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We compute the $\mathbb{C}^*$-equivariant quantum cohomology ring of Y, the minimal resolution of the DuVal singularity $\mathbb{C}^2/G$ where G is a finite subgroup of $SU(2)$. The quantum product is expressed in terms of an ADE root system canonically associated to G. We generalize the resulting Frobenius manifold to nonsimply laced root systems to obtain an n parameter family of algebra structures on the affine root lattice of any root system. Using the Crepant Resolution Conjecture, we obtain a prediction for the orbifold Gromov–Witten potential of $[\mathbb{C}^2/G]$.

1. Introduction

1.1. ***Overview.*** Let G be a finite subgroup of $SU(2)$, and let

$$Y \to \mathbb{C}^2/G$$

be the minimal resolution of the corresponding DuVal singularity. The classical McKay correspondence describes the geometry of Y in terms of the representation theory of G [McKay 1980; Gonzalez-Sprinberg and Verdier 1983; Reid 2002].

The geometry of Y gives rise to a Dynkin diagram of ADE type. The nodes of the diagram correspond to the irreducible components of the exceptional divisor of Y. Two nodes have a connecting edge if and only if the corresponding curves intersect.

Associated to every Dynkin diagram of ADE type is a simply laced root system. In this paper, we describe the $\mathbb{C}^*$-equivariant quantum cohomology of Y in terms of the associated root system. This provides a quantum version of the classical McKay correspondence.

1.2. ***Results.*** The set $\{E_1, \dots, E_n\}$ of irreducible components of the exceptional divisor of Y forms a basis of $H_2(Y, \mathbb{Z})$. The intersection matrix $E_i \cdot E_j$ defines a perfect pairing on $H_2(Y, \mathbb{Z})$. Let R be the simply laced root system associated to the Dynkin diagram of Y. We can identify $H_2(Y, \mathbb{Z})$ with the root lattice of R in a

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way so that $E_1, \ldots, E_n$ correspond to simple roots $\alpha_1, \ldots, \alpha_n$ and the intersection matrix is minus the Cartan matrix

$$E_i \cdot E_j = -\langle \alpha_i, \alpha_j \rangle.$$

Using the above pairing, we identify $H^2(Y, \mathbb{Z})$ with $H_2(Y, \mathbb{Z})$ (and hence with the root lattice). Since the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^2$ commutes with the action of G, $\mathbb{C}^*$ acts on $\mathbb{C}^2/G$ and this action lifts to an action on Y. The cycles $E_1, \ldots, E_n$ are $\mathbb{C}^*$ invariant, and so the classes $\alpha_1, \ldots, \alpha_n$ have natural lifts to equivariant (co)homology. Additively, the equivariant quantum cohomology ring is thus a free module generated by the classes $\{1, \alpha_1, \ldots, \alpha_n\}$. The ground ring is

$$\mathbb{Z}[t][\![q_1, \ldots, q_n]\!]$$

where t is the equivariant parameter and $q_1, \ldots, q_n$ are the quantum parameters associated to the curves $E_1, \ldots, E_n$. So additively we have

$$QH^*_{\mathbb{C}^*}(Y) \cong H^*(Y, \mathbb{Z}) \otimes \mathbb{Z}[t][\![q_1, \ldots, q_n]\!].$$

We extend the pairing $\langle \cdot, \cdot \rangle$ to a

$$\mathbb{Q}[t, t^{-1}][\![q_1, \ldots, q_n]\!]$$

valued pairing on $QH^*_{\mathbb{C}^*}(Y)$ by making 1 orthogonal to α_i and setting

$$\langle 1, 1 \rangle = \frac{-1}{t^2|G|}.$$

The product structure of $QH^*_{\mathbb{C}^*}(Y)$ is determined by our main theorem:

Theorem 1. *Let $v, w \in H^2(Y, \mathbb{Z})$ which we identify with the root lattice of R as above. Then the quantum product of v and w is given by*

$$v \star w = -t^2|G|\,\langle v, w\rangle + \sum_{\beta \in R^+} \langle v, \beta\rangle\, \langle w, \beta\rangle\, t\, \frac{1+q^\beta}{1-q^\beta}\, \beta,$$

where the sum is over the positive roots of R and for $\beta = \sum_{i=1}^n b_i\, \alpha_i$, q^β is defined by

$$q^\beta = \prod_{i=1}^n q_i^{b_i}.$$

The quantum product satisfies the Frobenius condition

$$\langle v \star w, u\rangle = \langle v, w \star u\rangle\,,$$

making $QH^*_{\mathbb{C}^*}(Y)$ a Frobenius algebra over $\mathbb{Q}[t, t^{-1}][\![q_1, \ldots, q_n]\!]$.

Note that by a standard fact in root theory [Bourbaki 1968, VI.1.1 Proposition 3 and V.6.2 Corollary to Theorem 1], the formula in Theorem 1 can alternatively be written as

$$v \star w = \sum_{\beta \in R^+} \langle v, \beta\rangle \langle w, \beta\rangle \Big(-t^2 \frac{|G|}{h} + t \frac{1+q^\beta}{1-q^\beta} \beta \Big),$$

where $h = \frac{|R|}{n}$ is the Coxeter number of R.

We remark that we can regard $H^0(Y) \oplus H^2(Y)$ as the root lattice for the affine root system and consequently, we can regard $QH^*_{\mathbb{C}^*}(Y)$ as defining a family of algebra structures on the affine root lattice depending on variables $t, q_1, \dots, q_n$. We also remark that even though the product in Theorem 1 is expressed purely in terms of the root system, we know of no root theoretic proof of associativity, even in the "classical" limit $q_i \to 0$.

In Section 4, which can be read independently from the rest of this paper, we will generalize our family of algebras to root systems which are not simply-laced (Theorem 6). We will prove associativity of the product in the nonsimply laced case by reducing it to the simply laced case. Our formula also allows us to prove that the action of the Weyl group induces automorphisms of the Frobenius algebra (Corollary 7).

Our theorem is formulated as computing *small* quantum cohomology, but since the cohomology of Y is concentrated in degree 0 and degree 2, the large and small quantum cohomology rings contain equivalent information. The proof of Theorem 1 requires the computations of genus 0 equivariant Gromov–Witten invariants of Y. This is done in Section 2.

In Section 5, we use the Crepant Resolution Conjecture [Bryan and Graber 2008] and our computation of the Gromov–Witten invariants of Y, to obtain a prediction for the orbifold Gromov–Witten potential of $[\mathbb{C}^2/G]$ (Conjecture 11).

1.3. ***Relationship to other work.*** A certain specialization of the Frobenius algebra $QH^*_{\mathbb{C}^*}(Y)$ appears as the quantum cohomology of the G-Hilbert scheme resolution of $\mathbb{C}^3/G$ for $G \subset SO(3)$ [Bryan and Gholampour 2008]. The equivariant Gromov–Witten theory of Y in higher genus has been determined by recent work of Maulik [2008].

2. Gromov–Witten theory of Y

In this section we compute the equivariant genus zero Gromov–Witten invariants of Y. The invariants of nonzero degree are computed by relating them to the invariants of a certain threefold W constructed as the total space of a family of deformations of Y. The invariants of W are computed by the method of

Bryan, Katz, and Leung [2001]. The degree zero invariants are computed by localization.

