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Operad of formal homogeneous spaces and Bernoulli numbers

Sergei A. Merkulov

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It is shown that for any morphism, $\phi : \mathfrak{g} \to \mathfrak{h}$, of Lie algebras the vector space underlying the Lie algebra $\mathfrak{h}$ is canonically a $\mathfrak{g}$-homogeneous formal manifold with the action of $\mathfrak{g}$ being highly nonlinear and twisted by Bernoulli numbers. This fact is obtained from a study of the 2-coloured operad of formal homogeneous spaces whose minimal resolution gives a new conceptual explanation of both Ziv Ran's Jacobi–Bernoulli complex and Fiorenza–Manetti's L_∞-algebra structure on the mapping cone of a morphism of two Lie algebras. All these constructions are iteratively extended to the case of a morphism of arbitrary L_∞-algebras.

1. Introduction

1.1. The theory of operads and props gives a universal approach to the deformation theory of many algebraic and geometric structures [Merkulov and Vallette 2007]. It also gives a conceptual explanation of the well-known "experimental" observation that a deformation theory is controlled by a differential graded (dg, for short) Lie algebra or, more generally, a L_∞-algebra. What happens is the following:

(I) an algebraic or a (germ of) geometric structure, $\mathfrak{s}$, on a vector space V (which is an *object* in the corresponding category, $\mathfrak{S}$, of algebraic or geometric structures) can often be interpreted as a *morphism*, $\alpha_{\mathfrak{s}} : \mathcal{O}^{\mathfrak{S}} \to \mathcal{E}nd_V$, in the category of operads (or props), where $\mathcal{O}^{\mathfrak{S}}$ and $\mathcal{E}nd_V$ are operads (or props) canonically associated to the category $\mathfrak{S}$ and the vector space V;

(II) the operad/prop $\mathcal{O}^{\mathfrak{S}}$ often admits a unique minimal[1] dg resolution, $\mathcal{O}^{\mathfrak{S}}_\infty$, which, by definition, is a free dg operad/prop generated by some $\mathbb{S}$-(bi)module E together with a epimorphism $\pi : \mathcal{O}^{\mathfrak{S}}_\infty \to \mathcal{O}^{\mathfrak{S}}$ which induces an isomorphism on cohomology; it was proven in [Merkulov and Vallette 2007] (in two different ways) that the set of *all* possible morphisms, $\mathcal{O}^{\mathfrak{S}}_\infty \to \mathcal{E}nd_V$, can be identified with the set of Maurer–Cartan elements of a uniquely defined Lie (or, more generally, filtered

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[1] In fact there is no need to work with *minimal* resolutions: any free resolution of $\mathcal{O}^{\mathfrak{S}}$ will do.

L_∞-) algebra $\mathcal{G} := \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1]$ whose Lie brackets can be read directly from the generators and differential of the minimal resolution $\mathcal{O}_\infty^{\mathfrak{S}}$;

(III) thus, to our algebraic or geometric structure $\mathfrak{s}$ there corresponds a Maurer–Cartan element $\gamma_{\mathfrak{s}} := \pi \circ \alpha_{\mathfrak{s}}$ in $\mathcal{G}$; twisting $\mathcal{G}$ by $\gamma_{\mathfrak{s}}$ one obtains finally a Lie (or L_∞-) algebra $\mathcal{G}_{\mathfrak{s}}$ which controls the deformation theory of the structure $\mathfrak{s}$ we began with.

Many important dg Lie algebras in homological algebra and geometry (such as Hochschild, Schouten and Frölicher–Nijenhuis algebras) are proven in [Kontsevich and Soibelman 2000; Merkulov 2006; 2005; Merkulov and Vallette 2007; van der Laan 2002] to be of this operadic or propic origin. For example, if $\mathfrak{s}$ is a structure of an associative algebra on a vector space V, then,

(i) there is an operad, $\mathcal{A}ss$, uniquely associated with the category of associative algebras, and the structure $\mathfrak{s}$ corresponds to a morphism, $\alpha_{\mathfrak{s}} : \mathcal{A}ss \to \mathcal{E}nd_V$, of operads;

(ii) there is a unique minimal resolution, $\mathcal{A}ss_\infty$, of $\mathcal{A}ss$ which is generated by the $\mathbb{S}$-module $E = \{\mathbb{K}[\mathbb{S}_n][n-2]\}$ and whose representations, $\pi : \mathcal{A}ss_\infty \to \mathcal{E}nd_V$, in a dg space V are in one-to-one correspondence with Maurer–Cartan elements in the Lie algebra,

$$\left(\mathcal{G} := \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1] = \bigoplus_{n\geq 1} \mathrm{Hom}_{\mathbb{K}}(V^{\bigotimes n}, V)[1-n],\ [\ ,\ ]_G\right),$$

where $[\ ,\ ]_G$ are Gerstenhaber brackets (see, for example, [Kontsevich and Soibelman 2000; Merkulov and Vallette 2007]);

(iii) therefore, the particular associative algebra structure $\mathfrak{s}$ on V gives rise to the associated Maurer–Cartan element $\gamma_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \circ \pi$ in $\mathcal{G}$; twisting $\mathcal{G}$ by $\gamma_{\mathfrak{s}}$ gives the Hochschild dg Lie algebra,

$$\mathcal{G}_{\mathfrak{s}} = \left(\bigoplus_n \mathrm{Hom}_{\mathbb{K}}(V^{\bigotimes n}, V)[1-n],\ [\ ,\ ]_G, d_H := [\gamma_{\mathfrak{s}},\ ]_G\right),$$

which indeed controls the deformation theory of $\mathfrak{s}$.

1.2. Recently Ziv Ran introduced a so-called *Jacobi–Bernoulli* deformation complex and used it to study deformations of pairs of geometric structures such as a given complex manifold X and the moduli space, $\mathcal{M}_X$, of vector bundles on X, a complex manifold X and its complex compact submanifold Y, and others [Ran 2006; 2004]. The differential in this complex is, rather surprisingly, twisted by Bernoulli numbers. Fiorenza and Manetti [2007] discovered independently the same thing under the name of L_∞-algebra structure on the mapping cone of a morphism of Lie algebras using completely different approach based on explicit homotopy transfer formulae of Kontsevich and Soibelman [2000] and Merkulov [1999]; they also showed its relevance to the deformation theory of complex submanifolds in complex manifolds using the earlier results of Manetti [2005].

In view of the above paradigm one can raise a question: which operad gives rise to a deformation complex with such an unusual differential?

We suggest an answer in this paper. Surprisingly, this answer is not a straightforward operadic translation of the notion of *Lie atom* introduced and studied in [Ran 2006; 2004] but is based instead on another algebraic+geometric structure which we call a *formal homogeneous space* and which is, by definition, a triple, $(\mathfrak{g}, \mathfrak{h}, F)$, consisting of a Lie algebra $\mathfrak{g}$, a vector space $\mathfrak{h}$, and a morphism,

$$F : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}}$$

of Lie algebras, where $\mathcal{T}_{\mathfrak{h}}$ is the Lie algebra of smooth formal vector fields on the space $\mathfrak{h}$. Let $\mathcal{HS}$ be the 2-coloured operad whose representations are formal homogeneous spaces, $(\mathfrak{g}, \mathfrak{h}, F)$, and let $\mathcal{LP}$ be the 2-coloured operad whose representations are *Lie pairs*, that is, the triples, $(\mathfrak{g}, \mathfrak{h}, \phi)$, consisting of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ as well as a morphism,

$$\phi : \mathfrak{g} \to \mathfrak{h}$$

of Lie algebras. We prove in Theorem 4.1.1 below that there exists a *unique* nontrivial morphism of coloured operads,

$$JB : \mathcal{HS} \longrightarrow \mathcal{LP},$$

which we call the *Jacobi–Bernoulli* morphism because it involves Bernoulli numbers and eventually explains the differential in Ziv Ran's Jacobi–Bernoulli complex. It means the following: given a morphism of Lie algebras, $\phi : \mathfrak{g} \to \mathfrak{h}$, there is a canonically associated morphism of other Lie algebras, $F_\phi : \mathfrak{g} \to \mathcal{T}_{\mathfrak{h}}$, which is determined by ϕ and the Lie algebra brackets [,] in $\mathfrak{h}$. It means, therefore, that there is always a canonically associated nonlinear action of $\mathfrak{g}$ on the space $\mathfrak{h}$ which is twisted by Bernoulli numbers (and is given in local coordinates by (11)).

Thus one can think of the deformation theory of any given Lie pair, $\phi : \mathfrak{g} \to \mathfrak{h}$, in two different worlds:

(1) the world of algebraic morphisms of Lie algebras which allows deformations of three things — of a Lie algebra structure on $\mathfrak{g}$, of a Lie algebra structure on $\mathfrak{h}$ and of a morphism ϕ — and which is governed by the well-known 2-coloured dg operad, $\mathcal{LP}_\infty$, describing pairs of L_∞-algebras and L_∞-morphisms between them, and

(2) the world of formal $\mathfrak{g}$-homogeneous spaces $\mathfrak{h}$ which allows deformations of two things — of a Lie algebra structure on $\mathfrak{g}$ and of its action, $F_\phi : \mathfrak{g} \to \mathcal{T}_\mathfrak{h}$, on $\mathfrak{h}$ — which is governed by the minimal resolution, $\mathcal{HS}_\infty$, of the 2-coloured operad of formal homogeneous spaces which we explicitly describe below in Theorem 2.6.1.

