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In an earlier paper by one of us (Behrend), Donaldson–Thomas type invariants were expressed as certain weighted Euler characteristics of the moduli space. The Euler characteristic is weighted by a certain canonical $\mathbb{Z}$-valued constructible function on the moduli space. This constructible function associates to any point of the moduli space a certain invariant of the singularity of the space at the point.

Here we evaluate this invariant for the case of a singularity that is an isolated point of a $\mathbb{C}^*$-action and that admits a symmetric obstruction theory compatible with the $\mathbb{C}^*$-action. The answer is $(-1)^d$, where d is the dimension of the Zariski tangent space.

We use this result to prove that for any threefold, proper or not, the weighted Euler characteristic of the Hilbert scheme of n points on the threefold is, up to sign, equal to the usual Euler characteristic. For the case of a projective Calabi–Yau threefold, we deduce that the Donaldson–Thomas invariant of the Hilbert scheme of n points is, up to sign, equal to the Euler characteristic. This proves a conjecture of Maulik, Nekrasov, Okounkov and Pandharipande.

Introduction

The first purpose of this paper is to introduce *symmetric obstruction theories*. In a nutshell, these are obstruction theories for which the space of infinitesimal deformations is the dual of the space of infinitesimal obstructions.

As an example of an obstruction theory, consider the intersection of two smooth varieties V, W inside an ambient smooth variety M. The intersection X carries an obstruction theory. This is the 2-term complex of vector bundles

$$E \quad = \quad [\,\Omega_M|_X \xrightarrow{\mathrm{res}_V - \mathrm{res}_W} \Omega_V|_X \oplus \Omega_W|_X\,]$$

considered as an object of the derived category $D(X)$ of X, taking up degrees -1 and 0. We see that infinitesimal deformations of X are classified by $h^0(E^\vee) = T_X$,

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the sheaf of derivations on X. Moreover, the obstructions to the smoothness of X are contained in $h^1(E^\vee)$, which is called the *obstruction sheaf*, notation $ob = h^1(E^\vee)$. Note that $h^0(E^\vee)$ is intrinsic to X, but $h^1(E^\vee)$ is not. In fact, if X is smooth, all obstructions vanish, but $h^1(E^\vee)$ may be nonzero, although it is always a vector bundle, in this case.

This obstruction theory E is *symmetric*, if M is a complex symplectic manifold, i.e., hyperkähler, and V, W are Lagrangian submanifolds. In fact, the symplectic form σ induces a homomorphism $T_X \to \Omega_M$, defined by $v \mapsto \sigma(v, -)$. The fact that V and W are Lagrangian, i.e., equal to their own orthogonal complements with respect to σ, implies that there is an exact sequence

$$0 \longrightarrow T_X \longrightarrow \Omega_M|_X \longrightarrow \Omega_V|_X \oplus \Omega_W|_X \longrightarrow \Omega_X \longrightarrow 0.$$

Assuming for simplicity that X is smooth and hence this is an exact sequence of vector bundles, we see that $ob = h^1(E^\vee) = \Omega_X$ and hence T_X is, indeed, dual to ob.

In more abstract terms, what makes an obstruction theory E symmetric is a nondegenerate symmetric bilinear form of degree 1

$$\beta : E \overset{L}{\otimes} E \longrightarrow \mathbb{O}_X[1].$$

If M is an arbitrary smooth scheme, then Ω_M is a symplectic manifold in a canonical way, and the graph of any closed 1-form ω is a Lagrangian submanifold. Thus the scheme theoretic zero locus $X = Z(\omega)$ of ω is an example of the above, the second Lagrangian being the zero section.

As a special case of this, we may consider the Jacobian locus $X = Z(df)$ of a regular function on a smooth variety M. It is endowed with a canonical symmetric obstruction theory. In Donaldson–Thomas theory, where the moduli space is heuristically the critical locus of the holomorphic Chern–Simons functional, there is a canonical symmetric obstruction theory; see [Thomas 2000].

Unfortunately, we are unable to prove that every symmetric obstruction theory is locally given as the zero locus of a closed 1-form on a smooth scheme, even though we see no reason why this should not be true.

The best we can do is to show that the most general local example of a symmetric obstruction theory is the zero locus of an *almost closed* 1-form on a smooth scheme. A form ω is almost closed if its differential $d\omega$ vanishes on the zero locus $Z(\omega)$.

For the applications we have in mind we also need equivariant versions of all of the above, in the presence of a $\mathbb{G}_m$-action.

Weighted Euler characteristics and $\mathbb{G}_m$-actions. In [Behrend 2005] a new (as far as we can tell) invariant of singularities was introduced. For a singularity (X, P) the notation was

$$\nu_X(P).$$

The function ν_X is a constructible $\mathbb{Z}$-valued function on any Deligne–Mumford stack X. In [Behrend 2005], the following facts were proved about ν_X:

• If X is smooth at P, then $\nu_X(P) = (-1)^{\dim X}$.

• $\nu_X(P)\,\nu_Y(Q) = \nu_{X\times Y}(P, Q)$.

• If $X = Z(df)$ is the singular locus of a regular function f on a smooth variety M, then

$$\nu_X(P) = (-1)^{\dim M}(1 - \chi(F_P)),$$

where F_P is the Milnor fibre of f at P.

• Let X be a projective scheme endowed with a symmetric obstruction theory. The associated Donaldson–Thomas type invariant (or virtual count) is the degree of the associated virtual fundamental class. In this case, $\nu_X(P)$ is the contribution of the point P to the Donaldson–Thomas type invariant, in the sense that

$$\#^{\mathrm{vir}}(X) = \chi(X, \nu_X) = \sum_{n\in\mathbb{Z}} n\,\chi(\{\nu_X = n\}).$$

We define the weighted Euler characteristic of X to be

$$\widetilde{\chi}(X) = \chi(X, \nu_X).$$

The last property shows the importance of $\nu_X(P)$ for the calculation of Donaldson–Thomas type invariants.

In this paper we calculate the number $\nu_X(P)$ for certain kinds of singularities. In fact, we will assume that X admits a $\mathbb{G}_m$-action and a symmetric obstruction theory, which are compatible with each other. Moreover, we assume P to be an isolated fixed point for the $\mathbb{G}_m$-action. We prove that

$$\nu_X(P) = (-1)^{\dim T_X|_P}, \tag{1}$$

in this case.

We get results of two different flavors from this:

• If the scheme X admits a globally defined $\mathbb{G}_m$-action with isolated fixed points and around every fixed point admits a symmetric obstruction theory compatible with the $\mathbb{G}_m$-action we obtain

$$\widetilde{\chi}(X) = \sum_P (-1)^{\dim T_X|_P}, \tag{2}$$

the sum extending over the fixed points of the $\mathbb{G}_m$-action. This is because nontrivial $\mathbb{G}_m$-orbits do not contribute, the Euler characteristic of $\mathbb{G}_m$ being zero, and ν_X being constant on such orbits.

• If X is projective, with globally defined $\mathbb{G}_m$-action and symmetric obstruction theory, these two structures being compatible, we get

$$\#^{\mathrm{vir}}(X) = \tilde{\chi}(X) = \sum_P (-1)^{\dim T_X|_P}. \tag{3}$$

An example. It may be worth pointing out how to prove (1) in a special case. Assume the multiplicative group $\mathbb{G}_m$ acts on affine n-space $\mathbb{A}^n$ in a linear way with nontrivial weights $r_1, \dots, r_n \in \mathbb{Z}$, so that the origin P is an isolated fixed point. Let f be a regular function on $\mathbb{A}^n$, which is invariant with respect to the $\mathbb{G}_m$-action. This means that $f(x_1, \dots, x_n)$ is of degree zero, if we assign to x_i the degree r_i. Let $X = Z(df)$ be the scheme-theoretic critical set of f. The scheme X inherits a $\mathbb{G}_m$-action. It also carries a symmetric obstruction theory which is compatible with the $\mathbb{G}_m$-action.

Assume that $f \in (x_1, \dots, x_n)^3$. This is not a serious restriction. It ensures that $T_X|_P = T_{\mathbb{A}^n}|_P$ and hence that $\dim T_X|P = n$.

Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$ and $\eta \in \mathbb{C}$, $\eta \neq 0$ be chosen such that the Milnor fiber of f at the origin may be defined as

$$F_P = \{x \in \mathbb{C}^n \mid f(x) = \eta \text{ and } |x| < \epsilon\}.$$

It is easy to check that F_P is invariant under the S^1-action on $\mathbb{C}^n$ induced by our $\mathbb{G}_m$-action. Moreover, the S^1-action on F_P has no fixed points. This implies immediately that $\chi(F_P) = 0$ and hence that $\nu_X(P) = (-1)^n$.

Even though we consider this example $(Z(df), P)$ to be the prototype of a singularity admitting compatible $\mathbb{G}_m$-actions and symmetric obstruction theories, we cannot prove that every such singularity is of the form $(Z(df), P)$. We can only prove that a singularity with compatible $\mathbb{G}_m$-action and symmetric obstruction theory looks like $(Z(\omega), P)$, where ω is an almost closed $\mathbb{G}_m$-invariant 1-form on $\mathbb{A}^n$, rather than the exact invariant 1-form df. This is why the proof of (1) is more involved, in the general case. Rather than the Milnor fiber, we use the expression of $\nu_X(P)$ as a linking number, found in Proposition 4.22 of [Behrend 2005].

Lagrangian intersections. One amusing application of (3) is the following formula. Assume M is a complex symplectic manifold with a Hamiltonian $\mathbb{C}^*$-action, all of whose fixed points are isolated. Let V and W be invariant Lagrangian submanifolds. Assume their intersection is compact. Finally, assume that the Zariski tangent space of the intersection at every fixed point is even-dimensional. Then we can express the intersection number as an Euler characteristic:

$$\deg([V] \cap [W]) = \chi(V \cap W).$$

Hilbert schemes. Our result is a powerful tool for computing weighted Euler characteristics. It is a replacement for the lacking additivity of $\tilde{\chi}$ over strata.

As an example of the utility of (1), we will show in this paper that

$$\widetilde{\chi}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y), \tag{4}$$

for every smooth scheme Y of dimension 3.

In particular, if Y is projective and Calabi–Yau (i.e., has a chosen trivialization $\omega_Y = \mathcal{O}_Y$), we get that

$$\#^{\mathrm{vir}}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y),$$

where $\#^{\mathrm{vir}}$ is the virtual count à la Donaldson–Thomas [2000]. This latter formula was conjectured by Maulik, Nekrasov, Okounkov and Pandharipande in [Maulik et al. 2003]. Using the McMahon function $M(t) = \prod_{n=1}^{\infty}(1-t^n)^{-n}$, we can also express this result as

$$\sum_{n=0}^{\infty} \#^{\mathrm{vir}}(\mathrm{Hilb}^n Y)\, t^n = M(-t)^{\chi(Y)}.$$

The strategy for proving (4) is as follows. We first consider the open Calabi–Yau threefold $\mathbb{A}^3$. We exploit a suitable $\mathbb{G}_m$-action on $\mathbb{A}^3$ to prove (4) for $Y = \mathbb{A}^3$, using Formula (2). At this point, we can drop all Calabi–Yau assumptions.

Let F_n be the *punctual* Hilbert scheme. It parameterizes subschemes of $\mathbb{A}^3$ of length n which are entirely supported at the origin. Let ν_n be the restriction of $\nu_{\mathrm{Hilb}^n \mathbb{A}^3}$ to F_n. Formula (4) for $Y = \mathbb{A}^3$ is equivalent to

$$\chi(F_n, \nu_n) = (-1)^n \chi(F_n). \tag{5}$$

Finally, using more or less standard stratification arguments, we express $\widetilde{\chi}(\mathrm{Hilb}^n Y)$ in terms of $\chi(F_n, \nu_n)$. This uses the fact that the punctual Hilbert scheme of Y at a point P is isomorphic to F_n. Then (5) implies (4).

