investigates the relationship between Painlevé equations and expressions similar to one of the expressions we obtain for $(\mathcal{M})(2k,r)$, in Theorem 8.2. Connections of this sort have been uncovered before (see [Forrester and Witte 2006a; 2006b] and works of Borodin), but an approach through Noumi's ideas would be original. One of our goals then would be to obtain analytic continuation for $(\mathcal{M})(2k,r)$ in r, which would again allow to compute $|\mathcal{V}|(2,1)$. We also leave this for further study.

Our techniques are quite disconnected from the original motivation, so we discuss them separately.

Techniques. As mentioned, our techniques lie mostly in representation theory and algebraic combinatorics. We look at the characteristic polynomials or the derivatives as symmetric functions of the eigenvalues of U, and express them in that way. We eventually express those symmetric functions in the most natural basis to use, the Schur functions. This basis is particularly suitable since those functions are also (irreducible) characters of unitary groups $\mathrm{U}(N)$. We find ourselves integrating irreducible characters over their support (groups), which is very enviable!

To express all the different functions in this basis of the Schur functions, we use ideas present in [Bump and Gamburd 2006] and the author's thesis [Dehaye 2006]. We will introduce those ideas as we need them.

For a more thorough discussion of why a similar approach should always be attempted and other examples of its applications, please see the author's thesis and the results in [Dehaye 2007b].

Once we have a concise expression for the various moments, we still have to evaluate it. This will involve sums over partitions of values of the Schur functions. After reparametrizing those sums over the Frobenius coordinates of the partitions, the results of El-Samra and King were immediately useful to obtain the Schur values, and the results of Borodin to handle the combinatorics of the sums. We then obtain a very big sum for the moments (Theorem 5.9), but that can directly be evaluated on computer (and thus checked against small N results). After taking asymptotics, our results start simplifying into Theorem 5.11, enough to prove Theorem 1.1 on the general shape of those moments. However, the best expression is probably obtained once we use Macdonald's ninth variation of the Schur functions (Theorem 8.2).

Organization of this paper. In Section 2, we introduce all the nonstandard notation we will be using. In Section 3, we present the basic relations satisfied by the integrands

$$|Z_U(0)|^{2k}\left|\frac{Z_U'(0)}{Z_U(0)}\right|^r,\quad |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r,\quad \text{and}\quad |V_U(0)|^{2k}\left|\frac{V_U'(0)}{V_U(0)}\right|^r.$$

The bulk of this paper is contained in Sections 4 and 5. In Section 4, we reexpress the integrands as a sum in the Schur basis, in a way similar to Bump and Gamburd (via the Dual Cauchy Identity). In Section 5, we engage in a long computation to evaluate the result obtained in the previous section, mostly using the results of El-Samra and King, and Borodin. Section 6 merely serves to tie what has been done in Sections 4 and 5 into the proof of Theorem 1.1. In Section 7 we present the data we are now able to compute, and particularly discuss the position of the roots of $|\mathscr{V}|(2k, 2h)$ starting on page 57. Section 8 describes two attempts to simplify our results further, one using Macdonald's ninth variation of the Schur functions, and the other imitating a proof of the Cauchy identity.

2. Notation

We let $\mathbb{N}_+$ be the set $\mathbb{N} \setminus 0$. To avoid confusion with the index i, we set $\mathrm{i}^2 = -1$.

We use $\mathbf{v}$ for a generic *vector* (of integers) $(v_1, \dots, v_d)$, and $\vec{\mathbf{v}}$ for a strictly decreasing sequence of integers $v_1 > v_2 > \dots > v_d$, which we call a *Frobenius sequence*. Frobenius sequences are thus a special type of vectors.

Sequences of weakly decreasing positive integers amount to partitions, and we stick with classical notation for those, $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$, which defines $l(\lambda)$. We also freely change our point of view to Young tableaux when discussing partitions. We denote by λ^t the conjugate of a partition λ of $|\lambda|$. Define two sequences $p_i := \lambda_i - i$, $q_i := \lambda_i^t - i$. They are strictly decreasing; λ_i and λ_i^t are eventually 0, and hence $p_i = -i$ and $q_i = -i$ eventually. There exists d such that $p_d \geq 0 > p_{d+1}$ and $q_d \geq 0 > q_{d+1}$. We call d the *rank* of λ. The vectors $\vec{\mathbf{p}} = (p_1, \dots, p_d)$ and $\vec{\mathbf{q}} = (q_1, \dots, q_d)$ are Frobenius sequences, and we call $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ the *Frobenius coordinates* of the partition λ. We write $\lambda = \{\vec{\mathbf{p}} : \vec{\mathbf{q}}\}$.

Given $\mathbf{p}$, we define $\sigma_{\mathbf{p}} \in \mathscr{S}_d$ such that $\operatorname{sort}(\mathbf{p}) := (p_{\sigma_{\mathbf{p}}(i)})$ is strictly decreasing (and hence a Frobenius sequence). This is thus not defined if $p_i = p_j$ while $i \neq j$. We set $\operatorname{sgn}(\mathbf{p}) := \operatorname{sgn}(\sigma_{\mathbf{p}})$, with the added convention that $\operatorname{sgn}(\mathbf{p}) := 0$ if $\sigma_{\mathbf{p}}$ is not defined.

If λ and μ are partitions, $\lambda \cup \mu$ is the partition obtained by taking the union of their parts. The partition $\langle X^Y \rangle$ has a $Y \times X$ rectangle for Young tableau.

We also use the notation $[1^R]$ for R copies of 1, used as argument to a (Schur) function.

3. Basic relations among the integrands

We logarithmically differentiate Equation (1) to obtain

$$\frac{V_U'(\theta)}{V_U(\theta)} = \frac{\mathrm{i}N}{2} + \frac{Z_U'(\theta)}{Z_U(\theta)}$$

and hence, when θ is real,

$$\left|\frac{Z_U'(\theta)}{Z_U(\theta)}\right|^2 = \left|\frac{V_U'(\theta)}{V_U(\theta)}\right|^2 + \frac{N^2}{4} = \left(\frac{V_U'(\theta)}{V_U(\theta)}\right)^2 + \frac{N^2}{4} = \left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right)^2 + \mathrm{i}N\left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right).$$

These basic relations give

$$|\mathcal{M}|_N\,(2k,2h) = \sum_{j=0}^{h} (\mathrm{i}N)^{h-j}\binom{h}{j}(\mathcal{M})_N(2k,h+j),$$

$$|\mathcal{M}|\,(2k,2h) = \sum_{j=0}^{h} \mathrm{i}^{h-j}\binom{h}{j}(\mathcal{M})(2k,h+j), \tag{5}$$

$$|\mathcal{V}|_N\,(2k,2h) = \sum_{j=0}^{h} \binom{h}{j}\left(\frac{-N^2}{4}\right)^{h-j}|\mathcal{M}|_N(2k,2j),$$

$$|\mathcal{V}|\,(2k,2h) = \sum_{j=0}^{h} \binom{h}{j}\left(\frac{-1}{4}\right)^{h-j}|\mathcal{M}|(2k,2j). \tag{6}$$

These formulas are initially valid only when h is a nonnegative integer, but the right-hand sides can be analytically continued by plugging in noninteger h and extending the sum to infinity.[3] Thus we see that computing $(\mathcal{M})_N(2k,r)$ would get us most of the way to $|\mathcal{M}|_N(2k,2h)$ or $|\mathcal{V}|_N(2k,2h)$, and we now focus on the integrand $|Z_U(0)|^{2k}\,(Z_U'(0)/Z_U(0))^r$.

4. Derivation into the Schur basis

The goal here is to follow ideas similar of Bump and Gamburd [2006] in order to prove Proposition 4.3, page 40. One of their main tools was the dual Cauchy identity. We encourage the reader to look at their first proposition and corollary for the unitary group, since this is all we really exploit from that paper.

Lemma 4.1 (Dual Cauchy identity). *If $\{x_i\}$ and $\{y_j\}$ are finite sets of variables,*

$$\prod_{i,j}(1+x_iy_j) = \sum_{\lambda} s_{\lambda^t}(x_i)s_\lambda(y_j),$$

where the sum is over all partitions λ and s_λ is the Schur polynomial.

[3]Getting the correct analytic continuation can be tricky. The relation

$$|\mathcal{V}|\,(2k,2h) = \sum_{j=0}^{2h}\binom{2h}{j}\left(\frac{\mathrm{i}}{2}\right)^j(\mathcal{M})(2k,2h-j)$$

is also valid for integers h, but here the right-hand side does not analytically continue in h to the left-hand side, since we exploit $|V_U'(\theta)/V_U(\theta)|^{2h} = (V_U'(\theta)/V_U(\theta))^{2h}$, where h must be an integer.

Apply this lemma setting $\{x_j := e^{\mathrm{i}\theta_j} \mid j \in [1,\dots,N]\}$ to be the set of eigenvalues of U, and $\{y_j := 1 \mid j \in [1,\dots,2k]\}$. We chose the notation $s_\lambda(U) := s_\lambda(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_N})$. This gives

$$\begin{aligned}\sum_\lambda \overline{s_{\lambda^t}(U)}\, s_\lambda([1^{2k}]) = \det(\mathrm{Id}+\overline{U})^{2k} &= \overline{\det U}^k\, |\det(\mathrm{Id}+U)|^{2k} \\ &= \overline{s_{\langle k^N\rangle}(U)}\, |\det(\mathrm{Id}+U)|^{2k}\end{aligned}$$

or (replacing U by $-U$)

$$|Z_U(0)|^{2k} = |\det(\mathrm{Id}-U)|^{2k} = (-1)^{kN} s_{\langle k^N\rangle}(U) \sum_\lambda (-1)^{|\lambda|}\overline{s_{\lambda^t}(U)}\, s_\lambda([1^{2k}]).$$

We can also reexpress

$$\frac{Z_U'(0)}{Z_U(0)} = \sum_{j=1}^N \frac{\mathrm{i}e^{\mathrm{i}\theta_j}}{1-e^{\mathrm{i}\theta_j}} = \sum_{j=1}^N \mathrm{i} \lim_{z\to 1^-} \sum_{m=1}^\infty z^m e^{\mathrm{i}m\theta_j} = \mathrm{i} \lim_{z\to 1^-} \sum_{m=1}^\infty z^m p_m(U), \tag{7}$$

where $p_m(x_1,\dots,x_N)$ is the m-th power sum $x_1^m+\cdots+x_N^m$ and we have used the same convention as for $s_\lambda(U)$ of inputting the eigenvalues. We will use the same convention soon for the power sums $p_\lambda := \prod_i p_{\lambda_i}$.

In practice, we want the reader to just ignore the variable z and set it to 1. This will be justified *a posteriori*.

Putting everything together, we thus get for $|Z_U(0)|^{2k}\, (Z_U'(0)/Z_U(0))^r$

$$(-1)^{kN} s_{\langle k^N\rangle}(U) \left(\mathrm{i} \sum_{m=1}^\infty p_m(U) \right)^r \sum_\lambda (-1)^{|\lambda|} s_\lambda([1^{2k}])\, \overline{s_{\lambda^t}(U)}. \tag{8}$$

At this point, we will soon want to use the fact that the s_λs are the characters of unitary groups.

Indeed, if $U \in \mathrm{U}(N)$ then when $l(\lambda) > N$, we have[4] $s_\lambda(U) \equiv 0$, but when $l(\lambda), l(\mu) \le N$, we have

$$\left\langle s_\lambda(U)\overline{s_\mu(U)}\right\rangle_{\mathrm{U}(N)} = \delta_{\lambda\mu};$$

that is, for large enough N, s_λ is an irreducible character of $\mathrm{U}(N)$. This orthogonality is obviously good for our purposes, but the only obstacle is the need to express $s_{\langle k^N\rangle}(U)\big(\sum_{m=1}^\infty p_m(U)\big)^r$ exclusively in terms of the Schur functions. This can be done and will require the Murnaghan–Nakayama rule.

Let a ribbon be a connected Young skew-tableau not containing any 2×2-block. If a ribbon contains m blocks, it is called a m-ribbon. A first approximation to one version of the M–N rule says that $s_\lambda p_m$ is given by a signed sum of s_μs, where μ runs through all partitions obtained by adding a m-ribbon to λ.

[4]This is a consequence of the fact that $s_\lambda(x_1,\dots,x_n) \equiv 0$ if $l(\lambda) > n$.

If we average Expression (8) over $\mathrm{U}(N)$, we could thus see λ as running through all partitions obtained by adding r ribbons to the rectangle $\langle N^k \rangle$ (this uses the fact that this lax version of the M–N rule is invariant under transpositions, since we have yet to discuss the signs). There are more conditions, however. We also need $l(\lambda^t) \leq N$ (since otherwise $s_{\lambda^t}(U) \equiv 0$, as in note 4), and we need $l(\lambda) \leq 2k$ (since otherwise $s_\lambda([1^{2k}]) = 0$, again just as in that note). In other words, λ contains $\langle N^k \rangle$ but is contained in $\langle N^{2k} \rangle$. There are only finitely many (ways to obtain) such partitions, which will make the sum over λs finite, and thus only finitely many sets of lengths of the r ribbons will contribute. This justifies *a posteriori* setting z to 1 in (7), but *only when we can apply the dominated convergence theorem*. This will only occur if we know of a bound on the integrand independent of z that is itself integrable. We can pick $|Z_U(0)|^{2k} \left| Z'_U(0)/Z_U(0) \right|^r$ whenever this is integrable, that is, only when $2k - r > -1$.

We now state a more precise version of the M–N rule.

Theorem 4.2 (Murnaghan–Nakayama). *Let λ be a partition and $\boldsymbol{\rho}$ be a vector with $|\lambda| = \sum_i \boldsymbol{\rho}_i$. If $\chi^\lambda_{\boldsymbol{\rho}}$ is the value of the irreducible character of $\mathcal{S}_{|\lambda|}$ associated to λ on the conjugacy class of cycle-type* $\mathrm{sort}(\boldsymbol{\rho})$, *then*

$$p_{\boldsymbol{\rho}} = \sum_\lambda \chi^\lambda_{\boldsymbol{\rho}} s_\lambda \tag{9}$$

and (*more importantly*)

$$\chi^\lambda_{\boldsymbol{\rho}} = \sum_S (-1)^{\mathrm{ht}(S)}$$

summed over all sequences of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $r := l(\lambda)$, $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$, and such that each $\lambda^{(i)} - \lambda^{(i-1)}$ is a ribbon of length $\boldsymbol{\rho}_i$, and $\mathrm{ht}(S) = \sum_i \mathrm{ht}(\lambda^{(i)} - \lambda^{(i-1)})$.

We have not defined the height ht of a ribbon, but rather than doing so or detailing the computation here, we only expose the idea. Equation (9) tells us that $(\sum_m p_m)^r$ can be computed using the character values of symmetric groups, which can be evaluated by summing over sequences of partitions $(\lambda^{(0)}, \ldots, \lambda^{(r)})$. For each such sequence, the sequence $(\tilde{\lambda}^{(0)}, \ldots, \tilde{\lambda}^{(r)})$, with $\tilde{\lambda}^{(i)} := \langle N^k \rangle \cup \lambda^{(i)}$, would be associated with the combinatorics of the expansion of the product in (8). Indeed the combinatorics of ribbon is unchanged under translations (down by k) as long as the partitions are kept within a rectangle (actually, a horizontally bounded region).

If the computation is explicitly carried out, we get the following result.

Proposition 4.3. *If $2k - r > -1$, we have*

$$(\mathcal{M})_N(2k, r) = (-\mathrm{i})^r \sum_{\boldsymbol{\mu} \in \mathbb{N}^r_+} \sum_{\substack{\lambda \text{within} \\ k \times N}} \chi^\lambda_{\boldsymbol{\mu}}\, s_{\langle N^k \rangle \cup \lambda}([1^{2k}]), \tag{10}$$

with the understanding that $\chi^\lambda_{\boldsymbol{\mu}} = 0$ *if* $|\lambda| \neq \sum_i \boldsymbol{\mu}_i$.

For this result, we have preferred to index all the partitions containing $\langle N^k \rangle$ but contained in $\langle N^{2k} \rangle$ as $\langle N^k \rangle \cup \lambda$, for $\lambda \subset \langle N^k \rangle$.

We are now left with the task of evaluating the right-hand side of (10), which will turn out to be a tedious process.

5. Main computation

We are left with two problems. The first one is due to the characters of the symmetric group. Those are of course desperately hard to evaluate directly and individually. We are helped here because we will actually only evaluate something close to

$$\sum_{\boldsymbol{\mu} \in \mathbb{N}^l_+} \chi^\lambda_{\boldsymbol{\mu}}$$

for given λ. This amounts to computing the sum of values of the character χ^λ over the permutations with l cycles. The second issue is evaluating $s_{\langle N^k \rangle \cup \lambda}([1^{2k}])$. The author had previously used the Weyl Dimension Formula to do this (see [Dehaye 2006]). A formula giving that dimension in terms of the Frobenius coordinates of λ is probably better adapted for our purposes.

In addition, both "problems" combine extremely well, in that both expressions should involve a sign, which turns out to be the same.

We will then sum our terms over all partitions, expressed in Frobenius coordinates. This amounts to summing over possible ranks $(1 \leq d)$ and then pairs of Frobenius sequences of length d.

The value of the Schur function in Frobenius coordinates.

Dimension formula in Frobenius coordinates. El Samra and King [1979] use the notation $D_R\{p:q\}$ for $s_{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}}([1^R])$.

Assume $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ has d Frobenius coordinates. They prove that

$$\begin{aligned} s_{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}}([1^R]) &= \left| \frac{(R+p_i)!}{(R-q_j-1)!\,p_i!\,q_j!\,(p_i+q_j+1)} \right|_{d\times d} \\ &= \prod_{i=1}^{d} \frac{(R+p_i)!}{(R-q_i-1)!\,p_i!\,q_i!} \prod_{1\le i<j\le d} (p_i-p_j)(q_i-q_j) \prod_{i,j=1}^{d} \frac{1}{p_i+q_j+1} \end{aligned} \quad (11)$$

where the first expression is also known as the reduced determinantal form (see [Foulkes 1951], as cited in [El Samra and King 1979]).

It is a consequence of Cauchy's Lemma that the two expressions in (11) are equivalent.

Lemma 5.1 (Cauchy).

$$\left|\frac{1}{p_i+q_j+1}\right|_{d\times d} = \prod_{1\le i<j\le d}(p_i-p_j)(q_i-q_j)\prod_{i,j=1}^{d}\frac{1}{p_i+q_j+1}.$$

Observe that Formula (11) is positive (as it should, given that it is also a dimension) because the p_i and q_i are strictly decreasing.

However, the right-hand side of (11) still makes sense if we plug in the unsorted vectors $\mathbf{p}$, $\mathbf{q}$ (with even the possibility of $i\neq j$ but $p_i=p_j$). Hence this can be used to define $s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R])$ as well, which is then skew-symmetric in both the p_is and the q_is separately. This can be written

$$s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R]) = \operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})s_{\{\mathrm{sort}(\mathbf{p})\,:\,\mathrm{sort}(\mathbf{q})\}}([1^R]). \tag{12}$$

Observe that Formula (12) is still valid when $\mathrm{sort}(\mathbf{p})$ or $\mathrm{sort}(\mathbf{q})$ is not defined (this happens when two of the entries of $\mathbf{p}$ or $\mathbf{q}$ are equal) thanks to $\operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})=0$! (See the conventions in Section 2.)

Finally, it is helpful to remark that Formula (11) for $s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R])$ can be seen as a product indexed by the sets $\mathbf{p}\cup\mathbf{q}$ and pairs in the set $(\mathbf{p}\times\mathbf{p})\cup(\mathbf{q}\times\mathbf{q})\cup(\mathbf{p}\times\mathbf{q})$.

Evaluation of $s_{\langle N^k\rangle\cup\lambda}([1^{2k}])$. We take $\lambda=\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ to have d Frobenius coordinates.

In total analogy with (12), we first extend the definition of $s_{\langle N^k\rangle\cup\lambda}$ and set

$$s_{\langle N^k\rangle\cup\{\mathbf{p}\,:\,\mathbf{q}\}} := \operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})\,s_{\langle N^k\rangle\cup\{\mathrm{sort}(\mathbf{p})\,:\,\mathrm{sort}(\mathbf{q})\}},$$

with the understanding (as before) that the value of the right-hand side is taken as 0 if $p_i=p_j$ or $q_i=q_j$ for some $i\neq j$. Again, this is skew-symmetric in the p_is and separately in the q_is.

We have the following lemma.

Lemma 5.2. *Let* $\mathbf{p}$, $\mathbf{q}$ *be vectors with* d *coordinates. Then*

$$\begin{aligned} &s_{\langle N^k\rangle\cup\{\mathbf{p}\,:\,\mathbf{q}\}}([1^{2k}]) \\ &\qquad = s_{\langle N^k\rangle}([1^{2k}])\left(\prod_{i=1}^{d}\frac{(N-p_i)^{(k)}(k-q_i)^{(k)}}{(p_i+k+1)^{(k)}(N+q_i+1)^{(k)}}\right)s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^{2k}]). \end{aligned} \tag{13}$$

Proof. By skew-symmetry, we really only have to check this for $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$. If we want to use (11), we should look at the Frobenius coordinates of $\langle N^k\rangle\cup\lambda$. This would be rather unpleasant (particularly because the number of Frobenius coordinates would change for fixed N and k according to the λ considered).

We look instead at

$$\begin{aligned}
\vec{\mathbf{x}} &:= (N+k-1, \ldots, N), \\
\vec{\mathbf{y}} &:= (2k-1, \ldots, k), \\
\vec{\boldsymbol{\alpha}} &:= \vec{\mathbf{x}} \cup \vec{\mathbf{p}} \quad \text{(sorted)}, \quad \text{and} \\
\vec{\boldsymbol{\beta}} &:= \vec{\mathbf{y}} \cup \vec{\mathbf{q}} \quad \text{(sorted)}.
\end{aligned}$$

Then $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ are strictly decreasing, so those are Frobenius coordinates. The partition corresponding to those coordinates is obtained geometrically by sticking a $\langle k^{2k}\rangle$ block to the left of $\langle N^k\rangle \cup \lambda$, or equivalently, shifting $\langle N^k\rangle \cup \lambda$ by k spots to the right, while considering $\lambda = (\lambda_1, \ldots, \lambda_k)$ to have exactly k parts (with some possibly empty).

Because of this, we have (as in [Bump and Gamburd 2006, page 6])

$$s_{\{\vec{\boldsymbol{\alpha}}\,:\,\vec{\boldsymbol{\beta}}\}}([1^{2k}]) = e_{2k}^{k}([1^{2k}])\, s_{\langle N^k\rangle\cup\lambda}([1^{2k}]) = s_{\langle N^k\rangle\cup\lambda}([1^{2k}]).$$

Additionally, $\{\vec{\mathbf{x}} : \vec{\mathbf{y}}\}$ are the Frobenius coordinates of $\langle (N+k)^k\rangle \cup \langle k^k\rangle$. Hence, for the same reason as above, we have

$$s_{\{\vec{\mathbf{x}}\,:\,\vec{\mathbf{y}}\}}([1^{2k}]) = s_{\langle (N+k)^k\rangle\cup\langle k^k\rangle}([1^{2k}]) = e_{2k}^{k}([1^{2k}])\, s_{\langle N^k\rangle}([1^{2k}]) = s_{\langle N^k\rangle}([1^{2k}]).$$

When evaluating the product described in (11) using the $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ coordinates, we have a big product taken over the sets $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$, $\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\beta}}$. We expand those index sets using $\vec{\boldsymbol{\alpha}} = \vec{\mathbf{x}} \cup \vec{\mathbf{p}}$ and $\vec{\boldsymbol{\beta}} = \vec{\mathbf{y}} \cup \vec{\mathbf{q}}$.

One can see that the products indexed by $\vec{\mathbf{p}}$, $\vec{\mathbf{q}}$, $\vec{\mathbf{p}} \times \vec{\mathbf{p}}$, $\vec{\mathbf{q}} \times \vec{\mathbf{q}}$ and $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ together give

$$s_{\{\vec{\mathbf{p}}\,:\,\vec{\mathbf{q}}\}}([1^{2k}]) = s_\lambda([1^{2k}]).$$

Similarly, the products indexed by $\vec{\mathbf{x}}$, $\vec{\mathbf{y}}$, $\vec{\mathbf{x}} \times \vec{\mathbf{x}}$, $\vec{\mathbf{y}} \times \vec{\mathbf{y}}$ and $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ give

$$s_{\{\vec{\mathbf{x}}\,:\,\vec{\mathbf{y}}\}}([1^{2k}]) = s_{\langle N^k\rangle}([1^{2k}]).$$

We are left with only "cross-products" to evaluate, for the index sets $\vec{\mathbf{x}} \times \vec{\mathbf{p}}$, $\vec{\mathbf{x}} \times \vec{\mathbf{q}}$, $\vec{\mathbf{y}} \times \vec{\mathbf{p}}$ and $\vec{\mathbf{y}} \times \vec{\mathbf{q}}$. The definitions of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ now give the result. □

Sums of characters over conjugacy classes with same number of cycles. Assume $f(\{\mathbf{p} : \mathbf{q}\})$ is a function of pairs of vectors of the same length (say d). One can set $f(\lambda) := f(\{\vec{\mathbf{p}} : \vec{\mathbf{q}}\})$, where $\lambda = \{\vec{\mathbf{p}} : \vec{\mathbf{q}}\}$.

The goal in this section is to evaluate sums of characters of the general form

$$\sum_{\mu \in \mathbb{N}_+^l} \chi_\mu^\lambda f(\lambda).$$

We will eventually take $f(\lambda) = s_{\langle N^k\rangle\cup\lambda}([1^{2k}])$ but there is no reason to limit ourselves in that way for a while.

We rely on a few results of Borodin, that give a slightly different version of the Murnaghan–Nakayama rule.

Definitions. This is based on [Borodin 2000, around page 15] and [Borodin and Olshanski 1998, around page 6]. The relevant definitions (not included here) are *fragment*, the different *block* types, the *filling numbers*, *filled structure*, *sign of a structure*.

Theorem 5.5 is almost in Borodin's work, and his definitions are used in Proposition 5.6. Both of those results are used for Theorem 5.9, which can be read without looking at Borodin's papers.

However, the first condition to have a fragment needs clarification in both papers, that is, we change

> "(1) there is exactly one hook block that precedes the others"

to

> "(1) there is exactly one hook block in each fragment. That hook block precedes any other block in the fragment".

We also would like to correct a statement in [Borodin 2000], in that linear horizontal or vertical blocks are *positive*, not just *nonnegative* integers (in agreement with the other cited paper of Borodin [Borodin and Olshanski 1998]).

We can highlight one of the definitions. Any filled structure T with d fragments produces a set of pairs

$$\{(p_1, q_1), \dots, (p_d, q_d)\}$$

which consists of the filling p- and q- numbers of the fragments.

The sign of T is defined as follows.

$$\operatorname{sgn}(T) = \operatorname{sgn}(\mathbf{p}) \operatorname{sgn}(\mathbf{q})(-1)^{\sum q_i + v(T)},$$

where, as a reminder, the sgn inside the formula is 0 if $p_i = p_j$ (respectively, $q_i = q_j$) for $i \neq j$.

Simplified Murnaghan–Nakayama rule. Although we haven't defined anything, we state Proposition 4.3, taken from the first paper of Borodin.

Proposition 5.3. *For any two partitions λ and ρ with $|\lambda| = |\rho|$, we have*

$$\chi^{\lambda}_{\rho} = \sum_{T} \operatorname{sgn} T,$$

where the sum is taken over all filled structures of cardinality $\rho = (\rho_1, \dots, \rho_l)$ such that the sequences $(p_1, \dots, p_d)$ and $(q_1, \dots, q_d)$ of filling p-numbers and q-numbers of the structure T coincide, up to a permutation, with the Frobenius p-coordinates and q-coordinates of the partition λ, that is, $\lambda = \{\operatorname{sort}(\mathbf{p}) : \operatorname{sort}(\mathbf{q})\}$.

The proof of this proposition is quite simple. Back to the original presentation of the Murnaghan–Nakayama rule in terms of hooks, Borodin analyzes what happens to Frobenius coordinates when subtracting hooks/ribbons. Each such subtraction corresponds to a block. There are three cases to distinguish: the hook/ribbon can be above or below the "Frobenius diagonal" or even overlap it. Those cases correspond respectively to linear horizontal blocks, linear vertical blocks, and hook blocks.

This proposition, as stated in Borodin's work, is slightly restrictive: there is no need for ρ to be a partition. Let $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_l)$ be a vector of positive integers and define (just as in Theorem 4.2) $\chi^\lambda_{\boldsymbol{\rho}} := \chi^\lambda_{\mathrm{sort}(\boldsymbol{\rho})}$. Then, by summing over all vectors $\boldsymbol{\rho}$, we get:

Proposition 5.4. *For any partition* λ,

$$\sum_{\boldsymbol{\rho}\in\mathbb{N}^l_+} \chi^\lambda_{\boldsymbol{\rho}} = \sum_T \operatorname{sgn} T,$$

where the sum is taken over all filled structures T *of* l *blocks and with filling* p-*numbers* $(p_1, \ldots, p_d)$ *and* q-*numbers* $(q_1, \ldots, q_d)$ *such that* $\lambda = \{\mathrm{sort}(\mathbf{p}) : \mathrm{sort}(\mathbf{q})\}$.

Observe that d, the rank of λ, has to be less than or equal to l in order to have a structure.

We now state the main theorem we will use, which originates in Borodin's work.

Theorem 5.5. *Assume* f *is skew-symmetric within its two vector entries (separately), that is* $f\big(\{\mathrm{sort}(\mathbf{p}) : \mathrm{sort}(\mathbf{q})\}\big) = \mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\, f\big(\{\mathbf{p} : \mathbf{q}\}\big)$. *Then*

$$\sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{\boldsymbol{\rho}\in\mathbb{N}^l_+} \chi^\lambda_{\boldsymbol{\rho}} f(\lambda) = \sum_{d=1}^{l} \sum_{\substack{\mathbf{p}\in[0,N-1]^d \\ \mathbf{q}\in[0,k-1]^d}} f\big(\{\mathbf{p} : \mathbf{q}\}\big) \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)},$$

where $T(\mathbf{p}, \mathbf{q})$ *goes trough all filled structures of* d *fragments,* l *blocks,* $v(T)$ *vertical blocks with filling* p-*numbers* $(p_1, \ldots, p_d)$ *and* q-*numbers* $(q_1, \ldots, q_d)$.

Proof. We start by summing Proposition 5.4 over λs fitting inside a $k \times N$ box:

$$\begin{aligned}\sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{\boldsymbol{\rho}\in\mathbb{N}^l_+} \chi^\lambda_{\boldsymbol{\rho}} f(\lambda) &= \sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)}\, \mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\, f(\lambda) \\ &= \sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)} f\big(\{\mathbf{p} : \mathbf{q}\}\big),\end{aligned}$$

where the second sum in each right-hand side is over all filled structures $T(\mathbf{p}, \mathbf{q})$ of l blocks and d fragments such that the sequences of filling p-numbers $(p_1, \ldots, p_d)$ and q-numbers $(q_1, \ldots, q_d)$ of the structure coincide, up to two permutations, with the sequences of Frobenius p-coordinates and q-coordinates of the partition $\lambda = \{\operatorname{sort}(\mathbf{p}) : \operatorname{sort}(\mathbf{q})\}$. Note that d changes with λ.

We then obtain the final result by seeing the double sum over λ then permuted Frobenius coordinates of λ as a sum over *all* vectors of appropriate lengths.

We should not be concerned about vectors having two identical coordinates (say $p_i = p_j$), since the corresponding term on the right-hand side vanishes by skew-symmetry of f. $\square$

Counting structures. We now need to compute the sum

$$\sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)},$$

which is taken over the structures described above, that is, for given l, d, $\mathbf{p}$, $\mathbf{q}$, v. It would help to know how many structures there are for each choice of those parameters. We prove the following proposition.

Proposition 5.6. *There are exactly* $\#T(l, d, \mathbf{p}, \mathbf{q}, v) :=$

$$\sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d \\ \sum t_i = v \\ d+\sum s_i + t_i = l}} \left[\frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \prod_{i=1}^d (d + 1 - i + \sum_{j=i}^d s_j + t_j)} \right] \times \left[\prod_i^d \binom{p_i}{s_i} \binom{q_i}{t_i} \right] \tag{14}$$

structures with d fragments, l blocks, filling numbers $\mathbf{p} = (p_1, \ldots, p_d)$ and $\mathbf{q} = (q_1, \ldots, q_d)$ and v vertical blocks. The indices in the sum s_i (respectively, t_i) count horizontal (respectively, vertical) blocks in the i-th fragment.

Proof. This is a purely combinatorial problem. Given the number of vertical blocks on each fragment, we essentially have a partial order on blocks that we want to extend to form a linear order (across fragments). Part of the rules in the initial partial order say that the hook-block in the i-th fragment precedes any other block in that fragment. We then need to fill the structure, that is, to choose filling numbers for each block.

We can reverse this process.

- We first choose the numbers of horizontal and vertical blocks s_i and t_i on the i-th fragment. We have the conditions that $\sum t_i = v(T)$ and $d + \sum s_i + t_i = l$ (that is, there are l blocks in total, d hook, s_i horizontal in the i-th fragment and t_i horizontal in the i-th fragment).
- Starting from the d-th fragment, we decide where to insert the horizontal and vertical blocks of the i-th fragment in the partial order that is established so far on the set of fragments from the $(i + 1)$-th to the d-th one.
- We decide how to cut up the i-th fragment into filled blocks, respecting the number of horizontal/vertical blocks decided upon earlier.

The equality in the statement is intended to reflect clearly the layering described above: the sum corresponds to the first layer, while the other two layers correspond to one square-bracketed factor each.

Observe that the relation $s_d + t_d + \cdots + s_1 + t_1 + d = l$ could be used to simplify the numerator in this expression.

The only hard part is to derive for the second step

$$\frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \prod_{i=1}^{d} (d + 1 - i + \sum_{j=i}^{d} s_j + t_j)} = \frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \begin{pmatrix} (s_d+t_d+1)\times \\ (s_d+t_d+s_{d-1}+t_{d-1}+2)\times \\ \cdots\times \\ (s_d+t_d+s_{d-1}+t_{d-1}+\cdots+s_1+t_1+d) \end{pmatrix}}.$$

This is obtained by simplifying

$$\prod_{i=0}^{d-1} \binom{i + \sum_{j=d-i}^{d} s_j + t_j}{s_{d-i} + t_{d-i}} \binom{s_{d-i} + t_{d-i}}{s_{d-i}},$$

where the i-th factor in the $\prod_{i=0}^{d-1}$-product counts the number of ways of choosing the linear order on the blocks of the $(d-i)$-th fragment, as we know that the linear order restricted on the blocks of the fragments $d-i+1$ to d.

The first binomial factor intersperses the set of blocks of the $(d-i)$-th fragment among the blocks of fragments $d-i+1$ to d, while the second factor decides which blocks are horizontal and which are vertical. □

We wish to insist on the fact that the summand in (14) is not symmetric in the p_is or the q_is, because the factor in the denominator,

$$\prod_{i=1}^{d} \Big(d + 1 - i + \sum_{j=i}^{d} s_j + t_j \Big),$$

is not symmetric in the s_js or the t_js. For instance, s_d appears d times while s_1 appears only once.

Sum of determinants. We now aim to put together all the results obtained so far in this section, but we first need a quick lemma.