2.1. ***Invariants of nonzero degree.*** Def (Y), the versal space of $\mathbb{C}^*$-equivariant deformations of Y, is naturally identified with the complexified root space of the root system R [Katz and Morrison 1992]. A generic deformation of Y is an affine variety and consequently has no compact curves. The hyperplane $D_\beta \subset \operatorname{Def}(Y)$ perpendicular to a positive root

$$\beta = \sum_{i=1}^{n} b_i \alpha_i$$

parameterizes those deformations of Y for which the curve

$$b_1 E_1 + \cdots + b_n E_n$$

also deforms. Moreover, for a generic point $t \in D_\beta$, the corresponding curve is a smooth $\mathbb{P}^1$ which generates the Picard group of the corresponding surface [Katz and Morrison 1992, Theorem 1; Bryan et al. 2001, Proposition 2.2].

Let

$$\iota : \mathbb{C} \to \operatorname{Def}(Y)$$

be a generic linear subspace. We obtain a threefold W by pulling back the universal family over Def (Y) by ι. The embedding ι can be made $\mathbb{C}^*$-equivariant by defining the action on $\mathbb{C}$ to have weight 2. This follows from [Katz and Morrison 1992, Theorem 1] after noting that the $\mathbb{C}^*$ action in that paper is the square of the action induced by the action on $\mathbb{C}^2/G$. Clearly $Y \subset W$ and the normal bundle $N_{Y/W}$ is isomorphic to $\mathcal{O}_Y$. However, the action of $\mathbb{C}^*$ is nontrivial of weight two and hence it has a nontrivial Chern class in equivariant cohomology:

$$c_1(N_{Y/W}) = 2t$$

(recall that t is the equivariant parameter).

The threefold W is Calabi–Yau and its Gromov–Witten invariants are well defined in the nonequivariant limit. This assertion follows from the fact that the moduli space of stable maps to W is compact. This in turn follows from the fact that W admits a birational map

$$W \to W_{\mathrm{aff}}$$

contracting $E_1 \cup \cdots \cup E_n$ such that W_{aff} is an affine variety [Katz and Morrison 1992; Bryan et al. 2001]. Consequently, all nonconstant stable maps to W must have image contained in the exceptional set of $W \to W_{\mathrm{aff}}$ and thus, in particular, all nonconstant stable maps to W have their image contained in Y.

There is a standard technique in Gromov–Witten theory for comparing the virtual class for stable maps to a submanifold to the virtual class for the stable maps to the ambient manifold when all the maps have image contained in the submanifold [Behrend and Fantechi 1997]. This allows us to compare the Gromov–Witten invariants of W and Y.

For any nonzero class

$$A \in H_2(Y) \cong H_2(W),$$

let

$$\langle\ \rangle^Y_A \quad \text{and} \quad \langle\ \rangle^W_A$$

denote the genus zero, degree A, zero insertion Gromov–Witten invariant of Y and W respectively. We have

$$\langle\ \rangle^W_A = \int_{[\overline{M}_{0,0}(Y,A)]^{vir}} e(-R^\bullet \pi_* f^* N_{Y/W})$$

where $\overline{M}_{0,0}(Y, A)$ is the moduli space of stable maps, $\pi : C \to \overline{M}_{0,0}(Y, A)$ is the universal curve, $f : C \to Y$ is the universal map, and e is the equivariant Euler class.

Since the line bundle $N_{Y/W}$ is trivial up to the $\mathbb{C}^*$ action, and π is a family of genus zero curves, we get

$$R^\bullet \pi_* f^* N_{Y/W} = R^0 \pi_* f^* N_{Y/W} = \mathbb{O} \otimes \mathbb{C}_{2t}$$

where $\mathbb{C}_{2t}$ is the $\mathbb{C}^*$ representation of weight 2 so that we have

$$c_1(\mathbb{O} \otimes \mathbb{C}_{2t}) = 2t.$$

Consequently, we have

$$e(-R^\bullet \pi_* f^* N_{Y/W}) = \frac{1}{2t}$$

and so

$$\begin{aligned}\langle\ \rangle^W_A &= \int_{[\overline{M}_{0,0}(Y,A)]^{vir}} \frac{1}{2t}\\ &= \frac{1}{2t} \langle\ \rangle^Y_A .\end{aligned}$$

To compute $\langle\ \rangle^W_A$, we use the deformation invariance of Gromov–Witten invariants. Although W is noncompact, the moduli space of stable maps is compact, and the deformation of W is done so that the stable map moduli spaces are compact throughout the deformation. The technique is identical to the deformation argument used in [Bryan et al. 2001] where it is presented in greater detail.

We deform W to a threefold W' as follows. Let

$$\iota' : \mathbb{C} \to \mathrm{Def}\,(Y)$$

be a generic affine linear embedding and let W' be the pullback by ι' of the universal family over $\mathrm{Def}(Y)$. The threefold W' is a deformation of W since ι' is a deformation of ι.

Lemma 2. *The compact curves of W' consist of isolated $\mathbb{P}^1$s, each having normal bundle*

$$\mathcal{O}(-1)\oplus\mathcal{O}(-1),$$

one in each homology class $\beta\in H_2(W')\cong H_2(Y)$ corresponding to a positive root.

Proof. The map ι' intersects each hyperplane D_β transversely in a single generic point t. The surface S_t over the point t contains a single curve $C_t\cong\mathbb{P}^1$ of normal bundle $N_{C_t/S_t}\cong\mathcal{O}(-2)$ and this curve is in the class β. There is a short exact sequence

$$0\to N_{C_t/S_t}\to N_{C_t/W'}\to\mathcal{O}\to 0$$

and since ι' intersects D_β transversely, C_β does not have any deformations (even infinitesimally) inside W'. Consequently, we must have

$$N_{C_\beta/W'}\cong\mathcal{O}(-1)\oplus\mathcal{O}(-1). \qquad \square$$

Since all the curves in W' are isolated $(-1,-1)$ curves, we can compute the Gromov–Witten invariants of W' using the Aspinwall–Morrison multiple cover formula. Combined with the deformation invariance of Gromov–Witten invariants, we obtain

Lemma 3. *For $A\neq 0$ we have*

$$\langle\ \rangle_A^Y = 2t\,\langle\ \rangle_A^W = 2t\,\langle\ \rangle_A^{W'} = \begin{cases} 2t\dfrac{1}{d^3} & \text{if } A=d\beta \text{ where } \beta \text{ is a positive root},\\ 0 & \text{otherwise.}\end{cases}$$

Since all the cohomology of Y is in $H^0(Y)$ and $H^2(Y)$, the n-point Gromov–Witten invariants of nonzero degree are determined from the 0-point invariants by the divisor and the fundamental class axioms.

2.2. ***Degree 0 invariants.*** The only nontrivial degree zero invariants have 3 insertions and are determined by classical integrals on Y. They are given in the following lemma.

