Algebra & Number Theory

Volume 2

2008

No. 2

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 2:2(2008)

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Ulrich Derenthal, Michael Joyce and Zachariah Teitler

We compute a naturally defined measure of the size of the nef cone of a Del Pezzo surface. The resulting number appears in a conjecture of Manin on the asymptotic behavior of the number of rational points of bounded height on the surface. The nef cone volume of a Del Pezzo surface Y with (-2)-curves defined over an algebraically closed field is equal to the nef cone volume of a smooth Del Pezzo surface of the same degree divided by the order of the Weyl group of a simply-laced root system associated to the configuration of (-2)-curves on Y. When Y is defined over an arbitrary perfect field, a similar result holds, except that the associated root system is no longer necessarily simply-laced.

1. Introduction

An *ordinary Del Pezzo surface* is a smooth projective rational surface X on which the anticanonical class $-K_X$ is ample. If X is defined over an algebraically closed field, then X is one of the following: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the blowup of $\mathbb{P}^2$ at up to 8 points in general position. Points are in general position if no three are collinear, no six lie on a conic, and no eight lie on a cubic with one of them a singular point of the cubic. Then X may contain (-1)-curves, but no (-2)-curves, where for $n \in \{1, 2\}$, a $(-n)$*-curve* is a smooth rational curve on X having self-intersection number $-n$.

A *generalized Del Pezzo surface* is a smooth projective rational surface Y on which $-K_Y$ is big and nef. If Y is defined over an algebraically closed field, then Y is one of the following: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$, or a surface obtained from $\mathbb{P}^2$ by a sequence of blowings-up at up to 8 points, possibly infinitely near, each not lying on any (-2)-curve. Over an algebraically closed field, a generalized Del Pezzo surface is ordinary if and only if it contains no (-2)-curves. See Section 3 for more details.

MSC2000: primary 14J26; secondary 14C20, 14G05.
Keywords: Del Pezzo surface, Manin's conjecture, nef cone, root system.
The first author was partially supported by a Feodor Lynen Research Fellowship of the Alexander von Humboldt Foundation.

The nef cone volume of a generalized Del Pezzo surface Y is equal to the volume of the cross-section of the nef cone of Y obtained by intersecting with the hyperplane consisting of those divisor classes whose intersection with the anticanonical class $-K_Y$ is equal to 1. The resulting cross-section is a polytope. Its volume is a rational number, denoted by $\alpha(Y)$. We give details of this definition in Section 2.

In this paper we compute $\alpha(Y)$ for generalized Del Pezzo surfaces Y.

The *degree* of Y is the self-intersection number

$$d = \langle -K_Y, -K_Y \rangle.$$

It satisfies $1 \le d \le 9$, and when Y is the blowup of $\mathbb{P}^2$ at r points in almost general position, $d = 9 - r$.

A generalized Del Pezzo surface Y defined over a field $\mathbb{K}$ is *split* if it is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_2$, or the blowup of $\mathbb{P}^2$ at $1 \le r \le 8$ $\mathbb{K}$-rational points in almost general position. See Definition 3.2. Otherwise, Y is said to be nonsplit (for example, the blowup of $\mathbb{P}^2$ at two conjugate points). We consider only split Y until Section 6, and so the reader may assume that $\mathbb{K}$ is algebraically closed until that point.

An investigation of $\alpha(Y)$ for split ordinary Del Pezzo surfaces was undertaken by the first author. He proved the following result.

Theorem 1.1 [Derenthal 2007, Theorem 4]. *Let X_d denote a split ordinary Del Pezzo surface of degree d obtained by blowing up $9 - d$ points in general position on $\mathbb{P}^2$ and let N_d denote the number of (-1)-curves on X_d. For $d \le 6$,*

$$\alpha(X_d) = \frac{N_d}{d(9-d)} \alpha(X_{d+1}).$$

Combining this with simple calculations that show that

$$\alpha(\mathbb{P}^2) = \tfrac{1}{3}, \quad \alpha(\mathbb{P}^1 \times \mathbb{P}^1) = \tfrac{1}{4}, \quad \alpha(X_8) = \tfrac{1}{6}, \quad \alpha(X_7) = \tfrac{1}{24},$$

this theorem allows for an inductive calculation of $\alpha(X)$ for any split ordinary Del Pezzo surface X. This calculation is summarized in Table 1.

We extend this result in two directions. First, we study split generalized Del Pezzo surfaces. In Section 4, we prove the following theorem by analyzing the nef cone of such a surface Y. It allows us to compute $\alpha(Y)$ by induction on the rank of the Néron–Severi group of Y.

d	8	7	6	5	4	3	2	1
N_d	1	3	6	10	16	27	56	240
$\alpha(X_d)$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{72}$	$\frac{1}{144}$	$\frac{1}{180}$	$\frac{1}{120}$	$\frac{1}{30}$	1

Table 1. Values of $\alpha(X_d)$ for ordinary Del Pezzo surfaces X_d.

Theorem 1.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. For each E in the set $\mathscr{C}$ of (-1)-curves on Y, let Y_E denote the split generalized Del Pezzo surface of degree $d+1$ obtained by contracting E. Then*

$$\alpha(Y) = \sum_{E \in \mathscr{C}} \frac{1}{d(9-d)} \alpha(Y_E).$$

As with Theorem 1.1, using the additional calculation that $\alpha(\mathbb{F}_2) = \frac{1}{8}$, this theorem allows us to compute $\alpha(Y)$ for any split generalized Del Pezzo surface Y.

The first author computed $\alpha(Y)$ for split generalized Del Pezzo surfaces Y of degree $d \geq 3$ directly, using computer programs to find a triangulation of the nef cone case by case [Derenthal 2007, Section 3]. This numerical data led us to formulate the following theorem; see Section 5C for its proof.

Theorem 1.3. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$ and let X be a split ordinary Del Pezzo surface of the same degree. Then*

$$\alpha(Y) = \frac{1}{\#W(R_Y)} \alpha(X),$$

where $W(R_Y)$ is the Weyl group of the root system R_Y whose simple roots are the (-2)-curves on Y.

When combined with Theorem 1.1 the computation of $\alpha(Y)$ for an arbitrary split generalized Del Pezzo surface Y of any degree is reduced to a determination of the (-2)-curves on the surface. See Section 5 for more information on the root system R_Y and its Weyl group.

We also consider the case of nonsplit surfaces. Suppose that Y is a generalized Del Pezzo surface and X is an ordinary Del Pezzo surface, both of the same degree and defined over the same perfect field $\mathbb{K}$. Then the Néron–Severi groups of X and Y coincide (Proposition 6.2) and the absolute Galois group of $\mathbb{K}$ acts as a finite group G of automorphisms of this group (Proposition 6.1). Assume that the Galois actions associated to X and Y coincide. The Galois action on the root system R_Y allows us to associate to Y an orbit root system $\mathbb{O}(R_Y, G)$ (Definition 6.5). Our third main result is that under these assumptions

$$\alpha(Y) = \frac{\alpha(X)}{\#W(\mathbb{O}(R_Y, G))}.$$

See Corollary 7.5. The integer appearing in the denominator is the order of a Weyl group and is straightforward to compute. Thus all that is left is to compute $\alpha(X)$. There are a finite number of cases in each degree d, one for each conjugacy class of subgroups of the Weyl group of a canonically defined root system R_d (Section 5B). We perform the computations for $d \geq 5$ in Section 7B.

Manin's conjecture. The primary motivation for our study of the nef cone volume α is its appearance in Manin's conjecture on the number of rational points of bounded height on Fano varieties defined over number fields, as described below. Although the conjecture is now known not to hold for all Fano varieties [Batyrev and Tschinkel 1996, Theorems 3.1–3.3], it has been verified in a large number of cases, including some varieties for which the anticanonical class is big but not ample.

Let X be a smooth projective variety defined over a number field $\mathbb{K}$ for which $-K_X$ is big and assume that the set $X(\mathbb{K})$ of rational points is Zariski dense. Equip $X(\mathbb{K})$ with an anticanonical height function H (consult [Hindry and Silverman 2000, Part B] for information on height functions) and for any constructible set $U \subset X$ let

$$\mathcal{N}_U(B) := \#\{P \in U(\mathbb{K}) : H(P) \leq B\}.$$

The original formulation of the Manin conjecture [Batyrev and Manin 1990, Conjecture B] posits the existence of a Zariski open set $U \subset X$ such that for any open set $V \subset U$

$$\mathcal{N}_V(B) \sim c(X) B (\log B)^{\rho - 1} \quad \text{asymptotically as } B \to \infty,$$

where ρ is the Néron–Severi rank of X. The conjecture was initially made for Fano varieties, but a more ambitious version of the conjecture relaxes the condition on $-K_X$ to merely being big. The leading constant was given a conjectural interpretation by Peyre [1995, Definition 2.4] and Batyrev and Tschinkel [1995, Theorem 4.4.4]. They predict that $c(X) = \alpha(X)\beta(X)\tau(X)$, where $\alpha(X) \in \mathbb{Q}$ is the constant of interest in this paper, $\beta(X) \in \mathbb{N}$ is a cohomological invariant of the Galois action on the Néron–Severi group of X, and $\tau(X) \in \mathbb{R}$ is a volume of adelic points on X.

2. Definition of the nef cone volume

We recall the definition of $\alpha(X)$, first introduced by Peyre [1995, Definition 2.4].