Lemma 5.7. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be vectors of integers. Then*

$$\sum_{\sigma,\tau\in\mathcal{S}_d} \frac{(\operatorname{sgn}\sigma \operatorname{sgn}\tau)}{\prod_{i=1}^{d}(d+1-i+\sum_{j=i}^{d} s_{\sigma(j)}+t_{\tau(j)})} = \prod_{1\le i<j\le d} (s_i-s_j)(t_i-t_j) \prod_{1\le i,j\le d} \frac{1}{1+s_i+t_j}.$$

Proof. The proof proceeds as for the classical computation for the Vandermonde determinant: the left-hand side is skew-symmetric in $\mathbf{s}$ and $\mathbf{t}$ separately, and has obvious poles as prescribed on the right-hand side (when $s_{i_0} + t_{j_0} = -1$), and the degrees on the right-hand side are appropriate. Up to a constant of proportionality, both sides are thus the same. This constant is shown to be 1 by looking at the rates of decrease when s_1 goes to infinity. □

Proposition 5.8. *Assume f is skew-symmetric within its two vector entries (separately), that is, $f\left(\{\mathrm{sort}(\mathbf{p}):\mathrm{sort}(\mathbf{q})\}\right)=\mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\,f\left(\{\mathbf{p}:\mathbf{q}\}\right)$. Then*

$$\sum_{\boldsymbol{\mu}\in\mathbb{N}_+^l}\sum_{\substack{\lambda\text{within}\\k\times N}}\chi_{\boldsymbol{\mu}}^{\lambda}f(\lambda)=l!\sum_{d=1}^{l}\sum_{\substack{\mathbf{p}\in[0,N-1]^d\\\mathbf{q}\in[0,k-1]^d\\v}}f\left(\{\mathbf{p}:\mathbf{q}\}\right)(-1)^{\sum q_i+v}$$
$$\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\\sum t_i=v\\d+\sum s_i+t_i=l}}\prod_i^d\binom{p_i}{s_i}\binom{q_i}{t_i}\prod_i\frac{1}{s_i!t_i!}\prod_{1\le i<j\le d}(s_i-s_j)(t_i-t_j)\prod_{1\le i,j\le d}\frac{1}{1+s_i+t_j}.$$

Proof. We first combine (10) with Theorem 5.5:

$$\sum_{\boldsymbol{\mu}\in\mathbb{N}_+^l}\sum_{\substack{\lambda\text{within}\\k\times N}}\chi_{\boldsymbol{\mu}}^{\lambda}f(\lambda)$$
$$=l!\sum_{d=1}^{l}\sum_{\substack{\mathbf{p}\in[0,N-1]^d\\\mathbf{q}\in[0,k-1]^d\\v}}f\left(\{\mathbf{p}:\mathbf{q}\}\right)(-1)^{\sum q_i+v}\times\sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d\\\sum t_i=v\\d+\sum s_i+t_i=l}}\frac{\prod_i^d\binom{p_i}{s_i}\binom{q_i}{t_i}}{\prod s_i!\prod t_i!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_j+t_j)}$$
$$=l!\sum_{d=1}^{l}\sum_{\substack{\mathbf{p}\in[0,N-1]^d\\\mathbf{q}\in[0,k-1]^d\\v}}f\left(\{\mathrm{sort}(\mathbf{p}):\mathrm{sort}(\mathbf{q})\}\right)(-1)^{\sum q_i+v}\times\sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d\\\sum t_i=v\\d+\sum s_i+t_i=l}}\frac{\mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\prod_i^d\binom{p_i}{s_i}\binom{q_i}{t_i}}{\prod s_i!\prod t_i!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_j+t_j)}$$
$$=l!\sum_{d=1}^{l}\sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\\text{within }k\times N\\v}}f\left(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\right)(-1)^{\sum q_i+v}$$
$$\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\\sum t_i=v\\d+\sum s_i+t_i=l}}\sum_{\sigma,\tau\in\mathscr{S}_d}\frac{\left[\sum_{\pi,\theta\in\mathscr{S}_d}\mathrm{sgn}(\pi)\,\mathrm{sgn}(\theta)\prod_i^d\binom{p_{\pi(i)}}{s_{\sigma(i)}}\binom{q_{\theta(i)}}{t_{\tau(i)}}\right]}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}.$$

Now it is crucial that for fixed $\vec{\mathbf{p}}$, $\vec{\mathbf{q}}$, $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$, the sign of this last numerator (bracketed) will depend on the parity of σ and τ. Hence we obtain for the preceding expression

$$l!\sum_{d=1}^{l}\sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\ \text{within } k\times N\\ v}} f\big(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\big)(-1)^{\sum q_i+v}\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}}\ \sum_{\pi,\theta\in\mathcal{S}_d}\operatorname{sgn}(\pi)\operatorname{sgn}(\theta)\prod_i^d\binom{p_{\pi(i)}}{s_i}\binom{q_{\theta(i)}}{t_i}$$
$$\times\sum_{\sigma,\tau\in\mathcal{S}_d}\frac{\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}$$

$$=l!\sum_{d=1}^{l}\sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\ \text{within } k\times N\\ v}} f\big(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\big)(-1)^{\sum q_i+v}$$
$$\times\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}}\ \sum_{\sigma,\tau\in\mathcal{S}_d}\frac{\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_i^d\binom{p_i}{s_{\sigma(i)}}\binom{q_i}{t_{\tau(i)}}}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}.$$

The last line is now perfectly set for the substitution using Lemma 5.7. After changing the range of summation on $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ within $k\times N$ to $\mathbf{p}\in[0,N-1]^d$, $\mathbf{q}\in[0,k-1]^d$, we obtain the announced result.

Admittedly, this is not very enlightening. It is thus worth highlighting what happens: the sums we deal with initially are sums over partitions. By using Frobenius coordinates, and sorting the partitions by their rank d, we are expressing the main sum into a sum over d of multisums in d variables. We thus now have sums over two sets of d strictly decreasing variables (the sets $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$) of different ways of building up this partition (the data encoded in $\mathbf{s}$ and $\mathbf{t}$). Using skew-symmetry, we can unsort the variables $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ to $\mathbf{p}$ and $\mathbf{q}$ and decide instead to sort the variables according to “building blocks”, that is, switch from $\mathbf{s}$ and $\mathbf{t}$ to $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$. □

Putting everything together. We combine all the information obtained so far, and simultaneously clear the restriction $d+\sum s_i+t_i=l$ in (14) by encoding all the moments at once into an exponential generating function.

Theorem 5.9. *For a fixed $k\in\mathbb{N}$, the two series $\sum_{r>0}(\mathcal{M})_N(2k,r)\dfrac{(\mathrm{i}z)^r}{r!}$ and*

$$s_{\langle N^k\rangle}([1^{2k}])\sum_{d=1}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left|\frac{z^{1+s_i+t_j}}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d}$$
$$\times\sum_{\substack{\mathbf{p}\in[0,N-1]^d\\ \mathbf{q}\in[0,k-1]^d}}\left|k\frac{\binom{p_i}{s_i}\binom{q_j}{t_j}\binom{k+p_i}{p_i}\binom{k-1}{q_j}(N-p_i)^{(k)}(-1)^{q_j}}{(N+q_j+1)^{(k)}(1+p_i+q_j)}\right|_{d\times d}\tag{15}$$

have equal z^r coefficients for $r < 2k+1$.

Proof. A first necessary remark is that as a formal power series, the second series is well defined: the sum to obtain the r-th coefficient in that series reduces to a finite sum (because $s_i \le p_i$ and $t_j \le q_j$).

We know from (12) that

$$s_{\langle N^k\rangle \cup \{\mathbf{p}:\mathbf{q}\}}([1^{2k}])$$

is skew-symmetric in $\mathbf{p}$ and $\mathbf{q}$ (separately). Hence we can combine the relations (10), (11) and (13) with Proposition 5.8 to obtain a huge sum. The main statement then follows from the recombinations of the main product into determinants, using Cauchy's Lemma (5.1). □

Remarks on Theorem 5.9.

• This is a hypergeometric multisum (at least for fixed d), when we expand the determinants using Cauchy's Lemma. However, not even small d's seem tractable on computer.

• A definite advantage of this formula is that it can be tested at finite N (by expanding the integral defining $(M_N)(2k, r)$ symbolically using the Haar measure). This is helpful to confirm the results obtained so far.

• We wish to insist on the idea behind this theorem: initially we had a combinatorial problem on structures — see (14) — that had no symmetry for its summands in the s_is or t_is. We have exploited some skew-symmetry in the as and bs in (13) to change this. In particular, we have now switched from a sum over $\vec{\mathbf{p}}, \vec{\mathbf{q}}, \mathbf{s}, \mathbf{t}$ to a sum over $\mathbf{p}, \mathbf{q}, \vec{\mathbf{s}}, \vec{\mathbf{t}}$. We have also simplified the denominator in (14).

• As a consequence of the previous point, we can now assume that the s_is are all different. The same is true for the t_i.

• This has useful consequences, especially for computational purposes. It is interesting to compute a bound on r such that partitions with d fragments will have a nonzero contribution to the final sum in $(\mathcal{M})_N(2k, r)$. We have $r \ge d + \sum s_i + t_i$, and the s_is (respectively, t_is) should be all different. We can take them to be $0, 1, \ldots, d-1$. We thus have $r \ge d + 2\frac{d(d-1)}{2} = d^2$.

We now define

$$H^{N,k,s,t} := s!t! \sum_{\substack{p\in[0,N-1]\\ q\in[0,k-1]}} \frac{k(N-p)^{(k)}(-1)^q}{(N+q+1)^{(k)}(1+p+q)} \binom{k+p}{p}\binom{k-1}{q}\binom{p}{s}\binom{q}{t},$$

where the right-hand side is taken to be similar to the entries in one of the determinants in (15).

I have not been able to obtain a much better expression for this with Mathematica. Normally, the package MultiSum [Wegschaider 2004] should be able to deal with multiple hypergeometric series, but this particular one is too complicated. We will thus focus on an easier problem from now on, the problem of asymptotics (that is, we switch from $(\mathcal{M})_N(2k,r)$ to $(\mathcal{M})(2k,r)$).

Asymptotics. We need to compute asymptotics for $H^{N,k,s,t}$ more precisely.

Proposition 5.10. *For a fixed integer $k \geq 1$, when $k > t$,*

$$H^{k,s,t} := \lim_{N\to\infty} \frac{H^{N,k,s,t}}{N^{1+s+t}} = k \sum_{i=0}^{k-t-1} \frac{\Gamma(k+i)\Gamma(s+i+t+1)}{\Gamma(i+1)\Gamma(k+s+t+i+2)}$$
$$= \frac{1}{1+s+t} \frac{\prod_{i=k}^{2k-1}(i-t)}{\prod_{j=k+1}^{2k}(j+s)} = \frac{1}{1+s+t} \frac{\Gamma(2k-t)\Gamma(k+s+1)}{\Gamma(k-t)\Gamma(2k+s+1)}. \tag{16}$$

(This last expression is well defined since $k > t$.)

Proof. Define

$$\tilde{H}^{N,k,s,t} := t! \sum_{\substack{p\in[0,N-1]\\ q\in[0,k-1]}} \frac{k(N-p)^k(-1)^q}{(N+q+1)^k(1+p+q)} \frac{p^k}{k!} \binom{k-1}{q} p^s \binom{q}{t},$$

that is to say, $H^{N,k,s,t}$ stripped of some of its terms of obviously lower order in p, N and q combined. We do this because we want to compute the leading order of $H^{N,k,s,t}$ and there will be lots of cancellations due to the sum over q, as shown by (18) below.

Thus we wish to compute

$$\lim_{N\to\infty} \frac{\tilde{H}^{N,k,s,t}}{N^{1+s+t}} = \lim_{N\to\infty} \frac{H^{N,k,s,t}}{N^{1+s+t}}.$$

The proof of the second equality in the proposition follows from two basic identities on formal series:

$$(1-rX+r^2X^2-\cdots)^k(1-sX+s^2X^2-\cdots) = \sum_j (-1)^j \sum_{i=0}^{j} \binom{k+i-1}{i} r^i s^{j-i} X^j \tag{17}$$

and

$$\sum_{\substack{0\leq j\leq k-1\\ 0\leq q\leq k-1}} (-1)^q \binom{k-1}{q} q^j X^j = (-1)^{k+1}(k-1)!\, X^{k-1}. \tag{18}$$

We expand the definition of $\tilde{H}^{N,k,s,t}$ as a power series in q. Identity (17) indicates that we should only look at the coefficient of q^{k-1}, which we obtain from (18), setting $r := 1/N,\ s := 1/(p+1)$. We then let N tend to infinity, so the sum over p becomes a Riemann sum. Its limit is a β-integral, and thus a β-function appears, which can be expanded into a product of Γ-functions, giving the desired equality.

The last equality in the statement of the proposition is immediate and is the only one to require the bound $k > t$.

For the third equality in the statement,[5] we define

$$H_a^{k,s,t} := k \sum_{i=0}^{\infty} \frac{\Gamma(k+a+i)\Gamma(s+a+i+t+1)}{\Gamma(a+i+1)\Gamma(k+s+t+a+i+2)}$$
$$= \frac{k\Gamma(a+k)\Gamma(a+s+t+1)}{\Gamma(a+1)\Gamma(a+k+s+t+2)} \, {}_3F_2\left({1, a+k, a+s+t+1 \atop a+1, a+k+s+t+2}; 1 \right),$$

where the second equality follows form the definition of ${}_3F_2$. Then

$$H^{k,s,t} = H_0^{k,s,t} - H_{k-t}^{k,s,t}. \tag{19}$$

Since

$${}_3F_2\left({1, c, d \atop e, c+d-e+2}; 1 \right) = \frac{c+d-e+1}{(c-e+1)(d-e+1)} \left(1-e+\frac{\Gamma(c+d-e+1)\Gamma(e)}{\Gamma(c)\Gamma(d)} \right),$$

(see [Mat 2001], for example), we have

$$H_a^{k,s,t} = \frac{1}{1+s+t} \left(1-\frac{a\Gamma(a+k)\Gamma(a+s+t+1)}{\Gamma(a+1)\Gamma(a+k+s+t+1)} \right),$$

which yields the desired equality thanks to (19). □

Let $G(\,\cdot\,)$ be the Barnes G-function [Hughes et al. 2000, Appendix]. It is a quick consequence of the Weyl dimension formula [Bump and Gamburd 2006, (18)] that

$$s_{\langle N^k \rangle}([1^{2k}]) \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}.$$

We use the previous proposition to give a relatively concise expression for $(\mathcal{M})(2k, r)$.

Theorem 5.11. *For a fixed $k \in \mathbb{N}$, the two series $\sum_{r>0} (\mathcal{M})(2k, r) \frac{(\mathrm{i}z)^r}{r!}$ and*

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{d=1}^{\infty} \sum_{\vec{\mathbf{s}}, \vec{\mathbf{t}} \in \mathbb{N}^d} \left| \frac{1}{s_i! t_j! (1+s_i+t_j)} \right|_{d\times d} \left| \frac{H^{k,s_i,t_j}}{s_i! t_j!} \right|_{d\times d} z^{d+\sum(s_i+t_i)} \tag{20}$$

have equal coefficients of z^r for $r < 2k+1$. For a fixed r, the coefficients of z^r for low values of k can be meromorphically continued into each other. The series (20)

[5]This equality was first proved using Mathematica. Paul Abbott observed that the hypergeometric function that appears is Saalschützian and extracted the following proof by tracing Mathematica's output.

can also be written as

$$\frac{G(k+1)^2}{G(2k+1)}\sum_{d=1}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|^2_{d\times d} \left(\prod_{i,j=1}^{d}\frac{\Gamma(2k-t_j)\Gamma(k+s_i+1)}{\Gamma(k-t_j)\Gamma(2k+s_i+1)}\right)z^{d+\sum s_i+t_i} \tag{21}$$

and

$$\frac{G(k+1)^2}{G(2k+1)}\sum_{\substack{\lambda=\{\vec{\mathbf{s}}:\vec{\mathbf{t}}\}\\ \operatorname{rank}\lambda=d}} s_\lambda([1^k])\left|\frac{\dfrac{\Gamma(2k-t_j)}{\Gamma(2k+s_i+1)}}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d} z^{|\lambda|}. \tag{22}$$

Furthermore, by using Cauchy's Lemma, one can switch to an expression involving products instead of determinants (*that is, a hypergeometric expression*).

Proof. For (20), we proceed by substitution into (15), and looking at terms of order N^{k^2+r}. Again, Cauchy's Lemma is used repeatedly to reorganize determinants.

To obtain (21) or (22), we reorganized yet again the determinants using Cauchy's Lemma into a form corresponding to (11). We also summed over the partitions λ instead of summing first over their rank d then their Frobenius coordinates $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$.

For a fixed r, both sides indeed admit meromorphic continuations in k, which are equal by Carlson's Theorem [Andrews et al. 1999, Theorem 2.8.1, p. 110]. (See also [Dehaye 2006, p. 86].) Indeed, the left-hand side is shown to admit a meromorphic continuation in k using a Pochhammer contour. The meromorphic continuation on the right-hand side is already written in (21), if we admit that what is meant there is the value of the meromorphic continuation in k *evaluated at* k. The difference of the two sides satisfies the hypotheses in Carlson's Theorem, in that its value is 0 at integers, it is of exponential type, and the type $<\pi$ along the axes parallel to the imaginary axis. similar statements are shown in [Dehaye 2006].

It is probably good to insist that the meromorphic continuation of

$$\frac{\Gamma(2k-t_j)\Gamma(k+s_i+1)}{\Gamma(k-t_j)\Gamma(2k+s_i+1)}$$

to the left has to be taken very carefully and cannot be obtained by just plugging in values of k, once $k\le t$. We will discuss similar issues in Section 9. □

We now aim to replace the determinant left in (22) by a friendlier expression, a rational function of k.

6. General shape of $(\mathcal{M})(2k,r)$, $|\mathcal{M}|(2k,2h)$ and $|\mathcal{V}|(2k,2h)$

Proof of Theorem 1.1. By (20), we know that (for fixed r and as meromorphic functions of k)

$$\frac{\mathrm{i}^r}{r!}(\mathcal{M})(2k,r) = \frac{G(k+1)^2}{G(2k+1)} \sum_{\substack{1\le d\\ \vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ d+\sum(s_i+t_i)=r}} C(d,\vec{\mathbf{s}},\vec{\mathbf{t}}) \left| H^{k,s_i,t_j} \right|_{d\times d},$$

with $C(d,\vec{\mathbf{s}},\vec{\mathbf{t}}) \in \mathbb{Q}$, while for s and t fixed (and nonnegative, of course), Equation (16) indicates that $H^{k,s,t}$ is a rational function of k:

$$H^{k,s,t} = \frac{1}{1+s+t} \prod_{i=-t}^{s} \frac{k+i}{2k+i}. \tag{23}$$

This already shows that we have a rational function of k and that the numerator and denominator have the same degree. Equations (23) and (20) together, along with the fact that $H^{k,s,t} = H^{-k,t,s}$, a consequence of (16), explain why X_r is even.

In order to determine the Y_rs a bit better, we need to investigate possible denominators in terms of $\left| H^{k,s_i,t_j} \right|_{d\times d}$. If a is positive, $\left| H^{k,s_i,t_j} \right|_{d\times d}$ will have a factor $(2k+a)^{\alpha_a(r)}$ in its denominator if and only if a is odd (because there is cancellation in (23)) and all of $s_1,\dots,s_{\alpha_a}$ are greater than a. For this to happen, we need

$$r = d + \sum s_i + \sum t_i \ge \alpha_a(r) + \sum_{i=1}^{\alpha_a(r)} (a+i-1) + \sum_{i=1}^{\alpha_a(r)} (i-1), \tag{24}$$

where the inequality is obtained by taking as small as possible values for d, that is $\alpha_a(r)$, for the s_is (while requiring them to be different and greater or equal to a) and for the t_is (all different). We turn this inequality around and get

$$\alpha_a(r) \le \left\lfloor \frac{-a+\sqrt{a^2+4r}}{2} \right\rfloor.$$

The case of negative a is the same, exchanging the roles played by $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$.

Finally, the constant $D(r)$ ensuring that both X_r and Y_r are monic can be found, thanks to (20) and (23), taking $\lim_{k\to\infty}$:

$$D(r) = \sum_{\substack{1\le d\\ \vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ d+\sum(s_i+t_i)=r}} \left| \frac{1}{s_i!\,t_j!\,(1+s_i+t_j)} \right|_{d\times d}^2 \frac{1}{2^{d+\sum(s_i+t_i)}} = \frac{1}{r!\,2^r}, \tag{25}$$

where the last equality is left to the reader.

Actually, this last equality is enough to also guarantee that $X_r(u)$ and $Y_r(u)$ both have integer coefficients: just substitute for $H^{k,s,t}$ in (20) the value

$$H^{k,s,t} = \frac{1}{1+s+t} \prod_{i=-t}^{s} \frac{1}{2k+i} \left(k \sum_{i=0}^{s+t} h_i k^i \right)$$

for the appropriate (integer) h_is (in particular, $h_{s+t} = 1$).

This proves (2), at least for large k.

Meromorphic continuation has already been obtained in Theorem 5.11. □

Theorem 6.1. *For $h \in \mathbb{N}$, there are polynomials $\tilde{X}_{2h}$, $\hat{X}_{2h}$, with integer coefficients and* $\deg \hat{X}_{2h} = \deg X_{2h} > \deg \tilde{X}_{2h}$ *such that as meromorphic functions of k,*

$$|\mathcal{M}|(2k, 2h) = \hat{C}(h)\frac{G(k+1)^2}{G(2k+1)}\frac{\hat{X}_{2h}(2k)}{Y_{2h}(2k)},$$

$$|\mathcal{V}|(2k, 2h) = \tilde{C}(h)\frac{G(k+1)^2}{G(2k+1)}\frac{\tilde{X}_{2h}(2k)}{Y_{2h}(2k)},$$

where $Y_r(u)$ is as defined in Theorem 1.1.

Moreover (*but this is conjectural*), *the numerators are additionally monic polynomials*[6] *when $\hat{C}(h) = 1/2^{2h}$, $\tilde{C}(h) = (2h)!/(h!2^{3h})$, and* $\deg X_{2h} - \deg \tilde{X}_{2h} = 2h$.

Proof. For fixed integer r and large integer k, most of this follows immediately from (5) and (6), combined with Theorem 1.1.

The fact that $\deg \tilde{X}_{2h} < \deg X_{2h}$ for instance is a consequence of

$$(\mathcal{M})\ (2k, r) \sim_k \left(-\frac{\mathrm{i}}{2}\right)^r \frac{G(k+1)^2}{G(2k+1)},$$

which we use in the equation from note 3 (page 38):

$$\sum_{j=0}^{2h} \binom{2h}{j} \left(\frac{\mathrm{i}}{2}\right)^j \left(-\frac{\mathrm{i}}{2}\right)^{2h-j} = 0.$$

We can similarly show that *if it exists*, $\hat{C}(h) = 1/2^{2h}$. The constant $\tilde{C}(h)$ is more mysterious, and involves the lower order terms in k of (23).

The meromorphic continuation is obtained as in the proof of Theorem 1.1. □

Remark. Unfortunately, within their degree restrictions, the $X_r(u)$, $\tilde{X}_{2h}(u)$ and $\hat{X}_{2h}(u)$ polynomials still look utterly random. We merely have an expression for them as a sum of determinants of rank $d \leq \sqrt{r}$ (respectively, $2h$). This expression is relatively quick and allows at least to compute a few of those polynomials.

7. Computational data

The polynomials $X_r(u)$, $\tilde{X}_{2h}(u)$ and $\hat{X}_{2h}(u)$. We present our data for $(\mathcal{M})(2k, r)$ in Table 2, followed by the data for $|\mathcal{M}|(2k, 2h)$ in Table 3 and finally those for $|\mathcal{V}|(2k, 2h)$ in Table 4. Everything extends numerical results previously published,

[6]This is the normalization we will keep later, when discussing data about those polynomials.

r	$X_r(u)$
1	1
2	u^2-2
3	u^2-4
4	u^4-16u^2+66
5	u^4-20u^2+114
6	$u^8-51u^6+864u^4-5554u^2+4860$
7	$u^8-57u^6+1134u^4-8758u^2+8520$
8	$u^{10}-113u^8+4620u^6-86332u^4+682844u^2-765660$
9	$u^{10}-121u^8+5460u^6-115564u^4+1053964u^2-1457820$
10	$u^{14}-220u^{12}+18897u^{10}-831010u^8+20196928u^6-260164440u^4+1428629724u^2-2060092440$
11	$u^{14}-230u^{12}+20997u^{10}-996820u^8+26447168u^6-374214600u^4+2270621484u^2-3994446960$
12	$u^{18}-363u^{16}+52929u^{14}-4083011u^{12}+183649422u^{10}-4906031274u^8+73323636100u^6$ $-512994314412u^4+1371835414728u^2-927651213720$
13	$u^{18}-375u^{16}+57141u^{14}-4663655u^{12}+224398746u^{10}-6467410170u^8+105010072036u^6$ $-806857605660u^4+2461218471576u^2-1755890884440$
14	$u^{22}-582u^{20}+141344u^{18}-18977780u^{16}+1571817537u^{14}-84339778978u^{12}$ $+2962887441370u^{10}-66386724069396u^8+884603961264548u^6-6212383525692744u^4$ $+19176051246319080u^2-13863690471430800$
15	$u^{22}-596u^{20}+149296u^{18}-20838716u^{16}+1807941481u^{14}-102286957136u^{12}$ $+3809004157906u^{10}-90891702433976u^8+1298188100828836u^6-9917808021410976u^4$ $+33986748108863880u^2-25682708695644000$
16	$u^{24}-836u^{22}+295486u^{20}-58491716u^{18}+7245863641u^{16}-593291868896u^{14}$ $+32861804018536u^{12}-1227084273320096u^{10}+29900504376591736u^8$ $-444180655702337856u^6+3616035044845449600u^4-13500165816324763200u^2$ $+10671545982659562000$
17	$u^{24}-852u^{22}+308606u^{20}-62999492u^{18}+8101703961u^{16}-692989945072u^{14}$ $+40321523165416u^{12}-1589469869122752u^{10}+41098203910503416u^8$ $-652694167393180032u^6+5757854141711318400u^4-23590053001525406400u^2$ $+19761261673907754000$
18	$u^{30}-1216u^{28}+641547u^{26}-195081042u^{24}+38335269063u^{22}-5171814422892u^{20}$ $+495753742037253u^{18}-34353739684203042u^{16}+1726507702228490928u^{14}$ $-62290017635596811632u^{12}+1575250938092261972152u^{10}-26886933063310680515376u^8$ $+293595553738705518511056u^6-1882598606626601433513600u^4$ $+5855125431247144869877200u^2-4699357338080820827412000$
19	$u^{30}-1234u^{28}+663111u^{26}-206226048u^{24}+41629109007u^{22}-5794171874298u^{20}$ $+575320671855777u^{18}-41443936954862628u^{16}+2171988993390059952u^{14}$ $-81956498940701768368u^{12}+2174685878160406187416u^{10}-39111313358222167862304u^8$ $+452360970074645727302736u^6-3084756281794829726025120u^4$ $+10210913321050424344698000u^2-8861284072193198189544000$
20	$u^{34}-1615u^{32}+1140143u^{30}-467224385u^{28}+124593557421u^{26}-22981261798995u^{24}$ $+3040237566735165u^{22}-294611133821587635u^{20}+21107532245623967310u^{18}$ $-1116405478738744697410u^{16}+43058312795636550000904u^{14}$ $-1183070247664529791035320u^{12}+22374172979188549647921632u^{10}$ $-277662183945403036368852000u^8+2095071747708073688848702224u^6$ $-8269151494407104768839910640u^4+12529695816553717113566335200u^2$ $-6099189940914050054558484000$

Table 2. The first polynomials $X_r(u)$ as the numerators in $(\mathcal{M})(u,r)$.

$2h$	$\hat{X}_{2h}(u)$
2	u^2
4	u^4-8u^2-6
6	$u^8-33u^6+198u^4+74u^2-360$
8	$u^{10}-81u^8+1740u^6-8284u^4-7716u^2+34020$
10	$u^{14}-170u^{12}+9597u^{10}-215560u^8+1846928u^6-4247400u^4-12317076u^2+42366240$
12	$u^{18}-291u^{16}+30177u^{14}-1379507u^{12}+28177518u^{10}-236602818u^8+604630084u^6+1570591476u^4-10008266040u^2+7829929800$
14	$u^{22}-484u^{20}+90384u^{18}-8378492u^{16}+415889897u^{14}-11196067680u^{12}+157699171570u^{10}-1023611526808u^8+1699483809828u^6+11589901952544u^4-62361799232760u^2+44754182272800$
16	$u^{24}-708u^{22}+198590u^{20}-28525892u^{18}+2275085529u^{16}-102837376096u^{14}+2598141390568u^{12}-34807690054560u^{10}+213458763180152u^8-261862022455104u^6-3402805264433280u^4+19256263380043200u^2-11718802173078000$
18	$u^{30}-1054u^{28}+460431u^{26}-109299828u^{24}+15577804767u^{22}-1394331670638u^{20}+79872695247657u^{18}-2932723486507728u^{16}+68022586503825552u^{14}-962308385613255088u^{12}+7682283932820069016u^{10}-26475220331016986304u^8-59889950570120914224u^6+976582356673028315040u^4-3441287004848413282800u^2+1366282646437284576000$
20	$u^{34}-1415u^{32}+840943u^{30}-275540385u^{28}+55049482221u^{26}-7022476724835u^{24}+584090828573565u^{22}-31869278744265555u^{20}+1134427249824868110u^{18}-25880772100948222330u^{16}+365485578445889268104u^{14}-2970099871666499086840u^{12}+10773785732163438366432u^{10}+24904735536575464181280u^8-474478390713139651278576u^6+1993984711160163968152080u^4-1770512318771949573760800u^2+2149673189987662499160000$

Table 3. The first polynomials $\hat{X}_{2h}(u)$ as the numerators in $|\mathcal{M}|(u,2h)$.

for instance in [Hall 2002a; 2004] (but those rely on [Hughes 2005]) or [Conrey et al. 2006] (which is limited to $k=h$).

Extensions of those tables up to $r=2h=60$ are available on this article's web page or (possibly further extended) at [Dehaye 2007a].

To obtain those tables, we have implemented (21), which is the most computationally accessible version of the formulas available in Theorem 5.11. A Magma implementation of this algorithm is also available as expanded content.

The roots of $\tilde{X}_{2h}(u)$. It has been suggested before, based on limited numerical data, that the polynomials $\tilde{X}_{2h}(u)$ have only real roots. In fact we list in Table 6 the number of real roots and degree for each such polynomial. One quickly observes that $\tilde{X}_{42}(u)$ (of course!) is actually the first polynomial to break the initial fluke and have nonreal roots; see Table 5 on the next page. (It is not clear if this is related to a similar observation on the last line of [Hall 2002a] and throughout [Hall 2004].) This polynomial has four nonreal roots ($\pm 18.8631835 \pm 0.0090603\mathrm{i}$) that show up at once, since they would have to come in pairs of conjugate pairs by evenness of $\tilde{X}_{2h}(u)$. One could wonder why nonreal roots show up so late.

$2h$	$\tilde{X}_{2h}(u)$
2	1
4	1
6	$u^2 - 9$
8	$u^2 - 33$
10	$u^4 - 90u^2 + 1497$
12	$u^6 - 171u^4 + 6867u^2 - 27177$
14	$u^8 - 316u^6 + 30702u^4 - 982572u^2 + 6973305$
16	$u^8 - 484u^6 + 76902u^4 - 4461348u^2 + 67692705$
18	$u^{12} - 766u^{10} + 215847u^8 - 27766980u^6 + 1653656895u^4 - 41530140126u^2 + 337968054585$
20	$u^{14} - 1055u^{12} + 421093u^{10} - 79486155u^8 + 7242179715u^6 - 290444510205u^4 + 4099101803991u^2$ $- 8381907513945$
22	$u^{16} - 1496u^{14} + 892108u^{12} - 272180808u^{10} + 45430344630u^8 - 4121412379560u^6$ $+ 189676636728876u^4 - 3674923533427896u^2 + 14539253947899345$
24	$u^{18} - 1961u^{16} + 1566628u^{14} - 658984788u^{12} + 157743552510u^{10} - 21750520014270u^8$ $+ 1678578114026196u^6 - 67707100461703716u^4 + 1235110338400818825u^2$ $- 6787336148294472225$
26	$u^{20} - 2610u^{18} + 2860437u^{16} - 1718473240u^{14} + 620475009522u^{12} - 139083336332460u^{10}$ $+ 19348398203611266u^8 - 1624490941247619480u^6 + 77190294570345945549u^4$ $- 1813095317449668401010u^2 + 15009483262024846096425$
28	$u^{22} - 3243u^{20} + 4462647u^{18} - 3407674501u^{16} + 1586340567882u^{14} - 466277764083726u^{12}$ $+ 86845227411024846u^{10} - 10042821279688179978u^8 + 688582088681764130469u^6$ $- 25698037955845496067927u^4 + 444470604942195922015755u^2$ $- 2654155080367803900605025$
30	$u^{28} - 4190u^{26} + 7631083u^{24} - 7953124300u^{22} + 5258554468937u^{20} - 2313326757869890u^{18}$ $+ 691451285514065259u^{16} - 141062107217586416040u^{14} + 19477099336547993586171u^{12}$ $- 1781103872658227723795970u^{10} + 103764470143371018680338137u^8$ $- 3607131084573924222894990540u^6 + 66647887693999747894954784187u^4$ $- 515514421669410774166185623070u^2 + 658183121944091618062137174225$

Table 4. The first polynomials $\tilde{X}_{2h}(u)$ as the numerators in $|\mathcal{V}|(u, 2h)$.

$$
\begin{aligned}
&u^{44} - 12302u^{42} + 69239935u^{40} - 236610412148u^{38} + 549459541784707u^{36} \\
&\quad - 919748248913270486u^{34} + 1148989069656897835213u^{32} \\
&\quad - 1094474723973849448826480u^{30} + 805533314533281755701371226u^{28} \\
&\quad - 461541928967718110253944237052u^{26} + 206514429127544387915748094513446u^{24} \\
&\quad - 72119441118339869972121541587076920u^{22} \\
&\quad + 19577196457693502603026719624834404502u^{20} \\
&\quad - 4099121776759328236737053383626986012604u^{18} \\
&\quad + 654170727960937096861203148250462720819850u^{16} \\
&\quad - 78212503734767115379758317319774926243800176u^{14} \\
&\quad + 6836980008003428572296900814856434321006155189u^{12} \\
&\quad - 422028250886223501142365592098345343850710857462u^{10} \\
&\quad + 17476800084974190439148752639441918166326024419531u^{8} \\
&\quad - 448540393629268182677088044978029477583305447285620u^{6} \\
&\quad + 6253526937210642323596984565394593401672539709730775u^{4} \\
&\quad - 37013087756228993438266827460643377628945508512487 50u^{2} \\
&\quad + 36216052456609571501642100973941635690472733838765625.
\end{aligned}
$$

Table 5. The first polynomial X_r with nonreal roots occurs for $r = 42$.

h	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\deg(\tilde{X}_{2h})$	0	0	2	2	4	6	8	8	12	14	16	18	20	22	28
# real roots	0	0	2	2	4	6	8	8	12	14	16	18	20	22	28

h	16	17	18	19	20	**21**	22	**23**	**24**	25	**26**	**27**	**28**	**29**	**30**
$\deg(\tilde{X}_{2h})$	28	30	34	36	38	44	46	48	50	54	56	62	64	66	72
# real roots	28	30	34	36	38	40	46	44	46	54	52	58	60	62	68

Table 6. The degree and the number of real roots of $\tilde{X}_{2h}$. The hs for which there are nonreal roots are highlighted.