Conventions. We will work over the field of complex numbers. All stacks will be of Deligne–Mumford type. All schemes and stacks will be of finite type over $\mathbb{C}$. Hence the derived category $D_{\mathrm{qcoh}}(\mathcal{O}_X)$, of complexes of sheaves of $\mathcal{O}_X$-modules with quasicoherent cohomology is equivalent to the derived category $D(\mathrm{Qcoh}\text{-}\mathcal{O}_X)$ of the category of quasicoherent $\mathcal{O}_X$-modules (see Proposition 3.7 in Exposé II of SGA6). To fix ideas, we will denote by $D(X)$ the latter derived category and call it the derived category of X. We will often write $E \otimes F$ instead of $E \overset{L}{\otimes} F$, for objects E, F of $D(X)$.

Let X be a Deligne–Mumford stack. We will write L_X for the cutoff at -1 of the cotangent complex of X. Thus, if $U \to X$ is étale and $U \to M$ a closed immersion into a smooth scheme M, we have, *canonically*,

$$L_X|_U = [I/I^2 \to \Omega_M|_X],$$

where I is the ideal sheaf of U in M and we think of the homomorphism $I/I^2 \to \Omega_M|_X$ of coherent sheaves on U as a complex concentrated in the interval $[-1, 0]$. We will also call L_X the cotangent complex of X, and hope the reader will forgive this abuse of language. The cotangent complex L_X is an object of $D(X)$.

We will often use homological notation for objects in the derived category. This means that $E_n = E^{-n}$, for a complex $\cdots \to E^i \to E^{i+1} \to \cdots$ in $D(X)$.

For a complex of sheaves E, we denote the cohomology sheaves by $h^i(E)$.

Let us recall a few sign conventions: If $E = [E_1 \xrightarrow{\alpha} E_0]$ is a complex concentrated in the interval $[-1, 0]$, then

$$E^\vee = [E_0^\vee \xrightarrow{-\alpha^\vee} E_1^\vee]$$

is a complex concentrated in the interval $[0, 1]$. Thus the shifted dual $E^\vee[1]$ is given by

$$E^\vee[1] = [E_0^\vee \xrightarrow{\alpha^\vee} E_1^\vee]$$

and concentrated, again, in the interval $[-1, 0]$.

If $\theta : E \to F$ is a homomorphism of complexes concentrated in the interval $[-1, 0]$, such that $\theta = (\theta_1, \theta_0)$, then the shifted dual $\theta^\vee[1] : F^\vee[1] \to E^\vee[1]$ is given by $\theta^\vee[1] = (\theta_0^\vee, \theta_1^\vee)$.

Suppose $E = [E_1 \xrightarrow{\alpha} E_0]$ and $F = [F_1 \xrightarrow{\beta} F_0]$ are complexes concentrated in the interval $[-1, 0]$ and $\theta : E \to F$ and $\eta : E \to F$ homomorphisms of complexes. Then a homotopy from η to θ is a homomorphism $h : E_0 \to F_1$ such that $h \circ \alpha = \theta_1 - \eta_1$ and $\beta \circ h = \theta_0 - \eta_0$.

1. Symmetric obstruction theories

Nondegenerate symmetric bilinear forms.

Definition 1.1. Let X be a scheme or a Deligne–Mumford stack. Let $E \in D^b_{coh}(\mathcal{O}_X)$ be a perfect complex. A *nondegenerate symmetric bilinear form of degree* 1 on E is a morphism

$$\beta : E \overset{L}{\otimes} E \longrightarrow \mathcal{O}_X[1]$$

in $D(X)$, which is

(i) *symmetric*, which means that

$$\beta(e \otimes e') = (-1)^{\deg(e)\deg(e')}\beta(e' \otimes e);$$

(ii) *nondegenerate*, which means that β induces an isomorphism

$$\theta : E \longrightarrow E^\vee[1].$$

Remark 1.2. The isomorphism $\theta : E \to E^\vee[1]$ determines β as the composition

$$E \otimes E \xrightarrow{\theta\otimes\mathrm{id}} E^\vee[1] \otimes E \xrightarrow{\mathrm{tr}[1]} \mathcal{O}_X[1].$$

Symmetry of β is equivalent to the condition

$$\theta^\vee[1] = \theta.$$

Usually, we will find it more convenient to work with θ, rather than β. Thus we will think of a nondegenerate symmetric bilinear form of degree 1 on E as an isomorphism $\theta : E \to E^\vee[1]$, satisfying $\theta^\vee[1] = \theta$.

Remark 1.3. Above, we have defined nondegenerate symmetric bilinear forms of degree 1. One can generalize the definition to any degree $n \in \mathbb{Z}$. Only the case $n = 1$ will interest us in this paper.

Example 1.4. Let F be a vector bundle on X and let $\alpha : F \to F^\vee$ a symmetric bilinear form. Define the complex $E = [F \to F^\vee]$, by putting $F^\vee$ in degree 0 and F in degree -1. Then $E^\vee[1] = E$. Define $\theta = (\theta_1, \theta_0)$ by $\theta_1 = \mathrm{id}_F$ and $\theta_0 = \mathrm{id}_{F^\vee}$:

$$\begin{array}{ccccc} E & = & [F & \xrightarrow{\alpha} & F^\vee] \\ \Big\downarrow{\scriptstyle\theta} & & {\scriptstyle 1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1} \\ E^\vee[1] & = & [F & \xrightarrow{\alpha} & F^\vee] \end{array}$$

Then E is a perfect complex with perfect amplitude contained in $[-1, 0]$. Moreover, θ is a nondegenerate symmetric bilinear form on E. Note that θ is an isomorphism, and hence the form it defines is nondegenerate, whether or not α is nondegenerate.

Example 1.5. Let f be a regular function on a smooth variety M. The Hessian of f defines a symmetric bilinear form on $T_M|_X$, where $X = Z(df)$. So there is an induced symmetric bilinear form on the complex $[T_M|_X \to \Omega_M|_X]$.

Lemma 1.6. *Let E be a complex of vector bundles on X, concentrated in the interval $[-1, 0]$. Let $\theta : E \to E^\vee[1]$ be a homomorphism of complexes. Assume that $\theta^\vee[1] = \theta$, as morphisms in the derived category. Then Zariski-locally on the scheme X (or étale locally on the stack X) we can represent the derived category morphism given by θ as a homomorphism of complexes (θ_1, θ_0):*

$$\begin{array}{ccccc} E & = & [E_1 & \xrightarrow{\alpha} & E_0] \\ \Big\downarrow{\scriptstyle\theta} & & {\scriptstyle \theta_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \theta_0} \\ E^\vee[1] & = & [E_0^\vee & \xrightarrow{\alpha^\vee} & E_1^\vee] \end{array}$$

such that $\theta_1 = \theta_0^\vee$.

Proof. Let us use notation $\theta = (\psi_1, \psi_0)$. Then the equality of derived category morphisms $\theta^\vee[1] = \theta$ implies that, locally, $\theta^\vee[1] = (\psi_0^\vee, \psi_1^\vee)$ and $\theta = (\psi_1, \psi_0)$ are homotopic. So let $h : E_0 \to E_0^\vee$ be a homotopy:

$$h\alpha = \psi_1 - \psi_0^\vee,$$
$$\alpha^\vee h = \psi_0 - \psi_1^\vee.$$

Now define

$$\theta_0 = \tfrac{1}{2}(\psi_0 + \psi_1^\vee),$$
$$\theta_1 = \tfrac{1}{2}(\psi_1 + \psi_0^\vee).$$

One checks, using h, that (θ_1, θ_0) is a homomorphism of complexes, and as such, homotopic to (ψ_1, ψ_0). Thus (θ_1, θ_0) represents the derived category morphism θ, and has the required property. □

The next lemma shows that for amplitude 1 objects, every nondegenerate symmetric bilinear form locally looks like the one given in Example 1.4. Again, locally means étale locally, but in the scheme case Zariski locally.

Lemma 1.7. *Suppose that $A \in D^b_{coh}(\mathcal{O}_X)$ is of perfect amplitude contained in $[-1, 0]$, and that $\eta : A \to A^\vee[1]$ is an isomorphism satisfying $\eta^\vee[1] = \eta$. Then we can locally represent A by a homomorphism of vector bundles $\alpha : E \to E^\vee$ satisfying $\alpha^\vee = \alpha$ and the isomorphism η by the identity.*

Proof. Start by representing the derived category object A by an actual complex of vector bundles $\alpha : A_1 \to A_0$, and the morphism $\eta : A \to A^\vee[1]$ by an actual homomorphism of complexes (η_1, η_0). Then pick a point $P \in X$ and lift a basis of $\mathrm{cok}(\alpha)(P)$ to A_0. replace A_0 by the free $\mathcal{O}_X$-module on this bases, and pull back to get a quasiisomorphic complex.

Now any representative of η has, necessarily, that η_0 is an isomorphism in a neighborhood of P. By Lemma 1.6, we can assume that $\eta_1 = \eta_0^\vee$. Then both η_0 and η_1 are isomorphisms of vector bundles. Now use η_0 to identify A_0 with $A_1^\vee$. □

Isometries.

Definition 1.8. Consider perfect complexes A and B, endowed with nondegenerate symmetric forms $\theta : A \to A^\vee[1]$ and $\eta : B \to B^\vee[1]$. An isomorphism $\Phi : B \to A$, such that the diagram

$$\begin{tikzcd} B \arrow[r, "\Phi"] \arrow[d, "\eta"'] & A \arrow[d, "\theta"] \\ B^\vee[1] & A^\vee[1] \arrow[l, "\Phi^\vee[1]"'] \end{tikzcd}$$

commutes in $D(X)$, is called an *isometry* $\Phi : (B, \eta) \to (A, \theta)$.

Note that because η and θ are isomorphisms, the condition on Φ is equivalent to $\Phi^{-1} = \Phi^{\vee}[1]$, if we use η and θ to identify A with B.

We include the following lemma on the local structure of isometries for the information of the reader. Since we do not use it in the sequel, we omit the (lengthy) proof.

Lemma 1.9. *Let A and B be perfect, of amplitude contained in* $[-1, 0]$. *Suppose $\theta : A \to A^{\vee}[1]$ and $\eta : B \to B^{\vee}[1]$ are nondegenerate symmetric forms. Let $\Phi : B \to A$ be an isometry.*

Suppose that (A, θ) and (B, η) are represented as in Example 1.4 or Lemma 1.7. Thus,

$$A = [E \xrightarrow{\alpha} E^{\vee}] \quad \textit{and} \quad B = [F \xrightarrow{\beta} F^{\vee}],$$

for vector bundles E and F on X. Moreover, θ and η are the respective identities.

Assume that $\mathrm{rk}(F) = \mathrm{rk}(E)$. *Then, étale locally in X (Zariski locally if X is a scheme), we can find a vector bundle isomorphism*

$$\phi : F \longrightarrow E,$$

such that $\alpha \circ \phi = \phi^{\vee -1} \circ \beta$, and $(\phi, \phi^{\vee -1})$ represents Φ:

$$\begin{tikzcd} B \arrow[d, "\Phi"'] & = & {[F} \arrow[r, "\beta"] \arrow[d, "\phi"'] & {F^{\vee}]} \arrow[d, "\phi^{\vee -1}"] \\ A & = & {[E} \arrow[r, "\alpha"] & {E^{\vee}]} \end{tikzcd}$$

In particular, $(\phi^{-1}, \phi^{\vee})$ represents $\Phi^{\vee}[1]$.

Symmetric obstruction theories. Recall from [Behrend and Fantechi 1997] that a *perfect obstruction theory* for the scheme (or Deligne–Mumford stack) X is a morphism $\phi : E \to L_X$ in $D(X)$, where E is perfect, of amplitude in $[-1, 0]$, we have $h^0(\phi) : h^0(E) \to \Omega_X$ is an isomorphism and $h^{-1} : h^{-1}(E) \to h^{-1}(L_X)$ is onto.