These two worlds have very different deformation theories. The first one is controlled by the L_∞-algebra associated with $\mathcal{LP}_\infty$ as explained in [Merkulov and Vallette 2007, § 5.8]. The second one, as we show in Section 4 below, naturally gives rise to Ziv Ran's Jacobi–Bernoulli complex. This part of our story develops as follows: with a given Lie pair, $\phi : \mathfrak{g} \to \mathfrak{h}$, the Jacobi–Bernoulli morphism *JB* associates a Maurer–Cartan element, γ_ϕ, in the Lie algebra, $\mathfrak{G}_{\mathfrak{g},\mathfrak{h}}$, which describes all possible morphisms, $\mathcal{HS}_\infty \to \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$, of 2-coloured operads; this algebra $\mathfrak{G}_{\mathfrak{g},\mathfrak{h}}$ is proven to be a Lie subalgebra of the Lie algebra of coderivations of the graded commutative coalgebra $\bigodot^\bullet(\mathfrak{g}[1] \oplus \mathfrak{h})$ (see Proposition 2.7.1); hence the Maurer–Cartan element γ_ϕ equips this coalgebra with an associated codifferential, d_ϕ, and the resulting complex coincides precisely with the Jacobi–Bernoulli complex of Ran [2004], or, equivalently, with L_∞-structure on $\mathfrak{g} \oplus \mathfrak{h}[-1]$ of Fiorenza and Manetti [2007].

We also briefly discuss in our paper a strong homotopy extension of all the above constructions. It is proven that there exits a morphism of 2-coloured dg operads,

$$JB_\infty : \mathcal{HS}_\infty \longrightarrow \mathcal{LP}_\infty,$$

which associates a formal homogeneous$_\infty$ space to any triple, $(\mathfrak{g}, \mathfrak{h}, \phi_\infty)$, consisting of L_∞-algebras $\mathfrak{g}$ and $\mathfrak{h}$ and a L_∞-morphism $\phi_\infty : \mathfrak{g} \to \mathfrak{h}$. Hence there exists an associated Jacobi–Bernoulli$_\infty$ complex which has the same graded vector space structure as Ziv Ran's Jacobi–Bernoulli complex but a more complicated differential (and hence a more complicated L_∞-algebra structure on the mapping cone of ϕ_∞). We first show an iterative procedure for computing JB_∞ in full generality and then, motivated by the deformation quantization of Poisson structures [Kontsevich 2003], give explicit formulae for the natural composition

$$JB_{\frac{1}{2}\infty} : \mathcal{HS}_\infty \xrightarrow{JB_\infty} \mathcal{LP}_\infty \to \mathcal{LP}_{\frac{1}{2}\infty},$$

where $\mathcal{LP}_{\frac{1}{2}\infty}$ is the 2-coloured operad describing L_∞-morphisms, $\phi_\infty : \mathfrak{g} \to \mathfrak{h}$, between ordinary dg Lie algebras.

1.3. In this paper we extensively use the language of (coloured) operads. For an introduction of the theory of operads we refer to [Markl et al. 2002; Merkulov 2008b] and especially to [Berger and Moerdijk 2007; Kontsevich and Soibelman 2000; Longoni and Tradler 2003]. Some key ideas of this language can be grasped by looking at the basic example of the 2-coloured *endomorphism* operad, $\mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$, canonically associated to an arbitrary pair of vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ as follows: (a) as an $\mathbb{S}$-module the operad $\mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$ is given, by definition, by a collection of vector spaces,

$$\Big\{ \bigoplus_{m+n=N} \mathbb{K}[\mathbb{S}_N] \otimes_{\mathbb{S}_m \times \mathbb{S}_n} \mathrm{Hom}\big(\mathfrak{g}^{\bigotimes m} \otimes \mathfrak{h}^{\bigotimes n},\ \mathfrak{g} \oplus \mathfrak{h}\big) \Big\}_{N \geq 1}$$

on which the permutation groups $\mathbb{S}_N$ naturally act; (b) the operadic compositions in $\mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$ are given, by definition, by plugging the output of one linear map into a particular input (of the same "colour" $\mathfrak{g}$ or $\mathfrak{h}$) of another map. These compositions satisfy numerous "associativity" conditions which, when axiomatized, are used as the definition of an arbitrary 2-coloured operad.

1.4. ***Notations.*** If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is the graded vector space with $V[k]^i := V^{i+k}$. For any pair of natural numbers $m < n$ the ordered set $\{m, m+1, \dots, n-1, n\}$ is denoted by $[m, n]$. The ordered set $[1, n]$ is further abbreviated to $[n]$. For a finite set J the symbol $(-1)^J$ stands for $(-1)^{\text{cardinality of } J}$. For a subdivison, $[n] = I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$, of the naturally ordered set $[n]$ into k disjoint naturally ordered subsets, we denote by $\sigma(I_1 \sqcup I_2 \sqcup \dots \sqcup I_k)$ the associated permutation $[n] \to I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$ and by $(-1)^{\sigma(I_1 \sqcup \dots \sqcup I_k)}$ the sign of the latter. We work throughout over a field $\mathbb{K}$ of characteristic zero.

2. Operad of Lie actions and its minimal resolution

2.1. ***Motivation.*** Ran [2006; 2004] introduced a notion of Lie atom as a means to describe relative deformation problems in which deformations (controlled by some Lie algebra, say, $\mathfrak{g}$) of a geometric object leave some (controlled by another Lie algebra, say, $\mathfrak{h}$) aspect invariant. More precisely, a *Lie atom* (for *algebra to module* [Ran 2006]) is defined as a collection of data $(\mathfrak{g}, \mathfrak{h}, \langle\ ,\ \rangle, \phi)$ consisting of

(i) a Lie algebra $\mathfrak{g}$ with Lie brackets $[\ ,\ ]$,

(ii) a vector space $\mathfrak{h}$ equipped with a $\mathfrak{g}$-module structure, that is, with a linear map,

$$\begin{aligned} \langle\ ,\ \rangle :\ \mathfrak{g} \otimes \mathfrak{h} &\longrightarrow \mathfrak{h}, \\ a \otimes m &\mapsto \langle a, m \rangle, \end{aligned}$$

satisfying the equation,

$$\langle [a, b], m \rangle = \langle a, \langle b, m \rangle \rangle - (-1)^{ab} \langle b, \langle a, m \rangle \rangle,$$

and

(iii) a morphism, $\phi : \mathfrak{g} \to \mathfrak{h}$, of $\mathfrak{g}$-modules, that is, a linear map from $\mathfrak{g}$ to $\mathfrak{h}$ satisfying the equation

$$\phi([a, b]) = \langle a, \phi(b) \rangle = -(-1)^{ab} \langle b, \phi(a) \rangle$$

for any $a, b \in \mathfrak{g}$.

According to a general philosophy of the deformation theory outlined in Section 1.1, one might attempt to introduce a 2-coloured operad of Lie atoms, resolve it and then study the associated deformation complex of Lie atoms. It is easy to see, however, that the resulting deformation complex must be much larger than the Jacobi–Bernoulli complex and the theory of operads, if pushed in that direction, does not explain the results of Ran [2006; 2004].

This fact forces us to work with different versions of atoms which we call *formal (affine) homogeneous spaces*.

2.2. *Definition.* An *affine homogeneous space* is a collection of data $(\mathfrak{g}, \mathfrak{h}, \langle\ ,\ \rangle, \phi)$ consisting of

(i) a Lie algebra $\mathfrak{g}$ with Lie brackets $[\ ,\ ]$,

(ii) a vector space $\mathfrak{h}$ equipped with a $\mathfrak{g}$-module structure, $\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$, and

(iii) a linear map, $\phi : \mathfrak{g} \to \mathfrak{h}$, satisfying the equation

$$\phi([a, b]) = \langle a, \phi(b)\rangle - (-1)^{ab}\langle b, \phi(a)\rangle$$

for any $a, b \in \mathfrak{g}$.

The only difference between the definition of Lie atom in Section 2.1 and the present one lies in item (iii). This difference is substantial: for example, a pair of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ together with a morphism, $\phi : \mathfrak{g} \to \mathfrak{h}$, of Lie algebras makes a Lie atom, $(\mathfrak{g}, \mathfrak{h}, \langle\ ,\ \rangle, \phi)$, with $\langle a, m\rangle := [\phi(a), m]$ but does *not* make an affine homogeneous space as the condition (iii) in Section 2.2 is not satisfied.

The terminology is justified by the following lemma.

Lemma 2.2.1. *An affine homogeneous space structure on a pair*, $(\mathfrak{g}, \mathfrak{h})$, *consisting of a Lie algebra* $\mathfrak{g}$ *and a vector space* $\mathfrak{h}$ *is the same as a morphism of Lie algebras*,

$$F : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}}^{\mathrm{aff}},$$

where $\mathcal{T}_{\mathfrak{h}}^{\mathrm{aff}}$ *is the Lie algebra of affine vector fields on* $\mathfrak{h}$.