Let X be a smooth complete variety for which $-K_X$ is big. We denote the intersection form on X by $\langle \cdot, \cdot \rangle$. Recall that a divisor class D on X is *numerically trivial* if $\langle D, C \rangle = 0$ for all curves (equivalently, all 1-cycles) C on X, and two divisor classes are *numerically equivalent* if their difference is numerically trivial. One similarly defines numerical equivalence of curves. Numerical equivalence classes of divisors on X form a finitely-generated torsion-free abelian group $N^1(X)$ whose dual group $N_1(X)$ consists of numerical equivalence classes of 1-cycles on X. Let $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated Euclidean spaces. Inside $N^1(X)_{\mathbb{R}}$ lies the *effective cone* $\mathrm{Eff}^1(X)$, the closed convex cone spanned by the classes of effective divisors.

Recall that for a finite-dimensional real inner product space V and a convex cone $\Gamma \subset V$, the dual convex cone $\Gamma^\vee \subset V$ is defined by

$$\Gamma^\vee = \{\, v \in V : \langle v, c\rangle \geq 0 \text{ for all } c \in \Gamma \,\}.$$

The cone $\Gamma^\vee$ is closed as a subspace of the Euclidean space V. The dual $\mathrm{Eff}^1(X)^\vee$ of the effective cone of X in $N^1(X)_{\mathbb{R}}$ is the movable cone of X (see [Boucksom et al. 2004, Theorem 2.2], [Lazarsfeld 2004, Section 11.4.C]). Note that when X is a surface, $N_1(X) = N^1(X)$ and $\mathrm{Eff}^1(X)^\vee$ is the *nef cone* of X, denoted by $\mathrm{Nef}(X)$.

Since the cone $\mathrm{Eff}^1(X)^\vee$ has infinite volume in $N_1(X)_{\mathbb{R}}$, a natural means of measuring its "size" is to truncate the cone in an (anti)canonical manner. To do this, consider the hyperplane

$$\mathscr{H}_X := \{C \in N_1(X)_{\mathbb{R}} : \langle -K_X, C\rangle = 1\}.$$

Note that since $-K_X$ is big by hypothesis, $\mathscr{H}_X$ intersects each ray of $\mathrm{Eff}^1(X)^\vee$. We endow $N_1(X)_{\mathbb{R}}$ with Lebesgue measure ds normalized so that $N_1(X)$ has covolume 1, and we endow $\mathscr{H}_X$ with the induced Leray measure $d\mu$ with respect to the linear form $\langle -K_X, \cdot\rangle$. That is, letting l be the linear form $l(v) = \langle -K_X, v\rangle$, we have $ds = d\mu \wedge dl$. We construct the polytope

$$\mathscr{P}_X := \mathrm{Eff}^1(X)^\vee \cap \mathscr{H}_X$$

and define

$$\alpha(X) := \mathrm{Vol}(\mathscr{P}_X) = \int_{\mathscr{P}_X} d\mu.$$

There are variants of this definition differing only by a dimensional factor. Let $\rho = \dim N_1(X)_{\mathbb{R}}$ and

$$\mathscr{C}_X := \{C \in \mathrm{Eff}^1(X)^\vee : \langle -K_X, C\rangle \leq 1\}$$

be the convex hull of $\mathscr{P}_X$ and the origin. Then a simple slicing argument shows that $\alpha(X) = \rho \cdot \mathrm{Vol}(\mathscr{C}_X)$. Additionally,

$$\alpha(X) = \frac{1}{(\rho-1)!} \int \cdots \int_{\mathrm{Eff}^1(X)^\vee} \exp\left(-\langle -K_X, s\rangle\right) ds,$$

with the bigness of $-K_X$ ensuring the convergence of the integral.

Example 2.1. Let us compute $\alpha(\mathbb{P}^2)$. We have $N^1(\mathbb{P}^2)_{\mathbb{R}} \cong \mathbb{R}^1$, with the real number $x \in \mathbb{R}$ corresponding to the (real) divisor class xL, where L is the class of a line in $\mathbb{P}^2$. Then the nef cone $\mathrm{Nef}(\mathbb{P}^2) = \{x \in \mathbb{R} : x \geq 0\}$ and the anticanonical class corresponds the real number 3. The hyperplane $\mathscr{H}_{\mathbb{P}^2}$ is just $\{\frac{1}{3}\}$. The polytope $\mathscr{P}_{\mathbb{P}^2}$ is also $\{\frac{1}{3}\}$ and the convex hull $\mathscr{C}_{\mathbb{P}^2} = [0, \frac{1}{3}]$. Thus $\mathscr{C}_{\mathbb{P}^2}$ has volume $\frac{1}{3}$ and so $\alpha(\mathbb{P}^2) = 1 \cdot \mathrm{Vol}(\mathscr{C}_{\mathbb{P}^2}) = \frac{1}{3}$.

Example 2.2. Let X_8 be the blowup of $\mathbb{P}^2$ at a single point. Let L be the class of the pullback of a line to X_8 and let E be the class of the exceptional divisor. Then $N^1(X_8)$ is generated by L and E. In $N^1(X_8)_{\mathbb{R}} \cong \mathbb{R}^2$, with (a, b) corresponding to $aL + bE$, the nef cone $\mathrm{Nef}(X_8)$ is equal to

$$\{(a, b) : a \geq 0, a + b \geq 0\},$$

that is, the cone with extremal rays spanned by L and $L - E$. The anticanonical class corresponds to the point $(3, -1)$. The hyperplane $\mathscr{H}_{X_8}$ is the line $3a + b = 1$. One checks that $\mathscr{P}_{X_8}$ is the segment joining the points $(\frac{1}{3}, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$. Then $\mathscr{C}_{X_8}$ is the triangle with vertices the above two points together with the origin. The area of this triangle is $\frac{1}{12}$, and so $\alpha(X_8) = 2 \ \mathrm{Vol}\,\mathscr{C}_{X_8} = \frac{1}{6}$.

Terminology. Peyre [1995] introduced the notation $\alpha(X)$, but did not give a name to this quantity. We will refer $\alpha(X)$ as the "nef cone volume of X" whenever X is a surface.

3. Generalized Del Pezzo surfaces

As stated in the introduction, a *generalized Del Pezzo surface* is a smooth projective rational surface Y on which $-K_Y$ is big and nef. If Y is defined over an algebraically closed field, Y is one of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$, or $\mathbb{P}^2$ blown up at $1 \leq r \leq 8$ points in almost general position [Demazure 1980, Definition III.2.1]. To blow up r points on $\mathbb{P}^2$ in almost general position is to construct a sequence of morphisms

$$Y = Y_r \to Y_{r-1} \to \cdots \to Y_1 \to Y_0 = \mathbb{P}^2,$$

where each map $Y_i \to Y_{i-1}$ is the blowup of Y_{i-1} at a point $p_i \in Y_{i-1}$ not lying on any irreducible curves of self-intersection number -2 on Y_i.

For $n \in \{1, 2\}$, a $(-n)$*-class* on Y is a divisor class D such that $\langle D, D\rangle = -n$ and $\langle D, -K_Y\rangle = 2 - n$. If such a class is effective, then there is necessarily a unique curve in that class. If this curve is irreducible, we use the term $(-n)$*-curve* both for this curve and its class. It follows from the genus formula that a $(-n)$-curve is a smooth rational curve. A simple calculation [Demazure 1980, Tables 2 and 3] shows that the sets of (-1)- and (-2)-classes on a generalized Del Pezzo surface are finite.

Let Y be a generalized Del Pezzo surface defined over a field $\mathbb{K}$. We denote $Y \times_{\mathbb{K}} \overline{\mathbb{K}}$ by $\overline{Y}$. Recall that a generalized Del Pezzo surface Y is an ordinary Del Pezzo surface if and only if the anticanonical class $-K_Y$ is ample. Equivalently, there are no (-2)-curves on $\overline{Y}$.

Convention 3.1. Throughout the paper, we will use X to refer to an ordinary Del Pezzo surface and Y to refer to a generalized (possibly ordinary) Del Pezzo surface.

The absolute Galois group $G_{\mathbb{K}} = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on $N^1(\overline{Y})$.

Definition 3.2. A generalized Del Pezzo surface Y is *split* if $Y(\mathbb{K}) \neq \varnothing$ and the action of $G_{\mathbb{K}}$ on $N^1(\overline{Y})$ is trivial.

Apart from the exceptional cases where $\overline{Y}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, the existence of a rational point assures that Y is a blowup of $\mathbb{P}^2$ and the triviality of the Galois action assures that each exceptional divisor is defined over $\mathbb{K}$, and thus the sequence of blown-up points must themselves be defined over $\mathbb{K}$.

In the remainder of this section, we prove that for a split Y, the effective cone $\mathrm{Eff}^1(Y)$ is generated by the set of (-1)- and (-2)-curves on Y, collecting a number of useful facts along the way.

By the following lemma, the group $N^1(Y)$ depends only on the degree of Y. We will make frequent use of this well-known result.

Lemma 3.3. *Let X be a split ordinary Del Pezzo surface and let Y be a split generalized Del Pezzo surface of the same degree $d \leq 7$. There is an isomorphism of $N^1(X)$ and $N^1(Y)$ which identifies the intersection forms and takes $-K_X$ to $-K_Y$.*

Proof. Say X is the blowup of $\mathbb{P}^2$ at points $p_1, \dots, p_r \in \mathbb{P}^2$, $r = 9-d$, with blowdown $\pi_X : X \to \mathbb{P}^2$, and say Y is obtained by blowing up $\mathbb{P}^2$ at points $q_1, \dots, q_r$:

$$\pi_Y : Y = Y_r \to Y_{r-1} \to \cdots \to Y_1 \to Y_0 = \mathbb{P}^2$$

where $Y_j = \mathrm{Bl}_{q_j}(Y_{j-1})$, $q_j \in Y_{j-1}$. Let $E_{X,j}$ be the exceptional divisor over p_j, and let $E_{Y,j}$ be the total transform in Y of the exceptional divisor over q_j. (That is, if $f_j : Y \to Y_{j-1}$, then $E_{Y,j} = f_j^{-1}(q_j)$, scheme-theoretically.)