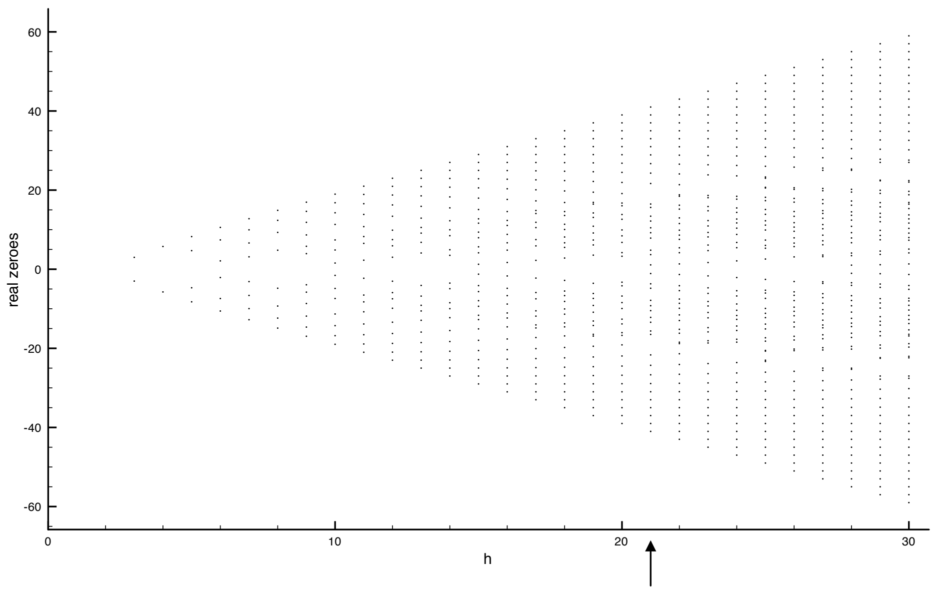

Figure 1. The roots of $\tilde{X}_{2h}(u)$. The line corresponding to $\tilde{X}_{42}(u)$, where the first real zeros go missing, has been indicated.

Fact. The polynomials $\tilde{X}_{2h}(u)$ tend to have many, *but not all*, of their roots real. For instance, for high h, $\tilde{X}_{2h}(u)$ has one root very close by to every odd integer between h and $2h$.

We first present graphical clues for this fact in Figure 1, which depicts the position of the real roots for $h = 1$ to $h = 30$. It thus omits the complex roots.

We now explain the fact. It helps at this point to remember that $\tilde{X}_{2h}(u)$ is obtained by summing various $X_r(u)$ for $r \leq 2h$, which are themselves obtained from (20), for instance. Furthermore, the summand in that equation associated to

$d, \mathbf{s}, \mathbf{t}$ (with $r = d + \sum_i s_i + t_i$) will have poles (as a function of $u = 2k$) at the odd integers a such that $-s_1 \leq a \leq t_1$ (this uses Lemma 5.1 to expand the determinant in H^{k,s_i,t_j}s). For each pole a, there are a few summands where this pole comes with multiplicity exactly $\alpha_a(r)$, but for most others the multiplicity is lower; see (24). So if we sum all of those terms, and multiply by $Y_{2h}(u)$ (the common denominator) to obtain $\tilde{X}_{2h}(u)$, a vast majority of terms factor a $(u-a)$ out. We thus have an expression of the form

$$\tilde{X}_{2h}(u) = (u-a)P_1(u) + P_2(u),$$

where the coefficients of $P_1(u)$ are expected to be much bigger than the coefficients of $P_2(u)$ (simply because much more terms are summed to obtain $P_1(u)$ than $P_2(u)$). Hence, we should expect $\tilde{X}_{2h}(u)$ to change sign when u travels along the real axis from below a to above a (because $|P_1(a)| > |P_2(a)|$ and $(u-a)$ changes sign) and we know that a root will be around $u = a$. This is especially true if $a > r/2$, because the restrictions impose then $s_1 > a > s_2$, and as a consequence $\alpha_r(a) = 1$ and the phenomenon described just now is accentuated. We present in Table 7 some numerical data associated to this phenomenon.

It is obvious from Figure 1 that a lot is yet to be understood about the polynomials $\tilde{X}_{2h}(u)$. For instance, it is not clear if asymptotically in h there is a positive proportion of real roots.

8. Alternative expressions

Using Macdonald's ninth variation of the Schur functions. Define, as in [Nakagawa et al. 2001] and [Noumi 2004], and similarly to [Macdonald 1992],

$$\tilde{s}_\lambda^{(R)} := \left| \tilde{h}^{(R-j+1)}_{\lambda_i - i + j} \right|_{l(\lambda)\times l(\lambda)}, \tag{26}$$

with

$$\tilde{h}_k^{(R)} := \frac{(R-1)!}{(R+k-1)!k!}.$$

We first prove that this variation of the Schur functions satisfies a *Giambelli identity*.

Proposition 8.1. *Let λ be a partition and $\{\vec{\mathbf{s}} : \vec{\mathbf{t}}\}$ its Frobenius coordinates, of rank d. Then,*

$$\tilde{s}_\lambda^{(R)} = \left| \tilde{s}^{(R)}_{(s_i|t_j)} \right|_{d\times d} = \left| \frac{\Gamma(R-t_j)/\Gamma(R+s_i+1)}{s_i!t_j!(1+s_i+t_j)} \right|_{d\times d}.$$

Note how this provides a second determinantal expression for this variation of Schur functions, but with a matrix of different rank.

Proof. We intend to use Exercise 3.21 of [Macdonald 1998], but to show that the exercise applies, we need to prove that

h	largest root of $\tilde{X}_{2h}(u)$	difference with $2h-1$	log. difference
1	no root	no root	no root
2	no root	no root	no root
3	3.0000000000000000000	2.000000000	0.69315
4	5.7445626465380286598	1.255437354	0.22748
5	8.2448923938491831987	0.7551076062	−0.28090
6	10.568920444013080343	0.4310795560	−0.84146
7	12.769459455674733521	0.2305405443	−1.4673
8	14.886048429155973920	0.1139515708	−2.1720
9	16.948550444560344620	0.05144955544	−2.9672
10	18.978943770872905688	0.02105622913	−3.8606
11	20.992206162055831068	0.007793837944	−4.8544
12	22.997383184072186530	0.002616815928	−5.9458
13	24.999198051064882757	0.0008019489351	−7.1285
14	26.999774030173017860	0.0002259698270	−8.3951
15	28.999941044846106152	$5.895515389 \times 10^{-5}$	−9.7388
16	30.999985671005722891	$1.432899428 \times 10^{-5}$	−11.153
17	32.999996738730003824	$3.261269996 \times 10^{-6}$	−12.633
18	34.999999301847217917	$6.981527821 \times 10^{-7}$	−14.175
19	36.999999858891343014	$1.411086570 \times 10^{-7}$	−15.774
20	38.999999972983353984	$2.701664602 \times 10^{-8}$	−17.427
21	40.999999995085836086	$4.914163914 \times 10^{-9}$	−19.131
22	42.999999999148595422	$8.514045781 \times 10^{-10}$	−20.884
23	44.999999999859167358	$1.408326421 \times 10^{-10}$	−22.683
24	46.999999999977712180	$2.228782021 \times 10^{-11}$	−24.527
25	48.999999999996618870	$3.381129731 \times 10^{-12}$	−26.413
26	50.999999999999507453	$4.925468142 \times 10^{-13}$	−28.339
27	52.999999999999930988	$6.901186254 \times 10^{-14}$	−30.304
28	54.999999999999990686	$9.313971788 \times 10^{-15}$	−32.307
29	56.999999999999998787	$1.212486889 \times 10^{-15}$	−34.346
30	58.999999999999999847	$1.524414999 \times 10^{-16}$	−36.420

Table 7. The largest root of $\tilde{X}_{2h}(u)$.

$$\tilde{s}^{(R)}_{(p|q)} := \det \begin{pmatrix} \tilde{h}^{(R)}_{p+1} & \tilde{h}^{(R-1)}_{p+2} & \cdots & \cdots & \cdots & \tilde{h}^{(R-q)}_{p+q+1} \\ 1 & \tilde{h}^{(R-1)}_{1} & \tilde{h}^{(R-2)}_{2} & \cdots & \cdots & \tilde{h}^{(R-q)}_{q} \\ 0 & 1 & \tilde{h}^{(R-2)}_{1} & \tilde{h}^{(R-3)}_{2} & \cdots & \tilde{h}^{(R-q)}_{q-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \tilde{h}^{(R-q)}_{1} \end{pmatrix}$$
$$= \frac{1}{p!q!(1+p+q)} \frac{\Gamma(R-q)}{\Gamma(R+p+1)}.$$

where the matrix has dimensions $(q+1)\times(q+1)$. This can be shown by expanding the determinant along the last column to obtain

$$\tilde{s}^{(R)}_{(p|q)} = (-1)^q \tilde{h}^{(R-q)}_{p+q+1} + \sum_{i=1}^{q} (-1)^{i+1} \tilde{h}^{(R-q)}_{i} \tilde{s}^{(R)}_{(p|q-i)}.$$

Subtract the left-hand side from the right-hand side, proceed by induction on q, and factor out

$$\frac{\Gamma(R-q)}{\Gamma(R+p+1)}.$$

The result then follows from the following equalities, for p and q positive integers,

$$\begin{aligned}
&\frac{(-1)^q}{(p+q+1)!} - \sum_{i=1}^{q} \frac{(-1)^i}{i!\,p!\,(q-i)!\,(p+q-i+1)} - \frac{1}{p!\,q!\,(1+p+q)} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{p+q+q \quad {}_2F_1\left(\begin{smallmatrix}1-q & -p-q\\ 1-p-q\end{smallmatrix};1\right)}{(p+q)(p+q+1)\,p!\,q!} - \frac{1}{p!\,q!\,(1+p+q)} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{q \quad {}_2F_1\left(\begin{smallmatrix}1-q & -p-q\\ 1-p-q\end{smallmatrix};1\right)}{(p+q)(p+q+1)\,p!\,q!} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{q}{(p+q)(p+q+1)\,p!\,q!}\,\frac{(q-1)!}{(1-p-q)^{(q+1)}} \\
&\quad = 0,
\end{aligned}$$

the last one being a consequence of Gauss's hypergeometric theorem.

The theorem now results directly from Exercise 3.21 in Macdonald's book. □

In essence, this proposition allows us to switch from a Giambelli-type expression to a Jacobi–Trudi expression. It immediately leads to a simplified version of Theorem 5.11:

Theorem 8.2. *With $G(\,\cdot\,)$ the Barnes G-function, and $\tilde{s}_\lambda$ defined as in (26),*

$$\sum_{r>0} (\mathcal{M})(2k,r)\frac{(iz)^r}{r!} = \frac{G(k+1)^2}{G(2k+1)} \sum_{\lambda} \tilde{s}^{(2k)}_{\lambda} s_\lambda([1^k]) z^{|\lambda|},$$

in the sense that their coefficients of z^r are equal for fixed r and large enough k so the coefficient on the left-hand side is defined.

Imitating the Cauchy identity. We can also give an alternative for the expression in (22), proceeding as in Gessel's theorem in its lead up to the Cauchy identity; see [Tracy and Widom 2001]. This uses Theorem 8.2.

Theorem 8.3.

$$
\begin{aligned}
\frac{G(k+1)^2}{G(2k+1)} &\sum_{r>0} (\mathcal{M})(2k,r)\frac{(\mathrm{i}z)^r}{r!} \\
&= \lim_{n\to\infty} \det\Big(\big(h_{j-i}([1^k])\big)_{n\times\infty}\cdot\big(\tilde{h}_{i-j}^{(2k-n+j)} z^{i-j}\big)_{\infty\times n}\Big) \\
&= \lim_{n\to\infty} \det\Big(\big(h_{j-i}([1^k]) z^{j-i}\big)_{n\times\infty}\cdot\big(\tilde{h}_{i-j}^{(2k-n+j)}\big)_{\infty\times n}\Big) \\
&= \lim_{n\to\infty} \det\Big(\sum_{l\geq 0} h_{l-i}([1^k])\tilde{h}_{l-j}^{(2k-n+j)} z^{l-j}\Big)_{n\times n} \\
&= \lim_{n\to\infty} \det\Big(\sum_{l\geq 0}\binom{l-i+k-1}{k-1}\frac{(2k-n+j-1)!}{(l-j)!\,(2k-n+l-1)!} z^{l-j}\Big)_{n\times n},
\end{aligned}
$$

in the sense that their coefficients of z^r are equal for fixed r and large enough k so the coefficient on the left-hand side is defined. The factorials on the last line should really be evaluated in groups, to give 0 *if $l<j$, and*

$$\frac{\Gamma(2k-n+j)}{\Gamma(2k-n+l)(l-j)!}$$

otherwise.

This can be truncated significantly when we are only after

$$\sum_{0<r\leq S} (\mathcal{M})(2k,r)\frac{(\mathrm{i}z)^r}{r!}$$

for a finite S, that is, when we are computing the head of the sequence of polynomials: we can drop the limit in n and settle for a sufficiently big n instead, and then cut the matrices in their infinite directions as well.

In Gessel's Theorem, in order to get to the other side of the Cauchy identity, one would then observe that the matrix on the last line is Toeplitz, and then use Szegö's theorem. Of course, that fails here because the matrix on the last line is not Toeplitz.

9. The result of Conrey and Ghosh

As explained on page 35, Conrey and Ghosh's theorem [1989] that $\mathcal{J}(2,1)=(e^2-5)/4\pi$ immediately leads to a conjecture that $|\mathcal{V}|(2,1)=(e^2-5)/4\pi$ as well. Our main concern is that we only know $|\mathcal{V}|(2k,2h)$ for integer h through (5) and (6) (while we would need $h=1/2$).

We offer in Figure 2 one way to circumvent this problem. The idea is to compute for each fixed integer h the values of the meromorphic continuation in k of

$(\mathcal{M})(2k, 2h)$ at $k = 1$, that is, at the crosses. This should be enough to know through (5) any value of the form $|\mathcal{M}|\,(2, 2h)$, which could then finally be used to meromorphically continue $|\mathcal{V}|\,(2, 2h)$ to $h = 1/2$.

Getting the meromorphic continuation of (21) to $k = 1$ is quite subtle.

Proposition 9.1. *Define* $(\mathcal{M})(2, r)$ *as the meromorphic continuation of* $(\mathcal{M})(2k, r)$ *in* k, *evaluated at* $k = 1$. *Then the exponential generating series of* $(\mathcal{M})(2, r)$ *is given by*

$$\sum_{r>0} (\mathcal{M})(2, r)\frac{(\mathrm{i}z)^r}{r!} = \sum_{d=1}^{\infty} \sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d} \left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|^2_{d\times d} \Big(\prod_{i,j=1}^{d} \frac{1}{2^{\nu(t_j)}}\frac{1-t_j}{2+s_j}\Big) z^{d+\sum s_i+t_i}, \quad (27)$$

where $\nu(0) = 0$ *when* $t = 0$, $\nu(t) = 1$ *when* $t \geq 2$. *The value* $\nu(1)$ *is free to choose.*

Proof. When looking for the analytic continuation in k, most of the formulas we have found so far are misleading. For instance, in light of the remark in note 4

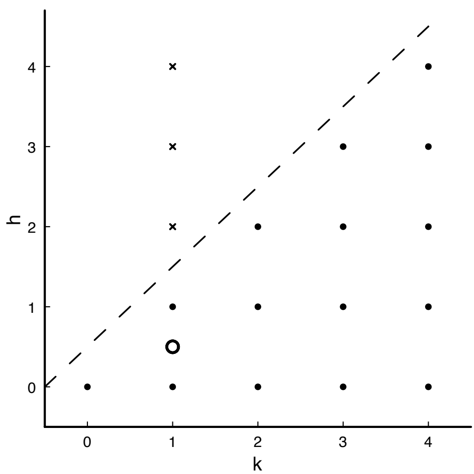

Figure 2. The real part of the situation in the Conrey–Ghosh case. The circle at $(1, 1/2)$ indicates the point for which the value of $\mathcal{J}(2k, 2h)$ is coveted. The dots indicate the locations where (21) applies, and the crosses indicate the points to which that expression is meromorphically continued (for a fixed h, that is, horizontally) thanks to (27). Note that for fixed integer h, this continuation hits a pole when crossing the dashed line (and many more before reaching $k = 1$, as h increases: see Figure 1).

(page 39), one could think that the sums over partitions λ in (22) or Theorem 8.2 immediately reduce when $k=1$ to sums over partitions λ of length 1, that is, the partitions indexed by a single variable. However, in those cases, the other factor in the summands, for instance $\tilde{s}_{\lambda}^{(2k)}$ in Theorem 8.2, might actually be undefined if we take $k=1$ (in that particular case, when $l(\lambda)\geq 3$, or equivalently when $t_1\geq 2$ if $\lambda=\{\vec{\mathbf{s}}:\vec{\mathbf{t}}\}$).

We can get a better intuition through Expression (21), which we use as a basis of our proof. We are clearly required to find the meromorphic continuation to $k=1$ and for fixed $s,t\geq 0\in\mathbb{N}$ of

$$\frac{\Gamma(2k-t)}{\Gamma(k-t)}\,\frac{\Gamma(k+s+1)}{\Gamma(2k+s+1)}.$$

The second factor is certainly not a problem and immediately gives $1/(s+2)$. For the first factor, we have to look at $\lim_{k\to 1}(\Gamma(2k-t)/\Gamma(k-t))$ for $t\geq 0$. Pick any integer a such that $1+a-t\geq 0$. Then, using the functional equation for Γ, we have

$$\lim_{k\to 1}\frac{\Gamma(2k-t)}{\Gamma(k-t)}=\lim_{k\to 1}\frac{\Gamma(2k+a-t)}{\Gamma(k+a-t+1)}\cdot\frac{(k-t)(k-t+1)\cdots(k-1)\cdots(k+a-t)}{(2k-t)\cdots(2k-2)\cdots(2k+a-t-1)}.$$

Note that the terms $(k-1)/(2k-2)$ only appear if $t\geq 2$. In that case we get

$$\lim_{k\to 1}\frac{\Gamma(2k-t)}{\Gamma(k-t)}=(1-t)\lim_{k\to 1}\frac{\Gamma(2k+a-t)}{\Gamma(k+a-t+1)}\cdot\frac{k-1}{2k-2}=\frac{1}{2}(1-t),$$

and in the case $t\leq 2$ the factor of 2 is missing. □

One can also check that the values recovered using Proposition 9.1 agree with the values obtained using $X_r(2)/Y_r(2)$ and thus Theorem 1.1.

For completeness, we give the beginning of the sequence of $X_r(2)$s, for $r=1$ to 15:

$$\begin{aligned}&1,2,0,18,50,-6540,-11760,852180,1228500,590126040,558613440,\\&\quad -39273224760,455842787400,5775116644337040,14904865051876800.\end{aligned}$$

Unfortunately, we fall short of actually finding the full meromorphic continuation of $(\mathcal{M})(2,r)$ and have to leave this for a further paper.

10. Conclusion

Our initial goal was to compute the $(\mathcal{M})\,(2k,r)$, $|\mathcal{M}|\,(2k,2h)$ and $|\mathcal{V}|(2k,2h)$ more effectively than previously done.

We feel that we have achieved this goal, since we have been able to shed some light (for instance in Theorem 1.1) on the structure of the results. This structure (rational functions with known denominators) underlines tables already available in

[Hughes 2005] or [Conrey et al. 2006]. We have also been able to use these results to obtain better algorithms to compute those rational functions, thereby extending the data that was available. As a corollary we have shown that for large(r) h the roots (in k) of $|\mathcal{V}|(2k, 2h)$ cease to all be real, a fluke only for the small-h cases available previously.

However, we have not obtained a formula for *all* $|\mathcal{V}|(2k, r)$. In particular, we cannot recover the value of $|\mathcal{V}|(2, 1)$, which can be conjectured from Conrey and Ghosh's result for $\mathcal{J}(2, 1)$.

Those methods should also give more general moments, for instance for expressions of the form

$$\left\langle |Z_U(\theta_1)|^{2k} \left| \frac{Z'_U(\theta_2)}{Z_U(\theta_2)} \right|^r \right\rangle_{\mathrm{U}(N)}$$

or

$$\left\langle |Z_U(\theta_1)|^{2k} \left| \frac{Z''_U(\theta_2)}{Z_U(\theta_2)} \right|^r \right\rangle_{\mathrm{U}(N)}. \tag{28}$$

An expression for those two extensions in the shape of (10) would definitely be available (for instance, in the case of Expression (28), we would most likely have to compute the equivalent of (10) by summing over $\vec{\mu} \in (2\mathbb{N}_+)^r$). However, the second part of the computation, the part covered here by Proposition 5.6, would probably be significantly worsened.

Acknowledgement

The structure in Theorem 1.1 had been guessed a few years ago by Chris Hughes based on computational evidence [Hughes 2005]. The author is deeply thankful to him for freely sharing and explaining all of his previous unpublished work.

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References

[Andrews et al. 1999] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, 1999. MR 2000g:33001 Zbl 0920.33001

[Borodin 2000] A. M. Borodin, "Characters of symmetric groups, and correlation functions of point processes", *Funktsional. Anal. i Prilozhen.* **34**:1 (2000), 12–28, 96. MR 2001h:20017 Zbl 0959.60037

[Borodin and Olshanski 1998] A. Borodin and G. Olshanski, "Point processes and the infinite symmetric group", *Math. Res. Lett.* **5**:6 (1998), 799–816. MR 2000i:20020 Zbl 1044.20501 arXiv math.RT/9804087

[Bump and Gamburd 2006] D. Bump and A. Gamburd, "On the averages of characteristic polynomials from classical groups", *Comm. Math. Phys.* **265**:1 (2006), 227–274. MR 2217304 Zbl 1107.60004

[Conrey 1988] J. B. Conrey, "The fourth moment of derivatives of the Riemann zeta-function", *Quart. J. Math. Oxford Ser.* (2) **39**:153 (1988), 21–36. MR 89e:11050 Zbl 0644.10028

[Conrey and Ghosh 1989] J. B. Conrey and A. Ghosh, "On mean values of the zeta-function. II", *Acta Arith.* **52**:4 (1989), 367–371. MR 90m:11125 Zbl 0688.10036

[Conrey et al. 2006] J. B. Conrey, M. O. Rubinstein, and N. C. Snaith, "Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function", *Comm. Math. Phys.* **267**:3 (2006), 611–629. MR 2007k:11130 Zbl 05207531

[Dehaye 2006] P.-O. Dehaye, *Averages over compact Lie groups, twisted by Weyl characters and application to moments of derivatives of characteristic polynomials*, PhD thesis, Stanford University, 2006, Available at http://www.maths.ox.ac.uk/~pdehaye/papers/thesis/thesis.pdf.

[Dehaye 2007a] P.-O. Dehaye, 2007, Available at http://www.maths.ox.ac.uk/~pdehaye/papers/joint_moments/.

[Dehaye 2007b] P.-O. Dehaye, "Averages over classical Lie groups, twisted by characters", *J. Comb. Theory Ser. A* **114**:7 (2007), 1278–1292. Zbl 05199328

[El Samra and King 1979] N. El Samra and R. C. King, "Dimensions of irreducible representations of the classical Lie groups", *J. Phys. A* **12**:12 (1979), 2317–2328. MR 80j:22012 Zbl 0445.22020

[Forrester and Witte 2006a] P. J. Forrester and N. S. Witte, "Boundary conditions associated with the Painlevé III′ and V evaluations of some random matrix averages", *J. Phys. A* **39**:28 (2006), 8983–8995. MR 2240469 Zbl 1098.15017

[Forrester and Witte 2006b] P. J. Forrester and N. S. Witte, "Random matrix theory and the sixth Painlevé equation", *J. Phys. A* **39**:39 (2006), 12211–12233. MR 2007g:34193 Zbl 1119.34070

[Foulkes 1951] H. O. Foulkes, "Reduced determinantal forms for S-functions", *Quart. J. Math., Oxford Ser.* (2) **2** (1951), 67–73. MR 12,794a Zbl 0045.15501

[Hall 2002a] R. R. Hall, "Generalized Wirtinger inequalities, random matrix theory, and the zeros of the Riemann zeta-function", *J. Number Theory* **97**:2 (2002), 397–409. MR 2003h:11108 Zbl 1066.11037

[Hall 2002b] R. R. Hall, "A Wirtinger type inequality and the spacing of the zeros of the Riemann zeta-function", *J. Number Theory* **93**:2 (2002), 235–245. MR 2003a:11114 Zbl 0994.11030

[Hall 2004] R. R. Hall, "Large spaces between the zeros of the Riemann zeta-function and random matrix theory", *J. Number Theory* **109**:2 (2004), 240–265. MR 2005h:11188 Zbl 1067.11055

[Hughes 2001] C. P. Hughes, *On the Characteristic Polynomial of a Random Unitary Matrix and the Riemann Zeta Function*, PhD thesis, University of Bristol, 2001.

[Hughes 2005] C. P. Hughes, "Joint moments of the Riemann zeta function and its derivative", 2005.

[Hughes et al. 2000] C. P. Hughes, J. P. Keating, and N. O'Connell, "Random matrix theory and the derivative of the Riemann zeta function", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **456**:2003 (2000), 2611–2627. MR 2002e:11117 Zbl 0996.11052

[Ingham 1928] A. E. Ingham, "Mean-value theorems in the theory of the Riemann zeta-function", *Proc. London Math. Soc.* (2) **27**:1 (1928), 273–300.

[Keating and Snaith 2000a] J. P. Keating and N. C. Snaith, "Random matrix theory and L-functions at $s = 1/2$", *Comm. Math. Phys.* **214**:1 (2000), 91–110. MR 2002c:11108 Zbl 1051.11047

[Keating and Snaith 2000b] J. P. Keating and N. C. Snaith, "Random matrix theory and $\zeta(1/2+it)$", *Comm. Math. Phys.* **214**:1 (2000), 57–89. MR 2002c:11107 Zbl 1051.11048

[Macdonald 1992] I. G. Macdonald, "Schur functions: theme and variations", pp. 5–39 in *Séminaire Lotharingien de Combinatoire* (Saint-Nabor, France, 1992), Publ. Inst. Rech. Math. Av. **498**, Univ. Louis Pasteur, Strasbourg, 1992. MR 95m:05245 Zbl 0889.05073

[Macdonald 1998] I. G. Macdonald, *Symmetric functions and orthogonal polynomials*, University Lecture Series **12**, American Mathematical Society, Providence, RI, 1998. MR 99f:05116 Zbl 0887.05053

[Mat 2001] *Generalized hypergeometric function 3F2: Specific values (formula 07.27.03.0020)*, 2001, Available at http://functions.wolfram.com/07.27.03.0020.01. The Wolfram Functions Site.

[Mezzadri 2003] F. Mezzadri, "Random matrix theory and the zeros of $\zeta'(s)$", *J. Phys. A* **36**:12 (2003), 2945–2962. MR 2004j:11100 Zbl 1074.11048

[Nakagawa et al. 2001] J. Nakagawa, M. Noumi, M. Shirakawa, and Y. Yamada, "Tableau representation for Macdonald's ninth variation of Schur functions", pp. 180–195 in *Physics and combinatorics* (Nagoya, 2000), edited by A. N. Kirillov and N. Liskova, World Sci. Publ., River Edge, NJ, 2001. MR 2002k:05242 Zbl 0990.05134

[Noumi 2004] M. Noumi, *Painlevé equations through symmetry*, Transl. Math. Monographs **223**, American Mathematical Society, Providence, RI, 2004. Translated from the 2000 Japanese original by the author. MR 2005b:34183 Zbl 1077.34003

[Tracy and Widom 2001] C. A. Tracy and H. Widom, "On the distributions of the lengths of the longest monotone subsequences in random words", *Probab. Theory Related Fields* **119**:3 (2001), 350–380. MR 2002a:60013 Zbl 0989.60012

[Wegschaider 2004] K. Wegschaider, *The Mathematica package MultiSum*, RISC Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria, 2004, Available at http://www.risc.uni-linz.ac.at/research/combinat/software/MultiSum.

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R-equivalence on three-dimensional tori and zero-cycles

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We prove that the natural map $T(F)/R \to A_0(X)$, where T is an algebraic torus over a field F of dimension at most 3, X a smooth proper geometrically irreducible variety over F containing T as an open subset and $A_0(X)$ is the group of classes of zero-dimensional cycles on X of degree zero, is an isomorphism. In particular, the group $A_0(X)$ is finite if F is finitely generated over the prime subfield, over the complex field, or over a p-adic field.

Let T be an algebraic torus over a field F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Let $A_0(X)$ be the subgroup of the Chow group $\mathrm{CH}_0(X)$ of classes of zero-dimensional cycles on X consisting of classes of degree zero. The map $T(F) \to A_0(X)$ taking a rational point t in $T(F)$ to $[t]-[1]$ factors through the R-equivalence on $T(F)$ (see Section 2C):

$$\varphi : T(F)/R \to A_0(X).$$

One can ask the following questions:

1. Is φ a homomorphism?
2. Is φ an isomorphism?

Note that φ is a homomorphism if and only if $[ts]-[t]=[s]-[1]$ for any two rational points $s, t \in T(F)$. If the translation action of T on itself extends to an action on X, the latter means that the natural action of $T(F)$ on $A_0(X)$ is trivial.

In the present paper we prove that φ is an isomorphism for all algebraic tori of dimension at most 3 (Theorem 4.4). All tori of dimension 1 and 2 are rational [Voskresenskiĭ 1998, § 4.9], therefore, φ is an isomorphism of trivial groups. Birational classification of 3-dimensional tori was given in [Kunyavskiĭ 1987].

We use the following notation in the paper:

The word "*variety*" will mean a separated scheme of finite type over a field.

F is a field.

F_{sep} is a separable closure of F.

MSC2000: 19E15.

Keywords: algebraic tori, R-equivalence, K-cohomology, zero-dimensional cycle.

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Γ is the Galois group of F_{sep}/F.
$X_L := X \times_F \operatorname{Spec} L$ for a scheme X over F and a field extension L/F.
X_{sep} is $X \times_F \operatorname{Spec} F_{\text{sep}}$.
T^* is the character group of an algebraic torus T over F_{sep} with Γ-action.
$T_* = \operatorname{Hom}(T^*, \mathbb{Z})$ is the cocharacter group of a torus T.
T° is the dual torus, $(T^\circ)^* = T_*$.
$K_*(X)$ is Quillen's K-group of a scheme X.
$H^*(X, K_*)$ is the K-cohomology group.
$\mathrm{CH}^i(X)$ is the Chow groups of cycles of codimension i on X.
$\mathrm{CH}_i(X)$ is the Chow groups of cycles of dimension i on X.
Fields$/F$ is the category of field extensions of F.
Ab is the category of abelian groups.
Sets is the category of sets.
$\mathbb{G}_m = \mathbb{G}_{m,F}$.

1. Preliminaries

1A. *R-equivalence.* Let F be a field. For a field extension L/F, we write H_L for the semilocal ring of all rational functions $f(t)/g(t) \in L(t)$ such that $g(0)$ and $g(1)$ are nonzero. Let A be a functor from the category of semisimple commutative F-algebras to the category *Sets*. If $i = 0$ or 1, we have a map $A(H_L) \to A(L)$, $a \mapsto a(i)$, induced by the L-algebra homomorphism $H_L \to L$ taking a function h to $h(i)$.

Two points $a_0, a_1 \in A(L)$ are called *strictly R-equivalent* if there is an $a \in A(H_L)$ with $a(0) = a_0$ and $a(1) = a_1$. The strict R-equivalence generates an equivalence relation R on $A(L)$, called the *R-equivalence relation*. The set of R-equivalence classes is denoted by $A(L)/R$.

Example 1.1. A scheme X over F defines the functor

$$X(A) := \operatorname{Mor}_F(\operatorname{Spec} A, X).$$

The notion of R-equivalence in $X(L)$ is classical and was introduced in [Manin 1986, Ch. 2, § 4]. If G is an algebraic group over F, then $G(L)/R = G(L)/RG(L)$, where $RG(L)$ is the subgroup of $G(L)$ consisting of all elements that are R-equivalent to the identity.

Example 1.2. Let G be an algebraic group over F. We can define the functor taking a commutative F-algebra A to the set of isomorphism classes $H^1_{\text{ét}}(A, G)$ of G-torsors over $\operatorname{Spec} A$.

Example 1.3. Let $1 \to S \to P \to T \to 1$ be an exact sequence of algebraic tori over F with P a quasitrivial torus, that is, $P \simeq R_{K/F}(\mathbb{G}_{m,K})$ for an étale F-algebra

K. As $H^1_{\text{ét}}(A, P) = H^1_{\text{ét}}(A \otimes_F K, \mathbb{G}_m) = 0$ for any semilocal commutative F-algebra A by Shapiro–Faddeev Lemma and Grothendieck's Hilbert Theorem 90, the sequence

$$P(A) \to T(A) \to H^1_{\text{ét}}(A, S) \to 0$$

is exact. Since P is an open subset in the affine space of K, we have $P(L)/R = 1$ for any field extension L/F. Hence the image of $P(L) \to T(L)$ consists of R-trivial elements in $T(L)$ and therefore,

$$T(L)/R \simeq H^1(L, S)/R.$$

If in addition S is a flasque torus (see [Voskresenskiĭ 1998, § 4.6]) then by [Colliot-Thélène and Sansuc 1977, Th. 2],

$$T(L)/R \simeq H^1(L, S).$$

1B. ***Category of Chow motives.*** Let $CM(F)$ be the category of Chow motives over F (see [Manin 1968]). Recall that $CM(F)$ is an additive category with objects formal finite direct sums $\coprod_k (X_k, i_k)$ (called *Chow motives*) where X_k are smooth proper varieties over F and $i_k \in \mathbb{Z}$. For a smooth proper variety X we write $M(X)(i)$ for the object (X, i) of $CM(F)$ and shortly $M(X)$ for $M(X)(0)$. If $M(X)$ and $M(Y)$ are objects in $CM(F)$ and X is irreducible of dimension d then

$$\mathrm{Mor}_{CM(F)}(M(X)(i), M(Y)(j)) = \mathrm{CH}_{d+i-j}(X \times Y).$$

We have the functor from the category $SP(F)$ of smooth proper varieties over F to $CM(F)$ taking a variety X to $M(X)$ and a morphism $f : X \to Y$ to the cycle of the graph of f.

We write $\mathbb{Z}(i)$ for $M(\mathrm{Spec}\, F)(i)$. A motive is called *split* if it is isomorphic to a motive of the form $\coprod_{i=1}^r \mathbb{Z}(d_i)$.

The functor taking an X to the K-cohomology groups $H^*(X, K_*)$ (see [Quillen 1973]) from the category $SP(F)$ to the category of (bigraded) abelian groups factors through the category $CM(F)$ as follows. Let $\alpha \in \mathrm{CH}(X \times Y)$ be a morphism $M(X)(i) \to M(Y)(j)$ in $CM(F)$. Then the functor takes α to the homomorphism $H^*(X, K_*) \to H^*(Y, K_*)$ defined by $\beta \mapsto (p_2)_*(\alpha \cdot p_1^*(\beta))$ where p_1^* and $(p_2)_*$ are the pull-back and the push-forward homomorphisms for the first and the second projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ respectively.

Recall that $H^p(X, K_p) = \mathrm{CH}^p(X)$ for a smooth X and every $p \geq 0$ by [Quillen 1973, § 7, Prop. 5.14].

Lemma 1.4. *Let M be a split motive. Then the product map*

$$\mathrm{CH}^p(M) \otimes K_q(F) \to H^p(M, K_{p+q})$$

is an isomorphism.