We denote the coherent sheaf $h^1(E^{\vee})$ by *ob* and call it the *obstruction sheaf* of the obstruction theory. It contains in a natural way the obstructions to the smoothness of X. Even though we do not include E in the notation, *ob* is by no means an intrinsic invariant of X.

Any perfect obstruction theory for X induces a virtual fundamental class $[X]^{\mathrm{vir}}$ for X. We leave the obstruction theory out of the notation, even though $[X]^{\mathrm{vir}}$ depends on it. The virtual fundamental class is an element of $A_{\mathrm{rk}\, E}(X)$, the Chow group of algebraic cycles modulo rational equivalence. The degree of $[X]^{\mathrm{vir}}$ is equal to the rank of E.

Definition 1.10. Let X be a Deligne–Mumford stack. A *symmetric obstruction theory* for X is a triple (E, ϕ, θ) where $\phi : E \to L_X$ is a perfect obstruction theory for X and $\theta : E \to E^\vee[1]$ a nondegenerate symmetric bilinear form.

We will often refer to such an E as a symmetric obstruction theory, leaving the morphisms ϕ and θ out of the notation.

Remark 1.11. It is shown in [Behrend 2005] that for symmetric obstruction theories, the virtual fundamental class is intrinsic to X, namely it is the degree zero Aluffi class of X.

Proposition 1.12. *Every symmetric obstruction theory has expected dimension zero.*

Proof. Recall that the expected dimension of $E \to L_X$ is the rank of E. If $E \to L_X$ is symmetric, we have $\operatorname{rk} E = \operatorname{rk}(E^\vee[1]) = -\operatorname{rk} E^\vee = -\operatorname{rk} E$ and hence $\operatorname{rk} E = 0$. □

By this proposition, the following definition makes sense.

Definition 1.13. Assume X is proper and we have given a symmetric obstruction theory for X. We define the *virtual count* of X to be the number

$$\#^{\mathrm{vir}}(X) = \deg[X]^{\mathrm{vir}} = \int_{[X]^{\mathrm{vir}}} 1.$$

If X is a scheme (or an algebraic space), the virtual count is an integer. In general it may be a rational number.

Proposition 1.14. *For a symmetric obstruction theory $E \to L_X$, the obstruction sheaf is canonically isomorphic to the sheaf of differentials*:

$$ob = \Omega_X.$$

Proof. We have $ob = h^1(E^\vee) = h^0(E^\vee[1]) = h^0(E) = \Omega_X$. □

Corollary 1.15. *For a symmetric obstruction theory*,

$$h^{-1}(E) = \mathcal{H}om(\Omega_X, \mathcal{O}_X) = T_X.$$

Proof. We always have $h^{-1}(E) = ob^\vee$. □

Definition 1.16. Let E and F be symmetric obstruction theories for X. An *isomorphism* of symmetric obstruction theories is an isometry $\Phi : E \to F$ commuting with the maps to L_X.

Remark 1.17. Let $f : X \to X'$ be an étale morphism of Deligne–Mumford stacks, and suppose that X' has a symmetric obstruction theory E'. Then f^*E' is naturally a symmetric obstruction theory for X.

Conversely, if we are given symmetric obstruction theories E for X and E' for X', we will say that the morphism f is compatible with the obstruction theories if E is isomorphic to f^*E' as symmetric obstruction theory.

Remark 1.18. If X and X' are Deligne–Mumford stacks with symmetric obstruction theories E and E', then $p_1^*E \oplus p_2^*E'$ is naturally a symmetric obstruction theory for $X \times X'$. We call it the product symmetric obstruction theory.

Example 1.19. Let M be smooth and ω a closed 1-form on M. Let $X = Z(\omega)$ be the scheme-theoretic zero locus of ω. Consider ω as a linear epimorphism $\omega^\vee : T_M \to I$, where I is the ideal sheaf of X in M. Let us denote the restriction to X of the composition of $\omega^\vee$ and $d : I \to \Omega_M$ by $\nabla\omega$. It is a linear homomorphism of vector bundles $\nabla\omega : T_M|_X \to \Omega_M|_X$. Because ω is closed, $\nabla\omega$ is symmetric and, as we have seen in Example 1.4, defines a symmetric bilinear form on the complex $E = [T_M|_X \to \Omega_M|_X]$.

The morphism $\phi : E \to L_X$ as in the diagram

$$\begin{tikzcd} E \arrow[d, "\phi"'] & = & {[T_M|_X} \arrow[r, "\nabla\omega"] \arrow[d, "\omega^\vee"'] & {\Omega_M|_X]} \arrow[d, "1"] \\ L_X & = & {[I/I^2} \arrow[r, "d"] & {\Omega_M|_X]} \end{tikzcd}$$

makes E into a symmetric obstruction theory for X. In particular, note that Example 1.5 gives rise to a symmetric obstruction theory on the Jacobian locus of a regular function.

Let us remark that for the symmetry of $\nabla\omega$ and hence the symmetry of the obstruction theory given by ω, it is sufficient that ω be *almost closed*, which means that $d\omega \in I\Omega_M^2$.

A remark on the lci case. We will show that the existence of a symmetric obstruction theory puts strong restrictions on the singularities X can have.

For the following proposition, it is important to recall that we are working in characteristic zero.

Proposition 1.20. *Let $E \to L_X$ be a perfect obstruction theory, symmetric or not. A criterion for the obstruction sheaf to be locally free is that X be a reduced local complete intersection.*

Proof. As the claim is local, we may assume that E has a global resolution $E = [E_1 \to E_0]$, that $X \hookrightarrow M$ is embedded in a smooth scheme M (with ideal sheaf I) and that $E \to L_X$ is given by a homomorphism of complexes $[E_1 \to E_0] \longrightarrow [I/I^2 \to \Omega_M|_X]$. We may even assume that $E_0 \to \Omega_M|_X$ is an isomorphism of vector bundles.

Under the assumption that X is a reduced local complete intersection, I/I^2 is locally free and that $I/I^2 \to \Omega_M|_X$ is injective. Then a simple diagram chase proves that we have a short exact sequence of coherent sheaves

$$0 \longrightarrow h^{-1}(E) \longrightarrow E_1 \longrightarrow I/I^2 \longrightarrow 0.$$

Hence, $h^{-1}(E)$ is a subbundle of E_1 and $ob = h^{-1}(E)^\vee$. In particular, ob is locally free. (We always have $h^{-1}(E) = ob^\vee$; the converse is generally false.) □

Corollary 1.21. *If X is a reduced local complete intersection and admits a symmetric obstruction theory, then X is smooth.*

Proof. Because $ob = \Omega_X$, the sheaf Ω_X is locally free. This implies that X is smooth. □

Examples.

Lagrangian intersections. Let M be an algebraic symplectic manifold and V, W two Lagrangian submanifolds. Let X be the scheme-theoretic intersection. Then X carries a canonical symmetric obstruction theory.

To see this, note first of all that for a Lagrangian submanifold $V \subset M$, the normal bundle is equal to the cotangent bundle, $N_{V/M} = \Omega_V$. The isomorphism is given by $v \longmapsto \sigma(v, -)$, where σ is the symplectic form, which maps $N_{V/M} = T_M/T_V$ to $\Omega_V = T_V^\vee$. It is essentially the definition of Lagrangian, that this map is an isomorphism of vector bundles on V.

Next, note that the obstruction theory for X as an intersection of V and W can be represented by the complex

$$E \quad = \quad [\, \Omega_M \xrightarrow{\mathrm{res}_V - \mathrm{res}_W} \Omega_V \oplus \Omega_W \,]\big|_X .$$

The shifted dual is

$$E^\vee[1] \quad = \quad [\, T_V \oplus T_W \longrightarrow T_M \,]\big|_X .$$

Define $\theta : T_M \to \Omega_V \oplus \Omega_W$ as the canonical map $T_M \to N_{V/M} \oplus N_{W/M}$ given by the projections, multiplied by the scalar factor $\frac{1}{2}$. Then $(\theta^\vee, \theta)$ defines a morphism of complexes $E^\vee[1] \to E^\vee$:

$$\begin{array}{ccccccc}
E^\vee[1] & = & [T_V \oplus T_W & \longrightarrow & T_M &] \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \theta^\vee} & & \Big\downarrow{\scriptstyle \theta} & \\
E & = & [\quad \Omega_M & \longrightarrow & \Omega_V \oplus \Omega_W &]
\end{array}$$

One checks that $(\theta^\vee, \theta)$ is a quasiisomorphism. Since $(\theta^\vee, \theta)^\vee[1] = (\theta^\vee, \theta)$, this morphism of complexes defines a symmetric bilinear form on $E^\vee[1]$, hence on E. Thus E is a symmetric obstruction theory on X.

Sheaves on Calabi–Yau threefolds. Let Y be an integral proper 3-dimensional Gorenstein Deligne–Mumford stack (for example a projective threefold). By the Gorenstein assumption, Y admits a dualizing sheaf ω_Y, which is a line bundle over Y, also called the *canonical* bundle. Let $\omega_Y \to \mathcal{O}_Y$ be a nonzero homomorphism, giving rise to the short exact sequence

$$0 \longrightarrow \omega_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

so that D is an anticanonical divisor on Y. In fact, D is a Cartier divisor. Of course, D may be empty (this case we refer to as the Calabi–Yau case). Finally, choose an arbitrary line bundle L on Y. Often we are only interested in the case $L = \mathcal{O}_Y$.

Now let us define a certain moduli stack $\mathfrak{M}$ of sheaves on Y. For an arbitrary $\mathbb{C}$-scheme S, let $\mathfrak{M}(S)$ be the groupoid of pairs $(\mathscr{E}, \phi)$. Here $\mathscr{E}$ is a sheaf of $\mathcal{O}_{Y\times S}$-modules, such that

(i) $\mathscr{E}$ coherent,

(ii) $\mathscr{E}$ is flat over S,

(iii) $\mathscr{E}$ is perfect as an object of the derived category of $Y \times S$, i.e., locally admits finite free resolutions, (by Cor. 4.6.1 of Exp. III of SGA 6, this is a condition which may be checked on the fibres of $\pi : Y \times S \to S$).

The second component of the pair $(\mathscr{E}, \phi)$ is an isomorphism $\phi : \det \mathscr{E} \to L$ of line bundles on $Y \times S$. Note that the determinant $\det \mathscr{E}$ is well-defined, by Condition (iii) on $\mathscr{E}$.

We require two more conditions on $\mathscr{E}$, namely that for every point $s \in S$, denoting the fibre of $\mathscr{E}$ over s by $\mathscr{E}_s$, we have

(iv) $\mathscr{E}_s$ is simple, i.e., $\kappa(s) \to \mathrm{Hom}(\mathscr{E}_s, \mathscr{E}_s)$ is an isomorphism,

(v) the map induced by the trace $R\,\mathscr{H}\mathrm{om}(\mathscr{E}_s, \mathscr{E}_s) \to \mathcal{O}_{Y_s}$ is an isomorphism in a neighborhood of D_s.

The last condition (v) is empty in the Calabi–Yau case. It is, for example, satisfied if $\mathscr{E}_s$ is locally free of rank 1 in a neighborhood of D.

We let X be an open substack of $\mathfrak{M}$ which is algebraic (for example, fix the Hilbert polynomial and pass to stable objects, but we do not want to get more restrictive than necessary). Then X is a Deligne–Mumford stack. We will now construct a symmetric obstruction theory for X.