Proof. A Lie algebra, $\mathcal{T}_{\mathfrak{h}}$, of smooth formal vector fields on $\mathfrak{h}$ is, by definition, the Lie algebra of derivations of the graded commutative ring,

$$\mathbb{O}_{\mathfrak{h}} := \prod_{n\geq 0} \bigodot^{n} \mathfrak{h}^*,$$

of smooth formal functions on $\mathfrak{h}$. Its subalgebra, $\mathcal{T}_{\mathfrak{h}}^{\mathrm{aff}}$, consists, by definition, of those vector fields, $V \in \mathcal{T}_{\mathfrak{h}}$, whose values, $V(\lambda)$, on arbitrary linear functions, $\lambda \in \mathfrak{h}^*$, lie in the subspace $\mathbb{K} \oplus \mathfrak{h}^* \subset \mathbb{O}_{\mathfrak{h}}$. Thus,

$$\mathcal{T}_{\mathfrak{h}}^{\mathrm{aff}} = \mathsf{End}(\mathfrak{h}) \oplus \mathfrak{h},$$

and the map $F : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}}^{\mathrm{aff}}$ gives rise to a pair of linear maps,

$$F_0 : \mathfrak{g} \to \mathfrak{h} \quad \text{and} \quad F_1 : \mathfrak{g} \to \mathsf{End}(\mathfrak{h}).$$

The map F_1 can be equivalently interpreted as a linear map $\hat{F}_1 : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$. Now it is a straightforward to check that the conditions for F to be a morphism of Lie algebras are precisely conditions (ii) and (iii) in Section 2.2 for the maps $\phi := F_0$ and $\langle\ ,\ \rangle := -\hat{F}_1$. □

Example 2.2.2. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. And $\psi_t : \mathfrak{g} \to \mathfrak{h}$ is a smooth 1-parameter family of morphisms of Lie algebras, with $-\varepsilon < t < \varepsilon$ and $\varepsilon > 0$. There is a naturally associated affine homogeneous space $(\mathfrak{g}, \mathfrak{h}, \langle\ ,\ \rangle, \phi)$ with

$$\langle a, m \rangle := [\psi_0(a), m] \quad \text{and} \quad \phi := \left.\frac{d\psi_t}{dt}\right|_{t=0}$$

for any $a \in \mathfrak{g}$, $m \in \mathfrak{h}$. Indeed, the condition (ii) in Section 2.2 is obviously satisfied, while the differentiation of the equality,

$$\psi_t([a, b]) = [\psi_t(a), \psi_t(b)],$$

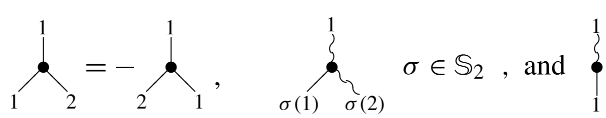
at $t = 0$ gives the condition (iii).

Example 2.2.3. Let $(\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i, [\ ,\ ], d)$ be a nilpotent dg Lie algebra. There is an associated *gauge* action of the nilpotent group $G_0 := \{e^g | g \in \mathfrak{g}^0\}$ on the subspace $\mathfrak{g}^1$ given by

$$\begin{array}{rccl} R: & G_0 \times \mathfrak{g}^1 & \longrightarrow & \mathfrak{g}^1, \\ & (e^g, \Gamma) & \mapsto & e^{ad_g}\Gamma - \dfrac{e^{ad_g} - 1}{ad_g} dg, \end{array}$$

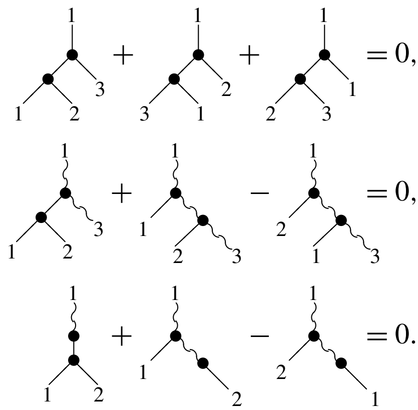
where ad_g stands for the adjoint action by g. This action makes the pair $(\mathfrak{g}^0, \mathfrak{g}^1)$ into an affine homogeneous space.

2.3. Operad of affine homogeneous spaces. This is a 2-coloured operad generated by the labelled corollas[2]



[2] All our graphs are by default directed with the flow running from the bottom to the top.

representing the operations $[\ ,\ ] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$, $-\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$ and $\phi : \mathfrak{g} \to \mathfrak{h}$, modulo the obvious relations



The interpretation in Lemma 2.2.1 of the notion of affine homogeneous space prompts us to introduce its generalization.

2.4. *Definition.* A *formal homogeneous space* is a triple, $(\mathfrak{g}, \mathfrak{h}, F)$, consisting of a Lie algebra $\mathfrak{g}$, a vector space $\mathfrak{h}$ and a morphism of Lie algebras,

$$F : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}},$$

where $\mathcal{T}_{\mathfrak{h}}$ is the Lie algebra of smooth formal vector fields on $\mathfrak{h}$.

Example 2.4.1. Let a Lie group G with the Lie algebra $\mathfrak{g}$ act on a vector space $\mathfrak{h}$ viewed as a smooth manifold (that is, the action may not necessarily preserve the linear structure on $\mathfrak{h}$). Then there is an associated formal homogeneous space, $\mathfrak{g} \to \mathcal{T}_{\mathfrak{h}}$.

Example 2.4.2. Let $\mathfrak{g}$ be the Lie algebra of formal vector fields on $\mathbb{R}^n$, and let $\mathbb{R}^{coor}$ be the space of infinite jets of smooth maps, $\mathbb{R}^n \to \mathbb{R}^n$. There is a canonical morphism of Lie algebras,

$$\mathfrak{g} \longrightarrow \mathcal{T}_{\mathbb{R}^{coor}},$$

which, for any point t in $\mathbb{R}^{coor}$, restricts to an isomorphism of vector spaces, $\mathfrak{g} \to (\mathcal{T}_{\mathbb{R}^{coor}})_t$, where $(\mathcal{T}_{\mathbb{R}^{coor}})_t$ is tangent vector space at t. This observation lies in the heart of the so-called *formal geometry* which provides us with a formal homogeneous space approach to many problems in differential geometry such as pseudogroup structures, foliations, characteristic classes, and so on (see [Bernšteĭn and Rosenfel′d 1973] and references cited there).

Example 2.4.3. Let $X \subset \mathfrak{h}$ be an analytic submanifold of $\mathfrak{h} = \mathbb{K}^n$. There is an associated formal homogeneous space $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ being the Lie subalgebra of $\mathcal{T}_{\mathfrak{h}}$ consisting of analytic vector fields on $\mathfrak{h}$ tangent to X along X.

Example 2.4.4. It will be proven below in Theorem 4.1.1 that for any morphism of Lie algebras, $\phi : \mathfrak{g} \to \mathfrak{h}$, there is a canonically associated formal homogeneous space $F_\phi : \mathfrak{g} \to \mathcal{T}_\mathfrak{h}$ with F_ϕ uniquely and rather nontrivially determined by both ϕ and the Lie algebra brackets $[\ ,\ ]$ in $\mathfrak{h}$.

In accordance with the general operadic paradigm [Merkulov and Vallette 2007, § 1], in order to obtain the deformation theory of formal homogeneous spaces one has to first describe the associated operad, $\mathcal{HS}$, and then compute its minimal dg resolution $\mathcal{HS}_\infty$. The first step is very easy.

2.5. *Operad of formal homogeneous spaces.* An arbitrary formal vector field, $h \in \mathcal{T}_\mathfrak{h}$, on a vector space $\mathfrak{h}$ is uniquely determined by its Taylor components, $\left\{h_n \in \mathrm{Hom}_\mathbb{K}(\mathfrak{h}^{\odot n}, \mathfrak{h})\right\}_{n\geq 0}$, with

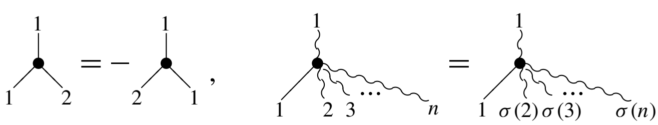
$$h = \sum_a h^a(x) \frac{\partial}{\partial x^a} \quad \overset{1-1}{\longleftrightarrow} \quad \left\{ h_n \simeq \frac{1}{n!} \frac{\partial^n h^a(x)}{\partial x^{b_1} \ldots \partial x^{b_n}} \Big|_{x=0} \right\}_{n\geq 0}$$

implying that an arbitrary linear map $F : \mathfrak{g} \to \mathcal{T}_\mathfrak{h}$ is uniquely described by a collection of its components $\left\{F_n \in \mathrm{Hom}_\mathbb{K}(\mathfrak{g} \otimes \mathfrak{h}^{\odot n}, \mathfrak{h})\right\}_{n\geq 0}$. Thus a 2-coloured operad, $\mathcal{HS}$, whose representations,

$$\rho : \mathcal{HS} \longrightarrow \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}},$$

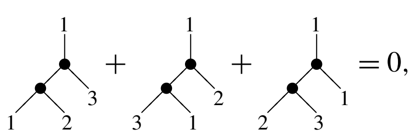
in an arbitrary pair of vector spaces $(\mathfrak{g}, \mathfrak{h})$ are the same as formal homogeneous space structures on $(\mathfrak{g}, \mathfrak{h})$, can be described as follows.

Definition 2.5.1. The *operad formal homogeneous spaces*, $\mathcal{HS}$, is a 2-coloured operad generated[3] by corollas



$\forall \sigma \in \mathbb{S}_{n-1}, n \geq 0,$ (1)

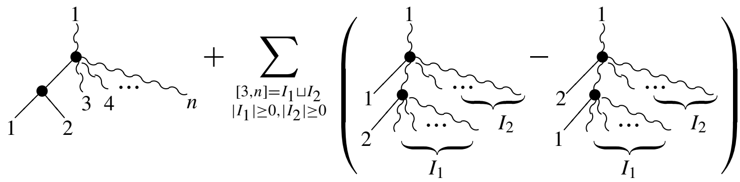
which correspond to the Lie brackets, $[\ ,\ ]$, in $\mathfrak{g}$ and, respectively, to the Taylor component, F_n, of the map F, modulo the relations,



(2)

[3] That is, spanned by all possible graphs built from the corollas described in (1) by gluing the output leg of one corolla to an input leg (with the *same* — "straight" or "wavy" — colour) of another corolla.

corresponding to the Jacobi identities for [,], and



$$= 0, \quad n \geq 2, \quad (3)$$

corresponding to the compatibility of F_n with the Lie algebra structures in $\mathfrak{g}$ and $\mathcal{T}_{\mathfrak{h}}$. Here the summation runs over all splittings of the ordered set $[3, n] := \{3, 4, \ldots, n\}$ into two (possibly empty) disjoint subsets I_1 and I_2.