Proof. The statement is obviously true for the motive $M = \mathbb{Z}(i)$. □

Let X be a smooth proper irreducible variety over F. The push-forward homomorphism

$$\deg : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} F) = \mathbb{Z}$$

with respect to the the structure morphism $X \to \operatorname{Spec} F$ is called the *degree homomorphism*. For every $i \geq 0$, we have the intersection pairing

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}_p(X) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta). \tag{1}$$

Proposition 1.5. *Let X be a smooth proper irreducible variety over F. Then the Chow motive of X is split if and only if*

(i) *the Chow group* $\mathrm{CH}(X)$ *is free abelian of finite rank and the map*

$$\mathrm{CH}(X) \to \mathrm{CH}(X_L)$$

is an isomorphism for every field extension L/F and

(ii) *the pairing (1) is a perfect duality for every p.*

Proof. Suppose that the motive of X is split. Mutually inverse isomorphisms between $M(X)$ and a split motive $\coprod_{i=1}^{r} \mathbb{Z}(d_i)$ are given by two r-tuples of elements $u_i \in \mathrm{CH}_{d_i}(X)$ and $v_i \in \mathrm{CH}^{d_i}(X)$ such that the tuple u (and also v) form a $\mathbb{Z}$-basis of $\mathrm{CH}(X)$ and $\deg(u_i v_j) = \delta_{ij}$ over any field extension of F.

Conversely, suppose that (i) and (ii) hold. Choose dual bases u_i and v_j of $\mathrm{CH}(X)$. They define morphisms α and β from a split motive N to $M(X)$ and back respectively so that $\beta \circ \alpha$ is the identity of N. By Yoneda Lemma, it suffices to prove that for every variety Y over F the morphism

$$u \otimes 1_Y : \mathrm{CH}\big(N \otimes M(Y)\big) \to \mathrm{CH}(X \times Y)$$

is an isomorphism. The injectivity follows from the fact that $\beta \circ \alpha = \mathrm{id}$. The surjectivity follows by induction on the dimension of Y using the localization and the fact that the map $u \otimes 1_Y$ is an isomorphism if Y is the spectrum of a field extension of F. □

1C. ***K-theory, K-cohomology and the Brown–Gersten–Quillen spectral sequence.*** Let X be a smooth variety over F. Let $K_*(X)^{(i)}$ denote the i-th term of the topological filtration on $K_*(X)$. Consider the Brown–Gersten–Quillen (BGQ) spectral sequence (see [Quillen 1973, § 7, Th. 5.4])

$$E_2^{p,q} = H^p(X, K_{-q}) \Rightarrow K_{-p-q}(X) \tag{2}$$

converging to the K-groups of X with the topological filtration. The K-cohomology groups $H^*(X, K_*)$ can be computed via Gersten complexes [Quillen 1973, § 7.5].

We have $E_2^{p,q} = 0$ if $p < 0$ or $p + q > 0$, or $p > \dim X$ and $E_2^{p,-p} = \mathrm{CH}^p(X)$. The E_2-term is as follows.

$$
\begin{tikzcd}
\mathrm{CH}^0(X) & 0 & & \\
H^0(X, K_1) \arrow[drr] & \mathrm{CH}^1(X) & 0 & \\
 & H^1(X, K_2) \arrow[drr] & \mathrm{CH}^2(X) & 0 \\
 & & H^2(X, K_3) & \mathrm{CH}^3(X)
\end{tikzcd}
$$

If in addition X is geometrically irreducible proper, we have $H^0(X, K_1) = F^\times$. The composition of the pull-back homomorphism $F^\times = K_1(F) \to K_1(X)$ for the structure morphism of X with the edge homomorphism $K_1(X) \to H^0(X, K_1)$ is the identity. Hence all the differentials starting at $E_*^{0,-1}$ are trivial. If in addition $\dim X = 3$, the spectral sequence yields an exact sequence

$$K_1(X)^{(1)} \to H^1(X, K_2) \to \mathrm{CH}^3(X) \xrightarrow{g} K_0(X), \tag{3}$$

where g is the edge homomorphism.

2. Zero cycles on toric models

2A. ***K-theory of toric models.*** Let T be an algebraic torus over a field F. Let X be a geometrically irreducible variety containing T as an open subset. We say that X is a *toric model of* T if the translation action of T on itself extends to an action on X. Every torus admits a smooth proper toric model [Brylinski 1979; Colliot-Thélène et al. 2005].

Let X be a smooth proper toric model of T. It follows from [Klyachko 1982, Prop. 3, Cor. 2] that X_{sep} satisfies the conditions (i) and (ii) of Proposition 1.5. Thus by Proposition 1.5, we have:

Proposition 2.1. *Let X be a smooth proper toric model of T. Then the Chow motive of X_{sep} is split.*

The proposition and Lemma 1.4 yield:

Corollary 2.2. *Let X be a smooth proper toric model of an algebraic torus T. Then the product map*

$$\mathrm{CH}^p(X_{\mathrm{sep}}) \otimes K_q(F_{\mathrm{sep}}) \to H^p(X_{\mathrm{sep}}, K_{p+q})$$

is an isomorphism.

The absolute Galois group Γ acts naturally on $K_0(X_{\mathrm{sep}})$ leaving each term $K_0(X_{\mathrm{sep}})^{(i)}$ invariant.

The following theorem was proven in [Merkurjev and Panin 1997].

Theorem 2.3. *Let X be a smooth proper toric model of an algebraic torus of dimension d over F. Then*

(1) *$K_0(X_{\mathrm{sep}})$ is a direct summand of a permutation Γ-module*;

(2) *the subgroup $K_0(X_{\mathrm{sep}})^{(d)}$ is infinite cyclic generated by the class of a rational point of X*;

(3) *the natural map $K_i(X) \to K_i(X_{\mathrm{sep}})^\Gamma$ is an isomorphism for $i \leq 1$*;

(4) *the product map $K_0(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^\times \to K_1(X_{\mathrm{sep}})$ is an isomorphism.*

Corollary 2.4. *Let X be a smooth proper toric model of a torus of dimension d over F. We have the following natural isomorphisms*:

(1) $K_i(X)^{(1)} \xrightarrow{\sim} \big(K_i(X_{\mathrm{sep}})^{(1)}\big)^\Gamma$ *for $i \leq 1$.*

(2) $K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^\times \xrightarrow{\sim} K_1(X_{\mathrm{sep}})^{(1)}$.

Proof. (1): The group $K_i(X)^{(1)}$ is the kernel of the restriction to the generic point $K_i(X) \to K_i F(X)$. The image of this map is equal to $H^0(X, K_i) = K_i(F)$ for $i = 0, 1$. Statement (1) follows from Theorem 2.3(3) applied to the exact sequence

$$0 \to (K_i(X_{\mathrm{sep}})^{(1)})^\Gamma \to K_i(X_{\mathrm{sep}})^\Gamma \to K_i(F_{\mathrm{sep}})^\Gamma$$

for $i = 0, 1$.

(2): Tensoring with $F_{\mathrm{sep}}^\times$ the split exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}}) \to \mathbb{Z} \to 0$$

we get (2) by Theorem 2.3(4). □

Corollary 2.5. *Let X be a smooth proper toric model of a torus of dimension d over F. Then*

(1) *$K_0(X_{\mathrm{sep}})^{(1)}$ is a direct summand of a permutation Γ-module.*

(2) *$K_0(X_{\mathrm{sep}})^{(d)}$ is a direct summand of the Γ-module $K_0(X_{\mathrm{sep}})$.*

Proof. (1): We have the canonical decomposition of Γ-modules via the structure sheaf $\mathbb{O}_X$:

$$K_0(X_{\mathrm{sep}}) = K_0(X_{\mathrm{sep}})^{(1)} \oplus \mathbb{Z} \cdot 1.$$

Hence $K_0(X_{\mathrm{sep}})^{(1)}$ is a direct summand of a permutation Γ-module by Theorem 2.3(1).

(2): For a rational point $x \in X(F)$, the composition of the push-forward homomorphism $K_0(F_{\mathrm{sep}}) = K_0(F_{\mathrm{sep}}(x)) \to K_0(X_{\mathrm{sep}})$ with the push-forward map $p_* : K_0(X_{\mathrm{sep}}) \to K_0(F_{\mathrm{sep}})$ induced by the structure morphism p of X_{sep} is the identity. It follows from Theorem 2.3(2) that the inclusion

$$K_0(X_{\mathrm{sep}})^{(d)} \to K_0(X_{\mathrm{sep}})$$

is split by p_* as a homomorphism of Γ-modules. □

We shall need the following property of K-cohomology groups of smooth proper toric models.

Proposition 2.6. *Let X be a smooth proper toric model of a torus of dimension d over F. Then the natural morphism $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)^{\Gamma}$ is an isomorphism.*

Proof. As X is geometrically rational and has a rational point, the statement follows from [Colliot-Thélène and Raskind 1985, Prop. 4.3] (if $\mathrm{char}(F) = 0$) and [Kahn 1996, Th. 1(a)] or [Garibaldi et al. 2003, Th. 8.9] (in general). □

2B. ***The group $A_0(X)$ of 3-dimensional toric models.*** Let T be an algebraic torus and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Let P and S be algebraic tori over F such that P^* is the permutation Γ-module with $\mathbb{Z}$-basis the set of irreducible components of $(X \setminus T)_{\mathrm{sep}}$ and $S^* = \mathrm{CH}^1(X_{\mathrm{sep}})$. We have natural Γ-homomorphisms $T^* \to P^*$ taking a character χ to $\mathrm{div}(\chi)$ (we consider χ as a rational function on X_{sep}) and $P^* \to S^*$ taking a component of $(X \setminus T)_{\mathrm{sep}}$ to its class in the Chow group. The sequence

$$0 \to T^* \to P^* \to S^* \to 0 \tag{4}$$

is a flasque resolution of T^* (see [Colliot-Thélène and Sansuc 1977, Prop. 6], [Voskresenskiĭ 1998, § 4.6]). Thus we have an exact sequence of algebraic tori

$$1 \to S \to P \to T \to 1, \tag{5}$$

a flasque resolution of T.

By [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3),

$$T(L)/R \simeq H^1(L, S) \tag{6}$$

for any field extension L/F.

The spectral sequence (2) for X_{sep} yields isomorphisms of Γ-modules

$$K_0(X_{\mathrm{sep}})^{(1/2)} \simeq \mathrm{CH}^1(X_{\mathrm{sep}}) = S^*$$

and

$$K_0(X_{\mathrm{sep}})^{(2/3)} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}).$$

Let T be a 3-dimensional torus and X a smooth proper toric model of T. By [Klyachko 1982, Prop. 3, Cor. 2], the pairing

$$\mathrm{CH}^1(X_{\mathrm{sep}}) \otimes \mathrm{CH}^2(X_{\mathrm{sep}}) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta)$$

is a perfect duality of Γ-lattices. It follows that $\mathrm{CH}^2(X_{\mathrm{sep}}) \simeq S_*$. Thus, the exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(2)} \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}})^{(1/2)} \to 0$$

yields an exact sequence of algebraic tori

$$1 \to S' \xrightarrow{\tau} Q \to S^\circ \to 1 \tag{7}$$

with $S'_* = K_0(X_{\mathrm{sep}})^{(2)}$ and $Q_* = K_0(X_{\mathrm{sep}})^{(1)}$ a direct summand of a permutation Γ-module by Corollary 2.5(1). By Theorem 2.3(2) and Corollary 2.5(2), we have isomorphisms of Γ-modules

$$S'_* = K_0(X_{\mathrm{sep}})^{(2)} \simeq K_0(X_{\mathrm{sep}})^{(2/3)} \oplus \mathbb{Z} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}) \oplus \mathbb{Z} \simeq S_* \oplus \mathbb{Z}.$$

Hence $S' \simeq S \times \mathbb{G}_m$ is a flasque torus. Let $\widetilde{Q}$ be a torus such that $Q \times \widetilde{Q}$ is a quasi-split torus. Then the exact sequence

$$1 \to S' \times \widetilde{Q} \xrightarrow{\tau \times 1_{\widetilde{Q}}} Q \times \widetilde{Q} \to S^\circ \to 1$$

is a flasque resolution of S°. By [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3) and (6), we have

$$S^\circ(L)/R \simeq H^1(L, S' \times \widetilde{Q}) \simeq H^1(L, S') \simeq H^1(L, S) \simeq T(L)/R \tag{8}$$

for any field extension L/F, and hence it follows from (7) that

$$\mathrm{Coker}(Q(F) \to S^\circ(F)) = S^\circ(F)/R. \tag{9}$$

As $K_0(X)$ injects into $K_0(X_{\mathrm{sep}})$ and $K_0(X_{\mathrm{sep}})^{(3)}$ is infinite cyclic group generated by the class of a rational point by Theorem 2.3, the kernel of the homomorphism g in (3) coincides with the kernel of the composition

$$\mathrm{CH}^3(X) \to \mathrm{CH}^3(X_{\mathrm{sep}}) \to K_0(X_{\mathrm{sep}})^{(3)} \simeq \mathbb{Z},$$

which is the degree map. Recall that we write $A_0(X)$ for the kernel of $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$. We then have

$$\mathrm{Ker}(g) = A_0(X). \tag{10}$$

The group $A_0(X)$ is 2-torsion, by [Merkurjev and Panin 1997, Cor. 5.11(4)].

By Corollary 2.4, we have isomorphisms

$$K_1(X)^{(1)} \simeq \bigl(K_1(X_{\mathrm{sep}})^{(1)}\bigr)^\Gamma \simeq \bigl(K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^\times\bigr)^\Gamma = (Q_* \otimes F_{\mathrm{sep}}^\times)^\Gamma = Q(F). \tag{11}$$

It follows from Corollary 2.2 and Proposition 2.6 that

$$H^1(X, K_2) \simeq H^1(X_{\mathrm{sep}}, K_2)^\Gamma \simeq \big(\mathrm{CH}^1(X_{\mathrm{sep}}) \otimes F^\times_{\mathrm{sep}}\big)^\Gamma = (S^* \otimes F^\times_{\mathrm{sep}})^\Gamma = S^\circ(F). \tag{12}$$

Remark 2.7. The referee has pointed out that using results from [Colliot-Thélène and Raskind 1985] one can deduce that $\mathrm{CH}^1(X) \otimes F^\times \simeq H^1(X, K_2)$ for a smooth projective rational variety X over an algebraically closed field F of characteristic zero.

Under the identifications (11) and (12), and the fact that the BGQ spectral sequence is compatible with products [Gillet 1981, § 7], the map $K_1(X)^{(1)} \to H^1(X, K_2)$ in (3) coincides with the homomorphism $Q(F) \to S^\circ(F)$ given by (7). It follows from (3), (9) and (10) that

$$\begin{aligned} S^\circ(F)/R &= \mathrm{Coker}(Q(F) \to S^\circ(F)) \\ &\simeq \mathrm{Coker}\big(K_1(X)^{(1)} \to H^1(X, K_2)\big) \simeq \mathrm{Ker}(g) = A_0(F). \end{aligned} \tag{13}$$

By (8), there are natural isomorphisms

$$T(F)/R \simeq S^\circ(F)/R \simeq A_0(X). \tag{14}$$

Similarly, over any field extension L/F we have an isomorphism

$$\boxed{\rho_L : T(L)/R \simeq A_0(X_L).} \tag{15}$$

We shall view ρ as an isomorphism of functors $L \mapsto T(L)/R$ and $L \mapsto A_0(X_L)$ from *Fields*$/F$ to *Ab*.

The following remark was suggested by J.-L. Colliot-Thélène.

Remark 2.8. The isomorphism (14) yields finiteness of $A_0(X)$ in all cases when $T(F)/R$ is known to be finite, that is, F a finitely generated over the prime subfield, over the complex field, over a p-adic field (see [Colliot-Thélène and Sansuc 1977, Th. 1 and Prop. 14] and [Colliot-Thélène et al. 2004, Th. 3.4]).

2C. ***The map $\varphi_L : T(L)/R \to A_0(X_L)$.*** Let T be an algebraic torus over F, X a smooth proper geometrically irreducible variety over F containing T as an open subset, and L/F a field extension. By [Colliot-Thélène and Sansuc 1977, Prop. 12, Cor.], the map

$$\boxed{\varphi_L : T(L)/R \to A_0(X_L)} \tag{16}$$

taking the R-equivalence class of an L-point $t \in T(L)$ to the class of the zero cycle $[t] - [1]$, is well defined. We view φ as a morphism of functors from *Fields*$/F$ to *Sets*.

Proposition 2.9. *The map φ_L does not depend* (*up to canonical isomorphism*) *on the choice of X.*

Proof. We may assume that $L = F$. Let X and X' be two smooth proper geometrically irreducible varieties containing T as an open subset. The closure of the graph of a birational isomorphism between X and X' that is identical on T yields morphisms between the motives $M(X)$ and $M(X')$ in $\mathit{CM}(F)$. These morphisms induce mutually inverse isomorphisms between $A_0(X)$ and $A_0(X')$ [Fulton 1984, 16.1.11]. □

Let X be a smooth proper toric model of T. Consider the flasque resolution (5). The S-torsor P_L over T_L can be extended to an S-torsor $q : U \to X_L$ (see [Colliot-Thélène and Sansuc 1977, Prop. 9] or [Merkurjev and Panin 1997, Prop. 5.4]). For any point $x \in X_L$, the fiber U_x of q over x is an S-torsor over $\operatorname{Spec} L(x)$. Denote by $[U_x]$ its class in $H^1(L(x), S)$. By [Colliot-Thélène and Sansuc 1977, Prop. 12], the map

$$\psi_L : \mathrm{CH}_0(X_L) \to H^1(L, S) = T(L)/R, \tag{17}$$

taking the class $[x]$ of a closed point $x \in X_L$ to $N_{L(x)/L}([U_x])$ extends to a well defined group homomorphism. The composition $\psi|_{A_0(X_L)} \circ \varphi$ is the identity. It follows that the map φ_L is injective.

3. Functors from *Fields*/F to *Sets*

We consider functors from the category *Fields*/F to the category *Sets*.

All functors we are considering take values in *Ab*, but some of the morphisms between such functors (namely, φ) may not be given by group homomorphisms.

In this section, we study compatibility properties for morphisms between functors with respect to norm and specialization maps.

3A. *Functors with norm maps.* Let $A : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. We say that A is a *functor with norms* if for any finite field extension E/F, there is given a *norm map* $N_{E/F} : A(E) \to A(F)$.

Example 3.1. Let T be an algebraic torus over F and E/F a finite field extension. There is an obvious norm map

$$N_{E/F} : T(E) = H^0(E, T_* \otimes E_{\mathrm{sep}}^{\times}) \to H^0(F, T_* \otimes F_{\mathrm{sep}}^{\times}) = T(F).$$

Thus the functor $L \mapsto T(L)$ is equipped with norms. Similarly, the functors $L \mapsto T(L)/R$, $L \mapsto H^1(L, T)$, and $L \mapsto A_0(X_L)$ also have norms.

A morphism $\alpha : A \to B$ of functors with norms from *Fields*$/F$ to *Sets* *commutes with norms* if for any field extension E/F, the diagram

$$\begin{array}{ccc} A(E) & \xrightarrow{\alpha_E} & B(E) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ A(F) & \xrightarrow{\alpha_F} & B(F) \end{array}$$

is commutative.

Example 3.2. Let T be a torus of dimension 3. The sequence (5) yields an isomorphism of functors $T(L)/R \xrightarrow{\sim} H^1(L, S)$ that commutes with norms. It follows that the isomorphism $T(L)/R \simeq S^\circ(L)/R$ in (8) commutes with norms.

Example 3.3. Let T be an arbitrary torus and $1 \to S \to P \to T \to 1$ a flasque resolution. Let $\mathrm{End}_F(S) = \mathrm{Hom}_\Gamma(S^*, S^*)$ be the endomorphism ring of S. For a field extension L/F, the group $T(L)/R = H^1(L, S)$ has a natural structure of an $\mathrm{End}_F(S)$-module. For any $\alpha \in \mathrm{End}_F(S)$, the endomorphism of the functor $L \mapsto T(L)/R$ taking a t to $\alpha(t)$ commutes with norms.

Proposition 3.4. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the morphism ψ in (17) commutes with norms.*

Proof. Let E/F be a finite field extension, $x \in X_E$ a closed point and x' the image of x under the natural morphism $X_E \to X$. We have $N_{E/F}([x]) = m[x']$ in $\mathrm{CH}_0(X)$, where $m = [E(x) : F(x')]$. The torsor U_x in the definition of ψ is the restriction of $U_{x'}$ to $E(x)$. By [Fulton 1984, Example 1.7.4], we have

$$N_{E(x)/F(x')}\big([U_{x'}]_{E(x)}\big) = m[U_{x'}].$$

Hence

$$\begin{aligned} N_{E/F}\big(\psi_E([x])\big) = N_{E(x)/F}([U_x]) &= N_{F(x')/F} N_{E(x)/F(x')}([U_{x'}]_{E(x)}) \\ &= m N_{F(x')/F}([U_{x'}]) = \psi_F\big(N_{E/F}([x])\big). \qquad \square \end{aligned}$$

Proposition 3.5. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the map $\varphi_F : T(F)/R \to A_0(X)$ in (16) is an isomorphism of groups if and only if the morphism φ commutes with norms.*

Proof. Suppose that φ commutes with norms. We show that φ is surjective. Every closed point in X is rationally equivalent to a zero-divisor with support in T. Let $x \in T$ be a closed point of degree n. It is sufficient to prove that $[x] - n[1]$ belongs to the image of φ_F. Let $E = F(x)$ and $x' \in T_E$ the canonical rational point over x. We have $\varphi_E(x') = [x'] - [1]$ and as φ commutes with norms,

$$[x] - n[1] = N_{E/F}([x'] - [1]) = N_{E/F} \circ \varphi_E(x') = \varphi_F(N_{E/F}(x')).$$

Thus, φ is a bijection. The inverse map given by (17) is a group homomorphism. Hence φ is a group isomorphism.

Conversely, if φ is an isomorphism, then φ commutes with norms as ψ does by Proposition 3.4. $\square$

Proposition 3.6. *Let T be an algebraic torus of dimension 3 over F and X a smooth proper toric model of T. Then the morphism of functors ρ in (15) commutes with norms.*

Proof. By Example 3.2, it suffices to prove that the morphism $S^\circ(L)/R \to A_0(X_L)$ given by (13) commutes with norms. Let E/F be a finite field extension. The statement follows from the commutativity of the diagram

$$\begin{array}{ccccc} S^\circ(E)/R & \longrightarrow & H^1(X_E, K_2) & \longrightarrow & \mathrm{CH}^3(X_E) \\ \big\downarrow{\scriptstyle N_{E/F}} & & \big\downarrow{\scriptstyle N_{E/F}} & & \big\downarrow{\scriptstyle N_{E/F}} \\ S^\circ(F)/R & \longrightarrow & H^1(X, K_2) & \longrightarrow & \mathrm{CH}^3(X). \end{array}$$

The exact direct image functor f_* takes the category $M^p(X_E)$ of coherent sheaves on X_E supported in codimension at least p to $M^p(X)$. Therefore, f_* yields a map of the BGQ spectral sequences for X_E and X. Hence the right square of the diagram is commutative.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, it suffices to prove commutativity of the left square in the split case. The left square coincides with

$$\begin{array}{ccc} S^* \otimes E^\times & \longrightarrow & H^1(X_E, K_2) \\ \big\downarrow{\scriptstyle 1\otimes N_{E/F}} & & \big\downarrow{\scriptstyle N_{E/F}} \\ S^* \otimes F^\times & \longrightarrow & H^1(X, K_2), \end{array}$$

where the horizontal maps are product maps after the identification of S^* with $\mathrm{CH}^1(X)$. The commutativity follows from the projection formula in K-cohomology [Rost 1996, § 14.5]. $\square$

3B. *Functors with specializations.* Let $A : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. We say that A is a *functor with specializations* if for any DVR (discrete valuation ring) over F of geometric type (a localization of an F-algebra of finite type) with quotient field L and residue field K there is given a map $s_A : A(L) \to A(F)$ called a *specialization map*.

Example 3.7. Let O be a DVR over F with quotient field L and residue field K

and X a variety over F. The specialization homomorphism

$$s : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(X_K)$$

is defined as follows. Let $\alpha \in \mathrm{CH}_0(X_L)$. As the restriction map $\mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_L)$ is surjective, we can choose $\alpha' \in \mathrm{CH}_1(X_O)$ such that $\alpha'_L = \alpha$. Then set $s(\alpha) = i^*(\alpha')$, the image of α' under the Gysin homomorphism $i^* : \mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_K)$, where $i : X_K \to X_O$ is the regular closed embedding of codimension one [Fulton 1984, § 2.6]. The map s is well defined as $i^* \circ i_* = 0$ for the principal divisor X_K in X_O by [Fulton 1984, Prop. 2.6(c)].

Example 3.8. (see [Gille 2004, Prop. 2.2]) Let T be a torus over F and O a DVR over F with quotient field L and residue field K. Let $1 \to S \to P \to T \to 1$ be a flasque resolution of T. The homomorphism

$$H^1_{\text{ét}}(O, S) \to H^1(L, S)$$

is an isomorphism by [Colliot-Thélène and Sansuc 1987, Cor. 4.2]. The composition

$$s : T(L)/R \simeq H^1(L, S) \simeq H^1_{\text{ét}}(O, S) \to H^1(K, S) \simeq T(K)/R$$

is called the *specialization homomorphism with respect to O*. One can easily see that the specialization homomorphism does not depend on the choice of a flasque resolution of T. It follows from the triviality of $H^1_{\text{ét}}(O, P)$ that the composition $T(O) \to T(L) \to T(L)/R$ is surjective.

$$\begin{tikzcd}
T(L)/R \arrow[d, "\wr"'] & T(L) \arrow[l, two heads] & T(O) \arrow[l] \arrow[r] \arrow[d, two heads] & T(K) \arrow[r, two heads] & T(K)/R \arrow[d, "\wr"'] \\
H^1(L, S) & & H^1_{\text{ét}}(O, S) \arrow[ll, "\sim"'] \arrow[rr] & & H^1(K, S)
\end{tikzcd}$$

Let $p \in T(L)/R$ and $q \in T(O)$ be a lift of p. Then it readily follows from the definition that $s(p)$ is the image of q under the composition $T(O) \to T(K) \to T(K)/R$.

Lemma 3.9. *Let T be an algebraic torus over F. Let $t, t' \in T$ be two points such that t belongs to the closure of t' and the local ring $O_{t',t}$ is a DVR. Let $s : T(F(t'))/R \to T(F(t))/R$ be the specialization homomorphism with respect to $O_{t',t}$. Then $s(t') = t$.*

Proof. In the ring $A := F[T]$ let P and P' be the prime ideals of y and y' respectively. Then O is the ring $A_P/P'A_P$. Let $\tilde{t} \in T(O) = \mathrm{Mor}(\mathrm{Spec}\, O, T)$ be the point given by the natural homomorphism of $A \to O$. Then the images of $\tilde{t}$

under the maps $T(O) \to T(F(t))$ and $T(O) \to T(F(t'))$ coincide with y and y' respectively. The statement follows now from Example 3.8. $\square$

Let $\theta : A \to B$ be a morphism of functors from *Fields*/F to *Sets* with specializations (for example, the functors $L \mapsto T(L)/R$ or $L \mapsto \mathrm{CH}_0(X_L)$). We say that θ *commutes with specializations* if for every DVR as above, the diagram

$$\begin{tikzcd} A(L) \arrow[r, "\theta_L"] \arrow[d, "s_A"'] & B(L) \arrow[d, "s_B"] \\ A(K) \arrow[r, "\theta_K"] & B(K) \end{tikzcd}$$

is commutative.

Proposition 3.10. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the morphism φ in (16) commutes with specializations.*

Proof. Let O be a DVR over F with quotient field L and residue field K. For an O-point p of T let $[p]$ denote the class of its graph in $\mathrm{CH}_1(X_O)$. Consider the diagram

$$\begin{tikzcd} T(K) \arrow[d, "\varphi_K"'] & T(O) \arrow[l] \arrow[r] \arrow[d, "\varphi_O"'] & T(L) \arrow[d, "\varphi_L"'] \\ \mathrm{CH}_0(X_K) & \mathrm{CH}_1(X_O) \arrow[l] \arrow[r] & \mathrm{CH}_0(X_L) \end{tikzcd}$$

where $\varphi_O(p) = [p] - [1]$ and the bottom maps are the pull-back homomorphisms. The statement follows from the commutativity property of the diagram. To prove commutativity let E be either K or L and $f : \operatorname{Spec} E \to \operatorname{Spec} O$, $g : X_E \to X_O$ the natural morphisms. Let $p \in T(O)$ be a point and $q \in T(E)$ its image. We view p and q as morphisms $p : \operatorname{Spec} O \to X_O$ and $q : \operatorname{Spec} E \to X_E$. By [Fulton 1984, Th. 6.2(a)], the diagram

$$\begin{tikzcd} \mathrm{CH}_1(\operatorname{Spec} O) \arrow[r, "f^*"] \arrow[d, "p_*"'] & \mathrm{CH}_0(\operatorname{Spec} E) \arrow[d, "q_*"] \\ \mathrm{CH}_1(X_O) \arrow[r, "g^*"] & \mathrm{CH}_0(X_E) \end{tikzcd}$$

is commutative. It follows that $[q] = q_*(1_E) = q_* f^*(1_O) = g^* p_*(1_O) = g^*([p])$ and the result follows. $\square$

Proposition 3.11. *Let T be an algebraic torus over F and $\theta, \theta' : T(?)/R \to B$ two morphisms of functors commuting with specializations. Suppose that $\theta_{F(T)}$ and $\theta'_{F(T)}$ coincide at the generic point of T. Then $\theta = \theta'$.*

Proof. Let $p : \operatorname{Spec} L \to T$ be a point of T over a field extension L over F. We need to prove that $\theta_L(p) = \theta'_L(p)$. Let $t \in T$ be the point in the image of p. We view t as a point of T over the residue field $F(t)$. As $F(t) \subset L$ and p is the image of t under the map $T(F(t)) \to T(L)$, it suffices to show that $\theta_{F(t)}(t) = \theta'_{F(t)}(t)$.

We prove this by induction on $\operatorname{codim}(t)$. By assumption, the statement holds if t is the generic point. Otherwise let $t' \in T$ be a point such that t is a direct specialization of t'. Then the local ring $O_{t',t}$ is a DVR with quotient field $F(t')$ and residue field $F(t)$. As θ and θ' commute with specializations, it follows from Lemma 3.9 that

$$\begin{aligned}\theta_{F(t)}(t) = \theta_{F(t)}(s(t')) &= s_B\big(\theta_{F(t')}(t')\big) \\ &= s_B\big(\theta'_{F(t')}(t')\big) = \theta'_{F(t)}(s(t')) = \theta'_{F(t)}(t).\end{aligned} \qquad \square$$

Proposition 3.12. *Let T be an algebraic torus of dimension 3 over F and X a smooth proper toric model of T. Then the morphism of functors ρ in (15) commutes with specializations.*

Proof. Let O be a DVR over F of geometric type with quotient field L and residue field K. The diagram

$$\begin{array}{ccccc} H^1(X_K, K_2) & \longleftarrow & H^1(X_O, K_2) & \longrightarrow & H^1(X_L, K_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CH}^3(X_K) & \longleftarrow & \mathrm{CH}^3(X_O) & \longrightarrow & \mathrm{CH}^3(X_L) \end{array}$$

where the middle vertical map is the differential in the E_2-term of the BGQ spectral sequence (2) for X_O. The right square is commutative since the morphism $X_L \to X_O$ is flat [Quillen 1973, § 7, Th. 5.4].

The pull-back homomorphism f^* for the morphism $f : X_K \to X_O$ in K-theory is defined as follows (see [Quillen 1973, § 7.2.5]). Let $\pi \in O$ be a prime element and $M(X_O, f)$ the full subcategory of the category $M(X_O)$ of coherent sheaves on X_O consisting of sheaves G with π a nonzero-divisor in G. Then f^* is the composition of the inverse of the isomorphism induced by the inclusion functor

$$\alpha : M(X_O, f) \to M(X_O)$$

on K-groups and the map induced by the restriction

$$\beta : M(X_O, f) \to M(X_K)$$

of the unverse image functor $M(X_O) \to M(X_K)$. Note that functors α and β take sheaves supported in codimension p into $M^p(X_O)$ and $M^p(X_K)$ respectively. Hence f induces a pull-back map of the BGQ spectral sequences for X_O and X_K. It follows that the left square of the diagram is commutative too.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, we may consider the split situation. In the diagram

$$\begin{tikzcd} S^\circ(K) \arrow[d] & S^\circ(O) \arrow[l] \arrow[r] \arrow[d] & S^\circ(L) \arrow[d] \\ H^1(X_K, K_2) & H^1(X_O, K_2) \arrow[l] \arrow[r] & H^1(X_L, K_2) \end{tikzcd}$$

the vertical maps are the product maps. The commutativity follows from the projection formula in K-cohomology [Rost 1996, § 14.5].

Finally, it follows from the definition that the isomorphism $T(L)/R \xrightarrow{\sim} S^\circ(L)/R$ of functors in (15) commutes with specializations. □

4. Main theorem

Let T be a torus over F and $1 \to S \to P \to T \to 1$ a flasque resolution.

4A. ***The group $T(F(T))/R$.*** Tensoring the exact sequence

$$0 \to F_{\mathrm{sep}}^\times \oplus T^* \to F_{\mathrm{sep}}(T)^\times \to \mathrm{Div}(T_{\mathrm{sep}}) \to 0$$

with S_* and applying Galois cohomology yields a surjective homomorphism

$$H^1(F, S) \oplus H^1(F, S_* \otimes T^*) \to H^1(F(T), S)$$

since $H^1\big(F, S_* \otimes \mathrm{Div}(T_{\mathrm{sep}})\big) = 0$ as S is flasque.

Tensoring (4) with S_* yields a surjective homomorphism

$$\mathrm{End}_F(S) = H^0(F, S_* \otimes S^*) \to H^1(F, S_* \otimes T^*)$$

as $H^1(F, S_* \otimes P^*) = 0$. Combining these two surjections we get another surjective homomorphism

$$(T(F)/R) \oplus \mathrm{End}_F(S) \to T(F(T))/R.$$

Note that the group $T(L)/R = H^1(L, S)$ is a left module over the ring $\mathrm{End}_F(S)$ for any field extension L/F. The image of an element $\alpha \in \mathrm{End}_F(S)$ in $T(F(T))/R$ is equal to $\alpha(\xi)$ (up to sign), where ξ is the generic point of T.

We have proven:

Proposition 4.1. *Every element of the group $T(F(T))/R$ is of the form $t \cdot \alpha(\xi)$ where $t \in T(F)/R$ and $\alpha \in \mathrm{End}_F(S)$.*

Now assume that $\dim T = 3$ and X is a smooth proper toric model of T.

Corollary 4.2. *There is an $\alpha \in \mathrm{End}_F(S)$ such that the composition $\rho^{-1} \circ \varphi$ takes every $t \in T(L)/R$ over a field extension L/F to $\alpha(t)$.*

Proof. By Propositions 3.10, 3.11 and 3.12, it is sufficient to prove the statement in the case when t is the generic point ξ of T. By Proposition 4.1, $(\rho^{-1} \circ \varphi)(\xi) = t \cdot \alpha(\xi)$ for some $\alpha \in \mathrm{End}_F(S)$ and $t \in T(F)/R$. As $(\rho^{-1} \circ \varphi)(1) = 1$, specializing at 1, we get $t = 1$. □

Example 3.3 then yields:

Corollary 4.3. *The composition $\rho^{-1} \circ \varphi$ commutes with norms.*

4B. ***Main theorem.***

Theorem 4.4. *Let T be an algebraic torus of dimension 3 and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the map $\varphi : T(F)/R \to A_0(X)$ is an isomorphism.*

Proof. In view of Proposition 2.9, we may assume that X is a smooth proper toric model of T. By Proposition 3.6 and Corollary 4.3, φ commutes with norms. It follows from Proposition 3.5 that φ is an isomorphism. □

Remark 4.5. The following is an alternative proof of Theorem 4.4. It avoids the machinery of Section 3, but it is based on deep, albeit classical, arithmetic-geometric result. We may assume that the field F is finitely generated over the prime subfield. By [Colliot-Thélène and Sansuc 1977, Th. 1], the group $T(F)/R$ is finite. It follows from (15) that $A_0(X)$ is also finite of the same order. As φ is injective, it is a bijection. Therefore, φ is an isomorphism of groups as we have a homomorphism of groups ψ with $\psi \circ \varphi = \mathrm{id}$.