Then $N^1(X)$ is the free abelian group on $L_X = \pi_X^* \mathbb{O}_{\mathbb{P}^2}(1)$, $E_{X,1}, \dots, E_{X,r}$. Similarly, $N^1(Y)$ is the free abelian group on $L_Y = \pi_Y^* \mathbb{O}_{\mathbb{P}^2}(1)$, $E_{Y,1}, \dots, E_{Y,r}$. The intersection form on $N^1(X)$ is given in this basis by the diagonal matrix with entries $(1, -1, \dots, -1)$; the intersection form on $N^1(Y)$ is given in this basis by the same matrix. We have $-K_X = 3L_X - \sum E_{X,j}$ and $-K_Y = 3L_Y - \sum E_{Y,j}$. $\square$

Remark 3.4. Note that the identification made in the proof of Lemma 3.3 is not necessarily unique; see [Harbourne 1985, Theorem 0.1].

The next lemma is a modest generalization of [Hassett and Tschinkel 2004, Proposition 4.5].

Lemma 3.5. *Let S be a surface and let $D_1, \dots, D_k$ be irreducible effective divisors on S. Let Γ denote the cone generated by $D_1, \dots, D_k$. Then the effective cone of S is equal to Γ if and only if $\Gamma^\vee \subset \Gamma$.*

Proof. If the effective cone of S is equal to Γ then it is a closed cone. The nef cone $\mathrm{Nef}(S) = \Gamma^\vee$ is contained in the closure of the effective cone, which is just Γ.

For the converse, it is clear that Γ is contained in the effective cone of S. Let D be an effective divisor. Then we can write $D = D' + a_1 D_1 + \cdots + a_k D_k$ with $a_i \geq 0$ and D' having none of the D_i as an irreducible component. It is clear that D' is contained in $\Gamma^\vee$, and by hypothesis, D' is consequently contained in Γ. Hence the same is true of D. □

Proposition 3.6. *If Y is a split generalized Del Pezzo surface, every (-1)-class in $N^1(Y)$ is effective. Indeed, if E is any (-1)-class, then either*

(1) *E is a (-1)-curve, or*

(2) *E can be written as the sum of a (-1)-curve and one or more (-2)-curves, or*

(3) *$d = 1$ and E can be written as the sum of $-K_Y$ and one or more (-2)-curves.*

Proof. See [Demazure 1980, Theorem III.2.c]. □

For a split generalized Del Pezzo surface Y of degree $d \geq 2$, this shows that every (-1)-class is a nonnegative integral linear combination of (-1)- and (-2)-curves. By the following lemma, this holds also in degree $d = 1$ if we allow rational instead of integral coefficients.

Lemma 3.7. *For a split generalized Del Pezzo surface Y of degree 1, the anti-canonical class $-K_Y$ is a linear combination of (-1)- and (-2)-curves with non-negative rational coefficients.*

Proof. Let X be an ordinary Del Pezzo surface of degree 1. It is easy to check that the sum of all (-1)-classes on X is $-240K_X$.

Using the identification of Lemma 3.3, the sum of all (-1)-classes on Y is $-240K_Y$. Using Proposition 3.6, we can write n of the (-1)-classes as the sum of a (-1)-curve and possibly some (-2)-curves, and the remaining $240 - n$ of the (-1)-classes as the sum of $-K_Y$ and some (-2)-classes. Note that $E_{Y,8}$ in the proof of Lemma 3.3 is a (-1)-curve on Y, so we have $n > 0$.

This gives us $-240K_Y$ as the sum of n (-1)-curves, $-(240-n)K_Y$, and some (-2)-curves. We transform this equation to write $-nK_Y$ as a sum of (-1)- and (-2)-curves. □

Lemma 3.8. *Let Y be a split generalized Del Pezzo surface and let E be a (-1)-class in $N^1(Y)$. Then E is irreducible if and only if $\langle E, C\rangle$ is nonnegative for every (-2)-curve C.*

Proof. See [Demazure 1980, Corollary on page 46]. □

In the case of ordinary Del Pezzo surfaces, the following result is well-known.

Proposition 3.9. *Let X be a split ordinary Del Pezzo surface of degree $d \leq 7$. Then the effective cone of X is minimally generated by the (-1)-classes on X, all of which are (-1)-curves.*

Proof. This can be proved directly (see [Hartshorne 1977, Theorem V.4.11] for a proof when $d = 3$) or can be taken as an immediate consequence of the calculation of generators for the Cox ring given in [Batyrev and Popov 2004, Theorem 3.2], making use of Lemma 3.7 in the case $d = 1$. □

We now reach our main goal for this section.

Theorem 3.10. *If Y is a split generalized Del Pezzo surface and has degree $d \leq 7$, the effective cone of Y is finitely generated by the set of (-1)- and (-2)-curves.*

Proof. Let Γ be the cone generated by the (-1)- and (-2)-curves on Y. To prove the theorem, it suffices by Lemma 3.5 to show that $\Gamma^\vee \subset \Gamma$. Let X be a split ordinary Del Pezzo surface of the same degree as Y. Identify $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3. Note that this identification takes (-1)-classes to (-1)-classes. By Proposition 3.9, $\mathrm{Eff}^1(X)$ is generated by (-1)-classes. Each (-1)-class lies in Γ by Proposition 3.6 and Lemma 3.7. Therefore $\mathrm{Eff}^1(X) \subset \Gamma$. It follows immediately that $\Gamma^\vee \subset \mathrm{Eff}^1(X)^\vee$. From Lemma 3.5 we have $\mathrm{Eff}^1(X)^\vee \subset \mathrm{Eff}^1(X)$. Thus $\Gamma^\vee \subset \Gamma$ and hence $\Gamma = \mathrm{Eff}^1(Y)$, again by Lemma 3.5. □

Remark 3.11. A generalization of Theorem 3.10 has already been proved by Lahyane and Harbourne [2005, Lemma 4.1]. We include our presentation both as a summary of results that we will use later and also because the approach here seems to have interest in its own right.

Corollary 3.12. *Let X be a split ordinary Del Pezzo surface and Y a split generalized Del Pezzo surface with $\deg(X) = \deg(Y) \leq 7$. Identifying $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3, we have $\mathrm{Eff}^1(X) \subset \mathrm{Eff}^1(Y)$ and $\mathrm{Nef}(X) \supset \mathrm{Nef}(Y)$.*

Let $\Gamma \subset N^1(Y)_{\mathbb{R}}$ be the cone spanned by the set of (-2)-curves on Y. Then $\mathrm{Eff}^1(Y)$ is the sum of $\mathrm{Eff}^1(X)$ and Γ, and $\mathrm{Nef}(Y) = \mathrm{Nef}(X) \cap \Gamma^\vee$. □

4. Inductive method

With these preliminaries in place, we now turn to proving Theorem 1.2. For a generalized Del Pezzo surface Y and any class $D \in N^1(Y)_{\mathbb{R}}$, we denote by $D^\perp$ the hyperplane

$$D^\perp := \{C \in N_1(Y)_{\mathbb{R}} : \langle D, C\rangle = 0\}.$$

Lemma 4.1. *Let Y be a split generalized Del Pezzo surface and E a (-1)-curve on Y. Let $\pi_E : Y \to Y_E$ be the contraction of E. Then*

$$\pi_E^* : N^1(Y_E) \longrightarrow E^\perp \cap N^1(Y)$$

is an isomorphism and induces an isomorphism of convex cones,

$$\pi_E^*(\mathrm{Nef}(Y_E)) = \mathrm{Nef}(Y) \cap E^\perp.$$

Proof. We have $N^1(Y) = \pi_E^*(N^1(Y_E)) \oplus \mathbb{Z}E$. We may identify $\mathrm{Nef}(Y_E)$ with $\pi_E^*(\mathrm{Nef}(Y_E)) \subset E^\perp$. The inclusion $\pi_E^*(\mathrm{Nef}(Y_E)) \subset \mathrm{Nef}(Y)$ follows immediately from the projection formula. This proves $\pi_E^*(\mathrm{Nef}(Y_E)) \subset \mathrm{Nef}(Y) \cap E^\perp$.

For the reverse inclusion, let $D \in \mathrm{Nef}(Y) \cap E^\perp$. Since $E^\perp = \pi_E^*(N^1(Y_E))$, we have $D = \pi_E^* \pi_{E*} D$. Again by the projection formula, for any curve $C \subset Y_E$,

$$\langle \pi_{E*} D, C\rangle_{Y_E} = \langle D, \pi_E^* C\rangle \geq 0,$$

since $D \in \mathrm{Nef}(Y)$. □

We now prove the first of our main theorems. We repeat it here for convenience.

