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Piecewise polynomials, Minkowski weights, and localization on toric varieties

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We use localization to describe the restriction map from equivariant Chow cohomology to ordinary Chow cohomology for complete toric varieties in terms of piecewise polynomial functions and Minkowski weights. We compute examples showing that this map is not surjective in general, and that its kernel is not always generated in degree one. We prove a localization formula for mixed volumes of lattice polytopes and, more generally, a Bott residue formula for toric vector bundles.

1. Introduction

Let Δ be a complete fan in $N_{\mathbb{R}}$, where N is a lattice of rank n, and let $X = X(\Delta)$ be the corresponding complete n-dimensional toric variety. See [Fulton 1993] for standard notation and general background on toric varieties. The equivariant operational Chow cohomology ring with integer coefficients $A^*_T(X)$ is naturally isomorphic to the ring of integral piecewise polynomial functions on Δ [Payne 2006a], and there is a canonical map to ordinary Chow cohomology with integer coefficients

$$\iota^* : A^*_T(X) \to A^*(X)$$

induced by inclusions of X in the finite dimensional approximations of the Borel mixed space [Edidin and Graham 1998a]. Now $A^*(X)$ is naturally isomorphic to the ring of Minkowski weights on Δ [Fulton and Sturmfels 1997], and ι^* has a natural interpretation in terms of localization and equivariant multiplicities, as follows.

Let $M = \mathrm{Hom}(N, \mathbb{Z})$, which is naturally identified with the character lattice of T, and let $\mathrm{Sym}^{\pm}(M)$ be the $\mathbb{Z}$-graded ring obtained by inverting all of the homogeneous elements in the ring $\mathrm{Sym}^*(M)$ of polynomials with integer coefficients. We refer to elements of $\mathrm{Sym}^{\pm}(M)$ as rational functions, and elements of

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the subring $\mathrm{Sym}^*(M)$ as polynomials. Each maximal cone $\sigma \in \Delta$ corresponds to a nondegenerate torus fixed point $x_\sigma \in X$, which has an "equivariant multiplicity" $e_{x_\sigma}[X] \in \mathrm{Sym}^{\pm}(M)$, which is a homogeneous rational function of degree $-n$. Since every rational polyhedral cone admits a unimodular subdivision, these equivariant multiplicities are determined by the following two properties.

(1) If $\sigma_1, \ldots, \sigma_r$ are the maximal cones of a rational polyhedral subdivision of a cone σ, then
$$e_{x_\sigma}[X] = e_{x_{\sigma_1}}[X] + \cdots + e_{x_{\sigma_r}}[X].$$

(2) If σ is a unimodular cone, spanned by a basis $e_1, \ldots, e_n$ for N, then
$$e_{x_\sigma}[X] = \frac{1}{e_1^* \cdots e_n^*}\,.$$

The fact that the sum of rational functions determined by (1) and (2) is independent of the choice of unimodular subdivision is not obvious from elementary considerations, though it follows directly from the theory of localization at torus fixed points in algebraic geometry [Edidin and Graham 1998b] and the theory of equivariant multiplicities developed by Rossmann [1989] and Brion [1997, Theorem 4.2 and Proposition 4.3, in particular]. Here we give a combinatorial proof of this independence; the techniques of this proof may be of independent interest. We view the multigraded Hilbert function $\mathrm{Hilb}(\sigma)$ of the affine toric variety U_σ, given by
$$\mathrm{Hilb}(\sigma) = \sum_{u \in (\sigma^* \cap M)} x^u\,,$$
as a rational function on the dense torus $T \subset X$. We define e_σ to be $(-1)^n$ times the quotient of the leading forms when $\mathrm{Hilb}(\sigma)$ is written as a quotient of two polynomials in local coordinates at the identity 1_T. We then show that e_σ satisfies properties analogous to (1) and (2) and therefore is equal to $e_{x_\sigma}[X]$. See Section 2 for details. Our approach is inspired by the presentation of multidegrees of multigraded modules over polynomial rings in [Knutson and Miller 2005, Sections 1.2 and 1.7] and [Miller and Sturmfels 2005, Chapter 8].

Recall that the ring of integral piecewise polynomial functions $PP^*(\Delta)$ is the ring of continuous functions $f : |\Delta| \to \mathbb{R}$ such that the restriction f_σ of f to each maximal cone $\sigma \in \Delta$ is a polynomial in $\mathrm{Sym}^*(M)$.

Proposition 1.1. *Let Δ be a complete n-dimensional fan, and let $f \in PP^k(\Delta)$ be a piecewise polynomial function. Then*
$$\sum_\sigma e_\sigma f_\sigma$$
is a homogeneous polynomial in $\mathrm{Sym}^*(M)$ *of degree $k - n$.*

In particular, if the degree of f is less than n then $\sum e_\sigma f_\sigma$ vanishes. If $\deg f = n$, then $\sum e_\sigma f_\sigma = d$ is an integer, which may be identified with the codimension n Minkowski weight $c(0) = d$ on Δ.

Minkowski weights of codimension less than n may be constructed similarly from piecewise polynomials using equivariant multiplicities, as follows. For any cone $\tau \in \Delta$, let Δ_τ be the fan in $(N/(N \cap \operatorname{span} \tau))_{\mathbb{R}}$ whose cones are the projections of the cones in Δ that contain τ. If σ is a maximal such cone, we define $e_{\sigma,\tau}$ to be $e_{\bar{\sigma}}$, where $\bar{\sigma}$ is the image of σ in Δ_τ. So $e_{\sigma,\tau}$ is a homogeneous rational function of degree $(\dim \tau - n)$ in the graded subring $\operatorname{Sym}^{\pm}(\tau^\perp \cap M)$ of $\operatorname{Sym}^{\pm}(M)$.

Proposition 1.2. *Let Δ be a complete fan, and let $f \in PP^k(\Delta)$ be a piecewise polynomial function. Then, for any $\tau \in \Delta$,*

$$c(\tau) = \sum_{\sigma \succeq \tau} e_{\sigma,\tau} f_\sigma$$

is a homogeneous polynomial in $\operatorname{Sym}^*(M)$ *of degree* $k + \dim \tau - n$.

If $k \leq n$ then $c(\tau)$ is an integer for every codimension k cone in Δ, and these integers are a Minkowski weight of codimension k. Propositions 1.1 and 1.2 are proved in Section 3 using elementary properties of generating functions for lattice points in polyhedral cones.

Remark 1.3. Proposition 1.1 is the special case of Proposition 1.2 where $\tau = 0$. The essential content of Propositions 1.1 and 1.2 is that the denominator of the sum must divide the numerator. In some special cases, this divisibility may be seen as a consequence of Brion's Formula, and its generalizations, in the theory of generating functions for lattice points in polyhedra. See Section 5 below. Other special cases of these cancellations appeared earlier in [Brion 1996]; in particular, Brion showed that $\sum e_{\sigma,\tau} f_\sigma$ is in $\operatorname{Sym}^*(M_{\mathbb{Q}})$ when Δ is simplicial.