Lemma 4. *Let 1 be the generator of $H^0_{\mathbb{C}^*}(Y)$ and let $\{\alpha_1,\dots,\alpha_n\}$ be the basis for $H^2_{\mathbb{C}^*}(Y)$ which is also identified with the simple roots of R as in Section 1. Then the* *degree* 0, 3*-point Gromov–Witten invariants of* Y *are given as follows*:

$$\langle 1,1,1\rangle_0 = \frac{1}{t^2|G|}, \tag{2-1}$$

$$\langle \alpha_i,1,1\rangle_0 = 0, \tag{2-2}$$

$$\langle \alpha_i,\alpha_j,1\rangle_0 = -\langle \alpha_i,\alpha_j\rangle, \tag{2-3}$$

$$\langle \alpha_i,\alpha_j,\alpha_k\rangle_0 = -t\sum_{\beta\in R^+}\langle \alpha_i,\beta\rangle\langle \alpha_j,\beta\rangle\langle \alpha_k,\beta\rangle. \tag{2-4}$$

Proof. The degree zero, genus zero, 3-point Gromov–Witten invariants are given by integrals over Y:

$$\langle x,y,z\rangle_0 = \int_Y x\cup y\cup z.$$

Because Y is noncompact, the integral must be defined[1] via $\mathbb{C}^*$ localization and takes values in $\mathbb{Q}[t,t^{-1}]$, the localized equivariant cohomology ring of a point:

$$\int_Y : \quad H^*_{\mathbb{C}^*}(Y) \longrightarrow \mathbb{Q}[t,t^{-1}],$$
$$\phi \mapsto \int_F \frac{\phi|_F}{e(N_{F/Y})}.$$

Here $F\subset Y$ is the (compact) fixed point locus of the action of $\mathbb{C}^*$ on Y.

By correspondence of residues [Bertram 2000], integrals over Y can be computed by first pushing forward to $\mathbb{C}^2/G$ followed by (orbifold) localization on $\mathbb{C}^2/G$. Equation (2-1) follows immediately:

$$\int_Y 1 = \int_{\mathbb{C}^2/G} 1 = \frac{1}{t^2|G|}.$$

The factor t^2 is the equivariant Euler class of the normal bundle of $[0/G]\subset[\mathbb{C}^2/G]$ and the factor $\frac{1}{|G|}$ accounts for the automorphisms of the point $[0/G]$.

Let $L_i\to Y$ be the $\mathbb{C}^*$ equivariant line bundle with

$$c_1(L_i)=\alpha_i.$$

Since α_i was defined to be dual to E_i via the intersection pairing, we have

$$\int_{E_j} c_1(L_i) = E_i\cdot E_j = -\langle \alpha_i,\alpha_j\rangle.$$

[1]This method of defining the Gromov–Witten invariants of a noncompact space does not affect the desired properties of quantum cohomology: the associativity still holds and the Frobenius structure still exists with the novelty that the pairing takes values in the ring $\mathbb{Q}[t,t^{-1}]$. See [Bryan and Graber 2008, section 1.4], for a discussion.

Computing the left hand side using localization, we see that the weight of the $\mathbb{C}^*$ action on L_i at a fixed point $p \in E_i$ must be the same as the weight of the $\mathbb{C}^*$ action on the normal bundle $N_{E_i/Y}$ at p, and the weight of the action on L_i is 0 over fixed points not on E_i.

Equations (2-3) and (2-2) then easily follow from localization.

To prove (2-4), we compute the left hand side by localization to get

$$\langle \alpha_i, \alpha_j, \alpha_k \rangle_0 = \begin{cases} 0 & \text{if } E_i \cup E_j \cup E_k = \varnothing, \\ -8t & \text{if } i = j = k, \\ w_{ijj} & \text{if } i \neq j = k \text{ and } E_i \cup E_j \neq \varnothing \end{cases}$$

where

$$w_{ijj} = c_1(N_{E_j/Y}|_{p_{ij}})$$

is the weight of the $\mathbb{C}^*$ action on the normal bundle of E_j at the point $p_{ij} = E_i \cup E_j$.

The normal weights w_{ijj} satisfy the following three conditions:

(1) Since K_Y is the trivial bundle with a $\mathbb{C}^*$ action of weight $2t$, the sum of the normal weights at $p = E_i \cap E_j$ is $2t$ and so

$$w_{ijj} + w_{jii} = 2t \quad \text{when } E_i \cap E_j \neq \varnothing \text{ and } i \neq j.$$

(2) Since E_i is $\mathbb{C}^*$ invariant, the sum of the tangent weights of any two distinct fixed points on E_i is zero. Combined with the above, we see that the sum of the normal weights at any two distinct fixed points is $4t$ so

$$w_{ikk} + w_{jkk} = 4t \quad \text{when } E_i \cap E_k \neq \varnothing,\ E_j \cap E_k \neq \varnothing, \text{ and } i \neq j \neq k.$$

(3) Since automorphisms of the Dynkin diagrams induce equivariant automorphisms of Y, the normal weights are invariant under such automorphisms.

The normal weights are completely determined by the above three conditions. Indeed, it is clear that once one normal weight is known, then properties (1) and (2) determine the rest. Moreover, in the case of Dynkin diagrams of type D_n or E_n, the curve corresponding to the trivalent vertex of the Dynkin graph must be fixed by $\mathbb{C}^*$ and so its tangent weights are zero. In the A_n case, condition (3) provides the needed extra equation.

To summarize the above, the three point degree zero invariants $\langle \alpha_i, \alpha_j, \alpha_k \rangle_0$ satisfy the following conditions and are uniquely determined by them.

(i) $\langle \alpha_i, \alpha_j, \alpha_k \rangle_0$ is symmetric in $\{i, j, k\}$;

(ii) $\langle \alpha_i, \alpha_j, \alpha_k \rangle_0$ is invariant under any permutation of indices induced by a Dynkin diagram automorphism;

(iii) $\langle \alpha_i, \alpha_j, \alpha_k \rangle_0 = 0$ if $\langle \alpha_j, \alpha_k \rangle = 0$;

(iv) $\langle \alpha_i, \alpha_j, \alpha_k \rangle_0 = -8t$ if $i = j = k$;

(v) $\langle \alpha_i, \alpha_i, \alpha_j \rangle_0 + \langle \alpha_j, \alpha_j, \alpha_i \rangle_0 = 2t$ if $\langle \alpha_i, \alpha_j \rangle = -1$;

(vi) $\langle \alpha_i, \alpha_k, \alpha_k \rangle_0 + \langle \alpha_j, \alpha_k, \alpha_k \rangle_0 = 4t$ if $i \neq j$ and $\langle \alpha_i, \alpha_k \rangle = \langle \alpha_j, \alpha_k \rangle = -1$.

So to finish the proof of Lemma 4, it suffices to show that the right hand side of (2-4) also satisfies all the above properties. This is precisely the content of Proposition 10, a root theoretic result which we prove in Section 4. □

3. Proof of the main theorem

Having computed all the Gromov–Witten invariants of Y, we can proceed to compute the quantum product and prove our main theorem.