Theorem 1.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. For each E in the set $\mathscr{C}$ of (-1)-curves on Y, let Y_E denote the split generalized Del Pezzo surface of degree $d+1$ obtained by contracting E. Then*

$$\alpha(Y) = \sum_{E \in \mathscr{C}} \frac{1}{d(9-d)} \alpha(Y_E).$$

Proof. We follow the argument used in [Derenthal 2007, Theorem 4]. Let $\mathscr{E}$ be the set of (-1)- and (-2)-curves on Y. Then $\mathscr{E}$ is exactly the set of generators for $\mathrm{Eff}^1(Y)$ described in Theorem 3.10. Recall that the hyperplane $\mathscr{H}_Y$ is defined as

$$\mathscr{H}_Y = \{C \in N_1(Y)_\mathbb{R} : \langle -K_Y, C\rangle = 1\}.$$

The intersection $\mathscr{P}_Y = \mathrm{Nef}(Y) \cap \mathscr{H}_Y$ is a polytope with faces corresponding to $E \in \mathscr{E}$. For $E \in \mathscr{E}$, let $\mathscr{P}_E \subset \mathscr{H}_Y$ be the convex hull of the vector $\frac{1}{d}(-K_Y)$ and the face $\mathscr{P}_Y \cap E^\perp$. (Note that $-K_Y$ is nef by the definition of generalized Del Pezzo surface and $\frac{1}{d}(-K_Y)$ is in $\mathscr{P}_Y$ since $\langle -K_Y, -K_Y\rangle = d$.) Then

$$\mathscr{P}_Y = \mathrm{Nef}(Y) \cap \mathscr{H}_Y = \bigcup_{E \in \mathscr{E}} \mathscr{P}_E.$$

The intersection of any two of the $\mathscr{P}_E$ has volume zero in $\mathscr{H}_Y$ because the intersection lies in a subspace of dimension strictly less than that of $\mathscr{H}_Y$. Therefore,

$$\alpha(Y) = \mathrm{Vol}\,\mathscr{P}_Y = \sum_{E \in \mathscr{E}} \mathrm{Vol}\,\mathscr{P}_E.$$

For each (-2)-curve E, $\langle K_Y, E\rangle = 0$ and hence $\frac{1}{d}(-K_Y) \in E^\perp$. Thus $\mathscr{P}_E$ lies in the hyperplane $\mathscr{H}_Y \cap E^\perp$ of dimension $\dim(\mathscr{H}_Y) - 1$, and so $\mathscr{P}_E$ has volume zero. We thus reduce to $\mathrm{Vol}\,\mathscr{P}_Y = \sum_{E \in \mathscr{C}} \mathrm{Vol}\,\mathscr{P}_E$.

For $E \in \mathscr{C}$, let $\pi_E : Y \to Y_E$ be the contraction. By Lemma 4.1 we have $\pi_E^* \mathscr{H}_{Y_E} = \mathscr{H}_Y \cap E^\perp$. This identifies the base of the cone $\mathscr{P}_E$ as $\mathscr{P}_Y \cap E^\perp = \pi_E^* \mathscr{P}_{Y_E}$. Thus $\mathscr{P}_E$ is a cone of dimension $9-d$ with height $1/d$ and base volume $\mathrm{Vol}(\pi_E^* \mathscr{P}_{Y_E})$. By Lemma 4.1, the sublattices $N^1(Y_E) \subset N^1(Y_E)_{\mathbb{R}}$ and $\pi_E^*(N^1(Y_E)) = E^\perp \cap N^1(Y) \subset E^\perp$ are isomorphic, so π_E^* is volume-preserving and $\mathrm{Vol}(\pi_E^* \mathscr{P}_{Y_E}) = \mathrm{Vol}\, \mathscr{P}_{Y_E} = \alpha(Y_E)$. Consequently,

$$\mathrm{Vol}\, \mathscr{P}_E = \frac{1}{d(9-d)} \mathrm{Vol}\, \mathscr{P}_{Y_E} = \frac{1}{d(9-d)} \alpha(Y_E).$$

Summing over $E \in \mathscr{C}$ gives the desired result. □

Remark 4.2. This generalization explains why Theorem 1.1 does not hold for $d = 7$. When one blows down a (-1)-curve on an ordinary Del Pezzo surface of degree d for $d \leq 7$ the result is an ordinary Del Pezzo surface of degree $d+1$. For $d \leq 6$, the resulting ordinary Del Pezzo surfaces all have the same nef cone volume. This is no longer true when $d = 7$. Let X_d denote an ordinary Del Pezzo surface of degree d obtained by blowing up $9-d$ points in general position on $\mathbb{P}^2$. Recall that $X_7 = \mathrm{Bl}_{p,q}(\mathbb{P}^2)$ contains three (-1)-curves: the exceptional divisors E_p and E_q, and the proper transform L_{pq} of the line through p and q. Contracting E_p or E_q results in an X_8, while contracting L_{pq} results in $\mathbb{P}^1 \times \mathbb{P}^1$. We have

$$\alpha(X_7) = \frac{1}{14}(2\alpha(X_8) + \alpha(\mathbb{P}^1 \times \mathbb{P}^1)) = \frac{1}{24}$$

since $\alpha(X_8) = \frac{1}{6}$ and $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{1}{4}$.

5. Root systems and Weyl groups

In this section, we recall some of the basic facts about the root system of (-2)-classes on a Del Pezzo surface and its associated Weyl group. We use this structure in our second main result which relates the nef cone volumes of split generalized and ordinary Del Pezzo surfaces of the same degree.

5A. *Root systems.*

Definition 5.1. A *root system* R is a finite collection of nonzero vectors in a finite-dimensional real vector space V with a nondegenerate definite inner product $\langle \cdot, \cdot \rangle$ satisfying the following conditions.

(1) The set R spans V, namely R is essential.

(2) For each $x \in R$, let $s_x : V \to V$ be the reflection through the hyperplane orthogonal to x:

$$s_x(v) = v - 2\frac{\langle x, v\rangle}{\langle x, x\rangle} x.$$

For each $x \in R$, it is required that s_x takes R to R.

(3) For every $x_1, x_2 \in R$,

$$2\frac{\langle x_1, x_2 \rangle}{\langle x_2, x_2 \rangle}$$

is an integer, that is, R is crystallographic.

(4) If $x \in R$ and $cx \in R$, then $c \in \{1, -1\}$, that is, R is reduced.

Definition 5.2. A morphism of root systems from $R \subset V$ to $R' \subset V'$ is a linear map $\Phi : V \to V'$ such that (1) $\Phi(R) \subset R'$, and (2) Φ preserves inner products up to a scalar multiple, that is, there is a $c \in \mathbb{R}$ such that $\langle \Phi(x), \Phi(y) \rangle = c \cdot \langle x, y \rangle$. Equivalently, the integers $2\langle x_1, x_2 \rangle / \langle x_2, x_2 \rangle$ are preserved for all $x_1, x_2 \in R$.

Remark 5.3. We will sometimes refer to a root system R in a vector space V even when R does not span V. Strictly speaking, R is only a root system in the subspace it spans, but this minor abuse of language should not cause any confusion.

We recall some standard notions; for details, see [Humphreys 1990, Section 1.3], [Bourbaki 2002, Section VI.1.2], [Hall 2003, Chapter 8]. Any hyperplane in V not containing any root of R divides R into two subsets, with *positive roots* on one side (and negative roots on the other side). Those positive roots which cannot be written as a sum of other positive roots with positive coefficients form a set of *simple roots*. Each set of simple roots (for each choice of a set of positive roots) is a linearly independent set such that every root in R is either a sum of simple roots with nonnegative coefficients or a sum of simple roots with nonpositive coefficients.

A decomposition of R is a disjoint union $R = R_1 \cup \cdots \cup R_k$ such that the span of R is the direct sum of the spans of the R_j, each R_j is a root system in its span, and the spans of the R_j are orthogonal to each other. If R admits no nontrivial decomposition, then R is an *irreducible root system*. If R is reducible, it has a unique (up to order) decomposition into irreducible root systems, called the *irreducible components* of R.

Recall the classification of root systems by *Dynkin diagrams*. For a root system R and a choice of a set R_0 of simple roots in R, the Dynkin diagram of R is the graph with vertex set R_0 and an edge joining two vertices if and only the corresponding roots are not perpendicular. One labels the edges of the graph according to the angle between the roots and their relative length; for details, see [Bourbaki 2002]. The Dynkin diagram is independent of the choice of a set of simple roots. The irreducible root systems correspond to connected graphs. The irreducible components of a reducible root system R correspond exactly to the connected components of the Dynkin diagram of R. One has the well-known classification of irreducible root systems corresponding to Dynkin diagrams of types $\mathbf{A}_n$ for $n \geq 1$, $\mathbf{B}_n$ for $n \geq 2$, $\mathbf{C}_n$ for $n \geq 3$, $\mathbf{D}_n$ for $n \geq 4$, $\mathbf{E}_n$ for $6 \leq n \leq 8$, $\mathbf{F}_4$ and $\mathbf{G}_2$.

The group of orthogonal transformations generated by all s_x, $x \in R$, is finite and is called the *Weyl group* $W(R)$. A *wall* in V is a hyperplane orthogonal to an

root system R	$\mathbf{A}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
$\#W(R)$	$(n+1)!$	$2^{n-1}\cdot n!$	$2^7\cdot 3^4\cdot 5$	$2^{10}\cdot 3^4\cdot 5\cdot 7$	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$

Table 2. The orders of simply laced Weyl groups.

root system R	$\mathbf{B}_n$	$\mathbf{C}_n$	$\mathbf{F}_4$	$\mathbf{G}_2$
$\#W(R)$	$2^n\cdot n!$	$2^n\cdot n!$	$2^7\cdot 3^2$	$2^2\cdot 3$

Table 3. The orders of nonsimply laced Weyl groups.

$x \in R$. Removing the walls from V leaves a finite set of open convex cones called *chambers*. The action of $W(R)$ permutes these chambers simply transitively.

Table 2 lists all of the *simply laced* root systems (those in which all roots have the same self-intersection) and the orders of their Weyl groups. Table 3 gives the same data for the nonsimply laced root systems.

5B. ***Root systems on Del Pezzo surfaces.*** Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. By [Manin 1986, Sections 23–25], the finite set R_d of (-2)-classes on Y is a root system in $N^1(Y)_{\mathbb{R}}$ and of course depends only on the degree d. For $d \leq 6$, the roots span the hyperplane $(-K_Y)^{\perp}$. The classification of this root system is shown in Table 4.

Not only is R_d a root system, but in fact the subset of (-2)-classes that are effective on Y gives rise to a root system [Demazure 1980, Theorem III.2.b].