Theorem 1.4. *The natural map $\iota^* : A^k_T(X) \to A^k(X)$ takes the equivariant Chow cohomology class corresponding to a piecewise polynomial function f to the ordinary Chow cohomology class corresponding to the Minkowski weight c given by*

$$c(\tau) = \sum_{\sigma \succeq \tau} e_{\sigma,\tau} f_\sigma,$$

for all codimension k cones $\tau \in \Delta$.

We prove Theorem 1.4 in Section 3 by interpreting Propositions 1.1 and 1.2 in terms of general localization formulas in equivariant Chow cohomology [Edidin and Graham 1998b].

We apply Theorem 1.4 to study the map $\iota^* : A_T^*(X) \to A^*(X)$. Recall that if X is smooth then capping with the fundamental class of X gives isomorphisms

$$A^*(X) \cong A_{n-*}(X) \quad \text{and} \quad A_T^*(X) \cong A_{n-*}^T(X).$$

Furthermore, the globally linear functions $u \in M$, identified with the equivariant first Chern classes of the toric line bundles $\mathcal{O}(\operatorname{div} \chi^u)$, act on $A_*^T(X)$ as homogeneous operators of degree -1, and there is a natural isomorphism to ordinary Chow homology [Brion 1997, Section 2.3],

$$A_*^T(X)/MA_*^T(X) \cong A_*(X).$$

It follows that if X is a smooth toric variety then ι^* is surjective and its kernel is generated by M in degree one. Similar arguments show that if X is simplicial, then ι^* becomes surjective after tensoring with $\mathbb{Q}$, with kernel generated by $M_{\mathbb{Q}}$ in degree one.

Theorem 1.5. *There exist projective toric surfaces X such that $\iota^* : A_T^2(X) \to A^2(X)$ is not surjective.*

In particular, even when the natural map $PP^*(\Delta)/(M) \to A^*(X)$ becomes an isomorphism after tensoring with $\mathbb{Q}$, it need not be an isomorphism over $\mathbb{Z}$.

Theorem 1.6. *There exist projective toric threefolds X such that $\iota^* : A_T^*(X)_{\mathbb{Q}} \to A^*(X)_{\mathbb{Q}}$ is not surjective and its kernel is not generated in degree one.*

It follows that the natural map from piecewise polynomials modulo linear functions to Minkowski weights is neither injective nor surjective in general. We prove Theorems 1.5 and 1.6 in Section 4 by computing the maps $A_T^*(X) \to A^*(X)$ for several examples of singular toric varieties using Theorem 1.4.

Remark 1.7. Minkowski weights on Δ, and classes in $A^*(X)$, correspond to tropical varieties supported on the cones of Δ, and are of significant interest in tropical geometry [Katz 2007, Section 9; Mikhalkin 2006, p. 10]. The desire to use piecewise polynomials to produce interesting examples of Minkowski weights was one of the main motivations for this research. We hope and expect that the combinatorial localization techniques developed here will be useful in tropical geometry.

2. Combinatorics of equivariant multiplicities

Let N be a lattice of rank n, and let $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice. Let $\operatorname{Poly}(N)$ denote the rational polytope algebra on $N_{\mathbb{R}}$, the subring of real-valued functions on $N_{\mathbb{R}}$ generated by the characteristic functions of closed rational polyhedra. We write $[Q] \in \operatorname{Poly}(N)$ for the characteristic function of a closed polyhedron Q. Recall that Q has a polar dual Q^*, which is a closed polyhedron in $M_{\mathbb{R}}$, defined by

$$Q^* = \{\, u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in Q \,\},$$

and there is a linear map from $\mathrm{Poly}(N)$ to $\mathrm{Poly}(M)$ given by $[Q] \mapsto [Q^*]$ [Lawrence 1988]. Furthermore, there is a linear map

$$\nu : \mathrm{Poly}(M) \to \mathbb{Q}(M),$$

to the quotient field $\mathbb{Q}(M)$ of the multivariate Laurent polynomial ring $\mathbb{Z}[M]$, that takes the class of a closed, pointed polyhedron P to the generating function $\sum_{u \in (P \cap M)} x^u$, expressed as a rational function, and takes the class of a polyhedron containing a line to 0 [Barvinok 2002, Theorem VIII.3.3]. In particular, for any closed polyhedral cone σ in $N_{\mathbb{R}}$, $\nu(\sigma^*) = \mathrm{Hilb}(\sigma)$, where

$$\mathrm{Hilb}(\sigma) = \sum_{u \in (\sigma^* \cap M)} x^u$$

is the multigraded Hilbert series of the affine toric variety U_σ. Composing polar duality with the valuation ν then gives a linear map

$$\nu^* : \mathrm{Poly}(N) \to \mathbb{Q}(M)$$

that takes $[\sigma]$ to $\mathrm{Hilb}(\sigma)$.

Lemma 2.1. *If $\sigma_1, \dots, \sigma_r$ are the maximal cones in a rational polyhedral subdivision of an n-dimensional cone σ, then*

$$\mathrm{Hilb}(\sigma) = \mathrm{Hilb}(\sigma_1) + \cdots + \mathrm{Hilb}(\sigma_r).$$

Proof. In the polytope algebra $\mathrm{Poly}(N)$,

$$[\sigma] = [\sigma_1] + \cdots + [\sigma_r] \pm \text{ classes of lower dimensional cones.}$$

Since the duals of lower dimensional cones contain lines, these terms are all in the kernel of ν^*. Therefore, $\nu^*([\sigma]) = \nu^*([\sigma_1]) + \cdots + \nu^*([\sigma_r])$, and the lemma follows. □

The generating function $\mathrm{Hilb}(\sigma)$, being an element of $\mathbb{Q}(M)$, is naturally interpreted as a rational function on the torus $T = \mathrm{Spec}\, \mathbb{Q}[M]$. Therefore, $\mathrm{Hilb}(\sigma)$ may be expanded as a quotient of two power series in local parameters at the identity 1_T. The principal part of this expansion, the quotient of the leading forms, which we denote by

$$\mathrm{Hilb}(\sigma)_\circ \in \mathrm{Sym}^{\pm}(M_{\mathbb{Q}}),$$

is a rational function on the tangent space of T at 1_T, which cuts out the tangent cone of zeros of $\mathrm{Hilb}(\sigma)$ minus the tangent cone of its poles. See the Appendix for details on principal parts of rational functions.

Definition 2.2. If σ is an n-dimensional rational polyhedral cone in $N_{\mathbb{R}}$ then

$$e_\sigma = (-1)^n \cdot \mathrm{Hilb}(\sigma)_\circ.$$

Lemma 2.3. *If σ is an n-dimensional rational polyhedral cone in $N_{\mathbb{R}}$, then e_σ is homogeneous of degree $-n$.*

Proof. The lemma follows directly from closed formulas for polyhedral generating functions, such as those given in [Payne 2007], as follows. Suppose Σ is a unimodular subdivision of σ^*, and $u_1, \dots, u_s$ are the primitive generators of the rays of Σ. Then every lattice point in σ^* lies in the relative interior of a unique cone $\tau \in \Sigma$, and the generating function for those in the relative interior of τ is $\prod_{u_i \in \tau} x^{u_i}/(1-x^{u_i})$. Therefore,

$$(1-x^{u_1})\cdots(1-x^{u_s})\cdot \mathrm{Hilb}(\sigma) = \sum_{\tau\in\Sigma}\Bigl(\prod_{u_i\in\tau} x^{u_i}\prod_{u_j\notin\tau}(1-x^{u_j})\Bigr).$$

If $\tau_1, \dots, \tau_r$ are the maximal cones of Σ, then taking leading forms at 1_T on both sides gives

$$u_1\cdots u_s\cdot(-1)^n\cdot\mathrm{Hilb}(\sigma)_\circ = \sum_{i=1}^{r}\prod_{u_j\notin\tau_i} u_j,$$

provided that the right hand side is nonvanishing. Since all of the u_j lie in σ^*, the right hand side is strictly positive on the interior of σ. In particular, it does not vanish, so the degree of $\mathrm{Hilb}(\sigma)_\circ$ is $-n$. □

Lemma 2.3 can also be seen as a special case of more general results on multigraded Hilbert series of modules. See [Miller and Sturmfels 2005, Definition 8.45 and Claim 8.54].