For this, denote the universal sheaf on $Y \times X$ by $\mathscr{E}$ and the projection $Y \times X \to X$ by π. Consider the trace map $R\,\mathscr{H}\mathrm{om}(\mathscr{E}, \mathscr{E}) \to \mathcal{O}$ and let $\mathscr{F}$ be its shifted cone, so that we obtain a distinguished triangle in $D(\mathcal{O}_{Y\times X})$:

$$\begin{tikzcd} & \mathcal{O} \arrow[dl, "+1"'] & \\ \mathscr{F} \arrow[rr] & & R\,\mathscr{H}\mathrm{om}(\mathscr{E}, \mathscr{E}) \arrow[ul, "tr"'] \end{tikzcd}$$

Note that $\mathcal{F}$ is self-dual: $\mathcal{F}^\vee = \mathcal{F}$, canonically.

Lemma 1.22. *The complex*

$$E = R\pi_* R\,\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_Y)[2]$$

is an obstruction theory for X.

Proof. This is well-known deformation theory; see [Thomas 2000]. □

The homomorphism $\omega_Y \to \mathcal{O}_Y$ induces an *isomorphism*

$$R\pi_*(\mathcal{F} \otimes \omega_Y) \longrightarrow R\pi_*\mathcal{F},$$

because the cone if this homomorphism is $R\pi_*(\mathcal{F} \otimes \mathcal{O}_D)$ and $\mathcal{F} \otimes \mathcal{O}_D = 0$, by Assumption (v), above. Dualizing and shifting, we obtain an isomorphism

$$(R\pi_*\mathcal{F})^\vee[-1] \longrightarrow \big(R\pi_*(\mathcal{F} \otimes \omega_Y)\big)^\vee[-1].$$

Exploiting the fact that $\mathcal{F}$ is self-dual, we may rewrite this as

$$(R\pi_*\mathcal{F})^\vee[-1] \longrightarrow \big(R\pi_* R\,\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_Y)\big)^\vee[-1],$$

or in other words

$$(R\pi_*\mathcal{F})^\vee[-1] \longrightarrow E^\vee[1]. \tag{6}$$

Now, relative Serre duality for the morphism $\pi : Y \times X \to X$ applied to $\mathcal{F}$ states that

$$R\pi_* R\,\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_Y[3]) = (R\pi_*\mathcal{F})^\vee,$$

or in other words

$$E = (R\pi_*\mathcal{F})^\vee[-1].$$

Thus, we see that (6) is, in fact, an isomorphism

$$\theta : E \longrightarrow E^\vee[1].$$

Lemma 1.23. *The isomorphism $\theta : E \to E^\vee[1]$ satisfies the symmetry property $\theta^\vee[1] = \theta$.*

Proof. This is just a derived version of the well-known fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, for endomorphisms A, B of a free module. □

Lemma 1.24. *The complex E has perfect amplitude contained in the interval $[-1, 0]$.*

Proof. Perfection is clear. To check the interval, note that by symmetry of E it suffices to check that the interval is $[-1, \infty]$. We have seen that $E = R\pi_*\mathcal{F}[2]$. So the interval is no wore than $[-2, \infty]$. But $h^{-2}(E) = 0$, by Assumption (iv), above. □

Corollary 1.25. *The Deligne–Mumford stack X admits, in a natural way, a symmetric obstruction theory, namely*

$$E = R\pi_* R\,\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_Y)[2] = R\pi_*\mathcal{F}[2].$$

We call this obstruction theory the *Donaldson–Thomas* obstruction theory.

In the next two propositions we mention two special cases. The first was originally introduced in [Thomas 2000], where the symmetry was pointed out, too.

Proposition 1.26. *Let Y be a smooth projective threefold with trivial canonical bundle, and let X be the fine moduli stack of stable sheaves on Y of rank $r > 0$, with fixed determinant L and with Chern classes c_2, c_3. Then X admits a symmetric obstruction theory.*

Proof. In fact, every trivialization $\omega_Y = \mathcal{O}_Y$ induces a symmetric obstruction theory. □

Proposition 1.27. *Let Y be a smooth projective threefold and D an effective anticanonical divisor on Y. Let X' be the scheme of torsion-free rank 1 sheaves with trivial determinant and fixed Chern classes c_2, c_3. Recall that such sheaves can be identified with ideal sheaves. Let $X \subset X'$ be the open subscheme consisting of ideal sheaves which define a subscheme of Y whose support is disjoint from D. Then X admits a symmetric obstruction theory.*

For example, $\mathrm{Hilb}^n(Y \setminus D)$, *the Hilbert scheme of length n subschemes of $Y \setminus D$ admits a symmetric obstruction theory.*

Proof. Again, we would like to point out that every homomorphism $\omega_Y \to \mathcal{O}_Y$ defining D gives rise to a symmetric obstruction theory on X. Even though the compactification is used in its construction, this symmetric obstruction theory does not depend on which compactification is chosen. □

Stable maps to Calabi–Yau threefolds.

Proposition 1.28. *Let Y be a Calabi–Yau threefold and let X be the open locus in the moduli stack of stable maps parameterizing immersions of smooth curves. Then the Gromov–Witten obstruction theory of X is symmetric, in a natural way.*

Proof. Let $\pi : C \to X$ be the universal curve and $f : C \to Y$ the universal map. Let F be the kernel of $f^*\Omega_Y \to \Omega_C$, which is a vector bundle of rank 2 on C. The Gromov–Witten obstruction theory on X is $E = R\pi_*(F \otimes \omega_{C/X})[1]$. By Serre duality for $\pi : C \to X$, we have $E^\vee[1] = R\pi_*(F^\vee)[1]$.

As F is of rank 2, we have $F = F^\vee \otimes \det F$. Because Y is Calabi–Yau, we have $\det F \otimes \omega_{C/X} = \mathcal{O}_C$. Putting these two facts together, we get $F \otimes \omega_{C/X} = F^\vee$ and hence $E = E^\vee[1]$. □

2. Equivariant symmetric obstruction theories

A few remarks on equivariant derived categories. Let X be a scheme with an action of an algebraic group G. Let $(\text{Qcoh-}\mathcal{O}_X)^G$ denote the abelian category of G-equivariant quasicoherent $\mathcal{O}_X$-modules. Thus, and object of $(\text{Qcoh-}\mathcal{O}_X)^G$ is a quasicoherent $\mathcal{O}_X$-module F together with descent data to the quotient stack $[X/G]$, in other words and isomorphism between p^*F and σ^*F satisfying the cocycle condition. Here p and σ are projection and action maps $X \times G \to X$, respectively. Denote by $D(X)^G$ the derived category of $(\text{Qcoh-}\mathcal{O}_X)^G$. Note that $\mathcal{O}_X$ is an object of $D(X)^G$, in a natural way.

There is the forgetful functor $D(X)^G \to D(X)$, which maps a complex of equivariant sheaves to its underlying complex of sheaves. It is an exact functor.

To simplify matters, let us make two assumptions:
(a) X admits a G-equivariant ample invertible sheaf $\mathcal{O}(1)$,
(b) G is a diagonalizable group, i.e., $G = \operatorname{Spec} \mathbb{C}[W]$ is the spectrum of the group ring of a finitely generated abelian group W. Then W is canonically identified with the character group of G.

The affine case. If $X = \operatorname{Spec} A$ is affine, A is W-graded. A G-equivariant $\mathcal{O}_X$-module is the same thing as a W-graded A-module.

We call a W-graded A-module *quasifree*, if it is free as an A-module on a set of homogeneous generators. Any quasifree W-graded A-module is isomorphic to a direct sum of shifted copies of A. Quasifree W-graded A-modules are projective objects in the abelian category $(\text{Qcoh-}\mathcal{O}_X)^G$ of W-graded A-modules. Hence this category has enough projective objects.

The global case. Let F be a G-equivariant $\mathcal{O}_X$-module. We can shift F by any character $w \in W$ of G. We denote the shift by $F[w]$. Every G-equivariant quasicoherent $\mathcal{O}_X$-module F can be written as a quotient of sheaf of the form

$$\bigoplus_{i \in I} \mathcal{O}(n_i)[w_i]. \tag{7}$$

Thus, every G-equivariant quasicoherent $\mathcal{O}_X$-module admits left resolutions consisting of objects of form (7). More generally, every bounded above complex in $D(X)^G$ can be replaced by a bounded above complex of objects of type (7). These resolutions are G-equivariant.

Since objects of the form (7) are locally free as $\mathcal{O}_X$-modules (forgetting the G-structure), we can use these resolutions to compute the derived functors of $\otimes$ and $\mathcal{H}om(-, F)$. Thus we see that for G-equivariant quasicoherent $\mathcal{O}_X$-modules E, F the quasicoherent $\mathcal{O}_X$-modules $\mathcal{T}or_i(E, F)$ and $\mathcal{E}xt^i(E, F)$ are again G-equivariant. More generally, we see that for bounded above objects E, F of $D(X)^G$, the objects $E \overset{L}{\otimes} F$ and $R\,\mathcal{H}om(E, F)$ are again in $D(X)^G$.

For a G-equivariant sheaf E, we write $E^\vee = \mathcal{H}\mathrm{om}(E, \mathcal{O}_X)$. For a bounded above object E of $D(X)^G$, we write $E^\vee = R\,\mathcal{H}\mathrm{om}(E, \mathcal{O}_X)$.

Let $\{U_i\}$ be an invariant affine open cover. Let F be a G-equivariant quasicoherent $\mathcal{O}_X$-module. Then, the Čech resolution $\mathcal{C}^\bullet(\{U_i\}, F)$ is a right resolution of F by G-equivariant quasicoherent $\mathcal{O}_X$-modules. It is an acyclic resolution for the global section functor, showing that the cohomology groups $H^i(X, F)$ are W-graded. More generally, let $f : X \to Y$ be a G-equivariant morphism. Then we see that $R^i f_* F$ are G-equivariant quasicoherent $\mathcal{O}_Y$-modules.

Moreover, if E is a bounded below complex in $D(X)^G$, we can construct the associated Čech complex $\mathcal{C}(\{U_i\}, E)$, which is a double complex. Passing to the associated single complex, we see that we may replace E by a bounded below complex of G-equivariant $\mathcal{O}_X$-modules which are acyclic for f_*, for any G-equivariant morphism $f : X \to Y$. Thus we see that the functor $Rf : D(X) \to D(Y)$ passes to a functor $Rf : D(X)^G \to D(Y)^G$.

The cotangent complex. If X is a G-scheme as above, the sheaf of Kähler differentials Ω_X and its dual $T_X = \Omega_X^\vee$ are G-equivariant.

We can use the equivariant ample line bundle L to construct a G-equivariant embedding $X \hookrightarrow M$ into a smooth G-scheme M. The cotangent complex $I/I^2 \to \Omega_M|X$ is then naturally an object of $D(X)^G$. The usual proof that L_X is a canonically defined object of $D(X)$ works equivariantly and proves that L_X is a canonically defined object of $D(X)^G$. By canonically defined, we mean that any two constructions are related by a canonical isomorphism.

Perfect objects. We call an object E in $D(X)^G$ *perfect* (of perfect amplitude in the interval $[m, n]$), if its underlying object of $D(X)$ is perfect (of perfect amplitude in the interval $[m, n]$).

Remark 2.1. If X is a scheme and E in $D(X)$ is a perfect complex, of perfect amplitude contained in $[m, n]$, then we can write E locally as a complex

$$[E^m \to \cdots \to E^n]$$

of free $\mathcal{O}_X$-modules contained in the interval $[m, n]$. This is essentially because if $E \to E''$ is an epimorphism of locally free coherent sheaves, the kernel is again locally free coherent.

In the equivariant context, we have to forgo this convenient fact. Suppose E in $D(X)^G$ is perfect, again of amplitude contained in $[m, n]$. We can, as we saw above, write E as a bounded above complex of sheaves of form (7), all of them coherent, i.e., with finite indexing set I. But when we cut off this infinite complex to fit into the interval $[m, n]$, we end up with a G-equivariant quasicoherent sheaf which is locally free coherent as an $\mathcal{O}_X$-module without the G-structure, but which is not locally quasifree and not locally projective in the category $(\mathrm{Qcoh}\text{-}\mathcal{O}_X)^G$.

