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We prove that the Coleman–Mazur eigencurve is proper (over weight space) at integral weights in the center of weight space.

1. Introduction

The eigencurve $\mathscr{E}$ is a rigid analytic space parameterizing overconvergent — and hence classical — modular eigenforms of finite slope. Since Coleman and Mazur's original work [1998], there have been numerous generalizations [Buzzard 2008; Chenevier 2004], as well as alternative constructions using modular symbols [Ash and Stevens ≥ 2008] and p-adic representation theory [Emerton 2006]. In spite of these advances, several elementary questions about the geometry of $\mathscr{E}$ remain. One such question was raised by Coleman and Mazur: does there exist a p-adic family of finite slope overconvergent eigenforms over a punctured disk, and converging, at the puncture, to an overconvergent eigenform of infinite slope? Another way of phrasing this question is to ask whether the projection $\pi : \mathscr{E} \to \mathscr{W}$ satisfies the valuative criterion for properness[1]. In [Buzzard and Calegari 2006], this was proved in the affirmative for the particular case of tame level $N = 1$ and $p = 2$. The proof, however, was quite explicit and required (at least indirectly) that the curve $X_0(Np)$ have genus zero. In this note, we work with general p and arbitrary tame level, although our result only applies at certain arithmetic weights in the center of weight space.

Recall that the $\mathbb{C}_p$-points of $\mathscr{W}$ are the continuous homomorphisms from the Iwasawa algebra

$$\Lambda := \mathbb{Z}_p[[\varprojlim(\mathbb{Z}/Np^k\mathbb{Z})^\times]]$$

to $\mathbb{C}_p$. Let χ denote the cyclotomic character. Our main theorem is:

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[1] The curve $\mathscr{E}$ has infinite degree over weight space $\mathscr{W}$, and so the projection $\pi : \mathscr{E} \to \mathscr{W}$ cannot technically be proper.

Theorem 1.1. *Let $\mathscr{E}$ be the p-adic eigencurve of tame level N. Let D denote the closed unit disk, and let $D^\times$ denote D with the origin removed. Let $h : D^\times \to \mathscr{E}$ be a morphism such that $\pi \circ h$ extends to D. Suppose, moreover, that $(\pi \circ h)(0) = \kappa$, where κ is of the form*

$$\kappa = \chi^k \cdot \psi,$$

for $k \in \mathbb{Z}$ and ψ a finite order character of conductor dividing N. Then there exists a map $\widetilde{h} : D \to \mathscr{E}$ making the following diagram commute:

$$\begin{tikzcd} D^\times \arrow[r, "h"] \arrow[d] & \mathscr{E} \arrow[d, "\pi"] \\ D \arrow[r] \arrow[ur, "\widetilde{h}"] & \mathscr{W} \end{tikzcd}$$

Our strategy in proving Theorem 1.1 follows that of [Buzzard and Calegari 2006]. We first try to prove that finite slope overconvergent eigenforms extend far into the supersingular region whereas forms of infinite slope do not. Then, since a limit of highly overconvergent forms is also highly overconvergent, this leads to a contradiction. The main technical improvement is Corollary 3.2, which we deduce from a lemma of Wan (who attributes the result to Coleman). It is plausible that the properness of the eigencurve is a global manifestation of a purely local theorem; such an idea was suggested to the author — at least at integral weights — by Mark Kisin. However, even with current advances in the technology of local Galois representations, a natural conjectural statement implying properness has not yet been formulated. One issue to bear in mind is that slightly stronger statements one may conjecture are false. For example, there exists a pointwise sequence of finite slope forms converging to an infinite slope form [Coleman and Stein 2004].

2. Overconvergent modular forms

Let $N \geq 5$ be an integer coprime to p; let $X = X_1(N)$; and let $X_0(p) = X(\Gamma_1(N) \cap \Gamma_0(p))$. Since $N \geq 5$, the curves X and $X_0(p)$ are the compactifications of smooth moduli spaces. The curve X comes equipped with a natural sheaf ω, which, away from the cusps, is the pushforward of the sheaf of differentials on the universal modular curve. If $p \geq 5$, let A be a characteristic zero lift of the Hasse invariant with coefficients in $W(\overline{\mathbb{F}}_p)[[q]]$, and thus, $A \in H^0(X/W(\overline{\mathbb{F}}_p), \omega^{\otimes(p-1)})$ by the q-expansion principle. We further insist that A has trivial character. Such an A always exists, for example, $A = E_{p-1}$. Let $X_0(p, r) \subseteq X_0^{\mathrm{an}}(p)$ denote the connected component containing ∞ of the affinoid $\{x \in X_0^{\mathrm{an}}(p);\ |A(x)| \geq |r|\}$. Standard arguments imply that $|A(x)|$ on $X_0(p, r)$ is independent of the choice of A, provided that $v(r) < 1$.