2.5.2. Dilation symmetry. For any $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$ let

$$\begin{array}{rccc} \psi_\lambda : & \mathfrak{h} & \longrightarrow & \mathfrak{h}, \\ & x & \mapsto & \lambda x, \end{array}$$

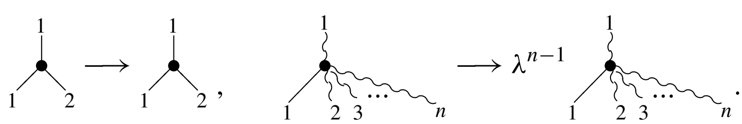
be the associated dilation automorphism of $\mathfrak{h}$. It induces an automorphism of the Lie algebra of formal vector fields,

$$d\psi_\lambda : \mathcal{T}_{\mathfrak{h}} \longrightarrow \mathcal{T}_{\mathfrak{h}}.$$

Therefore, the group $\mathbb{K}^*$ acts on the set of Lie action structures on a given pair, $(\mathfrak{g}, \mathfrak{h})$, of vector spaces,

$$\phi : \mathfrak{g} \to \mathcal{T}_{\mathfrak{h}} \quad \longrightarrow \quad \phi_\lambda := d\psi_\lambda \circ \phi : \mathfrak{g} \to \mathcal{T}_{\mathfrak{h}}.$$

It implies that the group $\mathbb{K}^*$ acts as an automorphism group of the operad $\mathcal{HS}$ as follows:



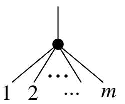
2.6. ***Minimal resolution of*** $\mathcal{HS}$**.** This is, by definition, a free[4] 2-coloured operad, $\mathcal{HS}_\infty$, equipped with a decomposable differential δ and with an epimorphism of dg operads,

$$\pi : (\mathcal{HS}_\infty, \delta) \longrightarrow (\mathcal{HS}, 0),$$

which induces an isomorphism in cohomology. Here we understand $(\mathcal{HS}, 0)$ as a dg operad with the trivial differential. A minimal resolution is defined uniquely up to an isomorphism.

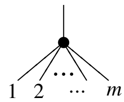
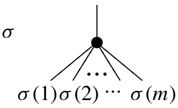
[4]That is, generated by a family of corollas with *no* relations.

Theorem 2.6.1. *The minimal resolution, $\mathcal{HS}_\infty$, is a free 2-coloured operad generated by m-corollas,*

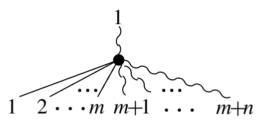


$$m \geq 2, \tag{4}$$

of degree $2-m$ with skewsymmetric input legs,

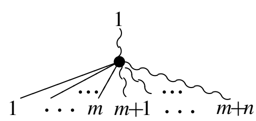
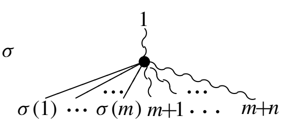
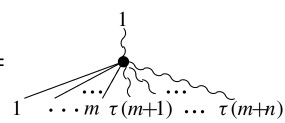

$= (-1)^\sigma$

$\forall\, \sigma \in \mathbb{S}_n,$

and (m,n)-corollas,

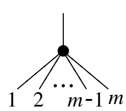
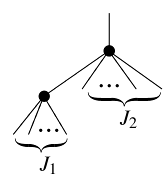


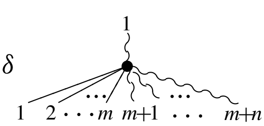
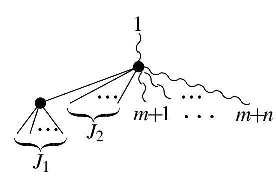
$$m \geq 1, n \geq 0, m+n \geq 2, \tag{5}$$

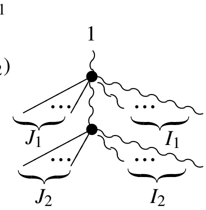
of degree $1-m$ with skewsymmetric m input legs in "straight" colour and symmetric n input legs in "wavy" colour,


$= (-1)^\sigma$

$=$


for any $\sigma \in \mathbb{S}_n$ and any $\tau \in \mathbb{S}_m$. The differential is given on the generating corollas by

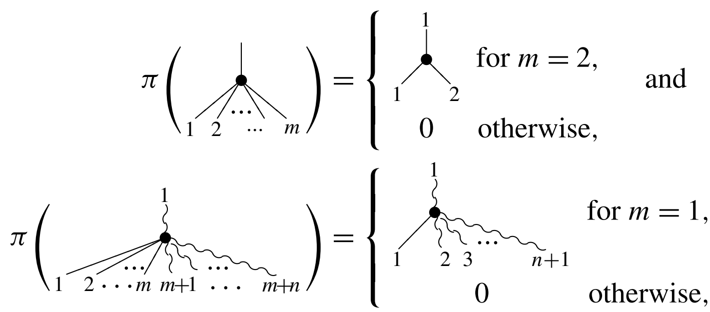
δ

$= \sum_{\substack{[m]=J_1\sqcup J_2\\ |J_1|\geq 2, |J_2|\geq 1}} (-1)^{J_1(J_2+1)+\sigma(J_1\sqcup J_2)}$

and

δ

$= \sum_{\substack{[m]=J_1\sqcup J_2\\ |J_1|\geq 2, |J_2|\geq 0}} (-1)^{(J_1+1)J_2+\sigma(J_1\sqcup J_2)}$


$- \sum_{\substack{[m]=J_1\sqcup J_2\\ [m+1,m+n]=I_1\sqcup I_2\\ |J_1|\geq 1, |J_2|\geq 1\\ |I_1|\geq 0, |I_2|\geq 0}} (-1)^{J_1(J_2+1)+\sigma(J_1\sqcup J_2)}$


where $(-1)^{\sigma(J_1\sqcup J_2)}$ is the sign of the permutation $[m] \to [J_1 \sqcup J_2]$.

Proof. It is a straightforward but tedious calculation to check that $\delta^2 = 0$. We define a projection $\pi : \mathcal{HS}_\infty \to \mathcal{HS}$ by its values on the generators,

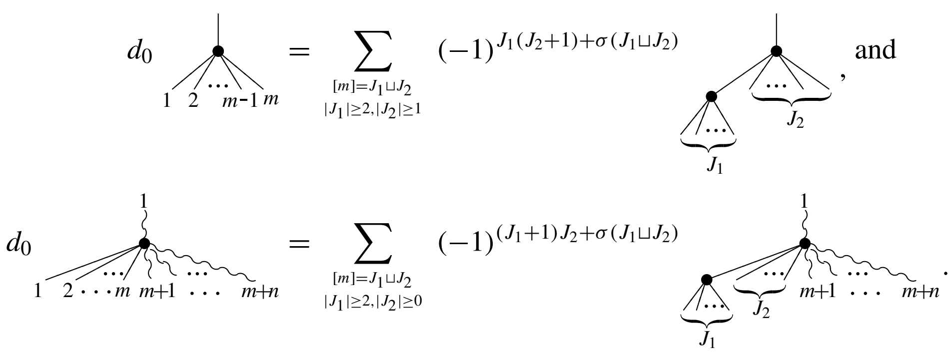


and notice that it commutes with the differentials and induces a surjection in cohomology. Thus to prove that π is a quasiisomorphism it is enough to show that the cohomology $H(\mathcal{HS}_\infty)$ is contained in $\mathcal{HS}$.

Let

$$\ldots \subset F_{-p} \subset F_{-p+1} \subset \ldots \subset F_0 = \mathcal{HS}_\infty$$

be a filtration with F_{-p} being a subspace of $\mathcal{HS}_\infty = \{\mathcal{HS}_\infty(n)\}_{n\geq 1}$ spanned by graphs with at least p wavy internal edges. This filtration is exhaustive and, as each $\mathcal{HS}_\infty(n)$ is a finite-dimensional vector space, bounded, and hence the associated spectral sequence $(E_r, d_r)_{r\geq 0}$ is convergent to $H(\mathcal{HS}_\infty)$. The 0-th term of this sequence has the differential given by

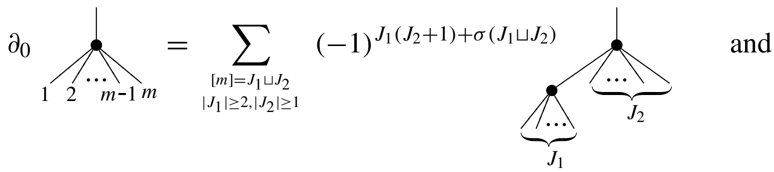


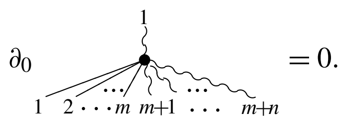
To compute the cohomology $H(E_0, d_0) = E_1$ we consider an increasing filtration,

$$0 \subset \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_p \subset \mathcal{F}_{p+1} \subset \ldots,$$

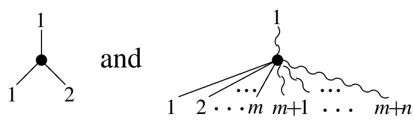
of E_0 with $\mathcal{F}_p$ being a subspace spanned by graphs whose vertices of type (5) have total homological degree $\geq -p$. It is again bounded and exhaustive so the associated spectral sequence, $\{\mathcal{E}_r, \partial_r\}_{r\geq 0}$, converges to E_1. The differential in $\mathcal{E}_0$

is given by

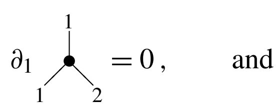


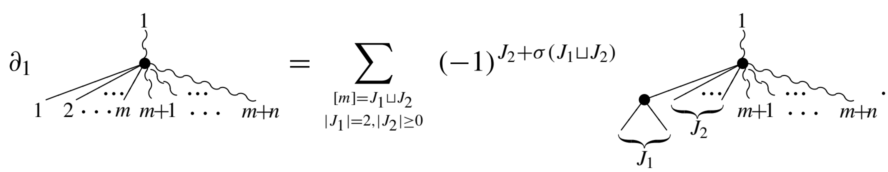


Thus modulo actions of finite groups, the complex $(\mathcal{E}_0, \partial_0)$ is isomorphic to the direct sum of tensor powers of the well-known complex $(\mathcal{L}_\infty, \delta)$, the minimal resolution of the operad of Lie algebras, tensored with trivial complexes. We conclude immediately that $\mathcal{E}_1 = H(\mathcal{E}_0, \partial_0)$ is a 2-coloured operad generated by corollas