Lemma 2.4. *Let σ be a unimodular cone, spanned by a basis $e_1, \dots, e_n$ for N. Then the principal part of* $\mathrm{Hilb}(\sigma)$ *at 1_T is*

$$\mathrm{Hilb}(\sigma)_\circ = \frac{(-1)^n}{e_1^*\cdots e_n^*},$$

where $e_1^*, \dots, e_n^*$ is the dual basis for M.

Proof. The generating function $\mathrm{Hilb}(\sigma)$ is given by

$$\mathrm{Hilb}(\sigma) = \frac{1}{(1-x^{e_1^*})\cdots(1-x^{e_n^*})}.$$

Now, $(1-x^{e_i^*})$ is a local parameter at 1_T, with principal part $(1-x^{e_i^*})_\circ = -e_i^*$. Since principal parts are multiplicative, it follows that the principal part of $1/(1-x^{e_i^*})$ is $-1/e_i^*$, and the lemma follows. □

Proposition 2.5. *Let σ be an n-dimensional rational polyhedral cone in $N_{\mathbb{R}}$.*

(1) *If $\sigma_1, \dots, \sigma_r$ are the maximal cones in a rational polyhedral subdivision of σ, then*

$$e_\sigma = e_{\sigma_1} + \cdots + e_{\sigma_r}.$$

(2) *If σ is unimodular, spanned by a basis $e_1, \ldots, e_n$ for N, then*

$$e_\sigma = \frac{1}{e_1^* \cdots e_n^*}.$$

In particular, the sum determined by (1) and (2) is independent of the choice of unimodular subdivision.

Proof. Part (1) follows from the additivity of $\mathrm{Hilb}(\sigma_i)$ (Lemma 2.1) and the fact that $\mathrm{Hilb}(\sigma)$ and the $\mathrm{Hilb}(\sigma_i)$ all have principal parts in degree $-n$ (Lemma 2.3). See Proposition A.1, in the Appendix. Part (2) is an immediate consequence of Lemma 2.4. □

Recall that for any cone $\tau \in \Delta$, Δ_τ is the fan in $(N/(N \cap \operatorname{span} \tau))_{\mathbb{R}}$ whose cones are the projections of the cones in Δ that contain τ. If σ is a maximal cone containing τ, we define $e_{\sigma,\tau}$ to be $e_{\overline{\sigma}}$, where $\overline{\sigma}$ is the image of σ in Δ_τ. So $e_{\sigma,\tau}$ is a homogeneous rational function of degree $(\dim \tau - n)$ in the graded subring $\mathrm{Sym}^{\pm}(\tau^{\perp} \cap M)$ of $\mathrm{Sym}^{\pm}(M)$. We write $V(\tau)$ for the T-invariant subvariety of X corresponding to τ.

Corollary 2.6. *If σ is an n-dimensional rational polyhedral cone in $N_{\mathbb{R}}$ and τ is a face of σ, then*

$$e_\sigma = e_{x_\sigma}[X] \ \text{ and } \ e_{\sigma,\tau} = e_{x_\sigma}[V(\tau)].$$

Lemma 2.7. *If σ is a unimodular cone spanned by a basis $e_1, \ldots, e_n$ for N and $\tau \preceq \sigma$ then*

$$e_{\sigma,\tau} = \prod_{e_i \notin \tau} \frac{1}{e_i^*}.$$

Proof. Apply part (2) of Proposition 2.5 to the fan Δ_τ. □

3. Localization and Minkowski weights

Here we use equivariant multiplicities to describe the natural map from piecewise polynomials on a complete fan to Minkowski weights. We then use localization to show that this map agrees with $\iota^* : A_T^*(X) \to A^*(X)$.

Lemma 3.1. *Let Δ be a complete n-dimensional fan. Then the sum of the rational functions e_σ for all maximal cones $\sigma \in \Delta$ is given by*

$$\sum_\sigma e_\sigma = \begin{cases} 0 & \text{for } n \geq 1, \\ 1 & \text{for } n = 0. \end{cases}$$

Proof. If $n = 0$, then Δ contains only one cone 0, and $e_0 = 1$. Suppose $n \geq 1$. In the polytope algebra ,

$$\sum_{\sigma} [\sigma] = [N_{\mathbb{R}}] \pm \text{ classes of smaller dimensional cones.}$$

Applying the linear transformation ν^* gives

$$\sum_{\sigma} \mathrm{Hilb}(\sigma) = 1.$$

Since each of the principal parts $\mathrm{Hilb}(\sigma)_\circ = (-1)^n \cdot e_\sigma$ is homogeneous of degree $-n$, it follows that the sum of these principal parts must vanish by Proposition A.1 in the Appendix, and the lemma follows. $\square$

Lemma 3.2. *Let τ be a cone in a complete n-dimensional fan Δ. Then*

$$\sum_{\sigma \succeq \tau} e_{\sigma,\tau} = \begin{cases} 0 & \textit{for } \dim \tau < n, \\ 1 & \textit{for } \dim \tau = n. \end{cases}$$

Proof. Apply Lemma 3.1 to the fan Δ_τ. $\square$

Piecewise polynomials are especially well-behaved on unimodular fans, that is, fans in which each maximal cone is spanned by a basis for the lattice. Suppose Δ is a unimodular fan, and $\rho_1, \ldots, \rho_s$ are the rays of Δ. Let v_i be the primitive generator of ρ_i. Then there is a unique piecewise linear function $\Psi_i \in PP^1(\Delta)$ whose values at the primitive generators of the rays are given by the Kronecker delta function

$$\Psi_i(v_j) = \delta_{ij},$$

and whose values elsewhere are given by extending linearly on each cone.

Then, for any k-dimensional cone $\tau \in \Delta$, we have a piecewise polynomial $\Psi_\tau \in PP^k(\Delta)$ that vanishes away from $\mathrm{Star}(\tau)$, the union of the cones in Δ that contain τ, defined by

$$\Psi_\tau = \prod_{v_i \in \tau} \Psi_i,$$

and $PP^*(\Delta)$ is generated by $\{\Psi_\tau\}_{\tau \in \Delta}$ as a $\mathrm{Sym}^*(M)$-module.