The statement of the following theorem (but not the proof) does not involve a toric model.

Theorem 4.6. *Let T be an algebraic torus of dimension 3. Then there is a natural isomorphism $T(F)/R \simeq H^1(F, T^\circ)/R$.*

Proof. The sequence dual to (5)

$$1 \to T^\circ \to P^\circ \to S^\circ \to 1$$

and [Colliot-Thélène and Sansuc 1977, Th. 2] (see Example 1.3) yield an isomorphism

$$S^\circ(F)/R \simeq H^1(F, T^\circ)/R.$$

On the other hand, by (8), $S^\circ(F)/R \simeq H^1(F, S) \simeq T(F)/R$. □

In the following examples we give two applications of Theorem 4.6.

Example 4.7. Let L/F be a degree 4 separable field extension and T the norm 1 torus for L/F, that is,

$$T = \mathrm{Ker}\big(R_{L/F}(\mathbb{G}_{m,L}) \xrightarrow{N_{L/F}} \mathbb{G}_m\big).$$

Then $T^\circ = R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$ and

$$H^1(F, T^\circ) = \mathrm{Br}(L/F),$$

the relative Brauer group of the extension L/F. Thus by Theorem 4.6, we have a canonical isomorphism

$$\mathrm{Br}(L/F)/R \simeq T(F)/R.$$

The case of a biquadratic extension L/F was considered in [Tignol 1981, p. 427].

Example 4.8. Let L and K be finite separable field extensions of a field F and set $M := K \otimes_F L$. Let T be the kernel of the norm homomorphism

$$N_{M/L} : R_{M/F}(\mathbb{G}_{m,M})/R_{K/F}(\mathbb{G}_{m,K}) \to R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m.$$

We have

$$T(F) = \{x \in M^\times \quad \text{such that} \quad N_{M/L}(x) \in F^\times\}/K^\times.$$

The dual torus T° is the kernel of the norm homomorphism

$$N_{M/K} : R_{M/F}(\mathbb{G}_{m,M})/R_{L/F}(\mathbb{G}_{m,L}) \to R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_m.$$

We have an exact sequence

$$K^\times \to H^1(F, T^\circ) \to \mathrm{Br}(M/L) \to \mathrm{Br}(K/F).$$

Now suppose that $[K : F] = 2$ and $[L : F] = 4$. Then T is a 3-dimensional torus and the last homomorphism in the exact sequence is isomorphic to the norm map

$$N_{L/F} : L^\times/N_{M/L}(M^\times) \to F^\times/N_{K/F}(K^\times).$$

Let U be the subtorus of $R_{L/F}(\mathbb{G}_{m,L}) \times R_{K/F}(\mathbb{G}_{m,K})$ consisting of all pairs (l, k) with $N_{L/F}(l) = N_{K/F}(k)$. It follows that

$$T(F)/R \simeq H^1(F, T^\circ)/R \simeq U(F)/R.$$

This isomorphism was known when L/F is a biquadratic extension (see [Shapiro et al. 1982, Cor. 1.13] and [Gille 1997, Prop. 3]).

5. Chow group of a 3-dimensional torus

Let T be an algebraic torus over a field F and X a smooth proper geometrically irreducible variety containing T as an open subset. Set $Z = X \setminus T$.

Lemma 5.1. (see [Colliot-Thélène and Sansuc 1977, Lemme 12], [Voskresenskiĭ 1998, Prop. 17.3] and [Gille 2004, Prop. 1.1]) *The torus T is isotropic if and only if $Z(F) \neq \varnothing$.*

Proof. Suppose T is isotropic. Then T contains a subgroup isomorphic to $\mathbb{G}_m$. The embedding of $\mathbb{G}_m$ into T extends to a regular morphism $f : \mathbb{P}^1 \to X$. Then $f(0)$ or $f(\infty)$ is a rational point of Z.

Conversely, suppose Z has a rational point z. Since z is regular on X, there is a geometric valuation v of $F(X)$ dominating z with residue field $F = F(z)$. Suppose that T is anisotropic. Then there is a proper geometrically irreducible variety X' containing T as an open subset such that $X' \setminus T$ has no rational points (see [Colliot-Thélène and Sansuc 1977, Lemme 12], [Voskresenskiĭ 1998, Prop. 17.3]). But v dominates a rational point on $X' \setminus T$, a contradiction. □

Write i_T (respectively n_Z) for the greatest common divisor of the integers $[L : F]$ for all finite field extensions L/F such that T is isotropic over L (respectively $Z(L) \neq \varnothing$).

Corollary 5.2. *The number i_T coincides with n_Z. In particular, the integer n_Z does not depend on the smooth proper geometrically irreducible variety X containing T as an open subset.*

Proposition 5.3. *The order of the class* $[1]$ *in* $\mathrm{CH}_0(T)$ *is equal to i_T.*

Proof. If T is isotropic, there is a subgroup H of T isomorphic to $\mathbb{G}_m$. As $\mathrm{CH}_0(\mathbb{G}_m) = 0$, we have $[1] = 0$ in $\mathrm{CH}_0(H)$ and therefore in $\mathrm{CH}_0(T)$. In the general case, let L be a finite field extension such that T_L is isotropic. By the first part of the proof, $[1]$ is trivial in $\mathrm{CH}_0(T_L)$; hence applying the norm map for the extension L/F yields $[L : F] \cdot [1] = 0$ in $\mathrm{CH}_0(T)$. Therefore, $i_T \cdot [1] = 0$.

Now let $m \cdot [1] = 0$ in $\mathrm{CH}_0(T)$ for some integer m. Hence the cycle $m \cdot [1]$ in $\mathrm{CH}_0(X)$ belongs to the image of the push-forward map $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$ [Fulton 1984, Prop. 1.8]. In particular, there is a zero-cycle on Z of degree m, hence $i_F = n_Z$ divides m. □

Consider the map

$$\alpha_T : T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$$

taking a pair (t, k) to the cycle $[t] + (k-1) \cdot [1]$.

Theorem 5.4. *Let T be a torus of dimension at most* 3. *Then the map α_T : $T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$ is an isomorphism.*

Proof. The Chow group $\mathrm{CH}_0(T)$ is the factor group of $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z} \cdot [1]$ by the image of $\mathrm{CH}_0(Z)$. Let $z \in Z$ be a closed point. By Lemma 5.1, the torus $T_{F(z)}$ is isotropic and hence is stably birational to a 2-dimensional torus. Therefore, $T_{F(z)}$ is rational, $A_0(X_{F(z)}) = 0$ and the image of the class of z in $A_0(X) \oplus \mathbb{Z} \cdot [1]$ is equal to $0 \oplus \deg(z) \cdot [1]$. Hence $\mathrm{CH}_0(T)$ is isomorphic to $A_0(X) \oplus \mathbb{Z}/i_T\mathbb{Z}$. The result follows from Theorem 4.4. □

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References

[Brylinski 1979] J.-L. Brylinski, "Décomposition simpliciale d'un réseau, invariante par un groupe fini d'automorphismes", *C. R. Acad. Sci. Paris Sér. A-B* **288**:2 (1979), A137–A139. MR 80f:14024 Zbl 0406.14022

[Colliot-Thélène and Raskind 1985] J.-L. Colliot-Thélène and W. Raskind, "$\mathcal{K}_2$-cohomology and the second Chow group", *Math. Ann.* **270**:2 (1985), 165–199. MR 86m:14005 Zbl 0536.14004

[Colliot-Thélène and Sansuc 1977] J.-L. Colliot-Thélène and J.-J. Sansuc, "La R-équivalence sur les tores", *Ann. Sci. École Norm. Sup.* (4) **10**:2 (1977), 175–229. MR 56 #8576 Zbl 0356.14007

[Colliot-Thélène and Sansuc 1987] J.-L. Colliot-Thélène and J.-J. Sansuc, "Principal homogeneous spaces under flasque tori: applications", *J. Algebra* **106**:1 (1987), 148–205. MR 88j:14059 Zbl 0597.14014

[Colliot-Thélène et al. 2004] J.-L. Colliot-Thélène, P. Gille, and R. Parimala, "Arithmetic of linear algebraic groups over 2-dimensional geometric fields", *Duke Math. J.* **121**:2 (2004), 285–341. MR 2005f:11063 Zbl pre02103584

[Colliot-Thélène et al. 2005] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov, "Compactification équivariante d'un tore (d'après Brylinski et Künnemann)", *Expo. Math.* **23**:2 (2005), 161–170. MR 2006c:14076 Zbl 1078.14076

[Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik **2**, Springer, Berlin, 1984. MR 85k:14004 Zbl 0541.14005

[Garibaldi et al. 2003] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, Providence, RI, 2003. MR 2004f:11034 Zbl pre01959122

[Gille 1997] P. Gille, "Examples of non-rational varieties of adjoint groups", *J. Algebra* **193**:2 (1997), 728–747. MR 98k:11038 Zbl 0911.14026

[Gille 2004] P. Gille, "Spécialisation de la R-équivalence pour les groupes réductifs", *Trans. Amer. Math. Soc.* **356**:11 (2004), 4465–4474. MR 2005k:20114 Zbl 1067.20063

[Gillet 1981] H. Gillet, "Riemann–Roch theorems for higher algebraic K-theory", *Adv. in Math.* **40**:3 (1981), 203–289. MR 83m:14013 Zbl 0478.14010

[Kahn 1996] B. Kahn, "Applications of weight-two motivic cohomology", *Doc. Math.* **1**:17 (1996), 395–416. MR 98b:14007 Zbl 0883.19002

[Klyachko 1982] A. A. Klyachko, "K-theory of Demazure models", pp. 61–72 in Исследования по теории чисел: арифметика алгебраических многообразий, edited by G. Chudakov, Saratov. Gos. Univ., 1982. In Russian; translated as pp. 37–46 in *Selected papers in K-theory*, Amer. Math. Soc. Transl. (2) **154**, Amer. Math. Soc., Providence, 1992. MR 947351 Zbl 0796.14031

[Kunyavskiĭ 1987] B. È. Kunyavskiĭ, "Three-dimensional algebraic tori", pp. 90–111 in Исследования по теории чисел: алгебраическая теория чисел, edited by N. G. Chudakov, Saratov. Gos. Univ., Saratov, 1987. In Russian; translated in Selecta Math. Soviet. **9**:1 (1990), 1–21. MR 91g:14050

[Manin 1968] J. I. Manin, "Correspondences, motifs and monoidal transformations", *Mat. Sb.* (*N.S.*) **77 (119)** (1968), 475–507. MR 41 #3482 Zbl 0199.24803

[Manin 1986] Y. I. Manin, *Cubic forms: Algebra, geometry, arithmetic*, 2nd ed., North-Holland Mathematical Library **4**, North-Holland, Amsterdam, 1986. MR 87d:11037 Zbl 0582.14010

[Merkurjev and Panin 1997] A. S. Merkurjev and I. A. Panin, "K-theory of algebraic tori and toric varieties", *K-Theory* **12**:2 (1997), 101–143. MR 98j:19005 Zbl 0882.19002

[Quillen 1973] D. Quillen, "Higher algebraic K-theory. I", pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, Berlin, 1973. MR 49 #2895 Zbl 0292.18004

[Rost 1996] M. Rost, "Chow groups with coefficients", *Doc. Math.* **1**:16 (1996), 319–393. MR 98a:14006 Zbl 0864.14002

[Shapiro et al. 1982] D. B. Shapiro, J.-P. Tignol, and A. R. Wadsworth, "Witt rings and Brauer groups under multiquadratic extensions. II", *J. Algebra* **78**:1 (1982), 58–90. MR 85i:11033 Zbl 0492.10015

[Tignol 1981] J.-P. Tignol, "Produits croisés abéliens", *J. Algebra* **70**:2 (1981), 420–436. MR 84f:16026 Zbl 0473.16004

[Voskresenskiĭ 1998] V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs **179**, American Mathematical Society, Providence, RI, 1998. MR 99g:20090

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Construction de (φ, Γ)-modules: représentations p-adiques et B-paires

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Soit $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. On étudie la catégorie des B-paires (W_e, W_{dR}^+) où W_e est un $\mathbf{B}_{\mathrm{e}}$-module libre muni d'une action semi-linéaire et continue de G_K et où W_{dR}^+ est un $\mathbf{B}_{\mathrm{dR}}^+$-réseau stable par G_K de $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$. Cette catégorie contient celle des représentations p-adiques, et est naturellement équivalente à la catégorie de tous les (φ, Γ)-modules sur l'anneau de Robba.

Let $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. We study the category of B-pairs (W_e, W_{dR}^+) where W_e is a free $\mathbf{B}_{\mathrm{e}}$-module with a semilinear and continuous action of G_K and where W_{dR}^+ is a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice in $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$. This category contains the category of p-adic representations and is naturally equivalent to the category of all (φ, Γ)-modules over the Robba ring.

Introduction

Dans tout cet article, p est un nombre premier, k est un corps parfait de caractéristique p, et K est une extension finie totalement ramifiée de $K_0 = W(k)[1/p]$. On s'intéresse aux représentations p-adiques du groupe de Galois $G_K = \mathrm{Gal}(\overline{K}/K)$. La théorie de Hodge p-adique [Fontaine 1994a ; 1994b] a pour but de décrire certaines de ces représentations, celles qui « proviennent de la géométrie », en termes d'objets plus maniables, les (φ, N)-modules filtrés. Le résultat le plus satisfaisant dans cette direction est le théorème de Colmez–Fontaine, qui dit que le foncteur $V \mapsto \mathrm{D}_{\mathrm{st}}(V)$ réalise une équivalence de catégories entre la catégorie des représentations p-adiques semi-stables et la catégorie des (φ, N)-modules filtrés admissibles.

Si D est un φ-module filtré qui provient de la cohomologie d'un schéma propre X sur $\mathbb{O}_K$, alors le φ-module sous-jacent ne dépend que de la fibre spéciale de X (c'en est la cohomologie cristalline) alors que la filtration ne dépend que de la fibre générique (c'est la filtration de Hodge de la cohomologie de de Rham, dans laquelle se plonge la cohomologie cristalline). Si $V = V_{\mathrm{cris}}(D)$ et $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$, alors

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on voit que $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V = (\mathbf{B}_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1}$ ne dépend que de la structure de φ-module de D et de plus, les φ-modules D_1 et D_2 sont isomorphes si et seulement si les $\mathbf{B}_{\mathrm{e}}$-représentations $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V_1$ et $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V_2$ le sont (voir la proposition 8.2 de [Fontaine 2003]). Par ailleurs, $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D)$ et les modules filtrés $K \otimes_{K_0} D_1$ et $K \otimes_{K_0} D_2$ sont isomorphes si et seulement si les $\mathbf{B}_{\mathrm{dR}}^+$-représentations $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V_1$ et $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V_2$ le sont.

L'idée de cet article est d'isoler les phénomènes liés à la « fibre spéciale » et à la « fibre générique » en considérant non pas des représentations p-adiques V, mais des *B-paires* $W = (W_e, W_{\mathrm{dR}}^+)$ où W_e est un $\mathbf{B}_{\mathrm{e}}$-module libre muni d'une action semi-linéaire et continue de G_K et où W_{dR}^+ est un $\mathbf{B}_{\mathrm{dR}}^+$-réseau stable par G_K de $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$. Rappelons que les anneaux $\mathbf{B}_{\mathrm{cris}}$ et $\mathbf{B}_{\mathrm{dR}}$ sont reliés, en plus de l'inclusion $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$, par la *suite exacte fondamentale* :

$$0 \to \mathbb{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0.$$

Si V est une représentation p-adique, alors on lui associe la B-paire $W(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$ et ce foncteur est pleinement fidèle car $\mathbf{B}_{\mathrm{e}} \cap \mathbf{B}_{\mathrm{dR}}^+ = \mathbb{Q}_p$ ce qui fait que $V = W_e(V) \cap W_{\mathrm{dR}}^+(V)$.

L'un des principaux outils dont on dispose pour étudier les représentations p-adiques est la théorie des (φ, Γ)-modules. De fait, on a une équivalence de catégories entre la catégorie des représentations p-adiques et la catégorie des (φ, Γ)-modules étales sur l'anneau de Robba (en combinant des résultats de Fontaine, Cherbonnier–Colmez et Kedlaya). Le premier résultat de cet article est que l'on peut associer à toute B-paire W un (φ, Γ)-module $\mathrm{D}(W)$ sur l'anneau de Robba, et que ce foncteur est alors une équivalence de catégories.

Théorème A. *Le foncteur $W \mapsto \mathrm{D}(W)$ réalise une équivalence de catégories entre la catégorie des B-paires et la catégorie des (φ, Γ)-modules sur l'anneau de Robba.*

La sous-catégorie pleine des B-paires de la forme $W(V)$ correspond à la sous-catégorie pleine des (φ, Γ)-modules étales. On peut alors se demander à quoi correspondent les (φ, Γ)-modules isoclines. Si $h \geqslant 1$ et $a \in \mathbb{Z}$ sont premiers entre eux, soit $\mathrm{Rep}(a, h)$ la catégorie dont les objets sont les $\mathbb{Q}_{p^h}$-espaces vectoriels $V_{a,h}$ de dimension finie, munis d'une action semi-linéaire de G_K et d'un Frobenius lui aussi semi-linéaire $\varphi : V_{a,h} \to V_{a,h}$ qui commute à G_K et qui vérifie $\varphi^h = p^a$. Si $V_{a,h} \in \mathrm{Rep}(a, h)$, alors on pose $W_e(V_{a,h}) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_{p^h}} V_{a,h})^{\varphi=1}$ et $W_{\mathrm{dR}}^+(V_{a,h}) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_{p^h}} V_{a,h}$.

Théorème B. *Si $V_{a,h} \in \mathrm{Rep}(a, h)$, alors $W(V_{a,h}) = (W_e(V_{a,h}), W_{\mathrm{dR}}^+(V_{a,h}))$ est une B-paire et le foncteur $V_{a,h} \mapsto W(V_{a,h})$ définit une équivalence de catégories entre* $\mathrm{Rep}(a, h)$ *et la catégorie des B-paires W telles que* $\mathrm{D}(W)$ *est isocline de pente a/h.*

Si la catégorie des B-paires est plus riche que la catégorie des représentations p-adiques, la morale de cet article est qu'elle est aussi plus maniable. Ceci est dû au fait qu'il est plus facile de travailler avec tous les (φ, Γ)-modules sur l'anneau de Robba que de se restreindre à ceux qui sont étales. La généralisation du théorème de Colmez–Fontaine aux B-paires devient alors un simple exercice d'algèbre linéaire. Si D est un (φ, N)-module filtré, et si $W_e(D) = (\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1, N=0}$ et $W_{\mathrm{dR}}^+(D) = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K)$, alors $W(D) = (W_e(D), W_{\mathrm{dR}}^+(D))$ est une B-paire, qui est semi-stable en un sens évident. Le foncteur $D \mapsto W(D)$ réalise alors une équivalence de catégories entre la catégorie des B-paires semi-stables et la catégorie des (φ, N)-modules filtrés.

Pour retrouver le théorème de Colmez–Fontaine à partir de cet énoncé, il faut identifier quelles sont les B-paires semi-stables qui proviennent des (φ, N)-modules filtrés admissibles. On peut faire cela en passant par les (φ, Γ)-modules, c'est ce qui est fait dans [Berger 2004a], où l'on démontre suffisamment pour impliquer aussi le théorème de monodromie p-adique pour les B-paires : toute B-paire de de Rham est potentiellement semi-stable.

Dans la suite de l'article, nous donnons quelques applications du théorème A. Par exemple, on montre que pour tout φ-module D sur l'anneau de Robba, il existe un φ-module étale $\mathrm{M} \subset \mathrm{D}[1/t]$ tel que $\mathrm{M}[1/t] = \mathrm{D}[1/t]$. Comme $\mathrm{D}(W)[1/t]$ « correspond » à W_e, on déduit de cet énoncé le résultat suivant, où

$$H_K = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty})).$$

Théorème C. *Si W_e est une $\mathbf{B}_{\mathrm{e}}$-représentation de G_K, alors il existe une représentation p-adique V de H_K telle que la restriction de W_e à H_K est isomorphe à $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V$.*

Enfin, nous donnons (quand $K \subset K_0(\mu_{p^\infty})$) une construction des (φ, Γ)-modules de *hauteur finie* (ce sont ceux qui ont une base dans laquelle les matrices de φ et de $\gamma \in \Gamma_K$ n'ont pas de dénominateurs en X) à partir de « φ-modules filtrés sur K_0 avec action de Γ_K ». Cela permet d'éclaircir la structure des B-paires de hauteur finie, et donc en particulier des représentations de hauteur finie. Un *φ-module filtré sur K_0 avec action de* Γ_K est un φ-module D sur K_0 muni d'une action de Γ_K commutant à φ et d'une filtration stable par Γ_K sur $D_\infty = K_\infty \otimes_{K_0} D$.

Théorème D. *Si D est un φ-module filtré sur K_0 avec action de Γ_K, alors la B-paire $W(D) = ((\mathbf{B}_{\max} \otimes_{K_0} D)^{\varphi=1}, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{K_\infty} D_\infty)$ est de hauteur finie.*

Toute B-paire de hauteur finie s'obtient de cette manière et $W(D_1) = W(D_2)$ si et seulement s'il existe un isomorphisme $K_0[t, t^{-1}] \otimes_{K_0} D_1 = K_0[t, t^{-1}] \otimes_{K_0} D_2$ compatible à φ et Γ_K, et compatible à la filtration quand on étend les scalaires à $K_\infty(\!(t)\!)$.

Des objets de nature similaire ont été étudiés, dans un cadre un peu différent, dans [Kisin 2006].

Pour terminer cette introduction, signalons que la notion de représentation trianguline de [Colmez 2005 ; 2007] s'écrit de manière très agréable en termes de B-paires. On dit qu'une B-paire est *trianguline* si elle est extension successive de B-paires de dimension 1. Par le théorème A, cela revient à dire que $\mathrm{D}(W)$ est extension successive de (φ, Γ)-modules de rang 1. On retrouve donc la définition de Colmez dans le cas où $W = W(V)$.

1. Rappels et compléments

Dans ce chapitre, nous donnons des rappels sur les anneaux de périodes et les (φ, Γ)-modules.

1.1. ***L'anneau $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ et ses petits camarades.*** Nous commençons par faire des rappels très succints sur les définitions (données dans [Fontaine 1994a] par exemple) des divers anneaux que nous utilisons dans cet article. Rappelons que

$$\widetilde{\mathbf{E}}^{+} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$$

est un anneau de caractéristique p, complet pour la valuation $\mathrm{val}_{\mathrm{E}}$ définie par $\mathrm{val}_{\mathrm{E}}(x) = \mathrm{val}_p(x^{(0)})$ et qui contient un élément ε tel que $\varepsilon^{(n)}$ est une racine primitive p^n-ième de l'unité. On fixe un tel ε dans tout l'article. L'anneau $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^{+}[1/(\varepsilon - 1)]$ est alors un corps qui contient comme sous-corps dense la clôture algébrique de $\mathbb{F}_p(\!(\varepsilon - 1)\!)$. On pose $\widetilde{\mathbf{A}}^{+} = W(\widetilde{\mathbf{E}}^{+})$ et $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ ainsi que $\widetilde{\mathbf{B}}^{+} = \widetilde{\mathbf{A}}^{+}[1/p]$ et $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$. L'application $\theta : \widetilde{\mathbf{B}}^{+} \to \mathbb{C}_p$ qui à $\sum_{k \gg -\infty} p^k [x_k]$ associe $\sum_{k \gg -\infty} p^k x_k^{(0)}$ est un morphisme d'anneaux surjectif et $\mathbf{B}^{+}_{\mathrm{dR}}$ est le complété de $\widetilde{\mathbf{B}}^{+}$ pour la topologie $\ker(\theta)$-adique, ce qui en fait un espace topologique de Fréchet. On pose $X = [\varepsilon] - 1 \in \widetilde{\mathbf{A}}^{+}$ et $t = \log(1 + X) \in \mathbf{B}^{+}_{\mathrm{dR}}$ et on définit $\mathbf{B}_{\mathrm{dR}}$ par $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}^{+}_{\mathrm{dR}}[1/t]$. Soit $\tilde{p} \in \widetilde{\mathbf{E}}^{+}$ un élément tel que $p^{(0)} = p$. L'anneau $\mathbf{B}^{+}_{\max}$ est l'ensemble des séries $\sum_{n \geqslant 0} b_n([\tilde{p}]/p)^n$ où $b_n \in \widetilde{\mathbf{B}}^{+}$ et $b_n \to 0$ quand $n \to \infty$ ce qui en fait un sous-anneau de $\mathbf{B}^{+}_{\mathrm{dR}}$ muni en plus d'un Frobenius φ qui est injectif, mais pas surjectif. On pose $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} = \bigcap_{n \geqslant 0} \varphi^n(\mathbf{B}^{+}_{\max})$ ce qui en fait un sous-anneau de $\mathbf{B}^{+}_{\max}$ sur lequel φ est bijectif. On pose enfin $\mathbf{B}_{\max} = \mathbf{B}^{+}_{\max}[1/t]$ et $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\max}[\log[\tilde{p}]]$ ce qui fait de $\mathbf{B}_{\mathrm{st}}$ un sous-anneau de $\mathbf{B}_{\mathrm{dR}}$ muni de φ et d'un opérateur de monodromie N. Remarquons que l'on travaille souvent avec $\mathbf{B}_{\mathrm{cris}}$ plutôt que $\mathbf{B}_{\max}$ mais le fait de préférer $\mathbf{B}_{\max}$ ne change rien aux résultats et est plus agréable pour des raisons techniques.

Rappelons que les anneaux $\mathbf{B}_{\max}$ et $\mathbf{B}_{\mathrm{dR}}$ sont reliés, en plus de l'inclusion $\mathbf{B}_{\max} \subset \mathbf{B}_{\mathrm{dR}}$, par la *suite exacte fondamentale* $0 \to \mathbb{Q}_p \to \mathbf{B}^{\varphi=1}_{\max} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^{+}_{\mathrm{dR}} \to 0$. Ce sont ces anneaux que l'on utilise en théorie de Hodge p-adique. Le point de départ de la théorie des (φ, Γ)-modules sur l'anneau de Robba (dont on parle au paragraphe

1.2) est la construction d'anneaux intermédiaires entre $\widetilde{\mathbf{B}}^+$ et $\widetilde{\mathbf{B}}$. Si $r > 0$, soit $\widetilde{\mathbf{B}}^{\dagger,r}$ l'ensemble des $x = \sum_{k \gg -\infty} p^k [x_k] \in \widetilde{\mathbf{B}}$ tels que $\mathrm{val}_{\mathrm{E}}(x_k) + k \cdot pr/(p-1)$ tend vers $+\infty$ quand k augmente. On pose $\widetilde{\mathbf{B}}^{\dagger} = \bigcup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}$, c'est le corps des éléments surconvergents, défini dans [Cherbonnier et Colmez 1998]. L'anneau $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ défini dans [Berger 2002, §2.3] est en quelque sorte la somme des anneaux $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ et $\widetilde{\mathbf{B}}^{\dagger}$; de fait, on a une suite exacte (d'anneaux et d'espaces de Fréchet) :

$$0 \to \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \oplus \widetilde{\mathbf{B}}^{\dagger,r} \to \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \to 0.$$

Rappelons que $K_0 = W(k)[1/p]$; pour $1 \leqslant n \leqslant +\infty$, on pose $K_n = K(\mu_{p^n})$ et $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ et $\Gamma_K = G_K/H_K$. Si R est un anneau muni d'une action de G_K (c'est le cas pour tous ceux que nous considérons), on note $R_K = R^{H_K}$. L'anneau $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ contient l'ensemble des séries $f(X) = \sum_{k \in \mathbb{Z}} f_k X^k$ avec $f_k \in K_0$ telles que $f(X)$ converge sur $\{p^{-1/r} \leqslant |X| < 1\}$. Cet anneau est noté $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K_0}$. Si K est une extension finie de K_0, il lui correspond par la théorie du corps de normes [Fontaine et Wintenberger 1979; Wintenberger 1983] une extension finie $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ qui s'identifie (si r est assez grand) à l'ensemble des séries $f(X_K) = \sum_{k \in \mathbb{Z}} f_k X_K^k$ avec $f_k \in K_0'$ telles que $f(X_K)$ converge sur $\{p^{-1/er} \leqslant |X_K| < 1\}$ où X_K est un certain élément de $\widetilde{\mathbf{B}}^{\dagger}_K$ et K_0' est la plus grande sous-extension non ramifiée de K_0 dans K_∞ et $e = [K_\infty : K_0(\mu_{p^\infty})]$. On pose

$$\mathbf{B}^{\dagger}_{\mathrm{rig},K} = \bigcup_{r>0} \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}, \quad \mathbf{B}^{\dagger,r}_K = \mathbf{B}^{\dagger,r}_{\mathrm{rig},K} \cap \widetilde{\mathbf{B}}^{\dagger}, \quad \mathbf{B}^{\dagger}_K = \bigcup_{r>0} \mathbf{B}^{\dagger,r}_K.$$

Les anneaux $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ et $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ coïncident avec les anneaux $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ et $\Gamma_{\mathrm{an,con}}$ définis dans [Kedlaya 2005, §2.2] (voir en particulier la convention 2.2.16 et la remarque 2.4.13 de [Kedlaya 2005]). Kedlaya les a étudiés en détail et nous rappelons à présent quelques uns des résultats que nous utilisons dans la suite.

Proposition 1.1.1. *Les anneaux $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ et l'anneau $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ sont de Bézout, ainsi que les anneaux $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ et l'anneau $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Si R est l'un de ces anneaux, et si M est un sous-R-module d'un R-module libre de rang fini, alors les affirmations suivantes sont équivalentes :*

(1) *M est libre ;*

(2) *M est fermé ;*

(3) *M est de type fini.*

Démonstration. Pour $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ ou $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$, c'est le théorème 2.9.6 de [Kedlaya 2005] et pour $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ ou $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, c'est une conséquence immédiate de ce que $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ et $\mathbf{B}^{\dagger}_{\mathrm{rig},K} = \bigcup_{r>0} \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$. L'affirmation quant aux sous-modules des modules libres est contenue dans le corollaire 2.8.5 et la définition 2.9.5 de [Kedlaya 2005]. □

Corollaire 1.1.2. *Si R est l'un des anneaux ci-dessus, et si $f : D \to E$ est un morphisme de R-modules libres de rang fini, alors* $\mathrm{im}\, f$ *et* $\ker f$ *sont libres de rang fini. De plus,* $\ker f$ *est saturé dans D.*

Remarque 1.1.3. *Les anneaux $\widetilde{\mathbf{B}}^+$ et $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ ne sont pas de Bézout.*

Démonstration. Commençons par $\widetilde{\mathbf{B}}^+$. Soit $\beta_1 > \beta_2 > \cdots$ une suite convexe décroissante d'éléments de $\mathbb{Z}[1/p]$ qui converge vers $\beta > 0$; si $r \in \mathbb{Z}[1/p]$, écrivons Y^r pour $[\tilde{p}^r]$ (rappelons que $\widetilde{\mathbf{E}}^+$ est un anneau parfait et donc que $\tilde{p}^r$ est bien déterminé). On pose $f = \sum_{i\geqslant 0} p^i Y^{\beta_{2i}}$ et $g = \sum_{i\geqslant 0} p^i Y^{\beta_{2i+1}}$. Supposons que l'idéal de $\widetilde{\mathbf{B}}^+$ engendré par f et g est principal, engendré par un élément h. Cet élément est nécessairement dans $Y^{\geqslant\beta}\widetilde{\mathbf{B}}^+$, puisque f et g le sont et comme $\widetilde{\mathbf{A}}^+/p$ est intègre, on peut supposer que $h \in \widetilde{\mathbf{A}}^+$ et que $I = (f,g) \cap \widetilde{\mathbf{A}}^+ = h\widetilde{\mathbf{A}}^+$. On a alors $\mathrm{val}_{\mathrm{E}}(\bar{h}) \geqslant \beta$. Par ailleurs, on a $(f - gY^{\beta_0-\beta_1})/p \in (f,g)$ et cet élément s'écrit $\sum_{i\geqslant 0} p^i Y^{\beta_{2i+2}}(1+O(Y))$ (c'est là qu'on utilise la convexité de la suite) et en itérant ce procédé, on voit que I contient un élément de la forme $\sum_{i\geqslant 0} p^i Y^{\beta_{2i+j}}(1+O(Y))$ pour tout $j \geqslant 0$ et donc que l'image de I dans $\widetilde{\mathbf{E}}^+$ contient des éléments de valuation β_j pour tout $j \geqslant 0$. Ceci entraîne que $\mathrm{val}_{\mathrm{E}}(\bar{h}) = \beta$. Si $\beta \notin \mathbb{Q}$, c'est impossible et donc $\widetilde{\mathbf{B}}^+$ n'est pas de Bézout.

Si $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ était de Bézout, alors il existerait $h \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ tel que $(f,g) = h\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ et en utilisant la théorie des polygones de Newton de [Kedlaya 2005, §2.5], on montre que $h \in \widetilde{\mathbf{B}}^+$. Comme ci-dessus, on peut supposer que $h \in \widetilde{\mathbf{A}}^+ \setminus p\widetilde{\mathbf{A}}^+$. Chacun des éléments $f_j = \sum_{i\geqslant 0} p^i Y^{\beta_{2i+j}}(1+O(Y))$ construits ci-dessus pour $j \geqslant 0$ peut donc s'écrire $f_j = hx_j$ et encore une fois, on a forcément $x_j \in \widetilde{\mathbf{A}}^+$. On a alors $\mathrm{val}_{\mathrm{E}}(\bar{h}) \leqslant \mathrm{val}_{\mathrm{E}}(\bar{f}_j) = \beta_j$ ce qui fait que $\mathrm{val}_{\mathrm{E}}(\bar{h}) \leqslant \beta$. Par ailleurs, si $\alpha \in \mathbb{Z}[1/p]$ et $\alpha < \beta$, alors Y^α divise f et g ce qui fait que $(f,g) \subset Y^\alpha\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ et donc aussi que $h \in Y^\alpha\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \cap \widetilde{\mathbf{A}}^+ = Y^\alpha\widetilde{\mathbf{A}}^+$ pour tout $\alpha < \beta$. Si $\beta \notin \mathbb{Q}$, alors $\mathrm{val}_{\mathrm{E}}(\bar{h}) < \beta$ et en choisissant $\mathrm{val}_{\mathrm{E}}(\bar{h}) < \alpha < \beta$, on trouve une contradiction, ce qui fait que $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ n'est pas de Bézout. □

Une grande partie de l'article [Kedlaya 2005] est consacré à l'étude des φ-modules sur l'anneau $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ ou sur l'anneau $\mathbf{B}^\dagger_{\mathrm{rig},K}$. On rappelle à présent quelques résultats quant au premier cas (pour le deuxième, voir le paragraphe 1.2).

Définition 1.1.4. Si $h \geqslant 1$ et $a \in \mathbb{Z}$ sont deux entiers premiers entre eux, alors le *φ-module élémentaire $M_{a,h}$* est le φ-module de base $e_0, \ldots, e_{h-1}$ avec $\varphi(e_0) = e_1, \ldots, \varphi(e_{h-2}) = e_{h-1}$ et $\varphi(e_{h-1}) = p^a e_0$ (cf. la définition 4.1.1 de [Kedlaya 2005] ; on utilise (a,h) plutôt que (c,d) pour être compatible avec les notations de Colmez et de Fontaine).