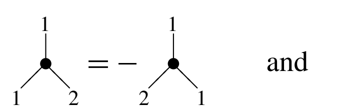


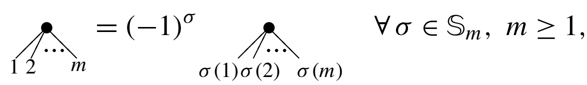
modulo relations (2). The differential ∂_1 in $\mathcal{E}_1$ is given on generators by



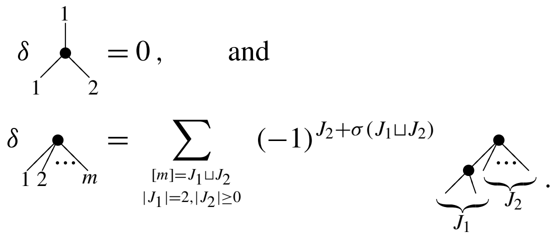


Thus, modulo actions of finite groups, the complex $(\mathcal{E}_1, \partial_1)$ is isomorphic to the direct sum of tensor products of trivial complexes with tensor powers of the dg properad $(\mathcal{S}, \delta)$ which is, by definition, generated by corollas





of degrees 0 and, respectively, $1-m$ modulo relation (2). The differential in $\mathcal{S}$ is given by



This complex (more precisely, a complex isomorphic to $\mathcal{S}$) was studied in [Merkulov 2008a, § 4.1.1] where it was proven that

$$H(\mathcal{S}, \delta) = \mathrm{span}\langle$$  $$\rangle.$$

Thus $\mathcal{E}_2$ is concentrated in degree 0 so all the other terms of both our spectral sequences degenerate and we get $\mathcal{E}_2 = \mathcal{E}_\infty = E_1 = E_\infty \simeq H(\mathcal{HS}_\infty)$. This fact implies that $H(\mathcal{HS}_\infty)$ is generated by corollas (1) modulo relations (2) and (3) completing thereby the proof. $\square$

Corollary 2.6.2. *The operad*, $\mathcal{HS}$, *of formal homogeneous spaces is Koszul.*

Proof. By Theorem 2.6.1, the operad $\mathcal{HS}$ admits a quadratic minimal model. The claim then follows from a straightforward analogue of [Merkulov and Vallette 2007, Theorem 34] (see also [Vallette 2007]) for coloured operads. $\square$

2.7. ***$\mathcal{HS}_\infty$-algebras as Maurer–Cartan elements.*** An $\mathcal{HS}_\infty$-*algebra* structure on a pair of dg vector spaces $(\mathfrak{g}, \mathfrak{h})$ is, by definition, a morphism of 2-coloured dg operads, $\rho : (\mathcal{HS}_\infty, \delta) \to (\mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}, d)$. First we give an explicit algebraic description of such a structure.

Proposition 2.7.1. *There is a one-to-one correspondence between $\mathcal{HS}_\infty$-algebra structures on a pair of dg vector spaces* $(\mathfrak{g}, \mathfrak{h})$ *and degree* 1 *codifferentials*, D, *in the free graded cocommutative coalgebra without counit*, $\bigodot^{\bullet \geq 1}(\mathfrak{g}[1] \oplus \mathfrak{h})$, *such that*

(a) *D respects the subcoalgebra $\bigodot^{\geq 1}(\mathfrak{g}[1])$, that is*

$$D\Big(\bigodot^{\geq 1}(\mathfrak{g}[1])\Big) \subset \bigodot^{\geq 1}(\mathfrak{g}[1]);$$

(b) *D respects the natural epimorphism of coalgebras*,

$$c : \bigodot^{\bullet \geq 1}(\mathfrak{g}[1] \oplus \mathfrak{h}) \to \bigodot^{\geq 1}(\mathfrak{g}[1]),$$

that is, $D \circ c = c \circ D$;

(c) *D is trivial on the subcoalgebra $\bigodot^{\geq 1}\mathfrak{h}$, that is*

$$D\Big(\bigodot^{\geq 1}\mathfrak{h}\Big)=0.$$

Proof. An arbitrary degree 1 coderivation, D, of $\bigodot^{\bullet\geq 1}(\mathfrak{g}[1]\oplus\mathfrak{h})$ is uniquely determined by two collections of degree 1 linear maps,

$$\Big\{D'_n : \bigodot^{n}(\mathfrak{g}[1]\oplus\mathfrak{h}) = \bigoplus_{p+q=n}\bigwedge^{p}\mathfrak{g}\otimes\bigodot^{q}\mathfrak{h}[p]\to\mathfrak{g}[1]\Big\}_{n\geq 1}\quad\text{and}$$

$$\Big\{D''_n : \bigodot^{n}(\mathfrak{g}[1]\oplus\mathfrak{h}) = \bigoplus_{p+q=n}\bigwedge^{p}\mathfrak{g}\otimes\bigodot^{q}\mathfrak{h}[p]\to\mathfrak{h}\Big\}_{n\geq 1}.$$

Conditions (a) and (b) say that D' is zero on all components $\bigwedge^{p}\mathfrak{g}\otimes\bigodot^{q}\mathfrak{h}[p]$ with $q\neq 0$, while condition (c) says that D'' is zero on all components $\bigwedge^{p}\mathfrak{g}\otimes\bigodot^{q}\mathfrak{h}[p]$ with $p=0$. Thus there is a one-to-one correspondence between degree 1 coderivations, D, in the coalgebra $\bigodot^{\bullet\geq 1}(\mathfrak{g}[1]\oplus\mathfrak{h})$, and morphisms of *non*-differential 2-coloured operads, $\rho:\mathcal{HS}_\infty\to\mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$, with D'_n being the values of ρ on corollas (4) and D''_n the values of ρ on corollas (5). Having established an explicit correspondence between coderivations D and morphisms ρ, it is now a straightforward computation (which we leave to the reader as an exercise) to check that the compatibility of ρ with the differentials, that is, the equation $\rho\circ\delta=d\circ\rho$, translates precisely into the equation $D^2=0$. □

Recall that a *L_∞-structure* on a vector space V is, by definition, a degree 1 codifferential μ in the free cocommutative coalgebra $\bigodot^{\geq 1}(V[1])$. It is often represented as a collection of linear maps,

$$\Big\{\mu_n:\bigwedge^{n}V\to V[2-n]\Big\}_{n\geq 1},$$

satisfying a system of quadratic equations which encode the relation $\mu^2=0$. Hence we can reformulate Proposition 2.7.1 in this language as follows.

Corollary 2.7.2. *There is a one-to-one correspondence between $\mathcal{HS}_\infty$-algebra structures on a pair of dg vector spaces $(\mathfrak{g},\mathfrak{h})$ and L_∞-structures, $\{\mu_n:\bigwedge^n V\to V[2-n]\}_{n\geq 1}$, on the vector space $V:=\mathfrak{g}\oplus\mathfrak{h}[-1]$ such that, for any $g_1,\dots,g_p\in\mathfrak{g}$ and $h_1,\dots,h_q\in\mathfrak{g}$ one has*

$$\pi_{\mathfrak{g}}\circ\mu_{p+q}(g_1,\dots,g_p,h_1,\dots,h_q)=0\quad\text{if } q\neq 1,\text{ and}$$
$$\pi_{\mathfrak{h}}\circ\mu_{p+q}(g_1,\dots,g_p,h_1,\dots,h_q)=0\quad\text{if } p=0,$$

where $\pi_{\mathfrak{g}}:V\to\mathfrak{g}$ and $\pi_{\mathfrak{h}}:V\to\mathfrak{h}[-1]$ are the natural projections.

It is straightforward to check that, for any dg spaces $\mathfrak{g}$ and $\mathfrak{h}$, the space of coderivations of the coalgebra $\bigodot^{\bullet\geq 1}(\mathfrak{g}[1]\oplus\mathfrak{h})$ which satisfy conditions (a)–(c) of Proposition 2.7.1 is closed with respect to the ordinary commutator, [,], of coderivations. Let us denote the Lie algebra of such coderivations by $(\mathscr{G}_{\mathfrak{g},\mathfrak{h}}, [\ ,\ ])$. As a vector space,

$$\mathscr{G}_{\mathfrak{g},\mathfrak{h}} \simeq \bigoplus_{n\geq 1} \mathrm{Hom}\Big(\bigwedge^n \mathfrak{g}, \mathfrak{g}\Big)[2-n] \;\oplus \bigoplus_{n\geq 1, p\geq 0} \mathrm{Hom}\Big(\bigwedge^n \mathfrak{g}\otimes \bigodot^p \mathfrak{h}, \mathfrak{h}\Big)[1-n].$$

Hence we get another useful reformulation of Proposition 2.7.1.

Corollary 2.7.3. *There is a one-to-one correspondence between $\mathscr{HS}_\infty$-algebra structures on a pair of dg vector spaces* $(\mathfrak{g},\mathfrak{h})$ *and Maurer–Cartan elements in the Lie algebra* $(\mathscr{G}_{\mathfrak{g},\mathfrak{h}}, [\ ,\ ])$.