Let $r \in \mathbb{C}_p$ be an element with $p/(p+1) > v(r) > 0$. Let χ denote the cyclotomic character; let ψ denote a finite order character of conductor dividing N; and let $k \in \mathbb{Z}$.

Definition 2.1. The overconvergent modular forms of weight $\chi^k \cdot \psi$, level N, and radius of convergence r are sections of $H^0(X_0(p, r), \omega^{\otimes k})$ on which the diamond operators act via ψ. We denote this space by $M(\mathbb{C}_p, N, \chi^k \cdot \psi, r)$. The space of overconvergent modular forms of weight $\chi^k \cdot \psi$ and level N is

$$M(\mathbb{C}_p, N, \chi^k \cdot \psi) := \bigcup_{|r|<1} M(\mathbb{C}_p, N, \chi^k \cdot \psi, r).$$

The space $M(\mathbb{C}_p, N, \chi^k \cdot \psi, r)$ has a natural Banach space structure. If $\chi^k = 1$, the norm $\|\cdot\|$ is the supremum norm.

Let $\kappa \in \mathcal{W}(\mathbb{C}_p)$ denote a point in weight space. Recall that the Eisenstein series $E(\kappa)$ is defined away from zeroes of the Kubota–Leopoldt zeta function $\zeta(\kappa)$ by the following formulas:

$$E(\kappa) = 1 + \frac{2}{\zeta(\kappa)} \sum_{n=1}^{\infty} \sigma^*_\kappa(n) q^n, \qquad \sigma^*_\kappa(n) = \sum_{(d,p)=1}^{d|n} \kappa(d) d^{-1}.$$

The coefficients of $E(\kappa)$ are rigid analytic functions on $\mathcal{W}$ away from the zeroes of ζ. If κ is trivial on the roots of unity in $\mathbb{Q}_p$, then, as a q-expansion, $E(\kappa)$ is congruent to 1 modulo the maximal ideal of $\overline{\mathbb{Z}}_p$. Coleman's idea is to define overconvergent forms of weight κ using the formal q-expansion $E(\kappa)$. Before we recall the definition, we also recall some elementary constructions related to weight space. If

$$\mathbb{Z}_{p,N} := \varprojlim (\mathbb{Z}/Np^k\mathbb{Z})^\times,$$

there is a natural isomorphism $\mathbb{Z}_{p,N} \simeq (\mathbb{Z}/N\boldsymbol{q}\mathbb{Z})^\times \times (1 + \boldsymbol{q}\mathbb{Z}_p)$, where $\boldsymbol{q} = p$ if p is odd, and $\boldsymbol{q} = 4$ otherwise. If $a \in \mathbb{Z}_{p,N}$, then $\langle\langle a \rangle\rangle$ denotes the projection of a onto the second factor, and $\tau(a) = a/\langle\langle a \rangle\rangle$ the projection onto the first. The rigid analytic space $\mathcal{W}$ has a natural group structure. Denote the connected component of the identity of $\mathcal{W}$ by $\mathcal{B}$; the component group of $\mathcal{W}$ is $(\mathbb{Z}/N\boldsymbol{q}\mathbb{Z})^\times$. If $\kappa \in \mathcal{W}(\mathbb{C}_p)$, then let $\langle \kappa \rangle$ denote the weight $a \mapsto \kappa(\langle\langle a \rangle\rangle)$ and $\tau(\kappa)$ the weight $a \mapsto \kappa(\tau(a))$; $\langle \kappa \rangle$ is the natural projection of κ onto $\mathcal{B}$. If χ denotes the cyclotomic character, then for any character ψ of $(\mathbb{Z}/\boldsymbol{q}N\mathbb{Z})^\times$, there is a unique congruence class modulo $p-1$ (or modulo 2 if $p = 2$) such that for any $k \in \mathbb{Z}$ in this congruence class, $\tau(\eta \cdot \chi^{-k})$ has conductor dividing N. We fix once and for all a choice of representative $k \in \mathbb{Z}$ for this congruence class.