Symmetric equivariant obstruction theories.

Definition 2.2. Let X be a scheme with a G-action. An *equivariant* perfect obstruction theory is a morphism $E \to L_X$ in the category $D(X)^G$, which is a perfect obstruction theory as a morphism in $D(X)$. (This definition is originally due to Graber and Pandharipande [1999].)

A *symmetric equivariant* obstruction theory, (or an *equivariant symmetric* obstruction theory) is a pair $(E \to L_X, E \to E^\vee[1])$ of morphisms in the category $D(X)^G$, such that $E \to L_X$ is an (equivariant) perfect obstruction theory and $\theta : E \to E^\vee$ is an isomorphism satisfying $\theta^\vee[1] = \theta$.

This is more than requiring that the obstruction theory be equivariant and symmetric, separately, as we can see in the following example.

Example 2.3. Let $\omega = \sum_i^n f_i dx_i$ be an almost closed 1-form on $\mathbb{A}^n$. Recall from Example 1.19 that ω defines a symmetric obstruction theory

$$H(\omega) = [T_M|_X \xrightarrow{\nabla\omega} \Omega_M|_X]$$

on the zero locus X of ω.

Define a $\mathbb{G}_m$-action on $\mathbb{A}^n$ by setting the degree of x_i to be r_i, where $r_i \in \mathbb{Z}$. Assume that each f_i is homogeneous with respect to these degrees and denote the degree of f_i by n_i. Then the zero locus X of ω inherits a $\mathbb{G}_m$-action.

If we let $\mathbb{G}_m$ act on T_M by declaring the degree of $\partial/\partial x_i$ to be equal to n_i, then $H(\omega)$ is $\mathbb{G}_m$-equivariant as well as the morphism $H(\omega) \to L_X$. Thus $H(\omega)$ is an equivariant obstruction theory.

But note that $H(\omega)$ is not equivariant symmetric. This is because the identity on $H(\omega)$ (which is θ in this case) is not $\mathbb{G}_m$-equivariant if we consider it as a homomorphism

$$H(\omega) \to H(\omega)^\vee[1].$$

Unless $n_i = -r_i$, because then the degree of $\partial/\partial x_i$ is equal to its degree as the dual of dx_i.

In the case $n_i = -r_i$, the form $\omega = \sum f_i dx_i$ is an invariant element of $\Gamma(M, \Omega_M)$, or an equivariant homomorphism $\mathbb{O}_M \to \Omega_M$. In this case we do get an equivariant symmetric obstruction theory.

The equivariant Donaldson–Thomas obstruction theory. Let G be a diagonalizable group as above. Consider a projective threefold Y, endowed with a linear G-action. Consider a G-equivariant nonzero homomorphism $\omega_Y \to \mathbb{O}_Y$, defining the G-invariant anticanonical Cartier divisor D.

Proposition 2.4. *Let X be as in Proposition 1.27. Then the Donaldson–Thomas obstruction theory of Corollary 1.25 on X is G-equivariant symmetric.*

Proof. Let X' be the compactification of X as in Proposition 1.27. Let $\mathscr{E}$ be the universal sheaf on $Y \times X$ and $Z \subset Y \times X$ be the universal subscheme. We have an exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_{Y\times X} \longrightarrow \mathscr{O}_Z \longrightarrow 0.$$

Let $\pi : Y \times X \to X$ be the projection. Note that $\mathscr{E}$ and $\mathscr{O}_Z$ are G-equivariant. This follows directly from the universal mapping property of $\mathscr{E}$.

The standard ample invertible sheaf on X' is $\det \pi_*(\mathscr{O}_Z(n))$, for n sufficiently large. It is G-equivariant, as all ingredients in its construction are. Hence X admits an equivariant ample invertible sheaf.

Next, notice that all the constructions involved in producing the obstruction theory $E = R\pi_* R\,\mathscr{H}om(\mathscr{F}, \omega_Y)[2]$ work equivariantly. Hence the symmetric obstruction theory is equivariant.

To prove that it is equivariant symmetric, we just need to remark that the bilinear form θ is induced from $\omega \to \mathscr{O}_Y$, which is equivariant, and that Serre duality is equivariant, because it is natural. $\square$

Local structure in the $\mathbb{G}_m$-case. Let $G = \mathbb{G}_m$. We will prove that Example 2.3 describes the unique example of a symmetric $\mathbb{G}_m$-equivariant obstruction theory, at least locally around a fixed point.

Lemma 2.5. *Let X be an affine $\mathbb{G}_m$-scheme with a fixed point P. Let n denote the dimension of $T_X|P$, the Zariski tangent space of X at P. Then there exists an invariant affine open neighborhood X' of P in X, a smooth $\mathbb{G}_m$-scheme M of dimension n and an equivariant closed embedding $X' \hookrightarrow M$*

Proof. Let A be the affine coordinate ring of X. The $\mathbb{G}_m$-action induces a grading on A. Let $\mathfrak{m}$ be the maximal ideal given by the point P. We can lift an eigenbasis of $\mathfrak{m}/\mathfrak{m}^2$ to homogeneous elements $x_1, \dots, x_n$ of $\mathfrak{m}$. Choose homogeneous elements $y_1, \dots, y_m$ in $\mathfrak{m}$ in such a way that $x_1, \dots, x_n, y_1, \dots, y_m$ is a set of generators of A as a $\mathbb{C}$-algebra. This defines a closed embedding $X \hookrightarrow \mathbb{A}^{n+m}$, which is equivariant if we define a $\mathbb{G}_m$-action on A^{n+m} in a suitable, obvious, way.

We have thus written A as a quotient of $\mathbb{C}[x, y]$. Let I denote the corresponding homogeneous ideal in $\mathbb{C}[x, y]$. Then we have

$$\mathfrak{m}/\mathfrak{m}^2 = (x, y)/\big(I + (x, y)^2\big).$$

Since this $\mathbb{C}$-vector space is generated by $x_1, \dots, x_n$, we have, in fact,

$$y_i \in I + (x, y)^2 + (x),$$

for $i = 1, \dots, m$. We can therefore find homogeneous elements $f_1, \dots, f_m \in I$ such that

$$y_i - f_i \in (x, y)^2 + (x) \quad \text{and} \quad \deg f_i = \deg y_i,$$

for all $i = 1, \dots, m$. Let $g \in \mathbb{C}[x, y]$ be the determinant of the Jacobian matrix $(\partial f_i/\partial y_j)$. We see that g is homogeneous of degree 0 and that $g(0, 0) = 1$. Let $U \subset \mathbb{A}^{n+m}$ be the affine open subset where g does not vanish. This is an invariant subset containing P. Let $Z \subset \mathbb{A}^{n+m}$ be the closed subscheme defined by $(f_1, \dots, f_m)$. It carries an induced $\mathbb{G}_m$-action. The intersection $M = Z \cap U$ is a smooth scheme of dimension n.

As $(f_1, \dots, f_m)$ is contained in I, X is a closed subscheme of Z. Let $X' = X \cap U$. □

Proposition 2.6. *Let X be an affine $\mathbb{G}_m$-scheme with a fixed point P and let $n = \dim T_X|_P$. Furthermore, let X be endowed with a symmetric equivariant obstruction theory $E \to L_X$. Then there exists an invariant affine open neighborhood X' of P in X, an equivariant closed embedding $X' \hookrightarrow M$ into a smooth $\mathbb{G}_m$-scheme M of dimension n and an invariant almost closed 1-form ω on M such that $X = Z(\omega)$. We can further construct an equivariant isometry $E \to H(\omega)$ commuting with the maps to L_X, but it will not be necessary for the purposes of this paper.*

Proof. We apply Lemma 2.5, to obtain the equivariant closed embedding $X' \hookrightarrow M$. Write X for X'. Let A be the affine coordinate ring of X and I the ideal of $\Gamma(\mathcal{O}_M)$ defining X.

Consider the object E of $D(X)^{\mathbb{G}_m}$. We can represent E by an infinite complex $[\cdots \to E_1 \to E_0]$ of finitely generated quasifree A-modules.

Because quasifree modules are projective, if E is represented by a bounded complex of quasifree modules as above and $E \to F$ is a morphism in $D(X)^G$, then $E \to F$ can be represented by an actual morphism of complexes, without changing E.

Thus we have morphisms of complexes of graded modules

$$[\cdots \to E_1 \to E_0] \to [I/I^2 \to \Omega_M|_X]$$

and

$$\theta : [\cdots \to E_1 \to E_0] \to [E_0^\vee \to E_1^\vee \to \cdots].$$

We can represent the equality of derived category morphisms $\theta^\vee[1] = \theta$ by a homotopy between $\theta^\vee[1]$ and θ, because E is a bounded above complex of quasifrees. Then, as in the proof of Lemma 1.6, we can replace θ_0 by $\frac{1}{2}(\theta_0 + \theta_1^\vee)$ and θ_1 by $\frac{1}{2}(\theta_1 + \theta_0^\vee)$, without changing the homotopy class of θ. Then $\theta_1 = \theta_0^\vee$.

Now we can replace E_1 by $\operatorname{cok}(E_2 \to E_1)$ and $E_1^\vee$ by $\ker(E_1^\vee \to E_2^\vee)$. Because of the perfection of E, both $\operatorname{cok}(E_2 \to E_1)$ and $\ker(E_1^\vee \to E_2^\vee)$ are projective A-modules (after forgetting the grading), which are, moreover, dual to each other.

Thus we have now represented E by a complex $[E_1 \to E_0]$ of equivariant vector bundles and $E \to L_X$ and $\theta : E \to E^\vee[1]$ by equivariant morphisms of complexes.

Moreover, $\theta = (\theta_0^\vee, \theta_0)$, for an equivariant morphism of vector bundles $\theta_0 : E_0 \to E_1^\vee$.

Now we remark that we may assume that the rank of E_0 is equal to n. Simply lift a homogeneous basis of $\Omega_X|_P$ to E_0 and replace E_0 by the quasifree module on these n elements of E_0. Then pass to an invariant open neighborhood of P over which both $E_0 \to \Omega_M|_X$ and $\theta_0 : E_0 \to E_1^\vee$ are isomorphisms. Use these isomorphisms to identify. Then our obstruction theory is given by an equivariant homomorphism

$$\begin{tikzcd} {[T_M|_X} \arrow[r, "\alpha"] \arrow[d, "\phi"'] & {\Omega_M|_X]} \arrow[d, "\mathrm{id}"] \\ {[I/I^2} \arrow[r] & {\Omega_M|_X]} \end{tikzcd}$$

such that $\alpha^\vee = \alpha$. Note that ϕ is necessarily surjective.

As we may assume that $\Omega_M|_X$ and hence $T_M|_X$ is given by a quasifree A-module, we may lift ϕ to an equivariant epimorphism $T_X \to I$. This gives the invariant 1-form ω. $\square$

3. The main theorem

Preliminaries on linking numbers. Here our dimensions are all *real* dimensions.

We work with orbifolds. Orbifolds are differentiable stacks of Deligne–Mumford type, which means that they are representable by Lie groupoids $X_1 \rightrightarrows X_0$, where source and target maps $X_1 \to X_0$ are étale (i.e., local diffeomorphisms) and the diagonal $X_1 \to X_0 \times X_0$ is proper. If a compact Lie group G acts with finite stabilizers on a manifold X, the quotient stack $[X/G]$ is an orbifold.

All our orbifolds will tacitly assumed to be *oriented*, which means that any presenting groupoid $X_1 \rightrightarrows X_0$ is oriented, i.e., X_0 and X_1 are oriented and all structure maps (in particular source and target $X_1 \to X_0$) preserve orientations.

Given an orbifold X, presented by the groupoid $X_1 \rightrightarrows X_0$, with proper diagonal $X_1 \to X_0 \times X_0$, the image of the diagonal is a closed equivalence relation on X_0. The quotient is the *coarse moduli space* of X.