Proof of Proposition 1.1. Since equivariant multiplicities are additive with respect to subdivisions, we may assume that Δ is unimodular. Say $\rho_1, \ldots, \rho_s$ are the rays of Δ and v_i is the primitive generator of ρ_i. Since $PP^*(\Delta)$ is generated as a $\mathrm{Sym}^*(M)$-module by the piecewise polynomials Ψ_τ, it suffices to prove that

$$\sum e_\sigma \cdot (\Psi_\tau)_\sigma$$

is in $\mathrm{Sym}^*(M)$ for all τ. Now, if σ is spanned by a basis $e_1, \dots, e_n$ for N and $\tau \preceq \sigma$, then $(\Psi_\tau)_\sigma = \prod_{v_i \in \tau} e_i^*$. It then follows from Lemma 2.7 that

$$e_\sigma \cdot (\Psi_\tau)_\sigma = \begin{cases} e_{\sigma,\tau} & \text{for } \sigma \succeq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 3.2, $\sum e_\sigma \cdot (\Psi_\tau)_\sigma$ vanishes unless τ is a maximal cone, in which case the sum is equal to one. In particular, $\sum e_\sigma \cdot (\Psi_\tau)_\sigma$ is in $\mathrm{Sym}^*(M)$, as required. □

Proof of Proposition 1.2. The sum in Proposition 1.2 is over the maximal cones in $\mathrm{Star}(\tau)$. Then the proof of Proposition 1.2 is similar to the proof of Proposition 1.1, since $PP^*(\mathrm{Star}(\tau))$ is generated as a $\mathrm{Sym}^*(M)$-module by the restrictions of the piecewise polynomial functions Ψ_γ, for $\gamma \in \mathrm{Star}(\tau)$. □

It remains to show that if f is a homogeneous piecewise polynomial of degree k, then the integer-valued function c on codimension k cones of Δ given by

$$c(\tau) = \sum_{\sigma \succeq \tau} e_{\sigma,\tau} f_\sigma$$

is a Minkowski weight of codimension k, and that $f \mapsto c$ agrees with the natural map $\iota^* : A_T^*(X) \to A^*(X)$. Although the entire statement can be proved using the general machinery of localization, the fact that the integers $c(\tau)$ give a Minkowski weight is purely combinatorial, and we include an elementary proof.

We recall the definition of Minkowski weights from [Fulton and Sturmfels 1997]. If γ is a codimension $k+1$ cone in Δ contained in a codimension k cone τ, we write $v_{\tau/\gamma} \in N/(N \cap \mathrm{span}\,\gamma)$ for the primitive generator of the image of τ in Δ_γ.

Definition 3.3. An integer valued function c on codimension k cones $\tau \in \Delta$ is a Minkowski weight if, for every codimension $k+1$ cone $\gamma \in \Delta$,

$$\sum_{\tau \succeq \gamma} c(\tau) \cdot v_{\tau/\gamma} = 0.$$

We will use the following basic property of equivariant multiplicities to show that the integer-valued function c coming from a piecewise polynomial function is a Minkowski weight. Let v_ρ denote the primitive generator of a ray ρ.

Proposition 3.4. *If σ is an n-dimensional rational polyhedral cone in $N_\mathbb{R}$ and u is in M then*

$$\sum_{\rho \preceq \sigma} \langle u, v_\rho \rangle \cdot e_{\sigma,\rho} = u \cdot e_\sigma.$$

We will prove the proposition by subdividing σ and reducing to the case where σ is unimodular.

Lemma 3.5. *If σ is an n-dimensional unimodular cone in $N_{\mathbb{R}}$ and u is in M then*

$$\sum_{\rho \preceq \sigma} \langle u, v_\rho \rangle \cdot e_{\sigma,\rho} = u \cdot e_\sigma .$$

Proof. Say σ is spanned by a basis $e_1, \ldots, e_n$ for N, and $u = u_1 e_1^* + \cdots u_n e_n^*$. Then

$$\sum \langle u, v_\rho \rangle \cdot e_{\sigma,\rho} = \sum_{i=1}^{n} \frac{u_i}{e_1^* \cdots \widehat{e_i^*} \cdots e_n^*},$$

which is equal to $u \cdot e_\sigma$. □

Lemma 3.6. *If $\sigma_1, \ldots, \sigma_s$ are the maximal cones in a subdivision of σ, and if ρ is a ray in this subdivision then*

$$\sum_{\sigma_i \succeq \rho} e_{\sigma_i,\rho} = \begin{cases} e_{\sigma,\rho} & \textit{if } \rho \preceq \sigma, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose ρ lies in the relative interior of a k-dimensional face $\tau \preceq \sigma$. Consider the fan Δ_ρ, whose maximal cones are the images of the $\sigma_i \succeq \rho$. The support $|\Delta_\rho|$ is a closed polyhedral cone in an $(n-1)$-dimensional vector space whose minimal face is $(k-1)$-dimensional, so the polar dual $|\Delta_\rho|^*$ has dimension $n-k$. It follows that the principal part of $\nu^*(|\Delta_\rho|)$ has degree $k-n$. Since each $e_{\sigma,\rho}$ has degree $1-n$, and $\sum e_{\sigma_i,\rho}$ is the principal part of $\pm\nu^*(|\Delta_\rho|)$ unless this sum vanishes (Appendix, Proposition A.1), the lemma follows. □

Proof of Proposition 3.4. Let $\sigma_1, \ldots, \sigma_r$ be the maximal cones of a unimodular subdivision of σ. Then $u \cdot e_\sigma = u \cdot e_{\sigma_1} + \cdots + u \cdot e_{\sigma_r}$. Since σ_i is unimodular,

$$u \cdot e_{\sigma_i} = \sum_{\rho \preceq \sigma_i} \langle u, v_\rho \rangle e_{\sigma_i,\rho} .$$

Therefore, by rearranging terms in the summation, we have

$$u \cdot e_\sigma = \sum_\rho \Big(\sum_{\sigma_i \succeq \rho} \langle u, v_\rho \rangle \cdot e_{\sigma_i,\rho} \Big).$$

By Lemma 3.6, the right hand side is equal to $\sum_{\rho \preceq \sigma} e_{\sigma,\rho}$, as required. □

Proposition 3.7. *Let $f \in PP^k(\Delta)$ be a homogeneous piecewise polynomial of degree k. Then the integers*

$$c(\tau) = \sum_{\sigma \succeq \tau} e_{\sigma,\tau} f_\sigma$$

are a Minkowski weight of codimension k on Δ.