We now recall the definition of overconvergent modular forms of weight κ:

Definition 2.2. Overconvergent modular forms of weight κ and tame level N are q-expansions of the form $VE_{\langle\kappa\cdot\chi^{-k}\rangle}\cdot F$, where $F\in M(\mathbb{C}_p, N, \chi^k\cdot\tau(\kappa\cdot\chi^{-k}))$.

This is not exactly the definition that occurs in Section 2.4 of [Coleman and Mazur 1998], since we have chosen to work with $\Gamma_0(p)$ structure rather than $\Gamma_1(p)$ structure. Yet both definitions are easily seen to be equivalent, using, for example, Theorem 2.2.2 of the same reference. We do not define the radius of convergence of an overconvergent form of general weight.

3. Hasse invariants

In this section, we prove some estimates for the convergence of certain overconvergent modular forms related to Hasse invariants. As in Section 2, let A be a characteristic zero lift of the Hasse invariant with coefficients in $W(\overline{\mathbb{F}}_p)[[q]]$ for $p\geq 5$.

Lemma 3.1. *Let $v(r)<1/(p+1)$, and let x be a point on $X_0(p,r)$. Then*

$$\frac{A(x)}{VA(x)}\equiv 1 \mod \frac{p}{A(x)^{p+1}}.$$

Proof. This follows directly from Lemma 2.1 of [Wan 1998], after noting that the argument remains unchanged if E_{p-1} is replaced by A. □

Corollary 3.2. *Suppose that $v(r)<1/(p+1)$. Then $\log(A/VA)\in M(\mathbb{C}_p,N,1,r)$. If $s\in\mathbb{C}_p$ is sufficiently small, then $(A/VA)^s\in M(\mathbb{C}_p,N,1,r)$.*

Proof. From Lemma 3.1, we deduce that $A/VA-1$ has norm less than one on $X_0(p,r)$, which implies the first claim. Moreover, $\|s\cdot\log(A/VA)\|\ll 1$ for sufficiently small s, and hence, if s is sufficiently small,

$$(A/VA)^s=\exp\bigl(s\cdot\log(A/VA)\bigr)$$

is well defined and lies in $M(\mathbb{C}_p,N,1,r)$. □

Remark. When $p=2$ or 3, the conclusions of Corollary 3.2 still hold with A replaced by the classical modular forms E_4 and E_6 respectively, as can be seen by a direct computation. To aid the reader in such a computation, let $f=\Delta(2\tau)/\Delta(\tau)$ and $g=(\Delta(3\tau)/\Delta(\tau))^{1/2}$ be uniformizers for $X_0(2)$ and $X_0(3)$ respectively. Then

$$\frac{E_4}{VE_4}=\frac{1+2^8f}{1+2^4f},\qquad \frac{E_6}{VE_6}=\frac{1-2\cdot 3^5g-3^9g^2}{1+2\cdot 3^2g-3^3g^2}.$$

4. Families of eigenforms

Let $h:D^\times\to\mathscr{E}$ denote an analytic family of overconvergent modular eigenforms of finite slope such that $\pi\circ h$ extends to D, and suppose that $(\pi\circ h)(0)=\kappa$, where κ is of the form $\kappa = \chi^k \cdot \psi$ with $k \in \mathbb{Z}$ and a finite order character ψ of conductor dividing N. We assume that the image of h lies in the cuspidal locus since the Eisenstein locus is finite and hence proper; see [Buzzard and Calegari 2006, Theorem 8.2]. Any weight in $\mathcal{W}(\mathbb{C}_p)$ sufficiently close to κ lies in the set $\kappa \cdot \mathcal{B}^*$, where $\mathcal{B}^*$ is defined as

$$\begin{aligned} &\left\{\eta(s) : a \mapsto \langle\langle a\rangle\rangle^{4s} \,\middle|\, s \in \mathbb{C}_p,\ v(s) > -3\right\} && \text{if } p = 2,\\ &\left\{\eta(s) : a \mapsto \langle\langle a\rangle\rangle^{6s} \,\middle|\, s \in \mathbb{C}_p,\ v(s) > -\tfrac{3}{2}\right\} && \text{if } p = 3,\\ &\left\{\eta(s) : a \mapsto \langle\langle a\rangle\rangle^{s(p-1)} \,\middle|\, s \in \mathbb{C}_p,\ v(s) > -1 + \frac{1}{p-1}\right\} && \text{if } p \geq 5. \end{aligned}$$