We call an orbifold *compact*, if its course moduli space is compact. More generally, we call a morphism $f : X \to Y$ of orbifolds *proper*, if the induced map on coarse moduli spaces is proper.

To fix ideas, let $H^*(X)$ denote de Rham cohomology of the orbifold X. For the definition and basic properties of this cohomology theory, see [Behrend 2004]. Note that homotopy invariance holds: the projection $X \times \mathbb{R} \to X$ induces an isomorphism $H^*(X) \to H^*(X \times \mathbb{R})$.

If $f : X \to Y$ is a proper morphism of orbifolds, there exists a wrong way map $f_! : H^i(X) \to H^{i-d}(Y)$, where $d = \dim X - \dim Y$ is the relative dimension of

f. If Y is the point, then we also denote $f_!$ by $\int_X$. We will need the following properties of $f_!$:

(i) Functoriality: $(g \circ f)_! = g_! \circ f_!$.

(ii) Naturality: if $v : V \subset Y$ is an open suborbifold and $u : U \subset X$ the inverse image of U under $f : X \to Y$, we have $v^* \circ f_! = g_! \circ u^*$, where $g : U \to V$ is the restriction of f.

(iii) Projection formula: $f_!\big(f^*(\alpha) \cup \beta\big) = \alpha \cup f_!(\beta)$.

(iv) Poincaré duality: if X is a compact orbifold, the pairing $\int_X \alpha \cup \beta$ between $H^i(X)$ and $H^{n-i}(X)$ is a perfect pairing of finite dimensional $\mathbb{R}$-vector spaces ($n = \dim X$).

(v) Long exact sequence: if $\iota : Z \subset X$ is a closed suborbifold with open complement U, there is a long exact sequence ($c = \dim X - \dim Z$)

$$\cdots \xrightarrow{\partial} H^{i-c}(Z) \xrightarrow{\iota_!} H^i(X) \longrightarrow H^i(U) \xrightarrow{\partial} H^{i-c+1}(Z) \longrightarrow \cdots$$

In the situation of (v), we call $\mathrm{cl}(Z) = \iota_!(1) \in H^c(X)$ the *class* of Z.

We could use any other cohomology theory with characteristic zero coefficients which satisfies these basic properties.

Remark 3.1. Let $T \subset \mathbb{R}$ be an open interval containing the points 0 and 1. Let Z and X be a compact orbifolds and $h : Z \times T \to X$ a differentiable morphism of orbifolds such that $h_0 : Z \times \{0\} \to X$ and $h_1 : Z \times \{1\} \to X$ are isomorphisms onto closed suborbifolds Z_0 and Z_1 of X. We call h a *differentiable homotopy* between Z_0 and Z_1. It is not difficult to see, using Poincaré duality and homotopy invariance, that the existence of such an h implies that $\mathrm{cl}(Z_0) = \mathrm{cl}(Z_1) \in H^*(X)$.

Linking numbers and S^1-actions. Let A and B be closed submanifolds, both of dimension p, of a compact manifold S of dimension $2p+1$. Assume that $H^{p+1}(S) = H^p(S) = 0$ and that $A \cap B = \varnothing$. For simplicity, assume also that p is odd.

Under these assumptions we can define the *linking number* $L_S(A, B)$ as follows. By our assumption, the boundary map $\partial : H^p(S \setminus B) \to H^0(B)$ is an isomorphism. Let $\beta \in H^p(S \setminus B)$ be the unique element such that $\partial\beta = 1 \in H^0(B)$. Via the inclusion $A \to S \setminus B$ we restrict β to A and set

$$L_S(A, B) = \int_A \beta.$$

Now assume A' is another closed submanifold of S of dimension p, and $A' \cap B = \varnothing$, too. Thus $L_S(A', B)$ is defined. We wish to compare $L_S(A', B)$ with $L_S(A, B)$.

Suppose $h : Z \times T \to S$ is a differentiable homotopy between A and A', as in Remark 3.1. It is an obvious, well-known fact, that if the image of h is entirely contained in $S \setminus B$, then $L_S(A', B) = L_S(A, B)$. We wish to show that in the presence of an S^1-action, $L_S(A', B) = L_S(A, B)$, even if $h(Z \times T)$ intersects B.

Proposition 3.2. *Let S^1 act on S with finite stabilizers. Assume that A, A' and B are S^1-invariant. Finally, assume that there exists an S^1-equivariant homotopy $h : T \times Z \to S$ from A to A'. Then $L_S(A', B) = L_S(A, B)$.*

Proof. The condition that h be equivariant means that S^1 acts on Z with finite stabilizers and that h is equivariant, i.e. $h(t, \gamma \cdot z) = \gamma \cdot h(t, z)$, for all $\gamma \in S^1$ and $(t, z) \in T \times Z$.

We form the quotient orbifold $X = [S/S^1]$, which is compact of dimension $2p$. It comes together with a principal S^1-bundle $\pi : S \to X$. Let $\widetilde{A}$, $\widetilde{A}'$, $\widetilde{B}$ and $\widetilde{Z}$ be the quotient orbifolds obtained from A, A', B and Z. The homotopy h descends to a differentiable homotopy $h : T \times \widetilde{Z} \to X$ between $\widetilde{A}$ and $\widetilde{A}'$, proving that $\mathrm{cl}(\widetilde{A}) = \mathrm{cl}(\widetilde{A}') \in H^{p+1}(X)$. This conclusion is all we need the homotopy h for.

Next we will construct, for a fixed B, an element $\eta \in H^{p-1}(X)$, such that

$$L_S(A, B) = \int_X \eta \cup \mathrm{cl}(\widetilde{A}),$$

for any A, such that $A \cap B = \varnothing$. This will conclude the proof of the proposition.

In fact, let $\beta \in H^p(S \setminus B)$, such that $\partial\beta = 1 \in H^0(B)$. The S^1-bundle $S \setminus B \to X \setminus \widetilde{B}$ induces a homomorphism $\pi_! : H^p(S \setminus B) \to H^{p-1}(X \setminus \widetilde{B})$. Note that the restriction $H^{p-1}(X) \to H^{p-1}(X \setminus \widetilde{B})$ is an isomorphism, since the codimension of $\widetilde{B}$ in X is $p+1$. Thus, there exists a unique $\eta \in H^{p-1}(X)$, such that

$$\eta|_{X \setminus \widetilde{B}} = \pi_!\beta.$$

Hence

$$L_S(A, B) = \int_A \beta = \int_{\widetilde{A}} \pi_!\beta = \int_{\widetilde{A}} \eta = \int_X \eta \cup \mathrm{cl}(\widetilde{A}),$$

as claimed. The last equality follows from naturality of the wrong way maps and the projection formula. □

The proof of $\nu_X(P) = (-1)^n$. We return to the convention that dimensions are complex dimensions.

Let X be a scheme with a $\mathbb{G}_m$-action. Let $P \in X$ be a fixed point of this action. The point P is called an *isolated* fixed point, if 0 is not a weight of the induced action of $\mathbb{G}_m$ on the Zariski tangent space $T_X|_P$.

Proposition 3.3. *Let M be a smooth scheme on which $\mathbb{G}_m$ acts with an isolated fixed point $P \in M$. Let ω be an invariant (homogeneous of degree zero) almost closed 1-form on M and $X = Z(\omega)$. Assume $P \in X$. Then*

$$\nu_X(P) = (-1)^{\dim M}.$$

Proof. We will use the expression of $\nu_X(P)$ as a linking number from Proposition 4.22 of [Behrend 2005]. We choose étale homogeneous coordinates $x_1, \ldots, x_n$

for M around P and the induced étale coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$ of Ω_M. Since the linking number in question is defined inside a sufficiently small sphere in Ω_M around P (and is a topological invariant), we may as well assume that $M = \mathbb{C}^n$ and P is the origin. Of course, ω is then a 1-form holomorphic (instead of algebraic) at the origin. We write $\omega = \sum_{i=1}^n f_i dx_i$.

As in [ibid.], for $\eta \in \mathbb{C}$, $\eta \neq 0$, we write Γ_η for the graph of the section $\frac{1}{\eta}\omega$ of Ω_M. It is defined as a subspace of Ω_M by the equations $\eta p_i = f_i(x)$. It is oriented so that $M \to \Gamma_\eta$ is orientation preserving.

For $t \in \mathbb{R}$, we write Δ_t for the subspace of Ω_M defined by the equations $t p_i = \bar{x}_i$. We orient Δ_1 in such a way that the map $\mathbb{C}^n \to \Delta_1$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n)$ preserves orientation. Then we orient all other Δ_t by continuity. This amounts to the same as saying that the map $(p_1, \ldots, p_n) \mapsto (t\bar{p}_1, \ldots, t\bar{p}_n, p_1, \ldots, p_n)$ from $\mathbb{C}^n$ to Δ_t preserves orientation up to a factor of $(-1)^n$.

Proposition 4.22 of [ibid.] says that for sufficiently small $\epsilon > 0$ there exists $\eta \neq 0$ such that $\Gamma'_\eta = \Gamma_\eta \cap S_\epsilon$ is a manifold disjoint from $\Delta'_1 = \Delta_1 \cap S_\epsilon$ and

$$\nu_X(P) = L_{S_\epsilon}(\Delta'_1, \Gamma'_\eta).$$

Here S_ϵ is the sphere of radius ϵ centered at the origin P in Ω_M. It has dimension $4n - 1$. Let us fix ϵ and η.

The given $\mathbb{G}_m = \mathbb{C}^*$-action on M induces an action on $\Omega_M = \mathbb{C}^{2n}$. Let us denote the degree of x_i by r_i. Then the degrees of p_i and f_i are both equal to $-r_i$. By restricting to $S^1 \subset \mathbb{C}^*$, we get an induced S^1-action on S_ϵ. This action has finite stabilizers, because none of the r_i vanish, P being an isolated fixed point of the $\mathbb{G}_m$-action. Note that Γ'_η is an S^1-invariant submanifold of S_ϵ.

Consider the map from $\mathbb{R} \times S^{2n-1} \to S_\epsilon$ given by

$$(t, p_1, \ldots, p_n) \mapsto \frac{\epsilon}{\sqrt{1+t^2}}(t\bar{p}_1, \ldots, t\bar{p}_n, p_1, \ldots, p_n).$$

This map is an S^1-equivariant homotopy between the invariant submanifolds $\Delta'_0 = \Delta_0 \cap S_\epsilon$ and Δ'_1.

The fact that Δ'_1 is disjoint from Γ'_η follows from the fact that ω is almost closed, as explained in [ibid.]. The fact that Δ'_0 is disjoint from Γ'_η is trivial: Δ_0 is (up to orientation) the fiber of the vector bundle $\Omega_M \to M$ over the origin and Γ_η is the graph of a section. But there is no reason (at least none apparent to the authors) why there shouldn't exist values of t other than 0 or 1, for which $\Delta'_t = \Delta_1 \cap S_\epsilon$ intersects Γ_η.

Still, Proposition 3.2 implies that

$$L_{S_\epsilon}(\Delta'_1, \Gamma'_\eta) = L_{S_\epsilon}(\Delta'_0, \Gamma'_\eta).$$

Let us denote the fiber of Ω_M over the origin by $\overline{\Delta}_0$, and its intersection with S_ϵ by $\overline{\Delta}'_0$. By the correspondence between linking numbers and intersection numbers (see [Fulton 1984], Example 19.2.4), we see that $L_{S_\epsilon}(\overline{\Delta}'_0, \Gamma'_\eta)$ is equal to the intersection number of $\overline{\Delta}_0$ with Γ_η at the origin. This number is 1, as the section Γ_η intersects the fiber $\overline{\Delta}_0$ transversally.