Proof. Let γ be a codimension $k+1$ cone in Δ. It will suffice to show that $\sum \langle u, v_{\tau/\gamma}\rangle c(\tau) = 0$ for any $u \in (M \cap \gamma^{\perp})$, where the sum is over all codimension k cones τ containing γ. To prove this, we will show that $\sum \langle u, v_{\tau/\gamma}\rangle c(\tau)$, which is an integer by Proposition 1.2, is divisible by the linear function u in $\mathrm{Sym}^*(M)$. Now,

$$\sum_{\tau} \langle u, v_{\tau/\gamma}\rangle\, c(\tau) = \sum_{\tau} \Big(\sum_{\sigma \succeq \tau} \langle u, v_{\tau/\gamma}\rangle\, e_{\sigma,\tau}\, f_{\sigma} \Big),$$

and the sum on the right hand side may be rearranged as

$$\sum_{\sigma} \Big(f_{\sigma} \cdot \sum_{\tau \preceq \sigma} \langle u, v_{\tau/\gamma}\rangle\, e_{\sigma,\tau} \Big).$$

Applying Proposition 3.4 to Δ_{γ} then gives

$$\sum_{\tau \preceq \sigma} \langle u, v_{\tau/\gamma}\rangle\, e_{\sigma,\tau} = u \cdot e_{\sigma,\gamma}.$$

It follows that the integer $\sum \langle u, v_{\tau/\gamma}\rangle c(\tau)$ is divisible by u in $\mathrm{Sym}^*(M)$, as claimed, and hence must vanish. □

Proof of Theorem 1.4. To show that $f \mapsto c$ agrees with the natural map $\iota^* : A_T^*(X) \to A^*(X)$, we must prove that

$$\int_{V(\tau)} \iota^* c_f = c(\tau),$$

where c_f denotes the equivariant Chow cohomology class whose restriction to a torus fixed point is $f_{\sigma} \in \mathrm{Sym}^*(M) \cong A_T^*(x_{\sigma})$. By Corollary 2.6, $e_{\sigma,\tau}$ is equal to the equivariant multiplicity $e_{x_{\sigma}}[V(\tau)]$ of the nondegenerate T-fixed point x_{σ} in $V(\tau)$. Therefore, by localization [Edidin and Graham 1998b], in equivariant Chow homology tensored with $\mathrm{Sym}^{\pm}(M)$, we have

$$\int_{V(\tau)} c_f = \sum_{\sigma} e_{\sigma}\, f_{\sigma}.$$

Since $\sum_{\sigma} e_{\sigma} f_{\sigma}$ is an integer, projecting to $A_*(X)$ gives $\int_{V(\tau)} \iota^* c_f = c(\tau)$. □

4. Applications to Chow cohomology of toric varieties

Here we use combinatorial computations with piecewise polynomials to study the map $\iota^* : A_T^*(X) \to A^*(X)$ for some specific complete toric varieties X. As discussed in the introduction, this map is known to be surjective with kernel generated by M in degree one if X is smooth, and similar statements hold over $\mathbb{Q}$ if X is simplicial. We give the first examples showing that ι^* is not surjective in general, and that its kernel is not always generated in degree one.

Example 4.1 (Mirror dual of $\mathbb{P}^1 \times \mathbb{P}^1$). Let $N = \mathbb{Z}^2$, and let Δ be the complete fan in $\mathbb{R}^2$ whose rays are generated by

$$v_1 = (1, 1), \;\; v_2 = (1, -1), \;\; v_3 = (-1, -1), \;\; v_4 = (-1, 1),$$

and whose maximal cones are

$$\sigma_1 = \langle v_1, v_2 \rangle, \;\; \sigma_2 = \langle v_2, v_3 \rangle, \;\; \sigma_3 = \langle v_3, v_4 \rangle, \;\; \sigma_4 = \langle v_1, v_4 \rangle.$$

Then $X = X(\Delta)$ is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$, which is the Fano surface that is "mirror dual" to $\mathbb{P}^1 \times \mathbb{P}^1$.

We claim that the image of $PP^2(X)$ under the map $f \mapsto \sum_{i=1}^4 e_{\sigma_i} f_{\sigma_i}$ is exactly $2\mathbb{Z}$. We compute $e(\sigma_1)$ using the unimodular subdivision of σ_1 along $v = (1, 0)$,

$$\sigma_1 = \langle v_1, v \rangle \cup \langle v, v_2 \rangle.$$

Then, writing $a = e_1^*$ and $b = e_2^*$, the dual cones of $\langle v_1, v \rangle$ and $\langle v, v_2 \rangle$ are $\langle b, a - b \rangle$ and $\langle b, a + b \rangle$, respectively, so

$$e(x_1) = \frac{1}{b(a-b)} - \frac{1}{b(a+b)} = \frac{2}{a^2 - b^2}.$$

Similarly, we compute $e(\sigma_3) = 2/(a^2 - b^2)$ and

$$e(\sigma_2) = e(\sigma_4) = \frac{-2}{a^2 - b^2}.$$

Therefore, since two divides every term in $\sum_{i=1}^4 e_{\sigma_i} f_{\sigma_i}$, the sum must be divisible by two. Also, the piecewise polynomial function f that vanishes on $\sigma_2 \cup \sigma_3 \cup \sigma_4$ and whose restriction to σ_1 is $a^2 - b^2$ maps to two. So the image of $PP^2(\Delta)$ is $2\mathbb{Z}$, as required.

Proof of Theorem 1.5. Applying Theorem 1.4 to Example 4.1 shows that the image of $\iota^* : A_T^2(X) \to A^2(X)$ is $2A^2(X)$, which is a proper subgroup of $A^2(X) \cong \mathbb{Z}$. $\square$

In the following examples, we consider fans in $\mathbb{R}^3$ with respect to the lattice $N = \mathbb{Z}^3$.

Example 4.2 (Mirror dual of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$). Consider the toric variety $X = X(\Delta)$, where Δ is the fan whose nonzero cones are the cones over the faces of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Then X is the Fano toric threefold that is "mirror dual" to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Recall that, since X is complete, the rank of $A^i(X)$ is equal to the rank of $A_i(X)$ [Fulton and Sturmfels 1997, Proposition 2.4], so $\operatorname{rk} A^0(X) = \operatorname{rk} A^3(X) = 1$. Furthermore, since $A_2(X)$ is the Weil divisor class group of X, we also have $\operatorname{rk} A^2(X) = 5$. The remainder of the following table can be filled in by straightforward linear algebra computations with piecewise polynomial functions.

From these computations, it is clear that $A^2_T(X)$ does not surject onto $A^2(X)$, since its image has rank at most two.

	$\operatorname{rk} A^i_T(X)$	$\operatorname{rk} M \cdot A^{i-1}_T(X)$	$\operatorname{rk} A^i(X)$
$i = 0$	1	0	1
$i = 1$	4	3	1
$i = 2$	11	9	5
$i = 3$	23	22	1

Example 4.3 (Fulton's threefold). Consider the toric variety $X' = X(\Delta')$, where Δ' is the fan combinatorially equivalent to the fan over the cube as in the preceding example, but with the ray through $(1, 1, 1)$ replaced by the ray through $(1, 2, 3)$. Then X' is complete and, as in the previous example, $\operatorname{rk} A^0(X') = \operatorname{rk} A^3(X') = 1$, and $\operatorname{rk} A^2(X') = 5$, but Fulton showed that X' has no nontrivial line bundles [Fulton 1993, pp. 25–26], so $A^1(X') = 0$. The remainder of the following table is filled in by linear algebra computations with piecewise polynomial functions.

	$\operatorname{rk} A^i_T(X')$	$\operatorname{rk} M \cdot A^{i-1}_T(X')$	$\operatorname{rk} A^i(X')$
$i = 0$	1	0	1
$i = 1$	3	3	0
$i = 2$	8	6	5
$i = 3$	20	16	1

Here, again, we see that $\iota^* : A^2_T(X') \to A^2(X')$ is not surjective, since its image has rank at most two. Furthermore, the kernel of ι^* is not generated in degree one, since the degree one part of the kernel is M, and $A^3_T(X')/M \cdot A^2_T(X')$ has rank four, and hence cannot map injectively into $A^3(X')$. However, X' is not projective, so to prove Theorem 1.6, it remains to give a projective example with similar properties.