Proof. The quantum product $\star$ is defined in terms of the genus 0, 3-point invariants of Y by

$$-\langle x \star y, z \rangle = \sum_{A \in H_2(Y,\mathbb{Z})} \langle x, y, z \rangle_A \, q^A$$

where the strange looking minus sign is due to the fact that the pairing $\langle \cdot, \cdot \rangle$, which coincides with the Cartan pairing on the roots, is the negative of the cohomological pairing.

To prove our formula for $v \star w$, it suffices to check that the formula holds after pairing both sides with 1 and with any $u \in H^2(Y)$.

By definition and Lemma 4 we have

$$\begin{aligned} -\langle v \star w, 1 \rangle &= \sum_{A \in H_2(Y)} \langle v, w, 1 \rangle_A \, q^A \\ &= \langle v, w, 1 \rangle_0 \\ &= -\langle v, w \rangle, \end{aligned}$$

which is in agreement with the right hand side of the formula in Theorem 1 when paired with 1 since 1 is orthogonal to $H^2(Y)$ and

$$\langle 1, 1 \rangle = -\frac{1}{t^2 |G|}.$$

For $u \in H^2(Y)$ we apply the divisor axiom to get

$$\begin{aligned} -\langle v \star w, u \rangle &= \sum_{A \in H_2(Y)} \langle v, w, u \rangle_A \, q^A \\ &= \langle v, w, u \rangle_0 - \sum_{A \neq 0} \langle v, A \rangle \langle w, A \rangle \langle u, A \rangle \langle \, \rangle_A \, q^A. \end{aligned}$$

Applying Lemmas 4 and 3 we get

$$\begin{aligned}
-\langle v\star w, u\rangle &= -t\sum_{\beta\in R^+}\langle v,\beta\rangle\,\langle w,\beta\rangle\langle u,\beta\rangle \\
&\qquad -\sum_{\beta\in R^+}\sum_{d=1}^{\infty}\langle v, d\beta\rangle\,\langle w, d\beta\rangle\,\langle u, d\beta\rangle\,\frac{2t}{d^3}q^{d\beta} \\
&= -t\sum_{\beta\in R^+}\langle v,\beta\rangle\,\langle w,\beta\rangle\,\langle u,\beta\rangle\left(1+\frac{2q^\beta}{1-q^\beta}\right) \\
&= -t\sum_{\beta\in R^+}\langle v,\beta\rangle\,\langle w,\beta\rangle\,\langle u,\beta\rangle\left(\frac{1+q^\beta}{1-q^\beta}\right).
\end{aligned}$$

Pairing the right hand side of the formula in Theorem 1 with u, we find agreement with the above and the formula for $\star$ is proved.

To prove that the Frobenius condition holds, we only need to observe that the pairing on $QH^*_{\mathbb{C}^*}(Y)$ is induced by the three point invariant with one insertion of 1:

$$-\langle x, y\rangle = \langle x, y, 1\rangle_0\,.$$

This indeed follows from (2-1), (2-2), and (2-3). □

4. The algebra for arbitrary root systems

In this section we construct a Frobenius algebra QH_R associated to any irreducible, reduced root system R (Theorem 6). This section can be read independently from the rest of the paper.

4.1. ***Root system notation.*** Let R be an irreducible, reduced, rank n root system. That is,

$$R = \{R, V, \langle\,\cdot\,,\,\cdot\,\rangle\}$$

consists of a finite subset R of a real inner product space V of dimension n satisfying

(1) R spans V;

(2) if $\alpha\in R$ then $k\alpha\in R$ implies $k=\pm 1$;

(3) for all $\alpha\in R$, the reflection s_α about $\alpha^\perp$, the hyperplane perpendicular to α leaves R invariant;

(4) for any $\alpha,\beta\in R$, the number $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is an integer; and

(5) V is irreducible as a representation of W, the Weyl group (that is, the group generated by the reflections s_α, for $\alpha\in R$).

We will also assume that the inner product $\langle \cdot , \cdot \rangle$ takes values in $\mathbb{Z}$ on R.

Let $\{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots, namely a subset of R spanning V and such that for every $\beta = \sum_{i=1}^{n} b_i \alpha_i$ in R the coefficients b_i are either all nonnegative or all nonpositive. As is customary, we define

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

We will also require a certain constant ϵ_R which depends on the root system and scales linearly with the inner product.

Definition 5. Let n_i be the i-th coefficient of the largest root

$$\widetilde{\alpha} = \sum_{i=1}^{n} n_i \alpha_i .$$

We define

$$\epsilon_R = \tfrac{1}{2} \langle \widetilde{\alpha}, \widetilde{\alpha} \rangle + \tfrac{1}{2} \sum_{i=1}^{n} n_i^2 \langle \alpha_i, \alpha_i \rangle .$$

Note that in the case where R is as in Section 1, namely of ADE type and the roots have a norm square of 2, then

$$\epsilon_R = 1 + \sum_{i=1}^{n} n_i^2$$

and we have that

$$\epsilon_R = |G|$$

where G is the corresponding finite subgroup of $SU(2)$. This is a consequence of the McKay correspondence, part of which implies that $1, n_1, \dots, n_n$ are the dimensions of the irreducible representations of G [Gonzalez-Sprinberg and Verdier 1983, page 411].

4.2. ***The algebra QH_R.*** Let

$$H_R = \mathbb{Z} \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$$

be the affine root lattice and let QH_R be the free module over $\mathbb{Z}[t][\![q_1, \dots, q_n]\!]$ generated by $1, \alpha_1, \dots, \alpha_n$,

$$QH_R = H_R \otimes \mathbb{Z}[t][\![q_1, \dots, q_n]\!].$$

We extend the pairing $\langle \cdot , \cdot \rangle$ to a $\mathbb{Q}[t, t^{-1}][\![q_1, \dots, q_n]\!]$ valued pairing on QH_R by making 1 orthogonal to α_i and setting

$$\langle 1, 1 \rangle = \frac{-1}{t^2 \epsilon_R}.$$

For $\beta = \sum_{i=1}^{n} b_i\alpha_i$, we use the notation

$$q^\beta = \prod_{i=1}^{n} q_i^{b_i}.$$

Theorem 6. *Define a product operation $\star$ on QH_R by letting 1 be the identity and defining*

$$\alpha_i \star \alpha_j = -t^2\epsilon_R\langle\alpha_i, \alpha_j\rangle + \sum_{\beta\in R^+} \langle\alpha_i, \beta\rangle\, \langle\alpha_j, \beta^\vee\rangle\, t\frac{1+q^\beta}{1-q^\beta}\beta.$$

Then the product is associative, and moreover, it satisfies the Frobenius condition

$$\langle x\star y, z\rangle = \langle x, y\star z\rangle$$

making QH_R into a Frobenius algebra over the ring $\mathbb{Q}[t, t^{-1}][\![q_1, \dots, q_n]\!]$.