Proposition 1.1.5. *Si M est un φ-module sur $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$, alors il existe des entiers a_i, h_i tels que $M = \bigoplus M_{a_i,h_i}$.*

Démonstration. Étant donnée la définition 4.5.1 de [Kedlaya 2005], c'est le (a) du théorème 4.5.7 du même article. □

Remarquons que la décomposition n'est pas canonique, mais que l'ensemble des pentes $s_i = a_i/h_i$ comptées avec multiplicités est canonique (cf. le (c) du théorème

4.5.7 de [Kedlaya 2005]). Les rationnels ainsi obtenus sont les *pentes* de M.

Corollaire 1.1.6. *Si M est un φ-module sur $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, alors $1-\varphi : M[1/t] \to M[1/t]$ est surjective.*

Démonstration. Si $n \geqslant 0$, alors $(1-\varphi)(t^{-n}x) = t^{-n}(1-p^{-n}\varphi)(x)$ et il suffit donc de montrer que si $n \gg 0$, alors $1-p^{-n}\varphi : M \to M$ est surjectif. Étant donnée la proposition 1.1.5 ci-dessus et le fait que $M_{a,h}(-n) = M_{a-nh,h}$, c'est une conséquence immédiate du (b) de la proposition 4.1.3 de [Kedlaya 2005]. □

L'anneau $\mathbf{B}_{\max}^{\varphi=1}$ (qui est égal à $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$) occupe une place centrale dans cet article ; il est traditionnellement noté $\mathbf{B}_{\mathrm{e}}$. Étant donnée la définition de $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$, il est clair que l'on a $\mathbf{B}_{\mathrm{e}} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t])^{\varphi=1}$.

Lemme 1.1.7. *On a $\mathbf{B}_{\mathrm{e}} = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi=1}$.*

Démonstration. Si $x \in \mathbf{B}_{\mathrm{e}}$, alors $x \in (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t])^{\varphi=1} \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi=1}$. Réciproquement, si $x \in (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi=1}$, alors il existe $n \geqslant 0$ tel que $x = t^n x_n$ avec $x_n \in (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{\varphi=p^n}$ et le lemme suit alors de la proposition 3.2 de [Berger 2002] qui nous dit que $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{\varphi=p^n} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{\varphi=p^n}$. □

Lemme 1.1.8. *On a $\mathbf{B}_{\mathrm{e}}^{\times} = \mathbb{Q}_p^{\times}$ et si $z \in \mathbf{B}_{\mathrm{e}}$ engendre un $\mathbf{B}_{\mathrm{e}}$-module de rang 1 stable par G_K, alors $z \in \mathbb{Q}_p^{\times}$.*

Démonstration. Soient x et y dans $\mathbf{B}_{\mathrm{e}}$ tels que $xy=1$ et $v(x)$ et $v(y)$ leurs valuations t-adiques dans $\mathbf{B}_{\mathrm{dR}}$. On a $v(x)+v(y)=0$ et comme $\mathbf{B}_{\mathrm{e}} \cap \mathrm{Fil}^1 \mathbf{B}_{\mathrm{dR}} = 0$, cela entraîne que $v(x)=v(y)=0$. Le fait que $\mathbf{B}_{\mathrm{e}}^{\times} = \mathbb{Q}_p^{\times}$ suit alors du fait que $\mathbf{B}_{\mathrm{e}} \cap \mathrm{Fil}^0 \mathbf{B}_{\mathrm{dR}} = \mathbb{Q}_p$.

Si $z \in \mathbf{B}_{\mathrm{e}}$ engendre un $\mathbf{B}_{\mathrm{e}}$-module de rang 1 stable par G_K, alors $g(z)/z \in \mathbb{Q}_p^{\times}$ si $g \in G_K$ et l'application $g \mapsto g(z)/z$ définit un caractère cristallin de G_K. Un résultat classique (cf. le lemme 5.1.3 de [Fontaine 1994b]) dit qu'un tel $z \in \mathbf{B}_{\max}$ est de la forme $t^n z_0$ avec $n \in \mathbb{Z}$ et $z_0 \in \widehat{\mathbb{Q}}_p^{\mathrm{nr}}$ et si en plus $\varphi(z) = z$, alors $z \in \mathbb{Q}_p^{\times}$. □

En première approximation, on peut d'ailleurs penser à $\mathbf{B}_{\mathrm{e}}$ comme à l'anneau des polynômes $P(Y) \in \mathbb{C}_p[Y]$ tels que $P(0) \in \mathbb{Q}_p$.

Proposition 1.1.9. *L'anneau $\mathbf{B}_{\mathrm{e}}$ est de Bézout.*

Démonstration. Il suffit de montrer que si $f, g \in \mathbf{B}_{\mathrm{e}}$, alors l'idéal qu'ils engendrent est principal. Soit $n \geqslant 0$ tel que $f, g \in t^{-n}(\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{\varphi=p^n}$. Comme $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ est de Bézout, il existe $h \in \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ tel que $t^n f \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} + t^n g \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = h \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$. En particulier, il existe $\alpha, \beta \in \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ tels que $t^n f \alpha + t^n g \beta = h$. En appliquant $\varphi^{\pm 1}$ à cette relation, on trouve que h et $\varphi(h)$ engendrent le même idéal de $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$. Par la proposition 3.3.2 de [Kedlaya 2005], on peut (quitte à multiplier h par une unité) supposer que $\varphi(h) = p^m h$ avec $m \in \mathbb{Z}$; on en déduit que l'idéal de $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t]$ engendré par f et g est engendré par $h/t^m \in \mathbf{B}_{\mathrm{e}}$.

Reste à voir que h/t^m est dans l'idéal de $\mathbf{B}_{\mathrm{e}}$ engendré par f et g. On se ramène à montrer que si $f, g \in \mathbf{B}_{\mathrm{e}}$ engendrent $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t]$, alors ils engendrent $\mathbf{B}_{\mathrm{e}}$. Soit $n \geqslant 0$

tel que $t^n f, t^n g \in \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ et I l'idéal (principal) de $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ qu'ils engendrent. Considérons la suite exacte $0 \to M \to t^n f \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \oplus t^n g \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \to I \to 0$. Le $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$-module M est fermé et donc libre de rang fini (égal à 1) et c'est un φ-module. En tensorisant la suite exacte par $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t]$ (qui est plat sur $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$) et en prenant les invariants par φ, on trouve un morceau de suite exacte : $f\mathbf{B}_{\mathrm{e}} \oplus g\mathbf{B}_{\mathrm{e}} \to \mathbf{B}_{\mathrm{e}} \to M[1/t]/(1-\varphi)$ et on conclut par le Corollaire 1.1.6. □

Remarque 1.1.10. L'anneau $\mathbf{B}_{\mathrm{e}}$ est la réunion pour $n \geqslant 0$ des $t^{-n}(\mathbf{B}^+_{\max})^{\varphi=p^n}$ et chacun d'entre eux est un espace de Banach. La topologie de $\mathbf{B}_{\mathrm{e}}$ est celle de la limite inductive, ce qui en fait un espace LF.

Si $h \geqslant 1$, soit $t_h \in \mathbf{B}^+_{\max}$ l'élément construit dans [Colmez 2002, §9] (c'est une période d'un groupe de Lubin–Tate associé à l'uniformisante p de $\mathbb{Q}_{p^h}$) et dont les propriétés sont rappelées au début de [Colmez 2003a, §2.4]. On a notamment $\varphi^h(t_h) = pt_h$ et $\prod_{j=0}^{h-1} \varphi^j(t_h) \in \mathbb{Q}_p^\times \cdot t$.

Lemme 1.1.11. *Si $h \geqslant 1$ et $a \in \mathbb{Z}$, alors $(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t])^{\varphi^h=p^a}$ est un $\mathbf{B}_{\mathrm{e}}$-module libre de rang h.*

Démonstration. L'application $x \mapsto x/t_h^a$ est un isomorphisme de $(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t])^{\varphi^h=p^a}$ sur $(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t])^{\varphi^h=1}$. On se ramène donc au cas $a=0$. Si $\omega \in \mathbb{Q}_{p^h}$ engendre une base normale de $\mathbb{Q}_{p^h}$ sur $\mathbb{Q}_p$, la matrice $(\varphi^{i+j}(\omega))_{1 \leqslant i,j \leqslant h}$ est inversible et l'application de $(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t])^{\varphi^h=1}$ dans $\mathbb{Q}_{p^h} \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{e}}$ donnée par $x \mapsto \sum_{i=0}^{h-1} \varphi^i(\omega) \otimes \varphi^i(x)$ est un isomorphisme, d'où le lemme. □

L'anneau $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ est muni d'une topologie de Fréchet, qui est définie par des valuations $V_{[r;s]}$ avec $s > r$. On appelle $\widetilde{\mathbf{B}}^{[r;s]}$ le complété de $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ pour la valuation $V_{[r;s]}$ et $\mathbf{B}_K^{[r;s]}$ le complété de $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$. Les valuations $V_{[r;s]}$ et les anneaux $\widetilde{\mathbf{B}}^{[r;s]}$ ont été définis dans [Berger 2002, §2.1] (où ils sont notés $\widetilde{\mathbf{B}}_{[r;s]}$) et étudiés dans [Colmez 2003a; Kedlaya 2005] (entre autres) mais il faut faire attention au fait que les notations sont différentes. Les valuations sont indexées par des intervalles et si l'on pose $\rho(r) = (p-1)/pr$, alors notre intervalle $[r;s]$ coïncide avec l'intervalle $[\rho(s);\rho(r)]$ de [Colmez 2003a] et de [Kedlaya 2005].

Voici un tableau récapitulatif de quelques unes des notations :

[Berger 2002]	$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$	$\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}}$	$\widetilde{\mathbf{B}}_{[\rho(s);\rho(r)]}$	$\widetilde{\mathbf{B}}^{\dagger,\rho(r)}$	$\mathbf{B}^{\dagger,\rho(r)}_{\mathrm{rig},K}$	$\mathbf{B}^{\dagger,\rho(r)}_K$
[Colmez 2003a]	$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$	$\widetilde{\mathbf{B}}^{]0;r]}$	—	$\widetilde{\mathbf{B}}^{(0;r]}$	$\mathbf{B}_K^{]0;r]}$	$\mathbf{B}_K^{(0;r]}$
[Kedlaya 2005]	$\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$	$\Gamma^{\mathrm{alg}}_{\mathrm{an,r}}$	$\Gamma^{\mathrm{alg}}_{[r;s]}$	$\Gamma^{\mathrm{alg}}_r[1/p]$	$\Gamma_{\mathrm{an},r}$	$\Gamma_r[1/p]$

L'anneau $\widetilde{\mathbf{B}}^{[r;s]}$ est muni d'une action de G_K et la méthode de Sen (cf. [Colmez 2003b] et [Berger et Colmez 2007]) permet de simplifier grandement l'étude des $\widetilde{\mathbf{B}}^{[r;s]}$-représentations de G_K.

Proposition 1.1.12. *L'anneau* $\widetilde{\Lambda} = \widetilde{\mathbf{B}}^{[r;s]}$ *vérifie les conditions* (TS1), (TS2) *et* (TS3) *de [Berger et Colmez 2007] avec* $\Lambda_{H_K,n} = \varphi^{-n}(\mathbf{B}_K^{[p^n r; p^n s]})$ *et* $\mathrm{val}_\Lambda = V_{[r;s]}$, *les constantes* $c_1 > 0$, $c_2 > 0$ *et* $c_3 > 1/(p-1)$ *pouvant être choisies arbitrairement.*

Démonstration. Ceci est démontré dans [Colmez 2003a] : la condition (TS1) résulte du lemme 10.1, la condition (TS2) résulte de la proposition 8.12 et du fait que $\widetilde{\mathbf{B}}_K^{]0;\rho(r)]}$ est dense dans $\widetilde{\mathbf{B}}_K^{[r;s]}$ et la condition (TS3) résulte de la proposition 9.10 et de la même densité. □

Lemme 1.1.13. *Si* $r \gg 0$ *et si* I *est un intervalle contenu dans* $[r; +\infty[$, *alors* $\mathbf{B}_K^I \cap \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} = \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$.

Démonstration. Le corollaire 2.5.7 de [Kedlaya 2005] nous dit que $x \in \mathbf{B}_K^J$ pour tout intervalle $I \subset J \subset [r; +\infty[$ ce qui fait que $x \in \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$. □

1.2. ***Les*** $(\boldsymbol{\varphi}, \boldsymbol{\Gamma})$***-modules sur l'anneau de Robba.*** Nous donnons quelques rappels et compléments concernant les résultats de Kedlaya (en essayant de renvoyer systématiquement à [Kedlaya 2005] quand c'est possible) sur les pentes des φ-modules sur $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$. Rappelons qu'un φ-module sur un anneau R est un R-module libre D muni d'un Frobenius φ tel que $\varphi^*(\mathrm{D}) = \mathrm{D}$. Si cet anneau est en plus muni d'une action de Γ_K, alors un (φ, Γ)-module est un φ-module muni d'une action semi-linéaire et continue de Γ_K qui commute à φ. Nous renvoyons à l'article de Kedlaya pour la notion de pentes des φ-modules. Contentons-nous de dire que si D est un φ-module sur $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ alors ses *pentes* sont les rationnels qui sortent de la proposition 1.1.5 appliquée à $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}$.

Lemme 1.2.1. *Si* D *est un* φ*-module de rang* 1 *sur* $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$, *alors la pente de* D *appartient à* $\mathbb{Z}$.

Démonstration. Cette pente appartient à l'image par val_p du corps des coefficients de $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ et le lemme suit du fait que ce corps est une extension non ramifiée de K_0. On peut aussi dire que la pente de $M_{a,1}$ est $a \in \mathbb{Z}$. □

Définition 1.2.2. Si D est un φ-module, son *degré* $\deg \mathrm{D}$ est la pente de $\det(\mathrm{D})$.

Lemme 1.2.3. *Si* $s \in \mathbb{Q}$, *alors un* φ*-module sur* $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ *qui est une extension de* φ*-modules isoclines de pente* s *est lui-même isocline de pente* s.

Démonstration. Ecrivons $s = n/d$; si on a une suite exacte $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2 \to 0$ avec D_1 et D_2 isoclines de pente s, alors on peut écrire $\mathrm{D}_i = \mathbf{B}_{\mathrm{rig},K}^{\dagger} \otimes_{\mathbf{B}_K^{\dagger}} \mathrm{D}_i^{\dagger}$ où $\mathrm{D}_i^{\dagger}$ est un φ-module sur $\mathbf{B}_K^{\dagger}$ admettant une base dans laquelle $p^{-n}\mathrm{Mat}(\varphi^d) \in \mathrm{GL}_d(\mathbf{A}_K^{\dagger})$. Le lemme résulte alors de la proposition 7.4.1 de [Kedlaya 2005] qui nous permet de trouver une base de D dans laquelle $p^{-n}\mathrm{Mat}(\varphi^d) \in \mathrm{GL}_d(\mathbf{A}_K^{\dagger})$. □

Rappelons le résultat principal de [Kedlaya 2004] (redémontré et généralisé dans [Kedlaya 2005 ; 2006]).

Théorème 1.2.4. *Si* D *est un φ-module sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, alors il existe une unique filtration* $0 = \mathrm{D}_0 \subset \mathrm{D}_1 \subset \cdots \subset \mathrm{D}_\ell = \mathrm{D}$ *par des sous-φ-modules saturés, telle que :*

(1) *pour tout* $1 \leqslant i \leqslant \ell$, *le quotient* $\mathrm{D}_i/\mathrm{D}_{i-1}$ *est isocline ;*

(2) *si l'on appelle s_i la pente de* $\mathrm{D}_i/\mathrm{D}_{i-1}$, *alors* $s_1 < s_2 < \cdots < s_\ell$.

Remarque 1.2.5. Plaçons-nous dans la situation du Théorème 1.2.4 ci-dessus ; par la proposition 1.1.5, il existe des entiers a_j, h_j tels que $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D} = \oplus M_{a_j,h_j}$. Cette décomposition n'est pas canonique, mais l'ensemble des a_j/h_j est bien défini et coïncide, si l'on compte les multiplicités, avec l'ensemble des s_i (cf. le corollaire 6.4.2 de [Kedlaya 2005]).

Définition 1.2.6. Si D est un φ-module de rang d sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, alors (cf. [Kedlaya 2004, p. 157]) son *polygone de Newton* NP(D) est la réunion des segments d'extrémités (i, y_i) et $(i+1, y_{i+1})$ pour $0 \leqslant i \leqslant d-1$ où $y_0 = 0$ et $y_{i+1} - y_i$ est la $(i+1)$-ième plus petite pente de D en comptant les multiplicités. Dans les notations du Théorème 1.2.4, NP(D) est la réunion des ℓ segments de longueur $\mathrm{rg}(\mathrm{D}_i/\mathrm{D}_{i-1})$ et de pente s_i pour $1 \leqslant i \leqslant \ell$.

Remarquons que par le Lemme 1.2.1, les sommets de NP(D) sont à coordonnées entières. Le dernier sommet est de coordonnées $(\mathrm{rg}(\mathrm{D}), \deg \mathrm{D})$.

Lemme 1.2.7. *Si* D *est un φ-module sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ et si* M *est un sous-φ-module de* D (*pas nécessairement saturé ni de même rang*), *alors* NP(M) *est au-dessus de* NP(D), *et si* NP(M) *et* NP(D) *ont même extrémité*, *alors* M = D.

Démonstration. Si l'on appelle $\sigma_1, \ldots, \sigma_d$ et $\tau_1, \ldots, \tau_m$ les pentes de D et M prises avec multiplicité, l'affirmation « NP(M) est au-dessus de NP(D) » est équivalente au fait que pour tout $1 \leqslant k \leqslant m$, on a $\sigma_1 + \sigma_2 + \cdots + \sigma_k \leqslant \tau_1 + \tau_2 + \cdots + \tau_k$, ce qui revient à dire que pour tout $1 \leqslant k \leqslant m$, la plus petite pente de $\bigwedge^k \mathrm{D}$ est inférieure ou égale à la plus petite pente de $\bigwedge^k \mathrm{M}$. Ceci suit, après extension des scalaires à $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, du (a) de la proposition 4.5.14 de [Kedlaya 2005].

Enfin si NP(M) et NP(D) ont même extrémité, alors M et D ont même rang (le rang étant la x-longueur du polygone de Newton) et $\det(\mathrm{M}) = \alpha \cdot \det(\mathrm{D})$ où $\alpha \cdot \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ est étale, NP(M) et NP(D) ayant la même extrémité. Par la proposition 3.3.2 de [Kedlaya 2005], α est une unité de $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$; c'est donc une unité de $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, ce qui fait que M = D. □

Corollaire 1.2.8. *Si* $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2$ *est une suite exacte de φ-modules, avec* $\mathrm{rg}(\mathrm{D}) = \mathrm{rg}(\mathrm{D}_1) + \mathrm{rg}(\mathrm{D}_2)$, *alors* $\deg \mathrm{D} \geqslant \deg \mathrm{D}_1 + \deg \mathrm{D}_2$ *et on a égalité si et seulement si* $\mathrm{D} \to \mathrm{D}_2$ *est surjective.*

Démonstration. Par le Lemme 1.2.7 ci-dessus, on a $\deg(\mathrm{im}(\mathrm{D} \to \mathrm{D}_2)) \geqslant \deg \mathrm{D}_2$ avec égalité si et seulement si $\mathrm{D} \to \mathrm{D}_2$ est surjective. On se ramène donc à montrer que si on a une suite exacte $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2 \to 0$, alors $\deg \mathrm{D} = \deg \mathrm{D}_2 + \deg \mathrm{D}_1$. Ceci suit du fait que $\det \mathrm{Mat}(\varphi \mid \mathrm{D}) = \det \mathrm{Mat}(\varphi \mid \mathrm{D}_1) \cdot \det \mathrm{Mat}(\varphi \mid \mathrm{D}_2)$. □

Disons encore quelques mots de la « localisation en $\varepsilon^{(n)}-1$ » ; si $n \geqslant 0$, alors on pose $r_n = p^{n-1}(p-1)$. Si $x = \sum_{k \gg -\infty} p^k[x_k] \in \widetilde{\mathbf{B}}^{\dagger,r_0}$ alors la série qui définit x converge dans $\mathbf{B}_{\mathrm{dR}}^+$ et on en déduit un morphisme injectif noté ι_0 de $\widetilde{\mathbf{B}}^{\dagger,r_0}$ dans $\mathbf{B}_{\mathrm{dR}}^+$. Ce morphisme s'étend par continuité en un morphisme toujours injectif (voir la proposition 2.25 de [Berger 2002]) $\iota_0 : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_0} \to \mathbf{B}_{\mathrm{dR}}^+$. Si $r > 0$, alors soit $n(r)$ le plus petit entier tel que $r_n \geqslant r$. Si $n \geqslant n(r)$, et si $x \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, alors $\varphi^{-n}(x) \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r/p^n} \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_0}$ et on en déduit un morphisme injectif $\iota_n = \iota_0 \circ \varphi^{-n} : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \to \mathbf{B}_{\mathrm{dR}}^+$. On pose $Q_1 = \varphi(X)/X$ et $Q_n = \varphi^{n-1}(Q_1)$ ce qui fait que si $n \geqslant 1$, alors Q_n est le polynôme minimal de $\varepsilon^{(n)} - 1$. Par la proposition 4.8 de [Berger 2002], l'application $\theta \circ \iota_n : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \to \mathbb{C}_p$ est surjective et son noyau est l'idéal engendré par Q_n. Par le lemme 5.11 de [Berger 2002], la restriction de ι_n à $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ a pour image un sous-anneau dense (pour la topologie t-adique) de $K_n[[t]]$. Si D^r est un $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$-module, cette application $\iota_n : \mathbf{B}_{\mathrm{rig},K}^{\dagger,r} \to K_n[[t]]$ nous permet de définir la localisation $K_n[[t]] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}}^{\iota_n} \mathrm{D}^r$ de D^r en $\varepsilon^{(n)} - 1$ pour $n \geqslant n(r)$. Cette construction est fondamentale autant dans cet article que dans [Berger 2004a].

2. Les B-paires

Une *B-paire* est un couple $W = (W_e, W_{\mathrm{dR}}^+)$ où W_e est un $\mathbf{B}_{\mathrm{e}}$-module libre de rang fini muni d'une action semi-linéaire et continue de G_K (c'est-à-dire une $\mathbf{B}_{\mathrm{e}}$-*représentation*) et W_{dR}^+ est un $\mathbf{B}_{\mathrm{dR}}^+$-réseau de $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ stable par G_K (rappelons que toute $\mathbf{B}_{\mathrm{dR}}$-représentation admet un $\mathbf{B}_{\mathrm{dR}}^+$-réseau stable par Galois, cf. par exemple le début du §3.5 de [Fontaine 2004]).

L'objet de ce chapitre est d'étudier la catégorie des B-paires, et le résultat principal est que cette catégorie est équivalente à la catégorie des (φ,Γ)-modules sur $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$.

2.1. ***La catégorie des B-paires.*** Ce paragraphe est principalement consacré à donner des définitions relatives à la catégorie des B-paires.

Définition 2.1.1. Si W est une B-paire, alors on appelle *dimension de W* le rang commun de W_e et de W_{dR}^+.

Si W et X sont deux B-paires, un *morphisme de B-paires* $f : W \to X$ est la donnée de deux applications (f_e, f_{dR}^+) de W_e dans X_e et de W_{dR}^+ dans X_{dR}^+ telles que les deux applications induites par extension des scalaires à $\mathbf{B}_{\mathrm{dR}}$ coïncident ; on appelle alors f_{dR} cette application.

Exemple 2.1.2. Voici deux classes importantes de B-paires.

(1) Si V est une représentation p-adique de G_K, alors $(\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$ est une B-paire notée $W(V)$;

(2) si D est un (φ, N)-module filtré sur K, alors

$$\big((\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1,N=0}, \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K)\big)$$

est une B-paire notée $W(D)$.

Lemme 2.1.3. *Si W est une B-paire de dimension* 1, *alors il existe un caractère $\eta : G_K \to \mathbb{Q}_p^\times$ et $i \in \mathbb{Z}$ tels que $W_e = \mathbf{B}_e(\eta)$ et $W_{\mathrm{dR}}^+ = t^i \mathbf{B}_{\mathrm{dR}}^+(\eta)$.*

Démonstration. La première assertion résulte du Lemme 1.1.8 et la deuxième du fait que les idéaux fractionnaires de $\mathbf{B}_{\mathrm{dR}}$ sont tous de la forme $t^i \mathbf{B}_{\mathrm{dR}}^+$. □

Lemme 2.1.4. *Si $f : W \to X$ est un morphisme de B-paires, alors $X_e/f_e(W_e)$ est sans torsion.*

Démonstration. Le $\mathbf{B}_e$-module $\mathrm{Sat} f_e(W_e)$ est libre de même rang que $f_e(W_e)$ et le Lemme 1.1.8 montre que $\det f_e(W_e) = \det \mathrm{Sat} f_e(W_e)$ ce qui fait que l'image de $f_e(W_e)$ est saturée dans X_e. □

Remarque 2.1.5. En revanche, $X_{\mathrm{dR}}^+/f_{\mathrm{dR}}^+(W_{\mathrm{dR}}^+)$ peut avoir de la torsion (considérer par exemple l'application naturelle $(\mathbf{B}_e, t\mathbf{B}_{\mathrm{dR}}^+) \to (\mathbf{B}_e, \mathbf{B}_{\mathrm{dR}}^+)$). On dit que f est *strict* si $X_{\mathrm{dR}}^+/f_{\mathrm{dR}}^+(W_{\mathrm{dR}}^+)$ est sans torsion. L'existence de morphismes non-stricts implique que la $\otimes$-catégorie additive des B-paires n'est pas abélienne.

Définition 2.1.6. Si W et X sont deux B-paires, on dit que W est un *sous-objet* de X s'il existe un morphisme injectif strict $f : W \to X$. Une *suite exacte* est une suite où les morphismes sont stricts et les conditions habituelles (image–noyau) sont satisfaites.

Lemme 2.1.7. *Si $f : W \to X$ est un morphisme de B-paires, alors $\ker f = (\ker f_e, \ker f_{\mathrm{dR}}^+)$ est un sous-objet de W et $\mathrm{im} f = (\mathrm{im} f_e, \mathrm{im} f_{\mathrm{dR}}^+) \subset X$ est une B-paire et la suite $0 \to \ker f \to W \to \mathrm{im} f \to 0$ est exacte.*

Définition 2.1.8. Si W et X sont deux B-paires, on dit que W est une *modification* de X si $W_e \simeq X_e$.

2.2. *Construction de (φ, Γ)-modules.* Dans ce paragraphe, on associe à toute B-paire un (φ, Γ)-module et on montre le théorème A de l'introduction. Les constructions qui permettent de relier la catégorie des B-paires et celle des (φ, Γ)-modules sont proches de celles qu'on trouve dans l'article [Colmez 2003a], notamment les §§2, 3 et 10.

Rappelons que si $n \geqslant 0$, alors on pose $r_n = p^{n-1}(p-1)$ et que si $r > 0$, alors $n(r)$ est le plus petit entier tel que $r_n \geqslant r$. Si W est une B-paire est si $n \geqslant n(r)$, alors l'application $\iota_n : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t] \to \mathbf{B}_{\mathrm{dR}}$ nous donne un morphisme $\iota_n : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t] \otimes_{\mathbf{B}_e} W_e \to \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e = W_{\mathrm{dR}}$.

Lemme 2.2.1. *Si W est une B-paire, et si $\widetilde{\mathrm{D}}^r(W) = \{y \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t] \otimes_{\mathbf{B}_e} W_e$ tels que $\iota_n(y) \in W_{\mathrm{dR}}^+$ pour tout $n \geqslant n(r)\}$, alors :*

(1) *$\widetilde{\mathrm{D}}^r(W)$ est un $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$-module libre de rang d ;*

(2) *$\widetilde{\mathrm{D}}^r(W)[1/t] = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t] \otimes_{\mathbf{B}_e} W_e$;*

(3) $\widetilde{\mathrm{D}}^r(W)$ *est stable par* G_K *et* $\varphi(\widetilde{\mathrm{D}}^r(W)) = \widetilde{\mathrm{D}}^{pr}(W)$.

Démonstration. Commençons par remarquer que si $n \geqslant n(r)$, alors l'image de $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ par l'application ι_n est dense dans W_{dR}. Soit

$$\widetilde{\mathrm{D}}^r_{n(r)}(W) = \{y \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e \text{ tels que } \iota_{n(r)}(y) \in W^+_{\mathrm{dR}}\};$$

c'est un $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$-module libre de rang d (il est engendré par d éléments dont les images forment une base de W^+_{dR}). Par ailleurs $\widetilde{\mathrm{D}}^r(W)$ est fermé dans $\widetilde{\mathrm{D}}^r_{n(r)}(W)$ et par la proposition 1.1.1, $\widetilde{\mathrm{D}}^r(W)$ est libre de rang $\leqslant d$.

Montrons que si $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$, il existe $m \geqslant 0$ tel que $t^m x \in \widetilde{\mathrm{D}}^r(W)$, ce qui implique que $\widetilde{\mathrm{D}}^r(W)$ est un $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$-module libre de rang d et que $\widetilde{\mathrm{D}}^r(W)[1/t] = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$. Si $e_1, \ldots, e_d$ est une base de W_e, alors il existe m_1 tel que pour tout $n \geqslant n(r)$, l'image par ι_n des $t^{m_1} e_i$ appartient à W^+_{dR} (si c'est vrai pour un n, c'est vrai pour tous car $\varphi(e_i) = e_i$). Comme $\mathbf{B}_{\mathrm{e}} = \bigcup_{j \geqslant 0} t^{-j} (\widetilde{\mathbf{B}}^+_{\mathrm{rig}})^{\varphi = p^j}$, il existe m_2 tel que $t^{m_2} x$ appartient au $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$-module engendré par les e_i. On peut alors prendre $m = m_1 + m_2$.

Ceci démontre les points (1) et (2), et le (3) est une évidence. □

Lemme 2.2.2. *Pour tout* $n \geqslant n(r)$, *l'image de* $\widetilde{\mathrm{D}}^r(W)$ *par* ι_n *contient une base de* W^+_{dR} *et si* $\widetilde{\mathrm{D}}'$ *est un sous-*$\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$*-module fermé stable par* G_K *de* $\widetilde{\mathrm{D}}^r(W)$, *dont l'image par* ι_n *contient une base de* W^+_{dR} *pour tout* $n \geqslant n(r)$, *alors* $\widetilde{\mathrm{D}}' = \widetilde{\mathrm{D}}^r(W)$.

Démonstration. Soient $n \geqslant n(r)$ et $x_1, \ldots, x_d$ des éléments de $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ dont les images par ι_n forment une base de W^+_{dR}. Si $\ell \geqslant 0$ est tel que $t^\ell x_i \in \widetilde{\mathrm{D}}^r(W)$ pour tout i, alors posons $y_i = (t/Q_n)^\ell x_i$. Pour tout $m \geqslant n(r)$ on a $\iota_m(y_i) \in W^+_{\mathrm{dR}}$ et par ailleurs $\iota_n(y_i) = \text{inversible} \cdot \iota_n(x_i)$ ce qui fait que l'image de $\widetilde{\mathrm{D}}^r(W)$ par ι_n contient bien une base de W^+_{dR}.

Pour montrer l'unicité, on se ramène au cas de rang 1 en prenant le déterminant. Il faut donc montrer que si $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ engendre un idéal stable par G_K et si $\iota_n(x)$ est une unité de $\mathbf{B}^+_{\mathrm{dR}}$ pour tout $n \geqslant n(r)$, alors x est une unité. Soit $\eta : G_K \to (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})^\times = (\widetilde{\mathbf{B}}^{\dagger,r})^\times$ le cocycle $g \mapsto g(x)/x$. Par la proposition 4.2.1 de [Berger et Colmez 2007], l'anneau $\widetilde{\mathbf{B}}^{\dagger,r}$ satisfait les conditions de Tate–Sen et il existe donc une extension finie L de K, un entier $m \geqslant 0$ et une unité $y \in (\widetilde{\mathbf{B}}^{\dagger,r})^\times$ tels que $h(xy) = xy$ si $h \in H_L$ et $g(xy)/xy \in \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_L)$ si $g \in G_K$. Le lemme 3.2.5 de [Berger et Colmez 2007] montre alors que, quitte à augmenter m, on a $xy \in \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_L)$. Si $L_0 \subset L$ est une sous-extension non ramifiée de L, alors $\mathrm{N}_{L/L_0}(\varphi^m(xy))$ est un élément de $\mathbf{B}^{\dagger,p^m r}_{L_0}$ qui engendre un idéal stable par un sous-groupe ouvert de Γ_L. Un raisonnement analogue à celui du lemme I.3.2 de [Berger 2004b] montre que cet idéal est engendré par un élément de la forme $\prod_{n \geqslant n(r)} (Q_{n+m}/p)^{\alpha_n}$ et la condition selon laquelle $\iota_n(x)$ est une unité de $\mathbf{B}^+_{\mathrm{dR}}$ pour tout $n \geqslant n(r)$ nous dit que les α_n sont tous nuls, ce qui fait que $\varphi^m(xy)$ est une unité, et donc que x est une unité. □

En particulier, si $s \geqslant r$, alors l'application naturelle $\widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}} \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}} \widetilde{\mathrm{D}}^r(W) \to \widetilde{\mathrm{D}}^s(W)$ est un isomorphisme.

Définition 2.2.3. On définit $\widetilde{\mathrm{D}}(W) = \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}} \widetilde{\mathrm{D}}^r(W)$ et si I est un intervalle contenu dans $[r; +\infty[$, alors on pose $\widetilde{\mathrm{D}}^I(W) = \widetilde{\mathbf{B}}^I \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}} \widetilde{\mathrm{D}}^r(W)$.

Le Lemme 2.2.2 ci-dessus montre que cela ne dépend pas du choix de $r \in I$. Remarquons en particulier que si $J \subset I$, alors $\widetilde{\mathrm{D}}^J(W) = \widetilde{\mathbf{B}}^J \otimes_{\widetilde{\mathbf{B}}^I} \widetilde{\mathrm{D}}^I(W)$.

Proposition 2.2.4. *Si W est une B-paire, et si I est un intervalle, alors il existe $j \geqslant 0$ et une extension finie L de K tels que pour tout $k \geqslant 0$, il existe un $\mathbf{B}_L^{p^{j+k}I}$-module $\mathrm{D}_L^{p^{j+k}I}$ libre de rang fini vérifiant :*

(1) $\widetilde{\mathbf{B}}^{p^{j+k}I} \otimes_{\mathbf{B}_L^{p^{j+k}I}} \mathrm{D}_L^{p^{j+k}I} = \widetilde{\mathrm{D}}^{p^{j+k}I}(W)$;

(2) $\varphi^*(\mathrm{D}_L^{p^{j+k}I}) = \mathrm{D}_L^{p^{j+k+1}I}$;

(3) *les images de $\mathrm{D}_L^{p^{j+k}I}$ et $\mathrm{D}_L^{p^{j+k+1}I}$ dans $\widetilde{\mathrm{D}}^{p^{j+k}I \cap p^{j+k+1}I}(W)$ engendrent le même $\mathbf{B}_L^{p^{j+k}I \cap p^{j+k+1}I}$-module.*

Démonstration. Si l'on se donne une base de $\widetilde{\mathrm{D}}^I(W)$, alors l'action de G_K est donnée par une application $G_K \to \mathrm{GL}_d(\widetilde{\mathbf{B}}^I)$ et il existe une extension finie L de K telle que l'image de G_L par cette application soit incluse dans l'ensemble des matrices M vérifiant $V_I(1-M) > c_1 + 2c_2 + 2c_3$. Par la proposition 1.1.12, $\widetilde{\mathbf{B}}^I$ satisfait les conditions de Tate–Sen et la proposition 3.2.6 de [Berger et Colmez 2007] nous fournit alors une nouvelle base de $\widetilde{\mathrm{D}}^I(W)$ et $n \geqslant 0$ tels que l'application $G_K \to \mathrm{GL}_d(\widetilde{\mathbf{B}}^I)$ est triviale sur H_L et à valeurs dans $\mathrm{GL}_d(\varphi^{-n}(\mathbf{B}_L^{p^nI}))$. Si $\mathrm{D}_L^{p^nI}$ est le $\mathbf{B}_L^{p^nI}$-module engendré par φ^n de cette base, alors $\widetilde{\mathrm{D}}^{p^nI}(W) = \widetilde{\mathbf{B}}^{p^nI} \otimes_{\mathbf{B}_L^{p^nI}} \mathrm{D}_L^{p^nI}$.