Example 4.4. Consider the toric variety $X'' = X(\Delta'')$, where Δ'' is the fan combinatorially equivalent to the fan over the cube as in Example 4.2, but with the ray through $(1, 1, 1)$ replaced by the ray through $(1, 1, 2)$ and with the ray through $(1, -1, 1)$ replaced by the ray through $(1, -1, 2)$. It is straightforward to check that $-3K_{X''}$ is Cartier and ample, so X'' is $\mathbb{Q}$-Fano and projective. We compute the following table as in the preceding examples.

	$\operatorname{rk} A^i_T(X'')$	$\operatorname{rk} M \cdot A^{i-1}_T(X'')$	$\operatorname{rk} A^i(X'')$
$i = 0$	1	0	1
$i = 1$	4	3	1
$i = 2$	10	9	5
$i = 3$	22	19	1

Proof of Theorem 1.6. From the computations in Example 4.4, we conclude that $\iota^* : A_T^*(X)_{\mathbb{Q}} \to A^*(X)_{\mathbb{Q}}$ is not surjective in degree two, and its kernel in degree three is not in the ideal generated by its kernel in degree one. □

To balance these negative results, we conclude by proving a positive statement: $\iota^* : A_T^*(X) \to A^*(X)$ is always surjective in degree one.

Theorem 4.5. *For any toric variety $X = X(\Delta)$, $\iota^* : A_T^1(X) \to A^1(X)$ is surjective, giving a natural isomorphism $A^1(X) \cong PP^1(\Delta)/M$.*

Proof. If X is smooth, then the statement is clear. Suppose X is singular, and let

$$X_r \to \cdots \to X_1 \xrightarrow{\pi} X_0 = X$$

be a resolution of singularities, where each $X_i = X(\Delta_i)$ is a toric variety and $X_{i+1} \to X_i$ is the blowup along a smooth T-invariant center. Say X_1 is the blowup of X along $V(\tau)$ and $V(\rho) \subset X_1$ is the exceptional divisor. By induction on r, we may assume $A^1(X_1) \cong PP^1(\Delta_1)/M$. Also, we may assume $A^1(V(\rho)) = PP^1(\mathrm{Star}(\rho))/M$ and $A^1(V(\tau)) = PP^1(\mathrm{Star}(\tau))/M$, by induction on dimension. Then $\pi^* : A^1(X) \to A^1(X_1)$ is injective, and $c \in A^1(X_1)$ is in the image of π^* if and only if $c|_{V(\rho)}$ is in the image of $A^1(V(\tau))$ [Kimura 1992, Theorem 3.1]. The theorem then follows, since $\mathrm{Star}(\rho)$ is a subdivision of $\mathrm{Star}(\tau)$, Δ_1 and Δ coincide everywhere else, and the class of a piecewise linear function $[\Psi] \in PP^1(\mathrm{Star}(\rho))/M$ is pulled back from $\mathrm{Star}(\tau)$ if and only if Ψ is given by a single linear function on each cone of $\mathrm{Star}(\tau)$. □

Corollary 4.6. *For any toric variety X, the canonical map* $\mathrm{Pic}(X) \to A^1(X)$ *is an isomorphism.*

Proof. The corollary follows from the canonical identification of $PP^1(X)/M$ with $\mathrm{Pic}(X)$ [Fulton 1993, pp. 65–66]. □

Corollary 4.6 was known previously in the case where X is complete [Brion 1989]. See also [Fulton and Sturmfels 1997, Corollary 3.4].

Remark 4.7. One can use Kimura's inductive method, as in the proof of Corollary 4.6 and [Payne 2006a, Theorem 1], to compute the Chow cohomology of an arbitrary toric variety in all degrees. However, the resulting induction is more subtle, as Theorems 1.5 and 1.6 suggest.

5. Localization formula for mixed volumes of lattice polytopes

Let $P_1, \ldots, P_n$ be lattice polytopes in $M_{\mathbb{R}}$. For nonnegative real numbers a_i, the euclidean volume of $a_1P_1 + \cdots + a_nP_n$ is a homogeneous polynomial function of $(a_1, \ldots, a_n)$. The *mixed volume* $V(P_1, \ldots, P_n)$ is defined to be the coefficient of $a_1 \cdots a_n$ in this polynomial. Let Δ be the inner normal fan to $P_1 + \cdots + P_n$, and let $u_i(\sigma) \in M$ be the vertex of P_i that is minimal on σ, for each maximal cone $\sigma \in \Delta$.

Theorem 5.1. *The mixed volume of the polytopes P_i is given by*

$$n! \cdot V(P_1, \ldots, P_n) = (-1)^n \sum_{\sigma \in \Delta} e_\sigma \cdot u_1(\sigma) \cdots u_n(\sigma).$$

Theorem 5.1 follows from Theorem 1.4 and the fact that $V(P_1, \ldots, P_n)$ is the degree of $D_1 \cdots D_n$, where D_i is the T-Cartier divisor on $X(\Delta)$ corresponding to P_i [Fulton 1993, p. 116]. However, the statement of the theorem is purely combinatorial, and we give a combinatorial proof based on Brion's formula for generating functions for lattice points in polyhedra. The methods used in this proof may be of independent interest.

Let P be a lattice polytope in $M_{\mathbb{R}}$. Let Δ be the normal fan to P, and let $u(\sigma) \in M$ be the vertex of P that is minimal on σ, for each maximal cone $\sigma \in \Delta$.

Brion's Formula. The generating function for lattice points in P is

$$\sum_{u \in (P \cap M)} x^u = \sum_{\sigma} x^{u(\sigma)} \cdot \mathrm{Hilb}(\sigma).$$

In addition to Brion's Formula, we will use the following formula for mixed volumes, which is a lattice point counting analogue of the alternating sum of volumes in formula (3) of [Fulton 1993, p. 116].

Proposition 5.2. *Let $P_1, \ldots, P_n$ be lattice polytopes in $M_{\mathbb{R}}$. Then*

$$n! \cdot V(P_1, \ldots, P_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{n-k} \#((P_{i_1} + \cdots + P_{i_k}) \cap M).$$

Proof. The number of lattice points in $a_1 P_1 + \cdots + a_n P_n$ is a polynomial in the a_i of degree at most n, and the degree n part of this polynomial is $n!$ times the volume of $a_1 P_1 + \cdots + a_n P_n$ [McMullen 1978/79, Theorem 7]. Therefore, $n! \cdot V(P_1, \ldots, P_n)$ is the coefficient of $a_1 \cdots a_n$ in this polynomial, and the proposition is an immediate consequence of the following lemma. □

Lemma 5.3. *Let $f \in \mathbb{R}[t_1, \ldots, t_n]$ be a polynomial function on $\mathbb{R}^n$ of degree at most n. The coefficient of $t_1 \cdots t_n$ in f is*

$$\sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{n-k} f(e_{i_1} + \cdots + e_{i_k})$$

where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$.