Since the orientations of Δ_0 and $\overline{\Delta}_0$ differ by $(-1)^n$, we conclude that

$$\nu_X(P) = L_{S_\epsilon}(\Delta'_1, \Gamma'_\eta) = L_{S_\epsilon}(\Delta'_0, \Gamma'_\eta) = (-1)^n L_{S_\epsilon}(\overline{\Delta}'_0, \Gamma'_\eta) = (-1)^n,$$

which is what we set out to prove. □

Theorem 3.4. *Let X be an affine $\mathbb{G}_m$-scheme with an isolated fixed point P. Assume that X admits an equivariant symmetric obstruction theory. Then*

$$\nu_X(P) = (-1)^{\dim T_X|_P}.$$

Proof. Let $n = \dim T_X|_P$. By Proposition 2.6, we can assume that X is embedded equivariantly in a smooth scheme M of dimension n and that X is the zero locus of an invariant almost closed 1-form on M. Note that the embedding $X \hookrightarrow M$ identifies $T_X|_P$ with $T_M|_P$, so that P is an isolated point of the $\mathbb{G}_m$-action on M. Thus Proposition 3.3 implies that $\nu_X(P) = (-1)^n$. □

Corollary 3.5. *Let X be a $\mathbb{G}_m$-scheme such that all fixed points are isolated and every fixed point admits an invariant affine open neighborhood over which there exists an equivariant symmetric obstruction theory. Then we have*

$$\widetilde{\chi}(X) = \sum_P (-1)^{\dim T_X|_P},$$

the sum extending over the fixed points. Moreover, if $Z \subset X$ is an invariant locally closed subscheme, we have

$$\widetilde{\chi}(Z, X) = \sum_{P \in Z} (-1)^{\dim T_X|_P},$$

the sum extending over the fixed points in Z.

Proof. The product property of ν implies that ν_X is constant on nontrivial $\mathbb{G}_m$-orbits. The Euler characteristic of a scheme on which $\mathbb{G}_m$ acts without fixed points is zero. These two facts imply that only the fixed points contribute to $\widetilde{\chi}(X) = \chi(X, \nu_X)$. □

Corollary 3.6. *Let X be a projective scheme with a linear $\mathbb{G}_m$-action. Let X be endowed with an equivariant symmetric obstruction theory. Assume all fixed points of $\mathbb{G}_m$ on X are isolated. Then we have*

$$\#^{\mathrm{vir}}(X) = \sum_P (-1)^{\dim T_X|_P},$$

the sum extending over the fixed points of $\mathbb{G}_m$ *on* X.

Proof. We use the fact that X can be equivariantly embedded into a smooth scheme to prove that every fixed point has an invariant affine open neighborhood. Thus Corollary 3.5 applies. The main result of [Behrend 2005], Theorem 4.18, says that $\#^{\mathrm{vir}} = \tilde{\chi}(X)$. □

Application to Lagrangian intersections. Let M be an algebraic symplectic manifold with a Hamiltonian $\mathbb{G}_m$-action. Assume all fixed points are isolated. Let V and W be invariant Lagrangian submanifolds, X their intersection.

Proposition 3.7. *We have*

$$\tilde{\chi}(X) = \sum_{P \in X} (-1)^{\dim T_X|_P},$$

the sum extending over all fixed points inside X.

Proof. One checks that the action of $\mathbb{G}_m$ being Hamiltonian, i.e., that $\mathbb{G}_m$ preserves the symplectic form, implies that the symmetric obstruction theory on X is equivariant symmetric. □

Proposition 3.8. *Assume* X *is compact. Then*

$$\deg([V] \cap [W]) = \sum_{P \in X} (-1)^{\dim T_X|_P},$$

the sum extending over the fixed points contained in X.

Proof. Note that, in fact, the virtual number of points of X is the intersection number of V and W. □

Corollary 3.9. *Assume that* X *is compact and that* $\dim T_X|_P$ *is even, for all fixed points* P. *Then we have*

$$\deg([V] \cap [W]) = \chi(X).$$

4. Hilbert schemes of threefolds

The threefold $\mathbb{A}^3$**.** Let $T = \mathbb{G}_m^3$ be the standard 3-dimensional torus with character group $\mathbb{Z}^3$. Let T_0 be the kernel of the character $(1, 1, 1)$. Thus,

$$T_0 = \{(t_1, t_2, t_3) \in T \mid t_1 t_2 t_3 = 1\}.$$

We let T act in the natural way on $\mathbb{A}^3$. Write coordinates on $\mathbb{A}^3$ as x, y, z, then, as elements of the affine coordinate ring $\mathbb{C}[x, y, z]$ of $\mathbb{A}^3$, the weight of x is $(1, 0, 0)$, the weight of y is $(0, 1, 0)$ and the weight of z is $(0, 0, 1)$.

We choose on $\mathbb{A}^3$ the standard 3-form $dx \wedge dy \wedge dz$ to fix a Calabi–Yau structure. The torus T_0 acts by automorphisms of $\mathbb{A}^3$ preserving the Calabi–Yau structure.

by Proposition 2.4 we obtain a T_0-equivariant symmetric obstruction theory on $X = \mathrm{Hilb}^n \mathbb{A}^3$.

Lemma 4.1. (a) *The T_0-action on X has a finite number of fixed points. These correspond to monomial ideals in $\mathbb{C}[x, y, z]$.*
(b) *If I is such an ideal, the T_0-action on the Zariski tangent space to X at I has no invariant subspace.*
(c) If I is such an ideal and d is the dimension of the Zariski tangent space to X at I, we have $(-1)^d = (-1)^n$, in other words, the integer d has the same parity as n.

Proof. (a) Since the T_0-action on $\mathbb{A}^3$ has the origin as unique fixed point, any invariant subscheme must be supported at the origin. Let $I \subset \mathbb{C}[x, y, z]$ be the corresponding ideal; I must be generated by eigenvectors of the torus action on the polynomial ring. Any eigenvector can be written uniquely in the form $m\, g(xyz)$ where m is a monomial and $g \in \mathbb{C}[t]$ is a polynomial with $g(0) \neq 0$. However, since the ideal is supported at the origin, the zero locus of $g(xyz)$ is disjoint from the zero locus of I, and so by Hilbert's Nullstellensatz, the monomial m is also in I. Hence every T_0-invariant ideal is generated by monomials.

(b) Let us write $A = \mathbb{C}[x, y, z]$. The tangent space in question is $\mathrm{Hom}(I, A/I)$. We will prove that none of the weights $w = (w_1, w_2, w_3)$ of T on $\mathrm{Hom}_A(I, A/I)$ can satisfy $w_1, w_2, w_3 < 0$ or $w_1, w_2, w_3 \geq 0$. In particular, none of these weights can be an integer multiple of $(1, 1, 1)$.

This will suffice, in view of the following elementary fact: Let $w_1, \ldots, w_n \in \mathbb{Z}^3$ be characters of T. If none of the w_i is an integer multiple of $(1, 1, 1)$, there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \hookrightarrow T_0$, such that $w_i \circ \lambda \neq 0$, for all $i = 1, \ldots, n$.

Suppose, then, that $\phi : I \to A/I$ is an eigenvector of T with weight (w_1, w_2, w_3), with $w_1 \geq 0$, $w_2 \geq 0$ and $w_3 \geq 0$. Then for a monomial $x^a y^b z^c \in I$ we have $\phi(x^a y^b z^c) \equiv x^{a+w_1} y^{b+w_2} z^{c+w_3} \mod I$, which vanishes in A/I, proving that $\phi = 0$.

Now suppose $\phi : I \to A/I$ is an eigenfunction whose weights satisfy $w_1 < 0$, $w_2 < 0$ and $w_3 < 0$. Let a be the smallest integer such that $x^a \in I$. Then let b be the smallest integer such that $x^{a-1} y^b \in I$. Finally, let c be the smallest integer such that $x^{a-1} y^{b-1} z^c \in I$. Then if a monomial $x^r y^s z^t$ is in I, it follows that $r \geq a$, $s \geq b$ or $t \geq c$.

We have

$$\phi(x^a y^b z^c) = xz^c \phi(x^{a-1} y^b) \equiv xz^c x^{a-1+w_1} y^{b+w_2} \equiv x^{a+w_1} y^{b+w_2} z^c \mod I.$$

We also have

$$\phi(x^a y^b z^c) = xy\phi(x^{a-1} y^{b-1} z^c) \equiv x^{a+w_1} y^{b+w_2} z^{c+w_3} \mod I.$$

We conclude that

$$x^{a+w_1} y^{b+w_2} z^c - x^{a+w_1} y^{b+w_2} z^{c+w_3} \in I.$$

Since the ideal I is monomial, each of these two monomials is in I. But the latter one cannot be in I.

(c) This is an immediate consequence of [Maulik et al. 2003], Theorem 2 in § 4.10. In fact, this theorem states that if $w_1, \dots, w_d$ are the weights of T on the tangent space V,

$$\frac{\prod_{i=1}^d (-w_i)}{\prod_{i=1}^d w_i} = (-1)^n$$

inside the field of rational functions on T. □

Proposition 4.2. *For any T_0-invariant locally closed subset Z of* $\operatorname{Hilb}^n \mathbb{A}^3$ *we have*

$$\widetilde{\chi}(Z, \operatorname{Hilb}^n \mathbb{A}^3) = (-1)^n \chi(Z).$$

Proof. Since there are only finitely many fixed points of T_0 on X, we can use the fact mentioned in the proof of Lemma 4.1 to find a one-parameter subgroup $\mathbb{G}_m \to T_0$ with respect to which all weights of all tangent spaces at all fixed points are nonzero. Thus, all $\mathbb{G}_m$-fixed points are isolated. Because $\operatorname{Hilb}^n \mathbb{A}^3$ admits an equivariant embedding into projective space (see the proof of Proposition 2.4), every fixed point has an invariant affine open neighborhood.

The symmetric obstruction theory on $\operatorname{Hilb}^n_{(n)} \mathbb{A}^3$ is equivariant symmetric with respect to the induced $\mathbb{G}_m$-action. We can therefore apply Corollary 3.5. We obtain:

$$\widetilde{\chi}(Z, \operatorname{Hilb}^n \mathbb{A}^3) = \sum_{P \in Z} (-1)^n,$$

where the sum extends over fixed points P contained in Z. Since $\chi(Z) = \#\{P \in Z, P \text{ fixed}\}$, the result follows. □

Let F_n denote the closed subset of $\operatorname{Hilb}^n \mathbb{A}^3$ consisting of subschemes supported at the origin. Let ν_n be the restriction of the canonical constructible function $\nu_{\operatorname{Hilb}^n \mathbb{A}^3}$ to F_n. Thus $\widetilde{\chi}(F_n, \operatorname{Hilb}^n \mathbb{A}^3) = \chi(F_n, \nu_n)$. Note that all T_0-fixed points of $\operatorname{Hilb}^n \mathbb{A}^3$ are contained in F_n.

Let $M(t) = \prod_{n=1}^\infty (1-t^n)^{-n}$ be the McMahon function. It is the generating series for 3-dimensional partitions. Hence, if we write $M(t) = \sum_{n=0}^\infty p_n t^n$, then p_n denotes the number of monomial ideals I in $A = \mathbb{C}[x, y, z]$, such that $\dim_{\mathbb{C}} A/I = n$. The number p_n is the number of T_0-fixed points in F_n or $\operatorname{Hilb}^n \mathbb{A}^3$. Thus, $p_n = \chi(F_n) = \chi(\operatorname{Hilb}^n \mathbb{A}^3)$.

Corollary 4.3. *We have*

$$\chi(F_n, \nu_n) = (-1)^n \chi(F_n) = (-1)^n p_n,$$

and hence

$$\sum_{n=0}^\infty \chi(F_n, \nu_n) t^n = M(-t).$$

Weighted Euler characteristics of Hilbert schemes. Let Y be a smooth threefold, and $n > 0$ an integer. Consider the Hilbert scheme of n points on Y, denoted $\operatorname{Hilb}^n Y$. The scheme $\operatorname{Hilb}^n Y$ is connected, smooth for $n \leq 3$ and singular otherwise, and reducible for large enough n.