Note that

$$\mathscr{G}_{\mathfrak{g},\mathfrak{h}} = \mathrm{Hom}_{\mathbb{S}}(E, \mathscr{E}nd_{\mathfrak{g},\mathfrak{h}})[-1]$$

where E is the $\mathbb{S}$-bimodule spanned as a vector space by corollas (4) and (5). The Lie algebra we got above in Corollary 2.7.3 is an independent confirmation of the general principle (II) in Section 1.1 (which is the same as [Merkulov and Vallette 2007, Theorem 58]). Hence, applying next principle (III) (or [Merkulov and Vallette 2007, Proposition 66]) we may conclude this subsection with the following observation.

Fact 2.7.4. Let γ be an $\mathscr{HS}_\infty$-algebra structure, $\mathscr{HS}_\infty \xrightarrow{\gamma} \mathscr{E}_{\mathfrak{g},\mathfrak{h}}$, on a pair of dg spaces $\mathfrak{g}$ and $\mathfrak{h}$. The deformation theory of γ is then controlled by the dg Lie algebra $(\mathscr{G}_{\mathfrak{g},\mathfrak{h}}, [\ ,\ ], d := [\gamma,\ ])$.

2.8. ***Geometric interpretations of $\mathscr{HS}_\infty$-algebras.*** There are two ways to understand $\mathscr{HS}_\infty$-algebras geometrically.

The first one uses the language of formal manifolds [Kontsevich 2003]. Let $\mathscr{X}$ be a formal manifold associated with the coalgebra $\bigodot^{\bullet\geq 1}(\mathfrak{g}[1])$ and let $\mathscr{E}$ be a formal manifold associated with the total space of the trivial bundle over $\mathscr{X}$ with typical fiber $\mathfrak{h}$. The structure sheaf of $\mathscr{E}$ is then the coalgebra $\bigodot^{\bullet\geq 1}(\mathfrak{g}[1]\oplus\mathfrak{h})$. We have a natural projection of formal manifolds $\pi : \mathscr{E}\to\mathscr{X}$ and an embedding, $\mathscr{X}\subset\mathscr{E}$, of $\mathscr{X}$ into $\mathscr{E}$ as a zero section. Then a $\mathscr{HS}_\infty$-algebra structure on a pair of vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ is the same as a homological vector field on $\mathscr{E}$ which is tangent to the submanifold $\mathscr{X}$ and vanishes on the fiber of the projection π.

Another geometric picture uses an idea of L_∞-homogeneous formal manifolds:

Proposition 2.8.1. *There is a one-to-one correspondence between representations*,

$$\rho : \mathscr{HS}_\infty \longrightarrow \mathscr{E}nd_{\mathfrak{g},\mathfrak{h}},$$

and the triples, $(\mathfrak{g}, \mathfrak{h}, F_\infty)$, *consisting of a* L_∞*-algebra* $\mathfrak{g}$, *a complex* $(\mathfrak{h}, d)$ *and a* L_∞*-morphism*,

$$F_\infty : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}},$$

where $\mathcal{T}_{\mathfrak{h}}$ *is viewed as a dg Lie algebra equipped with the ordinary commutator*, $[\ ,\ ]$, *of vector fields and with the differential* ∂ *defined by*

$$\partial V := [d, V], \quad \forall V \in \mathcal{T}_{\mathfrak{h}},$$

where d *is interpreted as a linear vector field on* $\mathfrak{h}$.

The proof is a straightforward calculation (see Section 2.5). We omit the details.

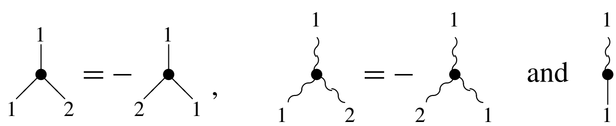
3. Operad of Lie pairs and its minimal resolution

3.1. ***Definition.*** A *Lie pair* is a collection of data $(\mathfrak{g}, \mathfrak{h}, \phi)$ consisting of

(i) Lie algebras $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}})$, and

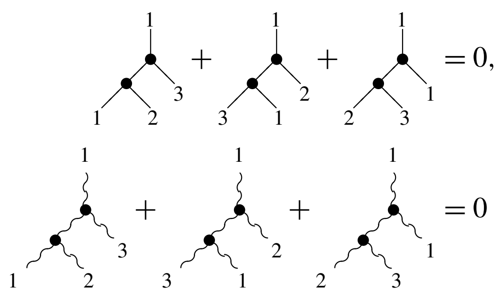
(ii) a morphism, $\phi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras.

Let $\mathcal{LP}$ be the 2-coloured operad whose representations, $\mathcal{LP} \to \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$, are structures of Lie pairs on the vector spaces $\mathfrak{g}$ and $\mathfrak{h}$. This operad *of Lie pairs*, $\mathcal{LP}$, is, therefore, generated by the corollas



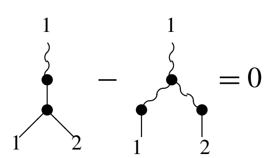
(6)

(which correspond, respectively, to the Lie brackets, $[\ ,\ ]_{\mathfrak{g}}$, Lie brackets $[\ ,\ ]_{\mathfrak{h}}$ and the morphism ϕ) modulo the relations



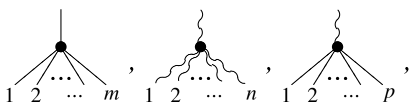
(7)

(corresponding to the Jacobi identities for $[\ ,\ ]_{\mathfrak{g}}$ and $[\ ,\ ]_{\mathfrak{h}}$), and



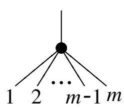
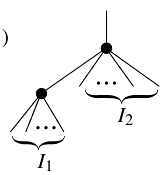
(corresponding to the compatibility of ϕ with Lie brackets). It is well-known [Markl et al. 2002] that the minimal resolution of $\mathcal{LP}$ is a dg free 2-coloured operad, $\mathcal{LP}_\infty$, whose representations, $\mathcal{LP}_\infty \to \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}$, describe L_∞-algebra structures in vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ together with a morphism, $\phi_\infty : \mathfrak{g} \to \mathfrak{h}$, of L_∞-algebras. For completeness of the paper we show below a new short proof of this result.

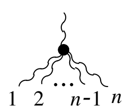
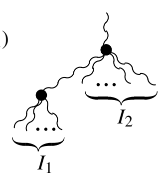
Theorem 3.1.1. *The minimal resolution, $\mathcal{LP}_\infty$, of the operad of Lie pairs is a free 2-coloured operad generated by three families of corollas with skewsymmetric input legs,*

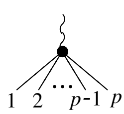
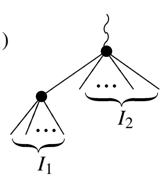


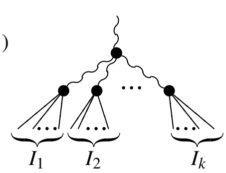
$m \geq 2, n \geq 2, p \geq 1,$

of degrees $2-m$, $2-n$ and $1-p$ respectively, and equipped with the differential given by

δ  $= \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 2, |I_2|\geq 1}} (-1)^{I_1(I_2+1)+\sigma(I_1\sqcup I_2)}$  ,

δ  $= \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 2, |I_2|\geq 1}} (-1)^{I_1(I_2+1)+\sigma(I_1\sqcup I_2)}$  ,

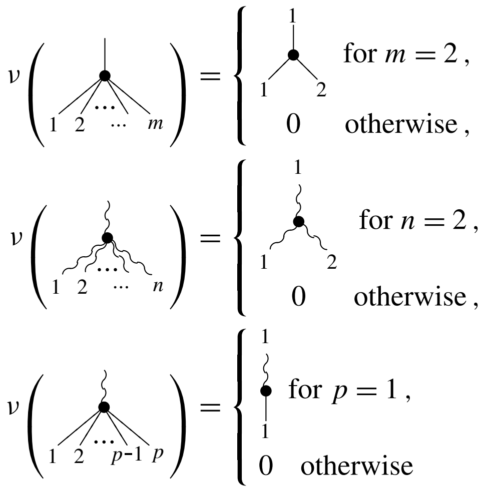
δ  $= \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 2, |I_2|\geq 1}} (-1)^{I_2(I_1+1)+\sigma(I_1\sqcup I_2)}$  (8)

$+ \sum_{\substack{[p]=I_1\sqcup\ldots\sqcup I_k\\ |I_i|\geq 1, k\geq 2}} (-1)^{\varepsilon+\sigma(I_1\sqcup\ldots\sqcup I_k)}$  ,

where

$$\varepsilon = 1 + \sum_{i=1}^{k-1} I_i(i-1+I_{i+1}+\ldots+I_k).$$

Proof. The projection $\nu : \mathcal{LP}_\infty \to \mathcal{LP}$ defined by

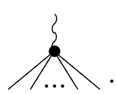


commutes with the differentials and is obviously surjective in cohomology. Thus to prove that π is a quasiisomorphism it is enough to show that $H(\mathcal{HS}_\infty) = \mathcal{HS}$ which in turn would follow if one proves that the cohomology $H(\mathcal{HS}_\infty)$ is concentrated in degree zero.

Let

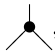
$$\ldots \subset F_{-q} \subset F_{-q+1} \subset \ldots \subset F_0 = \mathcal{HS}_\infty$$

be a filtration with F_{-q} being a subspace of $\mathcal{LP}_\infty = \{\mathcal{LP}_\infty(n)\}_{n\geq 1}$ spanned by graphs with at least q vertices of the form



This filtration is exhaustive and, as each $\mathcal{L}_\infty\mathcal{P}(n)$ is a finite-dimensional vector space, bounded, and hence the associated spectral sequence $(E_r, d_r)_{r\geq 0}$ is convergent to $H(\mathcal{HS}_\infty)$. The 0-th term of this sequence has the differential given by formulae (8) without the second sum. Hence the complex (E_0, d_0) is isomorphic, modulo actions of finite groups, to the tensor products of trivial complexes with two copies of the classical complex $\mathcal{L}_\infty$ and the complex $\mathcal{S}$ defined in the proof of Theorem 2.6.1. Hence its cohomology $E_1 = H(E_0, d_0)$ is generated by corollas (6) and is concentrated, therefore, in degree 0. This proves that $H(\mathcal{HS}_\infty)$ is concentrated in degree zero which in turn implies the required result. □

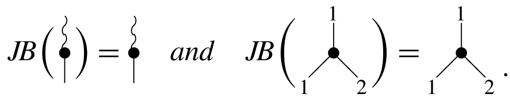
4. The Jacobi–Bernoulli morphism and its strong homotopy generalization

4.1. ***The Jacobi–Bernoulli morphism.*** The following result shows that, modulo actions of the dilation group $\mathbb{K}^*$ on the operad $\mathcal{HS}$ (see Section 2.5.2), there exists a *unique* nontrivial morphism of 2-coloured operads $\mathcal{HS} \to \mathcal{LP}$ which is identity on the generators, , with pure “straight” colour.