Corollary 7. *The Weyl group acts on QH_R (and thus on $QH^*_{\mathbb{C}^*}(Y)$) by automorphisms. Namely, if we define*

$$g(q^\beta) = q^{g\beta}$$

for $g \in W$, then for $v, w \in QH_R$ we have

$$g(v\star w) = (gv)\star(gw).$$

Proof. Let s_k be the reflection about the hyperplane orthogonal to α_k. By [Bourbaki 1968, VI.1.6 Corollary 1], s_k permutes the positive roots other than α_k. And since the terms

$$\frac{1+q^\beta}{1-q^\beta}\beta \quad \text{and} \quad \langle\alpha_i, \beta\rangle\langle\alpha_j, \beta^\vee\rangle$$

remain unchanged under $\beta \mapsto -\beta$, the effect of applying s_k to the formula for $\alpha_i\star\alpha_j$ is to permute the order of the sum:

$$\begin{aligned} s_k(\alpha_i\star\alpha_j) &= -t^2\epsilon_R\langle\alpha_i, \alpha_j\rangle + \sum_{\beta\in R^+} \langle\alpha_i, \beta\rangle\langle\alpha_j, \beta^\vee\rangle\, t\frac{1+q^{s_k\beta}}{1-q^{s_k\beta}}s_k\beta \\ &= -t^2\epsilon_R\langle s_k\alpha_i, s_k\alpha_j\rangle + \sum_{\beta\in R^+} \langle\alpha_i, s_k\beta\rangle\langle\alpha_j, s_k\beta^\vee\rangle\, t\frac{1+q^\beta}{1-q^\beta}\beta \\ &= s_k(\alpha_i)\star s_k(\alpha_j). \end{aligned}$$

The Corollary follows. □

4.3. ***The proof of Theorem 6.*** When R is of ADE type and the pairing is normalized so that the roots have a norm square of 2, then QH_R coincides with $QH^*_{\mathbb{C}^*}(Y)$ and so Theorem 6 for this case then follows from Theorem 1.

For any R, the Frobenius condition follows immediately from the formulas for $\star$ and $\langle\cdot,\cdot\rangle$.

So what needs to be established in general is the associativity of the $\star$ product. This is equivalent to the expression

$$\mathrm{Ass}^R_{xyuv} = \frac{1}{t^2}\,\langle (x \star y) \star u, v\rangle$$

being fully symmetric in $\{x, y, u, v\}$. Written out, we have

$$\mathrm{Ass}^R_{xyuv} = -\epsilon_R\,\langle x, y\rangle\,\langle u, v\rangle + \sum_{\beta,\gamma\in R^+} \langle x, \beta\rangle\,\langle y, \beta\rangle\,\langle u, \gamma\rangle\,\langle v, \gamma\rangle \left(\frac{1+q^\beta}{1-q^\beta}\right)\left(\frac{1+q^\gamma}{1-q^\gamma}\right)\langle \beta^\vee, \gamma^\vee\rangle.$$

Recalling that ϵ_R scales linearly with the pairing, we see that if Ass^R_{xyuv} is fully symmetric in $\{x, y, u, v\}$, then it remains so for any rescaling of the pairing.

To prove the associativity of QH_R for root systems not of ADE type, we reduce the nonsimply laced case to the simply laced case.

Let $\{R, V, \langle\cdot,\cdot\rangle\}$ be an ADE root system and let Φ be a group of *admissible automorphisms* of the Dynkin diagram. An automorphism g of a graph is admissible if there is no edge joining two vertices in the same g-orbit [Lusztig 1993, Definition 12.1.1]. We construct a new root system

$$\left\{R_\Phi, V^\Phi, \langle\cdot,\cdot\rangle_\Phi\right\}$$

as follows. A somewhat similar construction can be found in [Springer 1998, Section 10.3.1]. Let

$$V^\Phi \subset V$$

be the Φ invariant subspace equipped with $\langle\cdot,\cdot\rangle_\Phi$, the restriction of $\langle\cdot,\cdot\rangle$ to V^Φ, and let the roots of R_Φ be the Φ averages of the roots of R:

$$R_\Phi = \left\{\bar\alpha = \frac{1}{|\Phi|}\sum_{g\in\Phi} g\alpha,\ \alpha\in R\right\}.$$

Then it is easily checked that $\{R_\Phi, V^\Phi, \langle\cdot,\cdot\rangle_\Phi\}$ is an irreducible root system, specifically of type given by

R	A_{2n-1}	D_{n+1}	E_6	D_4
Φ	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
R_Φ	C_n	B_n	F_4	G_2

Thus all the irreducible, reduced root systems arise in this way.

We will frequently use the fact that if $y \in V^\Phi$, then

$$\langle x, y\rangle = \langle \bar x, y\rangle$$

which easily follows from the equalities $\langle x, y\rangle = \langle gx, gy\rangle = \langle gx, y\rangle$ for $g\in\Phi$.

We will also need the following two lemmas which we will prove at the end of the section.

Lemma 8. *The constants defined in Definition 5 coincide for the root systems R and R_Φ:*

$$\epsilon_{R_\Phi} = \epsilon_R.$$

Lemma 9. *Let $\beta \in R^+$ and let $\Phi\beta$ be the Φ orbit of β. Then*

$$\sum_{\beta' \in \Phi\beta} \beta' = \overline{\beta}^\vee.$$

The simple roots of R_Φ are given by $\overline{\alpha}_i$, the averages of the simple roots of R. Thus if $I = \{1, \dots, n\}$ is the index set for the simple roots of R, then Φ acts on I and

$$J = I/\Phi$$

is the natural index set for the simple roots of R_Φ. For $[i] \in J$, we let $\overline{\alpha}_{[i]} \in R_\Phi$ denote the simple root given by $\overline{\alpha}_i$.

We specialize the variables $\{q_i\}_{i\in I}$ to variables $\{\overline{q}_{[i]}\}_{[i]\in J}$ by setting

$$q_i = \overline{q}_{[i]} \tag{4-1}$$

and it is straightforward to see that under the above specialization,

$$q^\beta = \overline{q}^{\overline{\beta}}.$$

Now let R be an ADE root system whose roots have norm square 2. Then Ass^R_{xyuv} is fully symmetric in $\{x, y, u, v\}$. We specialize the q variables to the $\overline{q}$ variables as in (4-1) and we assume that $x, y, v, u \in V^\Phi$. Then