Our $\mathcal{B}^*$ is normalized slightly differently from that of [Coleman and Mazur 1998, p. 28], as we have included extra factors in the exponent, merely to avoid potentially troublesome notational issues later on. After shrinking D, if necessary, we may assume that $(\pi \circ h)(D^\times) \subset \kappa \cdot \mathcal{B}^*$. Given $t \in D$, we may consider $h(t)$ to be a normalized eigenform in $M(\mathbb{C}_p, N, \kappa \cdot \eta(s(t)))$, for some $\eta(s(t)) \in \mathcal{B}^*(\mathbb{C}_p)$ and analytic function $s(t)$. By assumption, $Uh(t) = \lambda(t)h(t)$ for some analytic function $\lambda(t)$ which does not vanish on $D^\times$. By considering q-expansions, we deduce that $h(0)$ exists as a p-adic modular form in the sense of Katz [1973]—for a more detailed proof, see [Buzzard and Calegari 2006, p. 229]. If $p \geq 5$, let A be as in Section 2, otherwise let $A = E_6$ if $p = 3$ or $A = E_4$ if $p = 2$. The modular form A has weight $\chi^{p-1} = \eta(1)$ if $p \geq 5$, and weights $\chi^6 = \eta(1)$ and $\chi^4 = \eta(1)$ if $p = 3$ and 2 respectively. Thus (shrinking D again if necessary), we may construct a map

$$g : D^\times \to M(\mathbb{C}_p, N, \kappa)$$

via the formula $g(t) = h(t)/A^{s(t)}$. This map is well defined as an easy consequence of Corollary B4.2.5 of [Coleman 1997], namely that E_s/A^s is overconvergent of weight zero where E_s is the Eisenstein series of weight $\eta(s)$.

Lemma 4.1. *Suppose that $v(r) < 1/(p+1)$. After shrinking D, if necessary, the image of g lands in $M(\mathbb{C}_p, N, \kappa, r^p)$.*

Proof. By construction, $g(t)$ lies in $M(\mathbb{C}_p, N, \kappa, \mu)$ for some μ with $v(\mu) > 0$. Since κ is of the form $\chi^k \cdot \psi$, we may therefore realize $g(t)$ as a section of $H^0(X_0(p, \mu), \omega^{\otimes k})$. Here we use the fact that ψ has conductor coprime to p. Consider the operator $U_t = U(A/VA)^{s(t)}$, where U is the usual operator on overconvergent modular forms [Coleman 1996; 1997]. If $s(t)$ is sufficiently small, then by Corollary 3.2, the factor $(A/VA)^{s(t)}$ lies in $M(\mathbb{C}_p, N, 1, r)$. On the other hand,

$$U_t(g(t)) = U(h(t)/VA^{s(t)}) = (\lambda(t)h(t)/A^{s(t)}) = \lambda(t)g(t).$$

If $g(t) \in M(\mathbb{C}_p, N, \kappa, \mu)$, then $(A/VA)^{s(t)}g(t) \in M(\mathbb{C}_p, N, \kappa, \max\{r, \mu\})$, and hence it follows that $U_t g(t)$ lies in $M(\mathbb{C}_p, N, \kappa, \max\{r^p, \mu^p\})$. Thus, since $\lambda(t) \neq 0$

for $t \in D^\times$, we deduce from the equality $g(t) = \lambda(t)^{-1}U_t(g(t))$ that $g(t)$ lies in $M(\mathbb{C}_p, N, \kappa, \max\{\mu^p, r^p\})$. By induction, we deduce that $g(t) \in M(\mathbb{C}_p, N, \kappa, r^p)$. $\square$

As remarked above, the q-expansion $g(0) = h(0)$ is a Katz p-adic modular form of weight κ, and moreover, by assumption, lies in the kernel of U. The argument now proceeds exactly as in Section 8 of [Buzzard and Calegari 2006]. Namely, as in the proof of Theorem 8.2 of loc. cit., we deduce that $h(0)$ is lies in $M(\mathbb{C}_p, N, \chi^k \cdot \psi, r^p)$ for any r satisfying the conditions of Lemma 4.1, namely, $v(r) < 1/(p+1)$. In particular, we may choose an r such that $h(0) \in M(\mathbb{C}_p, N, \chi^k \cdot \psi, r^p)$ and $v(r^p) > 1/(p+1)$. Yet this is in direct contradiction to Lemma 6.13 of [Buzzard and Calegari 2006] (note Remark 6.14), which says that modular forms in the kernel of U cannot converge beyond $1/(p+1)$. This completes the proof.

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