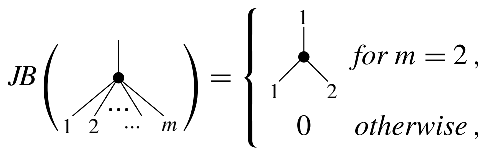
Theorem 4.1.1. *There is a unique morphism of* 2*-coloured dg operads*,

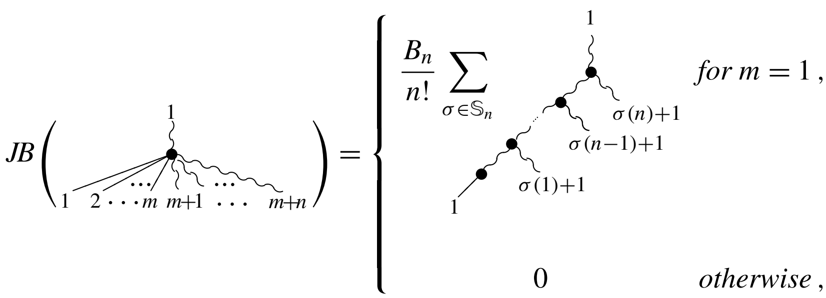
$$JB : (\mathcal{HS}_\infty, \delta) \longrightarrow (\mathcal{LP}, 0)$$

such that



It is given on the generators by





where B_n *are the Bernoulli numbers, that is,* $\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}$*, in particular,* $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$*, and so on.*

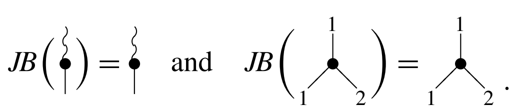
Proof. Since $\mathcal{LP}$ is concentrated in degree zero, the required morphism factors through the canonical projection,

$$JB : (\mathcal{HS}_\infty, \delta) \xrightarrow{\pi} (\mathcal{HS}, 0) \longrightarrow (\mathcal{LP}, 0),$$

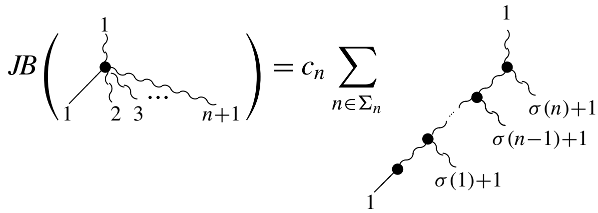
for some morphism of 2-coloured operads, $\mathcal{HS} \longrightarrow \mathcal{LP}$, which we denote by the same letter JB. Thus to prove Theorem 4.1.1 we have to show the existence of a unique morphism of operads,

$$JB : \mathcal{HS} \longrightarrow \mathcal{LP},$$

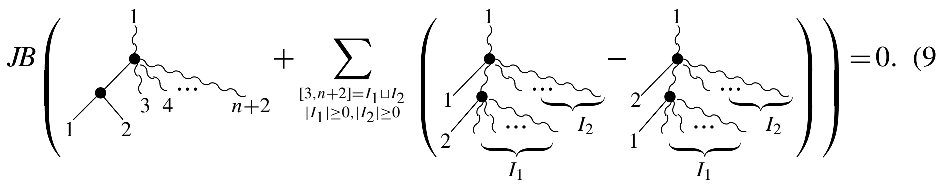
such that



For equivariance reasons it must be of the form



for some $c_n \in \mathbb{K}$ with $c_0 = 1$. Thus to prove the theorem we have to show that there exists a unique collection of numbers $\{c_n\} \in \mathbb{K}$ such that, for any $n \geq 0$,



We claim that the above equation gives an iterative procedure which uniquely specifies c_{n+1} in terms of $c_{\leq n}$ starting with $c_0 = 1$. Equation (9) is a sum of elements of the operad $\mathscr{LP}$ with all input legs of "wavy" colour being symmetrized; it is easier to control the relevant combinatorics by slightly changing the viewpoint: Equation (9) holds in $\mathscr{LP}$ if and only if it holds for an arbitrary representation $\rho_{\mathfrak{g},\mathfrak{h}} : \mathscr{LP} \to \mathscr{E}nd_{\mathfrak{g},\mathfrak{h}}$, that is, if for arbitrary pair of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and a morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ one has $\rho_{\mathfrak{g},\mathfrak{h}}(9) \equiv 0$, which, as it is not hard to see, is equivalent to the following system of equations (with $c_0 = 1$),

$$\sum_{\substack{0\leq k,l\leq n\\ k+l\leq n}} c_k c_{n+1-k}\big([\phi(a_1)@b^l, \phi(a_2)@b^k] - [\phi(a_2)@b^l, \phi(a_1)@b^k]\big)@b^{n-k-l}$$
$$+ c_n[\phi(a_1), \phi_2(a_2)]@b^n = 0 \tag{10}$$

for any $a_1, a_2 \in \mathfrak{g}$ and $b \in \mathfrak{h}$. Here we used the notation of Ran [2004]

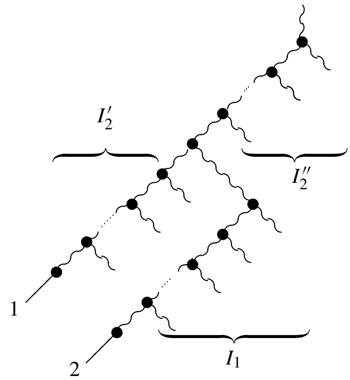
$$x@b^k := [\ldots[[x, \underbrace{b], b], \ldots, b]}_{k}, \qquad \forall\, x, b \in \mathfrak{h}.$$

The second summand in (10) corresponds to the first summand in (9) while the summand

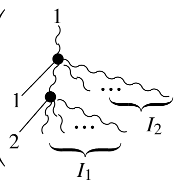
$$c_k c_{n+1-k}[\phi(a_1)@b^l, \phi(a_2)@b^k]@b^{n-k-l}$$

in (10) corresponds to the summand

$c_k c_{n+1-k}$



in the image

JB (), $|I_1| = k,\ |I_2| = n-k,$

which is uniquely determined by a decomposition, $I_2 = I_2' \sqcup I_2''$, of the indexing set I_2 into two disjoint subsets with

$$|I_2'| = l \text{ (and hence with } |I_2''| = n-k-l).$$

Equation (10) can be rewritten as follows,

$$\begin{aligned}
&c_{n+1}\sum_{k=0}^{n}\big([\phi(a_1),\phi(a_2)@b^k]@b^{n-k} - [\phi(a_2),\phi(a_1)@b^k]@b^{n-k}\big)\\
&= -c_n[\phi(a_1),\phi_2(a_2)]@b^n\\
&- \sum_{\substack{1\le k\le n\\ 0\le l\le n\\ k+l\le n}} c_k c_{n+1-k}\big([\phi(a_1)@b^l,\phi(a_2)@b^k] - [\phi(a_2)@b^l,\phi(a_1)@b^k]\big)@b^{n-k-l}.
\end{aligned}$$

implying that if system (9) has a solution, then it is unique. For example, for $n=0$, we have

$$2c_1 = -c_0 = -1,$$

while for $n=1$,

$$3c_2 = -c_1 - c_1^2 = \frac{1}{4}.$$

It was proven in [Ran 2004] (see Equation 1.2.4 there) that the collection $c_n = B_n/n!$, where B_n are the Bernoulli numbers, does solve system of equations (10), completing the proof of existence and uniqueness of the morphism JB. □

Corollary 4.1.2. *For every morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ there is a canonically associated structure of formal $\mathfrak{g}$-homogeneous space on $\mathfrak{h}$, that is, a morphism of Lie algebras,*

$$F_\phi : \mathfrak{g} \longrightarrow \mathcal{T}_{\mathfrak{h}},$$

given in local bases $\{e_a\}$ in $\mathfrak{g}$ and $\{e_\alpha\}$ in $\mathfrak{h}$ as follows,

$$F_\phi(e_a) = \sum_{n\geq 0} \frac{B_n}{n!} \phi_a^{\gamma_1} C_{\gamma_1\beta_1}^{\gamma_2} C_{\gamma_2\beta_2}^{\gamma_3} \dots C_{\gamma_n\beta_n}^{\alpha} t^{\beta_1} t^{\beta_2} \dots t^{\beta_n} \frac{\partial}{\partial t^\alpha}, \tag{11}$$

where $C_{\alpha\beta}^{\gamma}$ are the structure constants of Lie brackets in $\mathfrak{h}$, $[e_\alpha, e_\beta] = \sum_\gamma C_{\alpha\beta}^{\gamma} e_\gamma$, ϕ_a^α are the structure constants of the morphism ϕ, $\phi(e_a) = \sum_\alpha \phi_a^\alpha e_\alpha$, and $\{t^\alpha\}$ is the dual basis in $\mathfrak{h}^*$.