$$\begin{aligned}
&\mathrm{Ass}^R_{xyuv} + \epsilon_R \langle x, y\rangle \langle u, v\rangle \\
&\quad= \sum_{\beta,\gamma\in R^+} \langle x, \beta\rangle \langle y, \beta\rangle \langle u, \gamma\rangle \langle v, \gamma\rangle \left(\frac{1+q^\beta}{1-q^\beta}\right)\left(\frac{1+q^\gamma}{1-q^\gamma}\right)\langle \beta^\vee, \gamma^\vee\rangle \\
&\quad= \sum_{\beta,\gamma\in R^+} \langle x, \overline{\beta}\rangle \langle y, \overline{\beta}\rangle \langle u, \overline{g}\rangle \langle v, \overline{g}\rangle \left(\frac{1+\overline{q}^{\overline{\beta}}}{1-\overline{q}^{\overline{\beta}}}\right)\left(\frac{1+\overline{q}^{\overline{g}}}{1-\overline{q}^{\overline{g}}}\right)\langle \beta, \gamma\rangle \\
&\quad= \sum_{\overline{\beta},\overline{g}\in R^+_\Phi} \langle x, \overline{\beta}\rangle \langle y, \overline{\beta}\rangle \langle u, \overline{g}\rangle \langle v, \overline{g}\rangle \left(\frac{1+\overline{q}^{\overline{\beta}}}{1-\overline{q}^{\overline{\beta}}}\right)\left(\frac{1+\overline{q}^{\overline{g}}}{1-\overline{q}^{\overline{g}}}\right)\left\langle \sum_{\beta'\in\Phi\beta} \beta', \sum_{\gamma'\in\Phi\gamma} \gamma'\right\rangle \\
&\quad= \sum_{\overline{\beta},\overline{g}\in R^+_\Phi} \langle x, \overline{\beta}\rangle_\Phi \langle y, \overline{\beta}\rangle_\Phi \langle u, \overline{g}\rangle_\Phi \langle v, \overline{g}\rangle_\Phi \left(\frac{1+\overline{q}^{\overline{\beta}}}{1-\overline{q}^{\overline{\beta}}}\right)\left(\frac{1+\overline{q}^{\overline{g}}}{1-\overline{q}^{\overline{g}}}\right)\langle \overline{\beta}^\vee, \overline{g}^\vee\rangle_\Phi \\
&\quad= \mathrm{Ass}^{R_\Phi}_{xyuv} + \epsilon_{R_\Phi} \langle x, y\rangle_\Phi \langle u, v\rangle_\Phi
\end{aligned}$$

and thus

$$\mathrm{Ass}^{R_\Phi}_{xyuv} = \mathrm{Ass}^{R}_{xyuv}$$

is fully symmetric in $\{x, y, u, v\}$ and the theorem is proved once we establish Lemmas 8 and 9.

4.4. ***Proofs of Lemmas 8 and 9.*** We prove Lemma 9 first. If β is fixed by Φ, the lemma is immediate. We claim that if β is not fixed then $\langle \beta, g\beta\rangle = 0$ for nontrivial $g \in \Phi$. For simple roots, this follows from the admissibility condition: a node is never adjacent to a node in its orbit. For other roots this can also be seen from a direct inspection of the positive roots (listed, for example, in [Bourbaki 1968, Plates I, IV–VII]). For β not fixed by Φ we then have

$$\langle \overline{\beta}, \overline{\beta}\rangle = \frac{1}{|\Phi|^2}\Big\langle \sum_g g\beta, \sum_h h\beta \Big\rangle = \frac{1}{|\Phi|^2}\sum_g \langle g\beta, g\beta\rangle = \frac{2}{|\Phi|},$$

and Lemma 9 follows.

The preceding formula generalizes to all roots $\overline{\beta}$ by

$$\langle \overline{\beta}, \overline{\beta}\rangle = 2\frac{\mathrm{stab}(\beta)}{|\Phi|},$$

where $\mathrm{stab}(\beta)$ is the order of the stabilizer of the action of Φ on β.

To prove Lemma 8 we must find the coefficients of the longest root of R_Φ. Since the longest root of R is unique, it is fixed by Φ and so it coincides with the longest root of R_Φ:

$$\overline{\widetilde{\alpha}} = \widetilde{\alpha} = \sum_{i\in I} n_i\alpha_i = \sum_{[i]\in J} n_{[i]} \sum_{i'\in\Phi i}\alpha_i = \sum_{[i]\in J} n_{[i]}\,\overline{\alpha}^{\vee}_{[i]} = \sum_{[i]\in J}\frac{2n_{[i]}}{\langle\overline{\alpha}_{[i]}, \overline{\alpha}_{[i]}\rangle}\overline{\alpha}_{[i]}.$$

Thus we have

$$\begin{aligned}
2\epsilon_{R_\Phi} &= \langle \overline{\widetilde{\alpha}}, \overline{\widetilde{\alpha}}\rangle + \sum_{[i]\in J}\Big(\frac{2n_{[i]}}{\langle\overline{\alpha}_{[i]}, \overline{\alpha}_{[i]}\rangle}\Big)^2\langle\overline{\alpha}_{[i]}, \overline{\alpha}_{[i]}\rangle \\
&= \langle\widetilde{\alpha}, \widetilde{\alpha}\rangle + \sum_{i\in I}\frac{\mathrm{stab}(\alpha_i)}{|\Phi|}\frac{4n_i^2}{\langle\overline{\alpha}_i, \overline{\alpha}_i\rangle} \\
&= \langle\widetilde{\alpha}, \widetilde{\alpha}\rangle + \sum_{i\in I} 2n_i^2 \\
&= 2\epsilon_R
\end{aligned}$$

and Lemma 8 is proved.

4.5. ***The root theoretic formula for triple intersections.*** Here we prove the root theoretic result required to finish the proof of (2-4). Recall that R is a root system of ADE type normalized so that the roots have norm square 2. We write

$$g_{ij} = \langle \alpha_i, \alpha_j \rangle.$$

Proposition 10. *Let*

$$G_{ijk} = -\sum_{\beta \in R^+} \langle \alpha_i, \beta \rangle \langle \alpha_j, \beta \rangle \langle \alpha_k, \beta \rangle .$$

(i) G_{ijk} *is symmetric in* $\{i, j, k\}$.

(ii) G_{ijk} *is invariant under any permutation of indices induced by a Dynkin diagram automorphism.*

(iii) $G_{ijk} = 0$ *if* $g_{jk} = 0$.

(iv) $G_{ijk} = -8$ *if* $i = j = k$.

(v) $G_{iij} + G_{jji} = 2$ *if* $g_{ij} = -1$.

(vi) $G_{ikk} + G_{jkk} = 4$ *if* $i \neq j$ *and* $g_{ik} = g_{jk} = -1$.

Proof. From the definition of G_{ijk}, properties (i) and (ii) are clearly satisfied.

Let s_k be reflection about the hyperplane perpendicular to α_k so that

$$s_k \alpha_i = \alpha_i - g_{ik} \alpha_k .$$

Since s_k permutes the positive roots other than α_k [Bourbaki 1968, VI.1.6 Corollary 1], we get the following expression for G_{ijk}:

$$\begin{aligned} G_{ijk} &= -2 g_{ik} g_{jk} g_{kk} - \sum_{\beta \in R^+} \langle \alpha_i, s_k \beta \rangle \langle \alpha_j, s_k \beta \rangle \langle \alpha_k, s_k \beta \rangle \\ &= -4 g_{ik} g_{jk} - \sum_{\beta \in R^+} \langle \alpha_i - g_{ik} \alpha_k, \beta \rangle \langle \alpha_j - g_{jk} \alpha_k, \beta \rangle \langle -\alpha_k, \beta \rangle \\ &= -4 g_{ik} g_{jk} - G_{ijk} + g_{ik} G_{jkk} + g_{jk} G_{ikk} - g_{ik} g_{jk} G_{kkk} \end{aligned}$$

and so

$$G_{ijk} = -2 g_{ik} g_{jk} + \tfrac{1}{2} (g_{ik} G_{jkk} + g_{jk} G_{ikk} - g_{ik} g_{jk} G_{kkk}). \tag{4-2}$$

Setting $i = j = k = n$ we obtain property (iv):

$$G_{nnn} = -8$$

which we can substitute back into (4-2) and then specialize $i = j = a$ to get

$$G_{aak} = 2 g_{ak}^2 + g_{ak} G_{akk}. \tag{4-3}$$

Property (iii) then follows from (4-2) and (4-3) and property (v) follows from (4-3).