Theorem 5.4 (Demazure). *Let Y be a split generalized Del Pezzo surface of degree $d \leq 6$ and let R_Y^+ be the set of effective (-2)-classes on Y. Then $R_Y := R_Y^+ \cup -R_Y^+$ is a root system in $N^1(Y)$ whose simple roots are the (-2)-curves of Y and whose positive roots are R_Y^+. It is contained in R_d.* □

Remark 5.5. Urabe [1983, Main Theorem] has shown that every root system contained in R_d occurs as the root system R_Y of a generalized Del Pezzo surface Y of degree d as in Theorem 5.4, with four exceptions: the subsystem of type $7\mathbf{A}_1$ in R_2 and the subsystems of type $7\mathbf{A}_1$, $8\mathbf{A}_1$, and $\mathbf{D}_4 + 4\mathbf{A}_1$ in R_1.

Remark 5.6. The root system R_Y can have irreducible components of the following types: $\mathbf{A}_1, \ldots, \mathbf{A}_8$, $\mathbf{D}_4, \ldots, \mathbf{D}_8$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$.

For Y of degree $d \geq 3$, consider the anticanonical morphism ϕ defined by the linear series $|-K_Y|$ which maps Y to a projective space of dimension d. For $d = 2$

d	7	6	5	4	3	2	1
R_d	$\mathbf{A}_1$	$\mathbf{A}_1 \times \mathbf{A}_2$	$\mathbf{A}_4$	$\mathbf{D}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$

Table 4. Classification of root systems R_d.

(respectively, $d = 1$), let ϕ be the morphism defined by the linear series $|-2K_Y|$ (respectively, $|-3K_Y|$). Let Y' be the image of Y under ϕ. The map ϕ sends the union of (-2)-curves corresponding to any connected component of the Dynkin diagram to a singularity of Y', while it is an isomorphism between the complement of the (-2)-curves on Y and the complement of the singularities on Y'. Each singularity on Y' is a rational double point. Its type in the **ADE**-classification is given by the type of the corresponding irreducible Dynkin diagram. The surface Y' is a singular Del Pezzo surface, whose minimal desingularization is the generalized Del Pezzo surface Y.

5C. ***Weyl groups and nef cone volume.*** We proceed with the proof of our second main result, which we repeat here for the convenience of the reader.

Theorem 1.3. *Let Y be a split generalized Del Pezzo surface of degree $d \le 7$ and let X be a split ordinary Del Pezzo surface of the same degree. Then*

$$\alpha(Y) = \frac{1}{\#W(R_Y)}\alpha(X),$$

where $W(R_Y)$ is the Weyl group of the root system R_Y whose simple roots are the (-2)-curves on Y.

Proof. Identify $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3.

With notation as in the statement of Theorem 5.4, let C be the open convex cone in $N_1(Y)_{\mathbb{R}}$ dual to the cone spanned by the (-2)-curves of Y. That is,

$$C = \{\, v \in N_1(Y)_{\mathbb{R}} : \langle v, x\rangle > 0 \text{ for all } (-2)\text{-curves } x \text{ on } Y\}.$$

Since the (-2)-curves are a system of simple roots of R_Y, C is a single chamber for the Weyl group $W(R_Y)$. Recall that by Corollary 3.12, $\mathrm{Nef}(Y) = \mathrm{Nef}(X) \cap \overline{C}$. Intersecting with the hyperplane $\mathcal{H}_X$ gives $\mathcal{P}_Y = \overline{C} \cap \mathcal{P}_X$. We have

$$N^1(X)_{\mathbb{R}} = \bigcup_{w \in W(R_Y)} \overline{wC},$$

so

$$\mathcal{P}_X = \bigcup_{w \in W(R_Y)} \left(\overline{wC} \cap \mathcal{P}_X\right).$$

The sets $\overline{wC} \cap \mathcal{P}_X$, $w \in W(R_Y)$, are pairwise disjoint except along boundaries, which have zero volume. The action of $W(R_Y)$ preserves volume and fixes $\mathrm{Nef}(X)$ and $-K_X$. Therefore it fixes $\mathcal{P}_X$, and we have

$$\begin{aligned}\alpha(X) = \mathrm{Vol}\ \mathcal{P}_X &= \sum_{w \in W(R_Y)} \mathrm{Vol}(\overline{wC} \cap \mathcal{P}_X) = \#(W(R_Y))\ \mathrm{Vol}(\overline{C} \cap \mathcal{P}_X)\\ &= \#(W(R_Y))\ \mathrm{Vol}\ \mathcal{P}_Y = \#(W(R_Y)) \cdot \alpha(Y).\end{aligned}$$

□

Remark 5.7. As in Remark 5.6, let Y' be the singular Del Pezzo surface whose minimal desingularization is Y. The number $\#W(R_Y)$, and therefore $\alpha(Y)$, can be determined directly from the types of singularities on Y' as follows. The types R of the singularities of Y' coincide with the types of the irreducible components of R_Y. The orders of their Weyl groups $W(R)$ can be found in Table 2. Their product is $\#W(R_Y)$.

6. Nonsplit generalized Del Pezzo surfaces

We recall some facts about the geometry of generalized Del Pezzo surfaces that are not split and then introduce the notion of orbit root systems. The results collected here will be used in Section 7 to relate the nef cone volume of nonsplit generalized Del Pezzo surfaces to the nef cone volume of ordinary Del Pezzo surfaces.

6A. ***The Galois action.*** Throughout this section, we let Y be a generalized Del Pezzo surface of degree $d \leq 7$ defined over a perfect field $\mathbb{K}$ and we assume that Y contains a $\mathbb{K}$-rational point; we let $\overline{Y} = Y \times_{\mathbb{K}} \overline{\mathbb{K}}$. The Galois group $G_{\mathbb{K}} = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on $N^1(\overline{Y})$, and each automorphism of $N^1(\overline{Y})$ induced by an element of $G_{\mathbb{K}}$ preserves both the intersection form and the anticanonical class.

Proposition 6.1. *The group of automorphisms of $N^1(\overline{Y})$ which preserve the intersection form $\langle\cdot,\cdot\rangle$ and the anticanonical class $-K_Y$ is canonically isomorphic to $W(R_d)$.*

Proof. The result for ordinary Del Pezzo surfaces can be found in [Manin 1986, Theorem 23.9]. (The statement there is given only for $d \leq 6$, but the $d = 7$ case is an easy calculation.) The result holds for generalized Del Pezzo surfaces via the identification described in Lemma 3.3. □

Thus the action of $G_{\mathbb{K}}$ factors through (a subgroup of) the finite group $W(R_d)$.

Proposition 6.2. *Let Y be a generalized Del Pezzo surface defined over the field $\mathbb{K}$ containing a $\mathbb{K}$-rational point. Then $N^1(Y) = N^1(\overline{Y})^{G_{\mathbb{K}}}$.*

Recall that if S is a set on which the group G acts, the standard notation

$$S^G = \{s : gs = s \text{ for all } g \in G\}$$

denotes the set of fixed points of the action.

Proof. A result of Colliot-Thélène and Sansuc [1987, Theorem 2.1.2, Claim (iii)] assures that under the hypotheses of the proposition, $\mathrm{Pic}(Y) = \mathrm{Pic}(\overline{Y})^{G_{\mathbb{K}}}$. Since the intersection form on $\mathrm{Pic}(\overline{Y})$ is nondegenerate, we have $\mathrm{Pic}(\overline{Y}) = N^1(\overline{Y})$. Finally, to show $N^1(Y) = \mathrm{Pic}(Y)$ it suffices to prove that a divisor is numerically trivial on $\mathrm{Pic}(\overline{Y})$ if it is numerically trivial on $\mathrm{Pic}(Y)$.

Suppose $D \in \operatorname{Pic} Y$ is numerically trivial in $\operatorname{Pic} Y$. Let E be any divisor class on $\overline{Y}$. Recall that the action of $G_{\mathbb{K}}$ on $N^1(\overline{Y})$ factors through the finite Weyl group $W(R_d)$, so the $G_{\mathbb{K}}$-orbit of E is finite. Say this orbit is $\{E_1, \dots, E_s\}$. Since $G_{\mathbb{K}}$ preserves the intersection form on $\overline{Y}$ and D is $G_{\mathbb{K}}$-invariant,

$$\langle D, E\rangle = \frac{1}{s}\sum_i \langle D, E_i\rangle = \Big\langle D, \frac{1}{s}\sum_i E_i\Big\rangle = 0$$

because $(1/s)\sum E_i$ lies in $(\operatorname{Pic} \overline{Y})^{G_{\mathbb{K}}} = \operatorname{Pic} Y$.

With the above results put together, $N^1(Y) = \operatorname{Pic}(Y) = \operatorname{Pic}(\overline{Y})^{G_{\mathbb{K}}} = N^1(\overline{Y})^{G_{\mathbb{K}}}$, which proves the proposition. □

We now explain the relation between the effective cone of Y and that of $\overline{Y}$.