Proof. The function taking a polynomial g to $\sum (-1)^{n-k} g(e_{i_1} + \cdots + e_{i_k})$ vanishes on any monomial that does not contain all n variables, and its value on $t_1 \cdots t_n$ is 1. □

Proof of Theorem 5.1. For each σ, $(-1)^n \cdot e_\sigma \cdot u_1(\sigma) \cdots u_n(\sigma)$ is the principal part of

$$(x^{u_1(\sigma)} - 1) \cdots (x^{u_n(\sigma)} - 1) \cdot \mathrm{Hilb}(\sigma).$$

Expanding the product of the binomials, taking the sum over all σ, and applying Brion's Formula then gives

$$\sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{n-k} \cdot \sum_{u \in (P_{i_1} + \cdots + P_{i_k} \cap M)} x^u.$$

The theorem then follows from Proposition 5.2 by taking principal parts, since the leading form of x^u at 1_T is equal to one. □

6. Bott residue formula for toric vector bundles

The mixed volume $V(P_1, \ldots, P_n)$ is the degree of the top Chern class of the toric vector bundle $\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_n)$, where D_i is the T-Cartier divisor corresponding to P_i. Therefore, mixed volumes are a special case of Chern numbers of toric vector bundles, and Theorem 5.1 may be generalized as follows. Given a multiset of linear functions $\mathbf{u} \subset M$ let $\varepsilon_i(\mathbf{u}) \in \mathrm{Sym}^i(M)$ be the i-th elementary symmetric function in the elements of $\mathbf{u}$. For instance, if $\mathbf{u} = \{u_1, \ldots, u_r\}$, then $\varepsilon_1(\mathbf{u}) = u_1 + \cdots + u_r$ and $\varepsilon_r(\mathbf{u}) = u_1 \cdots u_r$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of n, let $\varepsilon_\lambda(\mathbf{u}) \in \mathrm{Sym}^n(M)$ be the product

$$\varepsilon_\lambda(\mathbf{u}) = \varepsilon_{\lambda_1}(\mathbf{u}) \cdots \varepsilon_{\lambda_s}(\mathbf{u}).$$

Recall that, for any toric vector bundle $\mathcal{E}$ on an arbitrary toric variety $X = X(\Delta)$ and any maximal cone $\sigma \in \Delta$, there is a unique multiset $\mathbf{u}(\sigma) \subset M$ such that the restriction of $\mathcal{E}$ to U_σ splits equivariantly as

$$\mathcal{E}|_{U_\sigma} \cong \bigoplus_{u \in \mathbf{u}(\sigma)} \mathcal{O}(\mathrm{div}\, \chi^u).$$

See [Klyachko 1989] or [Payne 2006b, Section 2] for this and other basic facts about toric vector bundles.

Theorem 6.1. *Let $\mathcal{E}$ be a toric vector bundle on a complete toric variety X, and let λ be a partition of n. Then the Chern number $c_\lambda(\mathcal{E})$ is given by*

$$c_\lambda(\mathcal{E}) = \sum_\sigma e_\sigma \cdot \varepsilon_\lambda(\mathbf{u}(\sigma)).$$

Proof. The Chern number $c_\lambda(\mathcal{E})$ is equal to the integral over $[X]$ of the equivariant Chow cohomology class corresponding to the piecewise polynomial whose restriction to σ is $\varepsilon_\lambda(\mathbf{u}(\sigma))$. Therefore, the theorem follows from Theorem 1.4. □

Theorem 6.1 has a straightforward generalization to top degree polynomials in the Chern classes of several toric vector bundles (we omit the details), and may be seen as a Bott residue formula for vector bundles on toric varieties with arbitrary singularities. This solves the toric case of the problem of proving residue formulas on singular varieties posed in [Edidin and Graham 1998b, Section 5]. Edidin and Graham handled the case of toric subvarieties of smooth toric varieties, but the extension to arbitrary toric varieties is nontrivial; there are singular toric varieties, such as Fulton's threefold (Example 4.3) that have no nontrivial line bundles, and hence admit no nonconstant morphisms to smooth varieties.

Example 6.2. We apply Theorem 6.1 to compute the Chern numbers of a specific nonsplit rank two toric vector bundle on the singular toric variety $X = X(\Delta)$ mirror dual to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Example 4.2). For this example, we assume that the base field has at least three elements. The primitive generators of the rays of Δ are

$$\begin{array}{llll} v_1 = (1, 1, 1), & v_2 = (1, 1, -1), & v_3 = (1, -1, 1), & v_4 = (1, -1, -1), \\ v_5 = (-1, 1, 1), & v_6 = (-1, 1, -1), & v_7 = (-1, -1, 1), & v_8 = (-1, -1, -1), \end{array}$$

and the maximal cones of Δ are

$$\begin{array}{lll} \sigma_1 = \langle v_1, v_2, v_3, v_4\rangle, & \sigma_2 = \langle v_1, v_2, v_5, v_6\rangle, & \sigma_3 = \langle v_1, v_3, v_5, v_7\rangle, \\ \sigma_4 = \langle v_2, v_4, v_7, v_8\rangle, & \sigma_5 = \langle v_3, v_4, v_7, v_8\rangle, & \sigma_6 = \langle v_5, v_6, v_7, v_8\rangle. \end{array}$$

Let ρ_i be the ray of Δ spanned by v_i, let $E = k^2$, fix four distinct lines L_1, L_2, L_3, and L_4 in E, and let $\mathscr{E}$ be the toric vector bundle determined by the filtrations

$$E^{\rho_1}(i) = \begin{cases} E & \text{for } i \le -1, \\ L_1 & \text{for } 0 \le i \le 3, \\ 0 & \text{for } i > 3, \end{cases} \qquad E^{\rho_4}(i) = \begin{cases} E & \text{for } i \le -1, \\ L_2 & \text{for } 0 \le i \le 3, \\ 0 & \text{for } i > 3, \end{cases}$$

$$E^{\rho_6}(i) = \begin{cases} E & \text{for } i \le -1, \\ L_3 & \text{for } 0 \le i \le 3, \\ 0 & \text{for } i > 3, \end{cases} \qquad E^{\rho_7}(i) = \begin{cases} E & \text{for } i \le -1, \\ L_4 & \text{for } 0 \le i \le 3, \\ 0 & \text{for } i > 3, \end{cases}$$

and

$$E^{\rho_j}(i) = \begin{cases} E & \text{for } i \le 1, \\ 0 & \text{for } i > 1, \end{cases}$$

for $j \in \{2, 3, 5, 8\}$. Since the lines L_i are distinct, the vector bundle $\mathscr{E}$ does not split as a sum of line bundles. It is straightforward to check that the multisets of linear functions $\mathbf{u}(\sigma_i)$ determined by $\mathscr{E}$ are as follows. For simplicity, we write a, b, and c, for e_1^*, e_2^* and e_3^*, respectively.

$$\begin{array}{ll} \mathbf{u}(\sigma_1) = \{(a+b+c), (a-b-c)\}, & \mathbf{u}(\sigma_2) = \{(a+b+c), (-a+b-c)\}, \\ \mathbf{u}(\sigma_3) = \{(a+b+c), (-a-b+c)\}, & \mathbf{u}(\sigma_4) = \{(a-b-c), (-a+b-c)\}, \\ \mathbf{u}(\sigma_5) = \{(a-b-c), (-a-b+c)\}, & \mathbf{u}(\sigma_6) = \{(-a+b-c), (-a-b+c)\}. \end{array}$$