For property (vi), observe that if $g_{ik} = g_{jk} = -1$ then $g_{ij} = 0$ and so $G_{ijk} = 0$ and (4-2) then simplifies to prove property (vi). □

5. Predictions for the orbifold invariants via the Crepant Resolution Conjecture

Let $G \subset SU(2)$ be a finite subgroup and let

$$\mathscr{X} = [\mathbb{C}^2/G]$$

be the orbifold quotient of $\mathbb{C}^2$ by G. Recall that

$$\pi : Y \to X$$

is the minimal resolution of X, the singular variety underlying the orbifold $\mathscr{X}$.

The Crepant Resolution Conjecture [Bryan and Graber 2008] asserts that F_Y, the genus zero Gromov–Witten potential of Y, coincides with $F_{\mathscr{X}}$, the genus zero orbifold Gromov–Witten potential of $\mathscr{X}$ after specializing the quantum parameters of Y to certain roots of unity and making a linear change of variables in the cohomological parameters.

Using the Gromov–Witten computations of Section 2, we obtain a formula for F_Y. By making an educated guess for the change of variables and roots of unity, and then applying the conjecture, we obtain a prediction for the orbifold Gromov–Witten potential of $\mathscr{X}$ (Conjecture 11). This prediction has been verified in the cases where G is $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ in [Bryan and Graber 2008; Bryan et al. 2008; Bryan and Jiang $\geq$ 2008] respectively, and recently it has been verified for all $\mathbb{Z}_n$ by Coates, Corti, Iritani, and Tseng [Coates et al. 2007].

5.1. ***The statement of the conjecture.*** The variables of the potential function F_Y are the quantum parameters

$$\{q_1, \dots, q_n\}$$

and cohomological parameters

$$\{y_0, \dots, y_n\}$$

corresponding the generators $\{1, \alpha_1, \dots, \alpha_n\}$ for $H^*_{\mathbb{C}^*}(Y)$.

The potential function is the natural generating function for the genus 0 Gromov–Witten invariants of Y. It is defined by

$$F_Y(q_1, \dots, q_n, y_0, \dots, y_n) = \sum_{k_0,\dots,k_n} \sum_{A \in H_2(Y)} \left\langle 1^{k_0} \alpha_1^{k_1} \cdots \alpha_n^{k_n} \right\rangle_A^Y \frac{y_0^{k_0}}{k_0!} \cdots \frac{y_n^{k_n}}{k_n!} \, q^A.$$

The potential function for the orbifold $\mathscr{X} = [\mathbb{C}^2/G]$ depends on variables

$$\{x_0, \dots, x_n\}$$

which correspond to a basis $\{1, \gamma_1, \dots, \gamma_n\}$ of $H^*_{\mathrm{orb}}(\mathscr{X})$, the orbifold cohomology of $\mathscr{X}$. The orbifold cohomology of $[\mathbb{C}^2/G]$ has a natural basis which is indexed by conjugacy classes of G. If $g \in G$ is an element of the group, we will write $x_{[g]}$ for the variable corresponding to the conjugacy class of g. There are no curve classes in $\mathscr{X}$ and hence no quantum parameters so the potential function is given by

$$F_{\mathscr{X}}(x_0, \dots, x_n) = \sum_{k_0,\dots,k_n} \left\langle 1^{k_0}\gamma_1^{k_1} \cdots \gamma_n^{k_n}\right\rangle^{\mathscr{X}} \frac{x_0^{k_0}}{k_0!} \cdots \frac{x_n^{k_n}}{k_n!}.$$

The conjecture states that there exists roots of unity $\omega_1, \dots, \omega_n$ and an analytic continuation of F_Y to the points

$$q_i = \omega_i$$

such that the equality

$$F_Y(\omega_1, \dots, \omega_n, y_0, \dots, y_n) = F_{\mathscr{X}}(x_0, \dots, x_n)$$

holds after making a (grading preserving) linear change of variables

$$x_i = \sum_{j=0}^{n} L_i^j y_j.$$

Thus to obtain a prediction for the potential $F_{\mathscr{X}}$, we must determine the roots of unity ω_i and the change of variables matrix[2] L.

5.2. ***The prediction.*** The only nontrivial invariants involving 1 are degree zero three point invariants. We split up the potentials $F_{\mathscr{X}}$ and F_Y into terms involving x_0 and y_0 respectively and terms without x_0 and y_0 respectively.

Let F_Y^0 be the part of F_Y with nonzero y_0 terms. It follows from Lemma 4 that F_Y^0 is given by

$$F_Y^0 = \frac{1}{t^2|G|}\frac{y_0^3}{3!} - \frac{y_0}{2}\sum_{i,j=1}^{n} \langle \alpha_i, \alpha_j\rangle y_i y_j.$$

Let $F_{\mathscr{X}}^0$ be the part of $F_{\mathscr{X}}$ with nonzero x_0 terms. An easy localization computation shows that $F_{\mathscr{X}}^0$ is given by

$$F_{\mathscr{X}}^0 = \frac{1}{t^2|G|}\frac{x_0^3}{3!} + \frac{x_0}{2}\frac{1}{|G|}\sum_{g\in G, g\neq Id} x_{[g]}x_{[g^{-1}]}.$$

Since the change of variables respects the grading, the terms in F_Y which are linear and cubic in y_0 must match up with the terms in $F_{\mathscr{X}}$ which are linear and

[2]Our matrix L here is the inverse of the matrix called L in [Bryan and Graber 2008].

cubic in x_0. Consequently we must have $x_0 = y_0$ and moreover, the change of variables must take the quadratic form

$$\frac{1}{|G|} \sum_{g\in G, g\neq Id} x_{[g]} x_{[g^{-1}]} \tag{5-1}$$

to the quadratic form

$$\sum_{i,j=1}^{n} -\langle \alpha_i, \alpha_j \rangle y_i y_j . \tag{5-2}$$

We can rewrite the above quadratic form in terms of the representation theory of G using the classical McKay correspondence [1980] as follows. The simple roots $\alpha_1, \dots, \alpha_n$, which correspond to nodes of the Dynkin diagram, also correspond to nontrivial irreducible representations of G, and hence to their characters $\chi_1, \dots, \chi_n$. Under this correspondence, the Cartan paring can be expressed in terms of $\langle \cdot | \cdot \rangle$, the natural pairing on the characters of G:

$$-\langle \alpha_i, \alpha_j \rangle = \langle (\chi_V - 2)\chi_i | \chi_j \rangle = \frac{1}{|G|} \sum_{g\in G} (\chi_V(g) - 2)\chi_i(g)\overline{\chi}_j(g),$$

where V is the two-dimensional representation induced by the embedding $G \subset SU(2)$.