Let us denote by ν_Y the canonical constructible function on $\operatorname{Hilb}^n Y$. Our goal is to calculate

$$\widetilde{\chi}(\operatorname{Hilb}^n Y) = \chi(\operatorname{Hilb}^n Y, \nu_Y).$$

Let us start with a useful general lemma on Hilbert schemes.

Lemma 4.4. *Let $f : Y \to Y'$ be a morphism of projective schemes and $Z \subset Y$ a closed subscheme. Assume that f is étale in a neighborhood of Z and that the composition $Z \to Y'$, which we will denote by $f(Z)$, is a closed immersion of schemes.*

Let X be the Hilbert scheme of Y which contains Z and P the point of X corresponding to Z. Let X' be the Hilbert scheme of Y' which contains $f(Z)$. Then there exists an open neighborhood U of P in X and an étale morphism $\phi : U \to X'$, which sends a subscheme $\widetilde{Z} \to Y$ to the composition $\widetilde{Z} \to Y'$.

Proof. For the existence of the open set U and the morphism ϕ, see for example Proposition 6.1, Chapter I of [Kollár 1996]. The fact that ϕ is étale in a neighborhood of P follows from a direct application of the formal criterion. □

The closed stratum. We start by recalling the standard stratification of $\operatorname{Hilb}^n Y$. The strata are indexed by partitions of n. Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a length r partition of n, i.e., $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r \geq 1$ and $\sum_{i=1}^r \alpha_i = n$. Let $\operatorname{Hilb}^n_\alpha Y$ be the locus of subschemes whose support consists of r distinct points with multiplicities $\alpha_1, \dots, \alpha_r$. The closed stratum is $\operatorname{Hilb}^n_{(n)} Y$. It corresponds to subschemes supported at a single point. To fix ideas, we will endow all strata with the reduced scheme structure.

Lemma 4.5. *For any threefold Y there is a natural morphism*

$$\pi_Y : \operatorname{Hilb}^n_{(n)} Y \to Y.$$

Proof. This is a part of the Hilbert–Chow morphism $\operatorname{Hilb}^n Y \to S^n Y$ to the symmetric product. A proof that this is a morphism of schemes can be found, for example, in [Lehn 2004]. □

Note that F_n is the fiber of $\pi_{\mathbb{A}^3}$ over the origin.

Lemma 4.6. *We have a canonical isomorphism*

$$\operatorname{Hilb}^n_{(n)} \mathbb{A}^3 = \mathbb{A}^3 \times F_n. \tag{8}$$

Moreover, $\nu_{\mathbb{A}^3} = p^*\nu_n$, where $p : \operatorname{Hilb}^n_{(n)} \mathbb{A}^3 \to F_n$ is the projection given by (8).

Proof. Consider the action of the group $\mathbb{A}^3$ on itself by translations. We get an induced action of $\mathbb{A}^3$ on $\operatorname{Hilb}^n \mathbb{A}^3$. Use this action to translate a subscheme supported at a point P to a subscheme supported at the origin. Obtain the morphism $p : \operatorname{Hilb}^n_{(n)} \mathbb{A}^3 \to F_n$ in this way. The product morphism $\pi_{\mathbb{A}^3} \times p : \operatorname{Hilb}^n_{(n)} \mathbb{A}^3 \to \mathbb{A}^3 \times F_n$ is an isomorphism.

It is a formal consequence of the general properties of the canonical constructible function, that it is constant on orbits under a group action. This implies the claim about $\nu_{\mathbb{A}^3}$. $\square$

Lemma 4.7. *Consider an étale morphism of threefolds $\phi : Y \to Y'$.*
(a) *Let $U \subset \operatorname{Hilb}^n Y$ be the open subscheme parameterizing subschemes $Z \subset Y$, which satisfy: if P and Q are distinct points in the support of Z, then $\phi(P) \neq \phi(Q)$. There is an étale morphism $\widetilde{\Phi} : U \to \operatorname{Hilb}^n Y'$ sending a subscheme of Y to its image under ϕ.*

$$\begin{tikzcd}
\operatorname{Hilb}^n_{(n)} Y \arrow[r] \arrow[d, "\Phi"'] & U \arrow[r] \arrow[d, "\widetilde{\Phi}"] & \operatorname{Hilb}^n Y \\
\operatorname{Hilb}^n_{(n)} Y' \arrow[r] & \operatorname{Hilb}^n Y' &
\end{tikzcd}$$

(b) *The restriction of $\widetilde{\Phi}$ to $\operatorname{Hilb}^n_{(n)} Y$ induces a cartesian diagram of schemes*

$$\begin{tikzcd}
\operatorname{Hilb}^n_{(n)} Y \arrow[r, "\Phi"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\square"] & \operatorname{Hilb}^n_{(n)} Y' \arrow[d, "\pi_{Y'}"] \\
Y \arrow[r, "\phi"] & Y'
\end{tikzcd}$$

Proof. The existence and étaleness of $\widetilde{\Phi}$ follows immediately from Lemma 4.4, applied to quasiprojective covers of Y and Y'. Part (b) is clear. $\square$

Let $\phi : Y \to Y'$ be an étale morphism with induced morphism $\Phi : \operatorname{Hilb}^n_{(n)} Y \to \operatorname{Hilb}^n_{(n)} Y'$. By Lemma 4.7, the morphism Φ extends to open neighborhoods in $\operatorname{Hilb}^n Y$ and $\operatorname{Hilb}^n Y'$, respectively. The extension $\widetilde{\Phi}$ is étale. Thus, we see that

$$\Phi^*(\nu_{Y'}) = \nu_Y.$$

Proposition 4.8. *Every étale morphism $\phi : Y \to \mathbb{A}^3$ induces an isomorphism $\operatorname{Hilb}^n_{(n)} Y = Y \times F_n$. The constructible function $\nu_Y|_{\operatorname{Hilb}^n_{(n)} Y}$ is obtained by pulling back ν_n via the induced projection $\operatorname{Hilb}^n_{(n)} Y \to F_n$.*

Proof. Combine Lemmas 4.6 and 4.7(b) with each other. $\square$

Corollary 4.9. *The morphism $\pi_Y : \operatorname{Hilb}^n_{(n)} Y \to Y$ is a Zariski-locally trivial fibration with fiber F_n. More precisely, there exists a Zariski open cover $\{U_i\}$ of Y, such* *that for every i, we have*

$$(\pi_Y^{-1}(U_i), \nu_Y) = (U_i, 1) \times (F_n, \nu_n).$$

This is a product of schemes with constructible functions on them.

Proof. Every point of Y admits étale coordinates, defined in a Zariski open neighborhood. □

Reduction to the closed stratum. From now on the threefold Y will be fixed and we denote $\operatorname{Hilb}^n_\alpha Y$ by X^n_α and $\operatorname{Hilb}^n Y$ by X^n.

Lemma 4.10. *Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a partition of n.*
(a) *Let V be the open subscheme of $\prod_{i=1}^r X^{\alpha_i}$ parameterizing r-tuples of subschemes with pairwise disjoint support. Then there is a morphism $f_\alpha : V \to X^n$ mapping $(Z_1, \dots, Z_r)$ to $Z = \bigcup_i Z_i$. The morphism f_α is étale. Its image U is open and contains X^n_α. Let $Z_\alpha = f_\alpha^{-1} X^n_\alpha$:*

$$\begin{tikzcd} Z_\alpha \arrow[r] \arrow[d, "\text{Galois}"'] \arrow[dr, phantom, "\square"] & V \arrow[r] \arrow[d, "f_\alpha"] & \prod_i X^{\alpha_i} \\ X^n_\alpha \arrow[r] & U \arrow[r] & X^n \end{tikzcd}$$

Moreover, the induced morphism $Z_\alpha \to X^n_\alpha$ is a Galois cover with Galois group G_α, where G_α is the automorphism group of the partition α.
(b) *The scheme Z_α is contained in $\prod_i X^{\alpha_i}_{(\alpha_i)}$ and has therefore a morphism $Z_\alpha \to Y^r$. There is a cartesian diagram*

$$\begin{tikzcd} Z_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i X^{\alpha_i}_{(\alpha_i)} \arrow[d] \\ Y^r_0 \arrow[r] & Y^r \end{tikzcd}$$

where Y^r_0 is the open subscheme in Y^r consisting of r-tuples with pairwise disjoint entries.

Proof. The existence of f_α and the fact that it is étale follows from Lemma 4.4 applied to the étale map $\coprod_{i=1}^r Y \to Y$ and the subscheme $Z_1 \amalg \ldots \amalg Z_r \subset \coprod_{i=1}^r Y$. All other facts are also straightforward to prove. □

Theorem 4.11. *Let Y be a smooth scheme of dimension 3. Then for all $n > 0$*

$$\tilde{\chi}(\operatorname{Hilb}^n Y) = (-1)^n \chi(\operatorname{Hilb}^n Y).$$

This implies

$$\sum_{n=0}^{\infty} \tilde{\chi}(\operatorname{Hilb}^n Y) t^n = M(-t)^{\chi(Y)}.$$

Proof. By formal properties of $\widetilde{\chi}$ as proved in [Behrend 2005], we can calculate as follows, using Lemma 4.10(a):

$$\begin{aligned}\widetilde{\chi}(X^n) &= \sum_{\alpha \vdash n} \widetilde{\chi}(X^n_\alpha, X^n) = \sum_{\alpha \vdash n} \widetilde{\chi}(X^n_\alpha, U) = \sum_{\alpha \vdash n} |G_\alpha| \, \widetilde{\chi}(Z_\alpha, V) \\ &= \sum_{\alpha \vdash n} |G_\alpha| \, \widetilde{\chi}\Big(Z_\alpha, \prod_i X^{\alpha_i}\Big).\end{aligned}$$

By Lemma 4.10(b) and Corollary 4.9, $Z_\alpha \to Y_0^{\ell(\alpha)}$ is a Zariski-locally trivial fibration with fiber $\prod_i F_{\alpha_i}$. Here we have written $\ell(\alpha)$ for the length r of the partition $(\alpha_1, \dots, \alpha_r)$. We conclude:

$$\widetilde{\chi}\Big(Z_\alpha, \prod_i X^{\alpha_i}\Big) = \chi(Y_0^{\ell(\alpha)}) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i})$$

Together with Corollary 4.3 this gives:

$$\widetilde{\chi}(X^n) = (-1)^n \sum_{\alpha \vdash n} |G_\alpha| \, \chi(Y_0^{\ell(\alpha)}) \prod_i \chi(F_{\alpha_i}). \tag{9}$$

Using the exact same arguments with the constant function 1 in place of ν gives the same answer, except without the sign $(-1)^n$. This proves our first claim. The second one follows then directly from the result of [Cheah 1996], which says that $\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n Y) t^n = M(t)^{\chi(Y)}$. □

The dimension zero MNOP conjecture. We now prove Conjecture 1 of [Maulik et al. 2003]. A proof of this result was also announced by J. Li at the workshop on Donaldson–Thomas invariants in Urbana–Champaign in March 2005.

Theorem 4.12. *Let Y be a projective Calabi–Yau threefold. Then, for the virtual count of* $\mathrm{Hilb}^n Y$ *with respect to the Donaldson–Thomas obstruction theory, we have*

$$\#^{\mathrm{vir}}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y).$$

In other words:

$$\sum_{n=0}^{\infty} \#^{\mathrm{vir}}(\mathrm{Hilb}^n Y)\, t^n = M(-t)^{\chi(Y)}.$$

Proof. By Theorem 4.18 of [Behrend 2005], we have

$$\#^{\mathrm{vir}}(\mathrm{Hilb}^n Y) = \widetilde{\chi}(\mathrm{Hilb}^n Y).$$

Thus the result follows from Theorem 4.11. □

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