Corollary 4.1.3. *For every morphism of dg Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ there is a canonically associated codifferential, D_ϕ, in the free coalgebra $J := \bigodot^\bullet(\mathfrak{g}[1] \oplus \mathfrak{h})$ making the data (J, D_ϕ) into the Jacobi–Bernoulli complex as defined in [Ran 2004].*

Proof. Morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ gives rise to an associated morphism of 2-coloured operads,

$$\rho_\phi : \mathcal{LP} \longrightarrow \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}}.$$

The composition,

$$\gamma_\phi : \mathcal{HS}_\infty \xrightarrow{JB} \mathcal{LP} \xrightarrow{\rho_\phi} \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}},$$

gives, by Proposition 2.7.1, rise to an associated codifferential D_ϕ. The rest of the proof is just a comparison of D_ϕ with the codifferential defined in [Ran 2004]. □

Corollary 4.1.4. *For every morphism of dg Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ there is a canonically associated L_∞-algebra structure on the vector $\mathfrak{g} \oplus \mathfrak{h}[-1]$.*

Proof. The claimed L_∞-structure is given by the morphism of operads γ_ϕ as above and Corollary 2.7.3. A straightforward inspection shows that this structure is identical to the one constructed in [Fiorenza and Manetti 2007] with the help of the explicit homotopy transfer formulae of [Kontsevich and Soibelman 2000; Merkulov 1999]. □

4.2. ***Morphisms of L_∞-algebras.*** The following theorem generalizes all the above constructions to the case of an arbitrary morphism, $\phi_\infty : \mathfrak{g} \to \mathfrak{h}$, of L_∞-algebras. The proof given below provides us with an iterative construction of the morphism JB_∞ (and hence with the associated differential in the Jacobi–Bernoulli complex or, equivalently, a L_∞-algebra structure on the mapping cone $\mathfrak{g} \oplus \mathfrak{h}[-1]$).

Theorem 4.2.1. *There exists a morphism of 2-coloured dg operads,*

$$JB_\infty : (\mathcal{HS}_\infty, \delta) \longrightarrow (\mathcal{LP}_\infty, \delta)$$

making the diagram,

$$\begin{array}{ccc} \mathcal{HS}_\infty & \xrightarrow{JB_\infty} & \mathcal{LP}_\infty \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \nu} \\ \mathcal{HS} & \xrightarrow[JB]{} & \mathcal{LP} \end{array}$$

commutative.

Proof. We have a solid arrow diagram,

$$\begin{array}{ccc} & & \mathcal{LP}_\infty \\ & {\scriptstyle JB_\infty}\nearrow & \downarrow{\scriptstyle \nu} \\ \mathcal{HS}_\infty & \xrightarrow[JB]{} & \mathcal{LP} \end{array}$$

with morphism ν being a surjective quasiisomorphism and the operad $\mathcal{HS}_\infty$ being cofibrant. Then the existence of the dotted arrow JB_∞ making the diagram above commutative follows immediately from the model category structure on operads. In fact, one can see it directly using an analogue of the classical Whitehead lifting trick (first used in the theory of CW complexes in algebraic topology): let ν^{-1} be an arbitrary section of the surjection ν in the category of dg spaces; as a first step in the inductive procedure we set $JB_\infty(C_0) := \nu^{-1} \circ JB(C_0)$ on degree 0 generating corollas, C_0, of the operad $\mathcal{HS}_\infty$. Assume by induction that the values of a morphism JB_∞ are already defined on all generating corollas of degrees $\geq -r$, and let C_{r+1} be a generating corolla of degree $-r-1$. As $\delta(C_{r+1})$ is a linear combination of graphs built from corollas of degrees $\geq -r$, $JB_\infty(\delta C_{r+1})$ is a well-defined element of $\mathcal{LP}_\infty$. Moreover, as JB_∞ commutes, by the induction assumption, with the differentials, we have an equation in the complex $(\mathcal{LP}_\infty, \delta)$,

$$\delta JB_\infty(\delta C_{r+1}) = 0.$$

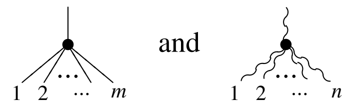
Since there are no nontrivial cohomology classes in $(\mathcal{LP}_\infty, \delta)$ of degree $-r$ for $r \geq 1$, we must have,

$$JB_\infty(\delta C_{r+1}) = \delta e_{r+1}$$

for some $e_{r+1} \in \mathcal{LP}_\infty$. We finally set $JB_\infty(C_{r+1}) := e_{r+1}$ completing thereby the inductive construction of the required morphism JB_∞. □

Corollary 4.2.2. *For every morphism of L_∞-algebras, $\phi_\infty : \mathfrak{g} \to \mathfrak{h}$, there is an associated codifferential, D_∞, in the free coalgebra $J := \bigodot^\bullet(\mathfrak{g}[1] \oplus \mathfrak{h})$ which coincides precisely with Ziv Ran's Jacobi–Bernoulli codifferential in the case of $\mathfrak{g}$, $\mathfrak{h}$ being dg Lie algebras and ϕ a morphism of dg Lie algebras.*

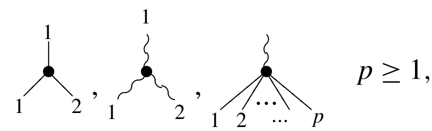
4.3. ***L_∞-morphisms of dg Lie algebras.*** Let I be the ideal in the free nondifferential operad $\mathcal{LP}_\infty$ generated by corollas



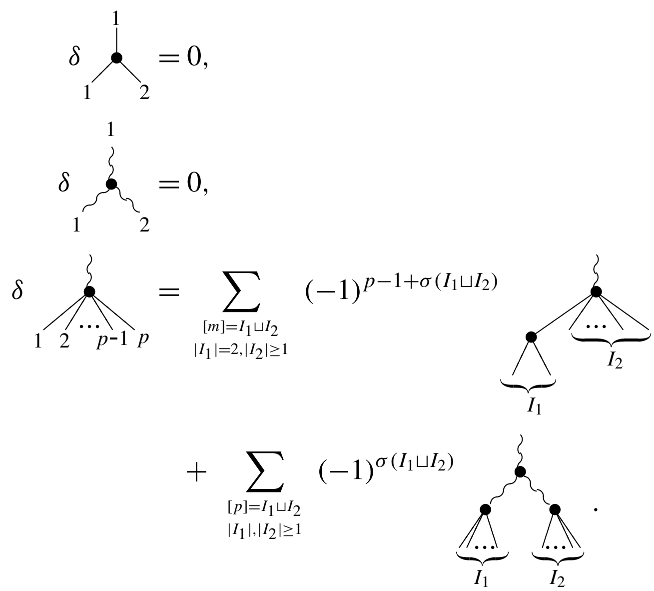
with $m \geq 3$ and $n \geq 3$, and let (I, dI) be the differential closure of I in the dg operad $(\mathcal{LP}_\infty, \delta)$. The quotient operad,

$$\mathcal{LP}_{\frac{1}{2}\infty} := \frac{\mathcal{LP}_\infty}{(I, dI)}$$

is a differential 2-coloured operad generated by corollas



modulo relations (7); the differential is given on the generators by



Representations,

$$\mathcal{LP}_{\frac{1}{2}\infty} \to \mathcal{E}nd_{\mathfrak{g},\mathfrak{h}},$$

of this dg operad are the same as triples, $(\mathfrak{g}, \mathfrak{h}, F_\infty)$, consisting of ordinary dg Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ together with a L_∞-morphism, $F_\infty : \mathfrak{g} \to \mathfrak{h}$, between them. Thus this operad describes a special class of representations of the operad $\mathcal{LP}_\infty$ which is, probably, the most important one in applications. For example, for any smooth manifold M, the triple, $(\bigwedge^\bullet \mathcal{T}_M, \mathcal{D}_M^{poly}, F_K)$, consisting of a Schouten Lie algebra of polyvector fields on M, the Hochschild dg Lie algebra, $\mathcal{D}_M^{poly}$, of polydifferential operators and Kontsevich's formality morphism

$$F_K : \bigwedge^{\bullet} \mathcal{T}_M \to \mathcal{D}_M^{poly}$$

is a representation of $\mathcal{LP}_{\frac{1}{2}\infty}$.

It is not hard to describe *explicitly* the quotient part,

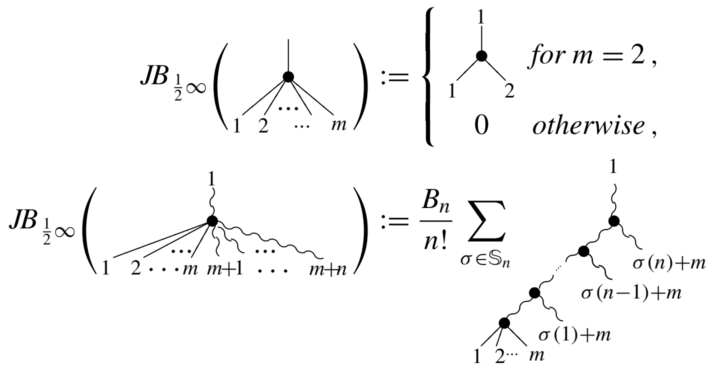
$$JB_{\frac{1}{2}\infty} : \mathcal{HS}_\infty \xrightarrow{JB_\infty} \mathcal{LP}_\infty \xrightarrow{proj} \mathcal{LP}_{\frac{1}{2}\infty},$$

of a morphism JB_∞.

Theorem 4.3.1. *The morphism of* 2*-coloured dg operads*,

$$JB_{\frac{1}{2}\infty} : (\mathcal{HS}_\infty, \delta) \longrightarrow \left(\mathcal{LP}_{\frac{1}{2}\infty}, \delta\right)$$

is given on the generators by



where B_n are the Bernoulli numbers.

The proof is similar to that of Theorem 4.1.1. We omit the details.

Acknowledgement

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sm@math.su.se *Department of Mathematics, Stockholm University, 10691 Stockholm, Sweden*