Proposition 6.3. *The effective cone of Y is equal to the cone of $G_{\mathbb{K}}$-invariant effective classes of $\overline{Y}$, that is,*

$$\operatorname{Eff}^1(Y) = \operatorname{Eff}^1(\overline{Y})^{G_{\mathbb{K}}}.$$

Proof. By Proposition 6.2, we have $N^1(Y) = N^1(\overline{Y})^{G_{\mathbb{K}}}$. It is clear that

$$\operatorname{Eff}^1(Y) \subseteq \operatorname{Eff}^1(\overline{Y})^{G_{\mathbb{K}}}.$$

To show the reverse inclusion, first note that if D is any effective divisor on $\operatorname{Eff}^1(\overline{Y})$, with $\mathbb{L}$ being a finite Galois extension of $\mathbb{K}$ over which D is defined, then $\sum_{\sigma\in\operatorname{Gal}(\mathbb{L}/\mathbb{K})}\sigma(D) \in \operatorname{Eff}^1(Y)$. For any $D \in \operatorname{Eff}^1(\overline{Y})^{G_{\mathbb{K}}}$ that is defined over a finite Galois extension $\mathbb{L}/\mathbb{K}$, we have

$$D = \frac{1}{\#\operatorname{Gal}(\mathbb{L}/\mathbb{K})}\sum_{\sigma\in\operatorname{Gal}(\mathbb{L}/\mathbb{K})}\sigma(D).$$

This completes the proof. □

The action of $G_{\mathbb{K}}$ on $N^1(\overline{Y})$ induces an action both on the set of (-1)-curves and on the set of (-2)-curves.

Corollary 6.4. *A set of generators for the effective cone of Y consists of, for each orbit of $G_{\mathbb{K}}$ on the sets of (-1)-curves and (-2)-curves, the sum of the classes in that orbit.* □

Note that this set of generators may fail to be minimal. (See rows 3, 6 and 9 of Table 8 for examples.)

6B. ***Orbit root systems.*** We will use the following construction in Section 7A in the case of $G_{\mathbb{K}}$ acting on the root system $R_{\overline{Y}} \subset N^1(\overline{Y})$ (as in Theorem 5.4) of $\overline{Y}$, in order to obtain a root system in $N^1(Y)$.

Definition 6.5. Let $R \subset V$ be a possibly reducible root system with a chosen set Π of positive roots. Suppose a group G acts linearly on V in such a way that it permutes the elements of R, preserves the inner product between elements of R and preserves positivity. In this case, we say that G acts on R. The set

$$\mathbb{O}(R, G) := \Bigl\{ \sum_{x \in \mathbb{O}} x : \mathbb{O} \text{ is a } G\text{-orbit of an element of } R \Bigr\}$$

is called the *orbit root system* of R with respect to G. (We show below that $\mathbb{O}(R, G)$ is indeed a root system.)

Proposition 6.6. *Let $R \subset V$ be an irreducible root system with a chosen positive system Π. Suppose G acts on R. Then $\mathbb{O}(R, G)$ is an irreducible root system as in Table 5. The simple (respectively, positive) roots of $\mathbb{O}(R, G)$ are the sums of elements of orbits of simple (respectively, positive) roots of R.*

Proof. Any group action which preserves inner products and positivity must necessarily act as an automorphism of the Dynkin diagram. Indeed, the group takes nonsimple roots to nonsimple roots, and thus takes simple roots to simple roots. Thus the group acts on the vertices of the Dynkin diagram; since the edges (and edge labelings) are determined by the inner product, they are preserved by the group. We check case by case that all nontrivial admissible group actions on irreducible Dynkin diagrams are listed in Table 5. In each case, a direct calculation shows that $\mathbb{O}(R, G)$ is indeed a root system of the listed type. □

A list similar to Table 5 has been compiled by Kac [1990, Propositions 7.9 and 7.10]. The main difference between our list and Kac's is that we use the sum of roots in an orbit, while he uses the average; because of this difference Kac's approach sometimes gives the dual root system to ours.

Lemma 6.7. *Let $R \subset V$ be a possibly reducible root system with a chosen positive system Π. Suppose G acts on R. Then G acts on the irreducible components of R in the following sense. If $R = \bigcup_{i=1}^{n} R_i$ is a decomposition of R into irreducible components and $g \in G$, the image $g(R_i)$ for any i is one of the irreducible components R_j.*

R	G	$\mathbb{O}(R, G)$
$\mathbf{A}_{2n}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{B}_n$
$\mathbf{A}_{2n+1}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{B}_{n+1}$
$\mathbf{D}_n$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{C}_{n-1}$
$\mathbf{D}_4$	$\mathbb{Z}/3\mathbb{Z}$ or $\mathfrak{S}_3$	$\mathbf{G}_2$
$\mathbf{E}_6$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{F}_4$

Table 5. Nontrivial irreducible orbit root systems.

Proof. One way to see this is by considering the Dynkin diagram D of R. Each component R_i corresponds to a connected component of the graph D. As noted above, the group G acts as a graph automorphism of D. Then each element of G must take connected components of D to connected components. $\square$

To avoid confusion between the actions of G on R and on the set of irreducible components of R, we refer to orbits in the latter set as "component orbits".

Proposition 6.8. *Let $R \subset V$ be a possibly reducible root system with a chosen positive system Π. Suppose G acts on R. Let $R_1, \ldots, R_k$ be irreducible components of R which form a set of component orbit representatives, that is, each component orbit contains exactly one of the R_i. For each i, let $G_i \subset G$ be the subgroup fixing R_i. Then $\mathbb{O}(R, G)$ is a root system and*

$$\mathbb{O}(R, G) \cong \bigcup_{i=1}^{k} \mathbb{O}(R_i, G_i). \qquad (*)$$

Proof. First, note the right-hand side is indeed a root system. For by Proposition 6.6, each $\mathbb{O}(R_i, G_i)$ is a root system contained in the subspace spanned by R_i (since each element of $\mathbb{O}(R_i, G_i)$ is a sum of one or more elements of R_i). Then if $i \neq j$, by assumption R_i and R_j are distinct irreducible components of R, so they span perpendicular subspaces of V. Therefore $\mathbb{O}(R_i, G_i)$ and $\mathbb{O}(R_j, G_j)$ are perpendicular. Hence the union on the right-hand side of $(*)$ is a perpendicular union of root systems.

Now, the spans of the component orbits are pairwise perpendicular, so we may treat them separately. We consider the orbit $i = 1$, the others being similar. Let the component orbit of R_1 consist of the components $R_{1,1} = R_1, R_{1,2}, \ldots, R_{1,p}$. Choosing elements $g_1 = \mathrm{id}_G, g_2, \ldots, g_p \in G$ such that $g_i R_1 = R_{1,i}$ for each i, we get isomorphisms

$$(\operatorname{span} R_1, R_1) \cong (\operatorname{span} R_{1,2}, R_{1,2}) \cong \cdots \cong (\operatorname{span} R_{1,p}, R_{1,p}).$$

Under this identification we have an isomorphism of the diagonal

$$\Delta \subset (\operatorname{span} R_1)^p \cong (\operatorname{span} R_{1,1}) \oplus \cdots \oplus (\operatorname{span} R_{1,p})$$

with $\operatorname{span}(R_1)$ by projection onto the first factor. Note that this projection preserves angles and ratios of lengths, but divides all lengths by a factor of $\sqrt{p}$. One can check that the projection takes $\mathbb{O}(R_{1,1} \cup \cdots \cup R_{1,p}, G)$ to $\mathbb{O}(R_1, G_1)$, as desired.

More precisely, if $\mathbb{O}$ is the orbit of $r \in R_1$ under G_1, then $\mathbb{O} \cup g_2\mathbb{O} \cup \cdots \cup g_p\mathbb{O}$ is the orbit of r under G. Then $g_i \sum_{x \in \mathbb{O}} x = \sum_{x \in g_i\mathbb{O}} x$ is an element of $\mathbb{O}(R_{1,i}, g_i G_1 g_i^{-1})$ where $g_i G_1 g_i^{-1}$ is the subgroup of G fixing $R_{1,i}$, while $\sum_{i=1}^{p} g_i \sum_{x \in \mathbb{O}} x$ is an element of $\mathbb{O}(R_{1,1} \cup \cdots \cup R_{1,p}, G)$, which lies in Δ. It is projected to the element $\sum_{x \in \mathbb{O}} x$ of $\mathbb{O}(R_1, G_1)$. $\square$

Corollary 6.9. *In the setting of Proposition 6.8,*

$$W(\mathbb{O}(R, G)) \cong \prod_{i=1}^{k} W(\mathbb{O}(R_i, G_i)). \qquad \square$$

7. Nef cone volume of nonsplit generalized Del Pezzo surfaces

Let Y be a nonsplit generalized Del Pezzo surface of degree at most 7, defined over a perfect field $\mathbb{K}$. As in Section 6 we continue to assume that Y contains a $\mathbb{K}$-rational point. Then $G_{\mathbb{K}} = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on the set of (-2)-curves on $\overline{Y}$ and on the associated root system. In this situation, we can construct an orbit root system as in Definition 6.5. As in the split case (Theorem 1.3), this allows us to relate the nef cone volume of Y to a volume associated to an ordinary Del Pezzo surface of the same degree. In Section 7B, we compute this volume for all nonsplit Del Pezzo surfaces of degree at least 5.

7A. ***Nef cone volume of pairs.*** Using Proposition 6.1, we can associate to a generalized Del Pezzo surface Y of degree $d \leq 7$ the pair $(\overline{Y}, H_Y)$, where $H_Y \subset W(R_d)$ is the image of $G_{\mathbb{K}}$ under the homomorphism

$$G_{\mathbb{K}} \to \mathrm{Aut}\big(N^1(\overline{Y}), \langle\cdot\,,\cdot\rangle, -K_Y\big) \cong W(R_d).$$

Note that $G_{\mathbb{K}}$ and therefore also H_Y acts on the set of (-2)-curves on $\overline{Y}$ and also on the set of its (-1)-curves.