Posons $J = p^nI \cap p^{n+1}I$ et $\mathrm{D}_L^{p^{j+k+1}I} = \varphi^*(\mathrm{D}_L^{p^{j+k}I})$ pour $k \geqslant 0$. Si J est vide, alors il suffit de prendre $j = n$, les conditions (1) et (2) étant vérifiées et la condition (3) vide. Supposons donc que J est non vide ; l'unicité dans la méthode de Sen (cf. la démonstration du (3) de la proposition 3.3.1 de [Berger et Colmez 2007]) ne nous donne pas la condition (3), mais nous dit qu'il existe $m \geqslant 0$ tel que :

$$\varphi^{-m}(\mathbf{B}_L^{p^mJ}) \otimes_{\mathbf{B}_L^{p^nI}} \mathrm{D}_L^{p^nI} = \varphi^{-m}(\mathbf{B}_L^{p^mJ}) \otimes_{\mathbf{B}_L^{p^{n+1}I}} \mathrm{D}_L^{p^{n+1}I}.$$

En appliquant φ^m à cette relation, on voit que (3) est satisfaite en prenant $j = m+n$. $\square$

Proposition 2.2.5. *Si $W = (W_e, W_{\mathrm{dR}}^+)$ est une B-paire de dimension d, alors il existe un unique (φ, Γ_K)-module $\mathrm{D}(W)$ sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ tel que $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}(W) = \widetilde{\mathrm{D}}(W)$.*

Démonstration. Choisissons un intervalle I tel que $I \cap pI$ est non vide. Le (3) de la proposition 2.2.4 nous fournit une collection compatible de $\mathbf{B}_L^{p^{j+k}I}$-modules, et par la définition 2.8.1 et le théorème 2.8.4 de [Kedlaya 2005], il existe un $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$-module $\mathrm{D}_L(W)$ libre de rang d et tel que $\mathrm{D}_L^{p^{j+k}I} = \mathbf{B}_L^{p^{j+k}I} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}} \mathrm{D}_L(W)$ pour $k \geqslant 0$. La condition (1) implique que $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}} \mathrm{D}_L(W) = \widetilde{\mathrm{D}}(W)$ et la condition (2) implique que $\mathrm{D}_L(W)$ est un φ-module. Par ailleurs $\mathrm{D}_L(W)$ est stable sous l'action de G_K et H_L agit trivialement dessus, puisque c'est vrai pour chaque $\mathrm{D}_L^{p^{j+k}I}$. Finalement, le lemme 4.2.5 de [Berger et Colmez 2007] montre que l'extension $\mathbf{B}^{\dagger}_{\mathrm{rig},L}/\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ vérifie les conditions de la proposition 2.2.1 de descente étale du même article, ce qui fait que l'on a $\mathrm{D}_L(W) = \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}_L(W)^{H_K}$ et on peut donc prendre $\mathrm{D}(W) = \mathrm{D}_L(W)^{H_K}$. Ceci montre l'existence de $\mathrm{D}(W)$.

Passons à l'unicité. Si D_1 et D_2 satisfont les conclusions de la proposition, choisissons des bases de D_1 et D_2, et appelons G_i la matrice de $\gamma \in \Gamma_K$ sur D_i et M la matrice de passage d'une base à l'autre, ce qui fait que $G_1 M = \gamma(M) G_2$. On se donne $r \gg 0$ tel que toutes ces matrices ont leurs coefficients dans $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ et I un intervalle contenu dans $[r;+\infty[$. La partie « unicité » de la méthode de Sen nous dit qu'il existe $n \geqslant 0$ tel que $M \in \mathrm{GL}_d(\varphi^{-n}(\mathbf{B}_K^{p^nI}))$. Comme $\mathbf{B}_K^{p^nI} \cap \widetilde{\mathbf{B}}^{\dagger,p^nr}_{\mathrm{rig}} = \mathbf{B}^{\dagger,p^nr}_{\mathrm{rig},K}$ par le Lemme 1.1.13, on trouve que $M \in \mathrm{GL}_d(\varphi^{-n}(\mathbf{B}^{\dagger,p^nr}_{\mathrm{rig},K}))$. Si P_1 et P_2 sont les matrices de φ sur D_1 et D_2, alors $P_1 M = \varphi(M) P_2$ et donc si $M \in \mathrm{GL}_d(\varphi^{-n}(\mathbf{B}^{\dagger}_{\mathrm{rig},K}))$, alors $M \in \mathrm{GL}_d(\mathbf{B}^{\dagger}_{\mathrm{rig},K})$, ce qui fait que $D_1 = D_2$. □

Rappelons que si D est un φ-module sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, alors pour $r \gg 0$ on note D^r le $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-module fourni par le théorème I.3.3 de [Berger 2004a]. En particulier

$$\mathrm{D}^{pr} = \mathbf{B}^{\dagger,pr}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}^r$$

et D^{pr} a une base contenue dans $\varphi(\mathrm{D}^r)$. Par exemple, $\mathrm{D}^r(W) = \widetilde{\mathrm{D}}^r(W) \cap \mathrm{D}(W)$ si $r \gg 0$.

Proposition 2.2.6. *Si* D *est un* (φ,Γ_K)*-module de rang* d *sur* $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$*, alors :*

(1) $W_e(\mathrm{D}) = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D})^{\varphi=1}$ *est un* $\mathbf{B}_{\mathrm{e}}$*-module libre de rang* d *stable sous l'action de* G_K *;*

(2) $W_{\mathrm{dR}}^{+}(\mathrm{D}) = \mathbf{B}_{\mathrm{dR}}^{+} \otimes^{\iota_n}_{\mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}} \mathrm{D}^{r_n}$ *ne dépend pas de* $n \gg 0$ *et c'est un* $\mathbf{B}_{\mathrm{dR}}^{+}$*-module libre de rang* d *stable sous l'action de* G_K *;*

(3) *le couple* $W(\mathrm{D}) = (W_e(\mathrm{D}), W_{\mathrm{dR}}^{+}(\mathrm{D}))$ *est une* B*-paire.*

Démonstration. Il est clair que $W_e(\mathrm{D})$ est un $\mathbf{B}_{\mathrm{e}}$-module stable sous l'action de G_K. Reste à montrer qu'il est libre de rang d. Pour cela, soit $\oplus M_{a_i,h_i}$ une décomposition de $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}$ en φ-modules élémentaires fournie par la proposition 1.1.5. On a $W_e(\mathrm{D}) = \bigoplus (M_{a_i,h_i}[1/t])^{\varphi=1}$. On vérifie que l'application $(M_{a,h}[1/t])^{\varphi=1} \to \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t]$ qui à $\sum_{i=0}^{h-1} x_i e_i$ associe x_0 est un isomorphisme entre $(M_{a,h}[1/t])^{\varphi=1}$ et $(\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t])^{\varphi^h=p^{-a}}$, et le (1) suit alors du Lemme 1.1.11.

Pour montrer le (2), remarquons que par le théorème I.3.3 de [Berger 2004a], on a $\varphi^*(\mathrm{D}^{r_n}) = \mathrm{D}^{r_{n+1}}$ pour $n \gg 0$ et donc le $\mathbf{B}_{\mathrm{dR}}^{+}$-module engendré par $\varphi^{-n}(\mathrm{D}^{r_n})$

ne dépend pas de $n \gg 0$. Comme D^{r_n} est libre de rang d et stable par G_K, il en est de même pour $W_{\mathrm{dR}}^+(\mathrm{D})$.

Montrons maintenant le (3) : $W_{\mathrm{dR}}(\mathrm{D}) = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e(\mathrm{D})$ est un $\mathbf{B}_{\mathrm{dR}}$-espace vectoriel de dimension d, réunion croissante des $\mathbf{B}_{\mathrm{dR}}$-espaces vectoriels

$$\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} \mathrm{D}^{r_n})^{\varphi=1}$$

ce qui fait que si $n \gg 0$, alors :

$$W_{\mathrm{dR}}(\mathrm{D}) = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} \mathrm{D}^{r_n})^{\varphi=1} \subset \mathbf{B}_{\mathrm{dR}} \otimes^{\iota_n}_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} \mathrm{D}^{r_n} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W_{\mathrm{dR}}^+(\mathrm{D}).$$

En comparant les dimensions, on voit que l'on a en fait égalité et donc que $W_{\mathrm{dR}}^+(\mathrm{D})$ est un réseau de $W_{\mathrm{dR}}(\mathrm{D})$. □

Théorème 2.2.7. *Les foncteurs* $W \mapsto \mathrm{D}(W)$ *et* $\mathrm{D} \mapsto W(\mathrm{D})$ *sont inverses l'un de l'autre et donnent une équivalence de catégories entre la catégorie des B-paires et la catégorie des (φ, Γ_K)-modules sur $\mathbf{B}_{\mathrm{rig},K}^\dagger$.*

Démonstration. On vérifie que ces deux foncteurs sont inverses l'un de l'autre en utilisant le fait que :

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}(W) = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W$$

dans la proposition 2.2.5 et que :

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{e}}} W_e(\mathrm{D}) = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}$$

dans la proposition 2.2.6, puis en identifiant les différents objets (c'est un exercice instructif que nous laissons au lecteur ; il faut utiliser l'unicité dans la proposition 2.2.5). □

Remarque 2.2.8. Les modifications de (φ, Γ_K)-modules correspondent à des modifications de B-paires. Par exemple :

(1) Si W et X sont deux B-paires, alors $W_e \simeq X_e$ si et seulement si $\mathrm{D}(W)[1/t] \simeq \mathrm{D}(X)[1/t]$;

(2) si $W = (W_e, W_{\mathrm{dR}}^+)$ est une B-paire, alors $\mathrm{D}((W_e, t^n W_{\mathrm{dR}}^+)) = t^n \mathrm{D}(W)$.

Proposition 2.2.9. *Le foncteur* $\mathrm{D} \mapsto W(\mathrm{D})$ *réalise une équivalence de catégories entre la catégorie des (φ, Γ_K)-modules étales et la catégorie des B-paires de la forme $W(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$ où V est une représentation p-adique de G_K.*

On retrouve alors (en appliquant le (b) du théorème 6.3.3 de [Kedlaya 2005], qui nous dit que le foncteur naturel de la catégorie des (φ, Γ_K)-modules étales sur $\mathbf{B}_K^\dagger$ vers la catégorie des (φ, Γ_K)-modules étales sur $\mathbf{B}_{\mathrm{rig},K}^\dagger$ est une équivalence de catégories) le résultat principal de [Cherbonnier et Colmez 1998], c'est-à-dire l'équivalence de catégories entre représentations p-adiques et (φ, Γ_K)-modules étales, avec une démonstration proche de celle de [Berger et Colmez 2007].

Dans le paragraphe 3.2, nous reviendrons sur le problème de la description « explicite » des B-paires dont le (φ, Γ)-module associé est isocline.

2.3. *Théorie de Hodge p-adique.* Dans ce paragraphe, nous généralisons les notions habituelles de théorie de Hodge p-adique aux B-paires.

Définition 2.3.1. Si $\heartsuit \in \{\text{cris}, \text{st}, \text{dR}\}$, et si W est une B-paire, alors on dit que W est *cristalline* (ou *semi-stable* ou *de de Rham*) si la $\mathbf{B}_\heartsuit$-représentation $\mathbf{B}_\heartsuit \otimes_{\mathbf{B}_\mathrm{e}} W_e$ est triviale. On pose $\mathrm{D}_\heartsuit(W) = (\mathbf{B}_\heartsuit \otimes_{\mathbf{B}_\mathrm{e}} W_e)^{G_K}$.

Remarquons que si V est une représentation p-adique, alors bien sûr V est cristalline (ou semi-stable, ou de de Rham, ou de Hodge–Tate) si et seulement si $W(V)$ l'est.

Lemme 2.3.2. *Si D est un (φ, N)-module sur K_0, alors $(\mathbf{B}_\mathrm{st} \otimes_{K_0} D)^{\varphi=1, N=0}$ est un $\mathbf{B}_\mathrm{e}$-module libre de rang $d = \dim D$.*

Démonstration. Si $u \in \mathbf{B}_\mathrm{st}$ vérifie $N(u) = 1$, alors on a $\mathbf{B}_\mathrm{st} = \mathbf{B}_\mathrm{max}[u]$ et si $x \in (\mathbf{B}_\mathrm{st} \otimes_{K_0} D)^{N=0}$, alors on peut écrire $x = x_0 + u x_1 + \cdots + u^e x_e$ avec $x_i \in \mathbf{B}_\mathrm{max} \otimes_{K_0} D$. Par la proposition 11.7 de [Colmez 2002], l'application $x \mapsto x_0$ de $(\mathbf{B}_\mathrm{st} \otimes_{K_0} D)^{N=0}$ dans $\mathbf{B}_\mathrm{max} \otimes_{K_0} D$ est un isomorphisme, et on se ramène donc à montrer que si D est un φ-module sur K_0, alors $(\mathbf{B}_\mathrm{max} \otimes_{K_0} D)^{\varphi=1}$ est un $\mathbf{B}_\mathrm{e}$-module libre de rang d. Par le théorème de Dieudonné–Manin, $\widehat{\mathbb{Q}}_p^\mathrm{nr} \otimes_{K_0} D$ se décompose en somme directe de φ-modules élémentaire M_{a_i, h_i}. L'application $x = \sum_{j=0}^{h-1} x_j e_j \mapsto x_0$ de $(\mathbf{B}_\mathrm{max} \otimes_{\widehat{\mathbb{Q}}_p^\mathrm{nr}} M_{a,h})^{\varphi=1}$ dans $\mathbf{B}_\mathrm{max}^{\varphi^h = p^{-a}}$ est un isomorphisme, et le lemme résulte alors du Lemme 1.1.11. □

Proposition 2.3.3. *Si W est une B-paire, alors* :

(1) $\mathrm{D}_\mathrm{st}(W)$ *est un (φ, N)-module sur K_0 et* $\mathrm{D}_\mathrm{cris}(W) = \mathrm{D}_\mathrm{st}(W)^{N=0}$;

(2) $\mathrm{D}_\mathrm{dR}(W)$ *est un K-espace vectoriel filtré avec* $\mathrm{Fil}^i(\mathrm{D}_\mathrm{dR}(W)) = \mathrm{D}_\mathrm{dR}(W) \cap t^i W_\mathrm{dR}^+$ *et l'application naturelle* $K \otimes_{K_0} \mathrm{D}_\mathrm{st}(W) \to \mathrm{D}_\mathrm{dR}(W)$ *est injective* ;

(3) *si W est semi-stable, alors*

$$W_e = (\mathbf{B}_\mathrm{st} \otimes_{K_0} \mathrm{D}_\mathrm{st}(W))^{\varphi=1, N=0} \quad \text{et } W_\mathrm{dR}^+ = \mathrm{Fil}^0(\mathbf{B}_\mathrm{dR} \otimes_K \mathrm{D}_\mathrm{dR}(W));$$

(4) *si D est un (φ, N)-modules filtré, et si* $W_e(D) = (\mathbf{B}_\mathrm{st} \otimes_{K_0} D)^{\varphi=1, N=0}$ *et* $W_\mathrm{dR}^+(D) = \mathrm{Fil}^0(\mathbf{B}_\mathrm{dR} \otimes_K D)$, *alors* $W(D) = (W_e(D), W_\mathrm{dR}^+(D))$ *est une B-paire semi-stable.*

On laisse la démonstration au lecteur à titre d'exercice (ce n'est pas différent du cas des représentations p-adiques semi-stables).

A la lumière du Théorème 2.2.7, l'étude des B-paires de de Rham revient à l'étude des (φ, Γ_K)-modules « localement triviaux » au sens de [Berger 2004a], ce qui est fait en détail dans le même article.

Si D est un (φ, N)-module filtré, on note $\mathcal{M}(D)$ le (φ, Γ_K)-module construit dans [Berger 2004a, §II.2]. Rappelons que :

$$\mathcal{M}(D)=\{y\in(\mathbf{B}^{\dagger}_{\mathrm{rig},K}[\log X,1/t]\otimes_{K_0}D)^{N=0}\mid \iota_n(y)\in\mathrm{Fil}^0(K_n((t))\otimes_K D_K)\ \forall n\gg 0\}.$$

Proposition 2.3.4. *Les foncteurs $W\mapsto \mathrm{D}_{\mathrm{st}}(W)$ et $D\mapsto W(D)$ sont inverses l'un de l'autre et donnent une équivalence de catégories entre la catégorie des B-paires semi-stables et la catégorie des (φ, N)-modules filtrés.*

Si W est une B-paire, alors $\mathrm{D}(W)=\mathcal{M}(\mathrm{D}_{\mathrm{st}}(W))$ et donc si D est un (φ, N)-module filtré, alors $W(D)=W(\mathcal{M}(D))$.

Démonstration. Ces affirmations ne présentent aucune difficulté. □

Théorème 2.3.5. *Le théorème de monodromie p-adique et le théorème « faiblement admissible implique admissible » sont vrais. De fait,*

(1) *toute B-paire de de Rham est potentiellement semi-stable ;*

(2) *le foncteur $W\mapsto\mathrm{D}_{\mathrm{st}}(W)$ réalise une équivalence de catégories entre la catégorie des objets de la forme $W(V)$ où V est une représentation p-adique semi-stable et la catégorie des (φ, N)-modules filtrés admissibles.*

Démonstration. Montrons tout d'abord le (1). Soit W une B-paire et $\mathrm{D}(W)$ le (φ,Γ_K)-module associé. Si W est de de Rham, alors pour $n\gg 0$, la $\mathbf{B}_{\mathrm{dR}}$-représentation

$$\mathbf{B}_{\mathrm{dR}}\otimes^{\iota_n}_{\mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}}\mathrm{D}^{r_n}(W)$$

est égale à $\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{B}_{\mathrm{e}}}W_e$ et donc triviale, ce qui fait par le théorème 3.9 de [Fontaine 2004] que le $K_\infty((t))$-module à connexion $K_\infty((t))\otimes^{\iota_n}_{\mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}}\mathrm{D}^{r_n}(W)$ est trivial. Étant donnée la définition III.1.2 de [Berger 2004a], on est en mesure d'en appliquer le théorème A du même article, qui nous dit qu'il existe une extension finie L de K et un (φ, N)-module filtré D sur L tels que $\mathrm{D}(W_{|L})=\mathcal{M}(D)$ et donc que $W_{|L}$ est semi-stable. Ceci montre le (1). Le (2) suit du théorème A de [Colmez et Fontaine 2000] ou bien (si l'on préfère passer par les (φ,Γ_K)-modules) du théorème B de [Berger 2004a]. □

Rappelons que si U est une $\mathbb{C}_p$-représentation de G_K, alors la réunion $U^{H_K}_{\mathrm{fini}}$ des sous-K_∞-espaces vectoriels de dimension finie stables par Γ_K de U^{H_K} a la propriété que l'application $\mathbb{C}_p\otimes_{K_\infty}U^{H_K}_{\mathrm{fini}}\to U$ est un isomorphisme [Sen 1980]. L'espace $U^{H_K}_{\mathrm{fini}}$ est muni de l'application K_∞-linéaire $\nabla_U=\log(\gamma)/\log_p(\chi(\gamma))$ avec $\gamma\in\Gamma_K\setminus\{1\}$ suffisamment proche de 1.

Définition 2.3.6. Si W est une B-paire, alors $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ est une $\mathbb{C}_p$-représentation de G_K et on pose $\mathrm{D}_{\mathrm{Sen}}(W)=(W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}})^{H_K}_{\mathrm{fini}}$. On pose $\Theta_{\mathrm{Sen}}=\nabla_W$ et on dit que W est *de Hodge–Tate* si Θ_{Sen} est diagonalisable à valeurs propres appartenant à $\mathbb{Z}$. Ces entiers sont les *poids de Hodge–Tate* de W.

3. Les (φ, Γ)-modules

Étant donné le Théorème 2.2.7, l'étude des B-paires revient à l'étude des (φ, Γ)-modules. Dans ce chapitre, on montre plusieurs résultats sur les (φ, Γ)-modules : modification, classification des objets isoclines, classification des objets de hauteur finie.

3.1. ***Modification de (φ, Γ)-modules.*** Si D est un φ-module sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ et si $n \geqslant n(r)$ avec $r \gg 0$, rappelons que D^r/Q_n est un K_n-espace vectoriel de dimension $\mathrm{rg}(D)$. Soit $M = \{M_n\}_{n \geqslant n(r)}$ une famille φ-compatible de sous-K_n-espaces vectoriels de D^r/Q_n, c'est-à-dire que pour tout $n \geqslant n(r)$, M_{n+1} est engendré par les $\varphi(y)$ où $y \in \mathrm{D}^r$ est tel que son image dans D^r/Q_n appartient à M_n. En d'autres termes, on un isomorphisme $K_{n+1} \otimes_{K_n} (\mathrm{D}^r/Q_n) \to \mathrm{D}^{pr}/Q_{n+1}$ obtenu en quotientant l'isomorphisme $\varphi^*(\mathrm{D}^r) \simeq \mathrm{D}^{pr}$ par $\varphi(Q_n) = Q_{n+1}$, et qui est donc donné par $\alpha \otimes \bar{m} \mapsto \alpha \otimes \overline{\varphi(m)}$ et on demande que M_{n+1} soit l'image de $K_{n+1} \otimes_{K_n} M_n$ par cet isomorphisme (voir [Berger 2004a, §II.1] pour une condition analogue).

Définition 3.1.1. Une telle famille $M = \{M_n\}_{n \geqslant n(r)}$ de sous-espaces vectoriels de D^r/Q_n est appelée une *donnée de modification* de D. On définit alors $\mathrm{D}[M] = \{y \in \mathrm{D}$ dont l'image dans D^r/Q_n appartient à $M_n\}$.

Proposition 3.1.2. *Si* D *est un φ-module de rang d sur* $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ *et si M est une donnée de modification, alors* $\mathrm{D}[M]$ *est un φ-module et de plus* :

(1) *si $M \subset N$, alors* $\mathrm{D}[M] \subset \mathrm{D}[N]$ *et si* $\mathrm{D}[M] = \mathrm{D}[N]$, *alors* $M = N$;

(2) $\mathrm{D}[M] = \mathrm{D}$ *si* $M_n = \mathrm{D}^r/Q_n$ *pour tout n, et* $\mathrm{D}[0] = t \cdot \mathrm{D}$;

(3) $\deg \mathrm{D}[M] = \deg \mathrm{D} + d - \dim M$;

(4) *si* D *est un (φ, Γ_K)-module et si M est stable par Γ_K, alors* $\mathrm{D}[M]$ *est un (φ, Γ_K)-module.*

Démonstration. Rappelons que $t = \log(1 + X) = X \cdot \prod_{n \geqslant 0} Q_n/p$ ce qui fait que $t \cdot \mathrm{D} \subset \mathrm{D}[M]$ quel que soit M. La définition de $\mathrm{D}[M]$ implique que c'est un sous-module fermé de D, et comme il contient $t \cdot \mathrm{D}$, il est libre de rang d. Le fait que c'est un φ-module résulte du fait que la famille des M_n est φ-compatible.

Le fait que si $M \subset N$, alors $\mathrm{D}[M] \subset \mathrm{D}[N]$ est évident. Remarquons que si l'image de $x \in \mathrm{D}$ appartient à M_n, alors $x \cdot t/Q_n \in \mathrm{D}[M]$ et son image dans D^r/Q_n est (un multiple de) x ce qui fait que l'image de $\mathrm{D}[M]$ dans D^r/Q_n est M_n et en particulier que si $\mathrm{D}[M] = \mathrm{D}[N]$, alors $M = N$. Ceci montre le (1). Le (2) est une évidence.

Pour montrer le (3), prenons un drapeau $\{0\} = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(d)}$ tel que $M^{(\dim M)} = M$. On en déduit une suite de φ-modules $t \cdot \mathrm{D} = \mathrm{D}^{(0)} \subset \cdots \subset \mathrm{D}^{(d)} = \mathrm{D}$ et donc une suite de φ-modules de rang 1 :

$$\det(t \cdot \mathrm{D}) = \det(\mathrm{D}^{(0)}) \subset \cdots \subset \det(\mathrm{D}^{(d)}) = \det(\mathrm{D}).$$

Leurs pentes forment une suite strictement décroissante (si on a égalité des pentes, on a égalité des modules par le Lemme 1.2.7) de $d+1$ nombres entiers (ils sont entiers par le Lemme 1.2.1) dont le premier est $d + \deg \mathrm{D}$ et le dernier est $\deg \mathrm{D}$, ce qui fait que $\deg(\mathrm{D}^{(m)})$ est forcément $\deg \mathrm{D} + d - \dim M$. Enfin, le (4) est une évidence. □

Proposition 3.1.3. *Si M est une donnée de modification de* D *et si l'on a une suite exacte* $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2 \to 0$, *alors* :

(1) $\{M_n \cap \mathrm{D}_1^r / Q_n\}$ *est une donnée de modification pour* D_1 *et* $\{\mathrm{im}(M_n \to \mathrm{D}_2^r / Q_n)\}$ *est une donnée de modification pour* D_2 ;

(2) *on a une suite exacte* $0 \to \mathrm{D}_1[M] \to \mathrm{D}[M] \to \mathrm{D}_2[M] \to 0$.

Démonstration. Le (1) suit du fait que les M_n sont φ-compatibles et que les D_i^r / Q_n le sont aussi. Passons au (2) ; on a clairement une suite exacte $0 \to \mathrm{D}_1[M] \to \mathrm{D}[M] \to \mathrm{D}_2[M]$. Par le (3) de la proposition 3.1.2, on a :

(1) $\deg \mathrm{D}_1[M] = \deg \mathrm{D}_1 + \dim \mathrm{D}_1 - \dim(M_n \cap \mathrm{D}_1^r / Q_n)$;

(2) $\deg \mathrm{D}[M] = \deg \mathrm{D} + \dim \mathrm{D} - \dim M_n$;

(3) $\deg \mathrm{D}_2[M] = \deg \mathrm{D}_2 + \dim \mathrm{D}_2 - \dim(\mathrm{im}(M_n \to \mathrm{D}_2^r / Q_n))$.

On en déduit que $\deg \mathrm{D}[M] = \deg \mathrm{D}_1[M] + \deg \mathrm{D}_2[M]$, et le Corollaire 1.2.8 implique alors que la suite $0 \to \mathrm{D}_1[M] \to \mathrm{D}[M] \to \mathrm{D}_2[M] \to 0$ est exacte. □

Remarque 3.1.4. Si M et N sont deux données de modification telles que $M \cap N = 0$, alors on un isomorphisme $\mathrm{D}[M][N] = \mathrm{D}[M \oplus N]$ (il y a une inclusion évidente ; comparer les degrés).

Théorème 3.1.5. *Si* D *est un φ-module sur* $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, *alors il existe un φ-module étale* $\mathrm{D}' \subset \mathrm{D}[1/t]$ *tel que* $\mathrm{D}'[1/t] = \mathrm{D}[1/t]$.

Démonstration. Il suffit de montrer qu'il existe un nombre fini de modifications successives (au sens de la proposition 3.1.2) de D dont le résultat est isocline de pente entière (si D est isocline de pente s, alors $t \cdot \mathrm{D}$ est isocline de pente $s+1$). Remarquons que si on a une suite exacte $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2 \to 0$, alors il est toujours possible de modifier D_1 ou D_2 sans toucher à l'autre, en choisissant M convenablement.

Etape 1 : si D est isocline, alors D se modifie en une extension successive de φ-modules de pentes entières. En effet, si D est isocline, modifions le par un M de codimension 1 ce qui augmente le degré de D de 1. Si $\mathrm{D}[M]$ est isocline de pente non entière, alors on répète cette opération. Le résultat est donc, après un nombre fini de modifications, que D devient isocline de pente entière (et on a terminé l'étape 1) ou bien que D se casse en deux morceaux. Dans ce dernier cas, on a terminé par récurrence sur la dimension de D.

Etape 2 : si D est extension successive de φ-modules de pentes entières, alors il se modifie en un φ-module isocline de pente entière. On se ramène au cas d'une extension de deux φ-modules de pentes entières $0 \to \mathrm{D}_1 \to \mathrm{D} \to \mathrm{D}_2 \to 0$ avec D_i isocline de pente s_i. Dans ce cas :

(1) si $s_2 = s_1$, alors on a terminé par le Lemme 1.2.3 ;

(2) si $s_2 < s_1$, alors la modification est facile à écrire : on remplace D par D$'$, l'ensemble des $y \in \mathrm{D}$ dont l'image dans D_2 appartient à $t^{s_1-s_2}\mathrm{D}_2$, ce qui fait que l'on a une suite exacte $0 \to \mathrm{D}_1 \to \mathrm{D}' \to t^{s_1-s_2}\mathrm{D}_2 \to 0$ et on a terminé par le Lemme 1.2.3 ;

(3) si $s_1 < s_2$, alors on modifie D par une donnée nulle sur D_1 et surjective sur D_2 ce qui augmente s_1 de 1 et on itère cette opération $s_2 - s_1$ fois. □

Corollaire 3.1.6. *Si W_e est une $\mathbf{B}_{\mathrm{e}}$-représentation de G_K, alors il existe une représentation p-adique V de H_K telle que la restriction de W_e à H_K est isomorphe à $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V$.*

Démonstration. Soit W_{dR}^+ un $\mathbf{B}_{\mathrm{dR}}^+$-réseau stable par G_K de $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ et $\mathrm{D} = \mathrm{D}(W)$ le (φ, Γ_K)-module associé à la B-paire $W = (W_e, W_{\mathrm{dR}}^+)$ par le Théorème 2.2.7. Par le Théorème 3.1.5, il existe un φ-module étale $\mathrm{D}' \subset \mathrm{D}[1/t]$ tel que $\mathrm{D}'[1/t] = \mathrm{D}[1/t]$. Comme D$'$ est étale, le $\mathbb{Q}_p$-espace vectoriel $V = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}')^{\varphi=1}$ est une représentation de H_K dont la dimension est le rang de D$'$. On a alors :

$$W_e = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D})^{\varphi=1} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}')^{\varphi=1} = \mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V,$$

et le corollaire est démontré. □

Remarque 3.1.7. (1) Dans le Corollaire 3.1.6 ci-dessus, on est loin d'avoir unicité. Par exemple, si D est un (φ, N)-module sur K_0 et si $W_e = (\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1, N=0}$, alors $W_e = \mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} V$ pour toute représentation V qui s'obtient comme $V_{\mathrm{st}}(D)$ à partir d'une filtration admissible sur $K \otimes_{K_0} D$;

(2) Dans la première version de cet article, j'affirmais que si D est un (φ, Γ_K)-module sur $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$, alors il existe une extension finie L de K et un (φ, Γ_L)-module étale D$'$ sur $\mathbf{B}_{\mathrm{rig},L}^{\dagger}$ tel que $\mathrm{D}' \subset \mathbf{B}_{\mathrm{rig},L}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}$ et $\mathbf{B}_{\mathrm{rig},L}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}' = \mathbf{B}_{\mathrm{rig},L}^{\dagger}[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}$. Ceci est incorrect, et Kiran Kedlaya et Ruochuan Liu ont construit le contre-exemple suivant : soient $p \neq 2$, $K = \mathbb{Q}_p$ et D une extension non-triviale de $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ par $\mathbf{B}_{\mathrm{rig},K}^{\dagger}(p^{-2})$ (ce qui veut dire que l'action de Γ_K est inchangée et que φ est multiplié par p^{-2}). En utilisant le théorème de filtration par les pentes de Kedlaya et les calculs de cohomologie de (φ, Γ)-modules de [Colmez 2007] et de [Liu 2007], on peut montrer qu'une telle extension non-triviale existe et que quelque soit l'extension finie L de K, $\mathbf{B}_{\mathrm{rig},L}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathrm{D}$ ne contient pas de sous-(φ, Γ_L)-module étale de rang 2 et donc qu'un D$'$ comme ci-dessus n'existe pas.

3.2. ***Classification des objets isoclines.*** Le Théorème 2.2.7 nous donne une filtration sur les B-paires, et il est intéressant de décrire les B-paires correspondant aux (φ, Γ_K)-modules isoclines. Le cas étale relève de la proposition 2.2.9.

Définition 3.2.1. Si $h \geqslant 1$ et $a \in \mathbb{Z}$ sont premiers entre eux, soit $\mathrm{Rep}(a, h)$ la catégorie dont les objets sont les $\mathbb{Q}_{p^h}$-espaces vectoriels $V_{a,h}$ de dimension finie, munis d'une action semi-linéaire de G_K et d'un Frobenius lui aussi semi-linéaire $\varphi : V_{a,h} \to V_{a,h}$ qui commute à G_K et qui vérifie $\varphi^h = p^a$. Les morphismes sont ceux que l'on imagine.

Remarque 3.2.2. (1) Si $V_{a,h} \in \mathrm{Rep}(a, h)$, alors $\dim_{\mathbb{Q}_{p^h}}(V_{a,h})$ est divisible par h (si e est cette dimension, alors $\varphi^h = p^{ae}$ sur $\det(V_{a,h})$) ;

(2) si $h = 1$ et $a = 0$, alors on retrouve simplement la catégorie des représentations p-adiques de G_K ;

(3) si $e \in \mathbb{Z}$, alors $\mathrm{Rep}(a, h)$ et $\mathrm{Rep}(a + eh, h)$ sont équivalentes de manière évidente ;

(4) si D est un φ-module isocline de pente a/h sur K_0, alors le théorème de Dieudonné–Manin implique que $V_{a,h} = (\widehat{\mathbb{Q}}_p^{\mathrm{nr}} \otimes_{K_0} D)^{\varphi^h = p^a}$ est un objet de $\mathrm{Rep}(a, h)$ dont la dimension en tant que $\mathbb{Q}_{p^h}$-espace vectoriel est $\dim_{K_0}(D)$. Dans ce cas, l'action de $I_K \subset G_K$ sur $V_{a,h}$ est d'ailleurs triviale.

Si $V_{a,h} \in \mathrm{Rep}(a, h)$, on pose $W_e(V_{a,h}) = (\mathbf{B}_{\max} \otimes_{\mathbb{Q}_{p^h}} V_{a,h})^{\varphi=1}$ et $W_{\mathrm{dR}}^+(V_{a,h}) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_{p^h}} V_{a,h}$.

Théorème 3.2.3. *Si $V_{a,h} \in \mathrm{Rep}(a, h)$, alors $W(V_{a,h}) = (W_e(V_{a,h}), W_{\mathrm{dR}}^+(V_{a,h}))$ est une B-paire et le foncteur $V_{a,h} \mapsto W(V_{a,h})$ définit une équivalence de catégories entre $\mathrm{Rep}(a, h)$ et la catégorie des B-paires W telles que $\mathrm{D}(W)$ est isocline de pente a/h.*

Démonstration. Il est clair que $W = W(V_{a,h})$ est une B-paire. Par ailleurs, la construction de $\mathrm{D}(W)$ fournie par le Lemme 2.2.1 et la proposition 2.2.5 montre que l'on a $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathrm{D}(W) = \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbb{Q}_{p^h}} V_{a,h}$ ce qui fait que $\mathrm{D}(W)$ est isocline de pente a/h.