To compute the Chern numbers of $\mathscr{E}$, we now need only to compute the equivariant multiplicities e_{σ_i}. First, σ_1^* is spanned by $u_1 = (1, 1, 0)$, $u_2 = (1, 0, 1)$, $u_3 = (1, 0, -1)$, and $u_4 = (1, -1, 0)$. Let $u = (1, 0, 0)$. We compute, as in [Payne 2007, Example 1.8],

$$\mathrm{Hilb}(\sigma_1) = \frac{(1+x^u)(1-x^{2u})}{(1-x^{u_1})(1-x^{u_2})(1-x^{u_3})(1-x^{u_4})}.$$

Since the principal parts of $1+x^u$, $1-x^{2u}$ and $1-x^{u_i}$ at 1_T are 2, $-2u$, and $-u_i$ respectively, it is then straightforward to compute $e_{\sigma_1} = -\mathrm{Hilb}(\sigma)_\circ$. Then

$$e_{\sigma_1} = \frac{4a}{(b^2-a^2)(c^2-a^2)}.$$

By symmetry, $e_{\sigma_6} = -e_{\sigma_1}$, and similarly

$$e_{\sigma_2} = \frac{4b}{(a^2-b^2)(c^2-b^2)} = -e_{\sigma_5},$$

and

$$e_{\sigma_3} = \frac{4c}{(a^2-c^2)(b^2-c^2)} = -e_{\sigma_4}.$$

Then, using Theorem 6.1 and combining the summands coming from σ_i and σ_{7-i}, we obtain

$$c_{111}(\mathscr{E}) = \frac{2\cdot(2a)^3\cdot 4a}{(b^2-a^2)(c^2-a^2)} + \frac{2\cdot(2b)^3\cdot 4b}{(a^2-b^2)(c^2-b^2)} + \frac{2\cdot(2c)^3\cdot 4c}{(a^2-c^2)(b^2-c^2)},$$

which simplifies to $c_{111}(\mathscr{E}) = 64$. Similarly,

$$c_{21}(\mathscr{E}) = \frac{16a^2(a^2-b^2-c^2)}{(b^2-a^2)(c^2-a^2)} + \frac{16b^2(-a^2+b^2-c^2)}{(a^2-b^2)(c^2-b^2)} + \frac{16c^2(-a^2-b^2+c^2)}{(a^2-c^2)(b^2-c^2)},$$

which simplifies to $c_{21}(\mathscr{E}) = 32$.

Appendix: Principal parts of rational functions

Associated graded rings and leading forms have been standard tools for about as long as commutative algebra has been applied to local algebraic geometry [Samuel 1953; 1955]. The generalization from leading forms of regular functions to principal parts of rational functions is straightforward but, since we have been unable to locate a reference, we include a brief account.

Let X be an algebraic variety over a field k, and let $x \in X(k)$ be a smooth point. Let $\mathfrak{m}$ be the maximal ideal in the local ring $\mathcal{O}_{X,x}$. Since x is smooth,

$$(\mathfrak{m}^d/\mathfrak{m}^{d+1}) \cong \mathrm{Sym}^d(\mathfrak{m}/\mathfrak{m}^2),$$

for all nonnegative integers d [Atiyah and Macdonald 1969, Theorem 11.22]. Suppose $g \in \mathcal{O}_{X,x}$ is a regular function whose order of vanishing at x is d. Then the *leading form* of g is its image

$$g_\circ \in \mathrm{Sym}^d(\mathfrak{m}/\mathfrak{m}^2).$$

In other words, if $x_1, \dots, x_n$ is a local system of parameters, then g can be expanded uniquely as a power series in $k[[x_1, \dots, x_n]]$, and the sum of the lowest degree terms in this power series is the homogeneous degree d polynomial in $x_1, \dots, x_n$ that maps to $g_\circ$ under the canonical isomorphism

$$k[x_1, \dots, x_n]_d \cong \mathrm{Sym}^d(\mathfrak{m}/\mathfrak{m}^2).$$

Now $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space of X at x, so $g_\circ$ is naturally a regular function on the tangent space $T_{X,x}$, and the zero locus of $g_\circ$ is the tangent cone of the divisor of zeros of g at x [Harris 1992, Lecture 20]. Note that leading forms are multiplicative; if $g, h \in \mathcal{O}_{X,x}$, then $(gh)_\circ = g_\circ h_\circ$ in $\mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$. For convenience, we define the leading form of zero to be $0 \in \mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$.

Suppose f is a rational function on X. Then f can be written as a fraction $f = g/h$, with $g, h \in \mathcal{O}_{X,x}$. We define the *principal part* of f to be

$$f_\circ = g_\circ / h_\circ,$$

which is a homogeneous element of $\mathrm{Sym}^\pm(\mathfrak{m}/\mathfrak{m}^2)$, the $\mathbb{Z}$-graded ring obtained by inverting all homogeneous elements in $\mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$. Note that $f_\circ$ is well-defined; if $g/h = g'/h'$, then $gh' = g'h$ (since $\mathcal{O}_{X,x}$ is a domain), and therefore

$$g_\circ h'_\circ = g'_\circ h_\circ,$$

since leading forms are multiplicative, so $g_\circ/h_\circ = g'_\circ/h'_\circ$. Also, $f_\circ$ is naturally a rational function on $T_{X,x}$, and its divisors of zeros and poles are the tangent cones of the zeros and poles of f, respectively.

Proposition A.1. *Suppose $f_1, \dots, f_s$ are rational functions on X with principal parts in degree d, and let $f = f_1 + \cdots + f_s$. Then either*

$$f_\circ = (f_1)_\circ + \cdots + (f_s)_\circ,$$

or $(f_1)_\circ + \cdots + (f_s)_\circ = 0$ and the principal part of f is in degree strictly greater than d.

Proof. If $f = 0$ then the proposition is clear. Suppose f is nonzero, and express each f_i as a fraction $f_i = g_i/h_i$, with $g_i, h_i \in \mathcal{O}_{X,x}$. Then we can write f as a fraction over a common denominator

$$f = \sum_{i=1}^{s} \frac{g_i \cdot h_1 \cdots \widehat{h_i} \cdots h_s}{h_1 \cdots h_s}.$$

Say h_i vanishes to order d_i at x. Then each summand in the numerator above vanishes to order exactly $d_1+\cdots+d_s+d$. Therefore, either the numerator vanishes to order exactly $d_1+\cdots+d_s+d$ and $f_\circ=(f_1)_\circ+\cdots+(f_s)_\circ$, or the numerator vanishes to some larger order and $f_\circ$ has degree greater than d. □

Corollary A.2. *Suppose $f_1,\ldots,f_s$ are rational functions on X with principal parts in degree d, and suppose $f=f_1+\cdots+f_s$ is regular at x. Then $(f_1)_\circ+\cdots+(f_s)_\circ\in \mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$ is regular on $T_{X,x}$.*

Proof. By Proposition A.1, if $(f_1)_\circ+\cdots+(f_s)_\circ$ does not vanish then it is equal to $f_\circ$, which is the principal part of a regular function. □

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