Remark 7.1. To every generalized Del Pezzo surface Y over $\mathbb{K}$ there is the associated pair $(\overline{Y}, H_Y)$, as described above. The "realization problem for pairs" is to describe which pairs $(\overline{Y}, H)$ are obtained in this manner. That is, for which pairs $(\overline{Y}, H)$, consisting of a split generalized Del Pezzo surface $\overline{Y}$ over $\overline{\mathbb{K}}$ of degree d and a subgroup $H \subset W(R_d)$ acting on the set of (-2)-curves, is there a Y defined over $\mathbb{K}$ such that $\overline{Y} = Y \times_{\mathbb{K}} \overline{\mathbb{K}}$ and $H = H_Y$ is the image of $G_{\mathbb{K}}$ in $W(R_d)$?

Corn has shown that every pair $(\overline{X}, H)$, with $\overline{X}$ a split *ordinary* Del Pezzo surface of degree 6 and $H \subset W(R_6)$ arbitrary, is realizable in the above sense [Corn 2005, Theorem 5.1].

We use pairs to circumvent this realization problem. This allows us to prove comparison theorems without having to address realization (see Corollary 7.5).

We now define the nef cone volume $\alpha(Y, H)$ of a pair (Y, H) where Y is any *split* generalized Del Pezzo surface of degree $d \leq 7$ and H is any subgroup of $W(R_d)$ that acts on the set of (-2)-curves on Y. It follows from Lemma 3.8 that H also acts on the set of (-1)-curves. Note that there is no restriction on H if Y is ordinary.

For such a pair (Y, H), define $N^1(Y, H)$ to be $N^1(Y)^H$. Motivated by Corollary 6.4, we define $\mathrm{Eff}^1(Y, H)$ to be the cone in $N^1(Y, H)_{\mathbb{R}}$ generated by the sum of the classes in each orbit of H acting on the sets of (-1)-curves and (-2)-curves of Y. We naturally get a dual cone

$$\operatorname{Nef}(Y, H) := \{C \in N^1(Y, H)_{\mathbb{R}} : \langle D, C\rangle \geq 0 \text{ for all } D \in \operatorname{Eff}^1(Y, H)\}.$$

We then have the hyperplane

$$\mathcal{H}_{Y,H} := \{C \in N^1(Y, H)_{\mathbb{R}} : \langle C, -K_Y\rangle = 1\}$$

and the polytope

$$\mathcal{P}_{Y,H} := \operatorname{Nef}(Y, H) \cap \mathcal{H}_{Y,H}.$$

And so we define $\alpha(Y, H) := \operatorname{Vol}(\mathcal{P}_{Y,H})$, with respect to the Leray measure $d\mu$ defined in the analogous manner to the way it was defined in Section 2.

It is immediate from Proposition 6.2 and Corollary 6.4 that if Y is any generalized Del Pezzo surface (not necessarily split), then $\alpha(Y) = \alpha(\overline{Y}, H_Y)$.

Lemma 7.2. *Assume that Y is split and let H_1, H_2 be two conjugate subgroups in $W(R_d)$. Then $\alpha(Y, H_1) = \alpha(Y, H_2)$.*

Proof. Let $w \in W(R_d)$ be such that $H_2 = wH_1w^{-1}$. Let $\mathcal{O}_i$, $i \in I$, denote the orbits of the (-1)- and (-2)-classes under H_1. By definition, $\operatorname{Eff}^1(Y, H_1)$ is generated by the sums $\sum_{D\in\mathcal{O}_i} D$, $i \in I$. A simple calculation shows that the orbits of these classes under H_2 are given by $w\mathcal{O}_i$, $i \in I$. We have

$$\alpha(Y, H_1) = \operatorname{Vol}\big(\{C \in N^1(Y, H_1)_{\mathbb{R}} : \langle -K_Y, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0 \text{ for all } i \in I\}\big).$$

Making use of the fact that elements of $W(R_d)$ preserve the intersection form and anticanonical class and noting that elements of $W(R_d)$ are orthogonal transformations and thus preserve volumes, we compute

$$\begin{aligned}\alpha(Y, H_2) &= \operatorname{Vol}\big(\{C \in N^1(Y, H_2)_{\mathbb{R}} : \langle -K_Y, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} wD\rangle \geq 0\ \forall i \in I\}\big)\\ &= \operatorname{Vol}\big(\{C \in N^1(Y, H_2)_{\mathbb{R}} : \langle -K_Y, w^{-1}C\rangle = 1, \langle w^{-1}C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0\ \forall i \in I\}\big)\\ &= \operatorname{Vol}\big(w\{C \in N^1(Y, H_1)_{\mathbb{R}} : \langle -K_Y, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0\ \forall i \in I\}\big)\\ &= \alpha(Y, H_1). \qquad \square\end{aligned}$$

Corollary 7.3. *Let Y_1 and Y_2 be generalized Del Pezzo surfaces of degree $d \leq 7$, defined over a perfect field $\mathbb{K}$, which are geometrically isomorphic, that is, $\overline{Y_1} \cong \overline{Y_2}$. Let H_1 and H_2 denote the images of $G_{\mathbb{K}}$ under the respective homomorphisms $G_{\mathbb{K}} \to W(R_d)$. If H_1 and H_2 are conjugate in $W(R_d)$, then $\alpha(Y_1) = \alpha(Y_2)$.* $\square$

We arrive at the following analogue of Theorem 1.3. That theorem provided a comparison between the nef cone volumes of a split generalized Del Pezzo surface and of a split ordinary Del Pezzo surface of the same degree. The following theorem generalizes this to the nef cone volumes of pairs.

Theorem 7.4. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$, X a split ordinary Del Pezzo surface of the same degree, and H a subgroup of $W(R_d)$ acting on the set of (-2)-curves on Y. Let R_Y be the root system whose simple roots are the (-2)-curves on Y, and let $\mathbb{O}(R_Y, H)$ be the orbit root system associated to the action of H on R_Y as in Definition 6.5. Then*

$$\alpha(Y, H) = \frac{\alpha(X, H)}{\#W(\mathbb{O}(R_Y, H))}.$$

Proof. The proof of this theorem is a generalization of the argument that proves Theorem 1.3. Using Lemma 3.3, we identify $N^1(X)$ and $N^1(Y)$. This gives an identification of $N^1(X, H)$ and $N^1(Y, H)$. As before, $\mathrm{Nef}(Y, H)$ is the intersection of $\mathrm{Nef}(X, H)$ with the closure of a chamber defined by the simple roots of $\mathbb{O}(R_Y, H)$. As in the proof of Theorem 1.3, the chambers of the Weyl group $W(\mathbb{O}(R_Y, H))$ intersect only along boundaries, which have zero volume. They fill $N^1(Y)$. There are $\#W(\mathbb{O}(R_Y, H))$ of the chambers. From here, the proof is completed by the same steps as in the proof of Theorem 1.3. □

We arrive at our third main result, the computation of the nef cone volume of a generalized Del Pezzo surface over an arbitrary perfect field.

Corollary 7.5. *Let Y be a generalized Del Pezzo surface of degree $d \leq 7$ over the perfect field $\mathbb{K}$ and X a split ordinary Del Pezzo surface of the same degree. Let $\overline{Y} = Y \times_{\mathbb{K}} \overline{\mathbb{K}}$, and identify $N^1(\overline{Y})$ with $N^1(X)$ as in Lemma 3.3. Let $H_Y \subset W(R_d)$ be the image of $G_{\mathbb{K}}$. Let $R_{\overline{Y}} \subset R_d$ be the root system whose simple roots are (-2)-curves on $\overline{Y}$. Then*

$$\alpha(Y) = \alpha(\overline{Y}, H_Y) = \frac{\alpha(X, H_Y)}{\#W(\mathbb{O}(R_{\overline{Y}}, H_Y))}. \qquad \square$$

Using Proposition 6.6 and Corollary 6.9, the integer appearing in the denominator is straightforward to compute. This reduces the computation of the nef cone volume of an arbitrary generalized Del Pezzo surface over a nonclosed field to the computation of the nef cone volume of a pair involving a split ordinary Del Pezzo surface.

7B. ***Pairs involving ordinary Del Pezzo surfaces of high degree.*** As examples, let us compute $\alpha(X)$ for the various possible nonsplit ordinary Del Pezzo surfaces X of degree $d \geq 5$.

For $d \geq 7$ there are very few possible nontrivial Galois actions on $\overline{X}$, and we list these cases briefly.

(1) There are no nontrivial possibilities with $d = 9$: we must have $\overline{X} \cong \mathbb{P}^2$, the Galois action is trivial, and $\alpha(X) = \frac{1}{3}$.

(2) For $d = 8$, the only nontrivial form occurs when X is a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ in which the Galois action permutes the two generating rulings. In this case $\alpha(X) = \frac{1}{2}$.

(3) For $d = 7$, the only possible nontrivial form occurs when X is the blowup of two conjugate rational points on $\mathbb{P}^2$, so the Galois action interchanges the points. In this case $\alpha(X) = \frac{1}{6}$.

For $d = 5, 6$ there are many more cases. For the remainder of this section, let X be a possibly nonsplit ordinary Del Pezzo surface of degree 5 or 6 defined over a nonclosed perfect field $\mathbb{K}$. Let $\overline{X} = X \times_{\mathbb{K}} \overline{\mathbb{K}}$. As above, we have $\alpha(X) = \alpha(\overline{X}, H_X)$ where H_X is the image of the Galois group in $W(R_d)$. We compute $\alpha(X)$ by finding the values of $\alpha(\overline{X}, H)$ for all subgroups H of $W(R_d)$. (As noted in Remark 7.1, it is not obvious which subgroups H of $W(R_d)$ arise as images of Galois groups, so a priori some values $\alpha(\overline{X}, H)$ might not correspond to any $\alpha(X)$.)