Réciproquement, si D est un (φ, Γ_K)-module isocline de pente a/h, alors par la proposition 1.1.5, on a une décomposition

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathrm{D} = \bigoplus_{i=1}^k M_{a,h} = \bigoplus_{i=1}^k \bigoplus_{j=0}^{h-1} \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \varphi^j(e_i)$$

où $e_i, \varphi(e_i), \ldots, \varphi^{h-1}(e_i)$ est une base de la i-ième copie de $M_{a,h}$. On voit alors que $\sum_{i=1}^k \sum_{j=0}^{h-1} \lambda_{ij} \varphi^j(e_i) \in (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathrm{D})^{\varphi^h = p^a}$ si et seulement si on a $\varphi^h(\lambda_{ij}) = \lambda_{ij}$ pour tous i, j et comme $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger)^{\varphi^h = 1} = \mathbb{Q}_{p^h}$, on trouve que $V_{a,h} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathrm{D})^{\varphi^h = p^a}$ est un $\mathbb{Q}_{p^h}$-espace vectoriel de dimension $\mathrm{rg}(\mathrm{D})$ qui hérite d'une action de G_K et d'un Frobenius tel que $\varphi^h = p^a$. Si W est une B-paire telle que $\mathrm{D}(W)$ est isocline de pente a/h, alors on lui associe l'espace $V_{a,h}$ construit à partir de $\mathrm{D}(W)$. Le lecteur vérifiera qu'on a ainsi défini un foncteur inverse de $V_{a,h} \mapsto W(V_{a,h})$. □

Il sera intéressant de calculer les extensions d'objets isoclines ; les extensions de (φ, Γ)-modules sont étudiées dans [Liu 2007].

3.3. ***Les (φ, Γ)-modules de hauteur finie.*** Comme l'anneau $\mathbf{B}^{+}_{\mathrm{rig},K} = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \cap \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ n'a de bonnes propriétés que si $K \subset K_0(\mu_{p^\infty})$, on suppose que cette condition est vérifiée dans tout ce paragraphe. Dans ce cas, $\mathbf{B}^{+}_{\mathrm{rig},K}$ s'identifie à l'ensemble des séries $f(X) = \sum_{k \geqslant 0} f_k X^k$ qui convergent sur le disque unité ouvert, ce qui en fait un anneau de Bézout, et si on pose $\mathbf{B}^{+}_{K} = \mathbf{B}^{+}_{\mathrm{rig},K} \cap \mathbf{B}^{\dagger}_{K}$, alors $\mathbf{B}^{+}_{K} = K_0 \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{K_0}[\![X]\!]$.

Définition 3.3.1. On dit qu'un (φ, Γ)-module D sur $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ est *de hauteur finie* s'il existe un (φ, Γ)-module D^{+} sur $\mathbf{B}^{+}_{\mathrm{rig},K}$ tel que $\mathrm{D} = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},K}} \mathrm{D}^{+}$ et on dit qu'une B-paire W est de hauteur finie si $\mathrm{D}(W)$ l'est.

Remarquons qu'un (φ, Γ)-module sur $\mathbf{B}^{+}_{\mathrm{rig},K}$ est un $\mathbf{B}^{+}_{\mathrm{rig},K}$-module D^{+} stable par φ et Γ tel que $\det \varphi$ est inversible dans $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ (et non dans $\mathbf{B}^{+}_{\mathrm{rig},K}$ ce qui serait trop restrictif).

Lemme 3.3.2. *Si $K \subset K_0(\mu_{p^\infty})$, alors la définition ci-dessus est compatible avec la définition habituelle quand $W = W(V)$.*

Démonstration. Si $W = W(V)$ avec V de hauteur finie au sens habituel (cf. [Colmez 1999]), alors il est évident que W est de hauteur finie. Montrons donc la réciproque. Par hypothèse, il existe une base de $\mathrm{D}(V)$ dans laquelle $\mathrm{Mat}(\varphi) = P^{+} \in \mathrm{M}_d(\mathbf{B}^{+}_{\mathrm{rig},K})$ et $\mathrm{Mat}(\gamma) = G^{+} \in \mathrm{GL}_d(\mathbf{B}^{+}_{\mathrm{rig},K})$ pour $\gamma \in \Gamma_K$ (puisque V est de hauteur finie) ainsi qu'une base de $\mathrm{D}^{\dagger}(V)$ dans laquelle $\mathrm{Mat}(\varphi) = P^{\dagger} \in \mathrm{GL}_d(\mathbf{B}^{\dagger}_{K})$ et $\mathrm{Mat}(\gamma) = G^{\dagger} \in \mathrm{GL}_d(\mathbf{B}^{\dagger}_{K})$ pour $\gamma \in \Gamma_K$.

Soit M la matrice de passage d'une base à l'autre. La proposition 6.5 de [Kedlaya 2004] montre que l'on peut écrire $M = M^{+} \cdot M^{\dagger}$ avec $M^{+} \in \mathrm{GL}_d(\mathbf{B}^{+}_{\mathrm{rig},K})$ et $M^{\dagger} \in \mathrm{GL}_d(\mathbf{B}^{\dagger}_{K})$. Dans la base de $\mathrm{D}^{\dagger}(V)$ obtenue en appliquant $M^{\dagger}$ à celle de $\mathrm{D}^{\dagger}(V)$, on a :

$$\mathrm{Mat}(\varphi) = \varphi(M^{+}) P^{+} (M^{+})^{-1} = \varphi(M^{\dagger})^{-1} P^{\dagger} (M^{\dagger}),$$
$$\mathrm{Mat}(\gamma) = \gamma(M^{+}) G^{+} (M^{+})^{-1} = \gamma(M^{\dagger})^{-1} G^{\dagger} (M^{\dagger}),$$

ce qui fait que ces matrices sont à coefficients dans $\mathbf{B}^{\dagger}_{K} \cap \mathbf{B}^{+}_{\mathrm{rig},K} = \mathbf{B}^{+}_{K}$. □

Lemme 3.3.3. *Si D^{+} est un (φ, Γ)-module sur $\mathbf{B}^{+}_{\mathrm{rig},K}$, alors il existe des entiers $\alpha_0, \ldots, \alpha_m$ tels que l'idéal engendré par* $\det(\varphi)$ *est engendré par $X^{\alpha_0} Q_1^{\alpha_1} \cdots Q_m^{\alpha_m}$ et si K est une extension finie de $\mathbb{Q}_p$, alors $\alpha_0 = 0$.*

Démonstration. Ce lemme se trouve dans la partie B.1.6 de [Wach 1996] mais nous en donnons une démonstration pour la commodité du lecteur. Comme l'action de φ commute à Γ_K, l'idéal engendré par $\det(\varphi)$ est stable par Γ_K et la première partie du lemme suit alors du lemme I.3.2 de [Berger 2004b] et du fait que si cet idéal est inversible dans $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, alors les α_j sont presque tous nuls.

Si $\delta = \det(\varphi)$ et $g = \det(\gamma)$ pour $\gamma \in \Gamma_K$, alors $\gamma(\delta)/\delta = \varphi(g)/g$ et en réduisant modulo X, on trouve que $\chi(\gamma)^{\alpha_0}$ s'écrit $\varphi(g_0)/g_0$ avec $g_0 \in K_0$. Si K est une extension finie de $\mathbb{Q}_p$, alors $\varphi(g_0)/g_0$ est de norme 1 et donc $\chi(\gamma)^{\alpha_0[K_0:\mathbb{Q}_p]} = 1$ ce qui fait que $\alpha_0 = 0$. □

Le lecteur vicieux montrera que si le corps résiduel de K est algébriquement clos, on peut effectivement avoir $\alpha_0 \neq 0$.

Définition 3.3.4. Un *φ-module filtré sur K_0 avec action de* Γ_K est un φ-module D sur K_0 muni d'une action de Γ_K commutant à φ et d'une filtration (décroissante, exhaustive et séparée) stable par Γ_K sur $D_\infty = K_\infty \otimes_{K_0} D$.

Il existe alors un entier $n \geqslant 0$ tel que la filtration de D_∞ est définie sur K_n, c'est-à-dire que si l'on pose $D_n = K_n \otimes_{K_0} D$, alors $\mathrm{Fil}^i D_\infty = K_\infty \otimes_{K_n} \mathrm{Fil}^i D_n$ pour tout $i \in \mathbb{Z}$ et on appelle $n(D)$ le plus petit entier ayant cette propriété.

Proposition 3.3.5. *Si D est un φ-module filtré sur K_0 avec action de Γ_K, alors la B-paire $W = ((\mathbf{B}_{\max} \otimes_{K_0} D)^{\varphi=1}, \mathbf{B}^+_{\mathrm{dR}} \otimes_{K_\infty} D_\infty)$ est de hauteur finie.*

Démonstration. Soit $h \geqslant 0$ tel que $\mathrm{Fil}^h D_\infty = 0$ et $\mathrm{Fil}^{-h} D_\infty = D_\infty$. Si $n \geqslant 1$, on écrit φ^{-n} pour ι_n. Posons alors comme dans [Colmez 2003a, §3.1] et [Kisin 2006] :

$$\mathcal{M}^+(D) = \{y \in t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D \mid \varphi^n(y) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)\ \forall n \in \mathbb{Z}\}.$$

C'est un sous-$\mathbf{B}^+_{\mathrm{rig},K}$-module fermé de $t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$ qui contient $t^h\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$ et qui est donc libre de rang $d = \dim D$. Il est de plus manifestement stable par φ et Γ_K.

Si $n \geqslant n(D)$, alors :

$$K_n[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D = t^{-h} K_n[\![t]\!] \otimes_{K_0} D \supset \mathrm{Fil}^0(K_n(\!(t)\!) \otimes_{K_n} D_n).$$

Si $y_n \in \mathrm{Fil}^0(K_n(\!(t)\!) \otimes_{K_n} D_n)$ et $w \geqslant 0$, il existe donc $y \in t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$ donc l'image par ι_n vérifie $\iota_n(y) - y_n \in \mathrm{Fil}^w(K_n(\!(t)\!) \otimes_{K_n} D_n)$. L'élément $z = y \cdot (t/Q_n)^{2h}$ a alors la propriété que $\iota_n(z)$ est un multiple de $\iota_n(y)$ et que pour tout $m \in \mathbb{Z}$ différent de $-n$, on a $\varphi^m(z) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$ ce qui fait que $z \in \mathcal{M}^+(D)$. On en déduit que si $n \geqslant n(D)$, alors l'application :

$$K_n[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D) \to \mathrm{Fil}^0(K_n(\!(t)\!) \otimes_{K_n} D_n)$$

est un isomorphisme. En particulier, l'application naturelle $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D) \to \widetilde{\mathrm{D}}(W)$ est un isomorphisme et donc, en utilisant l'unicité dans la proposition 2.2.5, on trouve que $\mathrm{D}(W) = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$ ce qui fait que W est de hauteur finie. Remarquons que dans les notations de [Berger 2004a], on a

$$\mathcal{M}(D) = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D).$$ □

Remarque 3.3.6. Le fait de ne considérer que les $y \in t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$ tels que l'on a $\varphi^n(y) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$ pour tout $n \in \mathbb{Z}$ n'est pas restrictif ; le lecteur pourra montrer que :

$$\mathcal{M}^+(D) = \{y \in \mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D \mid \varphi^n(y) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)\ \forall n \in \mathbb{Z}\}.$$

Si $a, b \in \mathbb{Z}$ sont tels que $\mathrm{Fil}^{-a+1} D = 0$ et $\mathrm{Fil}^{-b} D = D$ (c'est-à-dire que les *poids* de D sont dans l'intervalle $[a; b]$), alors on a $t^b \cdot \mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D \subset \mathcal{M}^+(D) \subset t^a \cdot \mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$.

En appliquant la proposition 3.3.5 à $\mathrm{D}_{\mathrm{cris}}(V)$, on retrouve le théorème principal de [Colmez 1999] généralisé dans [Berger et Breuil 2006].

Corollaire 3.3.7. *Si V est une représentation de G_K qui devient cristalline sur une extension K_n de K, alors V est de hauteur finie.*

On peut d'ailleurs se demander quand est-ce que $W(D)$ est cristalline.

Lemme 3.3.8. *Si V est une représentation de Γ_K, alors :*

(1) *V est cristalline si et seulement si $V = \bigoplus_{j \in \mathbb{Z}} V^{\Gamma_K = \chi^j}$;*

(2) *V est de Hodge–Tate si et seulement si elle est potentiellement cristalline.*

Démonstration. Le (1) est l'objet du lemme 3.4.3 de [Perrin-Riou 1994], mais nous en donnons une nouvelle démonstration. Pour cela, observons que $\mathrm{D}_{\mathrm{Sen}}(V) = K_\infty \otimes_{\mathbb{Q}_p} V$ et donc que V est de Hodge–Tate si et seulement si ∇_V est diagonalisable à valeurs propres entières sur $K_\infty \otimes_{\mathbb{Q}_p} V$; ceci est équivalent à demander qu'il existe $n \geqslant 0$ tel que $K_n \otimes_{\mathbb{Q}_p} V = \bigoplus_{j \in \mathbb{Z}} (K_n \otimes_{\mathbb{Q}_p} V)^{\Gamma_{K_n} = \chi^j}$. Comme $(K_n \otimes_{\mathbb{Q}_p} V)^{\Gamma_{K_n} = \chi^j} = K_n \otimes_{\mathbb{Q}_p} V^{\Gamma_{K_n} = \chi^j}$, et qu'une représentation d'image finie de Γ_K est cristalline si et seulement si elle est triviale, on en déduit le (1) et le (2). □

Corollaire 3.3.9. *Si D est un (φ, Γ_K)-module filtré sur K_0, la B-paire construite ci-dessus est cristalline si et seulement si $D = \bigoplus_{j \in \mathbb{Z}} D^{\Gamma = \chi^j}$.*

Démonstration. Comme $W_e = (\mathbf{B}_{\max} \otimes_{K_0} D)^{\varphi = 1}$, on a $\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e = \mathbf{B}_{\max} \otimes_{K_0} D$ ce qui fait que W est cristalline si et seulement si D est cristalline en tant que représentation de Γ_K. Le corollaire suit alors du Lemme 3.3.8 ci-dessus. □

Proposition 3.3.10. *Si* D *est un (φ, Γ_K)-module de hauteur finie, alors il existe un φ-module filtré D sur K_0 avec action de Γ_K tel que* $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$.

Démonstration. Soit D^+ un (φ, Γ_K)-module sur $\mathbf{B}^+_{\mathrm{rig},K}$ tel que $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$ et $D = \mathrm{D}^+/X$. Soit $\nabla = \log(\gamma)/\log_p(\chi(\gamma))$ pour $\gamma \in \Gamma_K$ proche de 1.

Commençons par montrer que si $P \in K_0[T]$ est un polynôme tel que $P(\nabla) : D(j) \to D(j)$ est bijectif pour tout $j \geqslant 1$, alors l'application $(\mathrm{D}^+)^{P(\nabla)=0} \to D^{P(\nabla)=0}$ est bijective. Pour cela, nous montrons d'abord que l'application :

$$(K_0[\![t]\!] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0} \to D^{P(\nabla)=0}$$

est bijective. L'injectivité ne pose pas de problème : si $y \in K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$ est dans son noyau, et si j est un entier $\geqslant 1$ tel que $y \in t^j K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$, alors $P(\nabla)(y) = 0$ dans $t^j K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+/t^{j+1} = D(j)$ et donc $y \in t^{j+1} K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$, ce qui fait en itérant que $y = 0$. Montrons à présent la surjectivité ; si $\bar{z} \in D^{P(\nabla)=0}$, alors il existe $z_0 \in \mathrm{D}^+$ qui relève $\bar{z}$ et tel que $P(\nabla)(z_0) \in X\mathrm{D}^+$ et l'hypothèse selon laquelle $P(\nabla) : D(j) \to D(j)$ est bijectif pour tout $j \geqslant 0$ nous permet de construire par récurrence $z_j \in z_{j-1} + X^j \mathrm{D}^+$ tel que $P(\nabla)(z_j) \in X^{j+1}\mathrm{D}^+$ et donc $z \in (K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0}$ relevant $\bar{z}$ ce qui fait que notre application est bien bijective.

Montrons à présent que $(K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0} = (\mathrm{D}^+)^{P(\nabla)=0}$. Pour cela, remarquons que $(K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0}$ est un K_0-espace vectoriel de dimension finie (puisqu'il s'injecte dans D) stable par φ. Si l'on choisit une base de D^+ et que l'on appelle Q la matrice de φ dans cette base, et si l'on choisit une base de $(K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0}$ et que l'on appelle P_0 la matrice de φ dans cette base et Y la matrice de passage entre les deux bases, alors on a $\varphi(Y)Q = P_0 Y$ et donc $Y = P_0^{-1}\varphi(Y)Q$. Ecrivons $Q = \sum_{k \geqslant 0} Q_k t^k$ et $Y = \sum_{k \geqslant 0} Y_k t^k$. Si M est une matrice à coeffcients dans K_0, notons $\mathrm{val}_p(M)$ le minimum des valuations de ses coefficients. Comme $\mathbf{B}^+_{\mathrm{rig},K} \subset \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t/p]]$ (n'oublions pas que $X = \exp(t) - 1$), on voit qu'il existe un entier h_1 tel que $\mathrm{val}_p(Q_k) \geqslant -h_1 - k$ et il existe par ailleurs un entier h_2 tel que $\mathrm{val}_p(P_0^{-1}) \geqslant -h_2$; posons $h = h_1 + h_2$. On déduit alors de l'équation $Y = P_0^{-1}\varphi(Y)Q$ que $Y = P_0^{-1}\varphi(P_0^{-1}) \cdot \varphi^2(Y) \cdot \varphi(Q)Q$ et (comme $\mathrm{val}_p(p^k Q_k) \geqslant -h_1$) que :

$$\begin{aligned}\mathrm{val}_p(Y_k) &\geqslant -2h + \min_{0 \leqslant \ell \leqslant k} (2\ell + \mathrm{val}_p(Y_\ell) - \ell) \\ &= -2h + \min_{0 \leqslant \ell \leqslant k} (\ell + \mathrm{val}_p(Y_\ell)).\end{aligned}$$

On en déduit par récurrence sur $k \geqslant 2h$ que si $k \geqslant 2h$, alors $\mathrm{val}_p(Y_k) \geqslant -2h + \min_{0 \leqslant \ell \leqslant 2h} \mathrm{val}_p(Y_\ell)$ et donc, comme $\mathcal{O}_{K_0}[[t]] \subset \mathcal{O}_{K_0}[[X/p]]$, que :

$$(K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0} = ((K_0 \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[[X/p]]) \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0}.$$

Pour terminer, on utilise de nouveau le fait que $((K_0 \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[[X/p]]) \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0}$ est un K_0-espace vectoriel de dimension finie stable par φ, et que si $n \geqslant 0$, alors :

$$\varphi(K_0 \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[[X^{p^n}/p]]) \subset K_0 \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[[X^{p^{n+1}}/p]]$$

pour conclure que $(K_0[[t]] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+)^{P(\nabla)=0} = (\mathrm{D}^+)^{P(\nabla)=0}$ puisque

$$\mathbf{B}^+_{\mathrm{rig},K} = \bigcap_{n \geqslant 0} K_0 \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[[X^{p^n}/p]].$$

Revenons à notre espace $D = \mathrm{D}^+/X$ et soit Q le polynôme minimal de ∇. On voit que l'on peut écrire $Q = PR$ où P est non trivial et a la propriété que $P(X+j) \wedge P(X) = 1$ pour tout $j \geqslant 1$, ce qui revient à dire que $P(\nabla) : D(j) \to D(j)$ est bijectif. Si $P \neq Q$, alors soit j le plus petit entier tel que $P(X+j) \wedge P(X) \neq 1$; on vient de montrer que l'application $(\mathrm{D}^+)^{P(\nabla)=0} \to D^{P(\nabla)=0}$ est un isomorphisme, et si l'on remplace D^+ par $\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} (\mathrm{D}^+)^{P(\nabla)=0} + X^j \mathrm{D}^+$, alors on a toujours $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$ mais on peut prendre P de degré plus grand. En itérant cette opération un nombre fini de fois, on voit donc que quitte à remplacer D^+ par un sous-module, on peut supposer que l'application $(\mathrm{D}^+)^{Q(\nabla)=0} \to D^{Q(\nabla)=0} = D$ est un isomorphisme.

On fait alors un léger abus de notation, et on pose $D = (\mathrm{D}^+)^{Q(\nabla)=0}$; c'est un φ-module sur K_0 avec action de Γ_K. Remarquons que $\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D \subset \mathrm{D}^+$ et que le déterminant de l'inclusion est un idéal de $\mathbf{B}^+_{\mathrm{rig},K}$ stable par Γ_K et par φ (car $\varphi : D \to D$ est bijectif) ce qui fait, par le lemme I.3.2 de [Berger 2004b], que $\mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D = \mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$. Par le Lemme 3.3.3, le déterminant de φ sur D^+ est de la forme $X^{\alpha_0} Q_1^{\alpha_1} \cdots Q_m^{\alpha_m}$. Si $n \geqslant m$, alors l'application déduite du Frobenius :

$$K_{n+1}[\![t]\!] \otimes^{\iota_{n+1}}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+ \to K_{n+1}[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$$

est un isomorphisme et la définition :

$$\mathrm{Fil}^i(D_\infty) = D_\infty \cap t^i K_\infty[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+$$

ne dépend donc pas de $n \geqslant m$. Si $y \in \mathrm{D}^+$ et $n \geqslant n(D) = m$, alors $\iota_n(y) \in \mathrm{Fil}^0(K_n(\!(t)\!) \otimes_{K_n} D_n)$ et si $n \geqslant -n(D)$, alors $\varphi^n(y) = \iota_{n(D)}(\varphi^{n+n(D)}(y))$ ce qui fait que pour tout $n \in \mathbb{Z}$, on a $\varphi^n(y) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$ et donc $\mathrm{D}^+ \subset \mathcal{M}^+(D)$. Il reste à constater que pour tout $n \geqslant n(D)$, on a :

$$K_n[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+ = K_n[\![t]\!] \otimes^{\iota_n}_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$$

ce qui fait que $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathrm{D}^+ = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$. □

Deux φ-modules filtrés sur K_0 avec action de Γ_K différents peuvent donner le même (φ, Γ_K)-module sur $\mathbf{B}^+_{\mathrm{rig},K}$. Par exemple, si on a moralement $D_1 = K_0$ et $D_2 = K_0 \cdot t$, alors $\mathcal{M}^+(D_1) = \mathcal{M}^+(D_2) = \mathbf{B}^+_{\mathrm{rig},K}$. Cet exemple est représentatif comme le montre la proposition ci-dessous :

Proposition 3.3.11. *Si D_1 et D_2 sont deux φ-modules filtrés sur K_0 avec action de Γ_K, alors $\mathcal{M}^+(D_1) = \mathcal{M}^+(D_2)$ si et seulement s'il existe un isomorphisme $K_0[t, t^{-1}] \otimes_{K_0} D_1 = K_0[t, t^{-1}] \otimes_{K_0} D_2$ compatible à φ et Γ_K, et compatible à la filtration quand on étend les scalaires à $K_\infty(\!(t)\!)$.*

Démonstration. La construction de $\mathcal{M}^+(D)$ montre que ce module ne dépend que de $\mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D$ et de la filtration sur $K_\infty(\!(t)\!) \otimes_{K_0} D$ ce qui fait que si les conditions de la proposition sont vérifiées, alors $\mathcal{M}^+(D_1) = \mathcal{M}^+(D_2)$. On voit réciproquement que si $\mathcal{M}^+(D_1) = \mathcal{M}^+(D_2)$, alors :

$$\mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D_1 = \mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D_2$$

et cet isomorphisme est compatible à la filtration quand on étend les scalaires à $K_\infty(\!(t)\!)$. Si G_1 et G_2 sont les matrices d'un élément $\gamma \in \Gamma_K$ qui n'est pas de torsion, et si M est la matrice de l'isomorphisme entre $\mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D_1$ et $\mathbf{B}^+_{\mathrm{rig},K}[1/t] \otimes_{K_0} D_2$ alors on a $\gamma(M)G_1 = G_2 M$, et si l'on écrit $M = \sum_{i \gg -\infty} t^i M_i$, alors on voit que l'on a $\chi(\gamma)^i M_i G_1 = G_2 M_i$ et donc que si $M_i \neq 0$, alors $\chi(\gamma)^i$ est quotient d'une valeur propre de G_1 par une valeur propre de G_2, ce qui n'est possible que pour un nombre fini de valeurs de i. On en déduit que M est à coefficients dans $K_0[t, t^{-1}]$ et donc que $K_0[t, t^{-1}] \otimes_{K_0} D_1 = K_0[t, t^{-1}] \otimes_{K_0} D_2$. □

Afin de terminer la démonstration du théorème D, il nous faut montrer que si $\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D_1) = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D_2)$, alors $\mathcal{M}^+(D_1) = \mathcal{M}^+(D_2)$. Cela suit de la proposition ci-dessous.

Proposition 3.3.12. *Si* $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$, *et si* D^+ *est un* (φ, Γ_K)*-module sur* $\mathbf{B}^+_{\mathrm{rig},K}$ *de rang fini et contenu dans* D, *alors* $\mathrm{D}^+ \subset \mathcal{M}^+(D)$.

Démonstration. Rappelons que $\mathrm{D}[1/t] = \mathbf{B}^\dagger_{\mathrm{rig},K}[1/t] \otimes_{K_0} D$; comme D^+ est de rang fini r, il existe $s > 0$ et $h \in \mathbb{Z}$ tels que $\mathrm{D}^+ \subset t^{-h}\mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \otimes_{K_0} D$. Choisissons des bases de D^+ et de D et appelons M, P^+ et P_0 la matrice de passage, la matrice de φ sur D^+ et la matrice de φ sur $t^{-h}D$ ce qui fait que $P_0\varphi(M) = MP^+$. Comme $P_0 \in \mathrm{GL}_d(K_0)$, le corollaire I.4.2 de [Berger 2004b] (qui s'étend verbatim aux matrices rectangulaires) nous donne que $M \in \mathrm{M}_{d\times r}(\mathbf{B}^+_{\mathrm{rig},K})$ et donc que $\mathrm{D}^+ \subset t^{-h}\mathbf{B}^+_{\mathrm{rig},K} \otimes_{K_0} D$.

Quitte à augmenter s, on a de plus que $\mathrm{D}^+ \subset \mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \otimes_{\mathbf{B}^+_{\mathrm{rig},K}} \mathcal{M}^+(D)$ et si $y \in \mathrm{D}^+$ et $n \geqslant n(s)$, alors $\iota_n(y) \in \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$ puisque $\iota_n(\mathbf{B}^{\dagger,s}_{\mathrm{rig},K}) \subset K_n[\![t]\!]$, et donc $\iota_n(\mathrm{D}^+) \subset \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$. Enfin, si $n \geqslant -n(s)$ et $y \in \mathrm{D}^+$, alors $\varphi^n(y) = \iota_{n(s)}(\varphi^{n(s)+n}(y))$ et donc $\varphi^n(\mathrm{D}^+) \subset \mathrm{Fil}^0(K_\infty(\!(t)\!) \otimes_{K_\infty} D_\infty)$ ce qui fait, étant donnée la définition de $\mathcal{M}^+(D)$ donnée dans la proposition 3.3.5, que $\mathrm{D}^+ \subset \mathcal{M}^+(D)$. □

En particulier, un (φ, Γ_K)-module sur $\mathbf{B}^+_{\mathrm{rig},K}$ de rang fini et contenu dans D est nécessairement de rang $\leqslant d$.

Annexe : Liste des notations

Voici une liste des principales notations dans l'ordre où elles apparaissent.

Introduction : k, K, K_0, G_K.

§1.1 : $\widetilde{\mathbf{E}}^+$, $\varepsilon^{(n)}$, $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}^+$, $\widetilde{\mathbf{B}}$, θ, $\mathbf{B}^+_{\mathrm{dR}}$, X, t, $\mathbf{B}_{\mathrm{dR}}$, $\tilde{p}$, $\mathbf{B}^+_{\max}$, $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, $\mathbf{B}_{\max}$, $\mathbf{B}_{\mathrm{st}}$, N, $\mathbf{B}_{\mathrm{cris}}$, $\widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$, K_n, H_K, Γ_K, $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$, K_0', $\mathbf{B}^{\dagger,r}_K$, $M_{a,h}$, $\mathbf{B}_{\mathrm{e}}$, t_h, $V_{[r;s]}$, $\widetilde{\mathbf{B}}^{[r;s]}$, $\mathbf{B}^{[r;s]}_K$, $\rho(r)$.

§1.2 : $\deg \mathrm{D}$, $\mathrm{NP}(\mathrm{D})$, r_n, $n(r)$, ι_n, Q_n.

§2 : W_e, W_{dR}^+, W_{dR}.

§2.2 : $\widetilde{\mathrm{D}}^r(W)$, $W(\mathrm{D})$.

§2.3 : $\mathrm{D}_{\mathrm{cris}}(W)$, $\mathrm{D}_{\mathrm{st}}(W)$, $\mathrm{D}_{\mathrm{dR}}(W)$, $\mathcal{M}(D)$, $\mathrm{D}_{\mathrm{Sen}}(W)$.

§3.2 : $\mathrm{Rep}(a, h)$.

§3.3 : $\mathbf{B}_{\mathrm{rig},K}^+$, $\mathbf{B}_K^+$, $\mathcal{M}^+(D)$, ∇.

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References

[Berger 2002] L. Berger, "Représentations p-adiques et équations différentielles", *Invent. Math.* **148**:2 (2002), 219–284. MR 2004a:14022 Zbl 01777245

[Berger 2004a] L. Berger, "Equations différentielles p-adiques et (φ, N)-modules filtrés", prépublication, 2004. À paraître dans *Astérisque*. arXiv math/0406601

[Berger 2004b] L. Berger, "Limites de représentations cristallines", *Compos. Math.* **140**:6 (2004), 1473–1498. MR 2006c:11138 Zbl 1071.11067

[Berger et Breuil 2006] L. Berger et C. Breuil, "Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$", prépublication, 2006. À paraître dans *Astérisque*. arXiv math/0601545

[Berger et Colmez 2007] L. Berger et P. Colmez, "Familles de représentations de de Rham et monodromie p-adique", 2007, http://www.umpa.ens-lyon.fr/~lberger/publications.html. À paraître dans *Astérisque*.

[Cherbonnier et Colmez 1998] F. Cherbonnier et P. Colmez, "Représentations p-adiques surconvergentes", *Invent. Math.* **133**:3 (1998), 581–611. MR 2000d:11146 Zbl 0928.11051

[Colmez 1999] P. Colmez, "Représentations cristallines et représentations de hauteur finie", *J. Reine Angew. Math.* **514** (1999), 119–143. MR 2001h:11147 Zbl 01343282

[Colmez 2002] P. Colmez, "Espaces de Banach de dimension finie", *J. Inst. Math. Jussieu* **1**:3 (2002), 331–439. MR 2004b:11160 Zbl 1044.11102

[Colmez 2003a] P. Colmez, "Espaces vectoriels de dimension finie et représentations de de Rham", 2003, www.institut.math.jussieu.fr/~colmez/publications.html. À paraître dans *Astérisque*.

[Colmez 2003b] P. Colmez, "Les conjectures de monodromie p-adiques", pp. 53–101, exposé 897 dans *Séminaire Bourbaki année 2001/2002*, Astérisque **290**, Soc. Math. de France, Paris, 2003. MR 2005f:12008 Zbl 02134851

[Colmez 2005] P. Colmez, "Série principale unitaire pour $GL_2(\mathbb{Q}_p)$ et représentations triangulines de dimension 2", 2005, http://www.institut.math.jussieu.fr/~colmez/publications.html.

[Colmez 2007] P. Colmez, "Représentations triangulines de dimension 2", prépublication, 2007, http://www.institut.math.jussieu.fr/~colmez/publications.html. À paraître dans *Astérisque*.

[Colmez et Fontaine 2000] P. Colmez et J.-M. Fontaine, "Construction des représentations p-adiques semi-stables", *Invent. Math.* **140**:1 (2000), 1–43. MR 2001g:11184 Zbl 1010.14004

[Fontaine 1994a] J.-M. Fontaine, "Le corps des périodes p-adiques", pp. 59–111 dans *Périodes p-adiques* (Bures-sur-Yvette, 1988), édité par J.-M. Fontaine, Astérisque **223**, Soc. Math. de France, Paris, 1994. MR 95k:11086 Zbl 0940.14012

[Fontaine 1994b] J.-M. Fontaine, "Représentations p-adiques semi-stables", pp. 113–184 dans *Périodes p-adiques* (Bures-sur-Yvette, 1988), édité par J.-M. Fontaine, Astérisque **223**, Soc. Math. de France, Paris, 1994. MR 95g:14024 Zbl 0865.14009

[Fontaine 2003] J.-M. Fontaine, *Presque $\mathbb{C}_p$-représentations*, édité par S. Bloch et al., Documenta Mathematica, Bielefeld, 2003. MR 2046593 Zbl 02028839

[Fontaine 2004] J.-M. Fontaine, "Arithmétique des représentations galoisiennes p-adiques", pp. xi, 1–115 dans *Cohomologies p-adiques et applications arithmétiques, III*, édité par P. Berthelot et al., Astérisque **295**, Soc. Math. de France, Paris, 2004. MR 2005i:11074 Zbl 1052.00008

[Fontaine et Wintenberger 1979] J.-M. Fontaine et J.-P. Wintenberger, "Le "corps des normes" de certaines extensions algébriques de corps locaux", *C. R. Acad. Sci. Paris Sér. A-B* **288**:6 (1979), A367–A370. MR 80b:12015 Zbl 0475.12020

[Kedlaya 2004] K. S. Kedlaya, "A p-adic local monodromy theorem", *Ann. of Math.* (2) **160**:1 (2004), 93–184. MR 2005k:14038 Zbl 1088.14005

[Kedlaya 2005] K. S. Kedlaya, "Slope filtrations revisited", *Doc. Math.* **10** (2005), 447–525. MR 2007c:14016 Zbl 1081.14028

[Kedlaya 2006] K. Kedlaya, "Slope filtrations for relative Frobenius", 2006. À paraître dans *Astérisque*. arXiv math/0609272

[Kisin 2006] M. Kisin, "Crystalline representations and F-crystals", pp. 459–496 dans *Algebraic geometry and number theory*, édité par V. Ginzburg, Progr. Math. **253**, Birkhäuser, Boston, 2006. MR 2007j:11163

[Liu 2007] R. Liu, "Cohomology and duality for (φ, Γ)-modules over the Robba ring", prépublication, 2007.

[Perrin-Riou 1994] B. Perrin-Riou, "Théorie d'Iwasawa des représentations p-adiques sur un corps local", *Invent. Math.* **115**:1 (1994), 81–161. MR 95c:11082 Zbl 0838.11071

[Sen 1980] S. Sen, "Continuous cohomology and p-adic Galois representations", *Invent. Math.* **62**:1 (1980), 89–116. MR 82e:12018 Zbl 0463.12005

[Wach 1996] N. Wach, "Représentations p-adiques potentiellement cristallines", *Bull. Soc. Math. France* **124**:3 (1996), 375–400. MR 98b:11119 Zbl 0887.11048

[Wintenberger 1983] J.-P. Wintenberger, "Le corps des normes de certaines extensions infinies de corps locaux; applications", *Ann. Sci. École Norm. Sup.* (4) **16**:1 (1983), 59–89. MR 85e:11098 Zbl 0516.12015

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