This discussion leads to an obvious candidate for the change of variables. That is, if we substitute

$$x_{[g]} = \sqrt{\chi_V(g) - 2} \; \sum_{i=1}^{n} \chi_i(g) y_i \tag{5-3}$$

into (5-1) we obtain (5-2). Since $\chi_V(g)$ is always real and less than or equal to 2, we can fix the sign of the square root by making it a positive multiple of i.

Thus we have seen that

$$F_Y^0 = F_{\mathscr{X}}^0$$

under the change of variables given by (5-3) and $x_0 = y_0$. So from here on, we set $x_0 = y_0 = 0$ and deal with just the part of the potentials $F_{\mathscr{X}}$ and F_Y not involving x_0 and y_0.

We apply the divisor axiom and the computations of Section 2:

$$\begin{aligned} &F_Y(y_1, \dots, y_n, q_1, \dots, q_n) \\ &= \frac{1}{6} \sum_{i,j,k=1}^{n} \langle \alpha_i, \alpha_j, \alpha_k \rangle_0 y_i y_j y_k + \sum_{\substack{A\in H_2(Y)\\ A\neq 0}} \langle\ \rangle_A\, q^A e^{\sum_{i=1}^n y_i \int_A \alpha_i} \\ &= \frac{-t}{6} \sum_{i,j,k=1}^{n} \sum_{\beta\in R^+} \langle \alpha_i, \beta \rangle \langle \alpha_j, \beta \rangle \langle \alpha_k, \beta \rangle\, y_i y_j y_k + \sum_{d=1}^{\infty} \sum_{\beta\in R^+} \frac{2t}{d^3} q^{d\beta} e^{\sum_{i=1}^n -d\langle \alpha_i, \beta\rangle y_i}. \end{aligned}$$

Taking triple derivatives we get

$$\frac{\partial^3 F_Y}{\partial y_i \partial y_j \partial y_k} = -t \sum_{\beta\in R^+} \langle\alpha_i,\beta\rangle\,\langle\alpha_j,\beta\rangle\,\langle\alpha_k,\beta\rangle \Big(1+\frac{2q^\beta e^{\sum_i -\langle\beta,\alpha_i\rangle y_i}}{1-q^\beta e^{\sum_i -\langle\beta,\alpha_i\rangle y_i}}\Big)$$
$$= -t \sum_{\beta\in R^+} \langle\alpha_i,\beta\rangle\,\langle\alpha_j,\beta\rangle\,\langle\alpha_k,\beta\rangle\,\frac{1+q^\beta e^{\sum_i -\langle\beta,\alpha_i\rangle y_i}}{1-q^\beta e^{\sum_i -\langle\beta,\alpha_i\rangle y_i}}.$$

We specialize the quantum parameters to roots of unity by

$$q_j = \exp\Big(\frac{2\pi i n_j}{|G|}\Big)$$

where n_j is the j-th coefficient of the largest root as in Definition 5. Note that n_j is also the dimension of the corresponding representation.

After specializing the quantum parameters, the triple derivatives of the potential F_Y can be expressed in terms of the function

$$H(u) = \frac{1}{2i}\Big(\frac{1+e^{i(u-\pi)}}{1-e^{i(u-\pi)}}\Big) = \frac{1}{2}\tan\Big(\frac{-u}{2}\Big)$$

as follows:

$$\frac{\partial^3 F_Y}{\partial y_i \partial y_j \partial y_k} = -2it \sum_{\beta\in R^+} \langle\alpha_i,\beta\rangle\,\langle\alpha_j,\beta\rangle\,\langle\alpha_k,\beta\rangle\, H(Q_\beta)$$

where for $\beta = \sum_{j=1}^n b_j\alpha_j$ we define

$$Q_\beta = \pi + \sum_{j=1}^n \Big(\frac{2\pi n_j b_j}{|G|} + i\langle\beta,\alpha_j\rangle y_j\Big).$$

It then follows that

$$F_Y(y_1,\dots,y_n) = 2t \sum_{\beta\in R^+} h(Q_\beta)$$

where $h(u)$ is a series satisfying

$$h'''(u) = \tfrac{1}{2}\tan\Big(\frac{-u}{2}\Big).$$

We can now make the change of variables given by (5-3):

$$\sum_{j=1}^n i\langle\beta,\alpha_j\rangle y_j = \sum_{j,k=1}^n ib_k\langle\alpha_k,\alpha_j\rangle y_j = \sum_{j,k=1}^n \frac{-ib_k}{|G|}\sum_{g\in G}(\chi_V(g)-2)\,\overline{\chi}_k(g)\chi_j(g)y_j$$
$$= \sum_{k=1}^n \frac{b_k}{|G|}\sum_{g\in G}\sqrt{2-\chi_V(g)}\;\;\overline{\chi}_k(g)x_{[g]}.$$

Substituting this back into Q_β we arrive at our conjectural formula for $F_{\mathscr{X}}$.

Conjecture 11. *Let $F_{\mathscr{X}}(x_1, \dots, x_n)$ denote the $\mathbb{C}^*$ equivariant genus zero orbifold Gromov–Witten potential of the orbifold $\mathscr{X} = [\mathbb{C}^2/G]$ where we have set the unit parameter x_0 equal to zero. Let R be the root system associated to G as in Section 1. Then*

$$F_{\mathscr{X}}(x_1, \dots, x_n) = 2t \sum_{\beta \in R^+} h(Q_\beta)$$

where $h(u)$ is a series with

$$h'''(u) = \tfrac{1}{2} \tan\left(\frac{-u}{2}\right)$$

and

$$Q_\beta = \pi + \sum_{k=1}^{n} \frac{b_k}{|G|}\left(2\pi n_k + \sum_{g \in G} \sqrt{2 - \chi_V(g)}\ \overline{\chi}_k(g) x_{[g]}\right)$$

where b_k are the coefficients of $\beta \in R^+$, n_k are the coefficients of the largest root, and V is the two-dimensional representation induced by the embedding $G \subset SU(2)$.

Note that the index set $\{1, \dots, n\}$ in the above formula corresponds to

(1) simple roots of R,

(2) nontrivial irreducible representations of G, and

(3) nontrivial conjugacy classes of G.

The index of a conjugacy class containing a group element g is denoted by $[g]$. Finally note that the terms of degree less than three are ill-defined for both the potential $F_{\mathscr{X}}$ and our conjectural formula for it.

The above conjecture has been proved in the cases where G is $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ in [Bryan and Graber 2008; Bryan et al. 2008; Bryan and Jiang $\geq$ 2008] respectively, and recently it has been verified for all $\mathbb{Z}_n$ by Coates, Corti, Iritani, and Tseng [2007].

We have also performed a number of checks of the conjecture for nonabelian G. Many of the orbifold invariants must vanish by monodromy considerations, and our conjecture is consistent with this vanishing. One can geometrically derive a relationship between some of the orbifold invariants of $[\mathbb{C}^2/G]$ and certain combinations of the orbifold invariants of $[\mathbb{C}^2/H]$ when H is a normal subgroup of G. This leads to a simple relationship between the corresponding potential functions. We have checked that this relationship is consistent with our conjecture.

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