For the case $d = 6$, recall that $\overline{X}$ is obtained by blowing up three noncollinear points in $\mathbb{P}^2$ and the cone $\mathrm{Eff}^1(\overline{X})$ is minimally generated by the (-1)-curves on $\overline{X}$. Let E_1, E_2, E_3 denote the exceptional curves and L denote the pullback of a line. The set of (-1)-curves is shown schematically in Figure 1: the vertices correspond to the generating classes for $\mathrm{Eff}^1(\overline{X})$, with the convenient shorthand $L_{ij} = L - E_i - E_j$. Two classes intersect if and only if the corresponding vertices in the graph are connected by an edge.

Table 6 lists the subgroups of $W(R_6) = W(\mathbf{A}_1) \times W(\mathbf{A}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3 \cong D_6$. By Lemma 7.2, it suffices to consider subgroups up to conjugacy. For each conjugacy class, we choose a representative subgroup H and give the order $\#H$ of H, the orbit structure of H on the generators of $\mathrm{Eff}^1(\overline{X})$, the rank ρ of $N^1(\overline{X}, H)$, the number m of generators in the minimal generating set of $\mathrm{Eff}^1(\overline{X}, H)$, and finally the nef cone volume $\alpha(\overline{X}, H)$. We describe H in terms of generators, using the generator

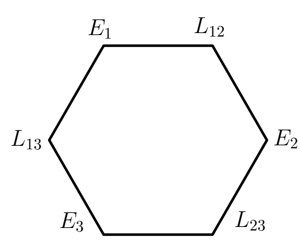

Figure 1. Configuration of (-1)-curves on an ordinary Del Pezzo surface of degree 6.

H	$\#H$	Orbit structure	ρ	m	$\alpha(\overline{X},H)$
$\langle s_{123}, s_{12}, s_{23}\rangle$	12				
$\langle s_{23}s_{123}, s_{23}s_{12}\rangle$	6	$\{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}\}$	1	1	1
$\langle s_{123}s_{12}s_{23}\rangle$	6				
$\langle s_{12}, s_{23}\rangle$	6	$\{E_1, E_2, E_3\}, \{L_{12}, L_{13}, L_{23}\}$	2	2	$\frac{1}{3}$
$\langle s_{12}s_{23}\rangle$	3				
$\langle s_{123}, s_{23}\rangle$	4	$\{E_1, L_{23}\}, \{E_2, E_3, L_{12}, L_{13}\}$	2	2	$\frac{1}{2}$
$\langle s_{123}s_{12}\rangle$	2	$\{E_1, L_{12}\}, \{E_2, L_{13}\}, \{E_3, L_{23}\}$	2	2	$\frac{1}{2}$
$\langle s_{123}\rangle$	2	$\{E_1, L_{23}\}, \{E_2, L_{13}\}, \{E_3, L_{12}\}$	3	3	$\frac{1}{8}$
$\langle s_{12}\rangle$	2	$\{E_1, E_2\}, \{E_3\}, \{L_{12}\}, \{L_{13}, L_{23}\}$	3	4	$\frac{1}{12}$
$\langle e\rangle$	1	$\{E_1\}, \{E_2\}, \{E_3\}, \{L_{12}\}, \{L_{13}\}, \{L_{23}\}$	4	6	$\frac{1}{72}$

Table 6. Values of $\alpha(\overline{X}, H)$ for a split ordinary Del Pezzo surface $\overline{X}$ of degree 6.

$s_{123} := s_{L-E_1-E_2-E_3}$ (180° rotation) of $W(\mathbf{A}_1)$ and the generators $s_{12} := s_{E_1-E_2}$ (flip swapping E_1 and E_2) and $s_{23} := s_{E_2-E_3}$ (flip swapping E_2 and E_3) of $W(\mathbf{A}_2)$.

Given H, we may compute $\alpha(\overline{X}, H)$ as follows. We explicitly compute the sums of elements in each orbit of the action of H on the generators of $\mathrm{Eff}^1(\overline{X})$, obtaining a set of generators of the cone $\mathrm{Eff}^1(\overline{X}, H)$. We compute the dual cone in $N^1(\overline{X}, H)$, obtaining $\mathrm{Nef}(\overline{X}, H)$. Intersecting with the hyperplane $\mathcal{H}_{\overline{X},H}$ gives the polytope $\mathcal{P}_{\overline{X},H}$, whose volume is $\alpha(\overline{X}, H)$. For the case when $d = 5$, recall that $\overline{X}$ is the blowup of $\mathbb{P}^2$ at 4 points in general position. Similarly to the case $d = 6$, the cone $\mathrm{Eff}^1(\overline{X})$ is generated by the (-1)-curves E_i for $1 \le i \le 4$ and $L_{ij} = L - E_i - E_j$ for $1 \le i < j \le 4$.

Figure 2 uses a different diagram to exhibit the full symmetry of the configuration of these 10 curves with respect to $W(R_5) = \mathfrak{S}_5$. (It seems impossible to make visible all of the symmetries in a diagram analogous to Figure 1). Here the minimal

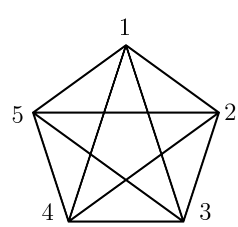

Figure 2. Configuration of (-1)-curves on an ordinary Del Pezzo surface of degree 5.

generators of $\mathrm{Eff}^1(\overline{X})$ correspond to *edges* of the graph, and two generating classes intersect if and only if the corresponding edges do *not* share a common vertex. The action of $W(R_5) = \mathfrak{S}_5$ corresponds to permuting the 5 vertices. Table 7 shows the

[12]	[13]	[14]	[15]	[23]	[24]	[25]	[34]	[35]	[45]
E_1	E_2	E_3	E_4	L_{34}	L_{24}	L_{23}	L_{14}	L_{13}	L_{12}

Table 7. Correspondence of edges in Figure 2 to generators of the effective cone of an ordinary Del Pezzo surface of degree 5.

H	$\#H$	Orbit structure	ρ	m	$\alpha(\overline{X},H)$
$\langle(12), (12345)\rangle$ $\langle(12)(34), (253)\rangle$ $\langle(1234), (13)(24), (12543)\rangle$ $\langle(12)(34), (13542)\rangle$	120 60 20 10	$\{E_1, E_2, E_3, E_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\}$	1	1	1
$\langle(12), (1234)\rangle$ $\langle(123), (12)(34), (14)(23)\rangle$	24 12	$\{E_1, E_2, E_3, L_{14}, L_{24}, L_{34}\}$, $\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	$\frac{2}{3}$
$\langle(12), (34), (345)\rangle$ $\langle(12)(34), (345)\rangle$ $\langle(12), (345)\rangle$	12 6 6	$\{E_1\}$, $\{L_{12}, L_{13}, L_{14}\}$, $\{E_2, E_3, E_4, L_{23}, L_{24}, L_{34}\}$	2	2	$\frac{1}{2}$
$\langle(12), (123)\rangle$ $\langle(123)\rangle$	6 3	$\{E_1, E_2, L_{34}\}$, $\{E_3, L_{14}, L_{24}\}$, $\{E_4, L_{13}, L_{23}\}$, $\{L_{12}\}$	3	4	$\frac{5}{24}$
$\langle(12345)\rangle$	5	$\{E_1, E_4, L_{12}, L_{14}, L_{34}\}$, $\{E_2, E_3, L_{13}, L_{23}, L_{24}\}$	1	1	1
$\langle(12)(34), (13)(24)\rangle$	4	$\{E_1\}$, $\{E_2, E_3, L_{24}, L_{34}\}$, $\{E_4, L_{23}\}$, $\{L_{12}, L_{13}\}$, $\{L_{14}\}$	3	4	$\frac{1}{6}$
$\langle(12), (34)\rangle$	4	$\{E_1, E_3, L_{14}, L_{34}\}$, $\{E_2, L_{24}\}$, $\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	$\frac{2}{3}$
$\langle(12), (34), (13)(24)\rangle$ $\langle(1234)\rangle$	8 4	$\{E_1, L_{14}\}$, $\{E_2, L_{24}\}$, $\{E_3, L_{34}\}$, $\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	$\frac{2}{3}$
$\langle(12)(34)\rangle$	2	$\{E_1\}$, $\{E_2, L_{24}\}$, $\{E_3, L_{34}\}$, $\{E_4\, L_{23}\}$, $\{L_{12}, L_{13}\}$, $\{L_{14}\}$	3	4	$\frac{1}{6}$
$\langle(12)\rangle$	2	$\{E_1\}$, $\{E_2, L_{34}\}$, $\{E_3, L_{24}\}$, $\{E_4\, L_{23}\}$, $\{L_{12}\}$, $\{L_{13}\}$, $\{L_{14}\}$	4	7	$\frac{1}{24}$
$\langle e\rangle$	1	$\{E_1\}$, $\{E_2\}$, $\{E_3\}$, $\{E_4\}$, $\{L_{12}\}$, $\{L_{13}\}$, $\{L_{14}\}$, $\{L_{23}\}$, $\{L_{24}\}$, $\{L_{34}\}$	5	10	$\frac{1}{144}$

Table 8. Values of $\alpha(\overline{X}, H)$ for a split ordinary Del Pezzo surface $\overline{X}$ of degree 7.

correspondence between the edges of the diagram and the generating classes, where we use the notation $[ij]$ to indicate the edge connecting vertex i with vertex j.

The enumeration of the conjugacy classes of subgroups of $\mathfrak{S}_5$ has been made by Götz Pfeiffer and is available online [Pfeiffer 2007]. Table 8 contains the values of $\alpha(\overline{X}, H)$ for the various possible conjugacy classes of subgroups of $\mathfrak{S}_5$.

Acknowledgments

We thank Brian Harbourne and Brendan Hassett for encouragement and helpful comments.

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Communicated by János Kollár
Received 2007-07-27 Revised 2007-10-19 Accepted 2007-12-11

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